# Quantifying Node Importance over Network Structural Stability

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## ABSTRACT

Quantifying node importance on engagement dynamics is critical to support network stability. We can motivate or retain the users in a social platform according to their importance s.t. the network is more sustainable. Existing studies validate that the *coreness* of a node is the "best practice" on network topology to estimate the engagement of the node. In this paper, the importance of a node is the effect on the engagement of other nodes when its engagement is strengthened or weakened. Specifically, the importance of a node is quantified via two novel concepts: the *anchor power* to measure the engagement effect of node strengthening (i.e., the overall coreness gain) and the *collapse power* to measure the engagement effect of node weakening (i.e., the overall coreness loss). We find the computation of the two concepts can be naturally integrated into a shell component-based framework, and propose a unified static algorithm to compute both the anchored and collapsed followers. When the network structure evolves, efficient maintenance techniques are designed to update the follower sets of each node, which is faster than redoing the static algorithm by around 3 orders of magnitude. Extensive experiments on real-life data demonstrate the effectiveness of our model and the efficiency of our algorithms.

## KEYWORDS

Cohesive subgraph, $k$-core, node importance, network stability

**ACM Reference Format:**
Fan Zhang, Qingyuan Linghu, Jiadong Xie, Kai Wang, Xuemin Lin, and Wenjie Zhang. 2023. Quantifying Node Importance over Network Structural Stability . In *Proceedings of 29th ACM SIGKDD Conference on Knowledge Discovery and Data Mining (KDD '23).* ACM, New York, NY, USA, 13 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

## 1 INTRODUCTION

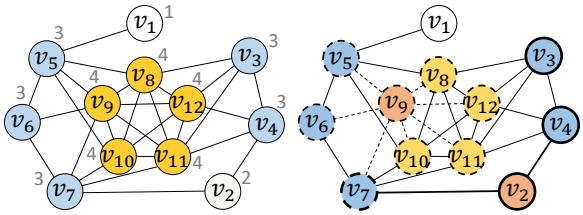
The structural stability of a network indicates the ability of the network to maintain a sustainable service and/or to defend the attacks from competitors. The leave of some nodes (aka the vertices or the users) in a network can be contagious, causing the continuous departure of other affected nodes and eventually breaking the stability of the network [17, 47]. For instance, Friendster was a popular social network with over 115 million users but it is suspended due to contagious leave of users with engagement decline [17, 47]. Thus, it is essential to quantify the importance of the nodes in a network, and then motivate or protect the nodes according to their importance to support the stability of the network [30, 36].

Complex networks are naturally modeled as graphs when studied. In graph theory, the $k$-core is defined as a (connected) maximal subgraph where the degree of each vertex in the subgraph is at least $k$, i.e., each vertex has at least $k$ neighbors in the subgraph [37, 46]. Accordingly, every vertex has a unique *coreness* value: the largest $k$ s.t. the $k$-core contains the vertex. Given a graph, *core decomposition* [2] iteratively removes every vertex with the smallest degree in current graph s.t. the $k$-cores and the coreness of each node can be computed. Existing works often adopt the $k$-core model on capturing node engagement, e.g., [3, 9, 30, 34, 36, 52–55], since its degeneration property can naturally capture the engagement dynamics in real-life networks. That is, the iterative removal of the nodes with the smallest degree well models the leaving sequence of users in the decline of a network. In [36], the coreness of a node is first demonstrated as the "best practice" on network structure for estimating its engagement. In [30], the engagement of a node (measured by its number of check-ins) is validated as positively correlated with its coreness in Gowalla social network. Thus, the structural stability of a network can be expressed by the coreness aggregation of all the nodes in the network.

Nevertheless, a node with high engagement is not certainly important for sustaining network stability, since motivating or protecting such a node may have a small effect on the engagement of other nodes. Thus, for node strengthening, the importance of a node $u$ is quantified by its *anchor power*, i.e., the coreness gain of all other nodes if $u$ is anchored (the degree of $u$ is regarded as $+\infty$ and it will not be deleted in any batch of core decomposition) [3]. We may give different incentives to the users according to their anchor powers to motivate overall user engagement. Besides, for node weakening, the importance of a node $u$ is quantified by its *collapse power*, i.e., the coreness loss of all other nodes if $u$ is collapsed (the degree of $u$ is regarded as 0 and it will be deleted in the first batch of core decomposition) [54]. The leave of the users with different collapse powers would break the network stability to different extends. We should protect (encourage) the participation of the users in a customized way according to their collapse powers. The


*KDD '23, August 6-10, 2023, Long Beach, CA, USA*

ACM ISBN 978-x-xxxx-xxxx-x/YY/MM.
https://doi.org/10.1145/nnnnnnn.nnnnnnn




**(a) Coreness of Every Vertex** **(b) Anchor $v_2$ / Collapse $v_9$**

**Figure 1: Node Engagement (a) and Node Importance (b)**

two powers are on different views for sustaining network stability: applying the anchor power is to actively strengthen the engagement of users, while applying the collapse power is to protect the user engagement from a defense view. To further motivate the study, the performance of above models on graph structure is validated with real-life node importance data (Section 3).

The anchor power (resp. collapse power) of a node $u$ equals the number of *anchored followers* (resp. *collapsed followers*) of $u$, where a follower is a node with coreness changed for anchoring $u$ (resp. collapsing $u$). It is because the anchoring/collapsing of one node can only change the coreness of another node by at most 1. Thus, in this paper, we aim to investigate and efficiently compute the importance of each node regarding its follower sets. Besides, many real-life networks are evolving (e.g., new friend relations, new web links, etc) [27, 29, 33] and the change of anchor/collapse power can be drastic for a simple graph update. Thus, we also aim to efficiently maintain the follower sets for each node against graph dynamics.

*Example 1.1.* Figure 1a shows a graph of 12 vertices and their connections, the coreness of each vertex is marked near the vertex, e.g., the coreness of $v_9$ is 4. The $k$-core is induced by all the vertices with coreness of at least $k$, e.g., the 3-core is induced by $v_3, v_4, ..., v_{11}$ and $v_{12}$. We can see that a vertex with higher coreness is better engaged in the graph. For node importance, as Figure 1b shows, the user $v_2$ has 3 anchored followers that are $v_3, v_4$ and $v_7$ (coreness increased by anchoring $v_2$, marked in bold circles). The user $v_9$ has 7 collapsed followers that are $v_5, v_6, v_7, v_8, v_{10}, v_{11}$ and $v_{12}$ (coreness decreased by collapsing $v_9$, marked in dashed circles). As the numbers of followers are relatively large, $v_2$ and $v_9$ are more important regarding anchor power and collapse power, respectively.

**Challenges.** To the best of our knowledge, no existing work focuses on the importance study of every single node on network stability. Core maintenance [56] is a streaming algorithm to update the coreness value of each vertex, after an edge is inserted into or removed from the graph. The computation of anchored/collapsed followers can be simulated to core maintenance by adding edges to the anchored vertex or removing edges from the collapsed vertex. However, it is cost-prohibitive because one execution of core maintenance can compute at most one follower set and we have to repeatedly execute core decomposition many times.

Recent works on user engagement study target at finding the optimal combination of $b$ anchored vertices [30] or $b$ collapsed vertices [52] and focus on pruning unpromising combinations, where $b$ is a given budget. Our problem is budget-free as we compute the importance of every single user, which is favored in the applications regarding motivating/protecting a large percent or all of the users in a network. Different actions may be applied to the users with different importances on network stability. Our problem also addresses the scenarios when there is no specified budget value or the budget is not fixed with time evolves. The advanced algorithms in above studies can be adapted as a baseline solution (Algorithm 1) to compute the follower set of every vertex on a static graph, while this is still not efficient enough on large graphs.

Regarding graph dynamics, after an edge insertion or removal, both the coreness values and the anchor/collapse powers of many vertices may change. As validated in our preliminary experiment, a large portion of the vertices for anchoring/collapsing in the graph will have their follower sets changed after a random removal of 1000 edges (up to 13% on the datasets in Section 6). Thus, it is non-trivial to locate the candidate vertices for updating their anchored/collapsed followers, and we need to carefully track the importances of all the vertices when the network evolves.

**Our Solution.** To address the challenges, we propose a novel framework to compute and maintain the follower sets of each vertex. We first divide the graph into multiple *shell components*[1] formed by all the maximal connected subgraphs where each vertex has the same coreness. Thus, a follower set consists of some vertex subsets from the shell components. For every vertex as a follower, the candidate sets of its anchored/collapsed vertices can be tracked oppositely. Therefore, we enumerate each shell component and compute the followers in the component for each candidate anchored/collapsed vertex (Algorithm 4) s.t. the proposed algorithm is more efficient and has a lower time complexity compared with the baseline. Due to the independent follower computation of each candidate on the corresponding shell components, any parallel architecture can be utilized to accelerate the computation.

We also propose a novel maintenance algorithm to efficiently update the follower sets of every vertex. When an edge is inserted into or removed from the graph, we first apply core maintenance [56] to update the coreness of each vertex. Then, the new induced shell components are carefully collected, and the follower sets of any vertex can be updated only based on the new shell components. The scale of new shell components against one inserted or removed edge is often small, and the parallelization can also be applied to further speed up the maintenance algorithm.

**Contributions.** The main contributions of this paper are as follows.

1) We motivate the anchor/collapse power on quantifying node importance over network structural stability, by validating its performance on real data (Section 3).

2) We formally prove that the candidate sets for anchoring or collapsing can be located in fine granularity and the follower sets can be independently computed on each shell component which can be efficiently parallelized (Sections 5.1-5.2).

3) An online maintenance algorithm is proposed to update the followers of each node against edge insertion/deletion, to correctly and efficiently handle real-life dynamic graphs (Section 5.3). We also further optimize the follower computation of one node (Section 5.4).

[1]The shell components are more fine-grained than the core tree used in existing studies [29, 30, 37, 44, 45, 52] where each tree node may correspond to multiple shell components.

4) The experiments conducted on 9 real-life datasets validate that the model of anchor/collapse power is effective and our proposed algorithms are efficient (Section 6).

## 2 RELATED WORK

The $k$-core [37, 46] is studied in various areas, e.g., community discovery [13–15, 25, 26], spreader identification [20, 28, 35, 49], and the applications of biology and ecology [1, 4, 41]. Core decomposition algorithms are studied under different computation environments [2, 8, 40, 51]. Network robustness metrics are surveyed in [16, 31, 42] where the focuses are different to our model, e.g., the centrality measures are built on information flow [10, 23, 39, 50] while our anchor/collapse power is based on user engagement. A node with a large centrality may be influential in information cascade while a node with a large anchor/collapse power is important on sustaining overall user engagement of a network. As validated in later experiments, the coreness-based metrics built on graph structure can well match the ground-truth node importance.

The anchor problems aim to find and enhance a small set of nodes to improve coreness aggregation level [3, 30]. The collapse problems hold a different view where a set of nodes are protected to sustain a certain level of coreness aggregation [52, 54]. Node anchoring is also studied in the scenario that motivates a smallest set of users s.t. an improvement quota on stability is satisfied [32]. A variant of coreness loss is to delete a user set leading to the maximum number of coreness-changed users [12]. Besides the anchoring/collapsing on nodes, edge manipulation is considered in network stability study to add promising new edges or protect key existing links, e.g., [6, 22, 38, 48, 57, 58]. Different to above network-level study, some works focus on the stability enhancement or protection on specified communities, e.g., [5, 7]. Nevertheless, none of the above works explore the importance of every node in a network and study its efficient computation on both the static and dynamic graphs.

## 3 ANALYSIS OF REAL DATA

To better motivate the study, we analyze the real data from Yelp [11] to check whether the coreness of a node can well estimate its engagement level and whether the anchor/collapse power of a node can well match the importance of the node on sustaining overall user engagement. The Yelp data contains a social network $G$ containing the users as the vertices and their friend relations as the edges, and the reviews (for Yelp POIs) written by the users with timestamps. The evaluation on Brightkite data is in Appendices, which also validates the effectiveness of the anchor/collapse power.

**Case Study on Node Engagement.** We first analyze the correlation between node engagement and different vertex ranking metrics on the structure of $G$ including coreness, degree, betweenness centrality [18] and closeness centrality [10]. The *ground-truth engagement* of a user is represented by his/her number of reviews for all the POIs in Yelp. Figure 2 depicts the average number of reviews for the vertices with the same coreness (resp. degree, betweenness or closeness). Since computing the betweenness or closeness centrality is costly, we randomly sample 1000 vertices to show their correlation with node engagement. For degree and coreness, we compute on all the vertices. We find that the engagement values may differ a lot for the vertices with close degrees (resp. betweenness or closeness values), while coreness is clearly in a positive correlation with node engagement. It is because core decomposition well models the leaving sequence of users in the degeneration of a network. The users with large corenesses are less likely to leave the network, compared with the users with small corenesses. Thus, the result is also worse than coreness for other metrics, e.g., the variants of centrality metrics [16, 31, 42]. The outperformance is similar if we normalize the scores to the same granularity for every metric.

**Case Study on Node Importance.** We further validate the match of anchor/collapse power (i.e., the number of anchored/collapsed followers) and node importance. To find the real importance data, we need to check the behavior of real anchored/collapsed users. Thus, for every two consecutive months in the Yelp data, we say a user is an *anchored user* (resp. *collapsed user*) if his/her number of reviews in the latter month is larger (resp. smaller) than the former month. Recall that the importance of a node is quantified by the effect on the engagement of other nodes if its engagement is strengthened/weakened, i.e., the engagement dynamic of its anchored/collapsed followers. Therefore, the *ground-truth importance* of an anchored user (resp. collapsed user) is measured by the average variation of review numbers of his/her anchored (resp. collapsed) followers between two consecutive months.

Then, we compare the anchored users (resp. collapsed users) with different scores on $G$ for each metric, i.e., the number of anchored followers, the number of collapsed followers, coreness, degree, betweenness, and closeness, respectively. For each metric, the first group contains the top 100 users[2] according to their scores, and the second group contains the rest users (793 users on average). The *metric effectiveness* (*ME*) is measured by the ground-truth importance of the first group divided by that of the second group.

Figure 3 shows the *ME* from different metrics where each bar is the average value from all consecutive months during one year and the error-bar shows the variance of *ME* from every month. Note that the anchor/collapse power (i.e., #Followers) performs the best for *every* two consecutive months among the 11-year real data. It shows that, for strengthening (resp. weakening) the users with large anchor (resp. collapse) powers, the effect on other users (i.e., the importance) is larger than strengthening (resp. weakening) the users from other metrics. Note that the result is also worse than the anchor/collapse power for other variant metrics. Thus, the metric (anchor/collapse power) studied in this paper is superior for quantifying node importance on network structural stability.

## 4 PRELIMINARIES

We consider an unweighted and undirected graph $G = (V, E)$. The notations are summarized in Table 1. We may omit $G$ in notations when the context is clear, e.g., using $deg(u)$ instead of $deg(u, G)$.

*Definition 4.1 ($k$-core[3]).* Given a graph $G$ and a positive integer $k$, a subgraph $G'$ is the $k$-core of $G$, denoted by $C_k(G)$, if $(i)$ $G'$ satisfies

[2] As the reviews with timestamps are "recommended reviews" selected by Yelp [11], the average number of anchored/collapsed users in one month is only 893.

[3] In this paper, we do not require the $k$-core be connected as in [37, 46] and use $k$-core to represent all the connected subgraphs satisfying Definition 4.1.

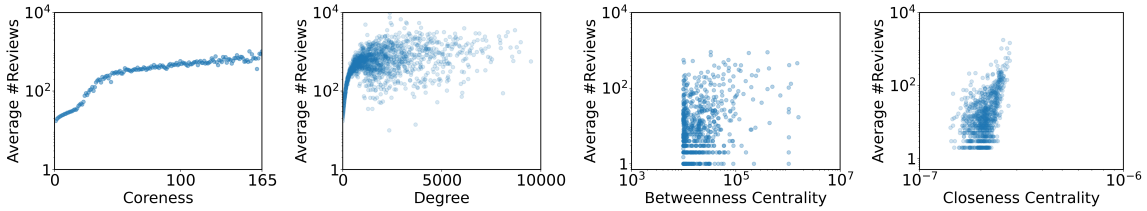
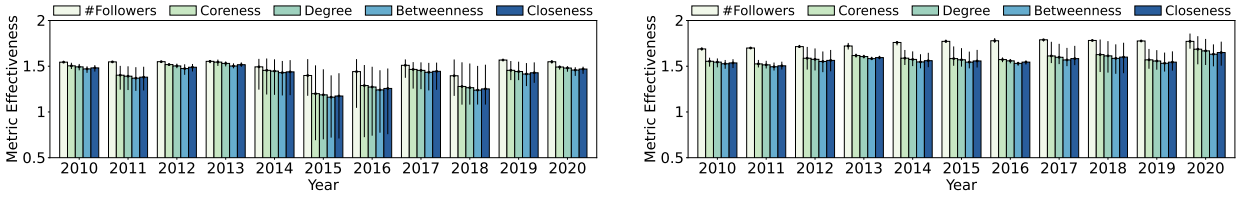



**Figure 2: Different Ranking Metrics v.s. Ground-truth Node Engagement (#Reviews in Yelp)**




**(a) Importance of Anchored Users (Top 100 / The Rest)** **(b) Importance of Collapsed Users (Top 100 / The Rest)**

**Figure 3: Effectiveness of Different Metrics by Relative Importance of the Selected Users (Dynamic of #Reviews in Yelp)**

the degree constraint, i.e., $deg(u, G') \geq k$ for each $u \in V(G')$; and (*ii*) $G'$ is maximal, i.e., any supergraph $G'' \supset G'$ is not a $k$-core.

Given two integers $k$ and $k'$ with $k \geq k'$, the $k$-core is always a subgraph of the $k'$-core, i.e., $C_k(G) \subseteq C_{k'}(G)$. In addition, each vertex in $G$ has a unique coreness value.

*Definition 4.2 (coreness).* Given a graph $G$, the coreness of a vertex $u \in V(G)$, denoted by $c(u, G)$, is the largest $k$ such that $C_k(G)$ contains $u$, i.e., $c(u, G) = max\{k \mid u \in C_k(G)\}$.

Given a graph $G$, *core decomposition* is to compute the coreness for each vertex $v \in V(G)$, which can be computed in $O(m)$ time by recursively deleting the vertex with the smallest degree in $G$ [19].

Now we introduce the concepts of *anchor* and *collapser*. In this paper, once a vertex $x$ is anchored, its degree is regarded as positive infinity (i.e., $deg(x, G) = +\infty$); once a vertex $x$ is collapsed, its degree is regarded as zero (i.e., $deg(x, G) = 0$). Note that the existence of anchored/collapsed vertices does not change the neighbor set of any vertex in the graph. An anchored vertex is also called an anchor (or an anchor vertex), and a collapsed vertex is also called a collapser (or a collapser vertex), respectively.

In core decomposition, an anchored vertex will not be removed as its degree is positive infinity; for every collapsed vertex, it will be deleted at the first iteration of core decomposition as its degree is zero. Thus, anchoring a vertex may increase the corenesses of other vertices, and collapsing a vertex may decrease the corenesses of other vertices, respectively.

When a vertex $x$ is anchored, we use $c_x^+(u, G)$ to denote the coreness of $u$ in $G$ with $deg(x, G) = +\infty$. A vertex with coreness increased is called the *anchored follower* of $x$.

*Definition 4.3 (anchored follower set).* Given a graph $G$ and an anchor vertex $x$, the anchored follower set of $x$ in $G$ is denoted by ${}^+\mathcal{F}(x, G)$, formed by every vertex except $x$ with its coreness increased after anchoring $x$ in $G$, i.e., ${}^+\mathcal{F}(x, G) = \{u \in V(G) \mid u \neq x \wedge c_x^+(u, G) > c(u, G)\}$.

**Table 1: Summary of Notations**

| Notation | Definition |
|---|---|
| $G$ | an unweighted and undirected graph |
| $V(G)$; $E(G)$ | the vertex set of $G$; the edge set of $G$ |
| $G[V]$ | the subgraph of $G$ induced by $V$ |
| $n$; $m$ | $\|V(G)\|$; $\|E(G)\|$ (assume $m > n$) |
| $N(u, G)$ | the set of neighbors of $u$ in $G$ |
| $deg(u, G)$ | $+\infty$ if $u$ is anchored, 0 if $u$ is collapsed, and $\|N(u, G)\|$ otherwise |
| $C_k(G)$ | the $k$-core of $G$ |
| $c(u, G)$ | the original coreness of $u$ in $G$ |
| $c_x^+(u, G)$; $c_x^-(u, G)$ | the coreness of $u$ in $G$ with anchoring/collapsing $x$ |
| ${}^+\mathcal{F}(x, G)$; ${}^-\mathcal{F}(x, G)$ | the anchored/collapsed follower set of $x$ in $G$ |
| $H_k(G)$ | the $k$-shell of $G$ |
| $\mathcal{SC}[v]$ | the only shell component containing $v$ |
| $S$; $S.V$; $S.E$; $S.c$ | a shell component and its vertex set, edge set and coreness value, respectively |
| $\mathcal{A}[S]$; $C[S]$ | the anchor/collapser candidate set of $S$ |
| ${}^+F[x][S]$; ${}^-F[x][S]$ | the anchored/collapsed follower set of $x$ in $S$ |

When a vertex $x$ is collapsed, we use $c_x^-(u, G)$ to denote the coreness of $u$ in $G$ with $deg(x, G) = 0$. A vertex with coreness decreased is called the *collasped followers* of $x$.

*Definition 4.4 (collapsed follower set).* Given a graph $G$ and a collapser vertex $x$, the collapsed follower set of $x$ in $G$ is denoted by ${}^-\mathcal{F}(x, G)$, formed by every vertex except $x$ with its coreness decreased after collapsing $x$ in $G$, i.e., ${}^-\mathcal{F}(x, G) = \{u \in V(G) : u \neq x \wedge c_x^-(u, G) < c(u, G)\}$.

In this paper, we aim to efficiently compute/maintain the anchored/collapsed follower sets for each vertex to capture its node importance over network structural stability.

**Problem Statement.** Given a graph $G$, for each $v \in V(G)$, we compute ${}^+\mathcal{F}(v, G)$ and ${}^-\mathcal{F}(v, G)$. When an edge is inserted into or removed from $G$, for each $v \in V(G)$, we maintain ${}^+\mathcal{F}(v, G)$ and ${}^-\mathcal{F}(v, G)$.

**Algorithm 1**: **Baseline**($G$)

**Input** : $G$ : the graph
**Output** : ${}^{+}\mathcal{F}(v, G)$ and ${}^{-}\mathcal{F}(v, G)$ for each $v \in V(G)$

1 $c(u) \leftarrow$ the coreness of each $u \in V(G)$;
2 **for** each $u \in V(G)$ in parallel **do**
3 Compute ${}^{+}\mathcal{F}(u)$ by Algorithm 4 of [30];
4 **for** each $u \in V(G)$ with at least 1 follower [52] in parallel **do**
5 Compute ${}^{-}\mathcal{F}(u)$ by Algorithm 3 of [52];
6 **return** ${}^{+}\mathcal{F}(v)$ and ${}^{-}\mathcal{F}(v)$ for each $v \in V(G)$

**Baseline Algorithm.** The baseline algorithm is based on the state-of-the-art algorithms on computing/maintaining the coreness of every vertex and computing the anchored/collapsed followers. Algorithm 1 shows the pseudo-code of the baseline. We first compute the coreness of each vertex, by core decomposition [19] in the static scenario, or by core maintenance in the dynamic scenario (Line 1). Then, we compute the anchored follower sets for each vertex $u$ (Lines 2-3) by Algorithm 4 of [30]. We enumerate each vertex with at least 1 collapsed follower (Lines 4-5) and compute by Algorithm 3 of [52] to find collapsed followers. Algorithm 1 can be easily parallelized since the computation on each vertex from Line 2 or Line 4 is independent.

As the worst-case time cost of Algorithm 4 in [30] is $O(m)$, and Algorithm 3 in [52] is $O(d_{max} \cdot m)$, the time complexity of Algorithm 1 is $O(n \cdot d_{max} \cdot m)$ where $d_{max}$ is the largest vertex degree in $G$. However, the baseline algorithm is still costly since it has redundant computations when searching for followers. Besides, for dynamic graphs, it needs to consider all the vertices in the graph as the candidates to compute their anchored/collapsed followers, which is cost-prohibitive.

## 5 OUR SOLUTION

In this section, we firstly introduce the concept of shell component in Section 5.1 and propose the algorithm for static computation utilizing the shell components in Section 5.2. Then, we propose the algorithm to maintain the follower sets against graph dynamics in Section 5.3. The optimized algorithms for computing the followers of one vertex are introduced in Section 5.4.

### 5.1 The Shell Component

Based on the concept of coreness, we define the shell component.

*Definition 5.1 (k-shell).* Given a graph $G$ and a positive integer $k$, the $k$-shell, denoted by $H_k(G)$, is the set of vertices in $G$ with their corenesses exactly equal to $k$, i.e., $H_k(G) = V(C_k(G)) \setminus V(C_{k+1}(G))$.

*Definition 5.2 (shell component).* Given a graph $G$ and the $k$-shell $H_k(G)$, a subgraph $S$ is a shell component of $H_k(G)$, if $S$ is a maximal connected component of $G[H_k(G)]$.

A $k$-shell is formed by the vertices of a series of non-overlapping shell components. For each vertex in $G$, there exists only one shell component $S$ containing $v$. Besides, in core decomposition, for an integer $k$, the deletion sequence of the shell components of $H_k(G)$ can be arbitrary.

**Algorithm 2**: **ShellDecomp**($G$)

**Input** : $G$ : the graph
**Output** : $\mathcal{SC}$ : the index of shell components in $G$

1 $c(u, G)$ of each $u \in V(G)$ by core decomposition [19];
2 **for** each *unassigned* $u \in V(G)$ **do**
3 $S \leftarrow$ a new shell component;
4 $S.c \leftarrow c(u, G)$;
5 $S.V \leftarrow S.V \cup \{u\}$;
6 $u$ is set *assigned*;
7 **ShellConnect**($u$, $S$, $\mathcal{SC}$);
8 $\mathcal{SC}[u] \leftarrow S$;
9 **return** $\mathcal{SC}$

**Algorithm 3**: **ShellConnect**($u$, $S$, $\mathcal{SC}$)

**Input** : $u$ : a vertex, $S$ : the shell component containing $u$, $\mathcal{SC}$ : the shell component index

1 **for** each $v \in N(u, G)$ with $c(v) = c(u)$ **do**
2 $S.E \leftarrow S.E \cup \{(u, v)\}$;
3 **if** $v$ is *unassigned* **then**
4 $S.V \leftarrow S.V \cup \{v\}$;
5 $v$ is set *assigned*;
6 **ShellConnect**($v$, $S$, $\mathcal{SC}$);
7 $\mathcal{SC}[v] \leftarrow S$;

*Example 5.3.* In Figure 4a, we have $H_1(G) = \{v_1\}$, $H_2(G) = \{v_2\}$, $H_3(G) = \{v_3, v_4, v_5, v_6, v_7\}$ and $H_4(G) = \{v_8, v_9, v_{10}, v_{11}, v_{12}\}$. As circled in the figure, we have 5 shell components from $S_1$ to $S_5$, e.g., $S_3$ and $S_4$ are two shell components of $H_3(G)$. In core decomposition of $G$, the deletion sequence of shell components can be either $(S_1, S_2, S_3, S_4, S_5)$ or $(S_1, S_2, S_4, S_3, S_5)$.

Note that the shell components are more fine-grained than the core tree/forest [29, 30, 37, 44, 45, 52] where each tree node also contains some vertices with a same coreness but it may correspond multiple shell components, e.g., a tree node will contain both $S_3$ and $S_4$ in Figure 4.

For a shell component $S$ of $H_k(G)$, we denote $S.V$, $S.E$ and $S.c$ as the vertex set, edge set and the coreness of the vertices in $S$, i.e., $S.V = V(S)$, $S.E = E(S)$ and $S.c = c(v, G)$ for any $v \in S.V$. We use the structure $\mathcal{SC}$ to index the shell components for all the vertices. For each $v \in V(G)$, $\mathcal{SC}[v]$ is the only shell component with $v \in \mathcal{SC}[v].V$.

**Shell Component Computation.** The shell components can be computed in $O(m)$ time by traversing the graph by a constant number of times. Algorithms 2 and 3 illustrate the process of decomposing each vertex into its shell component.

In Algorithm 2, firstly we need to conduct core decomposition [19] on $G$ to get the coreness of each vertex (Line 1). We traverse all the vertices with each vertex marked *unassigned* by default (Line 2). Each time meeting an *unassigned* vertex $u \in V(G)$, we create a new shell component $S$ (Line 3), set the related domains of $S$ and set $u$ as *assigned* (Lines 4-6). Then we call Algorithm 3 (detailed in next paragraph) to recursively collect all the vertices which should exist in $S$ (Line 7). After that, we set $\mathcal{SC}[u]$ by $S$ (Line 8). When all the vertices are set *assigned* (by Algorithms 2 or 3), we can get $\mathcal{SC}$.

**Algorithm 4: StaticFollowerComputation($G$)**

**Input** : $G$ : the graph
**Output** : ${}^{+}\mathcal{F}(v,G)$ and ${}^{-}\mathcal{F}(v,G)$ for each $v \in V(G)$

1 $\hat{\mathcal{S}}, \mathcal{A}[\cdot], C[\cdot] \leftarrow$ **ShellDecomp**($G$);
2 **for** each $S \in \hat{\mathcal{S}}$ in parallel **do**
3  **for** each $v \in \mathcal{A}[S] \cup C[S]$ in parallel **do**
4  **if** $v \in \mathcal{A}[S]$ **then**
5  ${}^{+}F[v][S] \leftarrow$ **FindAnchoredFollowers**$(v,S)$;
6  ${}^{+}\mathcal{F}(v,G) \leftarrow {}^{+}\mathcal{F}(v,G) \cup {}^{+}F[v][S]$;
7  **else if** $v \in C[S]$ **then**
8  ${}^{-}F[v][S] \leftarrow$ **FindCollapsedFollowers**$(v,S)$;
9  ${}^{-}\mathcal{F}(v,G) \leftarrow {}^{-}\mathcal{F}(v,G) \cup {}^{-}F[v][S]$;
10 **return** ${}^{+}\mathcal{F}(v,G)$ and ${}^{-}\mathcal{F}(v,G)$ for each $v \in V(G)$

In Algorithm 3, for the vertex $u$, we traverse each neighbor $v \in N(u,G)$. If $c(v) = c(u)$, we add the edge $(u,v)$ into $S.E$ (Line 2). Note that $(u,v)$ and $(v,u)$ are the same in our setting. Only if $v$ is *unassigned* (Line 3), we add $v$ into $S.V$ and recursively call Algorithm 3 to find all the vertices of $S$ (Lines 4-7).

## 5.2 Static Follower Computation

Here we show how to compute the anchored/collapsed follower sets with a better time complexity based on the shell components.

**Theorems for Candidate Sets.** For each shell component, its candidate anchor/collapser sets are limited by the following theorems.

Lemma 5.4. *If a vertex $x$ is anchored in $G$, the coreness of any $u \in V(G) \setminus \{x\}$ will not decrease and may increase by at most 1 [30].*

Lemma 5.5. *If a vertex $x$ is collapsed in $G$, the coreness of any $u \in V(G) \setminus \{x\}$ will not increase and may decrease by at most 1 [52].*

We define two affiliated structures for each shell component $S$, denoted by the anchor candidate set $\mathcal{A}[S]$ and the collapser candidate set $C[S]$. The anchor candidate set (resp. collapser candidate set) is the set of vertices that may affect the coreness of vertices in $S.V$ if they are anchored (resp. collapsed).

*Definition 5.6 (anchor candidate set).* Given a shell component $S$ in graph $G$, the anchor candidate set of $S$, denoted by $\mathcal{A}[S]$, is $\mathcal{A}[S] = S.V \cup \{v \mid v \in V(G) \wedge c(v,G) < S.c \wedge N(v,G) \cap S.V \neq \emptyset\}$.

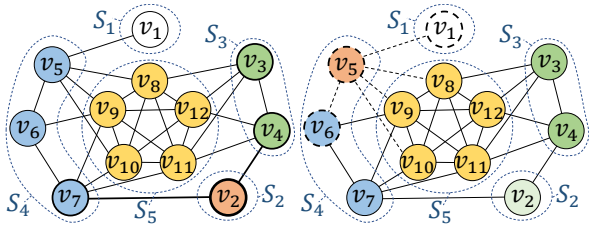
*Definition 5.7 (collapser candidate set).* Given a shell component $S$ in graph $G$, the collapser candidate set of $S$, denoted by $C[S]$, is $C[S] = S.V \cup \{v \mid v \in V(G) \wedge c(v,G) > S.c \wedge N(v,G) \cap S.V \neq \emptyset\}$.

Theorem 5.8. *If a vertex $u \in S.V$ is an anchored follower of vertex $x$, we have $x \in \mathcal{A}[S]$.*

Proof. The proofs are in Appendices for all the theorems. □

Theorem 5.9. *If a vertex $u \in S.V$ is a collapsed follower of vertex $x$, we have $x \in C[S]$.*

**Static Follower Computation.** Here we present our follower computation algorithm for static graphs in Algorithm 4. We firstly run Algorithm 2 to obtain all the shell components of $G$ and record them in $\hat{\mathcal{S}}$. For each $S \in \hat{\mathcal{S}}$, we also get $\mathcal{A}[S]$ and $C[S]$ accordingly,




**(a) Anchor $v_2$** **(b) Collapse $v_5$**

**Figure 4: Follower Computation**

which can be straightforward accumulated when traversing the neighbors of each vertex in computing the shell components.

Then, for each $v \in V(G)$ and a shell component $S$, we define ${}^{+}F[v][S] = S.V \cap {}^{+}\mathcal{F}(v,G)$ and ${}^{-}F[v][S] = S.V \cap {}^{-}\mathcal{F}(v,G)$ so as to compute $v$'s followers in each atom unit, i.e., the shell component $S$. We know that for each shell component $S$, its valid anchors and collapsers (with at least one follower in $S$) can only from $\mathcal{A}[S]$ (Theorem 5.8) and $C[S]$ (Theorem 5.9), respectively. Thus, in Algorithm 4, we compute ${}^{+}F[v][S]$ (resp. ${}^{-}F[v][S]$) w.r.t. every shell component $S$ in $G$. The optimized algorithms to find the followers of a vertex (Lines 5 and 8) will be introduced in Section 5.4.

As the shell component does not overlap with each other, we can directly utilize shared-memory parallelization techniques to compute ${}^{+}F[v][S]$ (resp. ${}^{-}F[v][S]$) for each vertex $v$ on each corresponding shell component $S$ without any conflicts. Note that we apply the parallelization at the vertex level rather than the shell component level to reduce the workload imbalance.

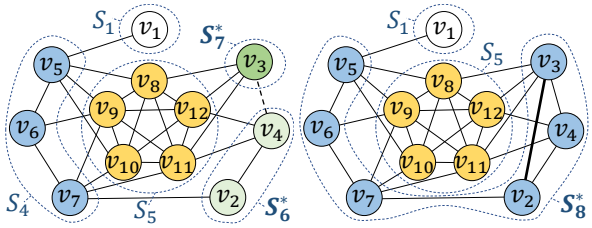
**Complexity Analysis.** In Algorithm 4 , the accumulated size of $\mathcal{A}[S] \cup C[S]$ for every $S$ in $G$ is no larger than $O(m)$ (Line 3). The time cost of follower computation on one vertex by Line 5 or 8 is $O(|S|_{max})$ where $|S|_{max}$ is the largest size (the number of edges) of a shell component. Thus, the time complexity of Algorithm 4 is $O(m \cdot |S|_{max})$. It is lower than the $O(n \cdot d_{max} \cdot m)$ time complexity of Algorithm 1 because $|S|_{max}$ is often smaller than $n$ and is certainly smaller than $n \cdot d_{max}$.

*Example 5.10.* Figure 4a shows a graph with 5 shell components. As there are 3 shell components $S_2$, $S_3$ and $S_4$ with $v_2 \in \mathcal{A}[S_2]$, $v_2 \in \mathcal{A}[S_3]$ and $v_2 \in \mathcal{A}[S_4]$, for the anchoring of $v_2$, we can compute ${}^{+}F[v_2][S_2]$, ${}^{+}F[v_2][S_3]$ and ${}^{+}F[v_2][S_4]$. Then, we have ${}^{+}F[v_2][S_2] = \emptyset$, ${}^{+}F[v_2][S_3] = \{v_3, v_4\}$ as $c^{+}_{v_2}(v_3) = c^{+}_{v_2}(v_4) = 4$, and ${}^{+}F[v_2][S_4] = \{v_7\}$ as $c^{+}_{v_2}(v_7) = 4$. So, ${}^{+}\mathcal{F}(v_2,G) = \{v_3, v_4, v_7\}$.

*Example 5.11.* In Figure 4b, for the collapsing of $v_5$, we compute ${}^{-}F[v_5][S_1]$ and ${}^{-}F[v_5][S_4]$, as $v_5 \in C[S_1]$ and $v_5 \in C[S_4]$. Since $v_5 \notin C[S_5]$, we know there is no follower of collapsing $v_5$ in $S_5$. Then, we have ${}^{-}F[v_5][S_1] = \{v_1\}$ as $c^{-}_{v_5}(v_1) = 0$, and ${}^{-}F[v_5][S_4] = \{v_6\}$ as $c^{-}_{v_5}(v_6) = 2$. So, ${}^{-}\mathcal{F}(v_5,G) = \{v_1, v_6\}$.

## 5.3 The Maintenance w.r.t. Edge Streaming

We consider the following two situations of edge streaming: 1) a single edge is removed from the graph; and 2) a single edge is inserted into the graph. Please note that, multiple edges and vertices streaming can be regarded as a sequence of situations 1) and 2). We maintain the anchored follower set ${}^{+}\mathcal{F}(v,G)$ and the collapsed




**(a) Remove** $(v_3, v_4)$ **(b) Insert** $(v_2, v_3)$

**Figure 5: Follower Maintenance**

follower set ${}^{-}\mathcal{F}(v, G)$ for each $v \in V(G)$ by Algorithm 5 which can be applied for either situation 1) or 2). Specifically, before inserting/removing the edge $(v_s, v_t)$, the current $\mathcal{SC}[\cdot]$, $\mathcal{A}[\cdot]$ and $C[\cdot]$ are stored in $\mathcal{SC}^*[\cdot]$, $\mathcal{A}^*[\cdot]$ and $C^*[\cdot]$, respectively (Line 1), and they will be properly updated. Then we compute the set $V^*$ which contains all the vertices to update corenesses for the insertion/removal of $(v_s, v_t)$ (Lines 2-4). We use $\hat{\mathcal{S}}$ to record all the new shell components and each vertex is set to *unassigned* initially.

After inserting/removing $(v_s, v_t)$ (Line 5), we adopt the state-of-the-art algorithm of core maintenance in [56] to update $c(u, G)$ for each $u \in V(G)$ (Line 6). Lines 7-14 collect the new shell components into $\hat{\mathcal{S}}$. Without the need to call Algorithm 2 again for the whole graph, the maintenance on shell components only starts from each vertex $u$ in $V^*$ (Line 7). Lines 8-13 do the same operations as Lines 3-8 of Algorithm 2 to compute the new shell component set $\hat{\mathcal{S}}$. Please note that $\mathcal{A}^*[\cdot]$ and $C^*[\cdot]$ can be updated straightforward during Lines 7-14 according to their definitions (Line 15), because only $\mathcal{A}^*[S^*]$ and $C^*[S^*]$ for each $S^* \in \hat{\mathcal{S}}$ need to be updated. We denote an updated vertex set as $U = \bigcup_{S^* \in \hat{\mathcal{S}}} S^*.V$ (Line 16). By traversing all the old shell components containing the vertices in $U$, we can remove all the expired anchored followers and collapsed followers (Lines 17-22). At last, for the new shell component set $\hat{\mathcal{S}}$, we call Algorithm 4 on $\hat{\mathcal{S}}$ along with the updated $\mathcal{A}^*[\cdot]$ and $C^*[\cdot]$ (Line 23) to compute the new followers. Finally, we put $\mathcal{SC}^*[\cdot]$, $\mathcal{A}^*[\cdot]$ $\mathcal{SC}[\cdot]$ back into $\mathcal{SC}[\cdot]$, $\mathcal{A}[\cdot]$ and $C[\cdot]$ for using in the next edge updating request. Note that as in Algorithm 4, Lines 17-24 of Algorithm 5 can be straightforward accelerated with shared-memory parallelization.

*Example 5.12.* In Figure 5a, we remove the edge $(v_3, v_4)$ from the graph. In the update of shell components (Lines 7-14), $S_2$ and $S_3$ are replaced (Line 13) with the new shell components, i.e., the new $S_6^*$ with $S_6^*.V = \{v_2, v_4\}$ and $S_6^*.c = 2$, and the new $S_7^*$ with $S_7^*.V = \{v_3\}$ and $S_7^*.c = 3$. The anchor and collapser candidate sets of $S_1$, $S_4$ and $S_5$ remain the same, and the vertices in the above two sets have the unchanged followers on $S_1$, $S_4$ and $S_5$. Thus, we only need to recompute the followers of $\mathcal{A}^*[S_6^*]$ and $C^*[S_6^*]$ on $S_6^*$, and the followers of $\mathcal{A}^*[S_7^*]$ and $C^*[S_7^*]$ on $S_7^*$, e.g., we compute ${}^{+}F[v_2][S_6^*] = \{v_4\}$ and we know ${}^{+}F[v_2][S_4]$ is not changed.

*Example 5.13.* In Figure 5b, we insert the edge $(v_2, v_3)$ into the graph. $S_2$, $S_3$ and $S_4$ are replaced with the new $S_8^*$. We have $S_8^*.V = \{v_2, v_3, v_4, v_5, v_6, v_7\}$ and $S_8^*.c = 3$. The anchor and collapser candidate sets of $S_1$ and $S_5$ remain the same, and the vertices in the above two sets have unchanged followers on $S_1$ and $S_5$. Only the followers of $\mathcal{A}^*[S_8^*]$ and $C^*[S_8^*]$ on $S_8^*$ need to be recomputed.

**Algorithm 5: FollowerMaintenance($(v_s, v_t)$, $G$)**

**Input** : $(v_s, v_t)$ : an edge to insert or remove, $G$ : the graph before inserting/removing $(v_s, v_t)$, $\mathcal{SC}[\cdot]$, $\mathcal{A}[\cdot]$ and $C[\cdot]$ of $G$
**Output** : ${}^{+}\mathcal{F}(v, G)$ and ${}^{-}\mathcal{F}(v, G)$ for each $v$ with changed followers

1 $\mathcal{SC}^*[\cdot]$, $\mathcal{A}^*[\cdot]$, $C^*[\cdot] \leftarrow \mathcal{SC}[\cdot]$, $\mathcal{A}[\cdot]$, $C[\cdot]$;
2 $\hat{\mathcal{S}} \leftarrow$ the new shell component set; $V^* \leftarrow \emptyset$;
3 **if** $c(v_s, G) = \min\{c(v_s, G), c(v_t, G)\}$ **then** $V^* \leftarrow V^* \cup \mathcal{SC}^*[v_s].V$;
4 **if** $c(v_t, G) = \min\{c(v_s, G), c(v_t, G)\}$ **then** $V^* \leftarrow V^* \cup \mathcal{SC}^*[v_t].V$;
5 $(v_s, v_t)$ is inserted into or removed from $G$;
6 Update $c(u, G)$ for each $u \in V(G)$ by core maintenance [56];
/* compute new shell components */
7 **for** each *unassigned* $u \in V^*$ **do**
8 $S^* \leftarrow$ a new shell component;
9 $S^*.c \leftarrow c(u, G)$;
10 $S^*.V \leftarrow S^*.V \cup \{u\}$;
11 $u$ is set *assigned*;
12 **ShellConnect**$(u, S^*, \mathcal{SC}^*)$;
13 Replace $\mathcal{SC}^*[u]$ by $S^*$ in $\mathcal{SC}^*$;
14 $\hat{\mathcal{S}} = \hat{\mathcal{S}} \cup \{S^*\}$;
15 $\mathcal{A}^*[\cdot]$ and $C^*[\cdot]$ are updated while doing Lines 8-15;
16 $U \leftarrow \bigcup_{S^* \in \hat{\mathcal{S}}} S^*.V$;
/* remove expired followers on old shell components */
17 **for** each $S \in \bigcup_{u \in U} \{\mathcal{SC}[u]\}$ in parallel **do**
18 **for** each $v \in \mathcal{A}[S] \cup C[S]$ in parallel **do**
19 **if** $v \in \mathcal{A}[S]$ **then**
20 ${}^{+}\mathcal{F}(v, G) \leftarrow {}^{+}\mathcal{F}(v, G) \setminus {}^{+}F[v][S]$;
21 **else if** $v \in C[S]$ **then**
22 ${}^{-}\mathcal{F}(v, G) \leftarrow {}^{-}\mathcal{F}(v, G) \setminus {}^{-}F[v][S]$;
/* compute new followers on new shell components */
23 Run **Lines 2-9 of Algorithm 4** with $\hat{\mathcal{S}}$, $\mathcal{A}^*[\cdot]$ and $C^*[\cdot]$;
24 $\mathcal{SC}[\cdot]$, $\mathcal{A}[\cdot]$, $C[\cdot] \leftarrow \mathcal{SC}^*[\cdot]$, $\mathcal{A}^*[\cdot]$, $C^*[\cdot]$;
25 **return** the result of Line 23

Theorem 5.14. *For the insertion or removal of $(v_s, v_t)$, Algorithm 5 correctly updates ${}^{+}\mathcal{F}(v, G)$ and ${}^{-}\mathcal{F}(v, G)$ for each $v \in V(G)$.*

**Complexity Analysis.** In Algorithm 5, Lines 1-22 can be finished in $O(m)$ time since they traverse $G$ by a constant number. So, the time complexity of Algorithm 5 is $O(m + m^* \cdot |S^*|_{max})$ where $m^*$ is the accumulated size of $\mathcal{A}^*[S^*] \cup C^*[S^*]$ for every $S^*$ in $\hat{\mathcal{S}}$ and $|S^*|_{max}$ is the largest size of a shell component in $\hat{\mathcal{S}}$.

## 5.4 Computation of Followers for One Vertex

Now we introduce our optimized algorithms of computing the collapsed followers (Algorithm 6) and the anchored followers (Algorithm 7) of one vertex in a shell component.

The *higher coreness support* of a vertex $u$, denoted by $HS(u)$, is the number of $u$'s neighbors with larger corenesses than $u$, i.e., $HS(u) = |\{v \mid v \in N(u, G) \wedge c(v) > c(u)\}|$.

**Collapsed Followers Computation.** We use Algorithm 6 to compute ${}^{-}F[x][S]$ which utilizes a queue $Q$ (Line 1) to explore the collapsed followers starting from the collapser vertex $x$. If $x \in S.V$, $x$ is set *discarded* and pushed into $Q$ (Line 2). Note that, all the

**Algorithm 6**: **FindCollapsedFollowers**($x$, $S$)

**Input** : $x$ : a collapser in $C[S]$, $S$ : a shell component
**Output** : ${}^{-}F[x][S]$ : the collapsed follower set of $x$ in $S$

1 $Q \leftarrow$ a queue;
2 **If** $x \in S.V$ **then** $x$ is set *discarded*; $Q.push(x)$;
3 **else**
4 $\quad HS(u) \leftarrow HS(u) - 1$; $Q.push(u)$ **for** each $u \in N(x, S)$
5 **while** $Q \neq \emptyset$ **do**
6 $\quad u \leftarrow Q.pop()$;
7 $\quad$ **if** $u \neq x$ **then**
8 $\quad\quad d^+(u) \leftarrow HS(u) + |\{v \mid v \in N(u, S) \wedge v \text{ is not } discarded\}|$;
9 $\quad\quad$ **If** $d^+(u) < S.c$ **then** $u$ is set *discarded*;
10 $\quad$ **if** $u$ is *discarded* **then**
11 $\quad\quad$ **for** each $v \in N(u, S)$ with $v$ is not *discarded* and $v \notin Q$ **do**
12 $\quad\quad\quad Q.push(v)$;
13 ${}^{-}F[x][S] \leftarrow$ *discarded* vertices in $S.V \setminus \{x\}$ ;
14 **return** ${}^{-}F[x][S]$

**Algorithm 7**: **FindAnchoredFollowers**($x$, $S$)

**Input** : $x$ : an anchor in $\mathcal{A}[S]$, $S$ : a shell component
**Output** : ${}^{+}F[x][S]$ : the anchored follower set of $x$ in $S$

1 $H \leftarrow$ a min heap w.r.t. the layer value of each vertex;
2 **If** $x \in S.V$ **then** $x$ is set *survived*; $H.push(x)$;
3 **else**
4 $\quad HS(u) \leftarrow HS(u) + 1$; $H.push(u)$ **for** each $u \in N(x, S)$;
5 **while** $H \neq \emptyset$ **do**
6 $\quad u \leftarrow H.pop()$;
7 $\quad V^{\leq} \leftarrow \{v \mid v \in N(u, S) \wedge l(v) \leq l(u) \wedge (v \text{ is survived} \vee v \in H)\}$;
8 $\quad V^{>} \leftarrow \{v \mid v \in N(u, S) \wedge l(v) > l(u) \wedge v \text{ is not } discarded\}$;
9 $\quad$ **if** $u \neq x$ **then**
10 $\quad\quad d^+(u) \leftarrow HS(u) + |V^{\leq}| + |V^{>}|$;
11 $\quad\quad$ **If** $d^+(u) \geq S.c + 1$ **then** $u$ is set *survived*;
12 $\quad$ **if** $u$ is *survived* **then**
13 $\quad\quad$ **for** each $v \in N(u, S)$ with $l(v) > l(u)$ and $v \notin H$ **do**
14 $\quad\quad\quad H.push(v)$;
15 $\quad$ **else**
16 $\quad\quad u$ is set *discarded* ;
17 $\quad\quad$ **Shrink**$(u, S)$ (Algorithm 8);
18 ${}^{+}F[x][S] \leftarrow$ *survived* vertices in $S.V \setminus \{x\}$ ;
19 **return** ${}^{+}F[x][S]$

vertices in $S.V$ are not *discarded* initially, and any *discarded* vertex (except $x$) becomes a collapsed follower. If $x \notin S.V$, for each $u \in N(x, S)$, we reduce $HS(u)$ by 1 and push $u$ into $Q$ (Lines 3-4). Then we traverse $Q$ until it becomes empty (Line 5). Each time when we pop a vertex $u$ (Line 6), if $u \neq x$, we need to decide whether it should be *discarded* (Lines 7-9). If $u$ is set *discarded*, we push each undiscarded $v \in N(u, S) \setminus Q$ into $Q$ (Lines 10-12). After traversing $Q$, all the *discarded* vertices in $S.V$ except $x$ form ${}^{-}F[x][S]$.

**Anchored Followers Computation**. We use Algorithm 7 to compute ${}^{+}F[x][S]$ which adapts Algorithm 4 of [30]. The main idea is utilizing the *layer value* of each vertex, i.e., the deletion batch (layer) in core decomposition. We use $L_k^i$ to denote the $i$-layer of the $k$-shell. Specifically, when $i = 1$, we have $L_k^1 = \{u \mid u \in C_k(G) \wedge deg(u, C_k(G)) < k+1\}$; when $i > 1$, we have $L_k^i = \{u \mid u \in G_k^i \wedge deg(u, G_k^i) < k+1\}$ where $G_k^i = G[V(C_k(G)) \setminus \bigcup_{1 \leq j \leq i-1} L_k^j]$. We denote $l(u) = i$ if $u \in L_k^i$ as the unique *layer value* of $u$.

**Algorithm 8**: **Shrink**($u$, $S$)

**Input** : $u$ : the vertex to shrink, $S$ : a shell component

1 **for** each *survived* vertex $v \in N(v, S)$ with $v \neq x$ **do**
2 $\quad d^+(v) \leftarrow d^+(v) - 1$;
3 $\quad$ **If** $d^+(v) < S.c + 1$ **then** $v$ is set *discarded*; $T \leftarrow T \cup \{v\}$;
4 **for** each $v \in T$ **do**
5 $\quad$ **Shrink**$(v)$;

In Algorithm 7, we utilize a min heap $H$ (Line 1) to traverse the potential followers (except $x$), where the key is the layer value of each vertex. Please note that, any vertex in $S.V$ is neither *discarded* or *survived* unless it is explicitly set so, and all the *survived* vertices form the follower set except $x$. Note that a *survived* vertex may still be *discarded* later due to the deletion cascade.

If $x \in S.V$, $x$ is set *survived* and pushed into $H$ (Line 2). If $x \notin S.V$, for each $u \in N(x, S)$, we increment $HS(u)$ by 1 and push $u$ into $H$ (Lines 3-4). Then we traverse $H$ until it becomes empty (Line 5). When we pop a vertex $u$ (Line 6), if $u \neq x$, we need to decide whether it can be set *survived* (Lines 7-11), where we compute a degree upper bound $d^+(u)$ at Line 10. The vertices in $V^{\leq}$ or $V^{>}$ are the neighbors of $u$ which may become its followers. If $u$ is *survived*, we push each unvisited potential follower into $H$ (Lines 12-14). Once a vertex is set *discarded*, it may cause a cascade of vertex discarding, computed by Algorithm 8 (Line 17). After traversing $H$, all the *survived* vertices in $S.V$ except $x$ form ${}^{+}F[x][S]$.

**Complexity Analysis.** The neighbor set $N(\cdot, S)$, higher coreness support $HS(\cdot)$ and layer value $l(\cdot)$ can be incidentally computed when traversing the neighbors of each vertex in Algorithm 2. Either Algorithm 6 or 7 traverses the graph by a constant number. Thus, the time complexity is $O(|S|)$ for both algorithms, where $|S|$ is the number of edges in $S$.

*Example 5.15.* To compute ${}^{+}F[v_2][S_3]$ on the graph in Figure 4a, we execute Algorithm 7. As $v_2 \notin S_3.V$, we have $HS(v_4) = 2 + 1 = 3$ (originally $HS(v_4) = 2$ w.r.t. $v_{11}$ and $v_{12}$) and $v_4$ is pushed into $H$ (Lines 5-6). After $v_4$ is popped (Line 8), we compute $d^+(v_4)$ (Line 12). Because $l(v_4) = 1, l(v_3) = 2$ and $v_3$ is not *discarded*, we have $V^{\leq} = \emptyset$ and $V^{>} = \{v_3\}$. So, $d^+(v_4) = 3 + 0 + 1 = 4$. Since $S_3.c = 3$, $v_4$ is set *survived* (Lines 13-14) and $v_3$ is pushed into $H$ (Lines 15-17). Then $v_3$ is popped and $d^+(v_3) = 3 + 1 + 0 = 4$ as $HS(v_3) = 3$, $V^{\leq} = \{v_4\}$ and $V^{>} = \emptyset$. So $v_3$ is set *survived* and no more vertex is pushed into $H$. Finally, we return ${}^{+}F[v_2][S_3] = \{v_3, v_4\}$.

## 6 EXPERIMENTAL EVALUATION

The experiment settings are as follows.

**Datasets.** We use 9 real-life datasets, where `NotreDame`, `Stanford` and `DBLP` are from [21], `Yelp` is from [11], and the others are from [24]. We clean self loops and multiple edges in the datasets. Each directed edge is regarded as undirected. Table 2 shows the statistics of the datasets, where $d_{max}$ is the largest vertex degree, $k_{max}$ is the

**Table 2: Statistics of Datasets**

| Dataset | Nodes | Edges | $d_{max}$ | $k_{max}$ | $\|S\|_{max}$ |
|---|---|---|---|---|---|
| Brightkite | 58,228 | 214,078 | 1,134 | 52 | 11,838 |
| Github | 37,700 | 289,003 | 9,458 | 34 | 20,976 |
| Gowalla | 196,591 | 950,327 | 14,730 | 51 | 14,060 |
| NotreDame | 325,729 | 1,090,108 | 10,721 | 155 | 215,052 |
| Stanford | 281,903 | 1,992,636 | 38,625 | 71 | 89,688 |
| Youtube | 1,134,890 | 2,987,624 | 28,754 | 51 | 72,726 |
| DBLP | 1,566,919 | 6,461,300 | 1,522 | 118 | 89,276 |
| Yelp | 1,032,416 | 17,971,548 | 6,367 | 165 | 269,238 |
| Orkut | 3,072,441 | 117,185,083 | 33,313 | 253 | 6,889,284 |


**Figure 6: Performance of Different Algorithms**

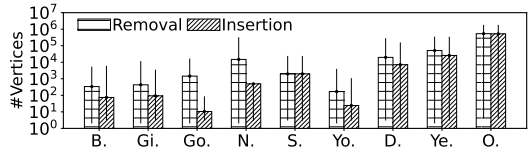



**Figure 7: #Candidate Vertices to Update Follower Sets**

largest vertex coreness and $|S|_{max}$ is the largest number of edges in a shell component. We may abbreviate the name of a dataset by the bold and underlined characters as shown in the table.

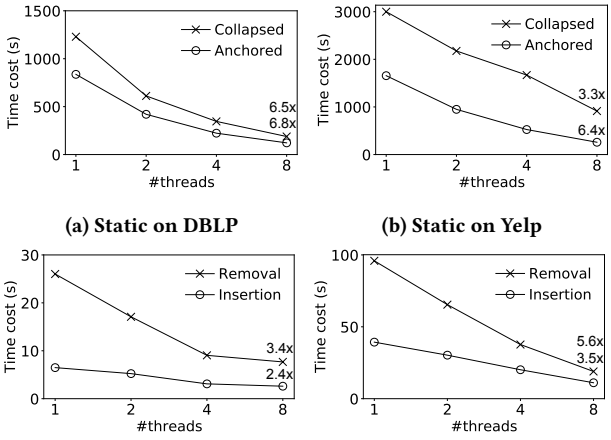
**Algorithms.** We evaluate the following algorithms: (i) `Baseline`: computing the follower sets of each vertex based on the state-of-the-art (Algorithm 1); (ii) `Static`: our new framework to compute the follower sets with shell components (Algorithm 4); (iii) `Maintenance`: our maintenance algorithm to update the follower sets of each vertex (Algorithm 5).

**Environments.** The experiments are performed on a CentOS Linux server (Release 7.5.1804) with Quad-Core Intel Xeon CPU (E5-2640 v4 @ 2.20GHz) and 128G memory. All the algorithms are implemented in C++. The source code is compiled by GCC (7.3.0) under the -O3 optimization. We adopt OpenMP for shared memory multi-processing programming.

## 6.1 Main Experiment Results

**Effectiveness of Anchor/Collapse Power.** For the evaluation of model effectiveness, please refer to the results in Section 3. Some additional results on effectiveness are in the appendices.

**Time Cost of Different Algorithms.** In Figure 6, we report the runtime of `Baseline`, `Static` and `Maintenance`. We use 8 threads for each algorithm. For sequential performance, the runtime gap between different algorithms is very similar to the gap with 8 threads. To test real structure difference, we randomly remove 100 edges and insert them back into each dataset, for the maintenance algorithms. Each time an edge is removed or inserted, we record the time cost




(a) Static on DBLP (b) Static on Yelp

(c) Maintenance on DBLP (d) Maintenance on Yelp

**Figure 8: Parallelization on `Static` and `Maintenance`**

of computing both the anchored and collapsed followers for all the vertices and report the average runtime of 100 edges.

Figure 6 shows that `Static` is faster than `Baseline` by around 1 order of magnitude. It is consistent with the lower time complexity of `Static` compared with `Baseline`, as analyzed at the end of Section 5.2. Besides, `Maintenance` outperforms the static algorithms by orders of magnitude. This is more significant than the difference in time complexities, because the search space of `Maintenance` is often much smaller than the worst case. TODO - The maintenance of edge insertion is generally faster than the maintenance of edge removal, because its update space is larger. We also add the error-bars to show the variance of time cost among the 100 edge removals and 100 edge insertions, respectively. We find that in the worst cases, the time cost of `Maintenance` is still less than other algorithms.

**Number of Candidate Vertices to Update Follower Sets.** We evaluate the number of candidate vertices to update their anchored and collapsed followers for both the edge insertion/deletion cases. On each dataset, we randomly remove 100 edges and then insert 100 edges back. Each time an edge is removed or inserted, we record the number of candidate vertices whose follower sets may be updated, i.e., $\mathcal{A}^*[\cdot]$ and $C^*[\cdot]$ in Algorithm 5. In Figure 7, the bars show the average number of candidate vertices w.r.t one edge insertion or removal. We can find that a single edge update can cause $10^1$ to $10^4$ vertices to update their followers. We also add the error-bar on each box to show the variance of the number of candidates to update among the removal and insertion of 100 edges, respectively. We can find that the minimum numbers of candidate vertices are generally close to 1 and the maximum numbers can reach up to around $10^5$ in extreme cases. The results are consistent with the time cost of the maintenance algorithm in Figure 6.

**Performance of Parallelization.** In Figure 8, we vary the number of threads from 1, 2, 4 to 8 and report the time cost of `Static` and `Maintenance`, for computing the anchored/collapsed follower sets of each vertex. Figures 8(a-b) report the performance of `Static` regarding collapsed follower computation (Algorithm 6) and anchored follower computation (Algorithm 7), respectively. We can

find that the time cost with 8 threads is much smaller than the sequential cost, where the speedup is up to 6.8x for anchoring and 6.5x for collapsing, respectively. Figures 8(c-d) show the performance of `Maintenance` regarding edge insertion and removal, respectively. The update cost of edge removal is usually larger than edge insertion, due to the larger number of candidate vertices to update their followers for edge removal. We mark the speedup ratio between 1 thread and 8 threads in each subfigure, which shows the advantage of our algorithms in separating the computation into atom units.

## 7 ACKNOWLEDGMENTS

This work is partially supported by the National Natural Science Foundation of China (62002073, U2241211) and the Guangdong Basic and Applied Basic Research Foundation (2023A1515012603, 2019B1515120048). We would like to thank the anonymous reviewers for their valuable and constructive feedback. Kai Wang is the corresponding author.

## 8 CONCLUSION

In this paper, we study the model of anchored/collapsed power built on the coreness value. We validate that the coreness value is positively correlated with node engagement, and the match between the anchored/collapsed power and node importance over network structural stability. A novel framework is proposed to compute the anchored/collapsed power of every node on both static and dynamic graphs, with well-designed optimizations. The experiments on 9 real-life datasets demonstrate the effectiveness of the model and the efficiency of our algorithms.

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# A APPENDICES

## A.1 Proofs of Theorems

**Proof of Theorem 5.8.** We prove it by contradiction. We assume $x \notin \mathcal{A}[S]$ and there is a vertex $u \in S.V$ with coreness increased by anchoring $x$ (i.e., $u$ is an anchored follower of $x$). According to the definition of $\mathcal{A}[S]$, we have the following possible situations: 1) $c(x,G) > S.c$; 2) $c(x,G) = S.c \wedge x \notin S.V$; 3) $c(x,G) < S.c \wedge N(x,G) \cap S.V = \emptyset$.

For situation 1), no matter $x$ is anchored or not, $x$ is always deleted after $u$ in core decomposition [19], i.e., the vertex deletion sequence before removing $u$ is same. So we have $c_x^+(u) = c(u)$ which contradicts with our assumption.

For situation 2), we first consider the vertex deletion sequence in core decomposition without anchoring $x$. After all the vertices with coreness less than $S.c$ are deleted, for each $S' \neq S$ and $S'.c = S.c$, it is feasible to delete $S.V$ before each $S'.V$ in core decomposition. After deleting $S.V$, we denote the deleted vertex set so far as $V^d$. Because $c(x) = S.c$ and $x \notin S.V$, we have $x \notin V^d$. Then, we consider the core decomposition with anchoring $x$. As $x \notin V^d$, we can still delete $V^d$ following the same vertex deletion sequence. Thus for each $v \in V^d$, $c_x^+(v) = c(v)$. Since $u \in S.V \subseteq V^d$, $u$ is not an anchored follower of $x$, which contradicts with our assumption.

For situation 3), we first consider the core decomposition without anchoring $x$. For each $v \in S.V$, we denote $N^<(v) = \{w \mid w \in N(v,G) \wedge c(w) < S.c\}$ and $N^>(v) = \{w \mid w \in N(v,G) \wedge c(w) > S.c\}$; we have $|N(v,S) \cup N^>(v)| < S.c + 1$ as $c(v) = S.c$. Then, we consider the core decomposition with anchoring $x$. For each $w \in N^<(v)$ subject to each $v \in S.V$, we have $w \neq x$ by situation 3) and $c_x^+(w) < S.c + 1$ by Lemma 5.4. Thus each above $w$ is not in the $(S.c+1)$-core. Now consider each $v \in S.V$. Let $W$ denote the set consists of each $w \in N^<(v)$ with $c_x^+(w) = S.c$. The vertices in $N^<(v) \setminus W$ is still deleted before the removal of $v$. Then, it is feasible to delete $W$ before $v$ since $W$ is not in the $(S.c+1)$-core. Then we still have $|N(v,S) \cup N^>(v)| < S.c + 1$. Thus, $v$ is not an anchored follower of $x$. As $u \in S.V$, this contradicts with our assumption. □

**Proof of Theorem 5.9.** We prove it by contradiction. We assume $x \notin C[S]$ and there is a vertex $u \in S.V$ with coreness decreased by collapsing $x$ (i.e., $u$ is an collapsed follower of $x$). According to the definition of $C[S]$, we have the following possible situations: 1) $c(x,G) < S.c$; 2) $c(x,G) = S.c \wedge x \notin S.V$; 3) $c(x,G) > S.c \wedge N(x,G) \cap S.V = \emptyset$.

For situation 1), no matter $x$ is collapsed or not, $x$ is always deleted before $u$ in core decomposition [19], so that $c_x^-(u) = c(u)$ which contradicts with our assumption.

For situation 2), we first consider the vertex deletion sequence in core decomposition without collapsing $x$. After all the vertices with coreness less than $S.c$ are deleted, for each $S' \neq S$ and $S'.c = S.c$, it is feasible to delete $S'.V$ before $S.V$ in core decomposition. After the deletion of every above $S'$, the remaining graph $G'$ still has $deg(v,G') \geq S.c$ for each $v \in S.V$. We denote the deleted (resp. remaining) vertex set so far as $V^d$ (resp. $V^r$). For each $v \in V^r \setminus S.V$, we have $c(v) > S.c$. Because $c(x) = S.c$ and $x \notin S.V$, we can ensure $x \in V^d$. Then, we consider the core decomposition with collapsing $x$. We first delete $V^d$ including $x$, and the remaining graph is still induced by $V^r$ satisfying $deg(v) \geq S.c$ for each $v \in V^r$. Since $u \in S.V \subseteq V^r$, $u$ is not a collapsed follower of $x$, which contradicts with our assumption.

For situation 3), we first consider the core decomposition without collapsing $x$. For each $v \in S.V$, we denote $N^>(v) = \{w \mid w \in N(v,G) \wedge c(w) > S.c\}$; we have $|N(v,S) \cup N^>(v)| \geq S.c$ as $c(v) = S.c$. Then, we consider the core decomposition with collapsing $x$, which firstly deletes each $v \in V(G)$ with $c_x^-(v) < S.c$. For each $w \in N^>(v)$ subject to each $v \in S.V$, we have $w \neq x$ by situation 3) and $c_x^-(w) \geq S.c$ by Lemma 5.5. For each $v \in S.V$, the vertices of $N^>(v)$ exist in current remaining graph, so we still have $|N(v,S) \cup N^>(v)| \geq S.c$. Thus, $v$ is not a collapsed follower of $x$. As $u \in S.V$, this contradicts with our assumption. □

**Proof of Theorem 5.14.** We first prove that Lines 1-14 correctly updates the shell components. Assume $c(v_s) \leq c(v_t)$ where the coreness is from the graph before inserting or removing $(v_s, v_t)$. For the insertion or removal of $(v_s, v_t)$, only the vertices with corenesses equal to $c(v_s)$ and are reachable from $v_s$ via a path consists of vertices with corenesses equal to $c(v_s)$ may have their corenesses changed by at most 1 [43, 56], i.e., only the vertices in $V^*$ may change their corenesses. After the insertion or removal, the new shell components in $\hat{\mathcal{S}}$ are correctly computed by Algorithm 3 at Lines 7-14. For any shell component $S'' \notin \hat{\mathcal{S}}$, i.e., $S''$ is not connected with any new shell component $S' \in \hat{\mathcal{S}}$ among the edges in the updated $\mathcal{SC}^*$, we have the coreness of each vertex in $S''$ keeps unchanged. It is because in previous and current core decompositions, the degree of each vertex in $S''$ keeps the same in the remaining graph when $S''$ is the next shell component to be deleted. Otherwise, $S''$ will be updated by Lines 7-14. Thus, each above $S''$ keeps the same and the shell components are correctly updated. By Theorems 5.8 and 5.9, the follower sets regarding the expired shell components should be removed (Lines 19-22) and the follower sets regarding the new shell components are correctly computed (Line 23). □

## A.2 Additional Experiments

**Different Metrics v.s. Ground-truth Node Engagement.** Similar to the engagement study in Section 3, we further validate the correlation between different metrics and node engagement in `Brightkite` data from [24]. Note that there are only a few datasets available for the validation, because we need the ground-truth engagement of each user and the engagement dynamic between different time periods to check the ground-truth node importance.

The setting on `Brightkite` is the same as in the study on `Yelp`, except that the ground-truth engagement of a user here is the number of his/her check-ins in `Brightkite`. As shown in Figures 9, the performance is similar to `Yelp`: the engagement values may differ a lot for the vertices with close degrees (resp. betweenness or closeness values), while coreness is clearly in a positive correlation with node engagement. Note that coreness also outperforms other variants of centrality metrics, because core decomposition well models the leaving sequence of users in the degeneration of a network. Thus, the users with large corenesses are less likely to leave the network, compared with the users with small corenesses. Therefore, it is promising to study the engagement dynamics with the coreness-based models, i.e., the anchor/collapse power.

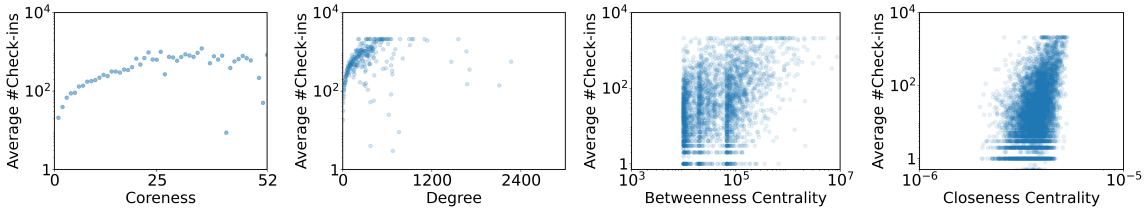
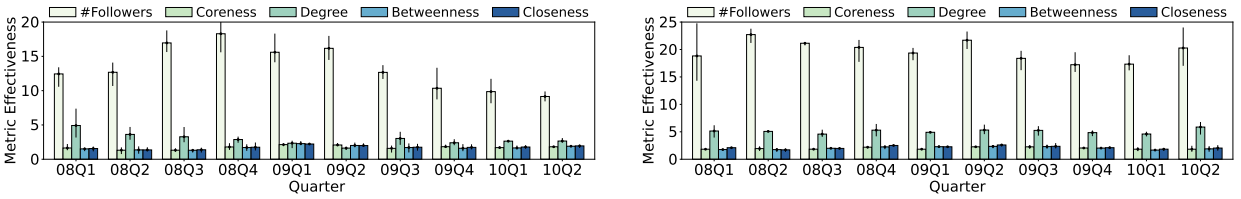



**Figure 9: Different Vertex Ranking Metrics v.s. Ground-truth Node Engagement (#Check-ins in `Brightkite`)**




**(a) Importance of Anchored Users (Top 500 / The Rest)** **(b) Importance of Collapsed Users (Top 500 / The Rest)**

**Figure 10: Effectiveness of Different Metrics by Relative Importance of the Selected Users (Dynamic of #Check-ins in Brightkite)**

**Different Metrics v.s. Ground-truth Node Importance.** Similar to the importance study in Section 3. The setting is the same as in the study on `Yelp`, except that 1) the ground-truth node importance here is represented by the effect on user check-ins, 2) the first group here contains 500 users with the highest scores, and 3) the second group contains the rest users (5737 on average). Compared to the case study of `Yelp` (Figure 3), the gaps between #Followers and other metrics are larger, because of the different settings on user groups and the large number of check-ins in `Brightkite` compared with the number of reviews in `Yelp`.