System(s) Theory

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Part I

Chapter 2

Overview

1. From DE $\rightarrow$ state

- Organization
- What is “systems theory”
- Example 2.1.1. & Thm. 2.2.4
- Controllability (§ 3.1–3.3)
    - Reachability
    - Reachable subspace
    - Controllability matrix $\mathscr{C}$
    - Controllability
    - *Kalman* Controllability decomposition
    - *Hautus* test

Organization

Gjerrit Meinsma

Felix Schwenninger

- Lecture / tutorial
- lecture notes (pdf or UnionShop 526?)
- Chapters 3,4,5 (and bits from 2)
- One standard written test
- Test includes a bit from NM
- (Three “challenges”)
- videos?

Difference between “DE’s” and “ST”

Inputs, diagrams, “plotter”

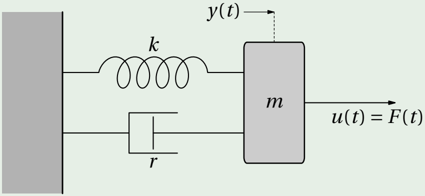
Applications

- Bio, glucose
- temperature control
- cruise control
- “war” (camera’s, targets)
- drones [youtube]
- robots, self-driving cars, platoons
- satellite [challenge of week ?]
- chemical,
- wafer-steppers
- ....

Have a look at the three YOUTUBE clips on CANVAS

State models (bits from § 2.1 & § 2.2)

Example (car–wall)



$$m\ddot{q}(t) + r\dot{q}(t) + kq(t) = u(t).$$

Can be turned into state model with $x_1 \doteq q$ and $x_2 \doteq \dot{q}$:

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{r}{m}x_2(t) + \frac{1}{m}u(t).$$

Linear state model/representation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Variations of constants:

$$x(t) = \mathrm{e}^{At} z(t)$$

$$\begin{aligned}
\dot{x} = Ax + Bu \iff & A\mathrm{e}^{At} z(t) + \mathrm{e}^{At} \dot{z}(t) = A\mathrm{e}^{At} z(t) + Bu(t) \\
\iff & \mathrm{e}^{At} \dot{z}(t) = Bu(t) \\
\iff & \dot{z}(t) = \mathrm{e}^{-At} Bu(t) \\
\iff & z(t) = z(t_0) + \int_{t_0}^{t} \mathrm{e}^{-A\tau} Bu(\tau)\,\mathrm{d}\tau
\end{aligned}$$

$$\begin{aligned}
x(t) &= \mathrm{e}^{At} z(t) \\
&= \mathrm{e}^{At} \mathrm{e}^{-At_0} x(t_0) + \int_{t_0}^{t} \mathrm{e}^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau
\end{aligned}$$

$$y(t) = C\mathrm{e}^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} C\mathrm{e}^{A(t-\tau)} Bu(\tau)\mathrm{d}\tau + Du(t)$$

$$
\begin{aligned}
y(t) &= C\,\mathrm{e}^{A(t-t_0)}\,x(t_0) + \int_{t_0}^{t} C\,\mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,\mathrm{d}\tau + Du(t) \\
&= \mathscr{H}\big(x(t_0), u(\tau)|_{\tau\in[t_0,t]}\big)
\end{aligned}
$$

The state $x(t_0)$ contains all the info from the past needed to continue into the future

DE

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = q_0 u(t)$$

Choose

$$x := \begin{bmatrix} y & y^{(1)} & \dots & y^{(n-2)} & y^{(n-1)} \end{bmatrix}^{\mathrm{T}}$$

Then

$$\begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-2)} \\ y^{(n-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ q_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} x$$

This does work for arbitrary DE's (with derivatives of u):

$$
\begin{aligned}
& y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\
= \quad & q_n u^{(n)}(t) + q_{n-1} u^{(n-1)}(t) + \cdots + q_1 u^{(1)}(t) + q_0 u(t)
\end{aligned}
$$

Equivalent state rep still exists:

Example

$$\ddot{y} + 5\dot{y} + 6y = 7\dot{u} + 8u$$

$$\ddot{y} = -5\dot{y} + 7\dot{u} - 6y + 8u$$

$$y = \int \left[-5y + 7u + \int [-6y + 8u] \right]$$

$$y = \underbrace{\int \left[-5y + 7u + \underbrace{\int [-6y + 8u]}_{x_1} \right]}_{x_2}$$

Example

$$y = \underbrace{\int \left[-5y + 7u + \underbrace{\int [-6y + 8u]}_{x_1} \right]}_{x_2}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

General case:

$$y^{(3)} = q_3 u^{(3)} + [q_2 u^{(2)} - p_2 y^{(2)}] + [q_1 u^{(1)} - p_1 y^{(1)}] + [q_0 u - p_0 y]$$

$$y = q_3 u + \underbrace{\int \left[q_2 u - p_2 y + \underbrace{\int \left[q_1 u - p_1 y + \underbrace{\int [q_0 u - p_0 y]}_{x_1} \right]}_{x_2} \right]}_{x_3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} q_0 & -p_0 \\ q_1 & -p_1 \\ q_2 & -p_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + q_3 u$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} q_0 & -p_0 \\ q_1 & -p_1 \\ q_2 & -p_2 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + q_3 u$$

Replace y with $x_3 + q_3 u$ to obtain state repr:

$$\dot{x} = \begin{bmatrix} 0 & 0 & -p_0 \\ 1 & 0 & -p_1 \\ 0 & 1 & -p_2 \end{bmatrix} x + \begin{bmatrix} q_0 - p_0 q_3 \\ q_1 - p_1 q_3 \\ q_2 - p_2 q_3 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + q_3 u$$

Is called observer canonical form

Example

DE

$$\dot{y} + 0y = \dot{u} + 0u$$

Then

$$\dot{y} = \dot{u} + 0y + 0u$$

$$y = u + \underbrace{\int 0}_{x}$$

Observer canonical form:

$$\dot{x} = 0$$

$$y = x + u$$

Hence y and u differ by a constant. Agreed.

Notice: u need not be differentiable now!

Polynomial notation for DE's

Operational Calculus (19th century. Arbogast, Boole, Heaviside): regard *differentiation* is as an *operation* on functions:

Example

$$\ddot{y} + 3\dot{y} + 5y = \ddot{u} - 6u$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3\frac{\mathrm{d}}{\mathrm{d}t} + 5\right) y = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 6\right) u$$

$$P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) y = Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) u$$

for the polynomials defined as

$$P(s) = s^2 + 3s + 5, \qquad Q(s) = s^2 - 6.$$

Properties of DE translate into properties of polynomials

$P(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$ has equilibrium $\bar{y} = 0$:

Definition (As.stability)

$P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$ is asymptotically stable if $\lim_{t\to\infty} y(t) = 0$ for all possible solutions of $P(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$.

$P(s)$ is the characteristic polynomial

Theorem (As. stability)

As.stable $\iff$ $P(s)$ *as.stable (= all zeros negative real part)*

Example

- $\dot{y} + 3y = 0$. Then $P(s) = s + 3$ so as.stable
- If $P(s) = (s+3)(s-2)$ then not as.stable

Proof.

- If $P(s)$ not as.stable. Then $P(s_0) \geq 0$ for some $\mathrm{re}(s_0) \geq 0$. Then $y(t) = \mathrm{e}^{s_0 t}$ does not go to 0 as $t \to \infty$, so not as.stable.
- If $P(s)$ as.stable then so is state model (observer canonical form):

$$\dot{x} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{bmatrix} x$$

$$y = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \end{bmatrix} x$$

because $\det(sI - A) = P(s)$ (A-matrix is compagnon matrix). Hence $\lim_{t \to \infty} y(t) = x_n(t) = 0$. So DE is as.stable

■

Also holds for system of DE's:

Lemma (As.stable)

If $P(s), Q(s)$ polynomial matrices, with $P(s)$ square, and $P(s), Q(s)$ having same # rows, then

$$P(\tfrac{\mathrm{d}}{\mathrm{d}t})y = Q(\tfrac{\mathrm{d}}{\mathrm{d}t})u \textit{ as.stable} \iff \det(P(s)) \textit{ as.stable}$$

Proof.

Suppose $\det(P)$ is not as.stable. Then $\det(P(s_0)) = 0$ for some $s_0 \in \mathbb{C}$ with $\mathrm{re}(s_0) \geq 0$. Let $v \in \mathbb{C}^m$ be a nonzero vector such that $P(s_0)v = 0$. Then $y(t) := v\mathrm{e}^{s_0 t}$ satisfies $P(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$. This $y(t)$ does not converge to zero, hence DE not as.stable.

Suppose $\det(P)$ is asymptotically stable. The *adjugate* R of P is polynomial and

$$RP = \det(P)I.$$

If $P(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$ then also $\det(P)Iy = R(\frac{\mathrm{d}}{\mathrm{d}t})P(\frac{\mathrm{d}}{\mathrm{d}t})y$ is zero. Therefore every y_i satisfies $\det(P(\frac{\mathrm{d}}{\mathrm{d}t}))y_i = 0$. Since $\det(P)$ is as.stable this implies that $\lim_{t\to\infty} y_i(t) = 0$. Hence DE is as.stable ■

Example

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}+1 & -1 \\ 2 & \frac{\mathrm{d}}{\mathrm{d}t}+3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}+1 \\ -2 \end{bmatrix} u,$$

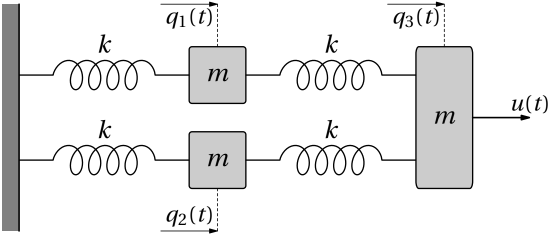
Then

$$P(s) := \begin{bmatrix} s+1 & -1 \\ 2 & s+3 \end{bmatrix}$$

and

$$\det(P(s)) = (s+1)(s+3)+2 = s^2+4s+5$$

It is as.stable.



Example

$$
\begin{aligned}
m\ddot{q}_1 &= -kq_1 + k(q_3 - q_1),\\
m\ddot{q}_2 &= -kq_2 + k(q_3 - q_2),\\
m\ddot{q}_3 &= -k(q_3 - q_1) - k(q_3 - q_2) \; + \; u.
\end{aligned}
$$

Example (... continued)

$$\begin{bmatrix} m\frac{\mathrm{d}^2}{\mathrm{d}t^2}+2k & 0 & -k \\ 0 & m\frac{\mathrm{d}^2}{\mathrm{d}t^2}+2k & -k \\ -k & -k & m\frac{\mathrm{d}^2}{\mathrm{d}t^2}+2k \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Then

$$\det P(s) = \Omega(s)(\Omega^2(s) - 2k^2)$$

in which $\Omega(s) = ms^2 + 2k$. All zeros are imaginary. Not as.stable.

Example (state repr as DE)

State repr:

$$\begin{aligned} \dot{y} &= Ay + Bu \\ \dot{y} - Ay &= Bu \\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t} I - A\right) y &= Bu \end{aligned}$$

As.stable $\iff$ $P(s) := (sI - A)$ as.stable. Correct!

Part II

Chapter 3

Overview

Chapter 3

Controllability & Observability

Definition (Reachability)

$\dot{x} = Ax + Bu$ is reachable if for every $x_1 \in \mathbb{R}^n$ and

$$x(0) = 0$$

there is a $t_1 > 0$ and $u : [0, t_1] \to \mathbb{R}^{n_u}$ such that

$$x(t_1) = x_1$$

- It does not fix t_1
- we say “the pair (A, B) is reachable”
- $x(t) = \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau$

Example

Not reachable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Is the next one reachable?:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u?$$

For instance can we steer the state to

$$\begin{bmatrix} x_1(t_1) \\ x_2(t_1) \end{bmatrix} = \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix}?$$

linear algebra:

Example (perpendicular to $x(t)$)

In this unreachable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad x(0) = 0$$

all solutions

$$\begin{bmatrix} x_1(t) = 0 \\ x_2(t) \end{bmatrix}$$

are perpendicular to $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2(t) \end{bmatrix} = 0$$

This generalizes:

Definition (Reachable subspace)

$$\mathbb{X}(t_1) := \left\{ \int_0^{t_1} \mathrm{e}^{A(t_1-\tau)} B\, u(\tau)\, \mathrm{d}\tau \;\middle|\; u : [0, t_1] \to \mathbb{R}^{n_u} \right\}$$

So $\mathbb{X}(t_1) \subseteq \mathbb{R}^n$. For now fix $t_1 > 0$:

Lemma

Let $t_1 > 0$ and $\eta \in \mathbb{R}^n$. The following statements are equivalent (TFSAE):

1. $\eta \perp \mathbb{X}(t_1)$
2. $\eta^{\mathrm{T}} \mathrm{e}^{At} B = 0$ *for all* $t \in [0, t_1]$
3. $\eta^{\mathrm{T}} A^k B = 0$ *for all* $k = 0, 1, \ldots$
4. $\eta^{\mathrm{T}} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$

Here n is het # of state components: $x(t) \in \mathbb{R}^{n}$

- $\eta^{\mathrm{T}}\, \mathbb{X}(t_1) = 0 \quad \Longleftrightarrow \quad \eta^{\mathrm{T}} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$
- $\mathbb{X}(t_1)$ and $\mathrm{im} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ same orthogonal complement
- Since $\mathbb{X}(t_1)$ is a subspace (of finite dimensional $\mathbb{R}^n$):
- $\mathbb{X}(t_1) = \mathrm{im}(\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix})$
- so reachable subspace $\mathbb{X}(t_1)$ does not depend on t_1 (only > 0)
- Define controllability matrix

$$\mathscr{C} := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times (n\, n_u)}$$

- conclusion:

$$\begin{aligned} \text{reachable} &\Longleftrightarrow \mathbb{X}(t_1) = \mathbb{R}^n \\ &\Longleftrightarrow \mathrm{im}(\mathscr{C}) = \mathbb{R}^n \\ &\Longleftrightarrow \mathscr{C} \text{ full row rank (rank } n\text{)} \end{aligned}$$

Example ($n = 2$)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \implies \mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$\mathscr{C}$ not full row rank, so not reachable, and "$\mathbb{X} = \left[\begin{smallmatrix} 0 \\ \mathbb{R} \end{smallmatrix}\right]$"

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \implies \mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$\mathscr{C}$ has full row rank, so reachable, $\mathbb{X} = \mathbb{R}^2$ (i.e. "$\mathbb{X} = \left[\begin{smallmatrix} \mathbb{R} \\ \mathbb{R} \end{smallmatrix}\right]$")

Theorem

The following statements are equivalent.

1. *the pair (A, B) is reachable*
2. $\mathrm{im}(\mathscr{C}) = \mathbb{R}^n$
3. *$\mathscr{C}$ has full row rank (rank n)*
4. *(if $\mathscr{C}$ square: $\mathscr{C}$ is invertible)*
5. *controllability gramian $P(t) = \int_0^t \mathrm{e}^{A\tau} B B^\mathrm{T} \mathrm{e}^{A^\mathrm{T}\tau}\, \mathrm{d}\tau$ is invertible for all $t > 0$*
6. *$P(t)$ is invertible for some $t > 0$.*

Then $x(t_1) = x_1$ if we apply

$$u_*(t) := B^\mathrm{T} \mathrm{e}^{A^\mathrm{T}(t_1 - t)} P^{-1}(t_1) x_1,$$

and, given t_1, it has the smallest possible norm:

$$\|u_*\|^2 := \int_0^{t_1} u_*^\mathrm{T}(t) u_*(t)\, \mathrm{d}t = x_1^\mathrm{T} P^{-1}(t_1) x_1 \le \|u\|^2$$

for all u that achieve $x(t_1) = x_1$

Definition (Controllability)

$\dot{x} = Ax + Bu$ is controllable if for every $x_0, x_1 \in \mathbb{R}^n$ with

$$x(0) = x_0$$

a $t_1 > 0$ exists and a $u : [0, t_1] \to \mathbb{R}^{n_u}$ such that

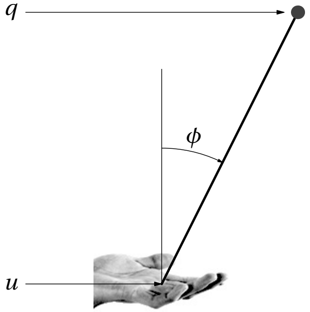
$$x(t_1) = x_1$$

controllable $\Longrightarrow$ reachable trivial.

reachable $\Longrightarrow$ controllable also true, because then:

$$x(t_1) = \mathrm{e}^{At_1}\, x_0 + \underbrace{\int_0^{t_1} \mathrm{e}^{A(t-\tau)}\, Bu(\tau)\,\mathrm{d}\tau}_{\in \mathbb{X}(t_1)=\mathbb{R}^n}$$

$$x(t_1) = \mathrm{e}^{At_1}\, x_0 + (x_1 - \mathrm{e}^{At_1}\, x_0) = x_1$$



Example (juggler)

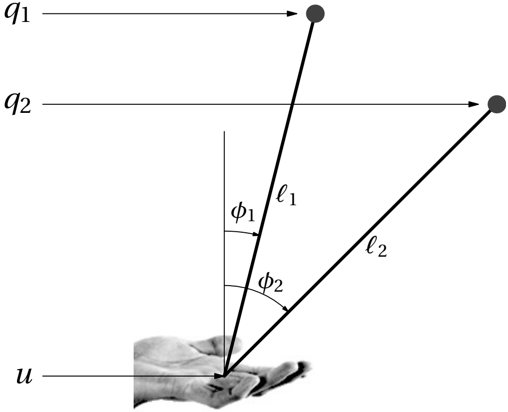
$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u.$$

Hence

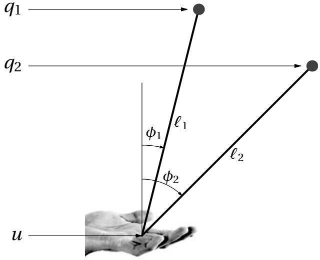
$$\mathscr{C} = \begin{bmatrix} 0 & -\frac{g}{\ell} \\ -\frac{g}{\ell} & 0 \end{bmatrix}.$$

Invertible, so controllable

So can achieve "any" $(q(t_1), v(t_1))$



When is this controllable?



Example (Juggler)

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot v_1 \\ \dot v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{g}{\ell_1} & 0 & 0 & 0 \\ 0 & \frac{g}{\ell_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{g}{\ell_1} \\ -\frac{g}{\ell_2} \end{bmatrix} u$$

For $\alpha := -\frac{g}{\ell_1}, \beta := -\frac{g}{\ell_2}$:

$$\mathscr{C} = \begin{bmatrix} 0 & \alpha & 0 & -\alpha^2 \\ 0 & \beta & 0 & -\beta^2 \\ \alpha & 0 & -\alpha^2 & 0 \\ \beta & 0 & -\beta^2 & 0 \end{bmatrix}.$$

Then $\det(\mathscr{C}) = [\alpha\beta(\beta-\alpha)]^2$?

Controllable iff $\ell_1 \neq \ell_2$!

If

$$\begin{bmatrix} \dot{x}_{\mathrm{c}} \\ \dot{x}_{\mathrm{uc}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}} \\ x_{\mathrm{uc}} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

then

$$\mathscr{C} = \begin{bmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n-1}B_1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and the reachable subspace $\mathrm{im}(\mathscr{C})$ is (part of)

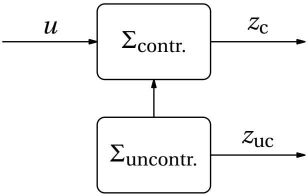
$$\begin{bmatrix} \mathbb{R}^q \\ 0 \end{bmatrix}$$

Lemma (Kalman Controllability decomposition)

Suppose first q columns of an invertible $T \in \mathbb{R}^{n\times n}$ span $\mathrm{im}(\mathscr{C})$. *Then* $z := T^{-1}x$ *gives*

$$\begin{bmatrix} \dot{z}_c \\ \dot{z}_{uc} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_c \\ z_{uc} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

with (A_{11}, B_1) *controllable, and* $\mathscr{C}_z = T^{-1}\mathscr{C}_x$, *and* $\mathrm{im}(\mathscr{C}_z) = \left[\begin{smallmatrix} \mathbb{R}^q \\ 0 \end{smallmatrix}\right]$



Example

$$\dot{x} = \begin{bmatrix} 4 & 3 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u \qquad \mathscr{C}_x = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -1 & -1 \\ -1 & -1 & -3 \end{bmatrix}$$

the reachable subspace has dimension 2

$$T = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 0 & 1 & .5 \\ 1 & 2 & .5 \\ 0 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad \mathscr{C}_z = T^{-1}\mathscr{C}_x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

It splits (the eigenvalues of) the A-matrix

Hautustest

Theorem (Hautustest — PBH-test)

$\dot{x} = Ax + Bu$ is controllable iff

$$\begin{bmatrix} sI - A & B \end{bmatrix}$$

has full row rank for all $s \in \mathbb{C}$.

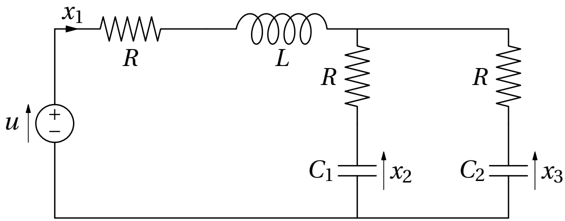
Equivalent: full row rank for all eigenvalues s of A

Example

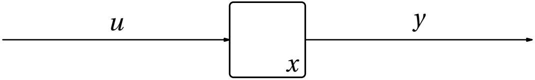
$$\dot{x} = \begin{bmatrix} 4 & 3 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u$$

$$\begin{aligned} \begin{bmatrix} sI - A & B \end{bmatrix} &= \begin{bmatrix} s-4 & -3 & -1 & 0 \\ 1 & s+1 & 0 & 1 \\ 2 & 1 & s & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} s-4 & -3 & -1 & 0 \\ 3 & s+2 & s & 0 \\ 2 & 1 & s & \boxed{-1} \end{bmatrix} \\ &\sim \begin{bmatrix} s-4 & -3 & \boxed{-1} & 0 \\ s^2-4s+3 & -2s+2 & 0 & 0 \\ 2 & 1 & s & \boxed{-1} \end{bmatrix} \end{aligned}$$

Row in the middle is zero for $s = 1$. Hence not controllable



- Suppose R, L, C_1, C_2 are all greater than zero
- state (x_1, x_2, x_3) (one current, two voltages)
- RLC circuit is not controllable if ??



- when controllable, we can force x to whatever
- ... using $u_*(t) = B^{\mathrm{T}} \mathrm{e}^{A^{\mathrm{T}}(t_1 - t)} P^{-1}(t_1)(x_1 - \mathrm{e}^{At_1} x_0)$
- this is not practical (think of glucose)
- for succesfull control we need to "look" at the system
- assume we "look" at y (the output)
- ... can we then figure out the state?

QUESTION:

Can we reconstruct / observe $x(t)$ on the basis of $u(t), y(t)$?

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

Definition (Observability)

A system is *observable* if there exists a $t_1 > 0$ such that for every triple of solutions (u_1, x_1, y_1), (u_2, x_2, y_2), with the same external behavior,

$$u_1(t) = u_2(t), \quad y_1(t) = y_2(t) \qquad \forall t \in [0, t_1],$$

also the state is the same,

$$x_1(t) = x_2(t) \qquad \forall t \in [0, t_1].$$

Then x follows uniquely from u, y....

Example

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

has (among others) this constant solution

$$x(t) = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

but then output is zero for all time:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = 0.$$

Hence system is not observable

Suppose first $u(t) = 0, y(t) = 0$:

$$\dot{x} = Ax + B\color{red}{0}$$
$$\color{red}{0} = Cx + D\color{red}{0}$$

So

$$\dot{x} = Ax$$
$$0 = Cx$$

So

$$C\,\mathrm{e}^{At}\,x_0 = 0$$

Definition (Unobservable subspace)

$$\mathbb{X}^{\mathrm{uo}}(t_1) := \{x_0 \in \mathbb{R}^n \mid C\,\mathrm{e}^{At}\,x_0 = 0 \ \forall\, t \in [0, t_1]\}$$

$$\mathbb{X}^{\mathrm{uo}}(t_1) := \{\eta \in \mathbb{R}^n \mid C\mathrm{e}^{At}\eta = 0 \ \forall\, t \in [0, t_1]\}$$

Lemma

Let $t_1 > 0$ and $\eta \in \mathbb{R}^n$. TFSAE:

1. $\eta \in \mathbb{X}^{nw}(t_1)$, *hence* $C\mathrm{e}^{At}\eta = 0$ *for all* $t \in [0, t_1]$
2. $CA^k\eta = 0$ *for all* $k = 0, 1, 2, \ldots$
3. $CA^k\eta = 0$ *for* $k = 0, 1, \ldots, n-1$

$$\underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}} \eta = 0$$

$\mathcal{O}$ is called observability matrix.

- $\eta \in \mathbb{X}^{uo}(t_1) \iff \mathscr{O}\eta = 0 \iff \eta \in \ker(\mathscr{O})$
- $\mathbb{X}^{ou}(t_1) = \ker(\mathscr{O})$
- $\mathbb{X}^{ou}(t_1)$ does not depend on $t_1 > 0$:

$$\{x_0 \in \mathbb{R}^n \,|\, C\,\mathrm{e}^{At}\,x_0 = 0 \,\forall\, t > 0\} = \ker(\mathscr{O})$$

- If $\ker(\mathscr{O})$ contains 2 or more entries, then not observable
- observability implies $\ker(\mathscr{O}) = \{0\}$
- In fact observability is equivalent to $\ker(\mathscr{O}) = \{0\}$ (next slides)

Theorem (Observability)

TFAE:

1. *system is observable (we say: "(A, C) is observable")*
2. $\ker(\mathscr{O}) = \{0\}$
3. *$\mathscr{O}$ has full column rank (rank n)*

if $\mathscr{O}$ is square, then: observable $\iff$ $\mathscr{O}$ invertible

Proof.

$1 \implies 2$ old. $2 \implies 3$ is linear algebra. $3 \implies 1$:

$$y(t) = C\mathrm{e}^{At}x_1(0) + \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}t + Du(t)$$

$$y(t) = C\mathrm{e}^{At}x_2(0) + \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)\,\mathrm{d}t + Du(t)$$

$$0 = C\mathrm{e}^{At}[x_1(0) - x_2(0)] \qquad t \in [0, t_1]$$

So $[x_1(0) - x_2(0)] \in \ker(\mathscr{O}) = 0$. ∎

Example (Unobservable)

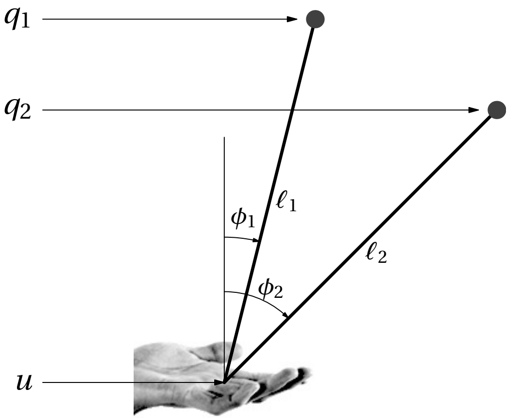
$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

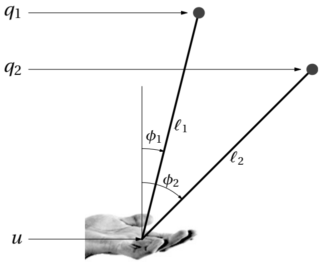
$$\mathscr{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$\mathscr{O}$ singular so system not observable, and unobservable subspace is:

$$\ker \mathscr{O} = \begin{bmatrix} 0 \\ \mathbb{R} \end{bmatrix}$$



with just one output: $y := \phi_1 - \phi_2$.
When is it observable? (with state (q_1, q_2, v_1, v_2))



Example (Juggler (with $y = \phi_1 - \phi_2$))

$$\begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot v_1 \\ \dot v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{g}{\ell_1} & 0 & 0 & 0 \\ 0 & \frac{g}{\ell_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{g}{\ell_1} \\ -\frac{g}{\ell_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{\ell_1} & \frac{-1}{\ell_2} & 0 & 0 \end{bmatrix} x + \left(\frac{1}{\ell_2} - \frac{1}{\ell_1}\right) u$$

$$\mathscr{O} = \begin{bmatrix} \frac{1}{\ell_1} & \frac{-1}{\ell_2} & 0 & 0 \\ 0 & 0 & \frac{1}{\ell_1} & \frac{-1}{\ell_2} \\ \frac{g}{\ell_1^2} & \frac{-g}{\ell_2^2} & 0 & 0 \\ 0 & 0 & \frac{g}{\ell_1^2} & \frac{-g}{\ell_2^2} \end{bmatrix}$$

Observable iff $\ell_1 \neq \ell_2$!

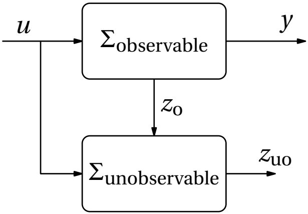
Lemma (Kalman Observability decomposition)

Suppose *final* *q columns of some invertible* T *span* $\ker(\mathscr{O}_x)$.
Then for $z := T^{-1}x$ *the system becomes*

$$\begin{bmatrix} \dot{z}_o \\ \dot{z}_{uo} \end{bmatrix} = \begin{bmatrix} A_{11} & 0_{(n-q)\times q} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_o \\ z_{uo} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & 0_{n_y \times q} \end{bmatrix} \begin{bmatrix} z_o \\ z_{uo} \end{bmatrix} + Du$$

with (C_1, A_{11}) *observable. Moreover* $\mathscr{O}_z = \mathscr{O}_x T$ *and* $\ker(\mathscr{O}_z) = \left[\begin{smallmatrix} 0 \\ \mathbb{R}^q \end{smallmatrix}\right]$.



Extra (💣): elegant proof of “Kalman”

Elegant (if you remember Lineaire Structures II):

- If $\mathcal{O}_x x = 0$ then $\mathcal{O}_x(Ax) = 0$.
- So $\ker(\mathcal{O}_x)$ is an A-invariant subspace
- Then $A|_{\ker(\mathcal{O}_x)}$ well defined
- we have $\ker(\mathcal{O}_z) = \{z | \mathcal{O}_z z = 0\} = \{z | \mathcal{O}_x T z = 0\} = \begin{bmatrix} 0 \\ \mathbb{R}^q \end{bmatrix}$
- So $A_z|_{\begin{bmatrix} 0 \\ \mathbb{R}^q \end{bmatrix}}$ well defined:

$$\begin{bmatrix} \dot{z}_o \\ \dot{z}_{uo} \end{bmatrix} = \begin{bmatrix} A_{11} & 0_{(n-q)\times q} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_o \\ z_{uo} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- finally: $\mathcal{O}_z = \mathcal{O}_x T = \begin{bmatrix} * & 0 \end{bmatrix}$, so $C_z = \begin{bmatrix} * & 0 \end{bmatrix}$

Example (construction of Kalman observability decomposition)

$$\dot{x} = \begin{bmatrix} 4 & -1 & -2 \\ 3 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} x, \qquad \mathscr{O}_x = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & -1 \\ 4 & -1 & -3 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} x$$

$\ker(\mathscr{O})$ has dimension 1, spanned by $(1,1,1)$:

$$\text{choose } T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad z = T^{-1}x,\ \ker(\mathscr{O}_z) = \begin{bmatrix} 0 \\ 0 \\ \mathbb{R} \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} z \qquad \mathscr{O}_z = \mathscr{O}_x T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 4 & -1 & 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} z$$

It splits A-matrix (splits eigenvalues)

Hautustest (for observability)

Hautus controllability test we know:

Theorem (controllability)

$\dot{x} = Ax + Bu$ is controllable iff the $n \times (n + n_u)$ matrix

$$\begin{bmatrix} sI - A & B \end{bmatrix}$$

has full row rank $\forall s \in \mathbb{C}$

Likewise for observability:

Theorem (observability)

$\dot{x} = Ax + Bu, y = Cx + Du$ is observable iff $(n + n_y) \times n$ matrix

$$\begin{bmatrix} sI - A \\ C \end{bmatrix}$$

full column rank $\forall s \in \mathbb{C}$

proof of "(not observable) $\iff$ (Hautus matrix loses rank)".

Transformation $z = T^{-1}x$ does not change observability.
if not observable then "Hautusmatrix"

$$\begin{bmatrix} sI - A_{11} & 0 \\ -A_{21} & sI - A_{22} \\ C_1 & 0 \end{bmatrix}$$

loses rank voor all eigenvalues of A_{22}. Conversely, if Hautusmatrix $\left[\begin{smallmatrix} sI-A \\ C \end{smallmatrix}\right] = 0$ loses rank at some $s_0 \in \mathbb{C}$, then nonzero x_0 exists such that

$$\begin{bmatrix} s_0 I - A \\ C \end{bmatrix} x_0 = 0.$$

This x_0 is a eigenvector van A, so

$$\mathcal{O} x_0 = \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ s_0 C x_0 = 0 \\ \vdots \end{bmatrix} = 0,$$

so then not observable. ∎

From DE to state space (see Chapter 2)

§ 3.5: Canonical Forms

Canonical forms are useful:

- $A \to D$ (diagonal)
- $A \to J$ (Jordan normal form)
- ....

Controllable & observable canonical forms:

With state transformation $z = T^{-1}x$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ y &= CTz + Du \end{aligned}$$

$$\begin{aligned} \mathscr{C}_z &= T^{-1}\mathscr{C}_x \\ \mathscr{O}_z &= \mathscr{O}_x T \\ \chi_{A_z}(\lambda) &= \chi_A(\lambda) \end{aligned}$$

we say:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{is isomorphic to} \quad \begin{bmatrix} T^{-1}AT & T^{-1}B \\ CT & D \end{bmatrix}$$

Lemma (A first form)

Suppose $n_u = 1$ *& define* $s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0 := \det(sI - A)$.
Every controllable $\dot{x} = Ax + Bu$ *via trafo* $v = \mathscr{C}^{-1}x$ *becomes*

$$\dot{v} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u$$

Proof.

So $T = \mathscr{C}_x$. Then $\mathscr{C}_v = T^{-1}\mathscr{C}_x = I$... ∎

Theorem (Controller canonical form)

Suppose $n_u = 1$ and define $s^n + p_{n-1}s^{n-1} + \cdots + p_0 := \det(sI - A)$.
Every controllable $\dot{x} = Ax + Bu$ is isomorphic to

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

(and $T = \mathscr{C}_x \mathscr{C}_z^{-1}$)

Proof.

Above (A_z, B_z) is controllable (says Hautus).
Hence (A_z, B_z) is isomorphic to first form (A_v, B_v) [via $v = \mathscr{C}_z^{-1} z$].
Also (A, B) is isomorphic to first form (A_v, B_v) [via via $v = \mathscr{C}_x^{-1} x$].
Hence $\mathscr{C}_z^{-1} z = \mathscr{C}_x^{-1} x$, that is, $x = \mathscr{C}_x \mathscr{C}_z^{-1} z$. ∎

Tedious to determine the transformation

$$T := \mathscr{C}_x \mathscr{C}_z^{-1}$$

we can bypass $\mathscr{C}_z$:

$$T = \begin{bmatrix} \eta \\ \eta A \\ \vdots \\ \eta A^{n-1} \end{bmatrix}^{-1} \quad \text{in which} \quad \eta := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathscr{C}_x^{-1}.$$

Makes the manipulation easier (well a little bit easier)

Example

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Then (verify this yourself)

$$T^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$$

and then (after the dust settles):

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -8 & 3 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} z$$

Only the new $C_z := CT$ requires hard work

Lemma (Observer canonical form)

Suppose $n_u = n_y = 1$ & $s^n + p_{n-1}s^{n-1} + \cdots + p_0 := \det(sI - A)$.
Every observable $\dot{x} = Ax + Bu, y = Cx + 0u$ *is isomorphic to*

$$\dot{z} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ \vdots \\ q_{n-1} \end{bmatrix} u,$$
$$y = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} z$$

and then $T = \mathscr{O}_x^{-1}\mathscr{O}_z$. *This* T *can be determined via*

$$T = \begin{bmatrix} \eta & A\eta & \cdots & A^{n-1}\eta \end{bmatrix} \quad \textit{met} \quad \eta := \mathscr{O}_x^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

But that is the observer canonical form (of Chapter 2), so:

- observer canonical form of $P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$ is, indeed, observable,
- Every observable $\dot{x} = Ax + Bu, y = Cx + Du$ is equivalent to a DE $P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$

Is the observer canonical form controllable?

$$\text{not controllable} \quad \Longleftrightarrow \quad \exists s, v \neq 0: \; v^{\mathrm{T}} \begin{bmatrix} sI - A & B \end{bmatrix} = 0$$

$$\Longleftrightarrow \exists s, v \neq 0: \; v^{\mathrm{T}} \begin{bmatrix} s & 0 & \cdots & 0 & p_0 & q_0 \\ -1 & \ddots & \ddots & \vdots & p_1 & q_1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & s & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & s + p_{n-1} & q_{n-1} \end{bmatrix} = 0$$

$$\Longleftrightarrow \exists s: \; \begin{bmatrix} 1 & s & s^2 & \cdots & s^{n-1} \end{bmatrix} \begin{bmatrix} s & 0 & \cdots & 0 & p_0 & q_0 \\ -1 & \ddots & \ddots & \vdots & p_1 & q_1 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & s & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & s + p_{n-1} & q_{n-1} \end{bmatrix} = 0$$

$$\Longleftrightarrow \exists s: \; \begin{bmatrix} 0 & \cdots & \cdots & 0 & P(s) & Q(s) \end{bmatrix} = 0$$

Hence controllable $\Longleftrightarrow$ P & Q have common zero!

Example

Suppose $P(s) = Q(s) = s + 2$,

$$\dot{y} + 2y = \dot{u} + 2u$$

then

$$\dot{x} = -2x$$
$$y = x + u$$

Not controllable.

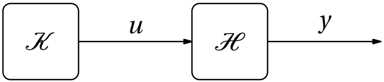
Part III

Chapter 4

Overview

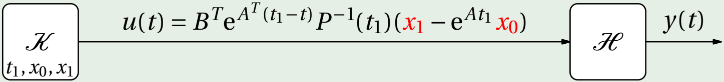
§ 4.0: Open loop versus closed loop

Open loop:



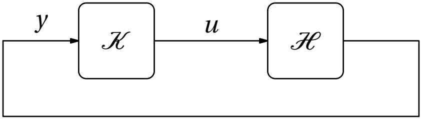
for example:

Example (blind juggler)



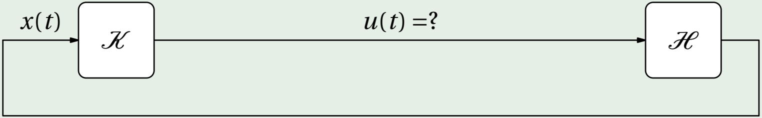
- It works in theory
- but not in practice

Closed loop:



This is natural (what a real juggler does):

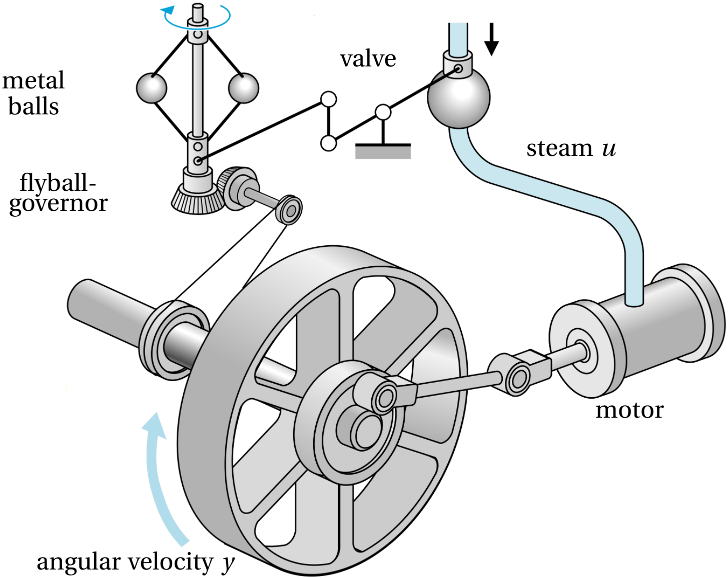
Example (juggler (with eyes!))



The juggler continuously looks at the pendulum, and uses it to determine u (the hand)

In this chapter: use u to stabilize the system
Today: assume the entire state $x(t)$ is available for feedback

flyball governor of James Watt (1788?) (C. Huygens, 1658)



Cornelis Drebbel (1572–1633) — egg incubator (1609)

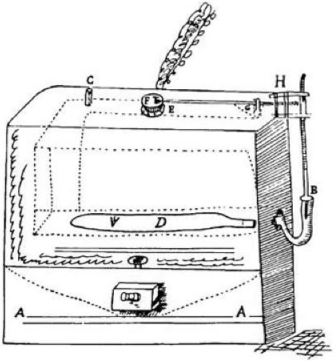


FIGURE 88.—Cornelis Drebbel's chicken incubator with temperature regulation, about 1620. Reprinted with permission of the Cambridge University Library from MS 2206, part 5, fol. 218.

He was known for his Perpetuum Mobile, built an incubator for eggs and a portable stove/oven with an optimal use of fuel, able to keep the heat on a constant temperature by means of a regulator/thermostat. *(one of the first recorded feedback-controlled devices)*

He designed a solar energy system for London (perpetual fire), demonstrated air-conditioning, made lightning and thunder “on command”, and developed fountains and a fresh water supply for the city of Middelburg.

Discovered that stannous chloride makes the colour of carmine much brighter and more durable. His daughters and sons-in-laws set up a very successful dye works. The recipe was kept a family secret, and the new bright red colour was very popular in Europe.

“The idea of Drebbel as a universal wonderworker was as widespread in the seventeenth century as the idea of Einstein as a genius is today.” [V. Keller, Princeton University]

Developed predecessors of the barometer and thermometer, and a harpsichords that played on solar energy.

Developed an automatic precision lens-grinding machine, build improved telescopes, constructs the first microscope ('lunette de Dreubells'), camera obscura, laterna magica.

Credited with the invention of the compound microscope. (In 1624 Galileo saw Drebbel's design for a microscope in Rome and created an improved version.)

... Drebbel went on to build two more submarines, each one bigger than the last. The final model had six oars and could carry 16 passengers. It was demonstrated to the king and thousands of Londoners on the Thames, and could stay submerged for three hours at a depth of 15 feet. How Drebbel maintained an air supply remains a mystery. (Might be an exaggeration.) [Wikipedia]

Example (open loop versus closed loop)

Suppose

$$\dot{x} = x + u.$$

The two inputs

$$\begin{aligned} \textit{open-loop:} \quad & u_o(t) = -3\,\mathrm{e}^{-2t}\,x(0), \\ \textit{closed-loop:} \quad & u_c(t) = -3x(t) \end{aligned}$$

stabilize & are identical in that $u_o(t) \equiv u_c(t)$ and identical state:

$$x(t) = \mathrm{e}^{-2t}\,x(0).$$

But they are very different if actual system is, say, $\dot{x} = 1.001x + u$:

$$\begin{aligned} \textit{open-loop:} \quad & x(t) = [\tfrac{3}{3.001}\,\mathrm{e}^{-2t} + \tfrac{0.001}{3.001}\,\mathrm{e}^{1.001t}]\,x(0), \\ \textit{closed-loop:} \quad & x(t) = \mathrm{e}^{-1.999t}\,x(0). \end{aligned}$$

Closed loop is much, much more robust against modeling errors (if given system unstable).

Definition (Stabilizability)

A system $\dot{x} = f(x,u)$ is *stabilizable* if for every

$$x(0) = x_0 \in \mathbb{R}^n,$$

there exists a $u : [0,\infty) \to \mathbb{R}^{n_u}$ such that

$$\lim_{t\to\infty} x(t) = 0.$$

It does not restrict u (open/closed loop, linear/nonlinear, ...)

Example (Three examples)

- $\dot{x} = +x + 1u$ is controllable and stabilizable
- $\dot{x} = -x + 0u$ is not controllable, yet stabilizable
- $\dot{x} = +x + 0u$ is not controllable, not stabilizable

Soon we see: stabilizability is weaker than controllability

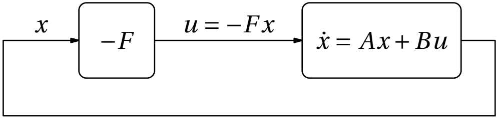
Of the many types, we focus on *(linear) static state feedback*:

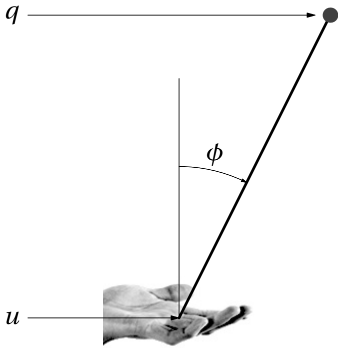
$$u(t) = -Fx(t).$$

Then

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax - BFx \\ &= (A - BF)x.\end{aligned}$$

This u is stabilizing (for every $x(0)$) iff $A - BF$ is asymptotically stable.





$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u.$$

Example (Juggler — pole placement)

Let $u = -Fx = -\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}$. Then

$$\begin{aligned}
\begin{bmatrix} \dot q \\ \dot v \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} u \\
&= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{g}{\ell} \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell}(1 + f_1) & +\frac{g}{\ell} f_2 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}.
\end{aligned}$$

$$\det(sI - (A - BF)) = s^2 - \tfrac{g}{\ell} f_2 s - \tfrac{g}{\ell}(1 + f_1)$$

This equals $(s+1)^2 = s^2 + 2s + 1$ iff

$$f_2 = -2\frac{\ell}{g}, \qquad f = -1 - \frac{\ell}{g}.$$

Now the eigenvalues of $A - BF$ are -1 (twice).

Example

Suppose system is in *controller canonical form*:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u.$$

Then $u = \begin{bmatrix} +p_0 - r_0 & \cdots & +p_{n-1} - r_{n-1} \end{bmatrix} z$ gives

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & \cdots & -r_{n-1} \end{bmatrix} z$$

This has characteristic polynomial $R(s)$.

Theorem (Pole placement)

Consider $\dot{x} = Ax + Bu$. For every polynomial

$$R(s) := s^n + r_{n-1}s^{n-1} + \cdots + r_0, \qquad r_k \in \mathbb{R},$$

there exists an $F \in \mathbb{R}^{n_u \times n}$ such that

$$\det(sI - (A - BF)) = R(s)$$

if and only if the system is controllable.

This implies:

Corollary

Every controllable system is stabilizable through $u(t) = -Fx(t)$

We may apply transformation $z = T^{-1}x$:

$$\begin{aligned} \dot{x} &= Ax + Bu & \dot{z} &= T^{-1}ATz + T^{-1}Bu \\ u &= -Fx & u &= -FTz \\ \chi_A(s) &= \det(sI - A) & \chi_{A_z}(s) &= \chi_A(s) \\ \chi_{A-BF}(s) &= \det(sI - (A - BF)) & \chi_{A_z - B_z F_z}(s) &= \chi_{A-BF}(s) \end{aligned}$$

Proof 1/2.

If not controllable then

$$(A,B) \quad \to \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right)$$

which after state feedback $u = -Fx = -\tilde{F}z$ gives

$$\begin{aligned} \det(sI-(A-BF)) &= \det(sI - (\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{F}_1 & \tilde{F}_2 \end{bmatrix})) \\ &= \det(\begin{bmatrix} ? & ? \\ 0 & sI - A_{22} \end{bmatrix}) \end{aligned}$$

Hence the eigenalvalues of A_{22} are fixed (can not be moved).
So then not "pole place-able" ∎

Proof continued.

If controllable (and $n_u = 1$) then controllable canonical form exists:

$$A_z - B_z F_z = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} F_z$$

for $F_z := \begin{bmatrix} -p_0 + r_0 & \cdots & -p_{n-1} + r_{n-1} \end{bmatrix}$ this becomes

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & \cdots & -r_{n-1} \end{bmatrix}$$

This has char.pol $R(s)$. So then "pole place-able" ∎

Useful but derivation not fun:

Lemma (Ackermann)

If $n_u = 1$ *then*

$$F = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathscr{C}_x^{-1} R(A)$$

Example

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u.$$

If we want closed loop "poles" at $-1, -4$ then $R(s) = (s+1)(s+4)$:

$$\begin{aligned}
F &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathscr{C}_x^{-1} R(A) \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}^{-1} (A+I)(A+4I) \\
&= \underbrace{\underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 1/2 & 1/2 \end{bmatrix}} \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}}_{\begin{bmatrix} 2 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 10 & 5 \end{bmatrix}
\end{aligned}$$

QUESTION: are there stabilizable systems $\dot{x} = Ax + Bu$ that are not stabilizable through $u(t) = -Fx(t)$?

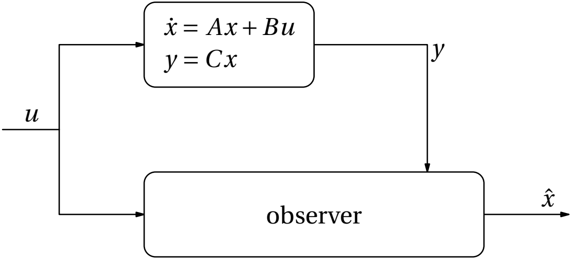
ANSWER: No (that's good news):

Theorem (4.2.4 — Stabilizability)

Consider $\dot{x} = Ax + Bu$. TFSAE:

1. *$\exists F$ such that $A - BF$ is asymptotically stable.*
   (So stabilizable through static state feedback $u = -Fx$.)
2. *The system is stabilizable.*
3. *In the Kalman controllability decomposition of $\dot{x} = Ax + Bu$, the eigenvalues of A_{22} have negative real part.*
4. $\begin{bmatrix} sI - A & B \end{bmatrix}$ *has full row rank for all $s \in \mathbb{C}$ with* $\mathrm{re}(s) \geq 0$.
5. $\begin{bmatrix} sI - A & B \end{bmatrix}$ *has full row rank for all eigenvalues $s \in \mathbb{C}$ of A with* $\mathrm{re}(s) \geq 0$.

Hence the choice $u(t) = -Fx(t)$ is not restrictive.



observer = mapping from signals (u, y) to signal $\hat{x}$.

Definition (Detectability)

A system is *detectable* if there exists an observer such that

$$\lim_{t\to\infty} \|\hat{x}(t) - x(t)\| = 0$$

for all initial conditions $x(0)$ and all inputs u.

Important: $x(t)$ and $\hat{x}(t)$ need not converge as $t \to \infty$!

$$\text{given system:} \quad \dot{x} = Ax + Bu, \quad y = Cx$$

$$\begin{aligned} \text{observer:} \quad \dot{\hat{x}} &= P\hat{x} + Qu + Ly \\ &= P\hat{x} + Qu + LCx \end{aligned}$$

Define estimation error $e := x - \hat{x}$:

$$\begin{aligned} \dot{e} &= (A - LC)x - P\hat{x} + (B - Q)u \\ &= (A - LC)(x - \hat{x}) + (A - LC - P)\hat{x} + (B - Q)u \end{aligned}$$

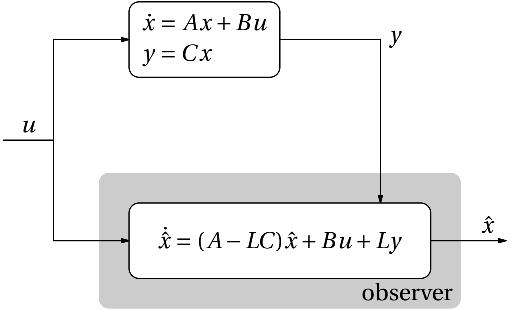
Choose $P := A - LC$ and $Q := B$:

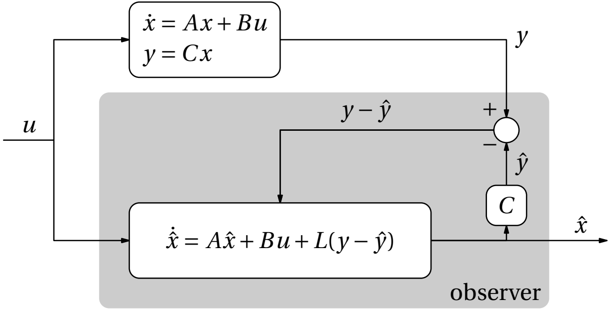
$$\dot{e} = (A - LC)e$$

$$\begin{aligned} \text{observer:} \quad \dot{\hat{x}} &= (A - LC)\hat{x} + Bu + Ly \\ &= A\hat{x} + Bu + L(y - C\hat{x}) \end{aligned}$$

Lemma

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ is an observer if $A - LC$ as.stable


$\dot{x} = Ax + Bu$
$y = Cx$
y
u
$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$
$\hat{x}$
observer


$\dot{x} = Ax + Bu$
$y = Cx$
y
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$\hat{y}$
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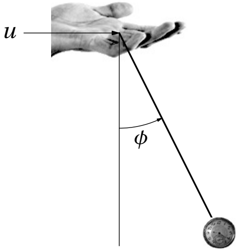
$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$
$\hat{x}$
observer



Example (Hypnotist)

Take $m = 0.1, \ell = 0.4, g = 10$ (and friction coef 0.05), then:

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2.5 & -0.5 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix}$$

It has eigenvalues $-0.25 \pm \mathrm{i}1.56$

Do you know “time constants”?

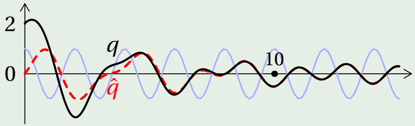
Example (Hypnotist — continued)

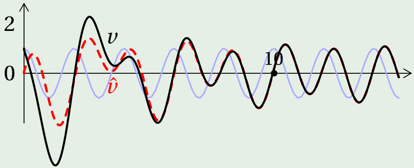
A possible "non-agressive" $L = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ gives

$$\begin{bmatrix} \dot{\hat{q}} \\ \dot{\hat{v}} \end{bmatrix} = \begin{bmatrix} -0.5 & 1.0 \\ -3.0 & -0.5 \end{bmatrix} \begin{bmatrix} \hat{q} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} u + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} y$$

having eigenvalues (aka "observer poles") $-0.5 \pm \mathrm{i}1.73$.

Simulation for $u(t) = \cos(\pi t)$, $\begin{bmatrix} q(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \hat{q}(0) \\ \hat{v}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:





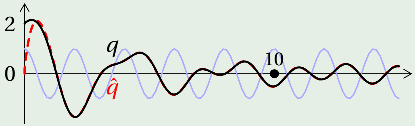
Example (Hypnotist — continued)

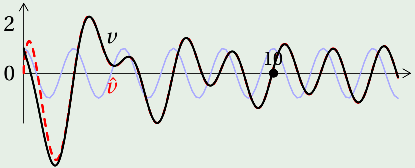
A possible "agressive" $L = \left[\begin{smallmatrix}5\\5\end{smallmatrix}\right]$ gives

$$\begin{bmatrix}\dot{\hat{q}}\\ \dot{\hat{v}}\end{bmatrix} = \begin{bmatrix}-5.0 & 1.0\\ -7.5 & -0.5\end{bmatrix}\begin{bmatrix}\hat{q}\\ \hat{v}\end{bmatrix} + \begin{bmatrix}0\\ 2.5\end{bmatrix}u + \begin{bmatrix}5.0\\ 5.0\end{bmatrix}y.$$

having eigenvalues (aka "observer poles") $-2.75 \pm \mathrm{i}1.56$.

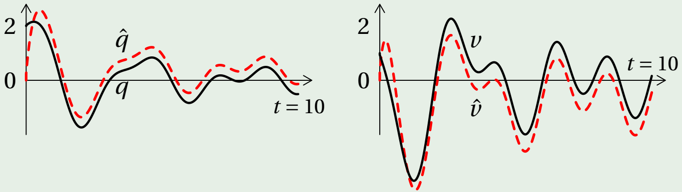
Simulation for $u(t) = \cos(\pi t)$, $\left[\begin{smallmatrix}q(0)\\ v(0)\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\ 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\hat{q}(0)\\ \hat{v}(0)\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$:



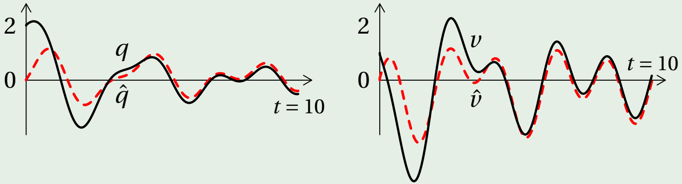


Example (Hypnotist — continued (final))

If there are (measurement) errors, say, $y_{\text{measured}} = y_{\text{real}} + 0.5$, then fast (agressive) observer not so good



Slow (non-agressive) is better:



Theorem (Observer pole placement)

Consider system $\dot{x} = Ax + Bu, y = Cx$. *For every real polynomial* $R(s) = s^n + r_{n-1}s^{n-1} + \cdots + r_0$, *there exists an* $L \in \mathbb{R}^{n \times n_y}$ *such that*

$$\det(sI - (A - LC)) = R(s)$$

iff (A, C) *is observable.*
If $n_y = 1$, *then* L *can be determined using Ackermann's formula*

$$L = R(A)\mathcal{O}_x^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

This implies:

Corollary

Every observable system has an observer of the special form
$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$ *(for some* L*).*

This special form is without loss of generality (very nice):

Lemma (Detectability)

Consider $\dot{x} = Ax + Bu, y = Cx$. TFSAE:

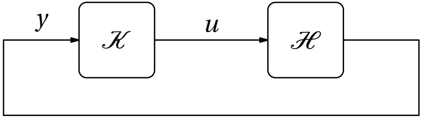
1. *There exists an L such that $A - LC$ is as.stable.*
   (hence an observer exists of the form $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$)
2. *The system is detectable*
3. *In Kalman Observability Decomp, the A_{22} is as.stable*
4. $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *has full column rank for all* $\mathrm{re}(s) \geq 0$
5. $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *has full column rank for all eigenvalues $s \in \mathbb{C}$ of A for which* $\mathrm{re}(s) \geq 0$

Hence the form $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ is not a restriction!

What we did so far:

$$\begin{aligned} \text{dynamic observer:}\quad & \dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \\ \text{static state feedback:}\quad & u = -Fx \end{aligned}$$

Now it is time to design a stabilizing controller $u = \mathcal{K}(y)$:



Bold idea: why not try this controller $u = \mathscr{K}(y)$:

$$\text{dynamic observer: } \dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$
$$\text{static state feedback with a twist: } u = -F\hat{x}$$

Eliminate Bu:

$$\dot{\hat{x}} = (A - LC - BF)\hat{x} + Ly$$
$$u = -F\hat{x}$$

This a system with input y and output u!

$$y \rightarrow \boxed{\begin{aligned}\dot{\hat{x}} &= (A - LC - BF)\hat{x} + Ly \\ u &= -F\hat{x}\end{aligned}} \xrightarrow{u} \boxed{\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}} \rightarrow y$$

Closed-loop is described by

$$\begin{bmatrix}\dot{x} \\ \dot{\hat{x}}\end{bmatrix} = \begin{bmatrix} A & -BF \\ LC & A - BF - LC\end{bmatrix}\begin{bmatrix}x \\ \hat{x}\end{bmatrix}, \qquad \begin{bmatrix}y \\ u\end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & -F\end{bmatrix}\begin{bmatrix}x \\ \hat{x}\end{bmatrix}$$

Example (Closed loop state transformation)

$$\underbrace{\begin{bmatrix} I & 0 \\ I & -I\end{bmatrix}}_{T}\begin{bmatrix}x \\ e\end{bmatrix} = \begin{bmatrix}x \\ \hat{x}\end{bmatrix}, \qquad \begin{bmatrix}x \\ e\end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ I & -I\end{bmatrix}}_{T^{-1}}\begin{bmatrix}x \\ \hat{x}\end{bmatrix}.$$

Similarity transformation:

$$\begin{bmatrix} I & 0 \\ I & -I\end{bmatrix}\begin{bmatrix} A & -BF \\ LC & A - BF - LC\end{bmatrix}\begin{bmatrix} I & 0 \\ I & -I\end{bmatrix} = \begin{bmatrix} A - BF & BF \\ 0 & A - LC\end{bmatrix}$$

The main theorem of Chapter 4:

Theorem (Stabilizing controller — separation principle)

If a system is stabilizable & detectable, then matrices F en L exist such that $A-BF$ & $A-LC$ asymptotically stable.

In that case the controller

$$\begin{aligned}\dot{\hat{x}} &= (A-LC-BF)\hat{x}+Ly\\ u &= -F\hat{x}\end{aligned}$$

stabilizes the given system (aka the "plant"), in the sense that

$$\lim_{t\to\infty} x(t)=0$$
$$\lim_{t\to\infty} \hat{x}(t)=0$$

for every initial condition $x(0)=x_0$ and $\hat{x}(0)=\hat{x}_0$.

By the way: the controller itself need not be stable!

Example (Juggler with $\ell = g/2$)

$$\begin{bmatrix} \dot q \\ \dot v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u,$$

$$y = q.$$

- determined earlier: $F = \begin{bmatrix} -3/2 & -1 \end{bmatrix}$, then $\chi_{A-BF}(s) = (s+1)^2$
- determined earlier: $L = \left[\begin{smallmatrix} 4 \\ 6 \end{smallmatrix}\right]$, then $\chi_{A-LC}(s) = (s+2)^2$

Gives this controller

$$\dot{\hat x} = \underbrace{\begin{bmatrix} -4 & +1 \\ -7 & -2 \end{bmatrix}}_{A-LC-BF} \hat x + \underbrace{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{L} y$$

$$u = \underbrace{\begin{bmatrix} 3/2 & 1 \end{bmatrix}}_{-F} \hat x$$

MATLAB code `simujuggler.m` (also in lec.notes appendix):

```
n=2;                            % number of states of system %%
g=10;                           % gravitational acceleration %%
l=0.5;                          % whatever length of pendulum %

A=[0 1; g/l 0];                 % A matrix for x=[q;v] %%%%%%%
B=[0; -g/l];                    % B matrix for u=position hand%
C=[1 0];                        % y=q %%%%%%%%%%%%%%%%%%%%%%%%
F=[-1-l/g -2*l/g];              % so A-BF has eigenv -1 (twice)
L=[4; 4+g/l];                   % so A-LC has eigenv -2 (twice)

eigABF=eig(A-B*F)               % check the stability of A-BF %
eigALC=eig(A-L*C)               % and of A-LC %%%%%%%%%%%%%%%%%

f =@(x,u) [x(2); +g/l*sin(x(1))]+[0; -g/l*u]; % nonlinear %
fl=@(x,u) [A*x+B*u];            % linearized %%%%%%%%%%%%%%%%%

FWX=@(t,z) [f(z(1:n),-F*z(n+(1:n)));       % f(x,u)=f(x,-Fx̂)
    L*C*z(1:n)+(A-L*C-B*F)*z(n+(1:n))]; % LCx+(A-LC+BF)x̂
FLW=@(t,z) [fl(z(1:n),-F*z(n+(1:n)));      % Ax+Bu=Ax+B(-Fx̂)
    L*C*z(1:n)+(A-L*C-B*F)*z(n+(1:n))]; % LCx+(A-LC+BF)x̂
```

```
%z0=[0;0.2;0;0];               % okay initial z(0) %%%%%%%%%
%z0=[.3;.3;0;0];               % too big (for nonlinear case)
z0=[0;0.3;0;0];                % scary   (for nonlinear case)
tspan =[0 20];
[tt,z]=ode45(FWX,tspan,z0);  % simulate nonlinear %%%%%%%%%
%[tt,z]=ode45(FLW,tspan,z0); % simulate linear %%%%%%%%%%%

% Plot all four state components against time: %%%%%%%%%%%
figure(1),   plot(tt,z'),    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%
legend('q','v','qh','vh'),   grid,  set(gca,'Fontsize',14);
% Simplistic movie: %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
figure(2);
for k=1:(length(tt)-1);
  u=-F*z(k,1:n)';
  q=z(k,1);
  plot([u,q],[0,sqrt(l^2-(q-u)^2)],[-1 1], ...
       [0 0],[0 0],[-0.1 .6]);
  axis off;
  drawnow;
  pause(tt(k+1)-tt(k));
end
```

Some remarks:

- Code of “juggler” is in Appendix A.11
- In the project you sometimes need more than 1 control input (so $u(t) \in \mathbb{R}^m$ with $m > 1$).

  In Appendix A.8 there is a quick summary of LQ-optimal control which can handle such cases.

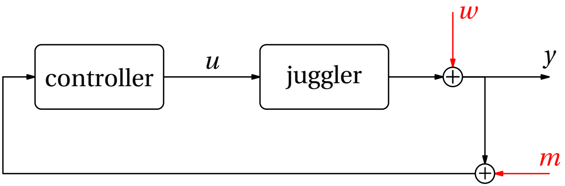
  Appendix A.8 also has some MATLAB code for this.
- Next chapter: walking juggler.

Part IV

Chapter 5

Overview

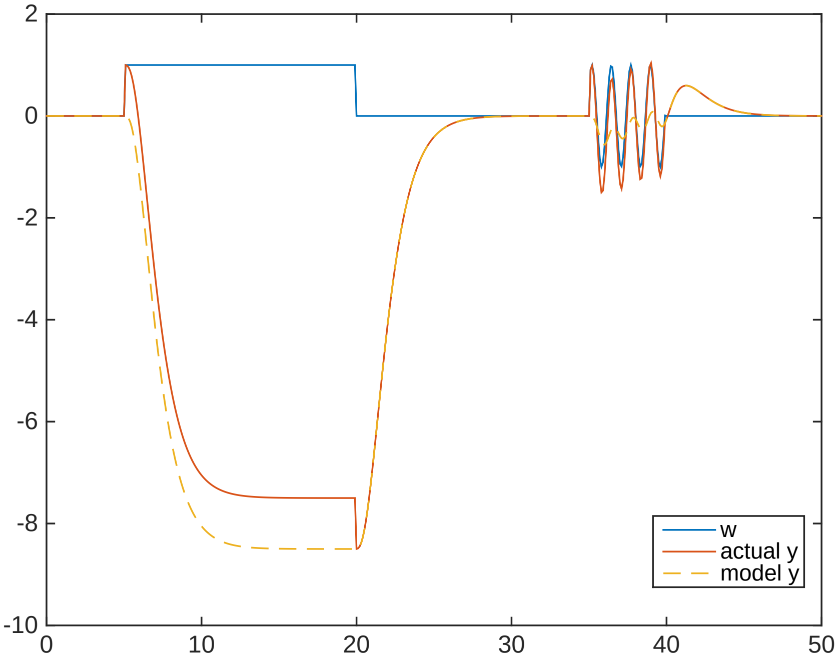
Where Chapter 5 is heading:

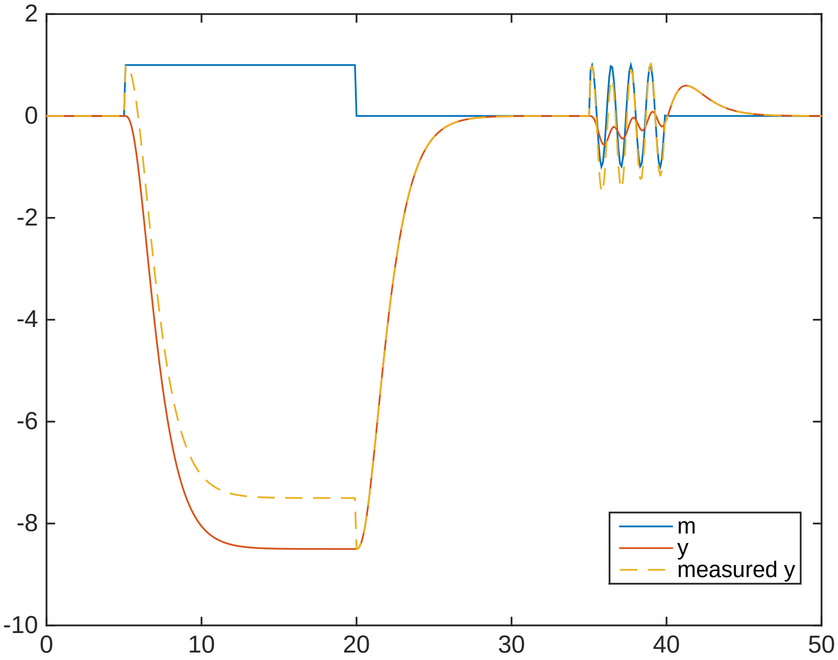


- what is the effect of w (positional disturbance / “wind”)?
- what is the effect of m (measurement error / drift)?
- How can we regulate y (= let y follow our command)?

Don't read it yet (just be impressed by how compact the code is)

```
s=tf('s');
P=-2/(s^2-2);                   % JUGGLER (for ℓ = g/2 = 5)
K=-(12*s+17)/(s^2+6*s+15); % CONTROLLER

Hyw= feedback(1,P*K);
Hym=-feedback(P*K,1);

t=0:.1:50;

w=0*t+(t>5 & t<20)+(t>35 & t<40).*cos(5*t);
y=lsim(Hyw,w,t);
plot(t,w,t,y,t,y-w','--');

m=0*t+(t>5 & t<20)+(t>35 & t<40).*cos(5*t);
y=lsim(Hym,m,t);
plot(t,m,t,y,t,y+m','--');
```


2
0
-2
-4
-6
-8
-10
0
10
20
30
40
50
w
actual y
model y

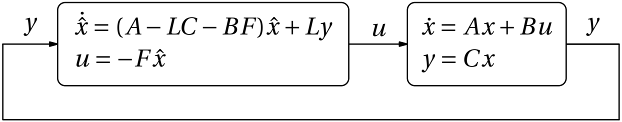


2
0
-2
-4
-6
-8
-10
0
10
20
30
40
50
m
y
measured y

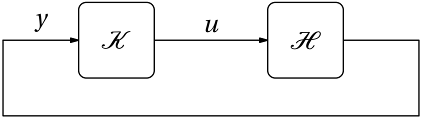
Chapter 5:

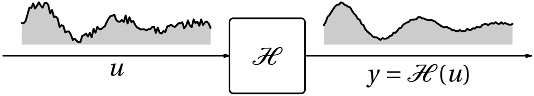
- set up a language for “such problems”
- language is “compact” (see MATLAB code)
- focus is on external behavior u, y (not the state)
- allows to design more practical controllers ...
- e.g. walking juggler, cruise controllers, many more...
- ... and then the course comes to an end :-(

Previously: focus on states:



Now focus on u, y:





- Focus on external signals u, y
- think of systems as mappings from u to y

Suppose $u, y : \mathbb{R} \to \mathbb{R}$ and $y = \mathscr{H}(u)$

Definition (LTI)

$y = \mathscr{H}(u)$ is LTI if

- $\mathscr{H}(u_1 + u_2) = \mathscr{H}(u_1) + \mathscr{H}(u_2)$
- $\mathscr{H}(\lambda u) = \lambda \mathscr{H}(u)$
- $\mathscr{H}(\sigma^\tau u) = \sigma^\tau \mathscr{H}(u)\ \forall \tau \in \mathbb{R}$.

Yeah, formally should include vector spaces $\mathbb{U}, \mathbb{Y}$.

Definition (Linearity and time invariance—LTI)

A system $y = \mathscr{H}(u)$ is linear if for all possible inputs u, u_1, u_2 and scalars λ we have

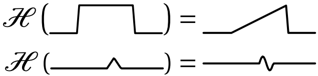
1. additivity: $\mathscr{H}(u_1 + u_2) = \mathscr{H}(u_1) + \mathscr{H}(u_2)$;
2. homogeneity: $\mathscr{H}(\lambda u) = \lambda \mathscr{H}(u)$.

A system is time invariant if “the response of the delay equals the delay of the response”, that is,

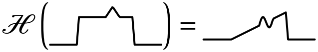
$$\mathscr{H}(\sigma^\tau u) = \sigma^\tau \mathscr{H}(u) \qquad \forall \tau \in \mathbb{R}$$

for all possible inputs u.
We call a system LTI if it is both linear and time invariant.

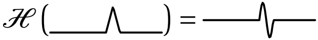
LINEARITY:
if



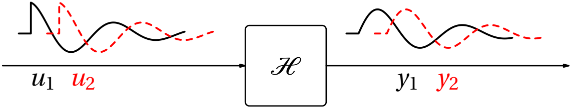
then additivity implies



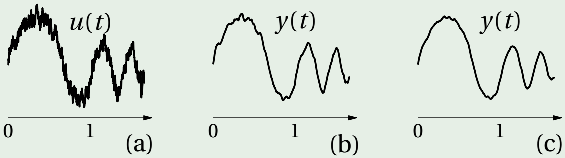
and homogeneity implies



TIME INVARIANCE:



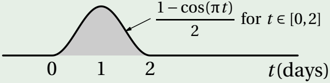
Example

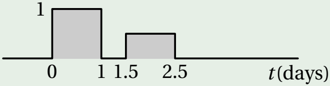


$$y = \mathscr{H}(u): \qquad y(t) = \frac{1}{P} \int_{t-P}^{t} u(\tau)\, \mathrm{d}\tau$$

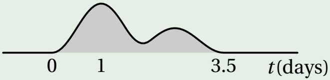
Example (Drainage system)

$u_0(t) =$ 

$y_0(t) =$ 

$u_1(t) = u_0(t) + \frac{1}{2} u_0(t-1.5)$ $=$ 

$$\begin{aligned}
y_1(t) &= \mathscr{H}(u_0 + \tfrac{1}{2}\sigma^{1.5} u_0)(t) \\
&= \mathscr{H}(u_0)(t) + \tfrac{1}{2}\mathscr{H}(\sigma^{1.5} u_0)(t) && \text{(linearity)} \\
&= \mathscr{H}(u_0)(t) + \tfrac{1}{2}\sigma^{1.5}\mathscr{H}(u_0)(t) && \text{(time inv.)} \\
&= y_0(t) + \tfrac{1}{2} y_0(t-1.5)
\end{aligned}$$



(Nice: $\int_{-\infty}^{\infty} u(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} y(t)\,\mathrm{d}t$)

Example (Prove that .. is (not?) LTI)

Consider system $\mathscr{H}(u)(t) = tu(t)$:

$$y(t) = tu(t)$$

It is not time-invariant (one counter example suffices): consider

$$u_0(t) := 1$$
$$y_0(t) = t$$

Then delayed copies

$$\tilde{u}(t) := u_0(t-1) = 1$$
$$\tilde{y}(t) := y_0(t-1) = t-1$$

do not satisfy the system equations:

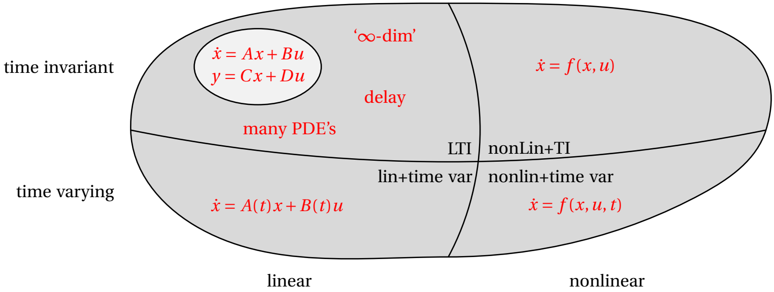
$$\mathscr{H}(\tilde{u})(t) = t \times 1 \neq \tilde{y}(t)$$

Compact: $\sigma(\mathscr{H}(1)(t)) = \sigma(t) = t-1 \neq \mathscr{H}(\sigma 1)(t) = \mathscr{H}(1)t = t$

Many systems are (approximately) LTI:

- Delays: $y(t) = u(t-1)$
- Echo: $y(t) = u(t-1) + \frac{1}{2}y(t)$
- heated beam
- state models: $\dot{x} = Ax + Bu, y = Cx + Du$

Although probably very few are truly linear



Impulse response & Convolution

Definition (Impulse response)

The *impulse response* $h : \mathbb{R} \to \mathbb{R}$ of a system $y = \mathscr{H}(u)$ is the response to the delta function,

$$h(t) = \mathscr{H}(\delta)$$

Yeah, h might contain delta functions as well

Theorem (LTI equals convolution)

$y = \mathscr{H}(u)$ is LTI $\iff$ $y = h * u$

In which case $h = \mathscr{H}(\delta)$

Also $\dot{x} = AxBu, y = Cx + Du$ can been as LTI mappings:

Suppose initially at rest

$$x(t) = 0,\ u(t) = 0,\ y(t) = 0 \quad \forall t < t_0$$

then

$$\begin{aligned} y(t) &= C\mathrm{e}^{A(t-t_0^-)}\,x(t_0^-) + \int_{t_0^-}^{t} C\mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,\mathrm{d}\tau + Du(t) \\ &= 0 + \int_{-\infty}^{t} C\mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,\mathrm{d}\tau + Du(t) \\ &= (h * u)(t) \end{aligned}$$

where

$$h(t) = C\mathrm{e}^{At}B\mathbb{1}(t) + D\delta(t)$$

Example (Integrator)

$$\begin{aligned} \dot{x} &= u \\ y &= x \end{aligned}$$

$$y(t) = \int_{-\infty}^{t} u(\tau)\,\mathrm{d}\tau$$

Example (Double integrator)

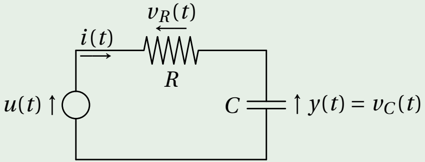
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} h(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbb{1}(t) \\ &= t\mathbb{1}(t) \end{aligned}$$

Example (*RC* circuit)

$$\dot{y} + \alpha y = \alpha u$$



$$\begin{aligned} \dot{x} &= -\alpha x + \alpha u \\ y &= x \end{aligned}$$

$$h(t) = 1\, e^{-\alpha t}\, \alpha \mathbb{1}(t)$$

$$y(t) = \int_{-\infty}^{t} e^{-\alpha(t-\tau)}\, \alpha u(\tau)\, d\tau$$

Example (delay)

$$y(t) = u(t-1)$$

$$h(t) = \delta(t-1)$$

$$y(t) = (h * u)(t) = \int_{-\infty}^{t} \delta((t-\tau)-1)\, u(\tau)\, \mathrm{d}\tau = u(t-1)$$

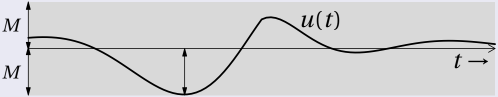
It is not of the form $\dot{x} = Ax + Bu, y = Cx + u$
(but can be approximated by one)

§ 5.2: BIBO stability (of linear mappings $y = \mathscr{H}(u)$)

- Previously: asymptotic stability
  (if you pull the plug, all signals should converge to equilibrium)
- For maps $y = \mathscr{H}(u)$ stability roughly speaking means:
  Bounded in implies Bounded out (BIBO)
- Useful for "thermostat"

Definition (Peak value – max-norm – sup-norm)

$$\|u\|_\infty := \sup_{t\in\mathbb{R}} |u(t)|$$



Definition (Maximal peak-to-peak gain — 1-norm)

$$\|\mathscr{H}\|_1 := \sup_u \frac{\|\mathscr{H}(u)\|_\infty}{\|u\|_\infty}$$

Definition (BIBO-stability)

BIBO-stable if $\|\mathscr{H}\|_1 < \infty$

Theorem ((Maximal) peak-to-peak gain)

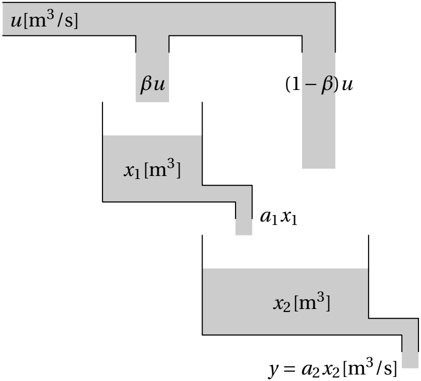
If $y = \mathscr{H}(u)$ LTI then

$$\|\mathscr{H}\|_1 = \int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t.$$

So BIBO stable iff $|h(t)|\,\mathrm{d}t < \infty$.

Also works for delta functions where $\int |\delta(t)| \mathrm{d}t = 1$.

$\|\mathscr{H}\|_1$ is easy if $h(t) \geq 0$ for all time.
Then every constant input achieves maximal peak-to-peak gain:



Example

$$\dot{x} = \begin{bmatrix} -a_1 & 0 \\ a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} \beta \\ 1-\beta \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & a_2 \end{bmatrix} x.$$

with $0 \leq \beta \leq 1$. Then

$$\|\mathscr{H}\|_1 = -CA^{-1}B = 1$$

Every $u(t) = c\mathbb{1}(t)$ achieves maximal peak-to-peak gain

Lemma (As.stable $\implies$ BIBO)

If A as.stable then initially-at-rest system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

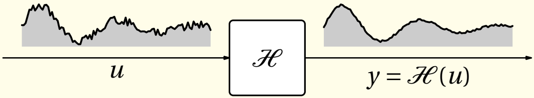
is BIBO-stable

Follows from $h(t) = C\,\mathrm{e}^{At}\,B\mathbb{1}(t) + D\delta(t)$

So “driving” an as.stable system with a bounded input results in a bounded output:

....

Summary



- $y, u : \mathbb{R} \to \mathbb{R},\ y = \mathscr{H}(u)$
- $y = \mathscr{H}(u)$ is LTI $\iff$ $y = h * u$. Here $h = \mathscr{H}(\delta)$
- $y = h * u$ is BIBO $\iff$ $\int |h(t)|\,\mathrm{d}t < \infty$
- $\dot{x} = Ax + Bu, y = Cx + Du$ initially-at-rest is LTI with $h(t) = C\mathrm{e}^{At}B\mathbb{1}(t) + D\delta(t)$.
- As.stable implies BIBO

§ 5.3 Step response

In LTI+BIBO systems the response to constant is constant:

Example

The response y to $u(t) = u_*$ (constant) is

$$\mathscr{H}(u_*)(t) = (h * u_*)(t) = \int_{-\infty}^{\infty} h(\tau) u_* \, \mathrm{d}\tau = \left(\int_{-\infty}^{\infty} h(\tau) \, \mathrm{d}\tau \right) \times u_*$$

The amplification factor, $\int_{-\infty}^{\infty} h(\tau) \, \mathrm{d}\tau$, is known as DC-gain.

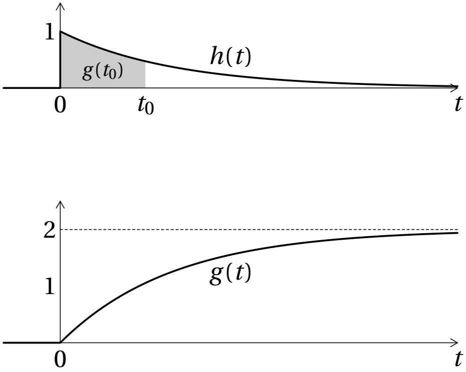
Response $g(t)$ to $u = \mathbb{1}(t)$ is known as step response:

Example (Step response)

In LTI+BIBO systems the step response converges to DC-gain:

$$\mathscr{H}(\mathbb{1})(t) = (h * \mathbb{1})(t) = \int_{-\infty}^{\infty} h(\tau) \mathbb{1}(t - \tau) \, \mathrm{d}\tau = \int_{-\infty}^{t} h(\tau) \, \mathrm{d}\tau$$

So step response $g := \mathscr{H}(\mathbb{1})$ is anti-derivative of $h := \mathscr{H}(\delta)$:



Example (As.stable 2nd-order system (p_2, p_1, p_0 same sign))

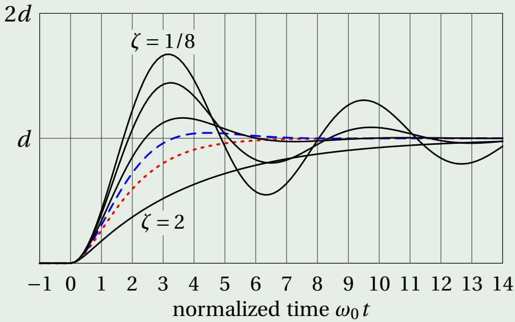
As.stable ODE

$$p_2\ddot{y}(t) + p_1\dot{y}(t) + p_0 y(t) = q_0 u(t)$$

is equivalent to (for some $\zeta > 0, \omega_0 > 0, d \in \mathbb{R}$)

$$\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + \omega_0^2 y(t) = d\omega_0^2 u(t)$$

Has this step response $g := \mathscr{H}(\mathbb{1})$:



§ 5.3 Frequency Response

Suppose $y = \mathscr{H}(u)$ is LTI & BIBO, then $y_\omega(t) := \mathscr{H}(\mathrm{e}^{\mathrm{i}\omega t})$ exists.
Now time invariance gives

$$\mathscr{H}(\mathrm{e}^{\mathrm{i}\omega(t-t_0)}) = y_\omega(t-t_0)$$

but time invariance gives

$$\begin{aligned}\mathscr{H}(\mathrm{e}^{\mathrm{i}\omega(t-t_0)}) &= \mathscr{H}(\mathrm{e}^{-\mathrm{i}\omega t_0}\,\mathrm{e}^{\mathrm{i}\omega t}) \\ &= \mathrm{e}^{-\mathrm{i}\omega t_0}\,\mathscr{H}(\mathrm{e}^{\mathrm{i}\omega t}) \\ &= \mathrm{e}^{-\mathrm{i}\omega t_0}\,y_\omega(t)\end{aligned}$$

Hence

$$\mathrm{e}^{-\mathrm{i}\omega t_0}\,y_\omega(t) = y_\omega(t-t_0)$$

For $t = t_0$ this says

$$e^{-i\omega t}\, y_\omega(t) = y_\omega(0).$$

Denote number $y_\omega(0)$ as $H(i\omega)$. Then:

$$y_\omega(t) = H(i\omega)\, e^{i\omega t}.$$

So output is again harmonic (with the same frequency)!

Theorem (Frequency response & eigenvalues..)

Every harmonic input $u(t) := e^{i\omega t}$ is eigenfunction of every LTI+BIBO system $y = \mathscr{H}(u)$, and its eigenvalue is denoted $H(i\omega)$

$H(i\omega)$ (as function of ω) is known as the frequency response of system

Example

$H(0)$ equals the DC-gain.

Example (Delay)

$$y(t) = u(t - t_0)$$

$$y(t) = \mathrm{e}^{\mathrm{i}\omega(t - t_0)} = \mathrm{e}^{-\mathrm{i}\omega t_0}\,\mathrm{e}^{\mathrm{i}\omega t}$$

$$H(\mathrm{i}\omega) = \mathrm{e}^{-\mathrm{i}\omega t_0}$$

$$\text{DC-gain} = H(0) = 1$$

Also follows from convolutions:

$$
\begin{aligned}
y(t) = \mathscr{H}(\mathrm{e}^{\mathrm{i}\omega\cdot})(t) &= (h * \mathrm{e}^{\mathrm{i}\omega\cdot})(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\,\mathrm{e}^{\mathrm{i}\omega(t-\tau)}\,\mathrm{d}\tau \\
&= \underbrace{\left(\int_{-\infty}^{\infty} h(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\right)}_{H(\mathrm{i}\omega):=}\mathrm{e}^{\mathrm{i}\omega t} \\
&= H(\mathrm{i}\omega)\,\mathrm{e}^{\mathrm{i}\omega t}
\end{aligned}
$$

We recognize $H(\mathrm{i}\omega)$ as the Fourier transform of $h(t)$.

Theorem (Initially-at-rest $\dot{x} = Ax + Bu$, $y = Cx + Du$)

From $h(t) = C\,\mathrm{e}^{At}B\mathbb{1}(t) + D\delta(t)$ it follows that

$$H(\mathrm{i}\omega) = C(\mathrm{i}\omega I - A)^{-1}B + D.$$

It exists if system is as.stable.

Example

The function

$$h(t) = \mathrm{e}^{-0.1t}\mathbb{1}(t)$$

is the impulse response of system

$$\dot{x} = -0.1x + u, \qquad y = x$$

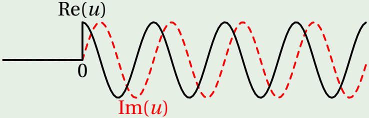
It is as.stable, so $H(\mathrm{i}\omega)$ exists and

$$H(\mathrm{i}\omega) = 1(\mathrm{i}\omega + 0.1)^{-1}1 + 0 = \frac{1}{\mathrm{i}\omega + 0.1}$$

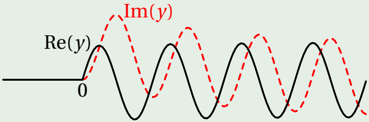
Example (Complex harmonische signalen)

Consider again $h(t)=\mathrm{e}^{-0.1\,t}\,\mathbb{1}(t)$, $H(\mathrm{i}\omega)=\frac{1}{\mathrm{i}\omega+0.1}$.
For $\omega=1$ gives:

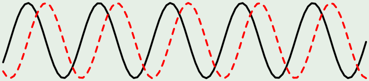
$$u(t)=\mathrm{e}^{\mathrm{i}\,t}\,\mathbb{1}(t)$$



$$y(t)$$



$$H(\mathrm{i})\,\mathrm{e}^{\mathrm{i}\,t}=\frac{1}{\mathrm{i}+0.1}\,\mathrm{e}^{\mathrm{i}\,t}$$



It seems $\mathscr{H}(\mathrm{e}^{\mathrm{i}\omega\cdot}\,\mathbb{1})$ converges to “steady state response” $\mathscr{H}(\mathrm{e}^{\mathrm{i}\cdot})$.

Final claim holds for all LTI+BIBO systems (see Lemma 5.4.5)

Frequency response for ODE's

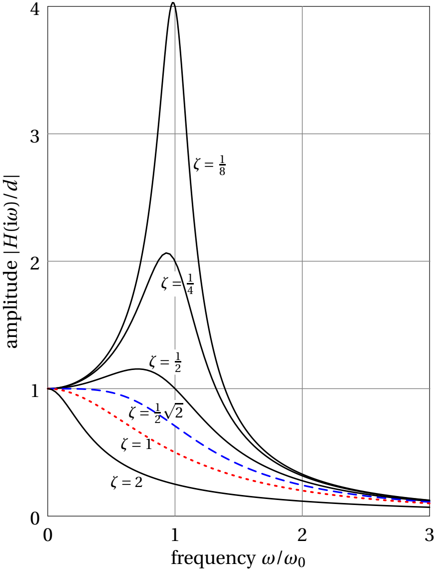
$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = q_n u^{(n)} + \cdots + q_0 u$$

$$\begin{aligned}\dot{z} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & -p_0 \\ 1 & \ddots & & \vdots & -p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1} \end{bmatrix} z + \begin{bmatrix} q_0 - p_0 q_n \\ q_1 - p_1 q_n \\ \vdots \\ \vdots \\ q_{n-1} - p_{n-1} q_n \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} z + q_n u,\end{aligned}$$

$$H(\mathrm{i}\omega) = \frac{q_n(\mathrm{i}\omega)^n + q_{n-1}(\mathrm{i}\omega)^{n-1} + \cdots + q_0}{(\mathrm{i}\omega)^n + p_{n-1}(\mathrm{i}\omega)^{n-1} + \cdots + p_0}.$$

Works if ODE is as.stable & initially-at-rest

$$\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + \omega_0^2 y(t) = d\omega_0^2 u(t), \quad H(\mathrm{i}\omega) = \frac{d}{(\mathrm{i}\omega/\omega_0)^2 + 2\zeta(\mathrm{i}\omega/\omega_0) + 1}.$$



Lemma (Frequency response (real version))

If LTI+BIBO then response y to

$$u(t) = \cos(\omega_0 t)$$

exists and is again harmonic with same frequency

$$y(t) = |H(\mathrm{i}\omega_0)| \cos(\omega t + \arg H(\mathrm{i}\omega_0))$$

Read it yourself...

§ 5.6 Transfer Function

By replacing $\mathrm{i}\omega$ by s we obtain transfer function of system:

$$H(s) := \int_{-\infty}^{\infty} h(t)\,\mathrm{e}^{-st}\,\mathrm{d}t.$$

1. If $\dot{x} = Ax + Bu, y = Cx + Du$ and if initially at rest then
$$H(s) = C(sI - A)^{-1}B + D.$$
Well defined if $\mathrm{re}(s) > \max_i \mathrm{re}(\lambda_i(A))$.
2. If ODE $P(\frac{\mathrm{d}}{\mathrm{d}t})y(t) = Q(\frac{\mathrm{d}}{\mathrm{d}t})u(t)$ initially at rest, then
$$H(s) = Q(s)/P(s).$$
Well defined if $\mathrm{re}(s) > \max_i \mathrm{re}(s_i)$ where s_i are the zeros of $P(s)$.
3. If $y = \mathscr{H}(u)$ is LTI and the Laplace transforms of $u(t)$ and $h(t)$ exist for all $\mathrm{re}(s) > \gamma$, then
$$Y(s) = H(s)U(s)$$
for all $\mathrm{re}(s) > \gamma$.

Example (Double integrator)

$$\ddot{y}(t) = u(t), \qquad \text{so } P(s) = s^2, \quad Q(s) = 1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} H(s) &= C(sI - A)^{-1} B + D \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2} = \frac{Q(s)}{P(s)} \end{aligned}$$

The eigenvalues of A are the poles of $H(s)$.
$H(s)$ exists iff $\mathrm{re}(s) > 0$ (to the right of its poles)

Because of this:

Definition

Rational transfer function is as.stable if all poles negative real part

But not all eigenvalues of A need to return as poles:

Example

$$\dot{x} = 3x + 0u$$
$$y = x + 2u$$

$$\begin{aligned} H(s) &= (s-3)^{-1}0 + 2 \\ &= 2 \end{aligned}$$

The (eigenvalue of) $A = 3$ in this case is not a pole of $H(s)$.
(By the way, the system is not controllable)

Example (Kalman)

$$\begin{bmatrix} \dot{z}_{\mathrm{o}} \\ \dot{z}_{\mathrm{uo}} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_{\mathrm{o}} \\ z_{\mathrm{uo}} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} z_{\mathrm{o}} \\ z_{\mathrm{uo}} \end{bmatrix} + Du$$

Then

$$\begin{aligned} H(s) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} sI - A_{11} & 0 \\ -A_{21} & sI - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \\ &= C_1 (sI - A_{11})^{-1} B_1 + D \end{aligned}$$

The unobservable part cancels in $H(s)$.

All poles of $H(s)$ are eigenvalues of A_{11}

Theorem (Minimal realization)

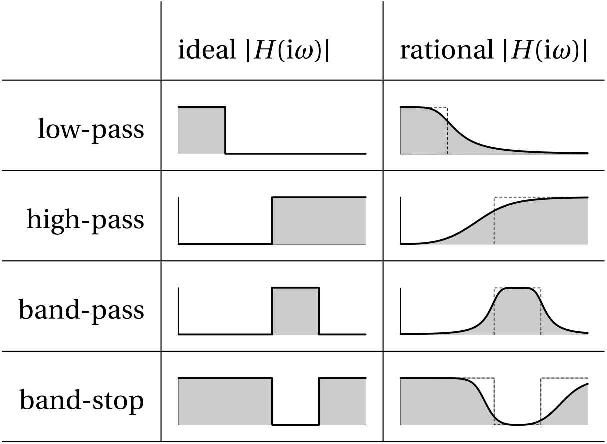
If (A, B) is controllable and (A, C) is observable, then the poles of $H(s) = C(sI - A)^{-1}B + D$ are exactly the eigenvalues of A.

Proof (outline).

(notice that $n_u = n_y = 1$.)

- Suppose $D = 0$ (without loss of generality)
- switch to observable canonical form
- From that define $P(s), Q(s)$.
- $P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$.
- Hence $H(s) = Q(s)/P(s) = Q(s)/\det(sI - A)$.
- controllable implies no common factors in $Q(s)/P(s)$..

∎



Can *design* rational (="simulatable") filters in frequency domain:

Example (Butterworth)

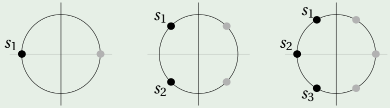
$$|H_n(\mathrm{i}\omega)|^2 = \frac{1}{1+\omega^{2n}}.$$

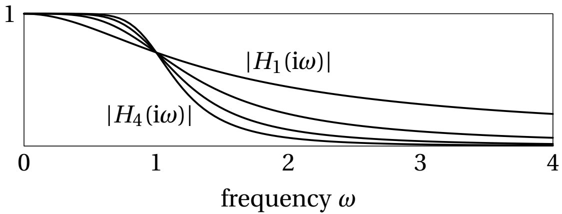
For $s = \mathrm{i}\omega$ this is

$$H_n(s)H_n(-s) = \frac{1}{1+(-s^2)^n} = (-1)^n \prod_{k=1}^{2n} \frac{1}{s-s_k}$$

$$\begin{aligned}
(-s_k^2)^n &= -1 \\
(-s_k^2) &= \sqrt[n]{-1} = \sqrt[n]{\mathrm{e}^{\mathrm{i}(2k-1)\pi}} = \mathrm{e}^{\frac{\mathrm{i}(2k-1)\pi}{n}} \\
s_k &= \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}(k-1/2)\pi}{n}}, \quad k = 1, 2, \ldots, 2n.
\end{aligned}$$

$$H_n(s) = \prod_{k=1}^{n} \frac{1}{s-s_k}.$$





Example (... continued)

$$H_1(s) = \frac{1}{s+1},$$

$$H_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}, \qquad \text{our blue dashed 2nd order system}$$

$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)},$$

$$H_4(s) = \frac{1}{(s^2 + \sqrt{2+\sqrt{2}}s + 1)(s^2 + \sqrt{2-\sqrt{2}}s + 1)}.$$

Encore: a bit of matlab (not exam material)

Butterworth

```
s=tf('s');
H=1/(s^2+sqrt(2)*s+1);
step(H);


t=0:.01:10;
u=cos(10*t);     % or cos(0.1*t) or ...
y=lsim(H,u,t);
plot(t,u,t,y);
```

Summary

- if LTI+BIBO then $u(t) = \mathrm{e}^{\mathrm{i}\omega t}$ implies $y(t) = H(\mathrm{i}\omega)\,\mathrm{e}^{\mathrm{i}\omega t}$
- DC gain: $H(0)$
- If LTI+BIBO then response $y(t)$ to

  $$u(t) = \mathrm{e}^{\mathrm{i}\omega t}\,\mathbb{1}(t)$$

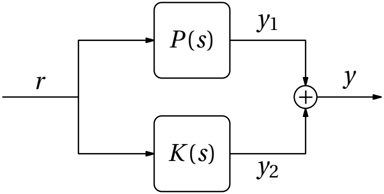
  converges to

  $$y_{\text{steady state}}(t) = |H(\mathrm{i}\omega)|\cos(\omega t + \phi)$$

- In particular, response to $u(t) = \mathbb{1}(t)$ converges to constant $H(0)$.
- $H(s) = C(sI - A)^{-1}B + D$ for all $\mathrm{re}(s) > \mathrm{re}(\lambda_i)$
- $H(s) = Q(s)/P(s)$ for all $\mathrm{re}(s) > \mathrm{re}(\lambda_i)$

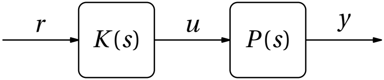
Now it is time to use it to analyze/design interconnections:

Parallel interconnection



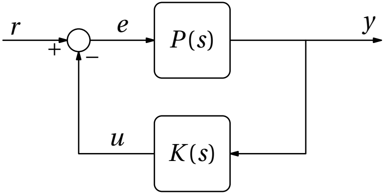
$$H_{y/r}(s) = P(s) + K(s)$$

Series interconnection



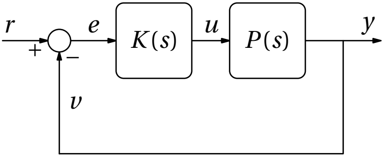
$$H_{y/r}(s) = P(s)K(s)$$

Feedback 1:

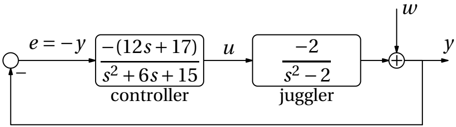


$$H_{y/r}(s) = \frac{P(s)}{1 + K(s)P(s)}$$

Feedback 2 (a very popular one):



$$H_{y/r}(s) = \frac{P(s)K(s)}{1 + P(s)K(s)}$$



Example (Juggler)

Juggler $P_{y/u}(s)$ and controller $K_{u/y}(s)$ designed in Chapter 4, with closed loop poles are -1 (twice) and -2 (twice):

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

$$P_{y/u}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -2 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \frac{-2}{s^2 - 2}.$$

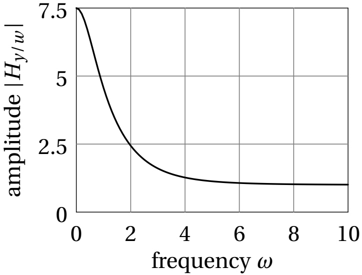
$$\dot{\hat{x}} = \begin{bmatrix} -4 & 1 \\ -7 & -2 \end{bmatrix} \hat{x} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} y, \qquad u = \begin{bmatrix} 3/2 & 1 \end{bmatrix} \hat{x},$$

$$\tilde{K}_{u/y}(s) = \begin{bmatrix} 3/2 & 1 \end{bmatrix} \begin{bmatrix} s+4 & -1 \\ 7 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{12s + 17}{s^2 + 6s + 15}.$$

Example (juggler continued)

$$H_{y/w}(s) = \frac{1}{1 + K(s)P(s)}$$

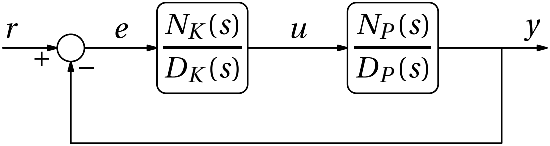
$$= \frac{(s^2 - 2)(s^2 + 6s + 15)}{s^4 + 6s^3 + 13s^2 + 12s + 4}$$



$$H_{y/w}(0) = \frac{-30}{4} = -7.5$$

To analyze closed loop stability we can bypass state models:

Closed loop (as).stability



Has input r, and output $(e), u, y$. For $r = 0$:

$$\begin{bmatrix} D_p(\frac{\mathrm{d}}{\mathrm{d}t}) & -N_p(\frac{\mathrm{d}}{\mathrm{d}t}) \\ N_k(\frac{\mathrm{d}}{\mathrm{d}t}) & D_k(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = 0$$

Lemma (Closed loop stability)

The closed loop is as.stable iff

$$\chi_{cl.}(s) = D_P(s) D_K(s) + N_P(s) N_K(s)$$

is as.stable polynomial.

From Chapter 2:

Lemma (As.stable)

If $P(s), Q(s)$ polynomial matrices, with $P(s)$ square, and $P(s), Q(s)$ having same # rows, then

$$P(\tfrac{\mathrm{d}}{\mathrm{d}t})y = Q(\tfrac{\mathrm{d}}{\mathrm{d}t})u \textit{ as.stable} \iff \det(P(s)) \textit{ as.stable}$$

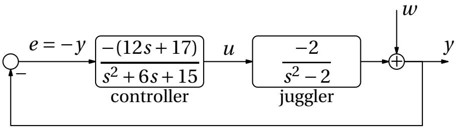
Proof.

Suppose $\det(P)$ is not as.stable. Then $\det(P(s_0)) = 0$ for some $s_0 \in \mathbb{C}$ with $\mathrm{re}(s_0) \geq 0$. Let $v \in \mathbb{C}^m$ be a nonzero vector such that $P(s_0)v = 0$. Then $y(t) := v\mathrm{e}^{s_0 t}$ satisfies $P(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$. This $y(t)$ does not converge to zero, hence DE not as.stable.

Suppose $\det(P)$ is asymptotically stable. The *adjugate* R of P is polynomial and

$$RP = \det(P)I.$$

If $P(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$ then also $\det(P)Iy = R(\frac{\mathrm{d}}{\mathrm{d}t})P(\frac{\mathrm{d}}{\mathrm{d}t})y$ is zero. Therefore every y_i satisfies $\det(P(\frac{\mathrm{d}}{\mathrm{d}t}))y_i = 0$. Since $\det(P)$ is as.stable this implies that $\lim_{t\to\infty} y_i(t) = 0$. Hence DE is as.stable ∎



Example (Juggler continued)

Juggler and controller are described by DE's:

$$\begin{bmatrix} \frac{d^2}{dt^2} - 2 & 2 \\ -(12\frac{d}{dt} + 17) & \frac{d^2}{dt^2} + 6\frac{d}{dt} + 15 \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = 0.$$

Claim: closed loop is as.stable iff

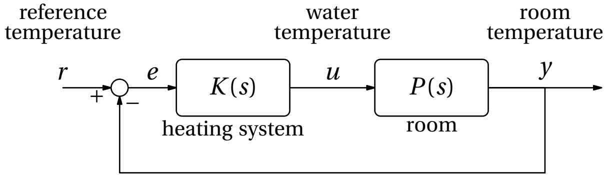
$$\chi_{\text{cl.}}(s) := \det \begin{bmatrix} s^2 - 2 & 2 \\ -(12s+17) & s^2 + 6s + 15 \end{bmatrix}$$

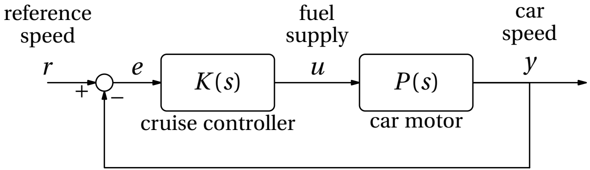
as.stable. Here that is the case:

$$\chi_{\text{cl}}(s) = s^4 + 6s^3 + 13s^2 + 12s + 4 = (s+1)^2(s+2)^2.$$

So closes loop poles are -1 (twice) and -2 (twice). No surprise.

Error feedback & reference input: $u = \mathcal{K}(e)$ & $e := r - y$





Closed loop (as).stability of error feedback

Example

- $K = 1/s, P = 1/(s+1)$
- $K = 1/s, P = s/(s+1)$
- $K = k, P = (s-1)/(s+1)$
- $K = k, P = -2/(s^2-2)$

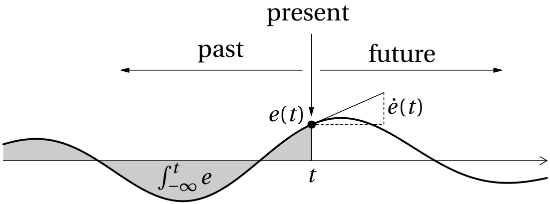
PID

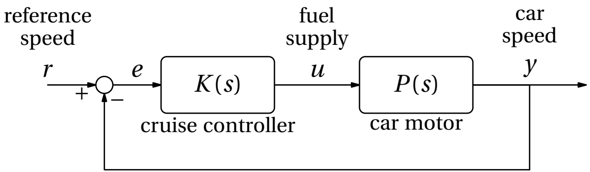
$$K_{\text{PID}}(s) = k_{\text{P}} + \frac{k_{\text{I}}}{s} + k_{\text{D}}\, s$$

P-action: $u(t) = k_{\text{P}}\, e(t)$

I-action: $\dot{u}(t) = k_{\text{I}}\, e(t)$

D-action: $u(t) = k_{\text{D}}\, \dot{e}(t)$





Example (Cruise controller – P)

car: $P(s) = b/(s+a)$ with $a, b > 0$

cruise controller: $K(s) = k$

closed loop char.pol: $s + a + kb$ cl.loop stable if $k > -a/b$

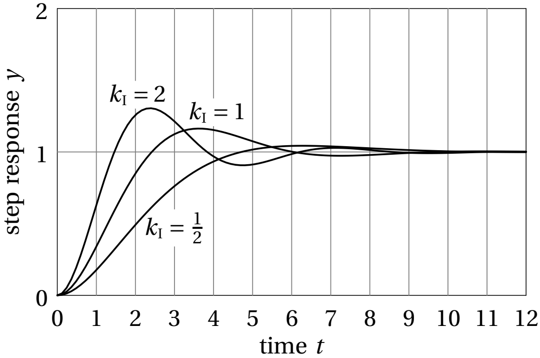
$$H_{y/r}(s) = kb/(s+a+kb)$$

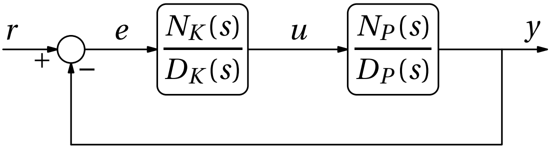
$$H_{y/r}(0) = kb/(a+kb)$$

Example (Cruise controller – I)

$$
\begin{aligned}
\text{car:}\quad & P(s) = b/(s+a) \qquad \text{with } a, b > 0\\
\text{controller:}\quad & K(s) = k_{\mathrm{I}}/s\\
\text{cl.loop pole:}\quad & s^2 + as + k_{\mathrm{I}} b, \qquad \text{cl.stable if } k_{\mathrm{I}} > 0\\
H_{y/r}(s)\quad & = k_{\mathrm{I}} b/(s^2 + as + k_{\mathrm{I}} b)\\
\text{DC-gain:}\quad & H_{y/r}(0) = 1 \text{ for all } a, b, k_{\mathrm{I}} > 0!
\end{aligned}
$$

For $a = b = 1$:





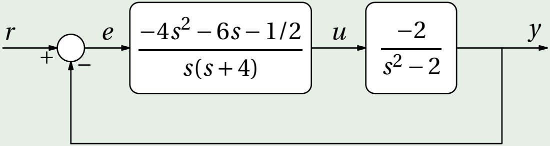
Lemma (Integrating action – zero steady-state error)

Suppose

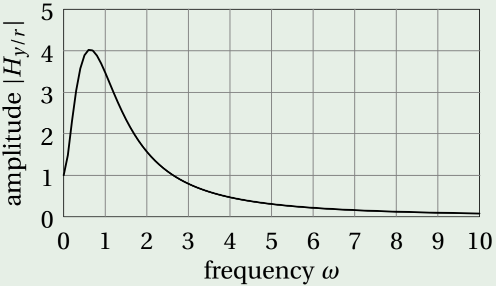
- $r(t) = r$ *(constant signal) or* $r(t) = r\mathbb{1}(t)$*,*
- *Closed loop is as.stable,*
- $P(s)K(s)$ *has a pole at* $s = 0$.

Then $y(t)$ *converges to* r *als* $t \to \infty$.

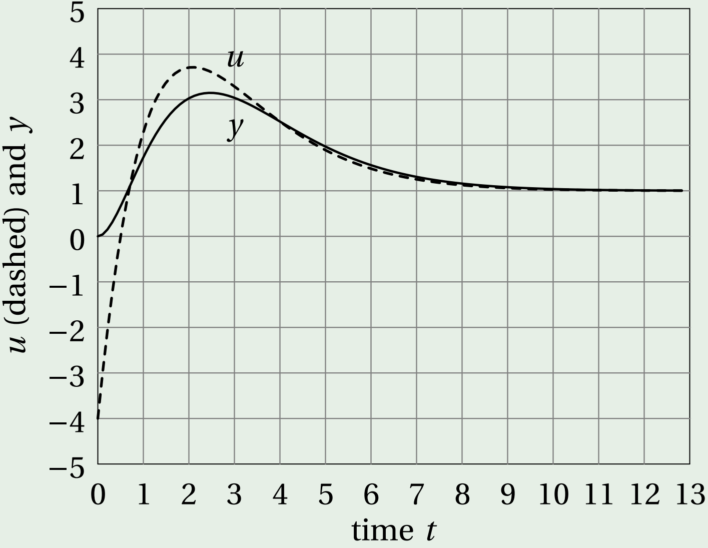
Example (Walking juggler)



$$\chi_{\text{closed}}(s) = (s+1)^4$$



Example (If reference input is $r(t) = \mathbb{1}(t)$)



Violent control signal u (can be resolved, see next slides)

A farewell example:

Example (Realistic relaxed walking juggler)

- $g = 10$
- $\ell = 1$
- $P(s) = -10/(s^2 - 10)$
- plant poles at $s = \pm\sqrt{10} = \pm 3.16234$
- Very unstable
- Requires “fast” control

Example

- Setting $\chi_{\text{cl}} = (s+3)^4$ with integrating action gives

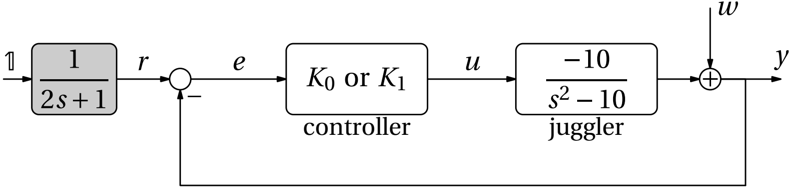
$$K_0 = \frac{64s^2 + 228s + 81}{-10\textcolor{red}{s}(s+12)}$$

  (Notice that K_0 is "proper")

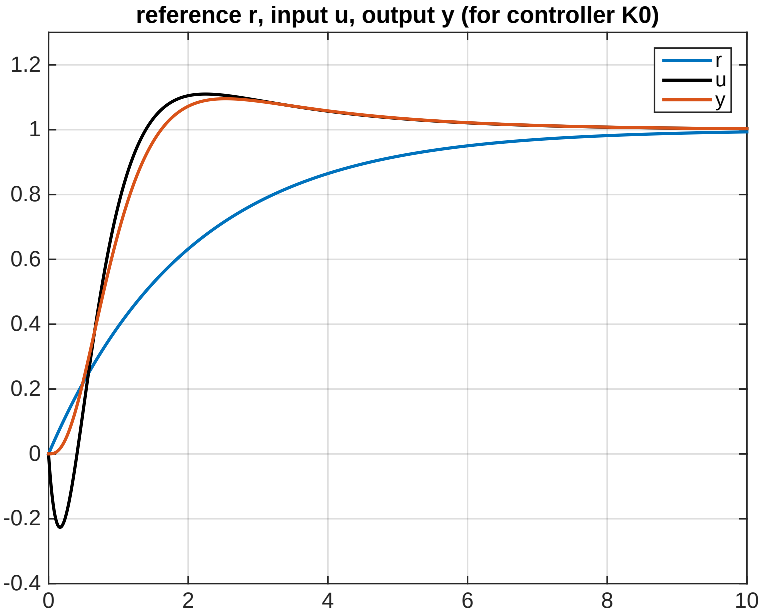
- Using some fancy control method (not in this course):

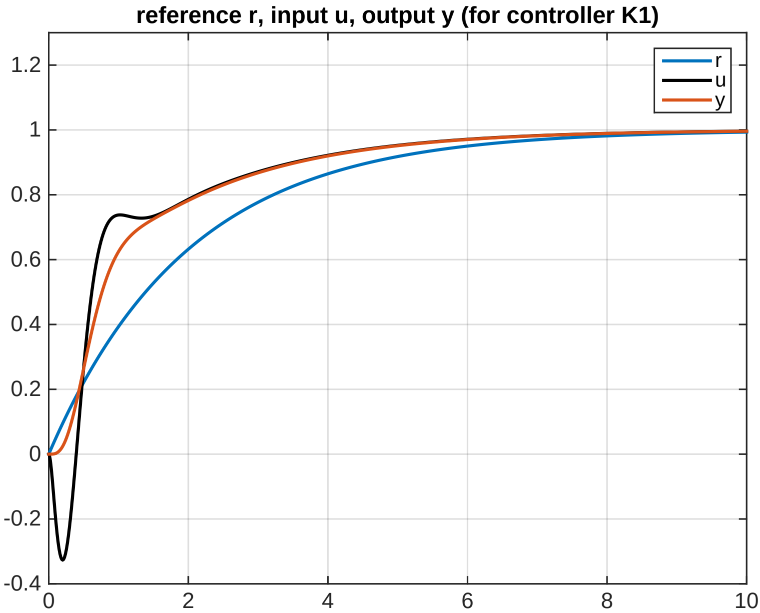
$$K_1 = \frac{-179.8s^2 - 729.1s - 515}{\textcolor{red}{s}(s^2 + 26.47s + 288.3)}$$

  (notice that K_1 is "strictly proper")



(by the way: $H_{y/r} = \frac{PK}{1+PK}$ as before; it does not depend on w.
We have $y = H_{y/r}\, r + H_{y/w}\, w$....)


reference r, input u, output y (for controller K0)
r
u
y
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
2
4
6
8
10


reference r, input u, output y (for controller K1)
r
u
y
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
2
4
6
8
10