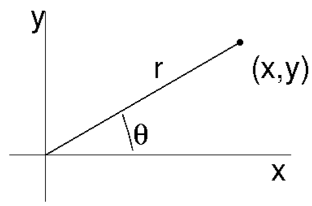
1 Selfstudy for Tutorial 5

Polking **§10.2:** 4, 7, 12, **§10.3:** 1, 3, 18; **§10.4:** 11, 14, 19

2 Exercises for Tutorial 5

1. Conversion to polar coordinates is convenient for some systems. Starting with a system for $x(t), y(t)$ we end up with a system for $r(t), \theta(t)$ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$, see the Figure. During the lecture you have seen one way to derive this transformation. Here we consider another, arriving at the same result.



   (a) By definition we have

$$r^2 = x^2 + y^2, \qquad \text{and} \qquad \tan(\theta) = \frac{y}{x}.$$

   Use these relations to prove

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}, \qquad \text{and} \qquad \frac{d\theta}{dt} = \frac{1}{r^2}\left(x\frac{dy}{dt} - \frac{dx}{dt}y\right).$$

   Separately consider the case $x = 0, y \neq 0$ for which $\tan(\theta)$ is undefined.

   (b) Transform the following system to polar coordinates.

$$\begin{aligned} \frac{dx}{dt} &= y + x(1 - x^2 - y^2)^2 \\ \frac{dy}{dt} &= -x + y(1 - x^2 - y^2)^2. \end{aligned}$$

   (c) Determine whether the origin is (asymptotically) stable or not.

   (d) Sketch the (complete!) phase portrait, first in the (r, θ)-plane and then next in the (x, y)-plane.

2. Consider the following system

$$\begin{aligned} r' &= r(1 - r)(3 - r)^2 \\ \theta' &= 1. \end{aligned}$$

   (a) Determine all possible ω-limit sets.

   (b) Sketch the (complete!) phase portrait in the (x, y)-plane.

3. For the three systems given below, sketch the phase portrait using nullclines. You may restrict yourself to $x, y \geq 0$.

(a) Show that the triangle with nodes $(0,0)$, $(1,0)$ and $(0,1)$ defines a positively invariant region for the system

$$\begin{aligned} x' &= x(y + 2x - 2) \\ y' &= y(y - 1) \end{aligned}$$

(b) The system below describes two species competing for food and space. Can these two species coexist?

$$\begin{aligned} x' &= x(2 - y - 2x) \\ y' &= y(3 - y - 4x) \end{aligned}$$

(c) Show that the triangle with nodes $(0,0)$, $(3,0)$ and $(0,3)$ defines a positively invariant region for the system

$$\begin{aligned} x' &= x(1 - y - x) \\ y' &= y(-2 + 2x - y) \end{aligned}$$

4. Consider the system

$$\begin{aligned} x' &= y \\ y' &= -x + y - y^3. \end{aligned}$$

(a) Sketch the vector field, i.e. some arrows, and the nullclines in the (x, y)-plane.

(b) Find a positively invariant region defined by a polygon for which the vector field points inwards (or along the boundary). Draw this polygon on top of your sketch of the vector field. Check that this region is indeed positively invariant, i.e. compute the inner-product of the flow with the inward normal for every piece of the boundary.

(c) Prove that the system has a periodic orbit.