

6.004 Tutorial Problems

L06 – Boolean Algebra and Logic Synthesis

Note: A small subset of essential problems are marked with a red star (★). We especially encourage you to try these out before recitation.

Problem 1. ★

Consider the truth table on the right, which defines two functions F and G of three input variables (A, B, and C).

For each function, write it in **normal form**, then find a **minimal sum of products** (minimal SOP) expression.

A	B	C	F	G
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	1
1	1	1	1	0

$$\text{Normal form for } F(A,B,C) = \bar{A} \cdot \bar{B} \cdot \bar{C} + \bar{A} \cdot \bar{B} \cdot C + \bar{A} \cdot B \cdot C + A \cdot \bar{B} \cdot \bar{C} + A \cdot B \cdot C$$

We look at all inputs that correspond to a 1 in the F column, which are:

1) 000 ($\bar{A} \cdot \bar{B} \cdot \bar{C}$), 001 ($\bar{A} \cdot \bar{B} \cdot C$), 011 ($\bar{A} \cdot B \cdot C$), 100 ($A \cdot \bar{B} \cdot \bar{C}$), and 111 ($A \cdot B \cdot C$)

The normal form is the sum (OR) of these components

$$\text{Minimal sum of products for } F(A,B,C) = \bar{A} \cdot \bar{B} + B \cdot C + \bar{B} \cdot \bar{C}$$

We group our normal form by common values, recalling that $\bar{A} + A = 1$, to simplify:

$$\begin{aligned}
 & \bar{A} \cdot \bar{B} \cdot \bar{C} + \bar{A} \cdot \bar{B} \cdot C + \bar{A} \cdot B \cdot C + A \cdot \bar{B} \cdot \bar{C} + A \cdot B \cdot C \\
 &= \bar{A} \cdot \bar{B} \cdot (\bar{C} + C) + (\bar{A} + A) \cdot B \cdot C + A \cdot \bar{B} \cdot \bar{C} \\
 &= \bar{A} \cdot \bar{B} + B \cdot C + A \cdot \bar{B} \cdot \bar{C} \\
 &= \bar{B}(\bar{A} + A \cdot \bar{C}) + B \cdot C \\
 &= \bar{B}((\bar{A} + A) \cdot (\bar{A} + \bar{C})) + B \cdot C \\
 &= \bar{B}(\bar{A} + \bar{C}) + B \cdot C = \bar{A} \cdot \bar{B} + \bar{B} \cdot \bar{C} + B \cdot C
 \end{aligned}$$

Normal form for $G(A,B,C) = \bar{A} \cdot \bar{B} \cdot \bar{C} + \bar{A} \cdot \bar{B} \cdot C + \bar{A} \cdot B \cdot \bar{C} + A \cdot \bar{B} \cdot \bar{C} + A \cdot B \cdot \bar{C}$

We look at all inputs that correspond to a 1 in the G column, which are:

1) 000 ($\bar{A} \cdot \bar{B} \cdot \bar{C}$), 001 ($\bar{A} \cdot \bar{B} \cdot C$), 010 ($\bar{A} \cdot B \cdot \bar{C}$), 100 ($A \cdot \bar{B} \cdot \bar{C}$), and 110($A \cdot B \cdot \bar{C}$)

The normal form is the sum (OR) of these components

Minimal sum of products for $G(A,B,C) = \bar{A} \cdot \bar{B} + \bar{C}$

This time, let's examine the truth table rows. We examine for multiple rows where G is 1, and can see the groupings (for ABC) 000 & 001, 100 & 110, 000 & 010 (we can reuse). Turning this to a SOP form yields us:

$$\begin{aligned} &= \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C} + A \cdot \bar{C} \\ &= \bar{A} \cdot \bar{B} + \bar{C} \cdot (A + \bar{A}) \\ &= \bar{A} \cdot \bar{B} + \bar{C} \end{aligned}$$

A smart observer would notice that in all of our literals, C is always 0, which leads to the term \bar{C} .

Problem 2. ★

Consider the 3-input Boolean function $G(A,B,C) = \bar{A} \cdot \bar{C} + A \cdot \bar{B} + \bar{B} \cdot \bar{C}$

1. How many 1's are there in the output column of G's 8-row truth table?

4 (000, 010, 100, 101)

A	B	C	G
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

2. Give a minimal sum-of-products expression for G.

$$\begin{aligned}
 & \bar{A} \cdot \bar{B} \cdot \bar{C} + \bar{A} \cdot B \cdot \bar{C} + A \cdot \bar{B} \cdot \bar{C} + A \cdot \bar{B} \cdot C \\
 &= \bar{A} \cdot \bar{C} \cdot (\bar{B} + B) + A \cdot \bar{B} \cdot (\bar{C} + C) \\
 &= \bar{A} \cdot \bar{C} + A \cdot \bar{B}
 \end{aligned}$$

3. There's good news and bad news: the bad news is that the stockroom only has G gates. The good news is that it has as many as you need. Using only combinational circuits built from G gates, one can implement (choose the best response):

- (A) Any Boolean function (G is functionally complete)
 (B) Only functions with 3 inputs or less
 (C) Only functions with the same truth table as G

$$\begin{aligned}
 G(A, B, 0) &= \bar{A} + A \cdot \bar{B} + \bar{B} \\
 &= \bar{A} + \bar{B} \cdot (A + 1) \\
 &= \bar{A} + \bar{B} \\
 &= \overline{A \cdot B}
 \end{aligned}$$

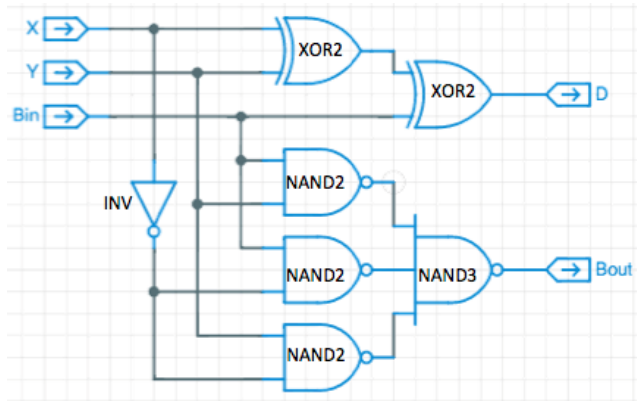
NAND is functionally complete. A universal gate is functionally complete, as all possible Boolean functions can be realized with the gate.

4. Can a sum-of-products expression involving 3 input variables with greater than 4 product terms *always* be simplified to a sum-of-products expression using fewer product terms?

Yes. There will always be at least two rows that differ in exactly one bit and those rows can be merged into one product term.

Problem 3. ★

Consider the logic diagram shown below, which includes XOR2, NAND2, NAND3, and INV (inverter) gates.



Gate	t_{PD}
INV	1.0ns
NAND2	1.5ns
NAND3	1.8ns
XOR2	2.5ns

- Using the t_{PD} information for the gate components shown in the table above, compute the t_{PD} for the circuit.

We determine the t_{PD} as the longest path after delay from input to output. The longest path to output D is the path that goes through both XORs. Similarly, the longest path to output B_{OUT} is through the INV to NAND2 and then to NAND3. We can calculate the t_{PD} of both these paths as:

$$X/Y \rightarrow \text{XOR2} \rightarrow \text{XOR2} \rightarrow D$$

$$t_{PD,D} = 2 * t_{PD,XOR} = 5ns$$

$$X/Y \rightarrow \text{INV} \rightarrow \text{NAND2} \rightarrow \text{NAND3} \rightarrow B_{OUT}$$

$$\begin{aligned} t_{PD,Bout} &= t_{PD,INV} + t_{PD,NAND2} + t_{PD,NAND3} \\ &= 1 + 1.5 + 1.8 = 4.3ns \end{aligned}$$

Path 1 is longer than 2, so t_{PD} is the max: 5ns

$$t_{PD} = \underline{\underline{5ns}}$$

- Find minimal sum-of-products expressions for both outputs, **D** and **Bout**.

NOTE: The gates implement the following functions:

- $NAND2(a, b) = \overline{a \cdot b}$
- $NAND3(a, b, c) = \overline{a \cdot b \cdot c}$
- $XOR2(a, b) = a \cdot \bar{b} + \bar{a} \cdot b$

The output of the first XOR2 is: $X \cdot \bar{Y} + \bar{X} \cdot Y$, and the output D is the XOR2 of that and B_{in}

$$D(X, Y, Bin) = (X \cdot \bar{Y} + \bar{X} \cdot Y) \cdot \overline{B_{in}} + \overline{(X \cdot \bar{Y} + \bar{X} \cdot Y)} \cdot B_{in}$$

Using DeMorgan's, we can simplify:

$$\begin{aligned} \overline{(X \cdot \bar{Y} + \bar{X} \cdot Y)} &= \overline{(X \cdot \bar{Y})} \cdot \overline{(\bar{X} \cdot Y)} = (\bar{X} + Y) \cdot (X + \bar{Y}) \\ &= \bar{X} \cdot X + \bar{X} \cdot \bar{Y} + X \cdot Y + Y \cdot \bar{Y} = X \cdot Y + \bar{X} \cdot \bar{Y} \end{aligned}$$

And then we substitute in:

$$= X \cdot \bar{Y} \cdot \overline{B_{in}} + \bar{X} \cdot Y \cdot \overline{B_{in}} + X \cdot Y \cdot B_{in} + \bar{X} \cdot \bar{Y} \cdot B_{in}$$

Our first 3 NAND2 are: $\overline{B_{in} \cdot Y}$, $\overline{B_{in} \cdot \bar{X}}$, $\overline{Y \cdot \bar{X}}$, so our NAND3 output is: $\overline{\overline{B_{in} \cdot Y} \cdot \overline{B_{in} \cdot \bar{X}} \cdot \overline{Y \cdot \bar{X}}}$
Using DeMorgan's we can simplify:

$$\overline{\overline{B_{in} \cdot Y} \cdot \overline{B_{in} \cdot \bar{X}} \cdot \overline{Y \cdot \bar{X}}} = \overline{\overline{B_{in} \cdot Y}} + \overline{\overline{B_{in} \cdot \bar{X}}} + \overline{\overline{Y \cdot \bar{X}}} = B_{in} \cdot Y + B_{in} \cdot \bar{X} + Y \cdot \bar{X}$$

$$B_{out}(X, Y, B_{in}) = \overline{\overline{B_{in} \cdot Y} \cdot \overline{B_{in} \cdot \bar{X}} \cdot \overline{Y \cdot \bar{X}}} = B_{in} \cdot Y + B_{in} \cdot \bar{X} + Y \cdot \bar{X}$$

Minimal sum of products for D(X,Y,Bin) = $X \cdot \bar{Y} \cdot \overline{B_{in}} + \bar{X} \cdot Y \cdot \overline{B_{in}} + X \cdot Y \cdot B_{in} + \bar{X} \cdot \bar{Y} \cdot B_{in}$

Minimal sum of products for Bout(X,Y,Bin) = $B_{in} \cdot Y + B_{in} \cdot \bar{X} + Y \cdot \bar{X}$

Problem 4.

Simplify the following Boolean expressions by finding a *minimal sum-of-products expression* for each one:

1. $\overline{ac + b + c}$
2. $(a + b)c + \bar{c}a + b(\bar{a} + c)$
3. $a\overline{(b + c)}(b + a(b + c))$
4. $a(b + c(d + ef))$

$$1. \quad \overline{ac + b + c} = \overline{b + c} = \bar{b} \cdot \bar{c}$$

Since c includes the literal ac , we can simplify: $ac + c = c$

$$2. \quad (a + b)c + \bar{c}a + b(\bar{a} + c) = ac + bc + \bar{c}a + b\bar{a} = a + bc + \bar{a}b$$

$$(a + \bar{a}b) + bc = (a + \bar{a})(a + b) + bc = a + (b + bc) = a + b$$

Note that we only write bc once

$$3. \quad a\overline{(b + c)}(b + a(b + c)) = a \cdot \bar{b} \cdot \bar{c} \cdot (b + ab + ac) = a \cdot \bar{b} \cdot \bar{c} \cdot (b + ac) = ab\bar{b}\bar{c} + abc\bar{c} = 0$$

Since $b\bar{b} = 0, c\bar{c} = 0$

$$4. \quad a(b + c(d + ef)) = a(b + cd + cef) = ab + acd + acef$$

Problem 5.

There are some Boolean expressions for which no assignment of values to variables can produce True (e.g., $a\bar{a}$). Those Boolean expressions are said to be *non-satisfiable*. Are the following Boolean expressions satisfiable? If the expression is satisfiable, give an assignment to variables that makes the expression evaluate to True. If the expression is non-satisfiable, prove it.

1. $(a + b)c + \bar{c}a + b(\bar{a} + c)$
2. $(x + y)(x + \bar{y})(z + \bar{y})(y + \bar{x})$
3. $(x + y + z)(x + y + \bar{z})(x + \bar{y} + z)(\bar{x} + y + z) \cdot (x + \bar{y} + \bar{z})(\bar{x} + y + \bar{z})(\bar{x} + \bar{y} + z)(\bar{x} + \bar{y} + \bar{z})$
4. $\overline{xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}}$

1. Satisfiable. Possible solution:

$$a=1, b=1, c=1$$

Alternatively, we can expand to $ac + bc + \bar{c}a + \bar{a}b + bc = ac + bc + \bar{c}a + \bar{a}b$, which has multiple solutions

2. Satisfiable. Possible solution:

$$x=1, y=1, z=1$$

As expanding would be tedious, we note that the expression is a POS, meaning that each of the four sums must be satisfied. We first set $x=1$, satisfying $(x + y)(x + \bar{y})$ and $(x + y)(x + \bar{y})$. Furthermore, the fourth sum $(y + \bar{x})$ indicates $y=1$. Finally, we set $z=1$, setting all the sums to 1, and showing $x=1, y=1, z=1$ as a solution.

3. Non-satisfiable. All 8 combinations of $x+y+z$ are present, so it is impossible to satisfy them all. We can also prove this by reducing the expression:

$$(x + y + z)(x + y + \bar{z})(x + \bar{y} + z)(\bar{x} + y + z)(x + \bar{y} + \bar{z})(\bar{x} + y + \bar{z})(\bar{x} + \bar{y} + z)(\bar{x} + \bar{y} + \bar{z}) \\ = (x + y)(x + \bar{y})(\bar{x} + y)(\bar{x} + \bar{y}) = x\bar{x} = 0$$

4. Satisfiable. Possible solution:

$$x=0, y=0, z=1$$

We have a SOP of 7 terms, which the entirety is negated. This means that all products must evaluate to 0 in order to satisfy the expression. Out of 7 three variable literals, we note that $\bar{x}\bar{y}z$ is omitted from the SOP. Therefore, we conclude $x=0, y=0, z=1$ – the missing term – is a solution to the expression.

Problem 6.

(A) Simplify the following Boolean expressions by finding a minimal sum-of-products expression for each one. (Note: These expressions can be reduced into a minimal SOP by repeatedly applying the Boolean algebra properties we saw in lecture.)

$$\begin{aligned} 1. & \overline{(a + b \cdot \bar{c})} \cdot d + c \\ &= \bar{a} \cdot \overline{(b \cdot \bar{c})} \cdot d + c \\ &= \bar{a} \cdot (\bar{b} + c) \cdot d + c \\ &= \bar{a} \cdot \bar{b} \cdot d + \bar{a} \cdot c \cdot d + c \\ &= \bar{a} \cdot \bar{b} \cdot d + c \cdot (\bar{a} \cdot d + 1) \\ &= \bar{a} \cdot \bar{b} \cdot d + c \end{aligned}$$

$$\begin{aligned} 2. & a \cdot \overline{(b + c)}(c + a) \\ &= a \cdot (\bar{b} \cdot \bar{c})(c + a) \\ &= a \cdot (\bar{b} \cdot \bar{c} \cdot c + a \cdot \bar{b} \cdot \bar{c}) \\ &= a \cdot (a \cdot \bar{b} \cdot \bar{c}) \\ &= a \cdot \bar{b} \cdot \bar{c} \end{aligned}$$

(B) There are Boolean expressions for which no assignment of values to variables can produce True (e.g., $a \cdot \bar{a}$). These Boolean expressions are said to be *non-satisfiable*.

Are the following Boolean expressions satisfiable? If the expression is satisfiable, give an assignment to variables that makes the expression evaluate to True. If the expression is non-satisfiable, explain why.

$$\begin{aligned} 1. & (\bar{x} + y\bar{z}) \cdot (\bar{y}x + z) \cdot (\bar{z}y + x) \\ & \quad \underbrace{\hspace{1.5cm}} \\ & \quad \bar{x}\bar{y}x + \bar{x}z + y\bar{z}\bar{y}x + y\bar{z}z \\ & \quad = \bar{x}z \\ & \quad \quad \underbrace{\hspace{1.5cm}} \\ & \quad \quad \bar{x}z\bar{z}y + \bar{x}zx = 0 \rightarrow \text{Unsatisfiable} \end{aligned}$$

$$\begin{aligned} 2. & (\bar{x} + y\bar{z}) \cdot (\bar{y}x + z) \cdot (\bar{z}y + x) + (\bar{x} + yz) \cdot (\bar{y}x + z) \cdot (\bar{z}y + x) \\ & \quad \underbrace{\hspace{2cm}} \hspace{1cm} \underbrace{\hspace{2cm}} \\ & \quad \hspace{1.5cm} 0 \text{ (same as above)} \hspace{1.5cm} xyz \\ & \hspace{10cm} \text{Satisfiable with } x=1 \ y=1 \ z=1 \end{aligned}$$

We could expand the right term, but instead we notice that $(\bar{x} + yz) = 1$ with $y=1$, $(\bar{y}x + z) = 1$ with $z=1$, and $(\bar{z}y + x) = 1$ with $x=1$.