SECTION A

(25 *marks*)

This section consists of **3** questions. Answer **all** questions.

1. Evaluate the following limits (if exist):

(a) $\lim_{x \to 3^-} \frac{x^2 - 9}{|x - 3|}$. **(3 marks)**

(b) $\lim_{x \to 5} \frac{2x - 10}{\sqrt{2x - 1} - 3}$. **(3 marks)**

(c) $\lim_{x \to +\infty} \frac{2\sqrt{x} - 4}{x - 4}$. **(3 marks)**

2. **(a)** Evaluate $\frac{dy}{dx}$ when $x = 0$ for each of the following

(i) $y = \ln\left(x^2 + \sqrt{x^2 + 1}\right)$. **(3 marks)**

(ii) $y = \frac{e^x \left(2x^2 + 1\right)}{\sqrt{x + 1}}$. **(4 marks)**

(b) Given $y = 3^{x^2}$, find $\frac{dy}{dx}$. **(3 Marks)**

3. The function $f(x) = x^3 - 3x^2 - 9x - 4$ is defined on the interval $[-2, 6]$. Find the critical points of $f(x)$ on the interval and determine whether each critical point is a minimum or maximum. **(6 marks)**

SECTION B

(75 *marks*)

This section consists of **7** questions. Answer **all** questions.

1. Given $z_1 = 3+2i$ and $z_2 = 1-3i$. Without using calculator, find $z_3 = \frac{z_2}{\bar{z}_1}$ and hence, evaluate $|z_3|$. **(5 marks)**

2. (a) Solve the equation $(\log x)^2 = \log x^3$. **(5 marks)**

(b) Find the interval notation of the inequality $2 + \left|\frac{5x+2}{x-3}\right| \geq 5$. **(8 marks)**

3. Given the functions f and g as follows

$$f(x) = x^2 + 3x + 1,$$
$$g(x) = x - 2.$$

(a) Find $f \circ g$ and $g \circ f$. **(4 marks)**

(b) State domain and range of $g \circ f$. **(3 marks)**

(c) Determine the value of x such that $f \circ g(x) = g \circ [g \circ f(x)]$. **(3 marks)**

4. The functions f and g are defined as $f(x) = \frac{x+1}{x-5}, x \neq 5$ and $g(x) = 4 - x$.

(a) Find $f^{-1}(x)$ and $g^{-1}(x)$. **(5 marks)**

(b) Evaluate $(f \circ g^{-1})(2)$. **(3 marks)**

5. (a) Given $f(x)=\begin{cases} 7-2x, & x \leq p \\ \dfrac{x^2+(q-2)x-2q}{x-2}, & p < x \leq 5 \\ 10-(x-7)^2, & x > 5 \end{cases}$

with $\lim\limits_{x \to p^+} f(x)=3$ and the function f is continuous for all real values of x.

Determine the values of p and q. **(7 marks)**

(b) A function f is defined by $f(x)=\begin{cases} \dfrac{2(1-x)}{x-2}, & x < \dfrac{3}{2} \\ 2\ , & x \geq \dfrac{3}{2} \end{cases}$

(i) Use the definition to show that f is continuous at $x=\dfrac{3}{2}$. **(1 mark)**

(ii) Sketch the graph of f. **(6 marks)**

6. **(a)** Find $\dfrac{dy}{dx}$ in terms of x and y if $x^2 \sin y + 2x = y$. **(7 marks)**

(b) Differentiate $\cos^3(\ln(2x-1))$ with respect to x. **(4 marks)**

(c) Given $y = 5\sin(3x)+\sqrt{x}$. Find the value of $\dfrac{dy}{dx}$ when $x=\dfrac{\pi}{2}$. **(4 marks)**

7. **(a)** Find the stationary points of the curve has an equation $y=\dfrac{1}{3}x^3+x^2-8x$.

(4 marks)

(b) Air is pumped into a spherical balloon at a rate $54\,cm^3s^{-1}$. Find the rate at which the radius is increasing when the volume of balloon is $36\pi\,cm^3$. **(6 marks)**

END OF QUESTIONS PAPER

SM015: SET KMS

ANSWER:

PART A

1. **(a)** − 6 **(b)** 6 **(c)** 0

2. **(a)** **(i)** 0 **(ii)** $\frac{1}{2}$

(b) $\frac{dy}{dx} = 3^{x^2}(\ln 3)(2x)$

3. (− 1,1) maximum point, (3, −31) minimum point

PART B

1. $z_3 = \frac{9}{13} - \frac{7}{13}i$, $|z_3| = 0.8771$

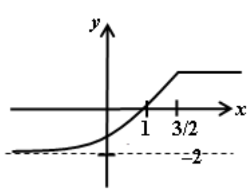
2. **(a)** x =1, 1000 **(b)** $\left(-\infty, -\frac{11}{2}\right] \cup \left[\frac{7}{8}, 3\right) \cup (3, \infty)$

3. **(a)** $f \circ g = x^2 - x - 1, \quad g \circ f = x^2 + 3x - 1$ **(b)** $D_{g \circ f} = (-\infty, \infty), R_{g \circ f} = [-\frac{13}{4}, \infty)$

(c) $x = \frac{1}{2}$

4. **(a)** $f^{-1}(x) = \frac{5x+1}{x-1}, g^{-1}(x) = 4 - x$ **(b)** − 1

5. **(a)** p =2, q = 1

(b) (ii) 

6. **(a)** $\frac{dy}{dx} = \frac{2(1 + x\sin y)}{1 - x^2 \cos y}$ **(b)** $\frac{dy}{dx} = -\frac{6}{2x-1}\cos^2(\ln(2x-1))\sin(\ln(2x-1))$

(c) 0.399

7. **(a)** $\left(2, -\frac{28}{3}\right)$ and $\left(-4, \frac{80}{3}\right)$. **(b)** $\frac{dr}{dt} = 0.4775 \ \text{cms}^{-1}$

PART A
(25 *marks*)

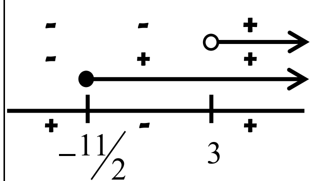
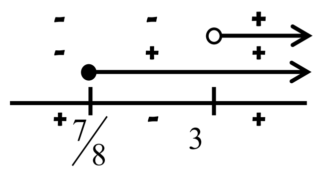
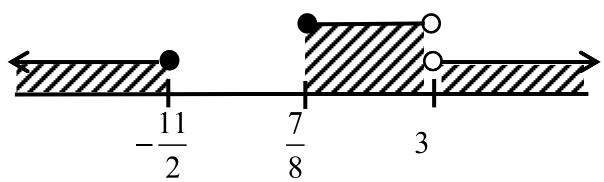
NO	ANSWER SCHEMES	REMARKS
1. (a)	$\lim_{x\to 3^-}\frac{x^2-9}{\lvert x-3\rvert}=\lim_{x\to 3^-}\frac{(x+3)(x-3)}{\lvert x-3\rvert}$ $=\lim_{x\to 3^-}\frac{(x+3)(x-3)}{-(x-3)}$ $=\lim_{x\to 3^-}\left[-(x+3)\right]$ $=-(3+3)$ $=-6$	**K1** Factorize and choose negative for absolute **K1** Substitute x=3 **J1**
(b)	$\lim_{x\to 5}\left(\frac{2x-10}{\sqrt{2x-1}-3}\right)=\lim_{x\to 5}\frac{(2x-10)\left(\sqrt{2x-1}+3\right)}{\left(\sqrt{2x-1}-3\right)\left(\sqrt{2x-1}+3\right)}$ $=\lim_{x\to 5}\frac{(2x-10)\left(\sqrt{2x-1}+3\right)}{\left(\sqrt{2x-1}\right)^2-3^2}$ $=\lim_{x\to 5}\left(\sqrt{2x-1}+3\right)$ $=\sqrt{2(5)-1}+3$ $=6$	**K1** **K1** **J1**
(c)	$\lim_{x\to +\infty}\frac{2\sqrt{x}-4}{x-4}=\lim_{x\to +\infty}\frac{2\left(\sqrt{x}-2\right)}{x-4}$ $=\lim_{x\to +\infty}\frac{2\left(\sqrt{x}-2\right)}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}$ $=\lim_{x\to +\infty}\frac{2}{\sqrt{x}+2}$ $=\lim_{x\to +\infty}\frac{\frac{2}{\sqrt{x}}}{1+\frac{2}{\sqrt{x}}}$ $=0$	**K1** **K1** **J1**
	TOTAL MARKS	**9 marks**

2.(a) (i)	$y=\ln\left(x^2+\sqrt{x^2+1}\right)$ $\frac{dy}{dx}=\frac{1}{x^2+\sqrt{x^2+1}}\left(2x+\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}(2x)\right)$ $=\frac{1}{x^2+\sqrt{x^2+1}}\left(2x+\frac{x}{\sqrt{x^2+1}}\right)$ *when* $x=0,\frac{dy}{dx}=\frac{1}{0+\sqrt{0+1}}\left[2(0)+\frac{0}{\sqrt{0+1}}\right]=0$	**K1** **K1** **J1**
(ii)	$y=e^x\left(2x^2+1\right)(x+1)^{-\frac{1}{2}}=\left(2e^xx^2+e^x\right)(x+1)^{-\frac{1}{2}}$ $\frac{dy}{dx}=\left(2e^xx^2+e^x\right)\left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}}+(x+1)^{-\frac{1}{2}}\left(4xe^x+2x^2e^x+e^x\right)$ *when* $x=0,$ $\frac{dy}{dx}=\left(2e^0(0)^2+e^0\right)\left(-\frac{1}{2}\right)(0+1)^{-\frac{3}{2}}+(0+1)^{-\frac{1}{2}}\left(4(0)e^0+2(0)^2e^0+e^0\right)$ $=\frac{1}{2}$	**K1** **K1** **K1** **J1**
(b)	$y=3^{x^2}$ $\ln y=x^2\ln 3$ $\frac{1}{y}\left(\frac{dy}{dx}\right)=(\ln 3)(2x)$ $\frac{dy}{dx}=(\ln 3)(2x)(y)$ $\frac{dy}{dx}=3^{x^2}(\ln 3)(2x)$	**K1** **K1** **J1**
	TOTAL MARKS	**10 marks**

3.	$f(x)=x^3-3x^2-9x-4$ $f'(x)=3x^2-6x-9$ when $f'(x)=0$ $3x^2-6x-9=0$ $x^2-2x-3=0$ $(x+1)(x-3)=0$ $x=-1$ or $x=3$ when $x=-1, f(x)=(-1)^3-3(-1)^2-9(-1)-4=1$ $x=3, f(x)=(3)^3-3(3)^2-9(3)=-31$ $\therefore$ Critical points are $(-1,1)$ and $(3,-31)$ $f''(x)=6x-6$ when $x=-1, f''(x)=6(-1)-6=-12<0$ $\therefore (-1,1)$ is a maximum point $x=3. f''(x)=6(3)-6=12>0$ $\therefore (3,-31)$ is a minimum point	**K1** Correct factorise and $f'(x)=0$ **K1** Subs both x values **J1** Both points correct **K1** Any method **J1** Any one point **J1** Both correct
	TOTAL MARKS	**6 marks**

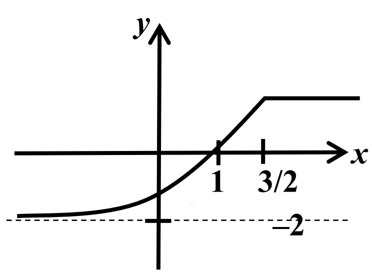
PART B
(*75 marks*)

NO	ANSWER SCHEMES	REMARKS
1.	$z_3 = \dfrac{z_2}{\bar{z}_1} = \dfrac{1-3i}{3-2i} = \dfrac{1-3i}{3-2i} \times \dfrac{3+2i}{3+2i}$	**K1** correct $\dfrac{z_2}{\bar{z}_1}$
	$= \dfrac{3+2i-9i-6i^2}{9+4} = \dfrac{9-7i}{13}$	**K1** correct conjugate and simplify
	$= \dfrac{9}{13} - \dfrac{7}{13}i$	**JI**
	$\lvert z_3 \rvert = \sqrt{\left(\dfrac{9}{13}\right)^2 + \left(-\dfrac{7}{13}\right)^2}$	**K1**
	$= \dfrac{\sqrt{130}}{13}$ or 0.8771	**J1**
	TOTAL MARKS	**5 marks**
2.(a)	$(\log x)^2 = \log x^3$	
	$(\log x)^2 = 3\log x$	**K1**
	let $u = \log x$	
	$u^2 - 3u = 0$	
	$u(u-3) = 0$	**K1** Quadratic form & factorize
	$u = 0 \qquad u = 3$	
	$\log x = 0 \qquad \log x = 3$	**J1**
	$x = 10^0 \qquad x = 10^3$	**K1** Attempt to solve
	$x = 1 \qquad x = 1000$	**J1**

	$2+\left\lvert\dfrac{5x+2}{x-3}\right\rvert \geq 5$	
	$\left(\dfrac{5x+2}{x-3}\right) \geq 3$ or $\left(\dfrac{5x+2}{x-3}\right) \leq -3$	**B1**
	$\left(\dfrac{5x+2}{x-3}\right) - 3 \geq 0$ $\left(\dfrac{5x+2}{x-3}\right) + 3 \leq 0$	**K1**
	$\dfrac{2x+11}{x-3} \geq 0$ $\dfrac{8x-7}{x-3} \leq 0$	**K1** Simplify
	 	**K1** Any method
(b)	$\left(-\infty, -\dfrac{11}{2}\right] \cup (3, \infty)$ $\left[\dfrac{7}{8}, 3\right)$	**J1J1**
		**K1**
	$\left(-\infty, -\dfrac{11}{2}\right] \cup \left[\dfrac{7}{8}, 3\right) \cup (3, \infty)$	**J1**
	TOTAL MARKS	**13 marks**

3.(a)	$f(x)=x^2+3x+1$ $g(x)=x-2$ $f\left[g(x)\right]=(f\left[x-2\right])$ $=(x-2)^2+3(x-2)+1$ $=x^2-x-1$ $\therefore f\circ g=x^2-x-1$ $g\left[f(x)\right]=g\left[x^2+3x+1\right]$ $=x^2+3x+1-2$ $=x^2+3x-1$ $\therefore g\circ f=x^2+3x-1$	**K1** **J1** **K1** **J1**
(b) **(c)**	$g\circ f=x^2+3x-1=(x+\frac{3}{2})^2-\frac{13}{4}$ $D_{gof}=(-\infty,\infty)\quad \mathrm{R}_{gof}=[-\frac{13}{4},\infty)$ $f\circ g(x)=g\circ[g\circ f(x)]$ $x^2-x-1=g[x^2+3x-1]$ $x^2-x-1=x^2+3x-1-2$ $x=\frac{1}{2}$	**K1** **J1JI** **K1** **K1** **JI**
	TOTAL MARKS	**10 marks**
4(a)	$f(x)=\frac{x+1}{x-5},x\neq 5$ $\frac{f^{-1}(x)+1}{f^{-1}(x)-5}=x$ $f^{-1}(x)+1=xf^{-1}(x)-5x$ $f^{-1}(x)-xf^{-1}(x)=-5x-1$ $f^{-1}(x)(1-x)=-5x-1$ $f^{-1}(x)=\frac{5x+1}{x-1}$	**K1** **K1** **JI**

	$g(x) = 4 - x$ $4 - g^{-1}(x) = x$ $g^{-1}(x) = 4 - x$	K1 JI
(b)	$\left(f \circ g^{-1}\right)(2) = f\left(g^{-1}\right)(2)$ $= f(4-2)$ $= f(2)$ $= \dfrac{2+1}{2-5}$ $= -1$	K1 K1 JI
	TOTAL MARKS	**8 marks**
5. (a)	$\lim\limits_{x \to p^-} 7 - 2x = \lim\limits_{x \to p^+} \dfrac{x^2 + (q-2)x - 2q}{x-2}$ $7 - 2p = 3$ $p = 2$ $\lim\limits_{x \to 2^+} \dfrac{x^2 + (q-2)x - 2q}{x-2} = 3$ $\lim\limits_{x \to 2^+} \dfrac{(x-2)(x+q)}{x-2} = 3$ $2 + q = 3$ $q = 1$	K1 K1 J1 K1 K1 K1 J1
(b) (i)	(i) $f\left(\dfrac{3}{2}\right) = 2$ (ii) $\lim\limits_{x \to \frac{3}{2}^-} \dfrac{2(1-x)}{x-2} = \lim\limits_{x \to \frac{3}{2}^+} 2$ $\dfrac{2\left(1 - \frac{3}{2}\right)}{\frac{3}{2} - 2} = 2$ $2 = 2$	

	$\lim_{x\to\frac{3}{2}^-} f(x) = \lim_{x\to\frac{3}{2}^+} f(x)$ $\lim_{x\to\frac{3}{2}} f(x)$ is exist. $f\left(\frac{3}{2}\right) = \lim_{x\to\frac{3}{2}} f(x)$ $\therefore\ f(x)$ is continuous at $x = \frac{3}{2}$. $\lim_{x\to+\infty} \frac{2(1-x)}{x-2} = \lim_{x\to+\infty} \frac{\frac{2}{x}-2}{1-\frac{2}{x}}$ $= \frac{0-2}{1-0}$ $= -2$ $\lim_{x\to-\infty} \frac{2(1-x)}{x-2} = \lim_{x\to-\infty} \frac{\frac{2}{x}-2}{1-\frac{2}{x}}$ $= -2$ $\therefore$ Horizontal Asymptote is $y = -2$	**J1** **K1** **K1** **J1**
(ii)		**R1** **R1** **R1**
	TOTAL MARKS	**14 marks**

6 (a)	$x^2 \sin y + 2x = y$ $2x\sin y + x^2 \cos y\left(\frac{dy}{dx}\right) + 2 = \frac{dy}{dx}$ $\frac{dy}{dx}(1 - x^2 \cos y) = 2(1 + x\sin y)$ $\frac{dy}{dx} = \frac{2(1 + x\sin y)}{1 - x^2 \cos y}$	**K1K1K1** Differentiate each term **K1K1** **K1J1**
(b)	$\frac{d}{dx}\left[\cos^3(\ln(2x-1))\right] = 3\left[\cos^2(\ln(2x-1)\right]\bullet\left[-\sin(\ln(2x-1)\right]\bullet\frac{2}{2x-1}$ $= -\frac{6}{2x-1}\cos^2(\ln(2x-1))\sin(\ln(2x-1))$	**K1K1** **K1J1**
(c)	$\frac{dy}{dx} = 15\cos(3x) + \frac{1}{2\sqrt{x}}$ $\left.\frac{dy}{dx}\right\|_{x=\frac{\pi}{2}} = 15\cos\left(3\left(\frac{\pi}{2}\right)\right) + \frac{1}{2\sqrt{\frac{\pi}{2}}}$ $= 15(0) + \frac{1}{2\frac{\sqrt{\pi}}{\sqrt{2}}\bullet\frac{\sqrt{2}}{\sqrt{2}}}$ $= \frac{1}{\sqrt{2\pi}}$ $= 0.399$	**K1K1** Differentiate both terms **K1** Substitute limit **J1**
	TOTAL MARKS	**15 marks**

7(a)	$y=\frac{1}{3}x^3+x^2-8x$ $\frac{dy}{dx}=x^2+2x-8$ $\frac{dy}{dx}=0$ $x^2+2x-8=0$ $(x-2)(x+4)=0$ $x=2;\ x=-4$ When $x=2;\ f(2)=\frac{1}{3}(2)^3+(2)^2-8(2)$ $=-\frac{28}{3}$ When $x=-4;\ f(-4)=\frac{1}{3}(-4)^3+(-4)^2-8(-4)$ $=\frac{80}{3}$ Stationary points are $\left(2,-\frac{28}{3}\right)$ and $\left(-4,\frac{80}{3}\right)$.	**K1** Differentiate **K1** **K1** **J1**
(b)	$\frac{dV}{dt}=54$ $\frac{dr}{dt}=\frac{dV}{dt}\times\frac{dr}{dV}$ $V=\frac{4}{3}\pi r^3=36\pi$ $r^3=27, r=3$ $\frac{dV}{dr}=4\pi r^2$ $\therefore \frac{dr}{dt}=54\times\frac{1}{4\pi(3)^2}=\frac{3}{2\pi}=0.4775\ \text{cms}^{-1}$	**B1** **K1** **J1** Value of r **K1** Differentiate **K1J1**
	TOTAL MARKS	**10 marks**