Cauchy Sequences: Problems and Solutions

Problem 1

Prove that the sequence (x_n) defined by $x_n = \frac{1}{n}$ is a Cauchy sequence in $\mathbb{R}$.

Solution:

To show that (x_n) is a Cauchy sequence, we need to demonstrate that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$|x_n - x_m| < \epsilon.$$

Step 1: Choose N Based on ϵ

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that:

$$N > \frac{2}{\epsilon}.$$

Step 2: Estimate $|x_n - x_m|$

For all $n, m \geq N$:

$$|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right|.$$

Step 3: Use Inequalities

Without loss of generality, assume $n \geq m$. Then:

$$|x_n - x_m| = \frac{1}{m} - \frac{1}{n} \leq \frac{1}{m} - \frac{1}{N}.$$

But since $m \geq N$, we have:

$$\frac{1}{m} \leq \frac{1}{N}.$$

Therefore:

$$|x_n - x_m| \leq \frac{1}{N} - \frac{1}{N} = 0.$$

This suggests that the difference is actually zero, which contradicts our earlier inequality. Let's correct this step.

Corrected Step 3:

Consider:

$$|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m - n}{mn} \right| \leq \frac{|m - n|}{N^2}.$$

Since $m, n \geq N$, $|m - n|$ is at least zero. However, regardless of m and n, $|m - n| \leq |m| + |n|$.

But this approach complicates things. A better estimation is:

$$|x_n - x_m| \leq \frac{1}{N} \leq \frac{\epsilon}{2} < \epsilon.$$

Conclusion:

Thus, for all $m, n \geq N$:

$$|x_n - x_m| < \epsilon.$$

Therefore, (x_n) is a Cauchy sequence in $\mathbb{R}$.

Problem 2

Prove that every convergent sequence in a metric space is a Cauchy sequence.

Solution:

Let (x_n) be a sequence in a metric space (M, d) that converges to some limit $L \in M$. We need to show that (x_n) is a Cauchy sequence.

Step 1: Definition of Convergence

By definition, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$d(x_n, L) < \frac{\epsilon}{2}.$$

Step 2: Use the Triangle Inequality

For all $m, n \geq N$:

$$d(x_n, x_m) \leq d(x_n, L) + d(L, x_m) = d(x_n, L) + d(x_m, L).$$

Step 3: Apply the Convergence Condition

Since $d(x_n, L) < \frac{\epsilon}{2}$ and $d(x_m, L) < \frac{\epsilon}{2}$:

$$d(x_n, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conclusion:

Therefore, (x_n) is a Cauchy sequence.

Problem 3

Show that in $\mathbb{R}$, every Cauchy sequence converges.

Solution:

The real numbers $\mathbb{R}$ form a complete metric space, meaning every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$.

Step 1: A Cauchy Sequence Is Bounded

Since (x_n) is Cauchy, set $\epsilon = 1$ or any fixed number,there exists N such that for all $m, n \geq N$:

$$|x_n - x_m| < 1.$$

This is a frequently used trick, contemplate on that. This implies that all x_n for $n \geq N$ are within a bounded interval.

Step 2: Apply the Bolzano-Weierstrass Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. Let (x_{n_k}) be a subsequence of (x_n) that converges to some $L \in \mathbb{R}$.

Step 3: Show the Entire Sequence Converges to L

Given $\epsilon > 0$, since (x_n) is Cauchy, there exists N_1 such that for all $m, n \geq N_1$:

$$|x_n - x_m| < \frac{\epsilon}{2}.$$

Since $x_{n_k} \to L$, there exists N_2 such that for all $k \geq N_2$:

$$|x_{n_k} - L| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, n_{N_2}\}$. For all $n \geq N$:

$$|x_n - L| \leq |x_n - x_{n_{N_2}}| + |x_{n_{N_2}} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Conclusion:

Thus, $x_n \to L$ in $\mathbb{R}$, and every Cauchy sequence in $\mathbb{R}$ converges.

Problem 4

Provide an example of a Cauchy sequence in $\mathbb{Q}$ (the rational numbers) that does not converge in $\mathbb{Q}$.

Solution:

Example Sequence:

Consider the sequence (x_n) where x_n is the decimal expansion of $\sqrt{2}$ truncated at n decimal places. Each x_n is rational, and the sequence approximates $\sqrt{2}$.

Step 1: Show (x_n) Is Cauchy in $\mathbb{Q}$

For m, n large:

$$|x_n - x_m| < \frac{1}{10^{\min\{n,m\}}} \to 0 \text{ as } n, m \to \infty.$$

Thus, for any $\epsilon > 0$, there exists N such that for all $m, n \geq N$:

$$|x_n - x_m| < \epsilon.$$

Step 2: (x_n) Does Not Converge in $\mathbb{Q}$

Since $\sqrt{2} \notin \mathbb{Q}$, the sequence cannot converge to a rational number.

Conclusion:

(x_n) is a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$, illustrating that $\mathbb{Q}$ is not complete.

Problem 5

Determine whether the sequence (x_n) defined by $x_n = (-1)^n$ is a Cauchy sequence in $\mathbb{R}$.

Solution:

Step 1: Examine the Behavior of (x_n)

The sequence alternates between 1 and -1:

$$x_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Step 2: Test the Cauchy Condition

Let $\epsilon = 1$. For any $N \in \mathbb{N}$, choose $n = 2N$ and $m = 2N + 1$:

$$|x_n - x_m| = |1 - (-1)| = 2 \geq 1.$$

Conclusion:

Since $|x_n - x_m| \geq 2$ infinitely often, (x_n) is not a Cauchy sequence.

Problem 6

Use the Cauchy Criterion to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

Solution:

Step 1: State the Cauchy Criterion for Series

A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$, there exists N such that for all $m > n \geq N$:

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.$$

Step 2: Apply to the Harmonic Series

Consider the partial sums:

$$S_m - S_n = \sum_{k=n+1}^{m} \frac{1}{k}.$$

Step 3: Estimate the Partial Sums

Using the integral test:

$$\sum_{k=n+1}^{m} \frac{1}{k} \geq \int_{n+1}^{m+1} \frac{1}{x}\, dx = \ln(m+1) - \ln(n+1).$$

Step 4: Analyze the Limit
As $m \to \infty$:

$$\lim_{m\to\infty}(S_m - S_n) \geq \lim_{m\to\infty}[\ln(m+1) - \ln(n+1)] = \infty.$$

Conclusion:
Since $(S_m - S_n)$ can be made arbitrarily large, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Problem 7

Let (x_n) be a sequence defined recursively by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$. Show that (x_n) is a Cauchy sequence.

Solution:
This sequence is a particular example of Babylonian method to approximate $\sqrt{x}$, here $x = 2$.
Step 1: Show the Sequence Is Monotonic and Bounded

(a) **Monotonicity:** We can show that x_n is increasing.

(b) **Boundedness:** We can show that $x_n \leq \sqrt{2} + \delta$ for some $\delta > 0$.

Step 2: Show Convergence
Since (x_n) is increasing and bounded above, it converges to some $L \geq 0$.
Step 3: Find the Limit
Taking limits on both sides of the recursive formula:

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right) \implies 2L = L + \frac{2}{L} \implies L^2 = 2.$$

Thus, $L = \sqrt{2}$.
Step 4: Conclude That (x_n) Is Cauchy
Since (x_n) converges in $\mathbb{R}$, it is a Cauchy sequence.

*Problem 8

Show that the open interval $(0, 1)$ with the usual metric is not complete by exhibiting a Cauchy sequence in $(0, 1)$ that does not converge in $(0, 1)$.

Solution:
Example Sequence:
Consider $x_n = 1 - \frac{1}{n}$. Each $x_n \in (0, 1)$.
Step 1: Show (x_n) Is Cauchy in $(0, 1)$
Given $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$. For all $m, n \geq N$:

$$|x_n - x_m| = \left|\left(1 - \frac{1}{n}\right) - \left(1 - \frac{1}{m}\right)\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{N} < \epsilon.$$

Step 2: Show (x_n) Does Not Converge in $(0, 1)$
The sequence converges to 1, which is not in $(0, 1)$.
Conclusion:
(x_n) is a Cauchy sequence in $(0, 1)$ that does not converge within $(0, 1)$. Thus, $(0, 1)$ is not complete.