SUGGESTED SOLUTION SET A

Section A

1.

(a)

$$\lim_{x\to 1} f(x) - \lim_{x\to 0} f(x) = \lim_{x\to 1}\frac{x^2-1}{x-1} - \lim_{x\to 0}\frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1}\frac{(x+1)(x-1)}{(x-1)} - \frac{0-1}{0-1}$$

$$= \lim_{x\to 1}(x+1) - (1)$$

$$= (1+1) - 1$$

$$= 1$$

(b)

$$\lim_{x\to +\infty}\frac{\sqrt{x^2-1}}{3x+5} = \lim_{x\to +\infty}\frac{\frac{\sqrt{x^2-1}}{\sqrt{x^2}}}{\frac{3x+5}{x}}$$

$$= \lim_{x\to +\infty}\frac{\sqrt{1-\frac{1}{x^2}}}{3-\frac{5}{x}}$$

$$= \frac{\sqrt{1-0}}{3-0}$$

$$= \frac{1}{3}$$

2. (a) $y = x\ln x - 3x$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + (\ln x)(1) - 3$$

$$\frac{dy}{dx} = \ln x - 2$$

$$0 = \ln x - 2$$

$$\ln x = 2$$

$$x = e^2$$

$$y = e^2 \ln e^2 - 3e^2$$

$$y = 2e^2 - 3e^2$$

$$y = -e^2$$

(b) $y = \alpha e^{2x} + \beta e^{-2x}$

$$\frac{dy}{dx} = \alpha(2e^{2x}) + \beta(-2e^{-2x})$$

$$\frac{dy}{dx} = 2\alpha e^{2x} - 2\beta e^{-2x}$$

$$\frac{d^2y}{dx^2} = 2\alpha(2e^{2x}) - 2\beta(-2e^{-2x})$$

$$\frac{d^2y}{dx^2} = 4\alpha e^{2x} + 4\beta e^{-2x}$$

$$\frac{d^2y}{dx^2} = 4(\alpha e^{2x} + \beta e^{-2x})$$

$$\frac{d^2y}{dx^2} = 4y$$ ***shown***

3. $f(x)=x^2-\frac{2}{x}$

$f'(x)=2x+\frac{2}{x^2}$

$0=2x+\frac{2}{x^2}$

$x^3=-1$

$x=-1$

When x=-1, $f(-1)=(-1)^2-\frac{2}{(-1)}=3$

the coordinate of stationary point $(-1,3)$

$f'(x)=2x+\frac{2}{x^2}$

$f''(x)=2-\frac{4}{x^3}$

$f''(-1)=2-\frac{4}{(-1)^3}=6>0$

(relative minimum)

Section B

1. (a) $z_1+z_2=2+3i+3+2i$

$=5+5i$

$|z_1+z_2|=\sqrt{5^2+5^2}$

$=\sqrt{50}$

$=5\sqrt{2}$

(b) $z=\frac{z_1 z_3}{z_2}=\frac{(2+3i)(a+bi)}{(3+2i)}\times\frac{(3-2i)}{(3-2i)}$

$$=\frac{(6-4i+9i-6i^2)(a+bi)}{(3+2i)(3-2i)}$$

$$=\frac{(12+5i)(a+bi)}{9+4}$$

$$=\frac{12a+12bi+5ai+5bi^2}{13}=\frac{12a-5b}{13}+\frac{(5a+12b)}{13}i$$

(c)(i) $z=\frac{17}{13}-\frac{7}{13}i$

$$\therefore \frac{17}{13}-\frac{7}{13}i=\frac{12a-5b}{13}+\frac{(5a+12b)}{13}i$$

$\frac{12a-5b}{13}=\frac{17}{13}$ $\qquad$ $\frac{5a+12b}{13}=-\frac{7}{13}$

$12a-5b=17---(1)$ $\qquad$ $5a+12b=-7---(2)$

$(1)\times 5,\quad 60a-25b=85---(3)$

$(2)\times 12,\quad 60a+144b=-84---(4)$

$(4)-(3),\ 169b=-169$

$\therefore b=-1 \quad , \quad \therefore a=1$

(ii) $\tan\theta=\frac{-\frac{7}{13}}{\frac{17}{13}}$

$$\theta=\tan^{-1}\left|\frac{-7}{17}\right|=0.3906\ rad$$

$$\arg z=-0.391\ rad\,(3d.p)$$

(iii) $|z|=\sqrt{\left(\frac{17}{13}\right)^2+\left(\frac{-7}{13}\right)^2}$

$$=\sqrt{\frac{338}{169}}=\sqrt{2}$$

The polar form

$$z=\sqrt{2}\left(\cos\left(-0.391\,rad\right)+i\sin\left(-0.391\,rad\right)\right)$$

2.(a) $\log_5 5x+\log_x 5=\log_4 64$

$$\log_5 5+\log_5 x+\frac{1}{\log_5 x}=\log_4 4^3$$

$$1+\log_5 x+\frac{1}{\log_5 x}=3$$

Let $a=\log_5 x$, $1+a+\frac{1}{a}=3$

$$a^2-2a+1=0$$

$$\left(a-1\right)^2=0$$

$$\therefore a=1$$

$$\therefore \log_5 x=1$$

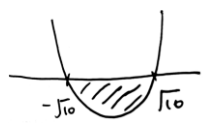
$$\therefore x=5$$

(b) $3-x \le x(x-3) < 10-3x$

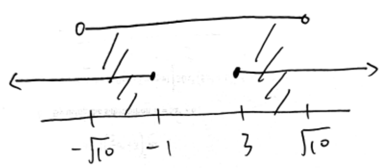
$3-x \le x(x-3)$ and $x(x-3) < 10-3x$

$$3-x \le x^2-3x \qquad\qquad x^2-3x<10-3x$$
$$x^2-2x-3 \ge 0 \qquad\qquad x^2-10<0$$
$$(x+1)(x-3) \ge 0 \qquad\qquad \left(x+\sqrt{10}\right)\left(x-\sqrt{10}\right)<0$$

$\therefore (-\infty, -1] \cup [3, \infty)$



$\therefore (-\sqrt{10}, \sqrt{10})$



$\therefore (-\sqrt{10}, -1] \cup [3, \sqrt{10})$

3. (a)

$$f(x)=\frac{x+2}{x-2},\ x \ne k$$
$$x-2 \ne 0$$
$$x \ne 2$$
$$k=2$$

b)

$$f(x_1)=f(x_2)$$
$$\frac{x_1+2}{x_1-2}=\frac{x_2+2}{x_2-2}$$
$$x_1x_2-2x_1+2x_2-4=x_1x_2+2x_1-2x_2-4$$
$$4x_1=4x_2$$
$$x_1=x_2$$

$\therefore f$ is a 1-1 function.

(c)

$$f\left[g\ \ x\ \right]=f\left(\frac{2x+2}{x-1}\right)$$

$$=\frac{\frac{2x+2}{x-1}+2}{\frac{2x+2}{x-1}-2}$$

$$=\frac{\frac{2x+2+2x-2}{x-1}}{\frac{2x+2-2x+2}{x-1}}$$

$$=\frac{4x}{4}$$

$$=x$$

$\because f\left[g\ \ x\ \right]=x, \therefore f$ and g are inverse each other.

4. (a)

$$f(x)=\ln(3x-1)$$

$$f(f^{-1}\ \ x\ )=\ln\ \ 3f^{-1}\ \ x\ \ -1$$

$$x=\ln\ \ 3f^{-1}\ \ x\ \ -1$$

$$e^{x}=3f^{-1}\ \ x\ \ -1$$

$$f^{-1}\ \ x\ \ =\frac{e^{x}+1}{3}$$

(b)

$$D_{f^{-1}}=R_f=\ -\infty,\infty\ \ ; \qquad R_{f^{-1}}=D_f=(\frac{1}{3},\infty)$$

(d)

$$g\ \ f\ \ x\ \ =g\ \ \ln(3x-1)$$

$$=e^{\ln(3x-1)}$$

$$=3x-1$$

5.

$$f(2) = p$$
$$8 - p = p$$
$$2p = 8$$
$$p = 4$$

$$f(2) = \lim_{x \to 2^-} r \;\; x + 6$$
$$p = r \;\; 2 + 6$$
$$4 = 8r$$
$$r = \frac{1}{2}$$

$$f(q) = \lim_{x \to q^-} \frac{2 - x^{\;2} - 4}{x}$$
$$\frac{1}{2} \;\; q + 6 \;\; = \frac{2 - q^{\;2} - 4}{q}$$
$$\frac{q+6}{2} = \frac{q^2 - 4q}{q}$$
$$q^2 + 6q = 2q^2 - 8q$$
$$q^2 - 14q = 0$$
$$q = 0 \;\; or \;\; q = 14$$
$$\because q \le x < 2, \therefore q = 0$$

6. (a) $x = 2; \;\; y^2 + 6 \;\; 2 \;\; = 3 \;\; 2 \;\; y + y$

$$y^2 - 7y + 12 = 0$$
$$(y-4)(y-3) = 0$$
$$y = 4 \;\; or \;\; y = 3$$

$\therefore$ Points are $(2,4)$ and $(2,3)$

$$y^2 + 6x = 3xy + y$$
$$2y\frac{dy}{dx} + 6 = 3x\frac{dy}{dx} + y(3) + \frac{dy}{dx}$$
$$2y\frac{dy}{dx} - 3x\frac{dy}{dx} - \frac{dy}{dx} = 3y - 6$$
$$\frac{dy}{dx}(2y - 3x - 1) = 3y - 6$$
$$\frac{dy}{dx} = \frac{3y-6}{2y-3x-1}$$
$$(2,4): \;\; \frac{dy}{dx} = \frac{3(4)-6}{2(4)-3(2)-1} = \frac{6}{1} = 6$$
$$(2,3): \;\; \frac{dy}{dx} = \frac{3(3)-6}{2(3)-3(2)-1} = \frac{3}{-1} = -3$$

(b) $x=\frac{1}{3}\sin^3 2\theta$; $y=2\cos^3 2\theta$

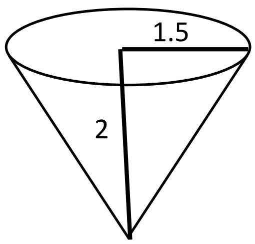
$\frac{dx}{d\theta}=\frac{3}{3}\sin^2 2\theta\left(2\cos 2\theta\right)$; $\frac{dy}{d\theta}=6\cos^2 2\theta\left(-2\sin 2\theta\right)$

$=2\sin^2 2\theta\cos 2\theta$; $=-12\cos^2 2\theta\sin 2\theta$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=\frac{-12\cos^2 2\theta\sin 2\theta}{2\sin^2 2\theta\cos 2\theta}=-6\cot 2\theta$$

$$\theta=\frac{\pi}{6}\ ;\ \frac{dy}{dx}=-6\cot 2\theta=\frac{-6}{\tan 2\theta}=\frac{-6}{\tan 2\left(\frac{\pi}{6}\right)}=\frac{-6}{\sqrt{3}}=-2\sqrt{3}$$

7.



$$\frac{r}{h}=\frac{1.5}{2}$$

$$r=\frac{3}{4}h$$

$$\frac{dV}{dt}=-0.25m^3/\min$$

$$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{3}{4}\right)^2h=\frac{3}{16}\pi h^3$$

$$\frac{dV}{dh}=\frac{9}{16}\pi h^2$$

(a) $\frac{dh}{dt}=\frac{dV}{dt}\times\frac{dh}{dV}$

$=-0.25\times\frac{16}{9\pi(1)^2}$

$=-0.1415m/\min$

(b) $r=\frac{3}{4}h$

$\frac{dr}{dh}=\frac{3}{4}$

$\frac{dr}{dt}=\frac{dh}{dt}\times\frac{dr}{dh}$

$=-0.1415\times\frac{3}{4}$

$=-0.1061m/\min$