1 Selfstudy for Tutorial 4

Polking **§9.6:** 9, 10, 37; **§9.9:** 2, 21; **§10.1:** 1, 7 (items ii,iii);

2 Exercises for Tutorial 4

1. Consider the model of a pendulum

$$\frac{d^2\theta}{dt^2} = -\sin(\theta).$$

   (a) Rewrite this model to a system of first-order ODEs.

   (b) Determine the equilibria.

   (c) Classify the equilibria using the eigenvalues of the linearisation.

2. In this exercise, we study how a change in a parameter a changes the phase portrait of a system qualitatively for

$$x' = x^2 + y,$$
$$y' = x - y + a.$$

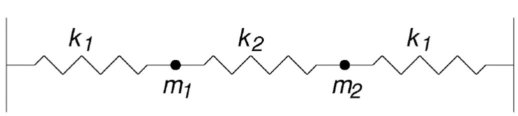
   (a) Determine the equilibria and the linearisation near the equilibria for $a = 1$ and $a = -1$.

   (b) Classify the type of both equilibria.

   (c) Determine the value of a such that there is a qualitative change, i.e., the number or type of equilibria changes. Note the coalescence condition is equivalent to finding an eigenvalue equal to zero.

   We say that there is a critical value a_0 such that the system has a `bifurcation`. Such a bifucation involves a change of stability of an equilibrium or the number of equilibria. The general study of such changes is known as bifurcation theory and the subject of a master course Nonlinear Dynamics.

3. Consider two coupled mass spring systems at positions x_1 and x_2 as in Figure 3.



Figuur 1: Coupled harmonic oscillators

   They are coupled via three springs attached to a wall on either side. Suppose the spring constants of the three springs are k_1, k_2 and k_1 and the masses are $m_{1,2} = 1$. If $x_1 = x_2 = 0$, the springs are at rest. Each separate spring exerts a force on a mass equal proportional to its displacement Δx_i, i.e., $F = km\Delta x_i$.

(a) Consider the two forces acting on each mass. Show that the equations of motion (=differential equations) for this system are given by

$$\begin{aligned} x_1'' &= -(k_1+k_2)x_1 + k_2x_2, \\ x_2'' &= k_2x_1 - (k_1+k_2)x_2. \end{aligned}$$

(b) Rewrite the two ODEs as a first-order system.

(c) Determine the eigenvalues and eigenvectors of the corresponding matrix. For the eigenvectors you may recall that in Tutorial 2, Exercise 2, we showed that for such 2D mechanical systems the eigenvectors are $(1, \lambda)$. Here you will have a similar structure.

(d) Determine the general solution $x(t)$ of the oscillations.

4. The (famous!) Lorenz system is given by

$$\begin{aligned} x' &= -ax + ay, \\ y' &= rx - y - xz, \\ z' &= -bz + xy, \end{aligned}$$

with a, b and r positive constants. Consider a, b to be fixed, but r may vary, i.e., it is a parameter. For more info, see §10.2 of Polking, p471-474. Here we deal with stability analysis of nontrivial equilibria that the book suggests would be too difficult to discuss (p473, line 4). Here we determine this condition by using the Routh-Hurwitz stability criterion. This criterion will return in a Challenge related to Systems Theory for general n-dimensional systems. Here, we have $n = 3$ and then the criterion states that the roots of a polynomial $p(x) = x^3 + p_2x^2 + p_1x + p_0$ have negative real part only if $p_2 > 0$, $p_1 > 0$, $p_0 > 0$ and $p_2p_1 > p_0$.

(a) Determine the linearisation for the equilibrium that exists for any value of r. Determine the eigenvalues and a critical value of r such that this equilibrium loses stability.

(b) Determine the equilibria. Give a condition such that nontrivial equilibria exist.

(c) Show the system has a symmetry, i.e., determine a linear transformation S such that $(S(x, y, z))' = f(S(x, y, z))$.
Hint: Look at the formula for the nontrivial equilibria.

(d) Consider the polynomial $p(x) = x^3 + p_2x^2 + p_1x + p_0$ and suppose $p_0 = p_2p_1$. Decompose p into two factors.

(e) Use the Routh-Hurwitz condition to show that the two nontrivial equilibria are asymptotically stable for $1 < r < r_{max} = \dfrac{a(3+b+a)}{a-b-1}$. Evaluate this expression numerically for $a = 10, b = 8/3$ and compare with the value mentioned on p473 of Polking. Steps: (1) Write down the characteristic polynomial, (2) Show $p_{0,1,2} > 0$, (3) Determine r from $p_0 = p_1p_2$.

(f) Using your decomposition of p, determine the eigenvalues when $r = r_{max}$. How would you classify the equilibria?