# Lecture 3

1) $x' = Ax$ try: $e^{tA}$

$x(t) = \sum_{i=1}^{n} c_i y_i(t)$ $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\downarrow$

fundamental solution $e^{tA}$

$A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$

$x(t) = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $\frac{A^n}{n!} = \begin{pmatrix} \frac{\lambda^n}{n!} & 0 \\ 0 & \ddots \end{pmatrix}$

and $\sum \frac{A^n}{n!} = \begin{pmatrix} \Sigma & 0 \\ 0 & \ddots \end{pmatrix}$

then $x(0) = Y(0)\mathbf{c}$ $\Rightarrow e^{At} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & \ddots \end{pmatrix}$

and so $\mathbf{c} = Y(0)^{-1} x(0)$

$Y(t)c = x(t) = Y(t)Y^{-1}(0)x(0)$

Call $\Phi(t) = Y(t)Y^{-1}(0) = e^{tA}$

$e^{tx=A} = \sum_{n=1}^{\infty} \frac{(tx)^{n \to A}}{n!}$

where $A$: $x' = Ax$

$Y(t) \to$ linearly independant

$\to Y^{-1}(t)$ exists $\to Y^{-1}(0)$ ✓

$J_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}}_{\alpha I} + \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} = R$ $IR = RI$

$NI = IN$

$J_3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{\lambda I} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N$

$e^{(a+b)t} = e^{at}e^{bt} \iff ab = ba$

$R = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ $R^2 = -\beta^2 I$

$e^{Rt} = \sum_{n=0}^{\infty} \frac{I t^n R^n}{n!} = \sum_{n=0}^{\infty} \frac{I(-1)^n \beta^{2n} I^{2n} t^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{I(-1)^n t^{2n-1} \beta^{2n-1} R^{2n-1}}{(2n-1)!}$

$= I(\cos(\beta t)) + \begin{pmatrix} 0 & \sin\beta t \\ -\sin\beta t & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$

$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $N^2 = 0I$

$e^{Nt} = I + tN + O(N^2)$

$= I + tN$

double eigenvalues $\to$ $e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

Jordan Normal form

$A = T^{-1} J T$, $e^{tA} = T^{-1} e^{tJ} T$

$= Y(t) Y(0)^{-1}$

$Y(t) = T^{-1} e^{tJ}$

$x(t) = Y(t)\mathbf{c}$

$y = \Phi(t)c = e^{At} y_0$

$c \to c(t)$

$y' = \Phi'(t)c + \Phi(t)c' = A\Phi(t)c + f(t) \Rightarrow c' = \Phi(t)^{-1} f(t)$

$y(t) = e^{At} y_0 + \int_{t_0}^{t} e^{A(t-s)} f(s)\, ds$