Answers Tutorial 1

1. (a) This is a linear inhomogeneous ODE. The homogeneous solution is $y_{hom} = ct^2$. Next variation of constants yields $c' = -1/t^2$, so for the particular solution we find $y_p = t$. With that we can write down the general solution $y(t) = t(ct+1)$ with $c \in \mathbb{R}$.

   (b) First rewrite the ODE to the standard form $y' = \dfrac{2y-t}{t}$. The right hand side is $f(y,t) = \frac{2y}{t} - 1$. This is not continuous at $t = 0$ and hence the existence theorem 7.6 (p78) does not apply as $t = 0 \in R$.

2. (a) $x(0) = 0$ and $x(t) = \frac{|t|^3}{3\sqrt{3}}$ which you may find using separation of variables. The parial derivative $\frac{df(t,x)}{dx} = tx^{-2/3}/3$ is not continuous for $x = 0$. Hence the conditions of theorem 7.16 (p82) are violated, so there is no uniqueness of solutions in a neighbourhood of $x = 0$. If you are curious about the absolute value signs in x, then draw this solution within a direction field.

   (b) We can define the following family of solutions for any $t_0 \geq 0$:

$$x = \begin{cases} x = 0, t < t_0 \\ x = |\frac{1}{3}(t^2 - t_0^2)|^{3/2}, t > t_0 \end{cases}$$

   This solution satisfies the initial condition $x(0) = 0$ and stays zero for some time including $x(t_0) = 0$. Each branch separately is a solution of the ODE. Using the difference quotient you may verify that these two branches together define a continuous and differentiable function.

3. (a) Choose $\alpha = 1 - n$, so that $z' = \alpha a(t) z + \alpha f(t)$

   (b) $z(t) = 1 - t + ce^{-t}$.

   (c) $x(t) = \frac{1}{z(t)} = \frac{1}{1-t+ce^{-t}}$

   (d) Sketch the direction field to make the following observations. First, for $c > 0$ we have $\lim_{t\to-\infty} x(t) = 0$ and $\lim_{t\to t_c} x(t) = +\infty$. Second, for $-1 < c < 0$ the functions $y = exp(-t)$ and $y = (t-1)/c$ intersect each other at two places. You cannot determine these points analytically, but it means there are two vertical asymptotes. So the domain of existence is of the form (t_-, t_+) for some lower bound t_- and upper bound t_+, i.e. $\lim_{t\to t_-} x(t) = +\infty$ and $\lim_{t\to t_+} x(t) = +\infty$. Finally for $c < -1$, there is a minimum and we find $\lim_{t\to\pm\infty} x(t) = 0$.

   (a) The condition is $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x}$; See theorem 6.20 (p66). Using $\mu = \mu(x)$ we find the following $\frac{\partial \mu}{\partial x} = \underbrace{\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}_{h}\mu$, which is an ODE (not a PDE) if the term h is independent of y.

   (b) Substitution yields $\mu' = -\frac{\mu}{x}$ with solution $\mu = 1/x$. Observe $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x} = 1$, so now we really have an exact ODE. To determine the function F we employ partial integration: $\frac{\partial F}{\partial x} = \mu P = y - \frac{1}{x}$ yields $F = yx - \log x + \phi(y)$ for some unknown function ϕ. Now ϕ is found using $\frac{\partial F}{\partial y} = \mu Q = x + \phi'(y) = x - y$, that is

$\phi(y) = -\frac{1}{2}y^2 + C$. Now we can solve $y(x)$ from $F = yx - \log x - \frac{1}{2}y^2 + C = 0$ to find $y = x \pm \sqrt{x^2 - 2\log(x) + C}$ with $C \in \mathbb{R}$.

4. (a) $\frac{dx}{x} - \frac{adv}{bv^2} = 0$.

   (b) $y = \frac{ax}{C - b\log(x)}$.