Answers Tutorial 3

1. (a) Using $v = x'$ we get the system $x' = v$ and $v' = -kx - bv$

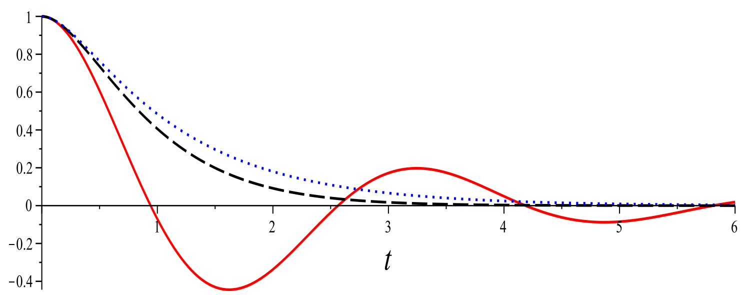
   (b) The eigenvalues are $\lambda_{1,2} = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4k}\right)$

   (c) The general solution for the displacement x is given by the following three cases

   $$\begin{array}{ll} b^2 - 4k > 0 & x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ b^2 - 4k = 0 & x(t) = e^{-bt/2}(c_1 + c_2 t) \\ b^2 - 4k < 0 & x(t) = e^{-bt/2}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right), \qquad \omega = \sqrt{(}4k - b^2) \end{array}$$

   Note you may find v from the relation $v = x'$.

   (d) The case $b^2 - 4k > 0$ corresponds to strong damping; the amplitude decays exponentially.
   For $b^2 - 4k = 0$, we have damping too, but weaker. Moreover, you may get one extremum.
   The case $b^2 - 4k < 0$ yields damped oscillations with frequency $\sqrt{(b^2 - 4k)/2}$. The amplitude of the oscillations (consider the minima/maxima) decay as e^{-bt}.



For illustration we set $k = 4$ and $b = 1$ (red,solid), $b = 4$ (black, dashed) and $b = 5$ (blue, dotted), with initial conditions $x(0) = 1$ and $v(0) = 0$.

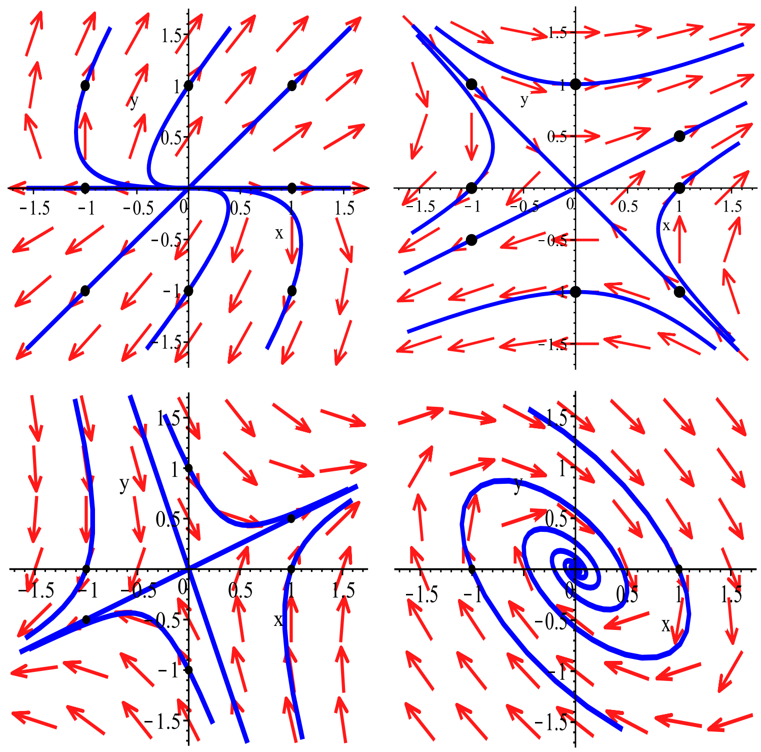
2. (a) Unstable node $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{t} + c_2 e^{3t} \\ c_2 e^{3t} \end{pmatrix}$

   (b) Saddle $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2c_1 e^{t} - c_2 e^{-t} \\ c_1 e^{t} + c_2 e^{-t} \end{pmatrix}$

   (c) Saddle $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -c_1 e^{-5t} + 2c_2 e^{2t} \\ 3c_1 e^{-5t} + c_2 e^{2t} \end{pmatrix}$

   (d) Stable spiral $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t} \begin{pmatrix} 5c_1 \cos(4t) + 5c_2 \sin(4t) \\ c_1(-3\cos(4t) - 4\sin(4t)) + c_2(-3\sin(4t) + 4\cos(4t)) \end{pmatrix}$

   You may use pplane (see Blackboard, DE-part, bottom) to verify these figures for yourself.



3. For A_1

$$v = (1, -1)^T \text{ bij } \lambda = -2, v = (1, 1)^T \text{ bij } \lambda = 4$$

so

$$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \text{ en } e^{tA_1} = \frac{1}{2} \begin{pmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{pmatrix}$$

For A_2

$$v = (\frac{1}{2}(1 \mp i), 1)^T \text{ bij } \lambda = 2 \pm i$$

so

$$A_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} \text{ en } e^{tA_2} = e^{2t} \begin{pmatrix} \cos(t) - \sin(t) & \sin(t) \\ -\sin(t) & \cos(t) + \sin(t) \end{pmatrix}$$

4. $\lambda = -1$ with $v = (-2, -1, 1)^T$ and $\lambda = -2$ (two times, i.e. double eigenvalue) with $v = (1, 1, 1)^T$. The generalised eigenvector w is defined as $Aw = \lambda w + v$. Solving this linear system we obtain $w = (1, 0, -1)^T + c(1, 1, 1)^T$ for some $c \in \mathbb{R}$. We choose $c = 0$.

$$x_1 = e^{-t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \quad x_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_3 = e^{-2t} \left(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right)$$