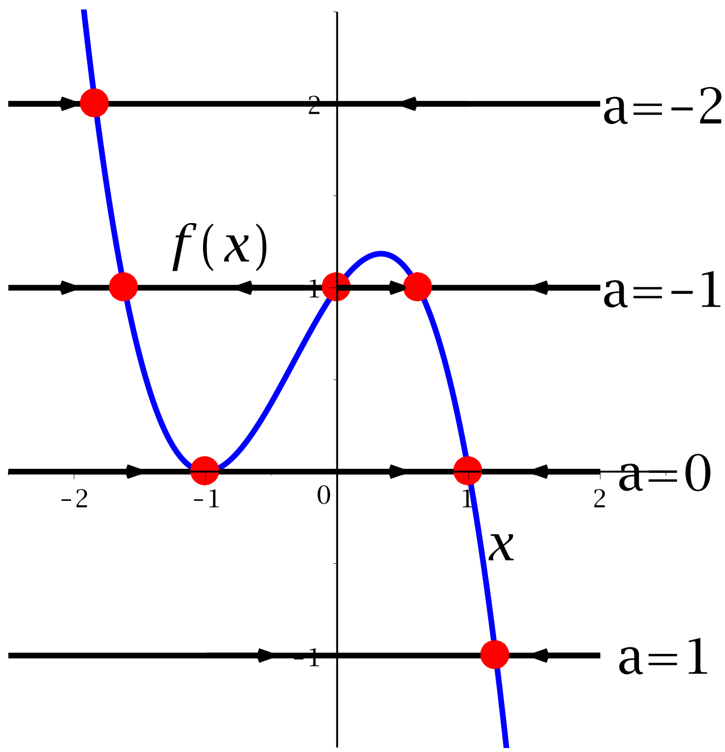
Answers Tutorial 2

1. The phase lines are shown in the figure. To create this figure, first draw the graph, next at each constant height the red dots represent the solutions $f(x) = -a$. (Note the minus-sign). Finally put arrows to the right if the line is above the horizontal line $y = -a$, and to the left if it is below.



[ht!]

We have $f' = (x + 1)(1 - 3x)$, so the extremes of f are $(-1, 0)$ and $(1/3, 32/27)$. So for $a \in (-32/27, 0)$ there are precisely three equilibria. When $a = 0$ and $a = -32/27$, we have a qualitative change in the behaviour of the system as the number of equilibria changes.

2. See where $x(t)$ and $y(t)$ are minimal or maximal, or equal to zero. First put some dots there, and then sketch a smooth curve through these points.

3. We present the argument for $n = 3$, but the idea is analogous for general $n \in \mathbb{N}$.

   (a) Introduce dummy variabeles $dx/dt = y$ and $dy/dt = z$ such that $dz/dt = -a_0x - a_1y - a_2z$. These three equations together may be written as a linear system $d/dt(x, y, z)^T = A(x, y, z)^T$ with matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$. Now for general n you will find that the elements of the matrix A are zero except for the upper-diagonal containing ones, and the lowest row where you have $-a_i$ in the ith column.

(b) There are two approaches to this part. First, you may insert the trial solution $x = e^{\lambda t}$ into the ODE. The second is to find the characteristic polynomial for the eigenvalues of the matrix A. We find $p(\lambda) = det(A - \lambda I) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.

(c) Note that $x = e^{\lambda t}$ is a solution, so we have $p(\lambda) = 0$, which may be rewritten as $-\lambda^3 = a_2\lambda^2 + a_1\lambda + a_0$
Next we substitute the given solution $v = v_1(1, \lambda, \lambda^2)^T$ into the system $y' = Ay$ to verify it is a solution of the system:

$$\lambda \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} \lambda \\ \lambda^2 \\ -a_0 - a_1\lambda - a_2\lambda^2 \end{pmatrix} e^{\lambda t}$$

Check that the terms on the left and right are the same, using $p(\lambda) = 0$ for the last row. The conclusion is that for the eigenvalue λ you have an explicit expression for the eigenvector v.

4. (a) Substituting $x = e^{\lambda t}$ yields a quadratic equation for λ with solution $\lambda_\pm = b \pm \sqrt{(\epsilon)}$.

(b) Here we recognize the difference quotient of the function $e^{\lambda t}$ differentiating w.r.t. λ. We differentiate and next set $\epsilon = 0$ to find $\lim_{\epsilon \to 0} x_\epsilon = te^{bt}$. Now as the system is linear, this is a solution too.

(c) In the previous exercise you have see what the expression is for the eigenvectors for first order systems corresponding to nth-order scalar ODEs. So for $n = 2$ we have $v = (1, \lambda)$. We observe that for $\epsilon > 0$ these eigenvectors are different, but as $\epsilon \to 0$ they align more and more until they collide for $\epsilon = 0$. And for $\epsilon < 0$ they are complex, and drawing them is meaningless. The matlab-script of lecture 3 also illustrates this point.

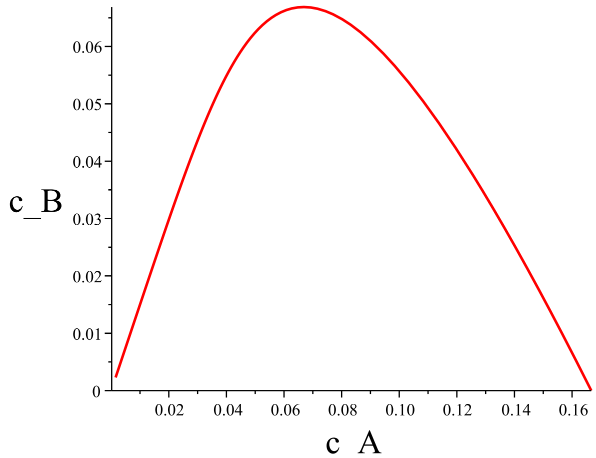
5. (a) $c_A = x_A/360$ en $c_B = x_B/360$.

(b) Tank A receives 9 liters per minute, but only 4 of them contain salt.

$$\begin{array}{c} \Delta x_A = \Delta t(-9c_A + 4c_B) \\ \Delta x_B = \Delta t(9c_A - 9c_B) \end{array} \longrightarrow \begin{pmatrix} c_A' \\ c_B' \end{pmatrix} = \begin{pmatrix} \frac{-9}{360} & \frac{4}{360} \\ \frac{9}{360} & \frac{-9}{360} \end{pmatrix} \begin{pmatrix} c_A \\ c_B \end{pmatrix}$$

(c) The eigenvalues are $\lambda = -1/120$ with eigenvector $v = (2, 3)$ and $\lambda = -1/24$ with $v = (2, -3)$. Hence we find $c_A(t) = \frac{1}{12}\left(e^{-t/24} + e^{-t/120}\right)$ en $c_B(t) = \frac{1}{8}\left(e^{-t/120} - e^{-t/24}\right)$

(d) We plot the solution $(c_A(t), c_B(t))$ as a parametric curve in the phase plane. Note we start at $(c_A, c_B) = (1/6, 0)$ and converge to $(0, 0)$.



c_B
0.06
0.05
0.04
0.03
0.02
0.01
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
c_A