ASSIGMENT 3. Matrix Inversion and Eigenvalue Problem

Score: 20 points
Deadline for submission: Week 4 before your group's practical lesson.
Requirements: Implement the algorithms for the specified methods (Iterative Method, LU factorization, Power method, Jacobi's method, Given's Method, House-Holder's Method).

1 General description

This assignment addresses key techniques for solving matrix inversion and eigenvalue problems. Matrix inversion is crucial for solving systems of linear equations and understanding the properties of linear transformations, while eigenvalue problems are fundamental in analyzing the stability and behavior of dynamic systems. These topics explore a range of methods designed to efficiently compute solutions to these problems, starting with iterative techniques suitable for large-scale matrices where exact solutions may be computationally expensive or unnecessary. Among the methods discussed are *LU factorization* (see Section (2.2)), which decomposes a matrix into lower and upper triangular forms, simplifying both matrix inversion and the solving of linear systems, and *the Power Method*, an iterative approach particularly useful for estimating the largest eigenvalue of a matrix. The topic also covers specialized algorithms for symmetric matrices, including *Jacobi's method* and *Given's method*, both of which simplify the matrix structure to aid in the computation of eigenvalues. Additionally, *Householder's method* is introduced as a powerful technique for reducing matrices to simpler forms, enhancing computational efficiency and numerical stability. To consolidate understanding, practical tasks are provided, allowing readers to apply these methods to real-world problems, and a list of further resources is included for those wishing to explore these topics in greater depth.

Matrix inversion and eigenvalue problems are fundamental topics in linear algebra with wide-ranging applications in various scientific and engineering disciplines. The ability to invert matrices and solve for eigenvalues is crucial in fields such as physics, engineering, computer science, economics, and data science, where systems of equations, stability analysis, and data transformation are commonplace.

2 Matrix inversion

Given a square matrix $\mathbf{A}$, its inverse $\mathbf{A}^{-1}$ is defined as the matrix that satisfies the equation

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}, \tag{0.1}$$

where $\mathbf{I}$ is the identity matrix. $\mathbf{A}$ matrix is invertible (non-singular) if its determinant is non-zero. Inverse matrices are essential for solving systems of linear equations $\mathbf{Ax}=\mathbf{b}$, where the solution $\mathbf{x}$ can be obtained as $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$.

2.1 Iterative method

Suppose we want to compute the inverse of a matrix $\mathbf{A}$, and we have an approximate inverse $\mathbf{B}$ such that:

$$\mathbf{A}^{-1} \approx \mathbf{B}.$$

To analyze the accuracy of this approximation, we can define an error matrix $\mathbf{E}$, which quantifies how far off our approximation is from the true inverse. The error matrix is defined as:

$$\mathbf{E} = \mathbf{AB} - \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. This leads us to the equation:

$$\mathbf{AB} = \mathbf{I} + \mathbf{E}.$$

Next, we can express the relationship between $\mathbf{A}$ and its approximate inverse $\mathbf{B}$ in terms of the error matrix:

$$(\mathbf{AB})^{-1} = (\mathbf{I} + \mathbf{E})^{-1}.$$

Using properties of inverses, we can rewrite this as:

$$(\mathbf{B})^{-1}(\mathbf{A})^{-1} = (\mathbf{I} + \mathbf{E})^{-1}.$$

From this equation, we can isolate the true inverse of $\mathbf{A}$:

$$\mathbf{A}^{-1} = \mathbf{B}(\mathbf{I} + \mathbf{E})^{-1}.$$

To find a more accurate approximation for $\mathbf{A}^{-1}$, we can use the series expansion for the inverse of a matrix. If the error matrix $\mathbf{E}$ is small enough (in terms of its norm), we can use the Neumann series expansion:

$$(\mathbf{I} + \mathbf{E})^{-1} = \mathbf{I} - \mathbf{E} + \mathbf{E}^2 - \cdots,$$

provided that this series converges. This gives us:

$$\mathbf{A}^{-1} = \mathbf{B}\left(\mathbf{I} - \mathbf{E} + \mathbf{E}^2 - \cdots\right).$$

If we have an approximate inverse $\mathbf{B}$ for a matrix $\mathbf{A}$, we can compute a more accurate approximation of the true inverse by considering the error matrix $\mathbf{E}$. The formula derived allows us to refine our approximation iteratively using the series expansion. This approach is particularly useful when direct computation of the inverse is difficult or when dealing with large matrices.

Example: Iterative method

Let's take the following matrix:

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

Suppose we have an approximate inverse:

$$\mathbf{B} = \begin{pmatrix} 0.75 & -0.50 \\ 0.25 & 1 \end{pmatrix}.$$

In the first iteration, we compute the error matrix $\mathbf{E}$:

$$\mathbf{E} = \mathbf{AB} - \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

we can compute $\mathbf{E}$:

$$\mathbf{E} = \mathbf{AB} - \mathbf{I} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0.75 & -0.50 \\ 0.25 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.5 & 0 \\ 1.5 & 1.5 \end{pmatrix}.$$

Using the relationship derived earlier, we can refine our approximation of $\mathbf{A}^{-1}$ to the second approximation, then

$$\begin{aligned} \mathbf{A}^{-1} &= \mathbf{B}(\mathbf{I} - \mathbf{E} + \mathbf{E}^2) = \\ &= \begin{pmatrix} 0.75 & -0.50 \\ 0.25 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2.5 & 0 \\ 1.5 & 1.5 \end{pmatrix} + \begin{pmatrix} 2.5 & 0 \\ 1.5 & 1.5 \end{pmatrix} \begin{pmatrix} 2.5 & 0 \\ 1.5 & 1.5 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0.75 & -0.50 \\ 0.25 & 1 \end{pmatrix} \begin{pmatrix} 4.75 & 0 \\ 4.5 & 1.75 \end{pmatrix} = \begin{pmatrix} 1.3125 & -0.875 \\ 5.6875 & 1.75 \end{pmatrix} \end{aligned}$$

In the second iteration, we assume the initial guess as follows

$$\mathbf{B} = \begin{pmatrix} 1.3125 & -0.875 \\ 5.6875 & 1.75 \end{pmatrix}$$

We will repeat the processes from the first iteration and continue this procedure until the error matrix $\mathbf{E}$ approaches zero $\mathbf{E} \approx 0$.

2.2 LU factorization

LU Factorization is the process of decomposing a square matrix $\mathbf{A}$ (with elements a_{ij}, where $i, j = \overline{1, n}$) into two matrices

$$\mathbf{A} = \mathbf{LU}, \tag{0.2}$$

where $\mathbf{L}$ is a lower triangular matrix, and $\mathbf{U}$ is an upper triangular matrix. For the n-dimensional case, we can express $\mathbf{L}$ and $\mathbf{U}$ as follows

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & u_{nn} \end{pmatrix}. \tag{0.3}$$

Here, the elements l_{ij} and u_{ij} are defined as follows

$$\begin{aligned} u_{1j} &= a_{1j}, \\ u_{ij} &= a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \quad \text{for} \quad j \geq i > 1 \\ l_{i1} &= \frac{a_{i1}}{u_{11}} \\ l_{ij} &= \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \quad \text{for} \quad i > j > 1 \right) \end{aligned} \tag{0.4}$$

The eqn.(0.2) gives $\mathbf{A}^{-1} = (\mathbf{LU})^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$. In order to find $\mathbf{L}^{-1}$, let $\mathbf{L}^{-1} = \mathbf{x}$, where $\mathbf{x}$ is a lower triangular matrix, where $\mathbf{Lx} = \mathbf{I}$:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 & \cdots & 0 \\ x_{21} & x_{22} & 0 & \cdots & 0 \\ x_{31} & x_{32} & x_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{0.5}$$

By multiplying the matrices on the left-hand side and equating the corresponding elements, we obtain

$$\begin{aligned} x_{ii} &= 1, \\ x_{ij} &= -\sum_{k=j}^{i-1} l_{ik} x_{kj}, \quad \text{for} \quad i > j. \end{aligned} \tag{0.6}$$

Thus, the inverse of the lower triangular matrix $\mathbf{L}^{-1} = \mathbf{x}$ is completely determined. We get $\mathbf{U}^{-1} = \mathbf{y}$, completely. Finally, combine these results to get the inverse matrix $\mathbf{A}^{-1} = \mathbf{U}^{-1}\mathbf{L}^{-1}$.

Example: LU factorisation

Find the inverse of the matrix $\mathbf{A}$ using the LU factorization method

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 6 & 8 & 3 \end{pmatrix}$$

We perform Gaussian elimination to get $\mathbf{U}$ and track the multipliers to form $\mathbf{L}$. After completing the operations, we get the LU decomposition as:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find the $\mathbf{L}^{-1} = x$, perform $\mathbf{Lx} = \mathbf{I}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then from the first row $x_{11} = 1$, from the second row $2x_{11} + x_{21} = 0$, $x_{22} = 1$ and from the third row $3x_{11} - x_{21} + x_{31} = 0$, $-x_{22} + x_{32} = 0$, $x_{33} = 1$. Now we can substitute and solve for the unknowns

$$\mathbf{x} = \mathbf{L}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 1 & 1 \end{pmatrix}$$

To find $\mathbf{U}^{-1} = \mathbf{y}$, perform $\mathbf{yU} = \mathbf{I}$:

$$\begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & 0 & y_{33} \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, the system of equations does not have a unique solution, which is why the matrix $\mathbf{U}$ does not possess an inverse. Consequently, the matrix $\mathbf{A}$ also does not have an inverse. ■

3 Eigenvalue Problems

Eigenvalues and eigenvectors play a critical role in understanding the dynamics of linear systems, especially in areas such as stability analysis, vibrations, and principal component analysis. For example, in structural engineering, eigenvalues correspond to the natural frequencies of a system, and in data analysis, eigenvectors are used to identify the principal directions of variance in a dataset.

For a square matrix $\mathbf{A}$, an eigenvalue λ and the corresponding eigenvector $\mathbf{v}$ satisfy the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{0.7}$$

where the scalar λ is called the eigenvalue, and the vector $\mathbf{v}$ is called the eigenvector. To have a non-trivial solution, the determinant of the following matrix must be zero

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

3.1 Power method

The Power method is an iterative technique for finding the dominant eigenvalue (the eigenvalue with the largest absolute value) and its corresponding eigenvector of a matrix $\mathbf{A}$. The basic idea is to repeatedly multiply a random vector by the matrix $\mathbf{A}$ and normalize it at each step. The vector will converge to the eigenvector corresponding to the dominant eigenvalue.

For a given matrix $\mathbf{A}$, the steps of the Power method are:

- *Step 1. Initialize.* Choose an initial non-zero vector $\mathbf{v_0}$.
- *Step 2. Iterate.* For $k = 1, 2, 3, \ldots$, perform the following:
  1. Multiply the current vector by the matrix:
  $$\mathbf{v_k} = \mathbf{A}\mathbf{v_{k-1}}$$
  2. Normalize the vector $\mathbf{v_k}$:
  $$\mathbf{v_k} = \frac{\mathbf{v_k}}{\|\mathbf{v_k}\|}$$
- *Step 3. Eigenvalue Estimation.* Estimate the dominant eigenvalue using the Rayleigh quotient:
  $$\lambda_k = \frac{\mathbf{v_k}^T\mathbf{A}\mathbf{v_k}}{\mathbf{v_k}^T\mathbf{v_k}}$$
- *Step 4. Check for convergence.*

1. Check if the vector has stopped changing significantly:

$$\|\mathbf{v_k} - \mathbf{v_{k-1}}\| < \epsilon$$

   where ϵ is a small tolerance value.

2. Alternatively, check if the eigenvalue estimates have stabilized:

$$|\lambda_k - \lambda_{k-1}| < \epsilon$$

Example: Power method

Let's apply the Power method to estimate the dominant eigenvalue and its corresponding eigenvector of the matrix:

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

First iteration. Start with an initial eigenvector $\mathbf{v_0} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then multiplication of $\mathbf{A}$ by $\mathbf{v_0}$:

$$\mathbf{v_1} = \mathbf{A}\mathbf{v_0} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Normalize $\mathbf{v_1}$:

$$\mathbf{v_1} = \frac{1}{\sqrt{5^2 + 5^2}} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Estimate the eigenvalue λ_1:

$$\lambda_1 = \frac{\mathbf{v_1}^T \mathbf{A} \mathbf{v_1}}{\mathbf{v_1}^T \mathbf{v_1}} = \frac{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}} \approx 5$$

Second Iteration. Multiply $\mathbf{A}$ by $\mathbf{v_1}$:

$$\mathbf{v_2} = \mathbf{A}\mathbf{v_1} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{pmatrix}$$

Normalize $\mathbf{v_2}$:

$$\mathbf{v_2} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Estimate the eigenvalue λ_2:

$$\lambda_2 = \frac{\mathbf{v_2}^T \mathbf{A} \mathbf{v_2}}{\mathbf{v_2}^T \mathbf{v_2}} \approx 5$$

Convergence Check. At this point, the vector has stabilized to:

$$\mathbf{v_2} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The eigenvalue estimate is stable at $\lambda \approx 5$, indicating that the method has converged.

3.2 Jacobi's method

The Jacobi method is an iterative algorithm used to find the eigenvalues and eigenvectors of a symmetric matrix. This method is particularly useful for large matrices where other techniques like QR decomposition or LU decomposition might be computationally expensive. The method works by successively diagonalizing the matrix through a series of orthogonal transformations (rotations), ultimately yielding the eigenvalues on the diagonal of the matrix.

A symmetric matrix is a square matrix $\mathbf{A}$ that satisfies the condition:

$$\mathbf{A} = \mathbf{A}^T, \quad (a_{ij} = a_{ji})$$

For example, the following matrix is symmetric:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix}$$

Properties:

- All eigenvalues of a symmetric matrix are real.
- Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

Algorithm of Jacobi's Method:

Step 1. Input the symmetric matrix. Start with a symmetric matrix $\mathbf{A}$ of size $n \times n$. Ensure that the matrix is symmetric ($\mathbf{A} = \mathbf{A}^T$). For example, let:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix}$$

Step 2. Initialize the eigenvalue and eigenvector matrices.

- The initial eigenvalue matrix $\mathbf{\Lambda}$ is simply the diagonal matrix of $\mathbf{A}$:

$$\mathbf{\Lambda} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

- The initial eigenvector matrix $\mathbf{V}$ is the identity matrix:

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

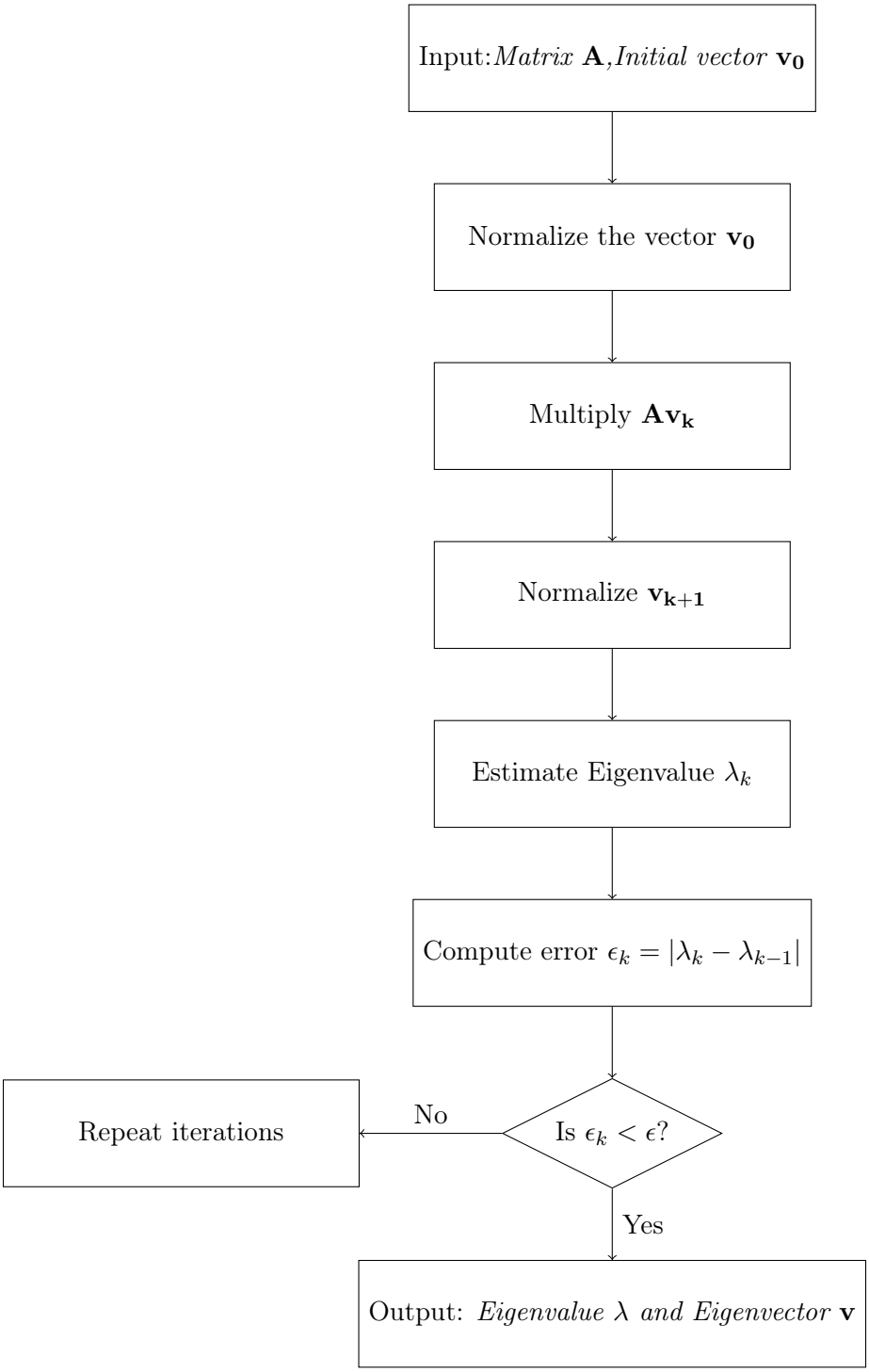

Figure 0.1: Block diagram of the power method

Step 3. Identify the largest off-diagonal element

At each iteration, identify the largest off-diagonal element $|a_{pq}|$, where $p \neq q$ (i.e., a_{pq} is not on the diagonal). This is the element that will be zeroed out using a Jacobi rotation.

Step 4. Compute the Jacobi rotation matrix

To eliminate the off-diagonal element a_{pq}, we apply a **Jacobi rotation**. The rotation matrix $\mathbf{G}$ is a 2x2 matrix used to zero out a_{pq}. The rotation matrix $\mathbf{G}$ is computed as:

$$\mathbf{G} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

where c is the cosine of the rotation angle, and s is the sine of the rotation angle. These are given by the following formulas:

$$c = \frac{a_{pp} - a_{qq}}{\sqrt{(a_{pp} - a_{qq})^2 + 4a_{pq}^2}}$$

$$s = \frac{2a_{pq}}{\sqrt{(a_{pp} - a_{qq})^2 + 4a_{pq}^2}}$$

Step 5. Apply the Jacobi rotation

Once the Jacobi rotation matrix $\mathbf{G}$ is computed, it is applied to the matrix $\mathbf{A}$ as follows:

$$\mathbf{A}' = \mathbf{G}^T \mathbf{A} \mathbf{G}$$

This operation zeros out the off-diagonal element a_{pq}, and the new matrix $\mathbf{A}'$ will have reduced off-diagonal elements.

Step 6. Update the eigenvector matrix

At each step, the matrix $\mathbf{V}$ is updated by multiplying it with the rotation matrix $\mathbf{G}$:

$$\mathbf{V}' = \mathbf{V}\mathbf{G}$$

The matrix $\mathbf{V}$ will eventually contain the eigenvectors of $\mathbf{A}$ as its columns.

Step 7. Repeat the process

Repeat steps 3 through 6 until the matrix $\mathbf{A}$ is sufficiently diagonal. The process is stopped when all off-diagonal elements are smaller than a predefined tolerance.

Convergence Criterion. The method stops when the largest off-diagonal element is sufficiently small. We use the following convergence criterion:

$$\max |a_{pq}| < \epsilon$$

where
- $|a_{pq}|$ is the off-diagonal element at position (p, q) of the matrix $\mathbf{A}$ at the k-th iteration.
- ϵ is the tolerance (e.g., 10^{-6}).

If $\max |a_{pq}| < \epsilon$, the method has converged, and the iterations stop.

Extracting Eigenvalues and Eigenvectors. Once the matrix is diagonal, the diagonal elements represent the eigenvalues of the original matrix $\mathbf{A}$. For example, if the final matrix is:

$$\mathbf{A}^{(\text{final})} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Then, the eigenvalues of A are λ_1 and λ_2.

The matrix $\mathbf{V}$, which has been updated throughout the rotations, contains the eigenvectors of $\mathbf{A}$. The columns of $\mathbf{V}$ correspond to the eigenvectors.

Example: Jacobi's method

Given the symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 5 \\ 2 & 5 & 6 \end{bmatrix}$$

Step 1. Initialization.

- Eigenvalue Matrix $\mathbf{\Lambda}$ = diagonal elements of $\mathbf{A}$:

$$\mathbf{\Lambda} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

- Eigenvector Matrix $\mathbf{V}$ = identity matrix:

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2. Identify the largest off-diagonal element. Largest off-diagonal element $|a_{23}| = 5$.

Step 3. Compute the Jacobi rotation.

- Compute the rotation matrix $\mathbf{G}$ to eliminate a_{23}.
- Compute rotation coefficients c and s:

$$c = \frac{a_{22} - a_{33}}{\sqrt{(a_{22} - a_{33})^2 + 4a_{23}^2}} \approx -0.287, \quad s = \frac{2a_{23}}{\sqrt{(a_{22} - a_{33})^2 + 4a_{23}^2}} \approx 0.957$$

- Build rotation matrix $\mathbf{G}$:

$$\mathbf{G} = \begin{bmatrix} -0.287 & 0.957 \\ -0.957 & -0.287 \end{bmatrix}$$

Step 4. Apply the rotation.

- Update $\mathbf{A}' = \mathbf{G}^T\mathbf{A}\mathbf{G}$:

$$\mathbf{A}' = \begin{bmatrix} 5.662 & 0.003 & 0 \\ 0.003 & 2.338 & 0 \\ 0 & 0 & 6.000 \end{bmatrix}$$

- Update $\mathbf{V}' = \mathbf{V}\mathbf{G}$:

$$\mathbf{V}' = \begin{bmatrix} -0.287 & 0.957 & 0 \\ -0.957 & -0.287 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 5. Check convergence.

- Largest off-diagonal element $|a_{23}| = 0.003$.

- Since $0.003 > 10^{-6}$, continue the iteration.

Step 6. Repeat until convergence.

After several iterations, the matrix A becomes approximately diagonal:

$$\mathbf{A}' = \begin{bmatrix} 10.000 & 0 & 0 \\ 0 & 2.000 & 0 \\ 0 & 0 & 1.000 \end{bmatrix}$$

Final Eigenvalues and Eigenvectors. - Eigenvalues: $\lambda_1 = 10.000, \lambda_2 = 2.000, \lambda_3 = 1.000$

- Eigenvectors: Columns of $\mathbf{V}$:

$$v_1 \approx \begin{bmatrix} -0.287 \\ -0.957 \\ 0 \end{bmatrix}, \quad v_2 \approx \begin{bmatrix} 0.957 \\ -0.287 \\ 0 \end{bmatrix}, \quad v_3 \approx \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

4 Tasks

1. Self-study of the following methods: Given's method and House-Holder's method

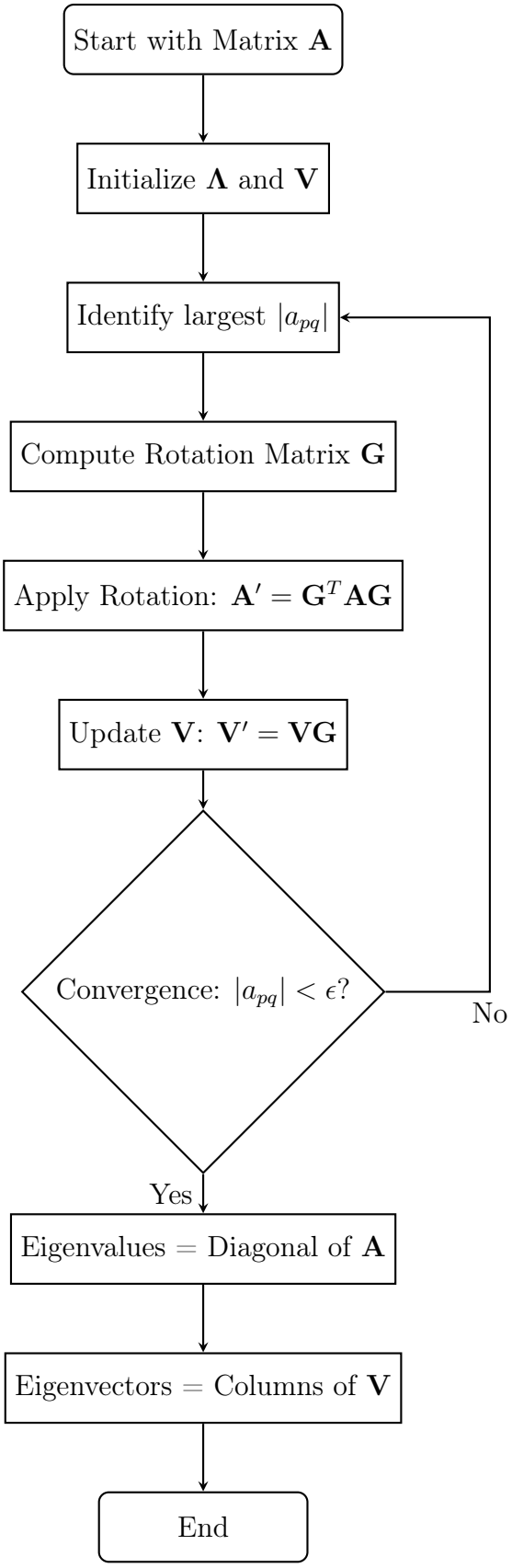

Figure 0.2: Block diagram for the Jacobi's method

2. Using the LU factorization method, find the inverse of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 50 & 107 & 36 \\ 35 & 54 & 20 \\ 31 & 66 & 21 \end{pmatrix}.$$

3. Apply iterative method to find more accurate inverse of $\mathbf{A}$, assuming the initial approximate inverse matrix $\mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 10 & 1 \\ 2 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0.4 & 2.4 & -1.4 \\ 0.14 & 0.14 & -0.14 \\ -0.85 & -3.8 & 2.8 \end{pmatrix}.$$

4. Find the largest eigenvalue and the corresponding eigenvector of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

using the power method. Take $[1, 0, 0]^T$ as the initial eigenvector.

5. Using Jacobi's method, find all the eigenvalues and the eigenvectors of the matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}.$$

5 Resources

- B.S. Grewal, "Numerical Methods in Engineering and Science", Khanna Publication, Ed. 9th., 2014 (Chapters 4)