Signals and Transforms

April 2020

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Why this course?

In 1807 Jean Baptiste Joseph Fourier, in his desire to determine the heat distribution in a metal plate, came up with the bold idea that every function can be written as a sum of harmonic functions. Understandably his claim was met with scepticism, especially because Fourier did not provide a proof of his claim. Several years later (still within Fourier's lifetime) Niels Henrik Abel settled the issue with a proper proof: Fourier was right after all, and this sum of harmonic functions since then carries his name: the *Fourier series*.

Nowadays it is difficult to imagine a world without Fourier series. Trillions of Fourier series are calculated routinely every second. For instance your laptop computes 250000 such series for every second that it is talking wirelessly to your modem, and your modem does the same. Ever wondered what JPEG files are? They are essentially just a collection of Fourier coefficients that your computer (by computing a Fourier series) can turn into a picture. Fourier is everywhere in audio and video processing (such as MP3, MPEG, et cetera) and the digital revolution would have been way less succesfull if it were not for the Fourier series and the discovery of the "fast fourier transform" which is an algorithm that can compute Fourier series very efficiently.

Fourier analysis is a standard tool in many engineering sciences. It offers an alternative representation of "signals" and "systems" providing lots of insight, and the above mentioned set of applications are just a few of them. Fourier analysis plays a very important role in signal processing applications, it is the work-horse behind efficient algorithms for computation of products of very long integers, and is used to solve partial differential equations, and much more.

In this course we analyze the Fourier series in detail and we use it to solve a number of "signal processing" problems and — which is how it all started — to solve certain differential equations and partial differential equations. The course also introduces the *Laplace transform*, which can be seen as an extension of the Fourier transform.

The material presented in Chapters 2–5 is quite common in engineering sciences. The first chapter, however, is of a different nature and might fit in an advanced course on linear algebra or a first course on functional analysis. It introduces Banach and Hilbert spaces and more, providing an abstract and general view of Fourier series.

Chapter 1

Introduction to Banach and Hilbert Space

This chapter assumes basis knowledge of linear algebra, in particular it assumes familiarity with real and complex vector space, with subspaces, basis and dimensions of finite dimensional vector space, and with linear transformations defined on vector space.

We make frequent use of the following notation:

$\mathbb{N}$	set of positive integers $\{1,2,3,\ldots\}$
$\mathbb{N}_0$	set of nonnegative integers $\{0,1,2,3,\ldots\}$
$\mathbb{Z}$	set of integers $\{\ldots,-2,-1,0,1,2,\ldots\}$
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	set of ordered n-tuples $(u_1,\ldots,u_n)$ with $u_k \in \mathbb{R}\ \forall k = 1,2,\ldots,n$
$\mathbb{C}^n$	set of ordered n-tuples $(u_1,\ldots,u_n)$ with $u_k \in \mathbb{C}\ \forall k = 1,2,\ldots,n$

We always assume that $\mathbb{R}^n$ is equipped with the standard vector addition and scalar multiplication, and this makes $\mathbb{R}^n$ a real vector space. Likewise $\mathbb{C}^n$ is a complex vector space. Both are vector spaces of finite sequences. For vector spaces of infinite sequences – with similar vector addition and scalar multiplication – we use the notation

ℓ	infinite sequence space $\{(u_1,u_2,\ldots) \mid u_k \in \mathbb{R}\,\forall k \in \mathbb{N}\}$

$\ell(A;B)$	general sequence space $\{u : A \to B\}$ with $A \subseteq \mathbb{Z}$. For instance $\ell = \ell(\mathbb{N};\mathbb{R})$
ℓ^{finite}	$\{u : \mathbb{N} \to \mathbb{R} \mid u_k \neq 0$ for finitely many $k \in \mathbb{N}\}$. It is a subset of ℓ.
ℓ^p	$\{u \in \ell \mid \sum_{k=1}^{\infty} \lvert u_k\rvert^p < \infty\}$. Assumes $0 < p < \infty$. For $p = \infty$ it is defined as $\{u \in \ell \mid \sup_{k\in\mathbb{N}} \lvert u_k\rvert < \infty\}$. More on this later.

The above assume that $A \subseteq \mathbb{Z}$. For arbitrary subsets A of $\mathbb{R}$ we use the following notation. Also these naturally are vector spaces.

$\mathscr{F}(A;B)$	Function space $\{f : A \to B\}$, i.e. the set of all functions that map from A to B. For instance $\mathbb{C}^n = \mathscr{F}(\{1,\dots,n\};\mathbb{C})$ and $\ell = \mathscr{F}(\mathbb{N};\mathbb{R})$. Typically, though, $\mathscr{F}$ is used for function spaces such as $\mathscr{F}([0,1];\mathbb{R})$.
$\mathscr{C}(A;B)$	The set of functions $f : A \to B$ that are continuous. It is a subset of $\mathscr{F}(A;B)$. We assume $A \subseteq \mathbb{R}$, and $B = \mathbb{R}^n$ or $B = \mathbb{C}^n$.
$\mathscr{L}^p(A;B)$	Set of functions $f : A \to B$ for which $\int_A \lvert f(t)\rvert^p \,\mathrm{d}t < \infty$. It is a subset of $\mathscr{F}(A;B)$. Assumes $1 \le p < \infty$ or $p = \infty$, and $A \subseteq \mathbb{R}$, and $B = \mathbb{R}^n$ or $B = \mathbb{C}^n$. More on this later.

1.1 Module 2 summary: normed vector space & convergence

A *normed vector space* loosely speaking is a vector space in which a “length” of a vector is available. This additional structure allows us to deal with convergence and optimal approximations within vector spaces. A “length” is called a *norm* if it has the following properties.

Definition 1.1.1 (norm). Let $\mathbb{X}$ be a real or complex vector space. A mapping $\|\cdot\|$ from $\mathbb{X}$ to $\mathbb{R}$ is a *norm* if for all $x, y \in \mathbb{X}$ and all scalars α it satisfies the three axioms:

1. $\|\alpha x\| = |\alpha|\|x\|$, (*positive homogeneous*)
2. $\|x + y\| \le \|x\| + \|y\|$, (*triangle inequality*)
3. $\|x\| > 0$ for every $x \neq 0$. (*positive definite*)

□

For $\alpha = 0$ the first axiom tells us that $\|0\| = 0$. So a norm $\|x\|$ is zero if and only if x is the zero vector. A *normed vector space* is a vector space on which a norm is defined.

Formally one should say "$(\mathbb{X}, \|\cdot\|)$ is a *normed vector space*" but we usually just say "$\mathbb{X}$ is a normed vector space" assuming that the choice of norm is clear from the problem at hand. Be aware, however, that a vector space can be equipped with different norms, see the next example.

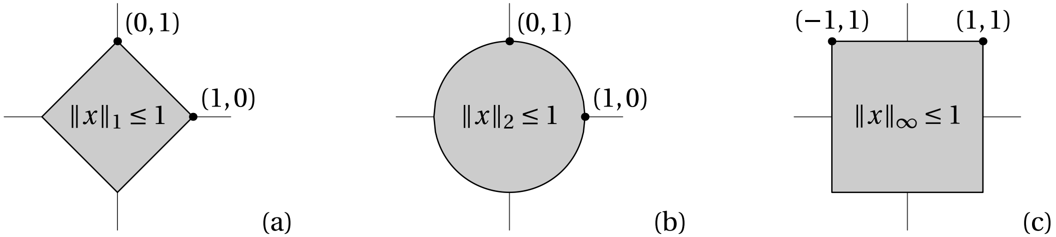


FIGURE 1.1: The unit balls $\{x \in \mathbb{R}^2 \mid \|x\|_p \le 1\}$ for $p = 1, 2, \infty$.

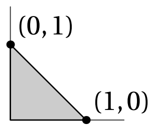
Example 1.1.2 (Three different norms on $\mathbb{R}^2$). Consider $\mathbb{R}^2$.

1. The 1-*norm* on $\mathbb{R}^2$ is defined as

$$\|x\|_1 = |x_1| + |x_2|.$$

In the first quadrant — where x_1 and x_2 are nonnegative — the 1-norm is just the the sum the entries, $\|x\|_1 = x_1 + x_2$. In the first quadrant therefore the norm is at most 1 iff $x_2 \le 1 - x_1$, which is the region

$$\{x \in \mathbb{R}^2 \mid x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 1\}$$



Combined with the other three quadrants we get that the unit ball $\{x \in \mathbb{R}^2 \mid \|x\|_1 \le 1\}$ is a polytope, see Fig. 1.1(a).

2. The *Euclidean norm* on $\mathbb{R}^2$, also known as the 2-*norm*, is defined as

$$\|x\|_2 := \sqrt{x_1^2 + x_2^2}.$$

In this norm the unit ball $\{x \in \mathbb{R}^2 \mid \|x\|_2 \le 1\}$ is the standard unit disc, see Fig. 1.1(b).

3. The *max-norm*, or ∞*-norm* on $\mathbb{R}^2$ is defined as

$$\|x\|_\infty = \max(|x_1|, |x_2|).$$

Now in this norm the unit ball $\{x \in \mathbb{R}^2 \mid \|x\|_\infty \le 1\}$ is a square with its axes parallel to the x_1- and x_2-axis, see Fig. 1.1(c).

In Exercise 1.1.2 we ask you to prove that all three are indeed norms. □

The 1-norm is sometimes called the *manhattan norm* because in a rectangular street grid — which is common in US cities — the 1-norm $\|x - y\|_1$ is the minimal Euclidean distance over the road required to travel from junction x to junction y, see Fig. 1.2.

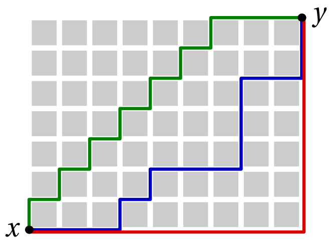
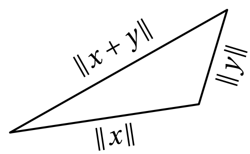


FIGURE 1.2: Manhattan norm: all three routes from x to y are equally long: $\|x - y\|_1$.

The triangle inequality $\|x + y\| \le \|x\| + \|y\|$ roughly speaking says that, in any norm, traveling from 0 to $x + y$ via x or y can only mean a detour. Moving the $\|y\|$ to the left-hand side of the inequality turns the triangle inequality into a statement that says that the length of any side in a triangle is at least the difference of that of the other two sides,

$$\|x + y\| - \|y\| \le \|x\|.$$



This is sometimes called the *reverse triangle inequality*, and it is commonly formulated in terms of $z = x + y$ as:

Lemma 1.1.3 (Reverse triangle inequality). $\big|\|z\| - \|y\|\big| \le \|z - y\|$. □

The reverse triangle inequality demonstrates that when two vectors z and y are "close" then their norms are "close" as well:

Lemma 1.1.4 (Norms are continuous). Every norm $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ is uniformly continuous.

Proof. Recall that a mapping $A : \mathbb{X} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{X}$ if for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that $\|x - x_0\| < \delta$ implies $|A(x) - A(x_0)| < \epsilon$. For $A := \|\cdot\|$ we can simply take $\delta_\epsilon = \epsilon$ because, by the reverse triangle inequality, $|A(x) - A(x_0)| = |\|x\| - \|x_0\|| \le \|x - x_0\|$. This works for every $x_0 \in \mathbb{X}$. (Continuity is *uniform* if we can choose δ_ϵ independent from x_0; that is the case here.) ∎

This is a very useful property, and we will exploit it later (be patient).

Example 1.1.5. Consider ℓ^{finite} (the vector space of infinite sequences $f = (f_1, f_2, f_3, \ldots)$ that have finitely many nonzero entries). On this space the 1-norm defined as

$$\|f\|_1 := \sum_{i=1}^{\infty} |f_i| \tag{1.1}$$

is a norm (see Exercise 1.2). □

Implicit in the definition of norm is that the norm is well defined (exists) for every element of the space. Hence on ℓ — which we use to denote the vector space of all real-valued sequences — the above 1-norm is *not* a norm because (1.1) is not convergent for the sequence $f = (1, 1, 1, \ldots) \in \ell$.

Example 1.1.6 (Continuous functions in max-norm). The standard norm on the vector space $\mathscr{C}([a, b]; \mathbb{R})$ of continuous functions on the real interval $[a, b]$ is the *max-norm*, also known as ∞-*norm*, defined as

$$\|f\|_\infty = \max_{t \in [a,b]} |f(t)|.$$

We now verify that this indeed satisfies the three axioms of norm, including existence:

0. Every continuous function on a finite interval is bounded and has a maximum, so $\|f\|_\infty$ exists for all $f \in \mathscr{C}([a, b]; \mathbb{R})$.

1. For every scalar α we have $\|\alpha f\|_\infty = \max_t |\alpha f(t)| = \max_t |\alpha||f(t)| = |\alpha| \max_t |f(t)| = |\alpha| \|f\|_\infty$.
2. The max norm inherits the triangle inequality from $\mathbb{R}$: since for every $p, q \in \mathbb{R}$ we have that $|p+q| \le |p| + |q|$, we also have for every $f, g \in \mathscr{C}([a,b];\mathbb{R})$ that
$$\begin{aligned}\|f+g\|_\infty &= \max_t |f(t)+g(t)| \\ &\le \max_t |f(t)| + |g(t)| \\ &\le \max_t |f(t)| + \max_t |g(t)| = \|f\|_\infty + \|g\|_\infty.\end{aligned}$$
3. If f is not the zero function then $f(t_0) \neq 0$ for at least one $t_0 \in [a,b]$. Now $\|f\|_\infty \ge |f(t_0)| > 0$.

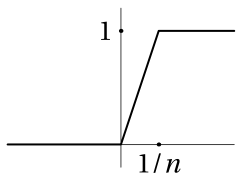
□

In the literature the vector space $\mathscr{C}([a,b];\mathbb{R})$ is often identified with the *normed* vector space $(\mathscr{C}([a,b];\mathbb{R}), \|\cdot\|_\infty)$. This is unfortunate since we might want to consider other norms on the space of continuous functions, for instance:

Example 1.1.7. Let $a, b \in \mathbb{R}$ and $a < b$. On $\mathscr{C}([a,b];\mathbb{R})$ the function $\|\cdot\|_1 : \mathscr{C}([a,b];\mathbb{R}) \to \mathbb{R}$ defined as
$$\|f\|_1 = \int_a^b |f(t)|\,\mathrm{d}t \tag{1.2}$$
is a norm (see Exercise 1.3). □

Notice that in this example the norm $\|f\|_1$ exists for every continuous function. For arbitrary functions in $\mathscr{F}([a,b];\mathbb{R})$ that need not be the case and this is the reason we restricted attention to $\mathscr{C}([a,b];\mathbb{R})$. However the space of continuous functions also has its drawbacks for this norm:

Example 1.1.8 (Limit does not exist in the space). Consider $\mathscr{C}([-1,1];\mathbb{R})$ and the 1-norm defined in (1.2). In this norm the sequence of functions
$$f_n(t) = \begin{cases} 0 & t \in [-1,0] \\ nt & t \in (0, \frac{1}{n}) \\ 1 & t \in [\frac{1}{n}, 1] \end{cases}$$



does not converge *in the space* $\mathscr{C}([-1,1];\mathbb{R})$ because no continuous function f_∞ exists for which $\lim_{n\to\infty}\|f_n - f_\infty\|_1 = 0$. (Convince yourself of this; Exercise 1.13 might be helpfull here.) Nevertheless the sequence of functions do approach one another in this norm, in the sense that

$$\sup_{n\geq N, m\geq N} \|f_n - f_m\|_1$$

goes to zero as $N\to\infty$. This follows from the fact that for any $n,m\geq N$ we have

$$\|f_n - f_m\|_1 = \int_{-1}^{1} |f_n(t) - f_m(t)|\,\mathrm{d}t = \int_0^{1/\min(n,m)} |f_n(t) - f_m(t)|\,\mathrm{d}t \leq \int_0^{1/N} 1\,\mathrm{d}t = \frac{1}{N}.$$

□

The above example demonstrates that there is a difference between converging sequences and sequences whose elements become closer and closer. The latter is called "Cauchy sequence," which we discuss next. Incidentally this difference is not specific to vector space. It also shows up in sets like the rational numbers $\mathbb{Q}$. Indeed, in $\mathbb{Q}$ we can construct sequences that approach one another in absolute value but that do not have a limit *in the set of rational numbers.* An example is the sequence of rational numbers $\{3, 3.1, 3.14, 3.141, \ldots\}$ that converges to the non-rational π.

Convergence and the Cauchy sequence

Quite often a vector x can be shown to have some desirable property by creating a sequence of easier-to-work-with vectors $\{x_n\}_{n\in\mathbb{N}}$ and that converge to this vector x. It is for this reason that the concept of convergence plays an important role in analysis. We define two types of convergence:

Definition 1.1.9 (Cauchy sequence and convergent sequence). Let $\mathbb{X}$ be a normed vector space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$ (that is, $x_n\in\mathbb{X}$ for every $n\in\mathbb{N}$).

- We say that $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if $\|x_n - x_m\|\to 0$ as $n,m\to\infty$. This means $\forall\epsilon>0\ \exists N_\epsilon\in\mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon \qquad \forall n\geq N_\epsilon, m\geq N_\epsilon.$$

- We say that $\{x_n\}_{n\in\mathbb{N}}$ *converges* in $\mathbb{X}$ if an $x \in \mathbb{X}$ exists such that $\|x - x_n\| \to 0$ as $n \to \infty$. This means $\forall \epsilon > 0\ \exists N_\epsilon > 0$ such that

$$\|x - x_n\| < \epsilon \qquad \forall n \geq N_\epsilon.$$

In that case x is called the *limit* of the sequence. □

If x_n converges then its limit x is unique (Exercise 1.13(a)). We denote this limit x as $x = \lim_{n\to\infty} x_n$. The Cauchy property can be expressed in terms of limits as well: a sequence x_n is a Cauchy sequence (on some vector space with norm $\|\cdot\|$) if and only if

$$\lim_{N\to\infty} \left(\sup_{n\geq N, m\geq N} \|x_n - x_m\| \right) = 0.$$

The term in between brackets is, so to say, the "maximal" distance between any two elements x_n and x_m in the "tail" of the sequence $\{x_N, x_{N+1}, x_{N+2}, \ldots\}$. For real sequences this is depicted in Fig. 1.3. It can be shown that for sequences $\{x_n\}$ of *real numbers* the two notions are equivalent[1]. I.e. a real sequence converges iff it is a Cauchy sequence. Figure 1.3 makes this plausible.

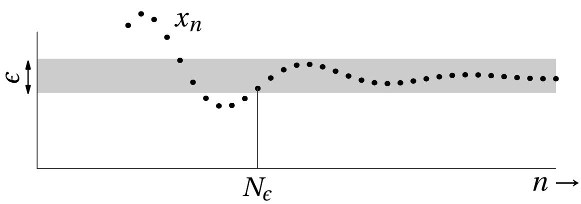


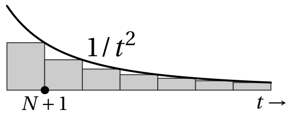
FIGURE 1.3: Cauchy criterion for real sequences x_n.

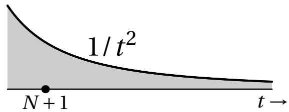
Example 1.1.10 (Integral test for real-valued sequences). Consider the real sequence

[1]See the course *Analysis 1*

$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \in \mathbb{R}$. Now for every $m \geq n \geq N$ we have

$$\begin{aligned} |x_m - x_n| &= \sum_{k=n+1}^{m} \frac{1}{k^2} \\ &< \sum_{k=N+1}^{\infty} \frac{1}{k^2} \\ &< \int_N^{\infty} \frac{1}{t^2}\,\mathrm{d}t \\ &= \frac{-1}{t}\Big|_N^{\infty} = \frac{1}{N}. \end{aligned}$$





Therefore $\sup_{n,m \geq N} |x_m - x_n|$ converges to zero as $N \to \infty$, which shows that it is a Cauchy sequence. But for real sequences being Cauchy is equivalent to being convergent, so $\lim_{n\to\infty} x_n$ exists. The beauty and strength of Cauchy is that we do not need to know the limit of a real sequence in order for us to know that the limit exists. (Cliff hanger: later on in this course we will see that the limit is $\pi^2/6 = 1.644934...$, see Example 3.5.5.) □

While on $\mathbb{R}$ every Cauchy sequence converges, Example 1.1.8 demonstrates that Cauchy sequences on other normed vector spaces need not always converge. The converse *does* hold on every vector space:

Theorem 1.1.11 (Convergence implies Cauchy). Every convergent sequence is a Cauchy sequence.

Proof. If $\lim_{n\to\infty} \|f_n - f\| = 0$ then $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon/2\ \forall n \geq N$. From the triangle inequality it now follows that $\|f_n - f_m\| = \|(f_n - f) - (f_m - f)\| \leq \|f_n - f\| + \|f_m - f\| < \epsilon/2 + \epsilon/2 = \epsilon$ for every $n, m \geq N$. So f_n is a Cauchy sequence. ■

1.2 Banach space

Normed sets on which every Cauchy sequence converges are so desirable that we give them a name:

Definition 1.2.1 (Complete set & Banach space). A normed set $\mathbb{X}$ is *complete* if every Cauchy sequence in $\mathbb{X}$ converges in $\mathbb{X}$. A *Banach space* is a complete normed vector space.

□

In a Banach space, therefore, a sequence converges *if and only if* it is a Cauchy sequence. This is convenient because the Cauchy property is often easier to check as it does not require knowledge of the limit, see Example 1.1.10. This will be of great help in the final section of this chapter.

Over the years many vector spaces have been shown to be Banach spaces, but also many fail to be. In this course we will not worry about completeness proofs too much because the proofs are often technical. We simply list a couple in the remainder of this section, and we prove only a few.

Theorem 1.2.2 (Continuous functions with max norm). $\mathscr{C}([a,b];\mathbb{R})$ is a Banach space in the max-norm $\|\cdot\|_\infty$.

Proof. Suppose f_n is a Cauchy sequence. Then $\forall\epsilon>0$ there is an $N_\epsilon>0$ such that $\|f_n - f_m\|_\infty<\epsilon$ for all $n,m\ge N_\epsilon$. Then at every $t\in[a,b]$ we have

$$|f_n(t)-f_m(t)|\le\|f_n-f_m\|_\infty<\epsilon \qquad \forall n,m\ge N_\epsilon. \tag{1.3}$$

So for every fixed $t\in[a,b]$ the sequence of real numbers $\{f_n(t)\}$ is Cauchy. Since $\mathbb{R}$ (in the absolute value as norm) is complete, we have for every t that the sequence $f_m(t)\in\mathbb{R}$ converges as $m\to\infty$. Denote this limit as $f(t)$. Letting $m\to\infty$ in (1.3) shows that

$$|f_n(t)-f(t)|\le\epsilon \qquad \forall n\ge N_\epsilon,$$

and that this N_ϵ does not depend on t. Hence $\|f_n-f\|_\infty\to 0$ as $n\to\infty$. Remains to show that this f is continuous. Fix an $n\ge N_{\epsilon/3}$. By continuity of f_n we have at each t that $|f_n(t)-f_n(t+h)|<\epsilon/3$ for all $h\in[-\delta_t,\delta_t]$ for some small enough δ_t. For all such h there holds

$$\begin{aligned}|f(t+h)-f(t)| &= |f(t+h)-f_n(t+h)+f_n(t+h)-f_n(t)+f_n(t)-f(t)|\\ &\le |f(t+h)-f_n(t+h)|+|f_n(t+h)-f_n(t)|+|f_n(t)-f(t)|\\ &< \epsilon/3+\epsilon/3+\epsilon/3=\epsilon.\end{aligned}$$

So f is continuous. ■

Notice that $\mathscr{C}([a,b];\mathbb{R})$ is *not* complete in the 1-norm (Example 1.1.8) thus completeness not only depends on the vector space, it also depends on the choice of norm. On finite dimensional space, however, it does not depend on the choice of norm:

Theorem 1.2.3 (Finite dimensional space). Every real or complex finite dimensional normed vector space is a Banach space. □

Proof (idea only). Let $S := \{v_1, \dots, v_m\}$ be a basis of the space. If f_n is a Cauchy sequence then it its coordinate vectors $f_{n,S}$ is a Cauchy sequence in $\mathbb{R}^m$ (or $\mathbb{C}^m$) in, for example, the Euclidean norm. This implies that each entry $f_{n,S,i}$ of these vectors is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) with norm $\|x_i\| = |x_i|$. Therefore $f_{\infty,S} := \lim_{n\to\infty} f_{n,S}$ exists. The corresponding $f_\infty := v_1 f_{S,1} + \cdots + v_m f_{S,m}$ is well defined, and one can show that $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$. ■

This theorem implies that $\mathbb{R}^2$ is a Banach space in each of the three norms as considered in Example 1.1.2. On infinite dimensional vector space matters are much more complicated. Let us have a look at the infinite sequence space ℓ defined as

$$\ell = \{(v_1, v_2, v_3, \dots) \,\big|\, v_i \in \mathbb{R} \forall i \in \mathbb{N}\}.$$

On this set the 1-norm, 2-norm and ∞-norm, that we defined on $\mathbb{R}^n$, become the series and supremum

$$\begin{aligned}
\|v\|_1 &:= |v_1| + |v_2| + |v_3| + |v_4| + \cdots \\
\|v\|_2 &:= \sqrt{|v_1|^2 + |v_2|^2 + |v_3|^2 + |v_4|^2 + \cdots} \\
\|v\|_\infty &:= \sup(|v_1|, |v_2|, |v_3|, |v_4|, \dots).
\end{aligned}$$

These, however, are *not* norms on ℓ for the simple reason that $\|v\|_1$ and $\|v\|_2$ are not always convergent, and $\|v\|_\infty$ does not always exist (i.e. might be ∞). For instance all three "norms" are not defined for the growing sequence

$$v = (1, 2, 3, 4, 5, \dots) \in \ell.$$

The way out of this problem is as simple as it is elegant. Merely restricting the sequence space ℓ to those elements that have finite norm will do the job, and the result is a Banach space:

Theorem 1.2.4 (Complete sequence spaces). All three sequence spaces

$$\begin{aligned}
\ell^1 &:= \{v \in \ell \mid \|v\|_1 < \infty\} \\
\ell^2 &:= \{v \in \ell \mid \|v\|_2 < \infty\} \\
\ell^\infty &:= \{v \in \ell \mid \|v\|_\infty < \infty\}
\end{aligned}$$

are complete vector spaces in their respective norms, i.e. they are Banach spaces in their respective norms. The same holds for the complex versions $\ell^1(\mathbb{N};\mathbb{C}), \ell^2(\mathbb{N};\mathbb{C}), \ell^\infty(\mathbb{N};\mathbb{C})$.

Proof 💣. We prove it only for ℓ^2, which is the most important case for our course. Exercise 1.27 shows that ℓ^2 is a vector space and that $\|v\|_2$ is a norm on this vector space. The completeness proof we do here. This proof follows the standard steps in completeness proofs:

- assume v_n is a Cauchy sequence,
- suggest a candidate limit w of v_n,
- show that w is in the space (i.e. in ℓ^2),
- show that $\lim_{n\to\infty} v_n = w$.

So assume that v_n is Cauchy, and realize that each v_n is itself a sequence, $v_n = (v_{n,1}, v_{n,2}, v_{n,3}, \ldots)$. For every fixed index $k_0 \in \mathbb{N}$, the entries $\{v_{n,k_0}\}_{n\in\mathbb{N}}$ is a Cauchy sequence because

$$|v_{n,k_0} - v_{m,k_0}| \le \sqrt{\sum_{k=1}^{\infty} |v_{n,k} - v_{m,k}|^2} = \|v_n - v_m\|_2 \to 0 \qquad \text{as } n, m \to \infty.$$

Since v_{n,k_0} is in $\mathbb{R}$, and $\mathbb{R}$ is complete, we conclude that $w_{k_0} := \lim_{n\to\infty} v_{n,k_0}$ exists. As candidate limit w of v_n we now propose $w := (w_1, w_2, w_3, \ldots)$. Next we show that this w is in ℓ^2. Since v_n is Cauchy, there is for every $\epsilon > 0$ an $N_\epsilon \in \mathbb{N}$ such that $\|v_n - v_m\|_2 < \epsilon$ for all $n, m \ge N_\epsilon$. Now fix an $L \in \mathbb{N}$ and note that then

$$\sum_{k=1}^{L} |v_{n,k} - v_{m,k}|^2 < \epsilon^2 \qquad \forall n, m \ge N_\epsilon.$$

For $m \to \infty$ this inequality says that $\sum_{k=1}^{L} |v_{n,k} - w_k|^2 \le \epsilon^2$ for all $n \ge N_\epsilon$. This holds for every L, and so as $L \to \infty$ we find that

$$\sum_{k=1}^{\infty} |v_{n,k} - w_k|^2 \le \epsilon^2 \qquad \forall n \ge N_\epsilon.$$

This shows that

$$\|v_n - w\|_2 \le \epsilon \qquad \forall n \ge N_\epsilon. \tag{1.4}$$

Hence $v_n - w \in \ell^2$ for every $n \ge N_\epsilon$. Since ℓ^2 is a vector space, and $v_n \in \ell^2$, we see that w is in ℓ^2 as well. Finally, (1.4) by definition means that $w = \lim_{n\to\infty} v_n$. ■

Example 1.2.5 (Cauchy or not Cauchy). Consider the infinite sequence

$$v_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, \tfrac{1}{n}, 0, 0, \ldots) \in \ell,$$

depending on $n \in \mathbb{N}$. For every n the sequence v_n has only finitely many nonzero entries, so it has finite 1-norm, finite 2-norm and finite ∞-norm, and thus is in all three vector spaces ℓ^1, ℓ^2 and ℓ^∞. The sequence v_n pointwise converges to the infinite sequence

$$w = (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \ldots).$$

This w is not in ℓ^1 because

$$\|w\|_1 = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

which diverges to ∞. But w is in ℓ^2 and ℓ^∞ because

$$\|w\|_2 = \sqrt{1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \cdots} < \infty$$
$$\|w\|_\infty = \sup_{k\ge 1}(1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots) = 1.$$

This is consistent with the observations that

- $\{v_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence in the 1-norm because no matter how large N is, the quantity

  $$\|v_n - v_m\|_1 = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

  can be taken arbitrary large by appropriate choice of $m \ge n \ge N$.

- $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the 2-norm because for all $n, m \geq N$ we have $\|v_n - v_m\|_2^2 < 1/N \to 0$ as $N \to \infty$ (See Example 1.1.10). Since ℓ^2 is a Banach space the v_n hence converges in ℓ^2. Indeed it does.

- $\{v_n\}_{n\in\mathbb{N}}$ is Cauchy in the ∞-norm because for all $n, m \geq N$ we have $\|v_n - v_m\|_\infty < 1/N \to 0$ as $N \to \infty$. Hence, by the Banach property of ℓ^∞ the v_n converges in ℓ^∞.

□

The function space equivalent of ℓ^1 we naively define as

$$\mathscr{L}^1([a,b];\mathbb{R}) := \{f : [a,b] \to \mathbb{R} \mid \|f\|_1 < \infty\},$$

where the 1-norm is now taken to be

$$\|f\|_1 = \int_a^b |f(t)|\,\mathrm{d}t.$$

We allow $a = -\infty$ and $b = +\infty$. This definition of $\mathscr{L}^1([a,b];\mathbb{R})$ is not precise because it still depends on the definition of integral $\int_a^b |f(t)|\,\mathrm{d}t$. The Riemann integral definition is not ideal because one can construct a Cauchy sequence of Riemann integrable functions whose limiting function is so crazy that its Riemann integral is no longer well defined. Hence the space $\mathscr{L}^1([a,b];\mathbb{R})$ would then fail to be complete. The desire of having a complete function space was so strong that it prompted mathematicians to look for alternative definitions of integration! In the beginning of the 20th century the issue was settled by Henri Lebesgue. He devised the *Lebesgue measure* and *Lebesgue integration* with respect to which the space $\mathscr{L}^1([a,b];\mathbb{R})$ *is* complete. The interested reader should follow a course on measure theory. The symbol $\mathscr{L}$ is standard in the math literature and it is in honor of Lebesgue. The difference between Riemann- and Lebesgue integration only shows up in really weird functions, and in this course we need not worry about such functions. We simply accept that:

Theorem 1.2.6 (Lebesgue space $\mathscr{L}^1$). $\mathscr{L}^1([a,b];\mathbb{R})$ and $\mathscr{L}^1([a,b];\mathbb{C})$ are Banach spaces in the 1-norm.

□

Built in in the definition of $\mathscr{L}^1$ is that its elements have a well defined 1-norm. This space contains all continuous functions but also many more, and they need not be bounded.

Example 1.2.7 (Several functions in $\mathscr{L}^1$). All functions of Fig. 1.4 are elements of $\mathscr{L}^1([0,1];\mathbb{R})$, except the last function: $f_9(t) = 1/t$. Indeed $\int_0^1 f_9(t)\,\mathrm{d}t = \log(t)\big|_0^1 = \infty$. □

Likewise, the counterpart of ℓ^2 is the space of square integrable functions:

Lemma 1.2.8 (Lebesgue space $\mathscr{L}^2$). The space of square integrable functions

$$\mathscr{L}^2([a,b];\mathbb{R}) := \{f : [a,b] \to \mathbb{R} \mid \|f\|_2 < \infty\}$$

is a Banach space with respect to the 2-norm defined as

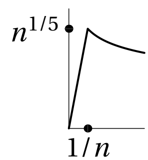
$$\|f\|_2 := \sqrt{\int_a^b |f(t)|^2\,\mathrm{d}t}. \tag{1.5}$$

□

Also here we allow $a = -\infty$ and $b = +\infty$. The top six functions of Fig. 1.4 are in $\mathscr{L}^2([0,1];\mathbb{R})$. The final three functions are not $\mathscr{L}^2([0,1];\mathbb{R})$.

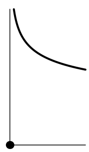
Example 1.2.9 (Complete in $\mathscr{L}^2$, not complete in $\mathscr{C}$). Consider the standard 2-norm of functions (1.5), and take $a = 0, b = 1$. For every $n > 0$ the function $f_n : [0,1] \to \mathbb{R}$ defined as

$$f_n(t) = \begin{cases} n^{6/5}t & 0 \le t \le 1/n \\ t^{-1/5} & 1/n < t \le 1 \end{cases}$$



is continuous. All f_n are therefore in $\mathscr{C}([0,1];\mathbb{R})$ as well as in $\mathscr{L}^2([0,1];\mathbb{R})$. As n goes to ∞ the pointwise limit of $f_n(t)$ is

$$f_\infty(t) = \begin{cases} 0 & t = 0, \\ t^{-1/5} & 0 < t \le 1. \end{cases}$$



This function is not in $\mathscr{C}([0,1];\mathbb{R})$ because it is not continuous and in fact it is not bounded. It is in $\mathscr{L}^2([0,1];\mathbb{R})$, however, because the improper integral converges:

$$\|f_\infty\|_2^2 = \int_0^1 f_\infty^2(t)\,\mathrm{d}t = \int_0^1 t^{-2/5}\,\mathrm{d}t = \tfrac{5}{3}t^{3/5}\Big|_0^1 = \frac{5}{3}.$$

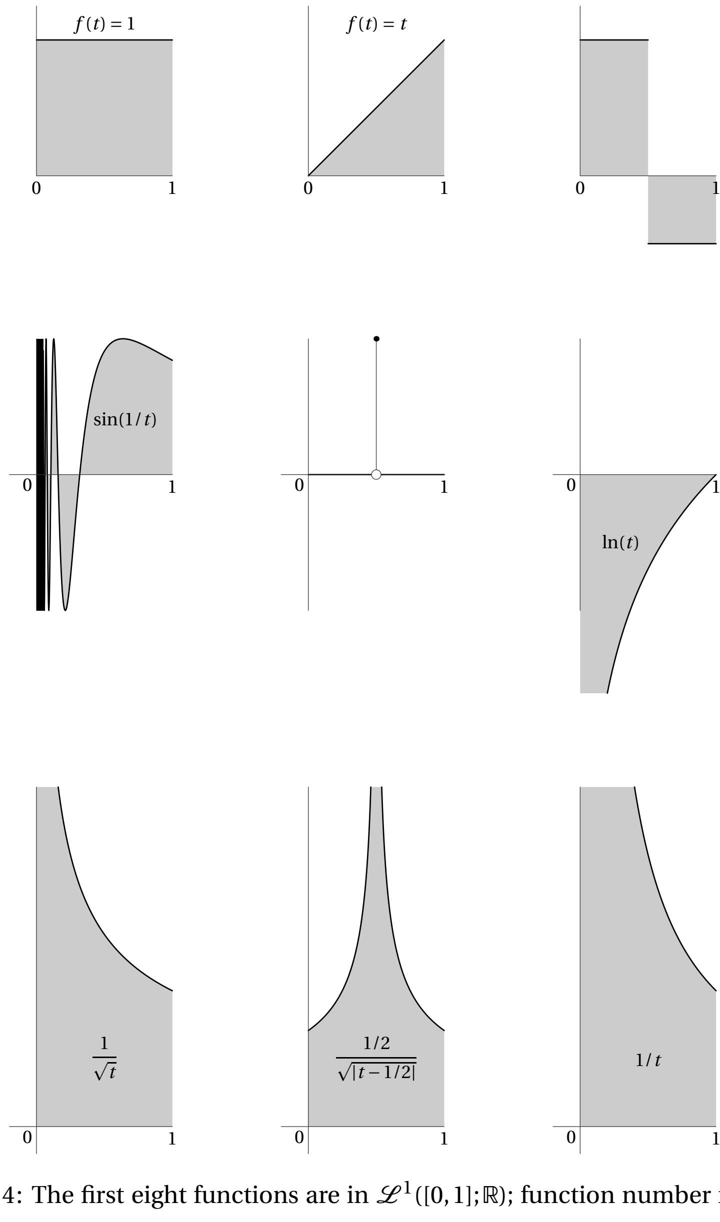


FIGURE 1.4: The first eight functions are in $\mathscr{L}^1([0,1];\mathbb{R})$; function number nine is not.

One can show that f_n is a Cauchy sequence in the 2-norm. Since the space $\mathscr{L}^2([a,b];\mathbb{R})$ is complete (says Lemma 1.2.8) in this norm, and since f_n is a Cauchy sequence (verify this yourself) it follows that $\lim_{n\to\infty} f_n$ exists in $\mathscr{L}^2([a,b];\mathbb{R})$. Indeed.

The space $\mathscr{C}([a,b];\mathbb{R})$ is not complete in this 2-norm, and hence even though f_n is a Cauchy sequence, its limit is not guaranteed to exist in the space $\mathscr{C}([a,b];\mathbb{R})$, and indeed in this case it does not exist. $\square$

But wait, we glossed over an unsettling problem: part of the definition of norm is that

$$\|f\| > 0 \text{ for all } f \neq 0,$$

but for $\mathscr{L}^1([a,b];\mathbb{R})$, and also for $\mathscr{L}^2([a,b];\mathbb{R})$, that is not the case! For example, function number five of Fig. 1.4 — the function in the middle — defined as

$$f(t) = \begin{cases} 1 & t = 1/2 \\ 0 & \text{elsewhere} \end{cases}$$

is not the zero function, yet its 1-norm is zero. The simplistic way out of this problem is to *identify* every function f with zero norm with the zero function. That is not far fetched because if $\|f\|_1 = 0$ then

$$\|f\|_1 = \int_a^b |f(t)|\,\mathrm{d}t = 0,$$

implying that $f(t)$ is zero "almost everywhere[2]". From now on we do not distinguish between functions $f, g \in \mathscr{L}^1$ whenever their difference has norm zero, so from now on *by definition* we say

$$f = g \quad \iff \quad \|f - g\|_1 = 0.$$

And likewise for $f, g \in \mathscr{L}^2$.

[2]In a course on measure theory this identification will be formalized through *equivalence classes* and then the notion of *almost everywhere* will be properly defined.

1.3 Module 2 summary: inner product

What is missing in a normed vector space is the notion of "angle" and "orthogonality" between vectors. We will see in this section that many (but not all) normed vector spaces can be equipped with an *inner product*. This generalizes the dot product of $\mathbb{R}^n$ and, like the dot product, allows to define orthogonality. We assume familiarity with inner products on finite dimensional vector space.

Definition 1.3.1 (Real inner product). Let $\mathbb{V}$ be a real vector space. A function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is a (real) *inner product* if for every $x, y, z \in \mathbb{V}$ and every $\alpha \in \mathbb{R}$ the following three axioms hold,

1. $\langle x, y \rangle = \langle y, x \rangle$, (*symmetric*)
2. $\langle \alpha x + z, y \rangle = \alpha \langle x, y \rangle + \langle z, y \rangle$, (*linear in first argument*)
3. $\langle x, x \rangle > 0$ for all $x \neq 0$. (*positive definite*)

□

That is all. The second axiom for $x = z$ and $\alpha = -1$ demonstrates that $\langle 0, y \rangle = 0$. This combined with the third axiom shows that $\langle x, x \rangle \geq 0$ for every x and that $\langle x, x \rangle = 0$ if and only if $x = 0$.

Once we settle on this definition we are forced to conclude that there are inner products on $\mathbb{R}^n$ that differ from the dot product. Indeed we could equip $\mathbb{R}^2$ for instance with the following inner product:

Example 1.3.2. Consider as inner product on $\mathbb{R}^2$

$$\langle x, y \rangle := x_1 y_1 + 3 x_2 y_2.$$

It is an inner product because it is well defined (in $\mathbb{R}$) for every $x, y \in \mathbb{R}^2$ and

1. $\langle x, y \rangle = x_1 y_1 + 3 x_2 y_2 = y_1 x_1 + 3 y_2 x_2 = \langle y, x \rangle$,
2. $\langle \alpha x + z, y \rangle = (\alpha x_1 + z_1) y_1 + 3(\alpha x_2 + z_2) y_2 = \alpha (x_1 y_1 + 3 y_2 x_2) + (z_1 y_1 + 3 z_2 y_2) = \alpha \langle x, y \rangle + \langle z, y \rangle$,

3. if x is nonzero then at least one of x_1, x_2 is nonzero. Then $\langle x, x\rangle = x_1^2 + 3x_2^2 > 0$.

□

As with norms, the definition of inner product implicitly says that it needs to exist for every two elements of the vector space.

Example 1.3.3. On $\mathscr{C}([a,b];\mathbb{R})$ the integral

$$\langle f, g\rangle := \int_a^b f(t)g(t)\,\mathrm{d}t \tag{1.6}$$

is an inner product. Let us verify (including existence):

0. If $f, g \in \mathscr{C}([a,b],\mathbb{R})$ then the product fg is continuous (and bounded) on $[a,b]$. So then the integral $\langle f, g\rangle := \int_a^b f(t)g(t)\,\mathrm{d}t$ exists in $\mathbb{R}$.

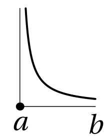
1. $\langle f, g\rangle = \int_a^b f(t)g(t)\,\mathrm{d}t = \int_a^b g(t)f(t)\,\mathrm{d}t = \langle g, f\rangle$.

2. $\langle \alpha f + g, h\rangle = \int_a^b (\alpha f(t) + g(t))h(t)\,\mathrm{d}t = \int_a^b \alpha f(t)h(t)\,\mathrm{d}t + \int_a^b g(t)h(t)\,\mathrm{d}t = \alpha\langle f, h\rangle + \langle g, h\rangle$.

3. If f is not the zero function then $|f(t_0)| > 0$ for at least one $t_0 \in [a,b]$. By continuity then $\langle f, f\rangle = \int_a^b f^2(t)\,\mathrm{d}t > 0$.

On $\mathscr{C}([a,b];\mathbb{R})$ the inner product exists for every two of its elements. On the bigger space $\mathscr{F}([a,b];\mathbb{R})$ — the space of all functions from $[a,b]$ to $\mathbb{R}$ — the integral (1.6) is *not* an inner product for the simple reason that the integral (1.6) is then not always convergent. Consider for instance the functions

$$f(t) = g(t) = \begin{cases} 0 & \text{if } t = a, \\ \frac{1}{t-a} & \text{if } a < t \le b. \end{cases}$$



See also Exercise 1.14. □

Also complex vector space can be equipped with an inner product. The inner product is then a function that maps to the complex numbers. We want to maintain, however, that $\langle x, x\rangle$ is a real number, in fact nonnegative:

Definition 1.3.4 (Inner product on a complex vector space). Let $\mathbb{V}$ be a complex vector space. A function $\langle\cdot,\cdot\rangle : \mathbb{V}\times\mathbb{V}\to\mathbb{C}$ is a *(complex) inner product* if for every $x, y, z \in \mathbb{V}$ and every $\alpha\in\mathbb{C}$ the following three axioms hold,

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$, (*conjugate symmetric*)
2. $\langle \alpha x + z, y\rangle = \alpha\langle x, y\rangle + \langle z, y\rangle$, (*linear in first argument*)
3. $\langle x, x\rangle > 0$ for all $x \neq 0$. (*positive definite*)

□

The conjugate symmetry property implies that $\langle x, x\rangle = \overline{\langle x, x\rangle}$, and this is another way of saying that $\langle x, x\rangle$ is real.

Example 1.3.5. On $\mathscr{C}([a, b]; \mathbb{C})$ the integral

$$\langle f, g\rangle := \int_a^b f(t)\overline{g(t)}\,\mathrm{d}t$$

is an inner product (Exercise 1.15). □

On complex vector space, linearity of an inner product in its first argument does not imply linearity in its second argument. Using the properties of inner product we find that

$$\langle x, \alpha y + z\rangle = \overline{\langle \alpha y + z, x\rangle} = \overline{\alpha\langle y, x\rangle + \langle z, x\rangle} = \overline{\alpha}\langle x, y\rangle + \langle x, z\rangle.$$

In particular, $\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle$. This property of inner products is called *conjugate linearity*.

Norm on inner product space & Cauchy-Schwarz inequality

On an *inner product space* (a vector space with inner product) we always define the norm via the inner product:

Definition 1.3.6 (norm associated with inner product). The *norm* $\|x\|$ on a vector space with inner product is defined as $\|x\| = \sqrt{\langle x, x\rangle}$. □

It is also possible to define norm without having inner product (see Definition 1.1.1); all that the above says is that *if* we have an inner product space then we *always* define the norm accordingly. Of course this assumes that the above satisfies the defining properties of norm. It does:

Lemma 1.3.7 (Norm). Let $\mathbb{V}$ be a real or complex vector space with inner product. The norm $\|\cdot\|$ associated with the inner product exists and satisfies the axioms of norm:

1. $\|\alpha x\| = |\alpha|\|x\|$, (*positive homogeneous*)
2. $\|x+y\| \leq \|x\| + \|y\|$, (*triangle inequality*)
3. $\|x\| > 0$ for all $x \neq 0$. (*positive definite*)

Proof. $\|x\|$ exists because $\langle x, x\rangle$ exists. The first axiom follows from $\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha|\|x\|$. The third axiom is obvious. The second (the triangle inequality) is more involved. It follows from Cauchy-Schwarz' inequality, which we will prove shortly (§ 1.3). ■

Since an inner product induces a norm, it is natural to ask whether *every* norm can be seen as induced by an inner product. That is not the case. In fact one can elegantly characterize which norms have an associated inner product.

Lemma 1.3.8 (Parallelogram law). Let $\|\cdot\|$ be a norm on a real or complex vector space $\mathbb{X}$. There is an inner product on $\mathbb{X}$ such that $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{X}$ iff the norm satisfies the *parallelogram law*

$$2(\|x\|^2 + \|y\|^2) = \|x-y\|^2 + \|x+y\|^2,$$

see Fig. 1.5. In that case the inner product follows uniquely from the norm as

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} \tag{1.7}$$

on real vector space, or as

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4} + \mathrm{i}\frac{\|x+\mathrm{i}y\|^2 - \|x-\mathrm{i}y\|^2}{4} \tag{1.8}$$

on complex vector space.

Proof. See Exercise 1.16. ■

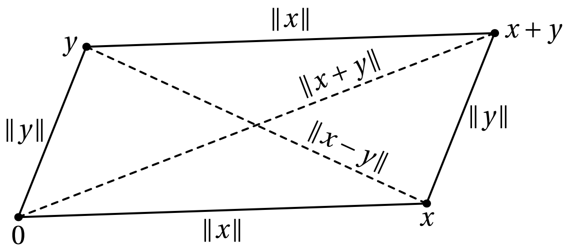


FIGURE 1.5: The parallelogram law on $\mathbb{R}^2$ with norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ says that the sum of squares of the lengths of the four outer sides of the parallelogram, $2(\|x\|^2 + \|y\|^2)$, equals the sum of squares of the lengths of the two diagonals, $\|x-y\|^2 + \|x+y\|^2$.

Example 1.3.9. Consider the three norms on $\mathbb{R}^2$ as used in Example 1.1.2:

$$\begin{aligned}
\|x\|_1 &= |x_1| + |x_2|, \\
\|x\|_2 &= \sqrt{x_1^2 + x_2^2}, \\
\|x\|_\infty &= \max(|x_1|, |x_2|).
\end{aligned}$$

The 1-norm has no associated inner product because for $x = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ the parallelogram law fails:

$$\underbrace{2(\|x\|_1^2 + \|y\|_1^2)}_{=4} \neq \underbrace{\|x-y\|_1^2 + \|x+y\|_1^2}_{=8}.$$

The 2-norm does have an inner product because $\|x\|_2$ equals $\sqrt{\langle x, x\rangle}$ for $\langle x, y\rangle := x_1 y_1 + x_2 y_2$. (For completeness we should show that the above is indeed an inner product. Do this yourself. Alternatively we could verify the parallelogram law for the 2-norm.)

The ∞-norm has no associated inner product because for $x = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ the parallelogram law fails:

$$\underbrace{2(\|x\|_\infty^2 + \|y\|_\infty^2)}_{=4} \neq \underbrace{\|x-y\|_\infty^2 + \|x+y\|_\infty^2}_{=2}.$$

□

Orthogonal complement

We use $x \perp y$ to mean that $\langle x, y\rangle = 0$, so orthogonality is inner product dependent. Likewise for any nonempty subset $\mathbb{S}$ of some inner product space $\mathbb{X}$ we use $\mathbb{S}^{\perp}$ to mean the *orthogonal complement* of $\mathbb{S}$, i.e. the set of all vectors in $\mathbb{X}$ that are orthogonal to all elements of $\mathbb{S}$,

$$\mathbb{S}^{\perp} = \{x \in \mathbb{X} \mid \langle x, s\rangle = 0 \, \forall s \in \mathbb{S}\}.$$

Example 1.3.10. Consider again the non-standard inner product of Example 1.3.2 on $\mathbb{R}^2$,

$$\langle x, y\rangle := x_1 y_1 + 3x_2 y_2.$$

In this norm the so-called *unit ball* $\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ is not a disc but an ellipse

$$\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\} = \{x \in \mathbb{R}^2 \mid x_1^2 + 3x_2^2 \leq 1\},$$

see Fig. 1.6(a). The orthogonal complement of the set $\mathbb{S}$ consisting of the single element,

$$\mathbb{S} := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is now

$$\mathbb{S}^{\perp} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\perp} = \{x \mid \langle \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right], x\rangle = 0\} = \{x \mid x_1 - 3x_2 = 0\} = \operatorname{span}\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\},$$

see Fig. 1.6(b). Orthogonality with respect to our inner product here does not mean "having an angle of $\pi/2$". □

The Pythagorean theorem remains valid on arbitrary inner product space.

Lemma 1.3.11 (Pythagoras). If $x \perp y$ then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Proof. If $\langle x, y\rangle = 0$ then

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x + y\rangle + \langle y, x + y\rangle \\
&= \langle x, x\rangle + \underbrace{\langle x, y\rangle}_{0} + \underbrace{\langle y, x\rangle}_{0} + \langle y, y\rangle = \|x\|^2 + \|y\|^2.
\end{aligned}$$

We used here that $\langle y, x\rangle = \overline{\langle x, y\rangle} = 0$. ■

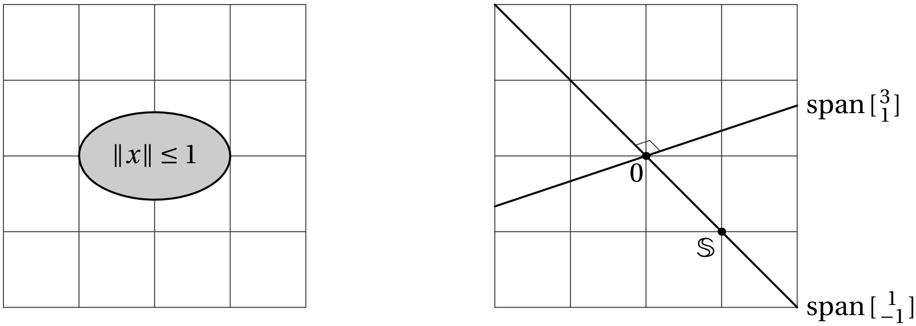


FIGURE 1.6: (a) Unit ball for inner product $\langle x, y\rangle = x_1 y_1 + 3x_2 y_2$; (b) two orthogonal vectors in this inner product.

In fact on *real* vector space, $x \perp y$ is *equivalent* to $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, but on complex vector space it is not equivalent, see Exercise 1.24.

Example 1.3.12 (Orthogonal sinusoids). Consider $\mathscr{C}([0,2\pi];\mathbb{R})$ with inner product $\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\,\mathrm{d}t$. On this space sine and cosine are orthogonal,

$$\sin \perp \cos.$$

Why? Because

$$\langle \sin, \cos\rangle = \int_0^{2\pi} \cos(t)\sin(t)\,\mathrm{d}t = \int_0^{2\pi} \tfrac{1}{2}\sin(2t)\,\mathrm{d}t = \frac{-\cos(2t)}{4}\Big|_0^{2\pi} = 0.$$

See Fig. 1.7(botttom left) for a "proof by picture" of orthogonality of these two functions. The Pythagorean theorem now implies that $\|\sin + \cos\|^2 = \|\cos\|^2 + \|\sin\|^2 = \pi + \pi = 2\pi$, see Fig. 1.7(right). □

Cauchy-Schwarz inequality

Let x and y be elements of some inner product space. Fig. 1.8 suggests that x can uniquely be written as a sum of two vectors where one is in $\operatorname{span}\{y\}$ and the other is orthogonal to this span, i.e. that

$$x = \alpha y + y_\perp$$

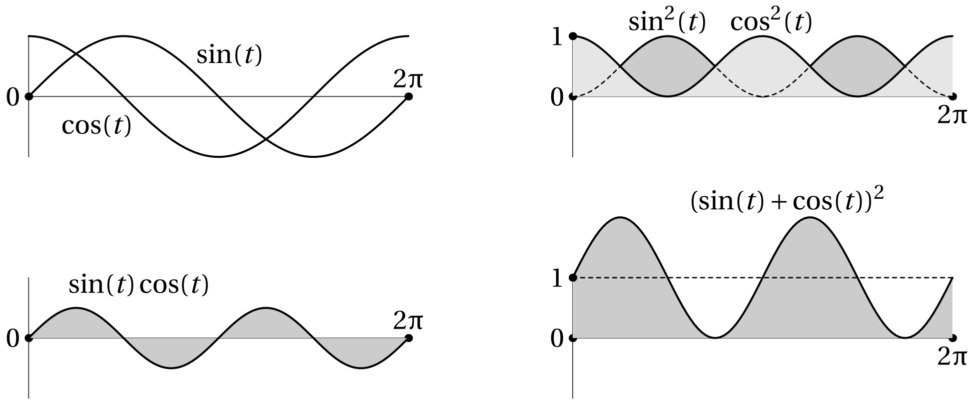


FIGURE 1.7: The graph shown in the bottom-left figure suggests that $\sin(t)\cos(t)$ is zero on average over $[0,2\pi]$. Hence $\langle \sin, \cos\rangle = 0$. The top-right figure shows that $\|\sin\|^2 = \|\cos\|^2 = \pi$, and then the bottom-right figure shows that $\|\sin+\cos\|^2 = 2\pi$.

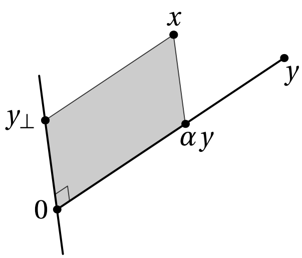


FIGURE 1.8: $x = \alpha y + y_\perp$.

for some unique αy and unique vector $y_\perp$ orthogonal to y. That is indeed the case because either $y=0$ in which case we have $\alpha y=0$ and $y_\perp = x$, or else

$$\begin{aligned} x=\alpha y+y_\perp &\iff (x-\alpha y)\perp y \\ &\iff \langle x-\alpha y, y\rangle = 0 \\ &\iff \langle x,y\rangle - \alpha\langle y,y\rangle = 0 \\ &\iff \alpha = \frac{\langle x,y\rangle}{\|y\|^2}. \end{aligned}$$

Hence α is unique, and therefore αy and $y_\perp := x-\alpha y$ are unique as well. Pythagoras says that then

$$\|x\|^2 = \|\alpha y\|^2 + \|y_\perp\|^2 = \frac{|\langle x,y\rangle|^2}{\|y\|^2} + \|y_\perp\|^2. \tag{1.9}$$

We derived here a one-dimensional version of the projection theorem (discussed later) but more importantly we derived that inner products can be bounded by norms. This is the fundamental inequality of Cauchy-Schwarz:

Theorem 1.3.13 (Cauchy-Schwarz inequality). Every inner product satisfies

$$|\langle x,y\rangle| \le \|x\|\,\|y\| \qquad \forall x,y,$$

and equality holds iff x and y are linearly dependent, i.e. iff $x=\alpha y$ or $y=\alpha x$ for some scalar α.

Proof. The result is trivial if $y=0$. For $y\neq 0$, Inequality (1.9) says that

$$\|x\|^2\|y\|^2 = |\langle x,y\rangle|^2 + \|y_\perp\|^2\|y\|^2 \ge |\langle x,y\rangle|^2,$$

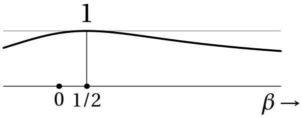
and so the result follows. We have equality iff $y_\perp = 0$ and that is the case iff x and y are linearly dependent. ■

Example 1.3.14 (Aligned vectors in $\mathbb{R}^4$). Let

$$x = \begin{bmatrix} 10\\10\\10\\5 \end{bmatrix}, \qquad y = \begin{bmatrix} 1\\1\\1\\\beta \end{bmatrix}.$$

Then in the standard inner product (dot product) on $\mathbb{R}^4$ we have

$$\frac{\langle x, y\rangle}{\|x\|\,\|y\|} = \frac{30+5\beta}{\sqrt{300+25}\,\sqrt{3+\beta^2}}.$$



The two vectors are linearly dependent (aligned) iff $\beta = 1/2$. Hence

$$\frac{\langle x, y\rangle}{\|x\|\,\|y\|} = \pm 1 \quad \text{iff} \quad \beta = 1/2.$$

The fact that the ratio is *plus* 1 at $\beta = 1/2$ means that x and y are *positively* aligned, i.e. that $x = \alpha y$ for some positive number α. □

The Cauchy-Schwarz inequality is very important and shows up in numerous applications, such as *signal detection* problems because we can use $\frac{\langle x,y\rangle}{\|x\|\|y\|}$ as a practical measure of "similarity" of x and y. A consequence of Cauchy-Schwarz is the triangle inequality.

Lemma 1.3.15 (Triangle inequality). For any inner product and associated norm we have

$$\|x+y\| \le \|x\| + \|y\|.$$

Proof.

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \overline{\langle x, y\rangle} + \langle y, y\rangle \\
&= \langle x, x\rangle + 2\,\mathrm{Re}(\langle x, y\rangle) + \langle y, y\rangle \\
&\le \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle \\
&\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

■

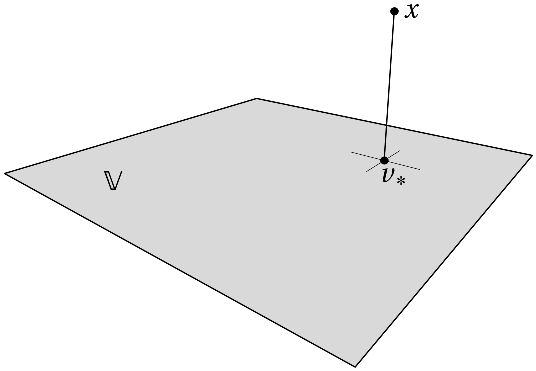


FIGURE 1.9: The best approximation $v_* \in \mathbb{V}$ of x has the property that $x - v_*$ is orthogonal to the subspace $\mathbb{V}$.

1.4 Orthogonal projection

With inner products comes along a notion of orthogonality and a notion of distance (norm), so we can talk about "orthogonal projections" and "best approximations".

Definition 1.4.1 (Best approximation). An element $v_* \in \mathbb{V}$ is said to be a *best approximation* in $\mathbb{V}$ of x if

$$\|x - v_*\| \le \|x - v\| \quad \forall v \in \mathbb{V}.$$

□

Figure 1.9 suggests that best approximations v_* in a subspace $\mathbb{V}$ are such that $x - v_*$ is orthogonal to $\mathbb{V}$. That is correct, in fact, the orthogonality property is both necessary and sufficient for optimality:

Theorem 1.4.2 (A projection theorem). Let $\mathbb{X}$ be a vector space with inner product, $\mathbb{V}$ a subspace of $\mathbb{X}$, and x an element of $\mathbb{X}$. Then

1. a $v_* \in \mathbb{V}$ is a best approximation in $\mathbb{V}$ of x iff $(x - v_*) \perp \mathbb{V}$,

2. if the best approximation $v_* \in \mathbb{V}$ exists then it is unique and it satisfies

$$\|x - v_*\|^2 = \|x\|^2 - \|v_*\|^2.$$

Proof. Suppose $x - v_* \perp \mathbb{V}$ for some $v_* \in \mathbb{V}$. Then for any $v \in \mathbb{V}$ the difference $v - v_*$ is in $\mathbb{V}$ by the subspace property, and so by Pythagoras we get

$$\|x - v\|^2 = \|\underbrace{(x - v_*)}_{\in \mathbb{V}^\perp} - \underbrace{(v - v_*)}_{\in \mathbb{V}}\|^2 = \|x - v_*\|^2 + \|v - v_*\|^2 \geq \|x - v_*\|^2.$$

Hence if $v \neq v_*$ then the norm of $x - v$ exceeds that of $x - v_*$, making v_* the unique best approximation.

Conversely, suppose $x - v_* \not\perp \mathbb{V}$. Then by definition there is a $v \in \mathbb{V}$ such that $x - v_* \not\perp v$ i.e. such that $\langle x - v_*, v\rangle \neq 0$. In particular this v is nonzero. We construct an improved approximation of x of the form $v_* + \alpha v$ with the scalar α yet to be determined.

$$\begin{aligned}\|x - (v_* + \alpha v)\|^2 &= \|(x - v_*) - \alpha v\|^2 \\ &= \|x - v_*\|^2 - 2\,\mathrm{Re}\langle x - v_*, \alpha v\rangle + \|\alpha v\|^2 \\ &= \|x - v_*\|^2 - 2\,\mathrm{Re}(\overline{\alpha}\langle x - v_*, v\rangle) + |\alpha|^2\|v\|^2.\end{aligned}$$

For $\alpha = \langle x - v_*, v\rangle / \|v\|^2$ this becomes

$$\begin{aligned} &= \|x - v_*\|^2 - 2\frac{|\langle x - v_*, v\rangle|^2}{\|v\|^2} + \frac{|\langle x - v_*, v\rangle|^2}{\|v\|^2} \\ &= \|x - v_*\|^2 - \frac{|\langle x - v_*, v\rangle|^2}{\|v\|^2} < \|x - v_*\|^2.\end{aligned}$$

This shows that v_* is not a best approximation.

The equality $\|x - v_*\|^2 = \|x\|^2 - \|v_*\|^2$ is a restatement of Pythagoras, see Fig. 1.9. ∎

With this theorem we can in many cases calculate best approximations. More on this soon, but first a puzzling example. It may happen that a best approximation does not exist:

Example 1.4.3 (No solution exists). Consider the inner product space $\mathbb{X} = \ell^2$ and its subspace ℓ^{finite} of finitely nonzero sequences. There is no best approximation $v_* \in \ell^{\text{finite}}$ of the infinite sequence

$$x = (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \dots)$$

because it can be approximated arbitrarily well by $v_n := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots)$ but the approximation error cannot be zero because $x \notin \ell^{\text{finite}}$. □

This example demonstrates that best approximations need not exist. For *finite* dimensional subspaces, however, best approximations always *do* exist, and in fact can be determined explicitly:

Lemma 1.4.4 (Projection onto finite dimensional subspace). Consider a real or complex inner product space $\mathbb{X}$, and subspace $\mathbb{V} \subseteq \mathbb{X}$, and assume that $\mathbb{V}$ is spanned by a finite set $\{v_1, \dots, v_n\}$. Then v_* is a best approximation in $\mathbb{V}$ of x if-and-only-if

$$v_* = \alpha_1 v_1 + \dots + \alpha_n v_n$$

where $\alpha \in \mathbb{R}^n$ (or $\mathbb{C}^n$) is a solution of

$$\underbrace{\begin{bmatrix} \langle v_1, v_1\rangle & \langle v_2, v_1\rangle & \cdots & \langle v_n, v_1\rangle \\ \langle v_1, v_2\rangle & \langle v_2, v_2\rangle & \cdots & \langle v_n, v_2\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle v_1, v_n\rangle & \langle v_2, v_n\rangle & \cdots & \langle v_n, v_n\rangle \end{bmatrix}}_{\text{Gram matrix } G} \alpha = \begin{bmatrix} \langle x, v_1\rangle \\ \langle x, v_2\rangle \\ \vdots \\ \langle x, v_n\rangle \end{bmatrix}. \tag{1.10}$$

The so defined *Gram matrix G* is square.

If $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{V}$ then the Gram matrix is invertible, and hence for every $x \in \mathbb{X}$ the solution α then exists and is unique.

Proof. Write the candidate best approximation v_* as $v_* = \sum_{i=1}^n \alpha_i v_i$. By the projection theorem, a vector $v_* \in \mathbb{V}$ is a best approximation of x iff

$$\begin{aligned}
(x - v_*) \perp \mathbb{V} &\iff \Big(x - \sum_{i=1}^{n} \alpha_i v_i\Big) \perp \operatorname{span}\{v_1, \ldots, v_n\} \\
&\iff \Big(x - \sum_{i=1}^{n} \alpha_i v_i\Big) \perp v_k && \forall k = 1, \ldots, n \\
&\iff \langle x, v_k\rangle - \sum_{i=1}^{n} \alpha_i \langle v_i, v_k\rangle = 0 && \forall k = 1, \ldots, n \\
&\iff \langle x, v_k\rangle - \begin{bmatrix} \langle v_1, v_k\rangle & \langle v_2, v_k\rangle & \cdots & \langle v_n, v_k\rangle \end{bmatrix} \alpha = 0 && \forall k = 1, \ldots, n \\
&\iff \text{Equation (1.10) holds.}
\end{aligned}$$

Remains to show that the Gram matrix is invertible if $\{v_1, \ldots, v_n\}$ is a linearly independent set. We use that the Gram matrix does not depend on x. Consider now the best approximation v_* of $x = 0$. Clearly this is $v_* = 0$ and it is unique. For $x = 0$, Equation (1.10) becomes $G\alpha = 0$ where G is the Gram matrix. If G would have been singular then more than one solution α would have existed, implying (by linear independence) that more than one best approximation $v_* = \sum_{i=1}^{n} \alpha_i v_i$ would have existed. That is not the case, i.e. G is not singular if $\{v_1, \ldots, v_n\}$ is independent. ∎

Equation (1.10) is also known as the *normal equation(s)*. For $n = 1$ the normal equations state that

$$v_* = \alpha v_1, \qquad \alpha = \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle}.$$

This recovers the one-dimensional projection formula that we used to prove the Cauchy-Schwarz inequality. Now with the Gram matrix we can determine orthogonal projections on subspaces of finite dimension:

Example 1.4.5. Let us redo the projection of Example 1.3.10, that is, we want find the best approximation in $\operatorname{span}\{v_1\}$ for $v_1 = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ of

$$x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

with the non-standard inner product $\langle x, y\rangle = x_1 y_1 + 3x_2 y_2$. According to the normal equations we have as best approximation

$$v_* = v_1 \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{2\times 1 + 3\times 2\times(-1)}{1+3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{-4}{4} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

See Fig. 1.10. □

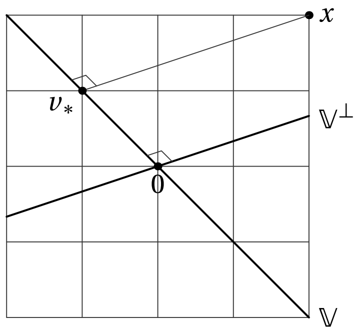


FIGURE 1.10: The best approximation in span$(1,-1)$ of $x = (2,2)$ with respect to inner product $\langle x, y\rangle = x_1 y_1 + 3x_2 y_2$ is $v_* = (-1,1)$. See Example 1.4.5.

Example 1.4.6. Consider $\mathscr{C}([-1,1];\mathbb{R})$ with standard inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t$. We determine the best approximation of the function e^t in the subspace of polynomials of degree at most 1. That is, we need to solve

$$\min_{\alpha_1,\alpha_2\in\mathbb{R}} \|\mathrm{e}^t - (\alpha_1 + \alpha_2 t)\|.$$

A basis of this subspace of polynomials is $\{1, t\}$, and with this choice the normal equations (1.10) become

$$\begin{bmatrix} \langle 1,1\rangle & \langle t,1\rangle \\ \langle 1,t\rangle & \langle t,t\rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \langle \mathrm{e}^t,1\rangle \\ \langle \mathrm{e}^t,t\rangle \end{bmatrix}.$$

The six inner products here equate to

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e} - \mathrm{e}^{-1} \\ 2\mathrm{e}^{-1} \end{bmatrix}.$$

Its solution is

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(\mathrm{e}-\mathrm{e}^{-1})\\ 3\mathrm{e}^{-1}\end{bmatrix}.$$

The best approximation on $[-1,1]$ of e^t by a degree one (or less) polynomial thus is

$$p_*(t) = \tfrac{1}{2}(\mathrm{e}-\mathrm{e}^{-1}) + 3\mathrm{e}^{-1}t,$$

see Fig. 1.11. □

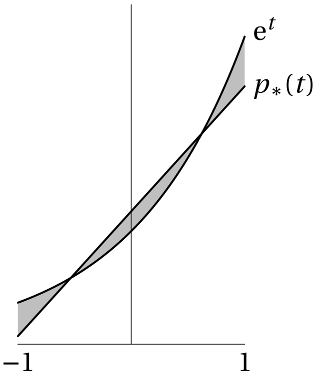


FIGURE 1.11: Best approximation of e^t by degree-1 polynomial $p_*(t)$. This polynomial minimizes the norm of the approximation error $\mathrm{e}^t - p_*(t)$ (shown in gray). See Example 1.4.6.

Orthonormal sequence and Parseval identity

A drawback of the Gram matrix is that it does not easily extend to infinite dimensional subspaces (its Gram matrix would be an infinite matrix). To avoid such problems it is customary to assume more structure, namely the case where the Gram matrix (1.10) is the identity matrix:

Definition 1.4.7 (Orthonormal sequence). A sequence $\{e_1, e_2, e_3, \ldots\}$ in some inner product space, is an *orthonormal sequence* if all elements have unit norm and are mutually orthogonal: $\|e_k\| = 1\ \forall k$ and $e_k \perp e_j\ \forall k \neq j$. □

Verify for yourself that the Gram matrix (1.10) is the identity matrix if $\{v_1,\dots,v_n\}$ is an orthonormal sequence $\{e_1,\dots,e_n\}$. As a result we immediately get that the solution α of (1.10) then is

$$\alpha_i = \langle x, e_i\rangle,$$

and, therefore, that the best approximation of x in $\operatorname{span}\{e_1,\dots,e_n\}$ is

$$v_* = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \cdots + \langle x, e_n\rangle e_n. \tag{1.11}$$

In the case of an orthonormal sequence, the coordinates $\alpha_i := \langle x, e_i\rangle$ are referred to as *Fourier coefficients.* The orthonormal property also gives us that

$$\|v_*\|^2 = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2.$$

If x happens to be an element of $\operatorname{span}\{e_1,\dots,e_n\}$ then, obviously, $v_* = x$ so then

$$\|x\|^2 = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2. \tag{1.12}$$

This is known as *Parseval's identity* and it should remind you of the fact that in $\mathbb{R}^n$ the Euclidean norm of the vector can be expressed by its coordinates with respect to the standard (orthonormal) basis as $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. Now an important example.

Example 1.4.8 (Finite real Fourier series). Consider $\mathscr{C}([-1,1];\mathbb{R})$ with standard inner product,

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t.$$

Similar to Example 1.3.12 we claim that $\cos(\pi t)$ and $\sin(\pi t)$ are orthogonal. In fact all cosines and sines with period π are orthogonal! We mean that for whatever n, the set of $2n+1$ harmonic functions

$$\{1/2,\ \cos(\pi t),\ \sin(\pi t),\ \cos(2\pi t),\ \sin(2\pi t),\dots,\ \cos(n\pi t),\ \sin(n\pi t)\}$$

is an orthonormal set (Exercise 1.36). In particular the set is therefore independent.

To keep matters simple, let us determine the best approximation of some $f \in \mathscr{C}([-1,1];\mathbb{R})$ in the span of only $1,\ \cos(\pi t),\ \sin(\pi t)$. That is we want to determine

$$\min_{f_*(t)=c_0+c_1\cos(\pi t)+c_2\sin(\pi t)} \|f - f_*\|.$$

Exploiting the orthonormality of $\{e_1, e_2, e_3\} := \{1/2, \cos(\pi t), \sin(\pi t)\}$ we find

$$\begin{aligned} f_*(t) &= \langle f, e_1\rangle e_1(t) + \langle f, e_2\rangle e_2(t) + \langle f, e_3\rangle e_3(t) \\ &= \langle f, \tfrac{1}{2}\rangle \tfrac{1}{2} + \langle f, \cos(\pi\cdot)\rangle \cos(\pi t) + \langle f, \sin(\pi\cdot)\rangle \sin(\pi t). \end{aligned}$$

For $f(t) = t^2$ the best approximation turns out to be $f_*(t) = \frac{1}{3} - \frac{4}{\pi^2}\cos(\pi t)$, see Fig. 1.12. $\square$

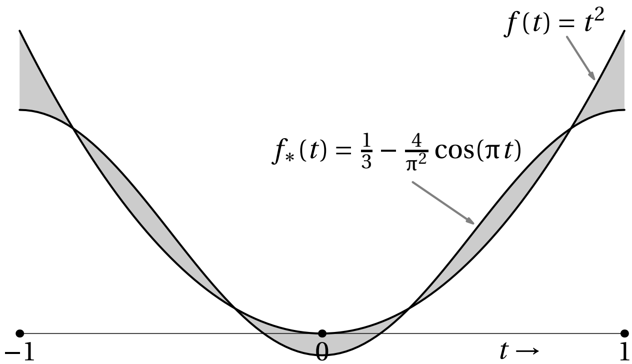


FIGURE 1.12: Function $f(t) = t^2$ on $[-1,1]$ and its best approximation $f_*(t)$ in $\operatorname{span}\{1/2, \cos(\pi t), \sin(\pi t)\}$. The error $f(t) - f_*(t)$ is shown in gray.

Example 1.4.9 (Legendre polynomials). Without proof we claim that the sequence of polynomials

$$\begin{aligned} p_0(t) &= 1, \\ p_1(t) &= t, \\ p_{k+1}(t) &= \frac{(2k+1)\,t\,p_k(t) - k\,p_{k-1}(t)}{k+1} \qquad \forall k \geq 1 \end{aligned}$$

defines an *orthogonal* sequence of polynomials on $\mathscr{C}([-1,1];\mathbb{R})$ with the standard inner product $\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)\,\mathrm{d}t$. Their norms are

$$\|p_k\| = \sqrt{\frac{2}{2k+1}}.$$

See Fig. 1.13. So then $e_k := \sqrt{\frac{2k+1}{2}}\,p_k$ is an orthonormal sequence. The p_k are known as the *Legendre polynomials* in honor of Adrien-Marie Legendre (1752-1833) who introduced them to describe Newtonian potentials. Legendre polynomials also play a role in numerical analysis (google for "Gaussian quadrature" if you want to know). □

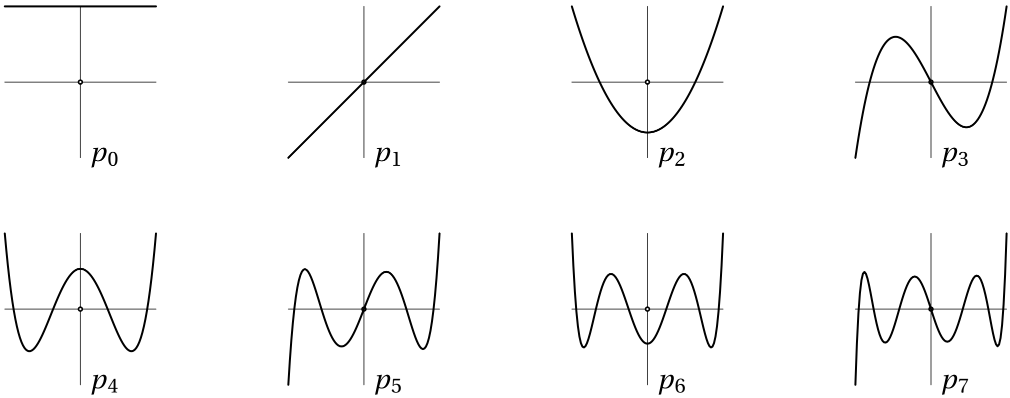


FIGURE 1.13: The first eight Legendre polynomials $p_k : [-1,1] \to \mathbb{R}$.

1.5 Hilbert space

Inner product spaces may be incomplete, and this is unfortunate since then all sorts of limits are not guaranteed to exist. For instance best approximations et cetera need not exist (Example 1.4.3). In this final section we focus on inner product spaces that *are* complete. These are known as *Hilbert spaces* in honor of the German mathematician David Hilbert (1862–1943). Hilbert spaces by definition enjoy all the properties of Banach spaces and in addition they has the rich structure brought about by the inner product. Most of our geometrical intuition for $\mathbb{R}^2$ and $\mathbb{R}^3$ generalizes to arbitrary Hilbert space.

Definition 1.5.1 (Hilbert space). A *Hilbert space* is an inner product space that is complete in the norm $\|x\| := \sqrt{\langle x, x\rangle}$. □

So a Hilbert space is a Banach space whose norm satisfies the parallelogram law. This law is usually easy to check. Every inner product space that is finite dimensional is a Hilbert space. For example

- $\mathbb{R}^n$ in whatever inner product, for instance the dot product, is a Hilbert space;
- The set of polynomials from $[a, b]$ to $\mathbb{R}$ of degree n or less, with the standard inner product $\langle f, g\rangle = \int_a^b f(t)g(t)\,\mathrm{d}t$, is a Hilbert space;
- $\mathbb{C}^n$ in whatever inner product, for instance $\langle x, y\rangle = \sum_{k=1}^n x_k \overline{y}_k$, is a Hilbert space.

Arguably the most interesting Hilbert space for our course is ℓ^2:

Example 1.5.2 (Square-summable sequence space). Theorem 1.2.4 claims that the space of real, square summable sequences

$$\ell^2 = \{(u_1, u_2, \ldots) \mid u_k \in \mathbb{R}, \|u\|_2 < \infty\},$$

is a Banach space in the standard 2-norm $\|u\|_2 := \sqrt{\sum_{k=1}^\infty u_k^2}$. This norm corresponds to the inner product defined as

$$\langle v, w\rangle_2 = \sum_{k=1}^\infty v_k w_k.$$

Hence ℓ^2 in this inner product is a Hilbert space. □

Similarly, $\ell^2(\mathbb{N}; \mathbb{C}) = \{(u_1, u_2, \ldots) \mid u_k \in \mathbb{C}, \sum_{k=1}^\infty |u_k|^2 < \infty\}$ is a Hilbert space in the complex inner product $\langle x, y\rangle_2 = \sum_{k=1}^\infty \overline{y}_k x_k$. The spaces ℓ^1 and ℓ^∞ are not Hilbert spaces because they are not even inner product spaces (Exercise 1.43). The Lebesgue space equivalent of ℓ^2 is $\mathscr{L}^2$ and this, again, is a Hilbert space:

Lemma 1.5.3 (Square integrable function space). Let $a, b \in \mathbb{R}$, $a < b$, possibly $a = -\infty$ and/or $b = \infty$. The set

$$\mathscr{L}^2([a,b];\mathbb{R}) := \{f : [a,b] \to \mathbb{R} \mid \int_a^b f^2(t)\,\mathrm{d}t < \infty\}$$

is a Hilbert space in the standard inner product

$$\langle f, g\rangle = \int_a^b g(t) f(t)\,\mathrm{d}t. \tag{1.13}$$

□

From one Hilbert space one can generate others.

Theorem 1.5.4 (Orthogonal complement). Let $\mathbb{S}$ be a non-empty subset of a Hilbert space $\mathbb{X}$. Then the orthogonal complement $\mathbb{S}^\perp$ is a Hilbert space.

Proof. First of all, $\mathbb{S}^\perp$ is a subspace and therefore a vector space. Remains to show that every Cauchy sequence in $\mathbb{S}^\perp$ converges in $\mathbb{S}^\perp$. Let f_n be a Cauchy sequence in $\mathbb{S}^\perp$. Since $\mathbb{X}$ is Hilbert, the sequence f_n has a limit f in $\mathbb{X}$. Now by continuity of inner product (Exercise 1.52) we have for every $s \in \mathbb{S}$ that $\langle f, s\rangle = \lim_{n\to\infty}\langle f_n, s\rangle = 0$. So $f \in \mathbb{S}^\perp$. ■

Example 1.5.5 (Even functions). The orthogonal complement of the set S of *odd* functions

$$S = \{f \in \mathscr{L}^2(\mathbb{R};\mathbb{R}) \mid f(t) = -f(-t)\ \forall t\}$$

are the even functions (verify this). So the set of even functions in $\mathscr{L}^2(\mathbb{R};\mathbb{R})$ is a Hilbert space. □

The projection theorem 1.4.2 claims that *if* a best approximation exists then it is unique. It does not say that a best approximation exists and in fact it need not exist (Example 1.4.3). In that example, we projected onto a space that is not a Hilbert space. Could it be? Yes:

Theorem 1.5.6 (Projection theorem, final version). Let $\mathbb{X}$ be an inner product space and suppose $\mathbb{V} \subseteq \mathbb{X}$ is a Hilbert space. Then every $x \in \mathbb{X}$ has a best approximation $v_* \in \mathbb{V}$, and the best approximation is unique.

Proof. Uniqueness we showed earlier (Thm. 1.4.2). Existence we show now. Let $\beta = \inf_{v\in\mathbb{V}} \|x-v\|$. This implies that a sequence $v_n \in \mathbb{V}$ exists such that

$$\|x-v_n\|^2 \leq \beta^2 + \frac{1}{n} \qquad \forall n \in \mathbb{N}.$$

Below we show that v_n is Cauchy. Then by completeness of Hilbert space we have that $v_* := \lim_{n\to\infty} v_n$ is in $\mathbb{V}$. Now by continuity of norm we have $\|x-v_*\| = \beta$, making v_* a best approximation. Done.

Proof that v_n is Cauchy. By the parallelogram law we have

$$\|v_n - v_m\|^2 + \|v_n + v_m - 2x\|^2 = 2\|v_n - x\|^2 + 2\|v_m - x\|^2,$$

see Fig. 1.14. Therefore

$$\begin{aligned}\|v_n - v_m\|^2 &= 2\|v_n - x\|^2 + 2\|v_m - x\|^2 - \|v_n + v_m - 2x\|^2 \\ &= 2\|v_n - x\|^2 + 2\|v_m - x\|^2 - 4\|x - \tfrac{v_n+v_m}{2}\|^2.\end{aligned}$$

Since $\frac{v_n+v_m}{2} \in \mathbb{V}$ and since β is the smallest possible error norm, we have that $\|x - \frac{v_n+v_m}{2}\| \geq \beta$. Therefore

$$\begin{aligned}\|v_n - v_m\|^2 &\leq 2\|v_n - x\|^2 + 2\|v_m - x\|^2 - 4\beta^2 \\ &\leq 2(\beta^2 + \tfrac{1}{n}) + 2(\beta^2 + \tfrac{1}{m}) - 4\beta^2 = 2(\tfrac{1}{n} + \tfrac{1}{m}).\end{aligned}$$

It follows that v_n is a Cauchy sequence. ■

Complete orthonormal system

Many familiar results from $\mathbb{R}^n$ carry over to arbitrary Hilbert space. For instance the following central and truly powerful theorem. It combines many results. For instance it generalizes the notion of basis to infinite dimensional Hilbert space.

Theorem 1.5.7 (Complete orthonormal system). Let $\mathbb{X}$ be a Hilbert space and let $\{e_1, e_2, \ldots\}$ be an orthonormal sequence (possibly infinite) in $\mathbb{X}$. The following conditions are equivalent.

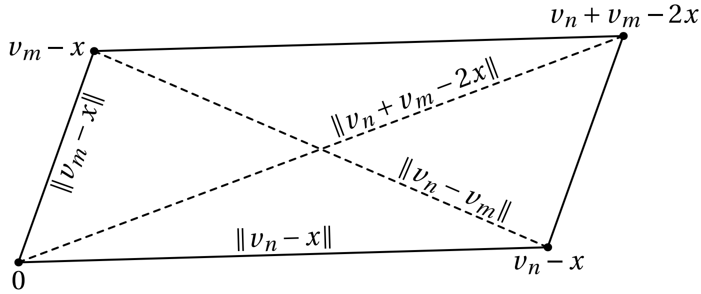


FIGURE 1.14: Parallelogram law.

1. $x \perp e_k$ for all e_k iff $x = 0$ (sequence is *complete*)
2. $x = \sum_k \langle x, e_k \rangle e_k \ \forall x \in \mathbb{X}$ (x equals *Fourier series*)
3. $\|x\|^2 = \sum_k |\langle x, e_k \rangle|^2 \ \forall x \in \mathbb{X}$ (*Parseval's identity*)

In that case we say that $\{e_1, e_2, \ldots\}$ is a *orthonormal basis* or a *complete orthonormal system* of $\mathbb{X}$. $\square$

Proof. We prove it for infinite sequences $\{e_1, e_2, \ldots\}$ (the finite case is simpler). (1) $\implies$ (2): for any n, the finite sum x_n defined as

$$x_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$$

is the best approximation of x in $\operatorname{span}\{e_1, \ldots, e_n\}$ hence by Thm. 1.4.2 the norm of the approximation x_n does not exceed that of x, so $\|\sum_{k=1}^{n} \langle x, e_k \rangle e_k\| \le \|x\| < \infty$. By Pythagoras this means that the positive real numbers,

$$\alpha_n := \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$$

are bounded from above by $\|x\|^2 < \infty$. This α_n hence is an increasing but bounded sequence. Therefore it converges, $\alpha_\infty = \lim_{n\to\infty} \alpha_n$. As a result x_n is a Cauchy sequence because for any $m \ge n \ge N$,

$$\|x_n - x_m\|^2 = \sum_{n<k\le m} |\langle x, e_k\rangle|^2 = \alpha_m - \alpha_n \le \alpha_\infty - \alpha_N$$

which converges to 0 as $N \to \infty$. By the Hilbert property then $x_\infty := \sum_{k=1}^\infty \langle x, e_k\rangle e_k$ exists in $\mathbb{X}$. This x_∞ equals x by the following argument: for every index k there holds

$$\begin{aligned}
\langle x - x_\infty, e_k\rangle &= \langle x, e_k\rangle - \langle \lim_{n\to\infty} x_n, e_k\rangle \\
&= \langle x, e_k\rangle - \lim_{n\to\infty} \langle x_n, e_k\rangle \\
&= \langle x, e_k\rangle - \lim_{n\to\infty} \langle \sum_{m=1}^n \langle x, e_m\rangle e_m, e_k\rangle \\
&= \langle x, e_k\rangle - \lim_{n\to\infty} \sum_{m=1}^n \langle x, e_m\rangle \langle e_m, e_k\rangle \\
&= \langle x, e_k\rangle - \langle x, e_k\rangle = 0 \qquad \forall k.
\end{aligned}$$

By completeness of the $\{e_1, e_2, \ldots\}$ this implies that $x - x_\infty = 0$, i.e. that $x = x_\infty$.

That (2) $\implies$ (3) is immediate from Pythagoras, continuity of norm and orthonormality of $\{e_1, e_2, \ldots\}$:

$$\|x\|^2 = \|\lim_{n\to\infty} x_n\|^2 = \lim_{n\to\infty} \|x_n\|^2 = \lim_{n\to\infty} \sum_{k=1}^n |\langle x, e_k\rangle|^2.$$

Finally the implication (3) $\implies$ (1): if $x \perp e_k$ for all k then $\|x\|^2 = \sum_k |\langle x, e_k\rangle|^2 = 0$ hence $x = 0$. ■

For finite dimensional space such as $\mathbb{R}^n$ the theorem is intuitive and says that $\{e_1, e_2, \ldots\}$ span the entire space iff the orthogonal complement of $\operatorname{span}\{e_1, e_2, \ldots\}$ is zero.

Lemma 1.5.8 (A polynomial orthonormal basis of $\mathscr{L}^2$). Consider the Hilbert space $\mathscr{L}^2([-1,1];\mathbb{R})$ with standard inner product $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)\,\mathrm{d}t$. There is an orthonormal sequence of polynomials $\{e_k\}_{k=0,1,2,\ldots}$ with e_k having degree k. Moreover this sequence is then an orthonormal basis of $\mathscr{L}^2([-1,1];\mathbb{R})$.

Proof. Let $v_{k-1}(t)$ be the best approximation of t^k in the space of polynomials of degree at most $k-1$. Then $e_k := (t^k - v_{k-1})/\|t^k - v_{k-1}\|$ is an orthonormal sequence with e_k having degree k. (See also Example 1.4.9.)

According to Thm. 1.5.7 it suffices to show that the only function $f \in \mathscr{L}^2([-1,1];\mathbb{R})$ orthogonal to all e_k is the zero function. So assume that $\langle f, e_k\rangle = 0$ for all $k = 0,1,\ldots$, i.e.

$$\int_{-1}^{1} f(t)t^k \,\mathrm{d}t = 0, \qquad k = 0,1,2,\ldots.$$

Assume, for the moment, that f is continuous. If f is nonzero then f is > 0 (or < 0) throughout some small enough interval $[c,d]$, see Fig. 1.15. Next let $h(t)$ be a parabola

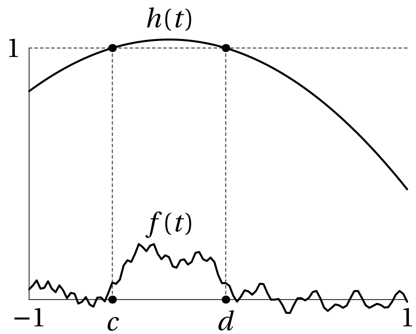


FIGURE 1.15: A parabola $h(t)$ so that $h(t) \geq 1$ on $[c,d]$ and $|h(t)| < 1$ elsewhere on $[0,1]$.

that exceeds 1 if $c < t < d$, and for all other $t < c$ abd $t > d$ is positive but less than 1. Such a parabola exists. Realize that every power h^n of h is a polynomial (in fact h^n has degree $2n$ so is an element of $\operatorname{span}\{e_0, e_1, \ldots, e_{2n}\}$. The higher the power, the larger h^n is in the interval $[c,d]$ and the smaller it is elsewhere. Therefore for some large enough n we have

$$\langle f, h^n\rangle \neq 0.$$

This means that f can not be orthogonal to all e_k. Hence the only continuous f that is orthogonal to all e_k is the zero function, $f = 0$.

Now let $f \in \mathscr{L}^2([0,1];\mathbb{R})$, not necessarily continuous, and assume that $f \perp e_k$ for all $k = 0,1,\ldots$. Again we show that f is then necessarily the zero function. Consider its antiderivative

$$F(t) := \int_{-1}^{t} f(\tau)\,\mathrm{d}\tau.$$

This function is continuous. It is also orthogonal to all e_k because

$$\langle F, e_k\rangle = \int_{-1}^{1} F(t)t^k\,\mathrm{d}t = \left[F(t)\frac{t^{k+1}}{k+1}\right]_{-1}^{1} - \int_{-1}^{1} f(t)\frac{t^{k+1}}{k+1}\,\mathrm{d}t = 0, \qquad \forall k = 0,1,2,\ldots.$$

Here we used that $F(0) = 0$ and $F(1) = \int_{-1}^{1} f(t)\,\mathrm{d}t = \langle f, 1\rangle = 0$. So F is orthogonal to e_k for all $k = 0,1,\ldots$. Since F is continuous and orthogonal to all e_k we have that F is the zero function. Its derivative f hence is zero as well, which is what we set out to prove. ∎

So every square integrable function $f : [-1,1] \to \mathbb{R}$ can be approximated (in the 2-norm) arbitrary well by polynomials e_k, and, in fact, $f = \sum_{k=0}^{\infty} f_k e_k$ for $f_k = \langle f, e_k\rangle$. Not bad. In Chapter 3 we will see that not only polynomials have this property but also *harmonic* functions (see Theorem 3.2.1).

1.6 Exercises

1.1 Consider the three norms on $\mathbb{R}^2$ as defined in Example 1.1.2.

(a) Show that the 1-norm is a norm.

(b) Show that the 2-norm is a norm. (The triangle inequality is a tricky one to verify; you probably want to read § 1.3 first.)

(c) Show that the ∞-norm is a norm.

1.2 Prove the claim of Example 1.1.5.

1.3 Prove the claim of Example 1.1.7.

1.4 Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on the same vector space. Is $\|\cdot\|_a + \|\cdot\|_b$ a norm?

1.5 Show that $\|\cdot\|$ is a norm on $\mathbb{R}$ if and only if

$$\|x\| = \gamma|x|$$

for some real number $\gamma > 0$. [Hint: use that a vector $x \in \mathbb{R}$ can also be seen as the scalar $x \in \mathbb{R}$ times the vector $1 \in \mathbb{R}$.]

1.6 Suppose $\mathbb{X}$ is a finite dimensional vector space with basis $V = \{v_1, \dots, v_n\}$. Denote the coordinate vector of $x \in \mathbb{X}$ as $x_V = (x_{V,1}, \dots, x_{V,n})$. Is

$$\|x\| := \sum_{k=1}^{n} |x_{V,k}|$$

a norm on $\mathbb{X}$?

1.7 Let $\mathbb{X}_A$ and $\mathbb{X}_B$ be two real vector spaces, and suppose $\|\cdot\|_A$ is a norm on $\mathbb{X}_A$ and $\|\cdot\|_B$ a norm on $\mathbb{X}_B$. Show that

$$\|(x_a, x_b)\| := \max(\|x_a\|_A, \|x_b\|_B)$$

is a norm on the cartesian product space

$$\mathbb{X}_A \times \mathbb{X}_B := \{(x_a, x_b) \mid x_a \in \mathbb{X}_A, x_b \in \mathbb{X}_B\}$$

1.8 Let $\mathscr{P}_2(\mathbb{R};\mathbb{R})$ be the set of polynomials $p : \mathbb{R} \to \mathbb{R}$ that have degree 2 or less. It is a subspace of $\mathscr{F}(\mathbb{R};\mathbb{R})$. Let $p_0, p_1, p_2 \in \mathbb{R}$. Is

$$\|p_0 + p_1 t + p_2 t^2\| := |p_0| + |p_1| + |p_2|$$

a norm on $\mathscr{P}_2(\mathbb{R};\mathbb{R})$?

1.9 If f_n is Cauchy, is $\|f_n\|$ then Cauchy as well?

1.10 Show that the f_n of Example 1.2.9 is a Cauchy sequence in the 2-norm.

1.11 Show that every Cauchy sequence is bounded (meaning: for every Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ there is an $M > 0$ such that $\|f_n\| \le M$ for all $n \in \mathbb{N}$).

1.12 *Bounded variation.* Functions of *bounded variation* roughly are those functions whose graph can be plotted up to any degree of precision, see Fig. 1.16. Formally, $f \in \mathscr{F}([a,b];\mathbb{R})$ is of bounded variation if there is an $M > 0$ (that may depend on f) such that

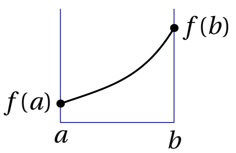
$$\sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| \leq M$$

for every N and every partitioning $a = x_0 < x_1 < \cdots < x_N = b$ of $[a,b]$. The *total variation* of f is defined as

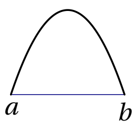
$$V(f) := \sup_{a=x_0<x_1<\cdots<x_N=b} \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})|.$$

Here the supremum is taken over all $N \in \mathbb{N}$ and all ordered x_k (with $x_0 = a, x_N = b$). Let $\mathscr{BV}([a,b];\mathbb{R})$ be the subset of $\mathscr{F}([a,b];\mathbb{R})$ of functions whose total variation is finite.

(a) Argue that $V(f) = f(b) - f(a)$ if f is a non-decreasing function, such as



(b) Determine $V(f)$ for $f(t) = (b-t)(t-a)$. A plausible argument suffices,



(c) Show that $\mathscr{BV}([a,b];\mathbb{R})$ is a subspace

(d) Show that $V(f)$ is not a norm on $\mathscr{BV}([a,b];\mathbb{R})$

(e) Show that $|f(a)| + V(f)$ is a norm on $\mathscr{BV}([a,b];\mathbb{R})$

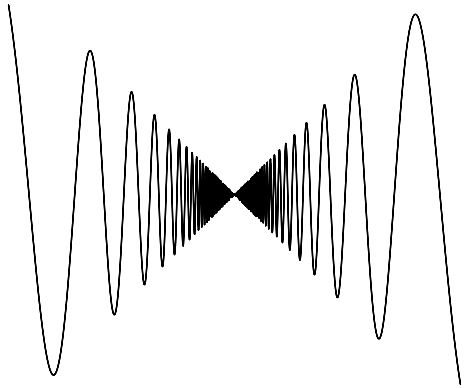


FIGURE 1.16: $t\cos(1/t)$ on $[-0.1, 0.1]$.

(f) Find a continuous $f \in \mathscr{F}([0,1];\mathbb{R})$ whose total variation is infinite.

1.13 *Uniqueness of limit.* Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in some normed vector space $\mathbb{X}$.

(a) Suppose f_n converges. Show that $\lim_{n\to\infty} f_n$ is unique.

(b) Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$ (with the norm as on $\mathbb{X}$). Suppose $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Y}$ is a sequence in $\mathbb{Y}$ (and hence in $\mathbb{X}$) and that it converges in $\mathbb{X}$ to some $f \in \mathbb{X}$. Show that $\{f_n\}_{n\in\mathbb{N}}$ converges in $\mathbb{Y}$ if and only if $f \in \mathbb{Y}$. [Hint: this is an easy problem.]

(c) Use the previous part to show that the f_n of Example 1.1.8 have a limit in $\mathscr{L}^1([-1,1];\mathbb{R})$ in the 1-norm, but not in the subspace $\mathscr{C}([-1,1];\mathbb{R})$ in the 1-norm.

1.14 Consider Example 1.3.3. Show that for the function space $\mathscr{F}([a,b];\mathbb{R})$ also the third axiom of inner product does not hold.

1.15 Prove the claim of Example 1.3.5

1.16 *Parallelogram law.*

(a) Prove that every inner product satisfies the *parallelogram law*:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

(b) Show that on real vector space we have (1.7)

(c) Show that on complex vector space we have (1.8)

1.17 Consider $\mathscr{C}([0,1];\mathbb{R})$ with the max-norm, $\|f\|_\infty = \max_{x\in[0,1]}|f(x)|$. Let

$$h(x) = 1, \qquad g(x) = x.$$

(a) Compute $\|h\|_\infty$, $\|g\|_\infty$, $\|h+g\|_\infty$, $\|h-g\|_\infty$

(b) Is there an inner product whose norm is the max-norm?

1.18 Let $\mathscr{P}_2([0,1];\mathbb{C})$ be the set of polynomials $p:[0,1]\to\mathbb{C}$ of degree 2 or less.

(a) Is

$$\langle p,q\rangle := p(0)\overline{q(0)} + p^{(1)}(0)\overline{q^{(1)}(0)}$$

an inner product on $\mathscr{P}_2([0,1];\mathbb{C})$?

(b) Show that

$$\langle p,q\rangle := p(0)\overline{q(0)} + p^{(1)}(0)\overline{q^{(1)}(0)} + p^{(2)}(0)\overline{q^{(2)}(0)} \tag{1.14}$$

is an inner product on $\mathscr{P}_2([0,1];\mathbb{C})$.

(c) Determine $\langle 1, \mathrm{i}x\rangle$ for the inner product of (1.14)

(d) What is the distance between $1+x+x^2$ and $2-2x$ in the inner product (1.14)? (The *distance* between two vectors f_1 and f_2 is defined as the norm $\|f_1-f_2\|$ of their difference.)

(e) Repeat 1.18c and 1.18d for the inner product

$$\langle p,q\rangle = \int_0^1 p(x)\overline{q(x)}\,\mathrm{d}x.$$

1.19 Prove the following properties of complex inner products.

(a) $\langle u_1+u_2+u_3, v\rangle = \langle u_1,v\rangle + \langle u_2,v\rangle + \langle u_3,v\rangle$

(b) $\langle u,\alpha v\rangle = \overline{\alpha}\langle u,v\rangle$

(c) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

1.20 Show that $\langle x, y\rangle = y^{\mathrm{T}} P x$ is an inner product on $\mathbb{R}^n$ if $P = Q^{\mathrm{T}} Q$ for some nonsingular $n \times n$ matrix Q.

1.21 Suppose $\langle x, y\rangle = 0$ for all $x \in \mathbb{X}$. Show that $y = 0$.

1.22 Is $\langle A, B\rangle := \operatorname{tr}(B^{\mathrm{T}} A)$ an inner product on $\mathbb{R}^{n\times k}$? Here tr is the *trace,* defined as the sum of the diagonal entries.

1.23 Is it $\mathbb{S}^{\perp\perp} \subseteq \mathbb{S}$ or $\mathbb{S} \subseteq \mathbb{S}^{\perp\perp}$?

1.24 Show that on real vector space $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ iff $x \perp y$ and find a counter example for this equivalence for the complex vector space $\mathbb{C}$.

1.25 Show that a $c > 0$ exists such that

$$\left|\int_0^{\pi} f(t)\cos(t)\,\mathrm{d}t\right| \le c\sqrt{\int_0^{\pi} |f(t)|^2\,\mathrm{d}t} \quad \forall f$$

and determine the smallest c_* possible and a function $f(t)$ that achieves equality for this smallest c_*.

1.26 Consider the vector space of continuous functions f defined on the complex unit circle,

$$f : \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C}.$$

Show that

$$\langle f, g\rangle = \int_0^{2\pi} \overline{g(\mathrm{e}^{\mathrm{i}\omega})} f(\mathrm{e}^{\mathrm{i}\omega})\,\mathrm{d}\omega$$

is an inner product.

1.27 Consider the set

$$\ell^2(\mathbb{N};\mathbb{C}) = \{u \in \ell(\mathbb{N};\mathbb{C}) \mid \sum_{n=1}^{\infty} |u_n|^2 < \infty\}.$$

Its elements are the infinite sequences $(u_1, u_2, \dots)$ whose entries $u_n \in \mathbb{C}$ converge to zero so fast that $\|u\|_2 := \sum_{n=1}^{\infty} |u_n|^2$ is convergent (is finite). In the exercise you will prove that ℓ^2 is a subspace of $\ell(\mathbb{N};\mathbb{C})$, and that ℓ^2 is, in fact, an inner product space.

(a) Show that $2|ab| \le |a|^2 + |b|^2 \ \forall a, b \in \mathbb{C}$.

(b) Show that $|a+b|^2 \le 2|a|^2 + 2|b|^2 \ \forall a, b \in \mathbb{C}$.

(c) Show that $\ell^2(\mathbb{N};\mathbb{C})$ is a subspace of $\ell(\mathbb{N};\mathbb{C})$.

(d) Show that $\|u\|_2 := \sqrt{\sum_{n=1}^{\infty} |u_n|^2}$ is a norm on $\ell^2(\mathbb{N};\mathbb{C})$.

Now consider the following mapping

$$\langle u, v\rangle := \sum_{i=1}^{\infty} u_i \overline{v_i} \qquad \text{where } u, v \in \ell^2(\mathbb{N};\mathbb{C}).$$

(e) Show that $\langle u, v\rangle$ is well defined[3] for every $u, v \in \ell^2(\mathbb{N};\mathbb{C})$.

(f) Show that this $\langle\cdot,\cdot\rangle$ is an inner product on $\ell^2(\mathbb{N};\mathbb{C})$.

(g) Show that this $\langle\cdot,\cdot\rangle$ is not an inner product on $\ell(\mathbb{N};\mathbb{C})$.

1.28 For which $\alpha \in \mathbb{R}$ is

$$\langle x, y\rangle := x_1 y_1 + \alpha(x_1 y_2 + x_2 y_1) + x_2 y_2$$

an inner product on $\mathbb{R}^2$?

1.29 Consider $\mathscr{L}^2([0,1];\mathbb{R})$. Does a best approximation of e^t in the subspace of all polynomial in $\mathscr{L}^2([0,1];\mathbb{R})$ exist?

1.30 Does $2|\langle x, y\rangle| \le \|x\|^2 + \|y\|^2$ hold for any inner product?

1.31 Is $\int_0^1 x(t)y(t)\,\mathrm{d}t + x(0)y(0)$ an inner product on $\mathscr{C}([0,1];\mathbb{R})$?

1.32 Consider $\mathscr{C}([0,1];\mathbb{R})$. Show that none of the expressions below are inner products and indicate which of the axioms of inner product fail:

[3] meaning that the infinite series $\sum_i u_i \overline{v}_i$ converges

(a) $\langle x,y\rangle := \int_0^1 x(t)y^2(t)\,\mathrm{d}t$

(b) $\langle x,y\rangle := \int_0^1 \frac{1}{t}x(t)y(t)\,\mathrm{d}t$

(c) $\langle x,y\rangle := \int_0^{0.5} x(t)y(t)\,\mathrm{d}t$

(d) $\langle x,y\rangle := \int_0^1\int_0^1 x(t)y(s)\,\mathrm{d}t\,\mathrm{d}s$

(e) $\langle x,y\rangle := x(0)y(0)+x(1)y(1)$

1.33 Consider $\mathscr{C}([0,1];\mathbb{R})$ with standard inner product. Determine the best approximation of t^2 in $\operatorname{span}\{1,t\}$.

1.34 Consider $\mathscr{C}([-1,1];\mathbb{R})$ with standard inner product. Determine the best approximation of t^4 in $\operatorname{span}\{1,t,t^2,t^3\}$.

1.35 Consider $\mathbb{C}^n$. Let $\alpha_k\in\mathbb{C}$. Show that

$$\langle x,y\rangle := \sum_{k=1}^{n}\alpha_k x_k \bar{y}_k$$

is an inner product if and only if all α_k are real and larger than zero.

1.36 Show that $\{1/2,\cos(\pi t),\sin(\pi t),\ldots,\cos(n\pi t),\sin(n\pi t)\}$ is an orthogonal set on $\mathscr{C}([-1,1];\mathbb{R})$ with standard inner product.

1.37 *Legendre polynomials.* Consider $\mathscr{C}([-1,1];\mathbb{R})$ and its subspace $\mathscr{P}_5([-1,1];\mathbb{R})$ (the subspace of polynomials $p:[-1,1]\to\mathbb{R}$ of degree 5 or less.) It can be shown that the degree-6 Legendre polynomial is

$$p_6(t) := \frac{1}{16}(231t^6-315t^4+105t^2-5)$$

and that its norm is $\|p_6\| = \sqrt{2/13}$.

(a) Determine the best approximation of t^6 in $\mathscr{P}_5([-1,1];\mathbb{R})$

(b) Determine the minimal approximation error $\min_{q\in\mathscr{P}_5([-1,1];\mathbb{R})}\|t^6-q\|$

(Hint: lengthy derivations can be avoided. The answer for part (b) is approximately 0.027)

1.38 Construct an orthonormal basis of the space $\operatorname{span}\{e^{-t}, e^{-2t}, e^{-3t}\}$ as subspace of $\mathscr{L}^2([0,\infty);\mathbb{R})$ with standard inner product $\langle f, g\rangle = \int_0^\infty f(t)g(t)\,\mathrm{d}t$.

1.39 Consider $\mathscr{L}^2([0,1];\mathbb{R})$ with standard inner product, and let $\mathscr{P}([0,1];\mathbb{R})$ be the set of all polynomials in $\mathscr{L}^2([0,1];\mathbb{R})$.

(a) Argue that the best approximation of e^t in the space of polynomials of any degree $\mathscr{P}([0,1];\mathbb{R})$ does not exist.

(b) Is the subspace $\mathscr{P}([0,1];\mathbb{R})$ a Hilbert space?

1.40 Consider the set of even square summable sequences

$$\ell^{2,\text{even}} = \{x \in \ell^2(\mathbb{Z},\mathbb{R}) \,|\, x_k = x_{-k}\}$$

and the standard inner product of $\ell^2(\mathbb{Z},\mathbb{R})$

(a) Show that $\ell^{2,\text{even}}$ is a Hilbert space

(b) Determine a complete orthonormal system for $\ell^{2,\text{even}}$

1.41 Show that ℓ^{finite} with standard inner product (that of ℓ^2) is not a Hilbert space.

1.42 Suppose $\mathbb{X}$ and $\mathbb{V} \subseteq \mathbb{X}$ are both Hilbert spaces. Show that every $x \in \mathbb{X}$ has a *unique* decomposition $x = v + v_\perp$ with $v \in \mathbb{V}$ and $v_\perp \in \mathbb{V}^\perp$.

1.43 Show that ℓ^1 and ℓ^∞ with standard norms are not inner product spaces.

1.44 *Hilbert matrix.* Let $\mathscr{P}_{n-1}$ be the subspace of $\mathscr{C}([0,1];\mathbb{R})$ spanned by the monomials $\{1, t, \ldots, t^{n-1}\}$. That is to say, $\mathscr{P}_{n-1}$ is the subspace formed by the polynomials $p(t) = \alpha_1 + \alpha_2 t + \cdots + \alpha_n t^{n-1}$ of degree at most $n-1$.

(a) Determine the normal equations (1.10) for the case that $\{v_1, \ldots, v_n\} = \{1, t, \ldots, t^{n-1}\}$ and $x(t) = t^n$.

(b) Determine the best approximation of t^2 in $\mathscr{P}_1$.

1.45 Show that $\langle f, g\rangle := \int_0^1 f(t)g(t) + f'(t)g'(t)\,\mathrm{d}t$ is an inner product on the vector space $\mathscr{C}^1([0,1];\mathbb{R})$ of continuously differentiable functions $f : [0,1] \to \mathbb{R}$.

1.46 Can the arguments of Exercise 1.27(a,b) also be used to prove that $\mathscr{L}^2([a,b];\mathbb{R})$ is a subspace?

1.47 Is $\ell^1 \subset \ell^2$?

1.48 Let $\mathbb{X}$ be some complex Hilbert space.

(a) Let $\{v_n\}_{n\in\mathbb{N}}$ be an orthogonal sequence in $\mathbb{X}$. Show that $\sum_{n\in\mathbb{N}} v_n$ converges iff $\sum_{n\in\mathbb{N}} \|v_n\|^2 < \infty$.

(b) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathbb{X}$, and $\alpha_n \in \mathbb{C}$. Show that $\sum_{n\in\mathbb{N}} \alpha_n u_n$ converges iff $\alpha := (\alpha_1, \alpha_2, \ldots)$ is in $\ell^2(\mathbb{N};\mathbb{C})$.

(c) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathbb{X}$, and $\alpha, \beta \in \ell^2(\mathbb{N};\mathbb{C})$. Show that

$$\langle \sum_{n\in\mathbb{N}} \alpha_n e_n, \sum_{n\in\mathbb{N}} \beta_n e_n \rangle_{\mathbb{X}} = \langle \alpha, \beta \rangle_{\ell^2}$$

Here $\langle \cdot,\cdot \rangle_{\mathbb{X}}$ is the inner product on $\mathbb{X}$, and $\langle \cdot,\cdot \rangle_{\ell^2}$ the standard inner product on $\ell^2(\mathbb{N};\mathbb{C})$.

(d) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in $\mathbb{X}$, and $x \in \mathbb{X}$. Derive *Bessel's inequality*:

$$\sum_{n\in\mathbb{N}} \langle x, e_n \rangle \le \|x\|^2.$$

Tougher problems

1.49 *Norm or not a norm.*

(a) Let $\mu \in [0,1]$. Show that every norm satisfies

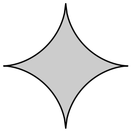
$$\|\mu x + (1-\mu) y\| \le \mu \|x\| + (1-\mu)\|y\|.$$

(b) Consider $\mathbb{R}^2$. Given two elements $x, y \in \mathbb{R}^2$, say,

draw all elements of $\{z \in \mathbb{R}^2 \mid z = \mu x + (1-\mu)y, \mu \in [0,1]\}$.

(c) Show that $\|z\| \le \max(\|x\|, \|y\|)$ for every element of $\{z \mid z = \mu x+(1-\mu)y, \mu \in [0,1]\}$

(d) Consider $\mathbb{R}^2$. Can this set



be the unit ball $\{x \mid \|x\| \le 1\}$ for some norm on $\mathbb{R}^n$?

1.50 Consider the sequence spaces $\ell^1, \ell^2, \ell^\infty$ and two of their continuous counterparts $\mathscr{L}^1, \mathscr{L}^2$. Prove or provide counter examples to the following claims:

(a) $\ell^2(\mathbb{N};\mathbb{R}) \subseteq \ell^\infty(\mathbb{N};\mathbb{R})$?

(b) $\ell^1(\mathbb{N};\mathbb{R}) \subseteq \ell^2(\mathbb{N};\mathbb{R})$? [Hint: first assume that $|x_n| < 1$ for all n.]

(c) $\mathscr{L}^1([0,1];\mathbb{R}) \subseteq \mathscr{L}^2([0,1];\mathbb{R})$?

(d) $\mathscr{L}^2([0,1];\mathbb{R}) \subseteq \mathscr{L}^1([0,1];\mathbb{R})$? [Hint: first assume that $|f(t)| > 1$ for all t.]

(e) $\mathscr{L}^2(\mathbb{R};\mathbb{R}) \subseteq \mathscr{L}^1(\mathbb{R};\mathbb{R})$?

(f) $\mathscr{L}^1(\mathbb{R};\mathbb{R}) \subseteq \mathscr{L}^2(\mathbb{R};\mathbb{R})$?

1.51 Consider two norms on $\mathbb{R}^m$: an arbitrary norm $\|\cdot\|$ and the max-norm $\|\cdot\|_\infty$ (also known as ∞-norm). In this problem we derive that al norms on $\mathbb{R}^n$ "equivalent" meaning that constants $\gamma, \delta > 0$ exist such that

$$\delta\|x\|_\infty \le \|x\| \le \gamma\|x\|_\infty \qquad \forall x. \tag{1.15}$$

(a) Show that there is a $\gamma > 0$ such that

$$\|x\| \le \gamma\|x\|_\infty.$$

[Hint: write x as $x = x_1 e_1 + \cdots + x_n e_n$ where $\{e_i\}_{i=1,\ldots,n}$ is the standard basis of $\mathbb{R}^n$ and express γ using the $\|e_i\|$.]

(b) The Bolzano-Weierstraß theorem says that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. This we can use to prove that a δ exists such that

$$\delta\|x\|_\infty \leq \|x\|$$

To this end, let

$$\delta_0 := \inf_{\|x\|_\infty=1} \|x\|.$$

If δ_0 is zero then a sequence $\{x_k\}$ of vectors in $\mathbb{R}^n$ exists for which $\|x_k\| \to 0$ as $k \to \infty$. Now, by Bolzano-Weierstraß this means that a convergent subsequence $\{x_{k_l}\}$ exists... Finish the proof, that is, prove that (1.15) holds.

1.52 *Continuity of inner product.* Let $\mathbb{X}$ be an inner product space and suppose that $x := \lim_{n\to\infty} x_n$ exists in $\mathbb{X}$. Use Cauchy-Schwarz to show that an inner product is continuous in the sense that $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$.

1.53 *Chebyshev.* Let $T_n : [-1,1] \to \mathbb{R}$ be the *Chebyshev* polynomials

$$T_n(t) = \cos(n \arccos(t)), \qquad n = 0, 1, 2, \ldots.$$

Soon we will see that the $T_n(t)$ are indeed polynomials in t. In what follows $\mathscr{P}_n([-1,1];\mathbb{R})$ denotes the set of polynomials $p : [-1,1] \to \mathbb{R}$ of degree n or less.

(a) Show that $\{T_n\}_{n=0,1,2,\ldots}$ are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t. \tag{1.16}$$

(Hint: substitute $t = \cos(\phi)$ and use the goniometric formula $\cos(t+s)+\cos(t-s) = 2\cos(t)\cos(s)$)

(b) What is $\|T_0\|$ and what is $\|T_n\|$ for $n > 0$?

(c) Derive the recursion

$$T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t) \qquad \forall n \geq 1,$$

and show that all T_n are polynomial. (Hint: again substitute $t = \cos(\phi)$ and now use that $\cos(\alpha)+\cos(\beta) = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$)

(d) Determine T_0, T_1, T_2, T_3.

(e) Use the previous part to determine the best approximation of t^3 in $\mathscr{P}_2([-1,1];\mathbb{R})$ with inner product (1.16).

(f) Let $n > 0$. Determine the approximation error $\min_{q\in\mathscr{P}_{n-1}([-1,1];\mathbb{R})} \|t^n - q\|$ in the norm defined by the inner product (1.16) (Hint: what is the leading coefficient of T_n?)

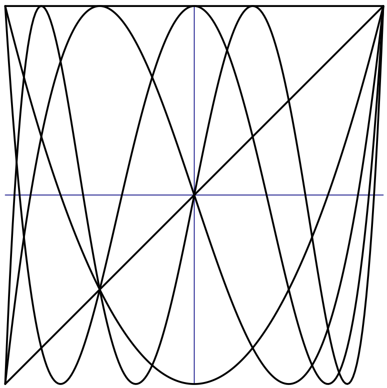


FIGURE 1.17: First five Chebyshev polynomials.

1.54 *Matlab.* Verify that on $\mathscr{C}([0,1];\mathbb{R})$ with its standard inner product we have the recursion

$$\langle t^k, \mathrm{e}^t\rangle = \mathrm{e} - k\langle t^{k-1}, \mathrm{e}^t\rangle$$

for any $k \in \mathbb{N}$. Use this and MATLAB to find the optimal degree 10 polynomial $p(t)$ approximation of e^t and plot the error $\mathrm{e}^t - p(t)$.

1.55 *Bases for images.* Let $e_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. This is the standard basis of $\mathbb{R}^2$. The standard basis for $\mathbb{R}^{2\times 2}$ can be expressed in terms of e_1, e_2 as

$$\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] = e_1 e_1^{\mathrm{T}}, \qquad \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] = e_1 e_2^{\mathrm{T}}, \qquad \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right] = e_2 e_1^{\mathrm{T}}, \qquad \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right] = e_2 e_2^{\mathrm{T}}.$$

This suggests that we can construct orthonormal bases for $\mathbb{R}^{n\times k}$ given *any* two bases for $\mathbb{R}^n$ and $\mathbb{R}^k$. We analyze this situation not for not $\mathbb{R}^{n\times k}$ but for "continuous square images", that is,

$$\mathscr{L}^2([0,1]\times[0,1]) := \{f : [0,1]^2 \to \mathbb{C} \mid \int_0^1 \int_0^1 |f(t,s)|^2 \,\mathrm{d}s\,\mathrm{d}t < \infty\}.$$

This space is a Hilbert space in the inner product

$$\langle x, y\rangle = \int_{t=0}^{t=1} \int_{s=0}^{s=1} x(t,s)\overline{y(t,s)}\,\mathrm{d}s\,\mathrm{d}t.$$

(a) Suppose $\{e_1(t), e_2(t), \dots\}$ is an orthonormal basis of $\mathscr{L}^2[0,1]$. Show that

$$\phi_{ij}(t,s) := e_i(t)e_j(s), \qquad i\in\mathbb{N},\ j\in\mathbb{N}$$

is an orthonormal set on $\mathscr{L}^2([0,1]\times[0,1])$.

(b) Show that $\langle x, \phi_{ij}\rangle = 0$ for all i, j implies that $x = 0$ (this might be a tough problem).

(c) Show that $\{\phi_{i,j}\}_{i,j\in\mathbb{N}}$ is an orthonormal basis of $\mathscr{L}^2([0,1]\times[0,1])$.

(d) Let $\phi_{m,n}(t,s) := \mathrm{e}^{\mathrm{i}2\pi(mt+ns)}$. Is $\{\phi_{m,n}\}_{m,n\in\mathbb{Z}}$ an orthonormal basis of $\mathscr{L}^2([0,1]\times[0,1])$?

1.56 Example 1.5.5 shows that the even functions $\mathscr{L}^2_{\text{even}}(\mathbb{R})$ in $\mathscr{L}^2(\mathbb{R})$ is a Hilbert space. Therefore, by Thm. 1.5.6, every $f\in\mathscr{L}^2(\mathbb{R})$ has a best approximation f_* in $\mathscr{L}^2_{\text{even}}(\mathbb{R})$. Show that

$$f_*(t) = \frac{f(t)+f(-t)}{2}$$

is this best approximation.

1.57 *Legendre polynomials & Rodrigues's formula.* Define the polynomials p_n as

$$p_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(t^2-1)}{\mathrm{d}t^n}, \qquad n = 0,1,2,\dots.$$

(a) Argue that p_n has degree n.

(b) Show that $\{p_n\}_{n\in\mathbb{N}}$ is an orthogonal sequence on $\mathscr{C}([-1,1];\mathbb{R})$ with respect to the standard inner product.
[Hint: derive first that $\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n \perp 1, t, \ldots, t^{n-1}$.]

(c) Show that $(n+1)p_n(t) = (2n+1)tp_n(t) - np_{n-1}(t)$ for all $n \geq 1$.

Chapter 2

Introduction to Signals and Convolutions

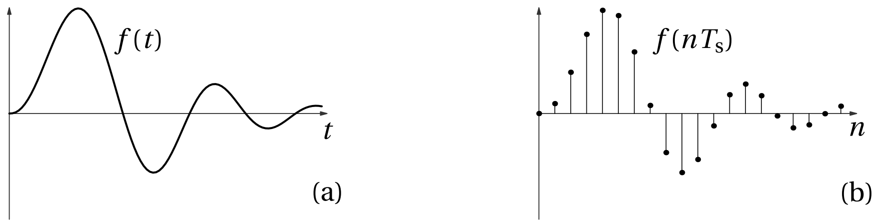


FIGURE 2.1: A continuous-time signal and a discrete-time signal

2.1 Continuous-time and discrete-time signals

Quantities that change with time, such as the voltage across a resistor or the outdoor temperature, may be regarded as functions $f(t)$ defined on $\mathbb{R}$ or on a subset of $\mathbb{R}$. Functions that depend on time are called *signals*. If a signal $f(t)$ is defined for all $t \in \mathbb{R}$ or for all t in some interval $(a, b) \subset \mathbb{R}$, then we say that $f(t)$ is a *continuous-time signal*. If $f(t)$ is defined only for a sequence of time instances

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots \qquad (t_n \in \mathbb{R}),$$

then $f(t)$ is said to be a *discrete-time signal.* A typical example of discrete-time signal is a signal obtained through sampling of a continuous-time signal. The sampled signal of a continuous-time signal $f(t)$ is the discrete-time signal $f(nT_s)$, $n \in \mathbb{Z}$, defined at integer multiples of the *sampling period* $T_s > 0$. Figure 2.1(a) shows the plot of a damped sinusoid (continuous-time) and Figure 2.1(b) shows the corresponding sampled signal (discrete-time) for a certain sampling period. In plots, discrete-time signals are represented by a series of stems on the real axis, such as in Figure 2.1(b).

In this chapter we introduce a couple of standard signals and provide a classification of signals. A second purpose of this chapter is to review some mathematical techniques that we will need in the rest of this course.

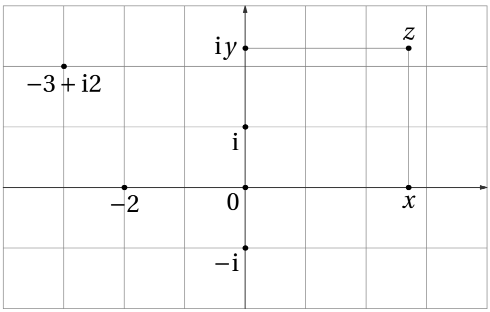


FIGURE 2.2: Several complex numbers in the complex plane

2.2 Review of complex numbers

Signals in the real world take real values. Such signals are therefore called *real* signals or *real-valued* signals. For a comprehensive theory of signals and their transforms, however, we need complex signals. This section recaps a couple of results about complex numbers. This is a concise recap and is not suited as a first contact with complex numbers.

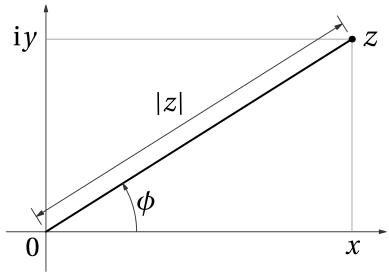


FIGURE 2.3: Modulus $|z|$ and argument ϕ of a point $z = x + \mathrm{i}y$ in the complex plane $\mathbb{C}$

A complex number z may be expressed in *canonical* form,

$$z = x + \mathrm{i}y, \qquad x, y \in \mathbb{R}.$$

Here i is the *imaginary unit* which is a number having the property that

$$\mathrm{i}^2 = -1.$$

The number x is known as the *real part* of $z = x + \mathrm{i}y$, and we denote it as

$$x = \mathrm{Re}(z).$$

Likewise the *imaginary part* of $x + \mathrm{i}y$ equals y, notation

$$y = \mathrm{Im}(z).$$

Note that the *imaginary* part is itself a *real* number (i.e. $y \in \mathbb{R}$).

A complex number z may also be expressed in *polar* form,

$$z = r\cos(\phi) + \mathrm{i}r\sin(\phi), \qquad \text{in which } r \geq 0 \text{ and } \phi \in \mathbb{R}. \tag{2.1}$$

Then $r = |z| := \sqrt{x^2 + y^2}$ is the *absolute value* or *modulus* of z, and $\phi = \arg(z)$ is the *argument* of z. The argument is the angle in radians that z makes with the positive real axis, and it is unique up to a multiple of 2π, see Fig. 2.3.

The set of complex numbers is denoted by $\mathbb{C}$ and it is commonly called the *complex plane*. The complex *conjugate* of $z \in \mathbb{C}$ will be denoted by z^* instead of the more common $\bar{z}$. So if $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ then $z^* = x - \mathrm{i}y$, see Fig. 2.4.

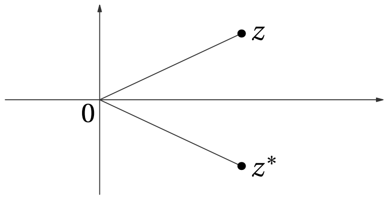


FIGURE 2.4: The complex conjugate of a $z \in \mathbb{C}$ is denoted z^* and is obtained by swapping the sign of its imaginary part

Complex sum and product in canonical form

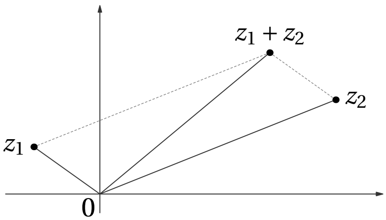


FIGURE 2.5: Addition of two complex numbers in the complex plane

Sum of complex numbers. The sum of two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ is obtained by adding both real parts and both imaginary parts,

$$z_1 + z_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2).$$

This corresponds to addition of vectors, see Fig. 2.5.

Product of complex numbers. To form the product of two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ we can work out the product using the rule that $\mathrm{i}^2 = -1$. So

$$\begin{aligned} z_1 z_2 &= (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) \\ &= x_1 x_2 + x_1 \mathrm{i}y_2 + \mathrm{i}y_1 x_2 + \underbrace{\mathrm{i}^2}_{-1} y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1). \end{aligned}$$

Quotient of complex numbers. There is useful trick to find the canonical form of a quotient z_1/z_2:

$$\begin{aligned} \frac{z_1}{z_2} = \frac{z_1}{z_2}\underbrace{\left(\frac{z_2^*}{z_2^*}\right)}_{1} &= \frac{(x_1 + \mathrm{i}y_1)(x_2 - \mathrm{i}y_2)}{(x_2 + \mathrm{i}y_2)(x_2 - \mathrm{i}y_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2 + \mathrm{i}(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + \mathrm{i}\left(\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right). \end{aligned}$$

The following five rules are now readily verified.

$$\begin{aligned} |z^*| &= |z|, \\ (z_1 + z_2)^* &= z_1^* + z_2^*, \\ (z_1 z_2)^* &= z_1^* z_2^*, \\ |z|^2 &= z z^*, \\ \arg(z^*) &= -\arg(z). \end{aligned} \tag{2.2}$$

Moreover, $z^* = z$ is a different way of saying that z is real.

The fundamental theorem of algebra

The familiar *abc*-formula states that the quadratic equation

$$az^2 + bz + c = 0, \qquad (a, b, c \in \mathbb{R}, a \neq 0)$$

has as solution

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It may happen that $b^2 - 4ac < 0$ in which case the quadratic equation has no real solutions. The formula however remains valid for complex numbers.

Example 2.2.1 (complex zeros). The two zeros of

$$z^2 + 2z + 3$$

are

$$z_{1,2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \mathrm{i}\sqrt{2}.$$

Both zeros are complex. $\square$

The example demonstrates that while second degree polynomials need not have zeros over the real numbers $z \in \mathbb{R}$, they *always* do have zeros over the complex numbers $z \in \mathbb{C}$. More generally there is the famous result:

Theorem 2.2.2 (Fundamental theorem of algebra). Every non-constant polynomial $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ ($p_i \in \mathbb{C}$) has a zero $z \in \mathbb{C}$. $\square$

A consequence is that every polynomial of degree $n \in \mathbb{N}$ has *precisely* n zeros in the sense that for any such polynomial $p(z)$, numbers $\lambda_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$ and $c \in \mathbb{C}, c \neq 0$ exist such that

$$p(z) = c \prod_{k=1}^{n} (z - \lambda_k).$$

Euler's formula

The exponential function e^z, with $z \in \mathbb{C}$, can be defined as the series

$$\mathrm{e}^z := 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots.$$

It is not hard to show that this series converges for every $z \in \mathbb{C}$. Without proof we state that the familiar rule $\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$ for real a and b remains valid for $a, b \in \mathbb{C}$. The exponential function is very interesting for imaginary $z = \mathrm{i}\phi$:

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}\phi} &= 1 + (\mathrm{i}\phi) + \frac{1}{2!}(\mathrm{i}\phi)^2 + \frac{1}{3!}(\mathrm{i}\phi)^3 + \frac{1}{4!}(\mathrm{i}\phi)^4 + + \frac{1}{5!}(\mathrm{i}\phi)^5 + \cdots \\
&= \left(1 + \frac{1}{2!}\mathrm{i}^2\phi^2 + \frac{1}{4!}\mathrm{i}^4\phi^4 + \cdots\right) \quad + \quad \left(\mathrm{i}\phi + \frac{1}{3!}\mathrm{i}^3\phi^3 + \frac{1}{5!}\mathrm{i}^5\phi^5 + \cdots\right) \\
&= \left(1 - \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 + \cdots\right) \quad + \quad \mathrm{i}\left(\phi - \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 + \cdots\right) \\
&= \cos(\phi) + \mathrm{i}\sin(\phi).
\end{aligned}
$$

In the last equality we used the real Taylor series of cosine and sine. This connection is known as *Euler's formula*:

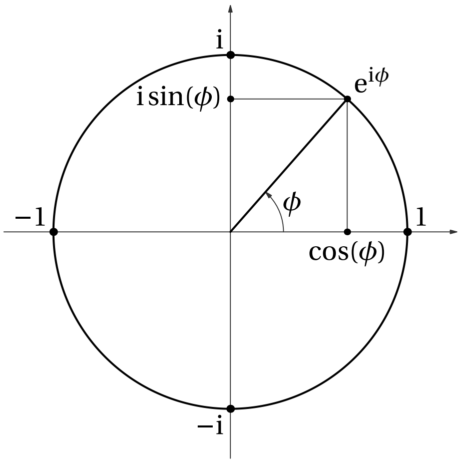


FIGURE 2.6: Euler's formula

Lemma 2.2.3 (Euler's formula). $\mathrm{e}^{\mathrm{i}\phi} = \cos(\phi) + \mathrm{i}\sin(\phi)$ for every $\phi \in \mathbb{R}$. $\square$

Since $\cos^2(\phi) + \sin^2(\phi) = 1$ we have that $|\mathrm{e}^{\mathrm{i}\phi}| = 1$. That is, $\mathrm{e}^{\mathrm{i}\phi}$ for every real ϕ is an element of the unit circle, see Fig. 2.6. As its real part is $\cos(\phi)$ and imaginary part is $\sin(\phi)$ it is immediate that $\arg(\mathrm{e}^{\mathrm{i}\phi}) = \phi$, see Fig. 2.6.

Euler's formula expresses an exponential as a sum of cosine and sine, but it may also used to express cosine and sine as sums of exponentials. Indeed from

$$\begin{aligned} e^{i\phi} &= \cos(\phi) + i\sin(\phi) \\ e^{-i\phi} &= \cos(\phi) - i\sin(\phi) \end{aligned}$$

we readily get that

$$\cos(\phi) = \operatorname{Re}(e^{i\phi}) = \frac{e^{i\phi} + e^{-i\phi}}{2}, \tag{2.3}$$

$$\sin(\phi) = \operatorname{Im}(e^{i\phi}) = \frac{e^{i\phi} - e^{-i\phi}}{2i}. \tag{2.4}$$

A byproduct of Euler's formula is that polar forms (2.1) can now be expressed more succinctly as

$$z = |z|e^{i\arg(z)}.$$

This, in turn, opens up another way of forming products or quotients of complex numbers.

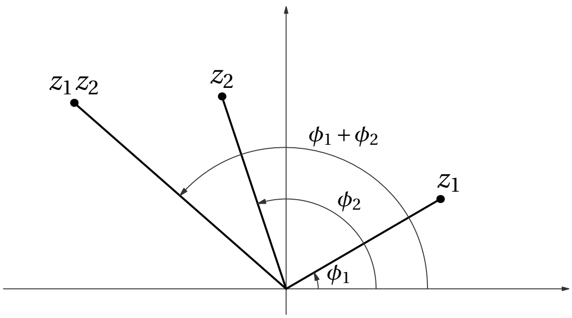


FIGURE 2.7: The argument of a products $z_1 z_2$ equals the *sum* of the arguments

For instance if z_1 and z_2 are given in polar form as

$$z_1 = r_1 e^{i\phi_1}, \qquad z_2 = r_2 e^{i\phi_2}$$

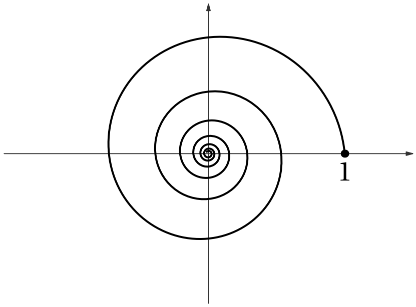


FIGURE 2.8: The complex exponential signal $\mathrm{e}^{(-\frac{1}{5}+2\mathrm{i})t}$ for $t \geq 0$ traces out this curve in the complex plane

then their product in polar form simply is

$$z_1 z_2 = r_1 \mathrm{e}^{\mathrm{i}\phi_1} r_2 \mathrm{e}^{\mathrm{i}\phi_2} = (r_1 r_2)\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)}.$$

The modulus of the product hence is the product of the moduli; on the other hand, the argument of the product is the *sum* of the arguments. The interpretation is that taking products in the complex plane means rotation about the origin. Similarly we have for quotients that

$$\frac{z_1}{z_2} = \frac{r_1 \mathrm{e}^{\mathrm{i}\phi_1}}{r_2 \mathrm{e}^{\mathrm{i}\phi_2}} = r_1 \mathrm{e}^{\mathrm{i}\phi_1} \frac{1}{r_2} \mathrm{e}^{-\mathrm{i}\phi_2} = \frac{r_1}{r_2} \mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)}.$$

2.3 Complex valued signals

The rules for sum, product and quotient also apply to complex-valued signals. For example, every complex signal $f(t)$ can be expressed as

$$f(t) = A(t)\mathrm{e}^{\mathrm{i}\phi(t)},$$

where now modulus $A(t) = |f(t)|$ and argument $\phi(t) = \arg(f(t))$ may depend on t.

Example 2.3.1 (Exponential signals).

- An important example of a complex-valued signal is the harmonic signal $f(t) = \mathrm{e}^{\mathrm{i}\omega t}$ (see also Section 2.4), where ω is a real-valued constant. The modulus $|f(t)|$ is equal to 1 for all t since

$$|\mathrm{e}^{\mathrm{i}\omega t}| = |\cos(\omega t) + \mathrm{i}\sin(\omega t)| = \sqrt{\cos^2(\omega t) + \sin^2(\omega t)} = 1 \quad \forall t \in \mathbb{R}.$$

- Let $a = u + \mathrm{i}v \in \mathbb{C},\ a \neq 0$ and consider $f(t) = \mathrm{e}^{-at}$. Then

$$\begin{aligned}
|f(t)| &= |\mathrm{e}^{-(u+\mathrm{i}v)t}| = |\mathrm{e}^{-ut}\mathrm{e}^{-\mathrm{i}vt}| \\
&= |\mathrm{e}^{-ut}||\mathrm{e}^{-\mathrm{i}vt}| = \mathrm{e}^{-ut} = \mathrm{e}^{-(\mathrm{Re}(a))t}, \\
\arg(f(t)) &= \arg(\mathrm{e}^{-(u+\mathrm{i}v)t}) = \arg(\mathrm{e}^{-ut}\mathrm{e}^{-\mathrm{i}vt}) \\
&= -vt = -\mathrm{Im}(a)t, \\
f^*(t) &= (\mathrm{e}^{-(u+\mathrm{i}v)t})^* = (\mathrm{e}^{-ut}\mathrm{e}^{-\mathrm{i}vt})^* \\
&= \mathrm{e}^{-ut}\mathrm{e}^{\mathrm{i}vt} = \mathrm{e}^{-(u-\mathrm{i}v)t} = \mathrm{e}^{-a^*t}.
\end{aligned}$$

- Let $a \in \mathbb{C}$ and suppose that $\mathrm{Re}(a) > 0$. Then

$$\lim_{t\to\infty} \mathrm{e}^{-at} = 0.$$

This is because its modulus $|\mathrm{e}^{-at}| = \mathrm{e}^{-\mathrm{Re}(a)t}$ decays exponentially to zero for $\mathrm{Re}(a) > 0$ as $t \to \infty$. As a consequence we have for every $\mathrm{Re}(a) > 0$ that

$$\int_0^\infty \mathrm{e}^{-at}\,\mathrm{d}t = \lim_{M\to\infty}\left[\frac{\mathrm{e}^{-at}}{-a}\right]_0^M = \frac{1}{a}. \tag{2.5}$$

- As t increases, the function

$$f(t) = \mathrm{e}^{(-\frac{1}{5}+2\mathrm{i})t}$$

traces out a curve in the complex plane that shrinks to zero (because $\mathrm{Re}(-\frac{1}{5}+2\mathrm{i}) < 0$) and that rotates counter-clockwise around the origin (because $\mathrm{Im}(-\frac{1}{5}+2\mathrm{i}) = 2 > 0$). See Fig. 2.8.

□

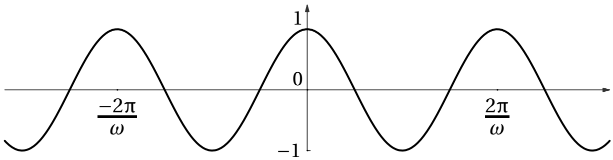


FIGURE 2.9: Graph of $\cos(\omega t)$

2.4 Periodic signals

A signal $f(t)$ is *periodic* if a $T > 0$ exists such that $f(t+T) = f(t)$ for all $t \in \mathbb{R}$. In that case T is known as a *period* of $f(t)$. Be aware, however, that if T is a period then so is $2T$ and $3T$ etcetera. A signal will be referred to as T*-periodic* if it is periodic with period T.

An important class of real-valued periodic signals are the *sinusoids* or *real harmonic signals.*

Definition 2.4.1 (Sinusoids). A real-valued signal $f(t)$ that can be written as

$$f(t) = A\cos(\omega t + \phi), \qquad A > 0,\ \phi \in \mathbb{R},\ t \in \mathbb{R}$$

is called a *sinusoid* or *real harmonic signal.* Then A is the *amplitude*, ω the *angular frequency* and ϕ the *initial phase* of the signal $f(t)$. □

It is easy to verify that a period of such sinusoids is $T = 2\pi/\omega$. If the time t expresses seconds, then the angular frequency ω is in units of radians per second (rad/s) and $\omega/(2\pi)$ is called the *frequency* and is in units of hertz (Hz). One hertz is one cycle per second. Similarly for the complex case we define the following.

Definition 2.4.2 (Harmonic signals). A signal $f(t)$ that can be written in the form

$$f(t) = c\mathrm{e}^{\mathrm{i}\omega t}$$

with $c \in \mathbb{C}$ and $\omega \in \mathbb{R}$, is called a *(complex) harmonic signal* with amplitude $|c|$, angular frequency ω and initial phase $\phi := \arg(c)$. □

The connection between a real and a complex harmonic signal lies in Euler's formula $\mathrm{e}^{\mathrm{i}\omega t} = \cos(\omega t) + \mathrm{i}\sin(\omega t)$. Every linear combination of sinusoids with the same frequency, is again a sinusoid. Indeed, if $a, b, \omega \in \mathbb{R}$, then

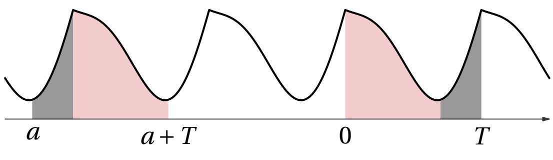
$$\begin{aligned}
a\cos(\omega t) + b\sin(\omega t) &= a\frac{\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}}{2} + b\frac{\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}}{2\mathrm{i}} \\
&= \frac{a - \mathrm{i}b}{2}\mathrm{e}^{\mathrm{i}\omega t} + \frac{a + \mathrm{i}b}{2}\mathrm{e}^{-\mathrm{i}\omega t} \\
&= \{\text{write } a - \mathrm{i}b \text{ in polar form } a - \mathrm{i}b = A\mathrm{e}^{\mathrm{i}\phi}\} \\
&= \frac{1}{2}A\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\omega t} + \frac{1}{2}A\mathrm{e}^{-\mathrm{i}\phi}\mathrm{e}^{-\mathrm{i}\omega t} \\
&= A\frac{\mathrm{e}^{\mathrm{i}(\omega t+\phi)} + \mathrm{e}^{-\mathrm{i}(\omega t+\phi)}}{2} \\
&= A\cos(\omega t + \phi).
\end{aligned}$$

Note that complex harmonic signals are allowed to have a negative frequency. In the above we assumed that both sinusoids have the same frequency. That is crucial. If the frequencies differ, then the sum need not even be periodic. For example $f(t) = \cos(\pi t) + \cos(t)$ is not periodic! In Chapter 3 we will see that practically every T-periodic signal can be written as a sum (possibly an infinite sum) of harmonic signals whose angular frequencies ω are integer multiples of $2\pi/T$.

The next result expresses that the integral of a periodic signal over one period does not depend on were the interval over which is integrated is situated. This is a minor and intuitive result; it will be of use later.

Lemma 2.4.3. Suppose that $f(t)$ is integrable on $[0, T]$ and that $f(t)$ is periodic with period $T > 0$. Then for every $a \in \mathbb{R}$, there holds

$$\int_a^{a+T} f(t)\,\mathrm{d}t = \int_0^T f(t)\,\mathrm{d}t.$$



***Proof*.** We write

$$\int_a^{a+T} f(t)\,\mathrm{d}t = \int_a^0 f(t)\,\mathrm{d}t + \int_0^T f(t)\,\mathrm{d}t + \int_T^{a+T} f(t)\,\mathrm{d}t.$$

The result now follows because the first and third integral on the right-hand side cancel each other, which follows by substitution $t = \tau + T$,

$$\int_T^{a+T} f(t)\,\mathrm{d}t = \int_0^a f(\tau + T)\,\mathrm{d}\tau = \int_0^a f(\tau)\,\mathrm{d}\tau = -\int_a^0 f(\tau)\,\mathrm{d}\tau. \tag{2.6}$$

■

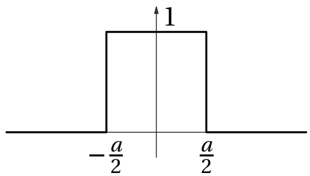
2.5 Standard signals

The following definition introduces four aperiodic signals that play a prominent role in in this course.

Definition 2.5.1 (Four standard signals).

- For a given $a > 0$ the *rectangular pulse* $\operatorname{rect}_a(t)$ is defined as

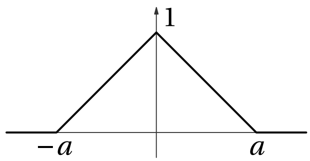
$$\operatorname{rect}_a(t) = \begin{cases} 0 & \text{if } |t| > a/2, \\ \frac{1}{2} & \text{if } |t| = a/2 \\ 1 & \text{if } |t| < a/2. \end{cases}$$



  In plots we normally use just one continuous line to represent the graph of the rectangular pulse, even though the function is discontinuous (at $t = \pm a/2$).

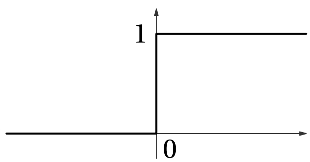
- For a given $a > 0$ the *triangular pulse* $\operatorname{trian}_a(t)$ is defined as

$$\operatorname{trian}_a(t) = \begin{cases} 0 & \text{if } |t| \geq a, \\ 1 - \dfrac{|t|}{a} & \text{if } |t| < a. \end{cases}$$



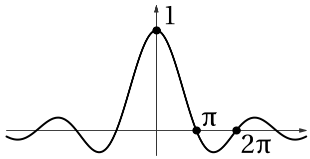
- The *unit step* $\mathbb{1}(t)$ is defined as

$$\mathbb{1}(t) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{2} & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$$



- The function sinc(t) (pronounced: "sink") is defined as

$$\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(t)}{t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$



□

Note that at the jump discontinuities the function value is here taken to be the average of the function immediately before and after the jump, $f(t) = (f(t^-)+f(t^+))/2$. This choice is arbitrary but circumvents certain technicalities when Fourier series and Fourier integrals are considered.

Warning: In signal processing (and in MATLAB) the sinc function is defined as sinc(πt). It is a scaled version of our sinc, and it is zero at the nonzero integers (as opposed to the nonzero integer multiples of π).

We end this section with the definition of the class of piecewise smooth signals. These are signals that we will keep coming back to. The piecewise smooth signals encompass practically all continuous-time signals one is likely to come across in practice.

Definition 2.5.2 (Piecewise smooth signals). A signal $f(t)$ is *piecewise smooth* on a finite interval $[a, b]$ if a finite partition, $a = c_0 < c_1 < c_2 < \cdots < c_m = b$, of the interval exists such that

1. $f(t)$ is continuously differentiable at every $t \in (c_i, c_{i+1})$, $(i = 0, \ldots, m-1)$.

2. At every $c_i, (i = 1, 2, \ldots, m-1)$ the following limits exist

$$f(c_i^+) := \lim_{h\downarrow 0} f(c_i + h), \qquad f'(c_i^+) := \lim_{h\downarrow 0} f'(c_i + h),$$
$$f(c_i^-) := \lim_{h\downarrow 0} f(c_i - h), \qquad f'(c_i^-) := \lim_{h\downarrow 0} f'(c_i - h),$$

3. $f(a^+)$, $f(b^-)$, $f'(a^+)$ and $f'(b^-)$ exist.

A signal $f(t)$ defined for all $t \in (-\infty, \infty)$ is said to be *piecewise smooth* if it is piecewise smooth on every finite interval $[a, b]$. The points c_i are sometimes called the *points of discontinuity*, see Fig. 2.10. □

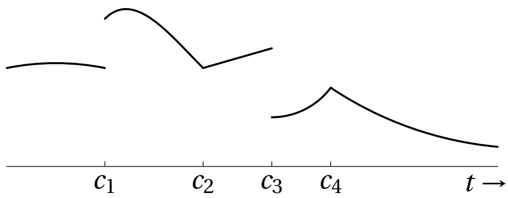


FIGURE 2.10: An example of a piecewise smooth signal

2.6 Energy and power

It is customary in signal analysis to use "energy" instead of norm:

Definition 2.6.1 (Energy content). The *energy (content)* E_f of a signal $f(t)$ is defined as

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t.$$

□

If $E_f < \infty$ (finite energy content), then the signal is said to be an *energy signal*. The rectangular and triangular pulses are examples of energy signals. Sinusoids and harmonic signals are not. For example, the harmonic signal $f(t) = c\mathrm{e}^{\mathrm{i}\omega_0 t}$ satisfies $|f(t)| = |c|$ and, hence, $E_f = \infty$. For a signal $f(t)$ to have a finite energy content it is necessary that $\lim_{t\to\infty} \int_a^t |f(\tau)|^2 \, \mathrm{d}\tau$ converges. Consequently, signals like sinusoids, periodic signals and the unit step and many others, are not energy signals. In such cases it is customary to consider instead its averaged energy *per unit time*, i.e., to look at its (averaged) power.

Definition 2.6.2 (Power). The *power* P_f of a signal $f(t)$ is defined as

$$P_f = \lim_{M\to\infty} \frac{1}{M} \int_{-M/2}^{M/2} |f(t)|^2 \, \mathrm{d}t.$$

□

Signals that have finite power are called *power signals*. The power of a bounded T-periodic signal $f(t)$ is finite, and you may wish to verify that its power then equals the average energy over one period,

$$P_f = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, \mathrm{d}t.$$

Example 2.6.3 (Power of a sinusoid). The power of the sinusoid $f(t) = A\cos(\omega_0 t + \phi)$ with period $T = 2\pi/\omega_0$, is

$$P_f = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} A^2\cos^2(\omega_0 t + \phi)\,\mathrm{d}t = \{\text{substitute } x = \omega_0 t\} = \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos^2(x+\phi)\,\mathrm{d}x = \frac{A^2}{2}.$$

The last equality uses that $\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$ which shows that the average of $\cos^2(\alpha)$ over a period is $\frac{1}{2}$. This example assumes that $\omega_0 > 0$. □

2.7 Convolutions

Loosely speaking a *convolution* is a linear combination of shifted copies of a signal. For instance

$$3g(t) - 41g(t-1) + 501g(t+100) \tag{2.7}$$

is an example of a convolution of a signal $g(t)$. Note that the convolution is itself a signal. More generally expressions like

$$\sum_\tau f_\tau g(t-\tau), \qquad f_\tau \in \mathbb{R} \tag{2.8}$$

are known as convolutions, and so is its integral version which we take to be its definition.

Definition 2.7.1 (Convolution). The *convolution* or *convolution product* of two signals $f(t)$ and $g(t)$ is denoted as $(f * g)(t)$ and is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau.$$

□

It is an interesting fact that convolution products commute,

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau = \{v = t-\tau\} = \int_{-\infty}^{\infty} g(v)f(t-v)\,\mathrm{d}v = (g * f)(t).$$

In the following section, where we introduce the delta function, we will see that the sum (2.8) is a convolution as well (i.e. we can write it as this integral expression). Convolutions are very common in applications, and are, for instance, useful if we want to remove noise from signals, detect edges in pictures, soften pictures, etcetera.

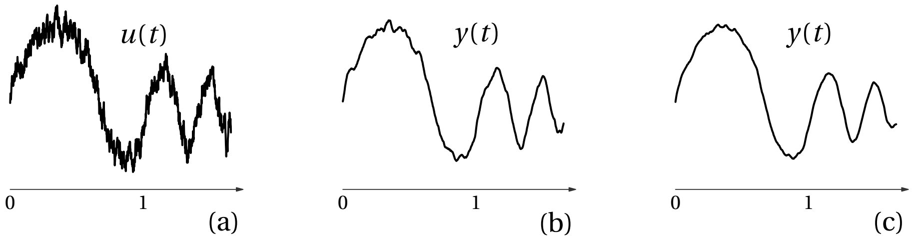


FIGURE 2.11: (a): A noisy signal; (b) averaged with $P = 0.05$; (c) averaged with $P = 0.1$

Example 2.7.2 (Sliding window averaging & noise reduction). For a given signal $f(t)$ we construct the signal $f_{\text{swa}}(t)$ by averaging $f(t)$ around t over an interval of a fixed length P, i.e., we consider

$$f_{\text{swa}}(t) = \frac{1}{P}\int_{t-P/2}^{t+P/2} f(\tau)\,\mathrm{d}\tau.$$

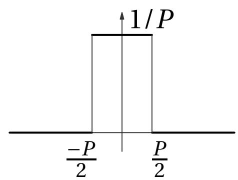
Averaging $f(t)$ this way filters out highly fluctuating noise. It is to be expected then, that $f_{\text{swa}}(t)$ is somewhat smoother than $f(t)$, but as long as P is not too large the graph of the averaged $f_{\text{swa}}(t)$ should retain roughly the same shape as the graph of $f(t)$. Figure 2.11(a) shows an example of a noisy signal $f(t)$. Figure 2.11(b) shows $f_{\text{swa}}(t)$ for the case that $P = 0.05$. In plot (c) of that figure the average was taken over a wider interval ($P = 0.1$) and as expected the plot is smoother than the one in (b).

The signal f_{swa} can be written as the convolution of f with a suitable function g:

$$f_{\text{swa}}(t) = \frac{1}{P}\int_{t-P/2}^{t+P/2} f(\tau)\,\mathrm{d}\tau = \{v = t-\tau\} = \frac{1}{P}\int_{-P/2}^{P/2} f(t-v)\,\mathrm{d}v = (f * g)(t),$$

for

$$g(t) = \frac{1}{P}\operatorname{rect}_P(t)$$



□

In order to determine a convolution it often is insightful to make plots of $f(\tau)$ and $g(t-\tau)$ as functions of τ. We demonstrate this on an example.

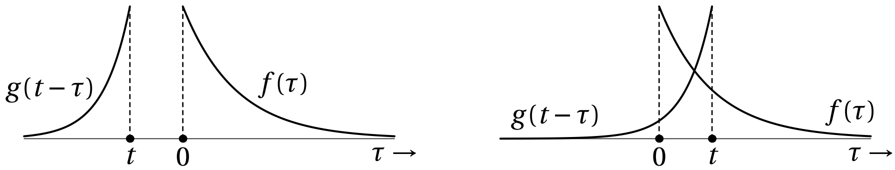


FIGURE 2.12: $f(\tau) = \mathrm{e}^{-\tau}\,\mathbb{1}(\tau)$ and $g(t-\tau) = \mathrm{e}^{-2(t-\tau)}\,\mathbb{1}(t-\tau)$ for $t < 0$ (left) and $t > 0$ (right)

Example 2.7.3 (Visualizing convolution). Let $f(t) = \mathrm{e}^{-t}\,\mathbb{1}(t)$ and $g(t) = \mathrm{e}^{-2t}\,\mathbb{1}(t)$. Convolving $f(t)$ and $g(t)$ amounts to integration over all τ of the product of $f(\tau)$ and $g(t-\tau)$. To compute the convolution it is helpful to fix t and then examine the plots of $f(\tau)$ and $g(t-\tau)$ as functions of τ. Figure 2.12 shows two cases; one for $t < 0$ and one for $t > 0$. Clearly for $t < 0$ the nonzero parts of the functions $f(\tau)$ and $g(t-\tau)$ do not overlap, hence

$$t < 0 \quad \Longrightarrow \quad (f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau = 0.$$

If $t > 0$ then there is an overlap on the time interval $[0, t]$. Therefore

$$\begin{aligned} t > 0 \quad \Longrightarrow \quad (f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t \mathrm{e}^{-\tau}\mathrm{e}^{-2(t-\tau)}\,\mathrm{d}\tau \\ &= \int_0^t \mathrm{e}^{-2t}\mathrm{e}^{\tau}\,\mathrm{d}\tau = \mathrm{e}^{-2t}\big[\mathrm{e}^{\tau}\big]_0^t = \mathrm{e}^{-t} - \mathrm{e}^{-2t}. \end{aligned}$$

With the unit step function we may combine the two cases $t < 0$ and $t > 0$:

$$(f * g)(t) = (\mathrm{e}^{-t} - \mathrm{e}^{-2t})\,\mathbb{1}(t) \qquad \forall t.$$

□

Sufficient for the existence of $(f * g)(t)$ is that $f(t)$ is bounded while $g(t)$ is absolutely integrable, or the other way around. Another important class of signals for which convolutions exist is the class of the so-called *causal* signals.

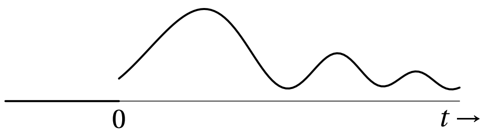


FIGURE 2.13: An example of a causal signal

Definition 2.7.4 (Causal signal). A signal $f(t)$ is *causal* if $f(t) = 0$ for all $t < 0$. (See Figure 2.13.) □

A signal of the form $f(t)\,\mathbb{1}(t)$ is causal because $\mathbb{1}(t) = 0$ for $t < 0$. If $f(t)$ and $g(t)$ are causal, then

$$
\begin{aligned}
(f * g)(t) = \big((f\,\mathbb{1}) * (g\,\mathbb{1})\big)(t) &= \int_{-\infty}^{\infty} f(\tau)\,\mathbb{1}(\tau) g(t-\tau)\,\mathbb{1}(t-\tau)\,\mathrm{d}\tau \\
&= \int_{0}^{\infty} f(\tau) g(t-\tau)\,\mathbb{1}(t-\tau)\,\mathrm{d}\tau \\
&= \begin{cases} 0 & \text{if } t < 0, \\ \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau & \text{if } t \geq 0, \end{cases} \\
&= \left(\int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau\right) \mathbb{1}(t).
\end{aligned}
$$

The convolution of two causal signals apparently is itself causal, and since for each t the integration above is over a finite interval $[0, t]$ it follows that the convolution exists for every t and every piecewise smooth $f(t)$ and $g(t)$.

Example 2.7.5 (Convolution with the unit step). Convolution with the unit step amounts to integration:

$$
(f * \mathbb{1})(t) = \int_{-\infty}^{\infty} f(\tau)\,\mathbb{1}(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau.
$$

□

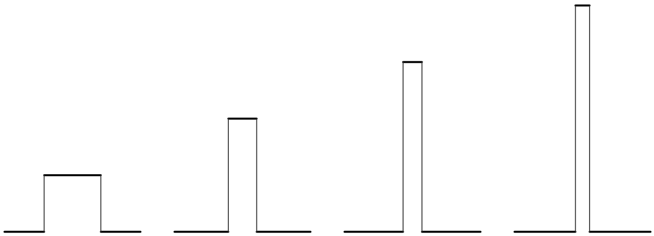
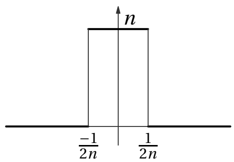


FIGURE 2.14: A series of $r_n(t)$ for $n=1$, $n=2$, $n=3$ and $n=4$

2.8 The delta function

In applications we often encounter signals that have a very short duration but nevertheless have a definite impact. An example is when you punch someone in the face (try it, it's fun). Such signals are called impulses. The standard impulse is the so-called Dirac *delta function* also known as the *unit impulse*. The delta function $\delta(t)$ is introduced as the limit as $n \to \infty$ of

$$r_n(t) := \begin{cases} n & \text{if } |t| < \frac{1}{2n} \\ 0 & \text{if } |t| > \frac{1}{2n} \end{cases} \tag{2.9}$$



As n goes to infinity, the rectangular pulses $r_n(t)$ become spikier and spikier, with their spike around $t=0$, see Figure 2.14. However, the area enclosed by the spike and the x-axis, $\int_{-\infty}^{\infty} r_n(t)\,\mathrm{d}t$, equals 1 independent of n. We now naively define the delta function $\delta(t)$ as the limit

$$\delta(t) = \lim_{n\to\infty} r_n(t) \tag{2.10}$$

and we think of the delta function as a "function" that is zero everywhere except at $t=0$ where it has a spike so large that

$$\int_{0^-}^{0^+} \delta(t)\,\mathrm{d}t = 1.$$

The delta function is usually depicted as done in Figure 2.15, i.e., depicted by the zero function with a fat arrow pointing upwards at $t = 0$. The idea to see the delta function as a spike in this sense is helpful, but mathematically it is far from sound. After all,

$$\lim_{n\to\infty} r_n(t) = \begin{cases} 0 & \text{if } t \neq 0, \\ \infty & \text{if } t = 0 \end{cases}$$

and the integral of a function that is zero everywhere except for one point, is zero. Still, for many applications with delta functions it is enough to see $\delta(t)$ as the limit (2.10). Many tentative problems in calculations involving delta functions may be avoided if we stick to the rule *the-last-limit-you-take*. By that is meant that in calculations with delta functions, first $\delta(t)$ is replaced with the well defined $r_n(t)$ and only at the very last step the limit $n \to \infty$ is taken. With this rule, for example, the following quintessential property of the delta function may be derived.

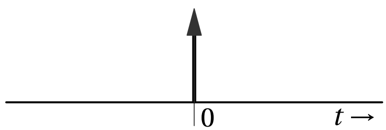


FIGURE 2.15: The delta function $\delta(t)$

Lemma 2.8.1 (delta function). If $f(t)$ is continuous at $t = 0$, then

$$\int_{-\infty}^{\infty} \delta(t) f(t)\,\mathrm{d}t = f(0).$$

***Proof*.** First replace $\delta(t)$ with $r_n(t)$,

$$\int_{-\infty}^{\infty} r_n(t) f(t)\,\mathrm{d}t = n \int_{-1/(2n)}^{1/(2n)} f(t)\,\mathrm{d}t.$$

The integral is bounded from above by

$$n \int_{-1/(2n)}^{1/(2n)} \max_{t\in[-\frac{1}{2n},\frac{1}{2n}]} f(t)\,\mathrm{d}t = \max_{t\in[-\frac{1}{2n},\frac{1}{2n}]} f(t)$$

and bounded from below by

$$n\int_{-1/(2n)}^{1/(2n)} \min_{t\in[-\frac{1}{2n},\frac{1}{2n}]} f(t)\,\mathrm{d}t = \min_{t\in[-\frac{1}{2n},\frac{1}{2n}]} f(t).$$

Now as $n\to\infty$ the interval $[-\frac{1}{2n},\frac{1}{2n}]$ shrinks to zero and the two bounds converge to $f(0)$. ■

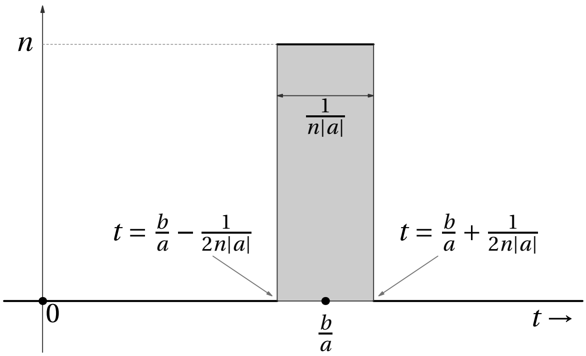


FIGURE 2.16: Shifted and scaled $r_n(t)$

Properties of the delta function

Delta functions can be added, they can be multiplied with regular functions, they can be integrated etcetera. In this subsection we review the more important properties and rules for delta functions.

The scaled and shifted delta function $\delta(at-b)$ we take to be defined as $\delta(at-b) = \lim_{n\to\infty} r_n(at-b)$. For $t=b/a$ the argument $at-b$ is zero, so $r_n(at-b)$ as a function of t is centered around $t=b/a$, see Fig. 2.16. This is very much like a shifted copy of $r_n(t)$ with the difference that the spike does not have a unit area. The width of the spike of $r_n(at-b)$ is easily seen to be $1/|an|$, so the area of the spike is $1/|a|$. In the limit as $n\to\infty$ the spike therefore approaches $1/|a|$ times the delta function that has its spike at $t=b/a$:

$$\delta(at-b) = \frac{1}{|a|}\delta(t-\frac{b}{a}). \tag{2.11}$$

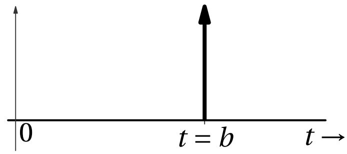


FIGURE 2.17: Shifted delta function $\delta(t-b)$

We can now generalize Lemma 2.8.1. If $f(t)$ is continuous, then

$$\int_{-\infty}^{\infty} \delta(at-b) f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(t-\frac{b}{a}) f(t)\,\mathrm{d}t \quad (\text{let } \tau = t - \frac{b}{a})$$
$$= \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau) f(\tau + \frac{b}{a})\,\mathrm{d}\tau = \frac{1}{|a|} f(\frac{b}{a}).$$

An immediate special case is that

$$\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t = f(b), \qquad (\text{if } f(t) \text{ is continuous at } t=b). \tag{2.12}$$

This property is known as the *sifting property* of the delta function. It is the property that out of all values $\{f(t) \mid t \in \mathbb{R}\}$ that $f(t)$ can take, the value at $t=b$ is sifted out. Note that $\delta(t-b)$ has its spike at $t=b$, see Figure 2.17, therefore another way to interpret the sifting property is that it says that $\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t$ equals $f(t)$ at that t where $\delta(t-b)$ has its spike. It is also possible to determine the convolution product $(f * \delta)(t)$ of a signal $f(t)$ and the delta function $\delta(t)$.

$$(f*\delta)(t) = \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau)\,\mathrm{d}\tau = \{\text{sifting property}\} = f(t). \tag{2.13}$$

Here we used the fact that $\delta(t-\tau) = \delta(\tau - t)$ as a function of τ has its spike at $\tau = t$. A final useful property to have is the following.

Lemma 2.8.2 (Products with delta functions). If $f(t)$ is continuous at $t=b$, then

$$f(t)\delta(t-b) = f(b)\delta(t-b). \tag{2.14}$$

Proof (idea). This is another instance were we will use the rule *the-last-limit-you-take*. The function $f(t)r_n(t-b)$ is zero for all $|t-b| > \frac{1}{2n}$. On the interval $[b-\frac{1}{2n}, b+\frac{1}{2n}]$ it generally has a spike. Since $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)r_n(t-b)\,\mathrm{d}t = f(b)$ it means that the area of this spike approaches $f(b)$ as $n \to \infty$, so that $f(t)r_n(t)$ approaches $f(b)$ times the delta function $\delta(t-b)$. ■

Example 2.8.3.

1. $t\delta(t) = 0 \times \delta(t) = 0$, i.e. the zero signal.

2. $\mathrm{e}^{\mathrm{i}\omega_0 t}\delta(t-b) = \mathrm{e}^{\mathrm{i}\omega_0 b}\delta(t-b)$.

3. $\delta(-t) = \delta(t)$. This is because of the scaling property for $a = -1$ (Table 2.1).

4. $\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = (\mathbb{1} * \delta)(t) = \mathbb{1}(t)$ for all $t \neq 0$. So the step is an indefinite integral of $\delta(t)$.

□

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b)f(t)\,\mathrm{d}t = f(b)$	$f(t)$ continuous at $t = b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t = b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a\rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

TABLE 2.1: Properties and rules of calculus for the delta function

Example 2.8.4 (Convolution with delta functions). Let

$$f(t) = 3\delta(t) - 41\delta(t-1) + 501\delta(t+100)$$

Then using the sifting property of delta functions we get that

$$
\begin{aligned}
(f * g)(t) &= \int_{-\infty}^{\infty} [3\delta(\tau) - 41\delta(\tau - 1) + 501\delta(\tau + 100)]\, g(t-\tau)\,\mathrm{d}\tau \\
&= 3g(t) - 41g(t-1) + 501g(t+100).
\end{aligned}
$$

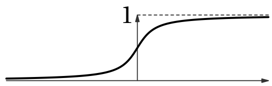
This example demonstrates that linear combinations of shifted copies of $g(t)$ are indeed convolutions as claimed in the previous section. □

Generalized derivatives

Now that we have the delta function at our disposal we have a way of differentiating discontinuous functions. Really! First an informal example.

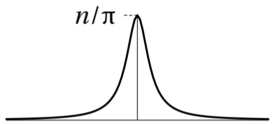
Example 2.8.5. The unit step $\mathbb{1}(t)$ may be seen as the limit as $n \to \infty$ of

$$
f_n(t) = \frac{1}{2} + \frac{\arctan(nt)}{\pi}.
$$



(Convince yourself of this.) A clear difference between these $f_n(t)$ and their limit $\mathbb{1}(t)$ is that all $f_n(t)$ are differentiable in the classical sense while their limit $\mathbb{1}(t)$ obviously is not. Let us examine the derivatives of $f_n(t)$. These are

$$
f_n^{(1)}(t) = \frac{1}{\pi}\frac{n}{1+(nt)^2}.
$$



And to what does it seem to converge as $n \to \infty$? To a function that is zero everywhere except at zero where it is so huge that if we integrate over this peak we get

$$
\int_{-\infty}^{\infty} f_n^{(1)}(t)\,\mathrm{d}t = f_n(\infty) - f_n(-\infty) = 1.
$$

In other words the derivative $f_n^{(1)}(t)$ converges to the delta function as $n \to \infty$. Now it is tempting to claim that the derivative of $\mathbb{1}(t)$ is $\delta(t)$. Weird. □

It is actually not so difficult to make the claim of this example more precise. We simply redefine slightly what we mean with *derivative*. If two continuous signals $f(t)$ and $g(t)$ satisfy

$$f(t) = f(a) + \int_a^t g(\tau)\,\mathrm{d}\tau$$

for a certain $a \in \mathbb{R}$ or $a = -\infty$, then we know that $f(t)$ is continuously differentiable with derivative $f'(t) = g(t)$. The integral equality, however, may also hold for non-differentiable functions $f(t)$, and, even in that case we will say that $f(t)$ is *differentiable* and that $g(t)$ is its *(generalized) derivative*, notation: $f'(t) = g(t)$. This generalization allows us to differentiate discontinuous functions.

Example 2.8.6.

- Let $f(t) = |t|$ and consider the signal $\operatorname{sgn}(t)$ defined as

  $$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

  The function $\operatorname{sgn}(t)$ is the generalized derivative of $f(t) = |t|$, because

  $$|t| = \int_0^t \operatorname{sgn}(\tau)\,\mathrm{d}\tau.$$

- Let $f(t) = \mathbb{1}(t)$. We showed earlier that

  $$\mathbb{1}(t) = \int_{-\infty}^t \delta(\tau)\,\mathrm{d}\tau.$$

  So $\delta(t)$ is the generalized derivative of $\mathbb{1}(t)$. Yes, we can differentiate discontinuous functions now.

□

It may be shown that the product rule and chain rule of differentiation remain valid for derivatives in the generalized sense.

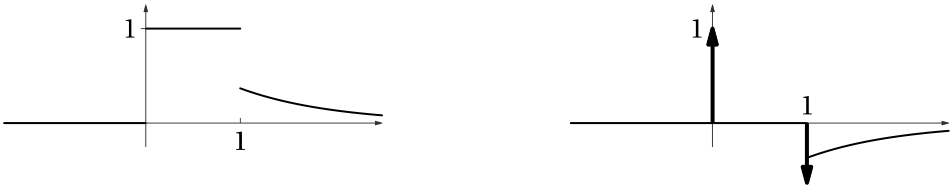


FIGURE 2.18: $f(t) = \mathbb{1}(t) - \mathbb{1}(t-1) + e^{-\alpha t}\,\mathbb{1}(t-1)$ and its generalized derivative $f'(t)$

Example 2.8.7. Consider the signal $f(t)$ depicted in Figure 2.18. It is the signal

$$f(t) = \mathbb{1}(t) - \mathbb{1}(t-1) + e^{-\alpha t}\,\mathbb{1}(t-1).$$

Its generalized derivative $f'(t)$ equals

$$\begin{aligned} f'(t) &= \delta(t) - \delta(t-1) - \alpha e^{-\alpha t}\,\mathbb{1}(t-1) + e^{-\alpha t}\delta(t-1) \\ &= \delta(t) - \delta(t-1) - \alpha e^{-\alpha t}\,\mathbb{1}(t-1) + e^{-\alpha}\delta(t-1) \\ &= \delta(t) - (1 - e^{-\alpha})\delta(t-1) - \alpha e^{-\alpha t}\,\mathbb{1}(t-1). \end{aligned}$$

The generalized derivative $f'(t)$ is depicted in Figure 2.18(b). □

We note that differentiation of a function at a point of discontinuity results in a delta function. If $f(t)$ at $t = b$ jumps from $f(b^-)$ to $f(b^+)$, then in the generalized derivative a delta function $(f(b^+) - f(b^-))\delta(t-b)$ shows up.

Example 2.8.8. The functions $y(t) = e^{-t}\,\mathbb{1}(t)$ and $u(t) = \mathbb{1}(t)$ are a (generalized) solution of the differential equation

$$y'(t) + y(t) = u'(t)$$

because

$$y'(t) + y(t) = \left(-e^{-t}\,\mathbb{1}(t) + e^{-t}\delta(t)\right) + e^{-t}\,\mathbb{1}(t) = e^{-t}\delta(t) = \delta(t) = u'(t).$$

□

2.9 Exercises

2.1 Plot z_1 and z_2 in the complex plane, and then compute and plot both sum $z_1 + z_2$ and product $z_1 z_2$ of

(a) $z_1 = 1 + \mathrm{i}$ and $z_2 = 2 + \mathrm{i}2$,

(b) $z_1 = a + \mathrm{i}b$ and $z_2 = b + \mathrm{i}a$ for arbitrary $a, b \in \mathbb{R}$.

2.2 Verify the equalities in Eqn. (2.2).

2.3 Show that $\lim_{t\to 0} \dfrac{\mathrm{e}^{2\mathrm{i}t} - 1}{t} = 2\mathrm{i}$.

2.4 Does $\lim_{t\to\infty} \mathrm{e}^{-\mathrm{i}t^2}$ exist? Motivate your answer.

2.5 Show that $\lim_{t\to\infty} \dfrac{\mathrm{e}^{\mathrm{i}t}}{t} = 0$.

2.6 Demonstrate that

$$f(t) = 220\sin(\frac{\pi}{25}t) + 110\cos(\frac{\pi}{25}t)$$

is a sinusoid, and find its amplitude.

2.7 Use Euler's formula to prove that

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta), \qquad \forall \alpha, \beta \in \mathbb{R}.$$

2.8 *De Moivre:* Show that $\big(\cos(\phi) + \mathrm{i}\sin(\phi)\big)^n = \cos(n\phi) + \mathrm{i}\sin(n\phi)$.

2.9 Demonstrate that

$$f(t) = \sin(\omega_0 t) + 2\cos(\omega_0 t) - \cos(\omega_0 t + \pi/4)$$

is a sinusoid.

2.10 Let $q \in \mathbb{Z}$. What is the smallest period of $\sin(2t) - \sin(qt)$?

2.11 A good understanding of a complex-valued signal $f(t)$ can often be had from a plot of the curve $(\mathrm{Re}(f(t)), \mathrm{Im}(f(t)))$ in the two-dimensional plane. Sketch by hand the curves of

(a) $f(t) = \mathrm{e}^{\mathrm{i}\omega_0 t}$, and

(b) $f(t) = \mathrm{e}^{\mathrm{i}\omega_0 t} + 0.1\mathrm{e}^{10\mathrm{i}\omega_0 t}$.

2.12 Calculate $\int_T^{2T} \mathrm{e}^{2\mathrm{i}\omega_0 t} \cos^2(\omega_0 t)\,\mathrm{d}t$, where $\omega_0 = 2\pi/T > 0$.

2.13 Suppose $T > 0$ and define $\omega_0 = 2\pi/T$. Let $f_1(t) = \sin(\omega_0 t)$ and $f_2(t) = f_1(t)\mathbb{1}(t)$. Sketch the graphs of the two signals $g_i(t)$ ($i = 1, 2$) given by

$$g_i(t) = \int_t^{t+T} f_i(\tau)\,\mathrm{d}\tau.$$

2.14 Determine the power of the harmonic signal $f(t) = 2\mathrm{e}^{-2\mathrm{i}\omega_0 t}$.

2.15 Let $f(t)$ be a T-periodic signal with power P_f.

(a) Show that for every $t_0 \in \mathbb{R}$ the power of $f(t - t_0)$ equals the power of $f(t)$.

(b) Express the power of $f(2t)$ in terms of the power P_f of $f(t)$.

2.16 Determine the energy content of

(a) $f(t) = 2\mathrm{e}^{-2\mathrm{i}\omega_0 t}\,\mathrm{rect}_a(t)$.

(b) $f(t) = \mathrm{rect}_2(t)\,\mathrm{trian}_2(t)$.

2.17 Show that for every $t_0 \in \mathbb{R}$ the energy content of $f(t - t_0)$ is the same as that of $f(t)$.

2.18 Show that the following signals are periodic, and determine their period.

(a) $\cos(\frac{5}{12}t)\cos(\frac{1}{6}t)$

(b) $\cos(\frac{2}{3}t) + \sin(\frac{4}{7}t)$

2.19 Sketch the following signals.

(a) $\mathbb{1}(t) - 3\,\mathbb{1}(t-2) + 2\,\mathbb{1}(t-5)$.

(b) $\mathrm{trian}_2(t)\,\mathrm{rect}_1(t)$.

(c) $\mathrm{e}^{-t}\,\mathbb{1}(t)$.

(d) $-\mathrm{e}^{t}\,\mathbb{1}(-t+1)$.

(e) $\mathrm{sinc}(t)\,\mathrm{rect}_{4\pi}(t)$.

(f) $\mathrm{sinc}(\pi(t-4))$.

2.20 Simplify the following expressions:

(a) $\sin(t)\delta(t)$,

(b) $\mathrm{e}^{-5t}\delta(t)$,

(c) $(t^2+2t)\delta(t)$,

(d) $(t^2+2t)\delta(t-5)$,

(e) $(t^2+2t)\delta(t+5)$,

(f) $(f*g)(t)$ where $f(t)=\mathbb{1}(t-5)$ and $g(t)=\delta(t)$,

(g) $(f*g)(t)$ where $f(t)=\mathrm{e}^{5t}$ and $g(t)=\delta(5t+5)$.

2.21 Given is a continuous function $f(t)$.

(a) Let $g_1(t)=\delta(2t+4)$. Determine $(f*g_1)(t)$.

(b) Let $g_2(t)=\delta(2t-1)$. Determine $f(t)g_2(t)$.

2.22 Determine the derivative in the generalized sense of the following signals.

(a) $t\,\mathrm{rect}_2(t)$,

(b) $\sin(t)\,\mathbb{1}(t)$,

(c) $t\,\mathrm{rect}_2(t-1)$,

(d) $\mathrm{e}^{\mathrm{i}t}\,\mathbb{1}(t-\pi)$,

(e) $\mathrm{rect}_1(t)\,\mathrm{trian}_1(t)$.

2.23 Let $f(t)$ and $g(t)$ be two continuously differentiable signals. Determine the generalized derivative of $f(t)\,\mathbb{1}(-t)+g(t)\,\mathbb{1}(t)$.

2.24 Let $f(t) = |\sin(t)|$. Compute the first and second order derivatives $f^{(1)}(t)$ and $f^{(2)}(t)$.

2.25 Determine the $c, \alpha \in \mathbb{R}$ for which $h(t) := c\mathrm{e}^{-\alpha t}\,\mathbb{1}(t)$ satisfies $h^{(1)}(t) + 2h(t) = 10\delta(t)$.

2.26 Determine the convolution $(f * g)(t)$ for the following signals.

(a) $f(t) = \mathrm{e}^{at}\,\mathbb{1}(-t)$ and $g(t) = \mathrm{e}^{-bt}\,\mathbb{1}(t)$ with $a > 0$ and $b > 0$,

(b) $f(t) = \mathrm{e}^{at}$ and $g(t) = \mathbb{1}(t-1) \quad (\mathrm{Re}(a) > 0)$,

(c) $f(t) = \mathrm{rect}_2(t)$ and $g(t) = \mathbb{1}(t)$.

(d) $f(t) = (1+t)\,\mathrm{rect}_2(t)$ and $g(t) = \mathrm{rect}_2(t)$.

2.27 Determine the convolution $(f * g)(t)$ for the cases that

(a) $f(t) = \mathrm{e}^{-t}\,\mathbb{1}(t),\ g(t) = \mathrm{sgn}(t)$,

(b) $f(t) = \mathbb{1}(2t+1),\ g(t) = \mathrm{e}^{-|t|}$,

(c) $f(t) = g(t) = \mathbb{1}(t)$,

(d) $f(t) = \mathbb{1}(t),\ g(t) = \mathbb{1}(t-1)$,

(e) $f(t) = \mathbb{1}(t-1),\ g(t) = \mathbb{1}(t-1)$,

(f) $f(t) = \mathrm{e}^{-t}\,\mathbb{1}(t),\ g(t) = \mathrm{e}^{t}\,\mathbb{1}(t)$,

(g) $f(t) = \mathrm{e}^{at}\,\mathbb{1}(t),\ g(t) = \mathrm{e}^{bt}\,\mathbb{1}(t)$ with $a, b \in \mathbb{C}$.

2.28 In the first part you will prove a property of the convolution which you will use in the second part.

(a) Prove that $(f * (g+h))(t) = (f * g)(t) + (f * h)(t)$.

(b) Use the previous result and Euler's formula to determine the convolution of $f(t) = \mathrm{e}^{-t}\,\mathbb{1}(t)$ and $g(t) = \cos(t)\,\mathbb{1}(t)$.

More involved exercises

2.29 Determine a closed-form expression for $\sum_{n=1}^{N} \sin(n\phi)$.

2.30 Let $n \in \mathbb{N}$, $n > 1$, n even, and define $z = \mathrm{e}^{\mathrm{i}t}$. Show that $\frac{\sin(nt)}{\sin(t)} = z^{n-1} + z^{n-3} + \cdots + z^{-(n-3)} + z^{-(n-1)} = 2\big(\cos((n-1)t) + \cos((n-3)t) + \cdots + \cos(t)\big)$.

2.31 *Dirichlet kernel.* Consider the *Dirichlet kernel*[1] defined as

$$D_N(\phi) = \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}k\phi}.$$

(a) Is $D_N(\phi)$ periodic?

(b) Show that for $\mathrm{e}^{\mathrm{i}\phi} \neq 1$ we have

$$D_N(\phi) = \frac{\mathrm{e}^{-\mathrm{i}N\phi} - \mathrm{e}^{\mathrm{i}(N+1)\phi}}{1 - \mathrm{e}^{\mathrm{i}\phi}}.$$

(c) Show that

$$\begin{aligned} D_N(\phi) &= \begin{cases} 2N+1 & \text{if } \phi \text{ is a multiple of } 2\pi \\ \frac{\sin((N+1/2)\phi)}{\sin(\phi/2)} & \text{if } \phi \text{ is not a multiple of } 2\pi \end{cases} \\ &= (2N+1)\frac{\operatorname{sinc}((N+1/2)\phi)}{\operatorname{sinc}(\phi/2)}. \end{aligned}$$

The plot of $D_N(t)$ is quite interesting, see Fig. 2.19. It consists of a series of peaks at integer multiples of 2π with oscillating behavior in between; oscillations that increase in number as N increases.

[1] Warning: other definitions of "Dirichlet kernel" exist, such as $\frac{1}{2\pi}D_N(\phi)$ and $\frac{1}{2N+1}D_N(\pi t)$.

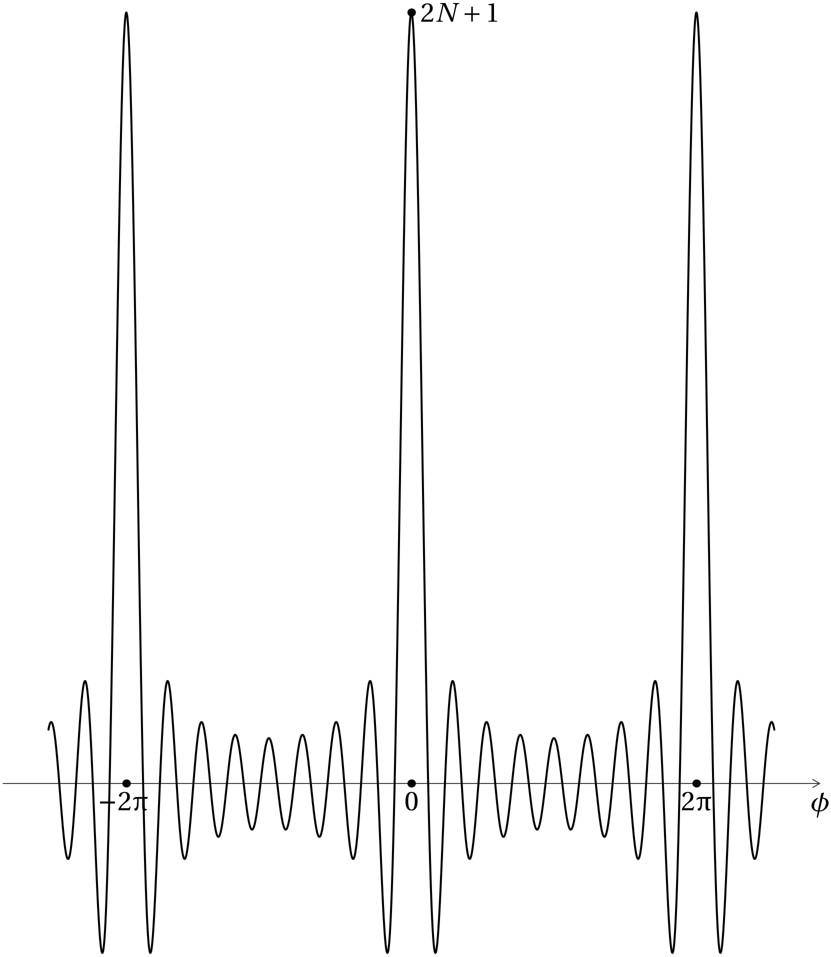


FIGURE 2.19: The Dirichlet kernel $D_N(\phi)$ for $N = 8$ (Exercise 2.31)

Chapter 3

Fourier Series

FIGURE 3.1: Jean Baptiste Joseph Fourier (1768–1830)

Why do we like Taylor series expansions such as

$$e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots .$$

We like them because they break down complicated functions, here e^t, into a sum of simple functions t^k. Ever wondered how calculators and programming languages calculate e^t and other functions? They rely on (a variation of) Taylor series expansions. Indeed, computation of the terms in the Taylor series require only simple operations, so are readily computed. The message of this short story is this:

The point of expansions is
to break down complicated signals
into a series of simpler signals.

Now what is considered complicated or simple depends very much on the application. If the application is computation, then Taylor series expansions might be useful. However, in signal analysis the simple signals are the harmonic signals. There are very convincing arguments from systems theory which we can not go into here, but recall that harmonic signals are easy to generate (think of the power grid) and that rainbows decompose visible light into a sum of harmonic signals (the visible colors).

Thus, the question becomes: which signals $f(t)$ expanded into a series of harmonic signals. Moreover, if this possible, how can these harmonic signals be determined from $f(t)$. That is the central question of this chapter. The discussion culminates in the famous result that says that practically every T-periodic signal $f(t)$ can be expressed as a *superposition*[1] of T-periodic harmonic signals,

$$f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ik\frac{2\pi}{T}t}, \qquad c_k \in \mathbb{C}. \tag{3.1}$$

The right-hand side is known as the *Fourier series expansion* of $f(t)$. Note that the harmonic signals in the expansion indeed are all T-periodic:

$$e^{ik\frac{2\pi}{T}(t+T)} = e^{ik\frac{2\pi}{T}t+ik2\pi} = e^{ik\frac{2\pi}{T}t}\underbrace{e^{ik2\pi}}_{1} = e^{ik\frac{2\pi}{T}t} \qquad \forall t \in \mathbb{R}.$$

If we define ω_0 as

$$\omega_0 = \frac{2\pi}{T}$$

[1]Superposition means sum.

then (3.1) is somewhat more succinctly described as

$$f(t) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}. \tag{3.2}$$

It shows that the frequencies $k\omega_0$ of the harmonic signals that together form $f(t)$ are integer multiples of ω_0. For this reason $\omega_0 := 2\pi/T$ is called the *fundamental frequency* of $f(t)$.

3.1 Convergence of Fourier series

In this short section we analyze convergence of the infinite Fourier series as on the right-hand side of (3.2). We will soon see that the definition of convergence of Fourier series differs from that of other series. The following example motivates the change of definition.

Example 3.1.1 (From a real to a complex Fourier series). Suppose $f(t)$ is a finite sum of sinusoids

$$f(t) = \sum_{k=0}^{N} a_k \cos(k\omega_0 t). \tag{3.3}$$

This can be rewritten as a sum of complex harmonic signals as follows.

$$\begin{aligned} f(t) &= \sum_{k=0}^{N} a_k \cos(k\omega_0 t) \\ &= \sum_{k=0}^{N} \frac{a_k}{2}(\mathrm{e}^{\mathrm{i}k\omega_0 t} + \mathrm{e}^{-\mathrm{i}k\omega_0 t}) \\ &= \sum_{k=0}^{N} \frac{a_k}{2}\mathrm{e}^{\mathrm{i}k\omega_0 t} + \sum_{k=-N}^{0} \frac{a_{|k|}}{2}\mathrm{e}^{\mathrm{i}k\omega_0 t} \\ &= \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}, \end{aligned} \tag{3.4}$$

with $c_0 = a_0$ and $c_k = \frac{1}{2}a_{|k|}$ for all $k = \pm 1, \pm 2, \ldots, \pm N$. □

It will be clear that in this example for $N = \infty$ one would end up with an infinite sum (aka “series”) of complex harmonics

$$\sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}. \tag{3.5}$$

A series like this is called a *(complex) Fourier series*, and the coefficients c_k are the *(complex) Fourier coefficients*. We sometimes refer to the c_k as the *line spectrum*. Recall that complex numbers c_k can be written in polar form,

$$c_k = A_k \mathrm{e}^{\mathrm{i}\phi_k},$$

where A_k is the amplitude and ϕ_k is the phase. In that case, we obtain

$$c_k \mathrm{e}^{\mathrm{i}k\omega_0 t} = A_k \mathrm{e}^{\mathrm{i}(k\omega_0 t + \phi_k)}.$$

We call A_k the *amplitude spectrum* and ϕ_k the *phase spectrum*.

The successive terms $c_k \mathrm{e}^{\mathrm{i}\omega_0 t}$ generally are complex-valued functions of t, even if their sum is a real-valued function. Note that whereas the index k in the real case (3.3) goes from 0 to N, in the complex case (3.4) the index k goes from $-N$ to N. Inspired by this we define convergence of the Fourier series (3.5) as a *symmetric* limit:

Definition 3.1.2 (Convergence of complex Fourier series). Given $t \in \mathbb{R}$, the *Fourier series* (3.5) is said to *converge* with limit $f(t)$ if

$$f(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}. \tag{3.6}$$

□

When we write $f(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$ we always mean (3.6). Note that the above is about pointwise convergence (at each point t) which is fundamentally different from convergence in $\mathscr{L}^2([0,T];\mathbb{C})$ (see Chapter 1).

If in a Fourier series, $f(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$, only finitely many coefficients c_k are nonzero, then obviously the Fourier series converges for every $t \in \mathbb{R}$. More generally, convergence for every $t \in \mathbb{R}$ is ensured if the c_k are absolutely summable, that is, if $\sum_{k=-\infty}^{\infty} |c_k|$ converges. In this case the sum $f(t)$ is in fact continuous everywhere:

Theorem 3.1.3 (Existence of infinite Fourier series). If $\sum_{k=-\infty}^{\infty}|c_k| < \infty$, then the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t}$ converges and moreover it is continuous at every $t \in \mathbb{R}$. □

Proof. Since $|e^{ik\omega_0}| = 1$ it follows that $|\sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t}| < \sum_{k=-\infty}^{\infty} |c_k| < \infty$. We conclude that the Fourier series converges absolutely for every t, and, hence, that the Fourier series itself converges for every t. That the Fourier series is also continuous is shown[2] in Appendix A.1. ■

Hence if $f(t)$ is *not* continuous then necessarily $\sum_{k=-\infty}^{\infty}|c_k| = \infty$, and in that case the partial sums $s_N(t) = \sum_{k=-N}^{N} c_k e^{ik\omega_0 t}$ may not provide a satisfactory approximation of $f(t)$ near the points of discontinuity. A famous phenomenon in this respect is the *Gibbs phenomenon* discussed in § 3.6.

3.2 The fundamental theorem of Fourier series

In the previous section we assumed the Fourier coefficients c_k given and the signal $f(t)$ to be the resulting sum. But what if we are given $f(t)$ and want to find the corresponding Fourier series (assuming one exists)?

We momentarily return to the Hilbert space theory of Chapter 1. Let us first observe that the set of all complex harmonic functions with period T,

$$e_k(t) := e^{i\frac{2\pi}{T}kt}, \qquad k \in \mathbb{Z}$$

is an orthonormal sequence in the complex inner product

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f(t)\overline{g(t)}\,dt.$$

This is readily verified: we have

$$\|e_k\|^2 = \langle e_k, e_k\rangle = \frac{1}{T}\int_0^T \underbrace{e^{i\frac{2\pi}{T}kt}\overline{e^{i\frac{2\pi}{T}kt}}}_{=1}\,dt = 1$$

[2] A nicer proof uses the fact that the partial sums $s_N(t) := \sum_{k=-N}^{N} c_k e^{ik\omega_0 t}$ are Cauchy in $\mathscr{C}([0,T];\mathbb{C})$ with the max-norm, and this space is a Banach space (Thm. 1.2.2), so s_N converges to a continuous function.

and, if $k \neq n$,

$$\langle e_k, e_n \rangle = \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}\frac{2\pi}{T}kt}\overline{\mathrm{e}^{\mathrm{i}\frac{2\pi}{T}nt}}\,\mathrm{d}t = \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}\frac{2\pi}{T}(k-n)t}\,\mathrm{d}t = \frac{\mathrm{e}^{\mathrm{i}\frac{2\pi}{T}(k-n)t}}{\mathrm{i}2\pi(k-n)}\bigg|_0^T = 0.$$

So immediately we get that the best approximation f_* in $\operatorname{span}\{\ldots, e_{-1}, e_0, e_1, \ldots\}$ of a function $f \in \mathscr{L}^2([0,T];\mathbb{C})$ is

$$f_* = \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle e_k,$$

that is,

$$f_*(t) = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}\frac{2\pi}{T}kt}$$

with

$$f_k := \langle f, e_k \rangle = \frac{1}{T}\int_0^T f(t)\mathrm{e}^{-\mathrm{i}2\pi kt}\,\mathrm{d}t.$$

The obvious question now is: is the Fourier series f_* of f equal to f? Or, to put it differently, is the sequence of harmonic functions $\{\ldots, e_{-1}, e_0, e_1, \ldots\}$ a *complete* orthonormal sequence in $\mathscr{L}^2([0,T];\mathbb{C})$? It is:

Theorem 3.2.1 (Fourier series in $\mathscr{L}^2$). Let $T > 0$. Every $f \in \mathscr{L}^2([0,T];\mathbb{C})$ equals its Fourier series,

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}k\frac{2\pi}{T}t}. \tag{3.7}$$

Here f_k are the *Fourier coefficients* of $f(t)$, defined as

$$f_k = \langle f, \mathrm{e}^{\mathrm{i}k\frac{2\pi}{T}\bullet} \rangle = \frac{1}{T}\int_0^T f(t)\mathrm{e}^{-\mathrm{i}k\frac{2\pi}{T}t}\,\mathrm{d}t. \tag{3.8}$$

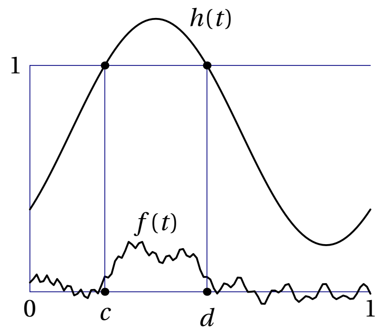


FIGURE 3.2: A shifted harmonic function $h(t)$ so that $h(t) \geq 1$ on $[c,d]$ and $0 \leq h(t) < 1$ elsewhere on $[0,1]$

Proof. For ease of exposition we prove it only for the case $T = 1$. According to Theorem 1.5.7 it suffices to show that the only function f orthogonal to all e_k is the zero function. Assume for the moment that f is real and continuous. If f is nonzero then f is > 0 (or < 0) throughout some small enough interval $[c,d]$, see Fig. 3.2. Next let $h(t)$ be the harmonic function

$$h(t) = b + \frac{1}{2}\cos(2\pi t + a) = b + \frac{1}{4}(\mathrm{e}^{\mathrm{i}(2\pi t+a)} + \mathrm{e}^{-\mathrm{i}(2\pi t+a)})$$

where a, b are chosen such that $h(t) \geq 1$ on the selected interval $[c,d]$ and $0 \leq h(t) < 1$ elsewhere, see Fig. 3.2. Realize that every power h^n of h is a sum of harmonic functions, in fact

$$h^n \in \operatorname{span}\{e_{-n}, \dots, e_0, \dots, e_n\}.$$

The higher the power, the larger $h^n(t)$ is in the interval $[c,d]$ and the smaller it is elsewhere. Therefore for some large enough n we have

$$\langle f, h^n \rangle \neq 0.$$

This means that f can not be orthogonal to all e_k. Hence the only continuous f that is orthogonal to all e_k is the zero function, $f = 0$.

Now let $f \in \mathscr{L}^2([0,1];\mathbb{R})$, not necessarily continuous, and assume that $f \perp e_k$ for all $k \in \mathbb{Z}$. Again we show that f is then necessarily the zero function. Consider its antiderivative

$$F(t) := \int_0^t f(\tau)\,\mathrm{d}\tau.$$

This function is continuous. It is also orthogonal to e_k for all $k \neq 0$ because

$$\langle F, e_k \rangle = \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi k t} F(t)\,\mathrm{d}t = \Big[\frac{\mathrm{e}^{-\mathrm{i}2\pi k t}}{-\mathrm{i}2\pi k} F(t)\Big]_0^1 - \frac{1}{-\mathrm{i}2\pi k}\langle f, e_k\rangle = 0 \quad \forall k \neq 0.$$

Here we used that $F(0) = 0$ and $F(1) = \int_0^1 f(t)\,\mathrm{d}t = \langle f, e_0\rangle = 0$. So $F - \langle F, e_0\rangle e_0$ is orthogonal to e_k for all $k \in \mathbb{Z}$, including $k = 0$. By continuity of F this implies that $F - \langle F, e_0\rangle e_0$ is the zero function, i.e. that F is a constant function. Its derivative f hence is the zero function, which is what we set out to prove.

For complex $f \in \mathscr{L}^2([0,1];\mathbb{C})$ we can apply the arguments to both its real and imaginary parts. ■

Be aware that equality in (3.7) is to be understood in $\mathscr{L}^2$-sense! I.e. $f(t)$ and its Fourier series $f_*(t)$ might differ at some values of t, as long as they are the same for "almost all t" in the sense that $\|f - f_*\| = 0$. Using (3.8) we can compute the Fourier series from a given function.

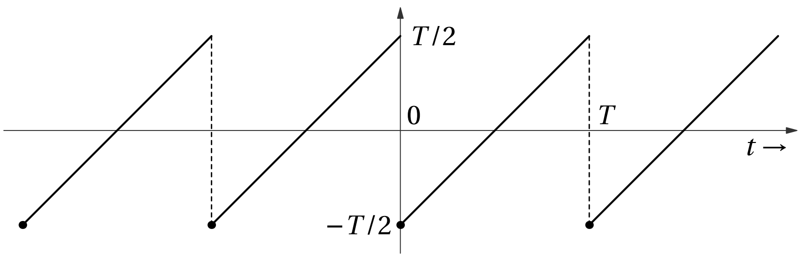


FIGURE 3.3: Graph of the sawtooth of period T

Example 3.2.2 (Sawtooth). Figure 3.3 shows the graph of the sawtooth with period T. It is the T-periodic signal $f(t)$ which on one period $[0, T)$ is given by

$$f(t) = t - T/2, \quad (0 \leq t < T).$$

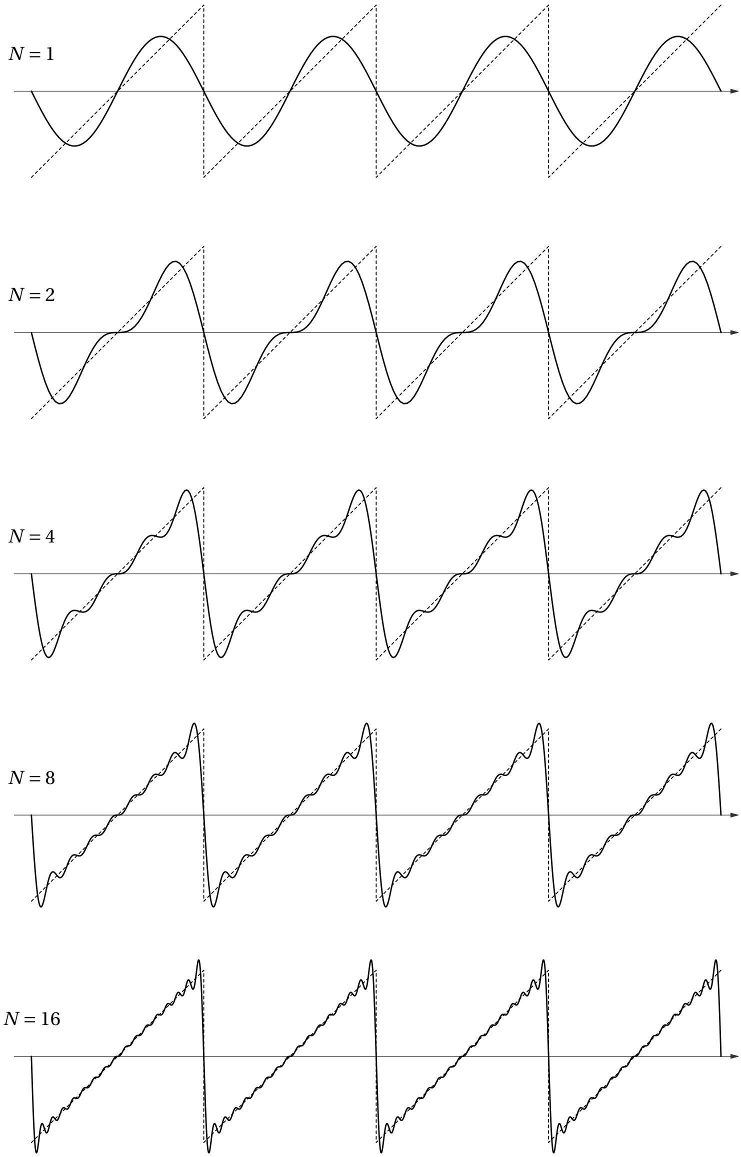


FIGURE 3.4: The graph of the sawtooth (dashed) and of its truncated Fourier series $\sum_{k=-N}^{N} f_k e^{ik\omega_0 t}$ for $N = 1, 2, 4, 8, 16$

The Fourier coefficients of $f(t)$ can be calculated explicitly from (3.8),

$$f_k = \frac{1}{T}\int_0^T f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \frac{1}{T}\int_0^T (t-T/2)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t.$$

Now we perform integration by parts to obtain, for $k \neq 0$, that

$$\begin{aligned} f_k &= \left.\frac{-(t-T/2)\mathrm{e}^{-\mathrm{i}k\omega_0 t}}{\mathrm{i}k\omega_0 T}\right]_0^T + \frac{1}{\mathrm{i}k\omega_0 T}\int_0^T \mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t \\ &= \frac{-T/2\mathrm{e}^{-\mathrm{i}k2\pi} - T/2}{\mathrm{i}k\omega_0 T} + \frac{1}{\mathrm{i}k\omega_0 T}\underbrace{\int_0^T \mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t}_{0} \\ &= \frac{\mathrm{i}}{k\omega_0}. \end{aligned}$$

This assumes that $k \neq 0$. Remains to compute f_0. It is defined as $f_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t$ which is the average of $f(t)$ over one period. The average of the sawtooth is zero,

$$f_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{d}t = 0.$$

Now we know all Fourier coefficients. Figure 3.4 shows plots of the partial sums $\sum_{k=-N}^{N} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$ for $N = 1, 2, 4, 8, 16$, and it seems that they indeed converge in some way to the sawtooth $f(t)$ as $N \to \infty$. □

Theorem 3.2.1 tells us that the Fourier series converges in $\mathscr{L}^2([0,T];\mathbb{C})$. However, this does not imply that for every fixed t the partial sums

$$\sum_{k=-N}^{N} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$$

converges to $f(t)$ as $N \to \infty$. In the example of the sawtooth the approximation is pretty good, except near the discontinuities of $f(t)$. In fact, precisely *at* the points of discontinuity the sawtooth, by definition, equals $-T/2$ whereas all truncated Fourier series are zero at these points. The following central result settles the issue. In this result we integrate over $[-T/2, T/2]$ instead of $[0, T]$, but that is not relevant since all functions involved are T-periodic.

Theorem 3.2.3 (The Fourier series theorem). Let $f(t)$ be a T-periodic signal and suppose it is piecewise smooth on $\mathbb{R}$. Then for every $t \in \mathbb{R}$ there holds that

$$\frac{f(t^+)+f(t^-)}{2} = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t},$$

where f_k are the Fourier coefficients of $f(t)$ defined as

$$f_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t. \tag{3.9}$$

Proof. The proof is technical. A complete proof is given in Appendix A.1. ∎

If $f(t)$ is, besides piecewise smooth, also continuous everywhere, then $\frac{f(t^+)+f(t^-)}{2}$ clearly equals $f(t)$ everywhere, and hence the function and its Fourier series then are identical:

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t} \qquad \forall t.$$

Example 3.2.4 (The sawtooth, continued). The sawtooth $f(t)$ is continuous everywhere except at $t = 0, \pm T, \pm 2T, \ldots$. At these values $t = nT$ the signal satisfies $f(nT^+) = f(0^+) = -T/2$ and $f(nT^-) = f(0^-) = T/2$, see Fig. 3.3. According to the Fourier theorem, at these points of discontinuity the Fourier series equals $(f(nT^+)+f(nT^-))/2 = 0$. The Fourier series therefore equals

$$\sum_{k\neq 0} \frac{\mathrm{i}}{k\omega_0}\mathrm{e}^{\mathrm{i}k\omega_0 t} = \begin{cases} 0 & \text{if } t = nT,\ n \in \mathbb{Z}, \\ f(t) & \text{elsewhere} \end{cases}$$

This is consistent with Fig. 3.4. Later, in § 3.6, we have more to say about the ripples around the points of discontinuity. □

It is interesting to recall at this point Theorem 3.1.3. It says that $\sum_{k=-\infty}^{\infty}|f_k| < \infty$ implies continuity of $f(t)$. Now obviously the sawtooth is not continuous at $t = nT$, and, hence, we must have that $\sum_{k=-\infty}^{\infty} |f_k| = \infty$. Indeed, $\sum_{k=-\infty}^{\infty} |f_k| = \sum_{k=1}^{\infty} 2/|k\omega_0| = \infty$. The interpretation is this: for small values of k the terms $f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$ only change gradually as a function of time, so if a Fourier series is to approximate the abrupt change at a discontinuity of $f(t)$ then it needs lots of high frequency components to accomplish this, i.e., the f_k need to be relatively large for large values of k, so large in fact that $\sum_{k=-\infty}^{\infty} |f_k|$ diverges to ∞.

Nevertheless, for every piecewise smooth $f(t)$ the Fourier coefficients f_k always converge to zero as $k \to \infty$. This is a byproduct of the Riemann–Lebesgue lemma:

Lemma 3.2.5 (Riemann–Lebesgue). If $f(t)$ is piecewise smooth on $[a, b]$, then

$$\lim_{|\omega|\to\infty} \int_a^b f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = 0.$$

Proof. Suppose first that $f(t)$ is continuously differentiable on $[a, b]$. Then we can use integration by parts to obtain

$$\int_a^b f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \frac{1}{\mathrm{i}\omega}\Big[f(t)\mathrm{e}^{\mathrm{i}\omega t}\Big]_a^b - \frac{1}{\mathrm{i}\omega}\int_a^b f'(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t.$$

Since $|\mathrm{e}^{\mathrm{i}\omega t}| = 1$ we can derive from this the bound

$$\left|\int_a^b f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\right| \le \frac{1}{|\mathrm{i}\omega|}(|f(b)| + |f(a)|) + \frac{1}{|\mathrm{i}\omega|}\int_a^b |f'(t)|\,\mathrm{d}t.$$

It is immediate that the right-hand side goes to zero as $|\omega| \to \infty$, which proves the claim.

If $f(t)$ is not continuously differentiable, then, since $f(t)$ is piecewise smooth, we may split $[a, b]$ into a finite set of subintervals $[t_i, t_{i+1}]$ $(i = 1, \ldots)$ such that $f(t)$ is continuously differentiable on each of these subintervals. Similarly as done above (using integration by parts) it follows that $\lim_{|\omega|\to\infty} \int_{t_i}^{t_{i+1}} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = 0$. Hence $\lim_{|\omega|\to\infty} \int_a^b f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = 0$. ■

An immediate consequence of the Riemann-Lebesgue lemma is that piecewise smooth signals $f(t)$ satisfy

$$\lim_{|k|\to\infty} \int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = 0,$$

i.e., the Fourier coefficients f_k converge to zero as $|k| \to \infty$. In fact, looking at the proof of the Riemann–Lebesgue lemma, we may conclude that $|f_k| \le A/|k|$ for some A (depending in f but not on k).

Remark. There is also a Fourier series expansion for *two*-dimensional signals $g(x, y)$, i.e., *images*, and like $f(t)$ may be represented by its Fourier coefficients, also pictures may be represented by Fourier coefficients. Simply discarding high frequency Fourier coefficients then gives a *finite* set of Fourier coefficients that roughly capture the original signal or picture. This is the essence of jpeg-encoding. (It also explains why jpeg pictures typically are softer than the original: sharp edges need high frequency components but these are discarded in the encoding.)

3.3 Real Fourier series

For real-valued signals $f(t)$ the Fourier coefficients obey the symmetry rule $f_{-k} = f_k^*$. This follows from

$$\begin{aligned} f_{-k} &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \{f(t) \text{ is real}\} = \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\mathrm{e}^{\mathrm{i}k\omega_0 t}\,\mathrm{d}t \\ &= \left(\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t\right)^* = f_k^*. \end{aligned}$$

The complex Fourier series of a real-valued function can then be rewritten as a real Fourier series, that is, as a sum of real sinusoids. This goes as follows. First define real a_k and b_k via

$$a_k = 2\operatorname{Re} f_k, \qquad b_k = -2\operatorname{Im} f_k \qquad \forall k = 0, 1, 2, \ldots. \tag{3.10}$$

That is to say

$$f_0 = \frac{a_0}{2}, \qquad f_k = \frac{a_k - \mathrm{i}b_k}{2} \qquad \forall k > 0.$$

This rather odd looking definition of a_k and b_k will turn out to be useful. Then

$$
\begin{aligned}
f(t) &= \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t} \\
&= f_0 + \sum_{k=1}^{\infty} (f_{-k}\mathrm{e}^{-\mathrm{i}k\omega_0 t} + f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}) \\
&= \{f_{-k} = f_k^*\} = f_0 + \sum_{k=1}^{\infty} \big((f_k \mathrm{e}^{\mathrm{i}k\omega_0 t})^* + f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}\big) \\
&= f_0 + \sum_{k=1}^{\infty} \operatorname{Re}(2 f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}). && (3.11) \\
&= \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} \operatorname{Re}\big[\underbrace{(a_k - \mathrm{i}b_k)}_{2f_k} \underbrace{(\cos(k\omega_0 t) + \mathrm{i}\sin(k\omega_0 t))}_{\mathrm{e}^{\mathrm{i}k\omega_0 t}}\big] \\
&= \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} \big(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\big). && (3.12)
\end{aligned}
$$

This finally establishes that $f(t)$ is indeed a sum of sinusoids. It will be clear that any $f(t)$ that is of the form (3.12) is a real-valued T-periodic signal with $T = 2\pi/\omega_0$. The series (3.12) is known as the *real Fourier series* and the coefficients a_k and b_k are the *real Fourier coefficients.* The term $a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$ is sometimes referred to as the k-th *harmonic* of $f(t)$. Summary:

Theorem 3.3.1 (Fourier series theorem, real-valued case). Let $f(t)$ be a real-valued T-periodic signal and suppose it is piecewise smooth. Then for every $t \in \mathbb{R}$ we have

$$
\frac{f(t^-) + f(t^+)}{2} = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} \big(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\big),
$$

where a_k and b_k are the real Fourier coefficients of $f(t)$ defined as

$$
a_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(k\omega_0 t)\,\mathrm{d}t, \qquad k = 0, 1, \ldots, \tag{3.13}
$$

$$
b_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(k\omega_0 t)\,\mathrm{d}t, \qquad k = 1, 2, \ldots. \tag{3.14}
$$

Proof. Let f_k be the complex Fourier coefficients of $f(t)$. In the above we showed that $a_k = 2\operatorname{Re} f_k$ and $b_k = -2\operatorname{Im} f_k$. Therefore

$$a_k = 2\mathrm{Re}\, f_k = 2\mathrm{Re}\,\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega_0 t)\,\mathrm{d}t,$$

and

$$b_k = -2\mathrm{Im}\, f_k = -2\mathrm{Im}\,\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(k\omega_0 t)\,\mathrm{d}t.$$

■

Example 3.3.2 (Sawtooth, real Fourier series). In Example 3.2.2 we found that the Fourier coefficients of the saw tooth of Fig. 3.3 are

$$f_0 = 0, \qquad f_k = \frac{\mathrm{i}}{k\omega_0} \quad \forall k \neq 0.$$

Hence the real Fourier coefficients are

$$a_k = 2\mathrm{Re}\,(f_k) = 0, \qquad b_k = -2\,\mathrm{Im}(f_k) = -\frac{2}{k\omega_0} = -\frac{2}{k\frac{2\pi}{T}} = -\frac{T}{k\pi}.$$

Surprisingly perhaps, all a_k are zero, so the real Fourier series consists of sines only,

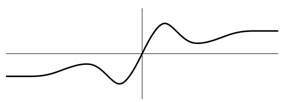
$$\frac{f(t^-) + f(t^+)}{2} = \sum_{k=1}^{\infty} \frac{-T}{k\pi}\sin(k\omega_0 t).$$

All coefficients $\frac{-T}{k\pi}$ are negative. This is consistent with Fig. 3.4. □

It is not uncommon that Fourier series only consists of sine terms (or only cosine terms). This is due to symmetry properties that signals $f(t)$ often have. The sawtooth is an *odd* signal.

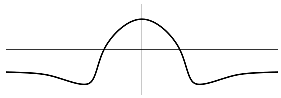
Definition 3.3.3 (Even and odd signals). A signal $f(t)$ is *odd* if

$$f(t) = -f(-t) \qquad \forall t.$$



A signal $f(t)$ is *even* if

$$f(t) = f(-t) \qquad \forall t.$$



□

Another way to think of even and odd is that the graph of an even signal is symmetric with respect to the 'y-axis' whereas odd means it is point-symmetric with respect to the origin. Clearly cosines are even and sines are odd. It is then no surprise (and easily proved in general, see the next example) that Fourier series of odd signals, such as the sawtooth, only consist of sines, and Fourier series of even signals only consist of cosines.

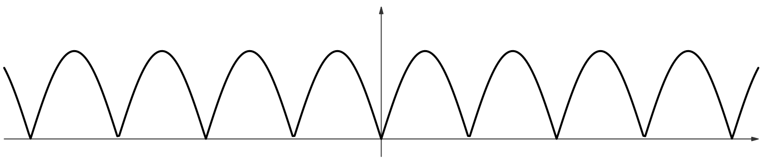


FIGURE 3.5: $f(t) = |\sin(\pi t)|$

Example 3.3.4 (Real Fourier coefficients). Consider the even signal $f(t) = |\sin(\pi t)|$, (see Figure 3.5). This signal is periodic with period $T = 1$. In addition the signal is even so that the $g_k(t) := f(t)\sin(k\omega_0 t)$ are odd. Hence the integrals for b_k,

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)\sin(k\omega_0 t)}_{g_k(t)}\,\mathrm{d}t, \qquad k = 1, 2, \ldots$$

are all zero. The Fourier series of $f(t)$ entails only cosines.

$$\begin{aligned}
a_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(k\omega_0 t)\,\mathrm{d}t = 2\int_{-1/2}^{1/2} |\sin(\pi t)|\cos(2k\pi t)\,\mathrm{d}t \\
&= \{\text{ integrand is even, hence } \textstyle\int_{-y}^{y} = 2\int_0^y \} \\
&= 4\int_0^{1/2} \sin(\pi t)\cos(2k\pi t)\,\mathrm{d}t \\
&= \{\ \sin(\alpha)\cos(\beta) = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta),\ \text{see Exercise 2.7 }\}, \\
&= 2\int_0^{1/2} (\sin((2k+1)\pi t) - \sin((2k-1)\pi t))\,\mathrm{d}t \\
&= \left.\frac{-2\cos((2k+1)\pi t)}{(2k+1)\pi}\right|_0^{1/2} - \left.\frac{-2\cos((2k-1)\pi t)}{(2k-1)\pi}\right|_0^{1/2} \\
&= \frac{2}{\pi}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right) = \frac{4}{\pi}\frac{1}{1-4k^2}.
\end{aligned}$$

The signal $f(t)$ is continuous and is piecewise smooth on $\mathbb{R}$, so we conclude that

$$|\sin(\pi t)| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{1-4k^2}\cos(2k\pi t)$$

for every $t \in \mathbb{R}$. Funny formula. For $t = 0$ it says that $\sum_{k=1}^{\infty} \frac{1}{1-4k^2} = -1/2$. For $t = 1/2$ it also gives a funny formula (try it yourself). □

Time and frequency domain. According to the Fourier theorem, any piecewise smooth periodic signal $f(t)$ may be reconstructed from its Fourier coefficients f_k, except at its points of discontinuity. Possibly after redefining $f(t)$ at the discontinuities, one can say that $f(t)$ is fully determined once its Fourier coefficients are known.

We say that f_k describes $f(t)$ in the *frequency domain*, and that $f(t)$ itself describes the signal in the *time domain.*

The values that a line spectrum takes are usually complex numbers. These may be expressed in polar form as $f_k = |f_k|\mathrm{e}^{\mathrm{i}\phi_k}$. The plot of $|f_k|$ versus frequency index k (or versus frequency $k\omega_0$) is then referred to as the *amplitude spectrum* of $f(t)$, and the plot ϕ_k versus k (or versus $k\omega_0$) as the *phase spectrum* of $f(t)$. The phase for a nonzero coefficient, is unique up to an integer multiple of 2π. For zero coefficients we often set the phase equal to zero.

If $f(t)$ is real-valued, then $f_{-k} = f_k^*$, hence, $|f_k| = |f_{-k}|$ and $\phi_k = -\phi_{-k}$. Real-valued signals hence have an amplitude spectrum that is even and a phase spectrum that is odd (up to multiples of 2π), see Fig. 3.6.

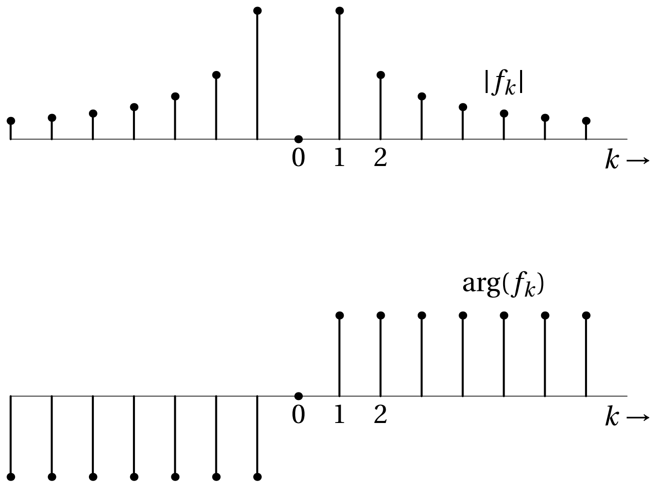


FIGURE 3.6: Amplitude spectrum (top) and phase spectrum (bottom) of the sawtooth, see Example 3.3.5

Example 3.3.5 (Amplitude and phase spectrum). In Example 3.2.2 we found the Fourier coefficients of the sawtooth to be $f_k = \mathrm{i}/(k\omega_0)$ for $k \neq 0$ and $f_k = 0$ for $k = 0$. The amplitude of the Fourier coefficients is therefore $|f_k| = 1/|k\omega_0|$, except for $k = 0$ where it is zero. The phase $\phi_k = \arg f_k$ equals $\pi/2$ for positive k, and $-\pi/2$ for negative k. The phase of $f_0 = 0$ is not really defined, but in such cases we take it to be zero. Figure 3.6 shows the amplitude and phase spectrum of the sawtooth (for a certain ω_0). The sawtooth is a real-valued function and this is in accordance with the fact that the amplitude spectrum is an even function of k and that the phase spectrum is an odd function of k. $\square$

3.4 Fourier series properties

It is convenient to express the connection between a function $f(t)$ and its Fourier coefficients f_k as a pair with arrows:

$$f(t) \longleftrightarrow f_k.$$

Table 3.1 collects a number of properties of Fourier series. We discuss them here.

Linearity. The linearity property says that for every two constants α, β we have

$$\alpha f(t) + \beta g(t) \longleftrightarrow \alpha f_k + \beta g_k.$$

This is a consequence of linearity of integration: the Fourier coefficients d_k of $d(t) = \alpha f(t) + \beta g(t)$ equals

$$\begin{aligned} d_k &= \frac{1}{T}\int_{-T/2}^{T/2} (\alpha f(t) + \beta g(t))\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t \\ &= \alpha\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t + \beta\frac{1}{T}\int_{-T/2}^{T/2} g(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \alpha f_k + \beta g_k. \end{aligned}$$

Time-shift. The time-shift property says

$$f(t-\tau) \longleftrightarrow \mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k.$$

To see this let $d(t) = f(t-\tau)$. Its Fourier coefficients d_k satisfy

$$\begin{aligned} d_k &= \frac{1}{T}\int_{-T/2}^{T/2} f(t-\tau)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t && \text{let } v = t-\tau \\ &= \frac{1}{T}\int_{-\frac{T}{2}-\tau}^{\frac{T}{2}-\tau} f(v)\mathrm{e}^{-\mathrm{i}k\omega_0(v+\tau)}\,\mathrm{d}v && \text{apply Lemma 2.4.3:} \\ &= \mathrm{e}^{-\mathrm{i}k\omega_0\tau}\frac{1}{T}\int_{-T/2}^{T/2} f(v)\mathrm{e}^{-\mathrm{i}k\omega_0 v}\,\mathrm{d}v = \mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k. \end{aligned}$$

Note that the amplitude of the Fourier coefficients are invariant under time shifts: $|d_k| = |\mathrm{e}^{-\mathrm{i}k\omega_0\tau}||f_k| = |f_k|$. This is not surprising because $f(t)$ and $f(t-\tau)$ clearly contain the same frequencies and with the same amplitude.

Time reversal. This says that

$$f(-t) \longleftrightarrow f_{-k}.$$

Proof: the Fourier coefficients d_k of $d(t) := f(-t)$ are

$$\begin{aligned} d_k &= \frac{1}{T}\int_{-T/2}^{T/2} f(-t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t && \text{substitute } v = -t \\ &= -\frac{1}{T}\int_{T/2}^{-T/2} f(v)\mathrm{e}^{\mathrm{i}k\omega_0 v}\,\mathrm{d}v \\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(v)\mathrm{e}^{-\mathrm{i}(-k)\omega_0 v}\,\mathrm{d}v = f_{-k}. \end{aligned}$$

Conjugation. This rule claims that

$$f^*(t) \longleftrightarrow f^*_{-k}.$$

Proof: the Fourier coefficients d_k of $f^*(t)$ follows as

$$d_k = \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \left(\frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{\mathrm{i}k\omega_0 t}\,\mathrm{d}t\right)^* = f^*_{-k}.$$

Frequency shift. The final property of Table 3.1 states

$$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t) \longleftrightarrow f_{k-n},$$

provided $n \in \mathbb{Z}$. Indeed, for a fixed n, the signal with Fourier coefficients f_{k-n} is

$$\begin{aligned} \sum_{k=-\infty}^{\infty} f_{k-n}\mathrm{e}^{\mathrm{i}k\omega_0 t} &= \{\text{substitute } m = k-n\} \\ &= \sum_{m=-\infty}^{\infty} f_m \mathrm{e}^{\mathrm{i}(m+n)\omega_0 t} \\ &= \mathrm{e}^{\mathrm{i}n\omega_0 t}\sum_{m=-\infty}^{\infty} f_m \mathrm{e}^{\mathrm{i}m\omega_0 t} = \mathrm{e}^{\mathrm{i}n\omega_0 t} f(t). \end{aligned}$$

This rule looks similar to the time-shift rule. Loosely speaking these two rules say that a shift in one domain corresponds to a multiplication with a harmonic term in the other domain.

TABLE 3.1: Properties of the Fourier series

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t)+\beta g(t)$	$\alpha f_k+\beta g_k$
Time-shift	$f(t-\tau),\quad (\tau\in\mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f^*_{-k}
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t),\quad (n\in\mathbb{Z})$	f_{k-n}

3.5 Convolution and Parseval's theorem

For periodic signals, convolution is defined differently:

Definition 3.5.1. The *convolution* or *convolution product* of two T-periodic signals $f(t)$ and $g(t)$ is the T-periodic signal $(f*g)(t)$ defined as

$$(f*g)(t)=\frac{1}{T}\int_{-T/2}^{T/2} f(\tau)g(t-\tau)\,\mathrm{d}\tau. \tag{3.15}$$

□

Convolution products in the time domain reduce to ordinary products of the respective line spectra in the frequency domain:

Theorem 3.5.2 (Convolution theorem for periodic signals). Let $f(t)$ and $g(t)$ be two T-periodic piecewise smooth signals with line spectra f_k and g_k respectively. Then $(f*g)(t)$ is piecewise smooth, continuous and its line spectrum $(f*g)_k$ satisfies

$$(f*g)_k = f_k g_k, \qquad k\in\mathbb{Z}.$$

Proof. We omit the proof that $(f*g)(t)$ is piecewise smooth and continuous, since the proof is technical but otherwise straightforward.

The line spectrum $(f*g)_k$ obey

$$\begin{aligned}
(f*g)_k &= \frac{1}{T}\int_{-T/2}^{T/2}\left(\frac{1}{T}\int_{-T/2}^{T/2} f(\tau)g(t-\tau)\,\mathrm{d}\tau\right)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t \\
&= \frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} f(\tau)g(t-\tau)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \{\text{change order of integration}\} \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\left(\frac{1}{T}\int_{-T/2}^{T/2} g(t-\tau)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,dt\right)\mathrm{d}\tau \\
&= \{\text{see Table 3.1, time-shift}\} \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\mathrm{e}^{-\mathrm{i}k\omega_0 \tau} g_k\,\mathrm{d}\tau \\
&= \left(\frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\mathrm{e}^{-\mathrm{i}k\omega_0 \tau}\,\mathrm{d}\tau\right) g_k = f_k g_k.
\end{aligned}$$

■

Remark. Since $f_k g_k = g_k f_k$ we see once again that $f*g = g*f$, i.e., that convolution products commute.

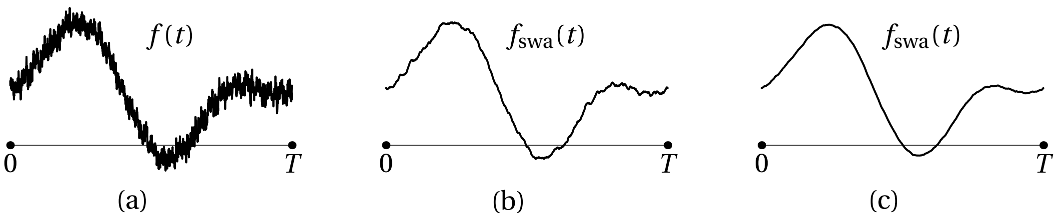


FIGURE 3.7: (a): A noisy periodic signal; (b) averaged with $\epsilon = 0.025$; (c) averaged with $\epsilon = 0.1$, see Example 3.5.3

Example 3.5.3 (Sliding window averaging). For a given T-periodic signal $f(t)$ we construct the signal $\hat{f}(t)$ by averaging $f(t)$ around t over an interval of a fixed length ϵT, $\epsilon \in (0,1)$ i.e., we consider

$$\hat{f}(t) = \frac{1}{\epsilon T}\int_{t-\epsilon T/2}^{t+\epsilon T/2} f(\tau)\,\mathrm{d}\tau.$$

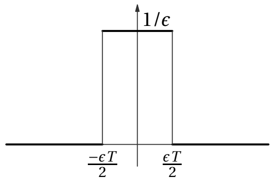
Averaging $f(t)$ this way filters out high-frequency noise. It is to be expected, then, that $\hat{f}(t)$ is somewhat smoother than $f(t)$, but as long as ϵ is not too large the graph of the averaged $\hat{f}(t)$ should retain roughly the same shape as the graph of $f(t)$. Figure 3.7(a) shows an example of a jumpy signal $f(t)$. Figure 3.7(b) shows $\hat{f}(t)$ for the case that $\epsilon = 0.03$. In plot (c) of that figure the average was taken over a wider interval ($\epsilon = 0.09$) and as expected the plot is smoother than the one in (b).

The signal $\hat{f}(t)$ can be considered as the convolution of $f(t)$ with a suitable function $g(t)$:

$$\begin{aligned}\hat{f}(t) &= \frac{1}{\epsilon T}\int_{t-\epsilon T/2}^{t+\epsilon T/2} f(\tau)\,\mathrm{d}\tau = \{v = t-\tau\} = \frac{1}{\epsilon T}\int_{-\epsilon T/2}^{\epsilon T/2} f(t-v)\,\mathrm{d}v \\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(t-v)g(v)\,\mathrm{d}v = (f * g)(t),\end{aligned}$$

with $g(t)$ a T-periodic function defined by:

$$g(t) = \begin{cases}\frac{1}{\epsilon} & \text{if } |t| \le \epsilon T/2, \\ 0 & \text{elsewhere.}\end{cases}$$



In frequency domain the process of averaging hence means multiplying the line spectrum with the line spectrum g_k of $g(t)$.

$$\begin{aligned}g_k &= \frac{1}{T}\int_{-T/2}^{T/2} g(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t \\ &= \frac{1}{\epsilon T}\int_{-\epsilon T/2}^{\epsilon T/2} \mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t = \{k \neq 0\} = \frac{-1}{\mathrm{i}k\omega_0\epsilon T}(\mathrm{e}^{-\mathrm{i}k\omega_0\frac{\epsilon T}{2}} - \mathrm{e}^{\mathrm{i}k\omega_0\frac{\epsilon T}{2}}) \\ &= \frac{\sin(k\omega_0\frac{\epsilon T}{2})}{k\omega_0\frac{\epsilon T}{2}} = \{\omega_0\tfrac{\epsilon T}{2} = \tfrac{2\pi}{T}\tfrac{\epsilon T}{2} = \pi\epsilon\} = \frac{\sin(k\epsilon\pi)}{k\epsilon\pi}, \\ g_0 &= \frac{1}{\epsilon T}\int_{-\epsilon T/2}^{\epsilon T/2} \mathrm{d}t = 1.\end{aligned}$$

Therefore

$$\hat{f}_k = \frac{\sin(k\epsilon\pi)}{k\epsilon\pi} f_k.$$

Note that $\sin(k\epsilon\pi)/(k\epsilon\pi)$ equals $\operatorname{sinc}(k\epsilon\pi)$. It tends to zero as $k \to \infty$. The high-frequency harmonics $f_k e^{i\omega_0 t}$ are therefore more attenuated than the lower frequency harmonics. This agrees with our understanding of averaging. Also, the greater the averaging interval, the smaller is $\operatorname{sinc}(k\epsilon\pi)$ for large k, i.e., the more are the high-frequency harmonics attenuated. Again this agrees with our understanding of averaging. □

Since T-periodic signals are fully determined by its Fourier coefficients, it should be possible to express any property that $f(t)$ may have in terms of its Fourier coefficients. It is for example possible to express the power

$$P_f = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt \tag{3.16}$$

of a T-periodic signal $f(t)$ in terms of the f_k's.

Theorem 3.5.4 (Parseval identity for periodic signals). If $f(t)$ is a piecewise smooth T-periodic signal, then

$$P_f = \sum_{k=-\infty}^{\infty} |f_k|^2.$$

Proof. This is a consequence of Parseval's identity as presented in Thm 1.5.7 (page 41) applied to Thm. 3.2.1. Indeed, in the inner product of Thm. 3.2.1 the power is the same as $\|f\|^2 := \langle f, f\rangle$, and according to Parseval and Thm. 3.2.1 the latter equals $\sum_{k=-\infty}^{\infty} |f_k|^2$.

(It can also be proved directly from Thm. 3.5.2: use $g(t) = f^*(-t)$ and then compare $(f * g)(t)$ at $t = 0$ with its Fourier series at $t = 0$.) ■

A curious by-product of the Parseval identity is that we can now explicitly compute certain classic series:

Example 3.5.5 (Power). Let $f(t)$ be the T-periodic sawtooth signal of Example 3.2.2. That is, $f(t) = t - T/2$ on $[0, T)$ and periodically continued elsewhere. In Example 3.2.2 we found that $f_k = \mathrm{i}/(k\omega_0)$ for $k \neq 0$ and that $f_0 = 0$. So on the one hand the power P_f equals

$$P_f = \frac{1}{T}\int_0^T (t - T/2)^2 \,\mathrm{d}t = \frac{1}{12}T^2,$$

and on the other hand, by Parseval's theorem,

$$P_f = \sum_{k\in\mathbb{Z},\, k\neq 0} \frac{1}{(k\omega_0)^2} = \frac{T^2}{4\pi^2} \sum_{k\in\mathbb{Z},\, k\neq 0} \frac{1}{k^2} = \frac{T^2}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

We conclude that

$$\frac{T^2}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{12}T^2,$$

in other words,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Nice. □

3.6 Gibbs phenomenon

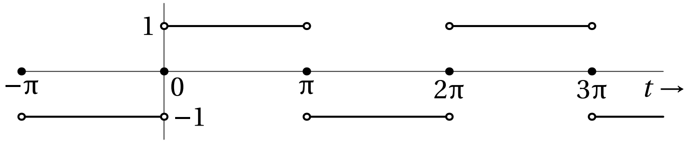
Since $f(t) = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$, it is tempting to think that partial sums $s_N(t)$ defined as

$$s_N(t) = \sum_{k=-N}^{N} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$$

form a "good" approximation of $f(t)$ if N is "large". In $\mathscr{L}^2$-sense that is correct, but the maximal difference, $\max_t |f(t) - s_N(t)|$, need *not* converge to zero! Let us illustrate it with the square wave.

Example 3.6.1 (Square wave). The T-periodic square wave $f(t)$, with $T = 2\pi$, is defined on $[0, 2\pi)$ as

$$f(t) = \begin{cases} 0 & \text{if } t = 0 \text{ or } t = \pi \\ 1 & \text{if } 0 < t < \pi \\ -1 & \text{if } \pi < t < 2\pi. \end{cases}$$



The square wave is real-valued, hence $f_k = f_{-k}^*$. Since the square wave is odd, the Fourier series consists of sines only (i.e. no cosines). Verify for yourself that the Fourier series is

$$f_*(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + \cdots\right).$$

The $(2N-1)$-th partial sum $s_{2N-1}(t)$ hence is

$$s_{2N-1}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots + \frac{1}{2N-1}\sin((2N-1)t)\right).$$

The derivative of $s_{2N-1}(t)$ clearly equals $\frac{4}{\pi}(\cos(t) + \cos(3t) + \cdots + \cos((2N-1)t))$ and from that it can be shown that $s_{2N-1}(t)$ is maximal at $t_N := \pi/(2N)$. This t_N converges to zero as $N \to \infty$ but, surprisingly, its peak value $s_N(t_N)$ does *not* converges to 1: see Table 3.2 (p. 137). Figure 3.8 shows plots of the partial sums $s_{2N-1}(t)$ for $N = 4, 8, 12, 16, 32$ and this confirms once more that the peak value does not converge to 1. Instead it seems to converges to 1.17898. □

In above example the amount overshoot is close to 0.17898. This type of overshoot phenomenon is called the *Gibbs phenomenon.* Without proof we claim that for every piecewise smooth function, near whatever point t of discontinuity, the overshoot, as $N \to \infty$, converges to[3]

$$\int_0^1 \operatorname{sinc}(\pi t)\,\mathrm{d}t - \frac{1}{2} = 0.08948987223608$$

times the magnitude of the jump $|f(t^+) - f(t^-)|$. In the previous example the magnitude of the jump at $t = 0$ is 2, so the overshoot converges to $2 \times 0.08948987223608 = 0.178979744472167$.

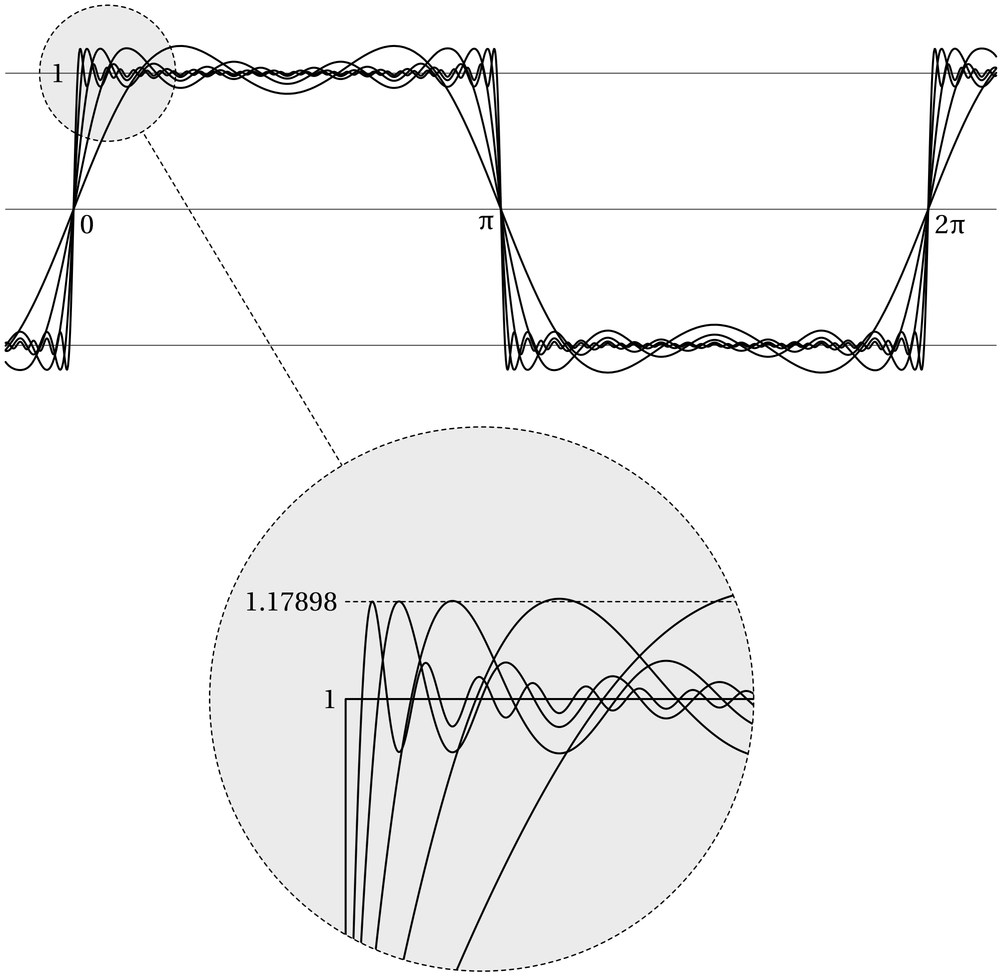


FIGURE 3.8: The partial sums $s_{2N-1}(t)$ of the Fourier series of the square wave, for $N = 2, 4, 8, 16, 32$. A part near the discontinuity at $t = 0$ is magnified. See Example 3.6.1

If the Fourier coefficients f_k are absolutely summable, i.e., if $\sum_{k=-\infty}^{\infty}|f_k|<\infty$, then the Gibbs phenomenon does not occur. Indeed, if $\sum_{k=-\infty}^{\infty}|f_k|<\infty$, then the maximal approximation error equals

$$\max_{t\in\mathbb{R}}|f(t)-s_N(t)| = \max_{t\in\mathbb{R}}\Big|\sum_{|k|>N} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}\Big| \le \sum_{|k|>N}|f_k|,$$

and the rightmost side is independent of t and converges to zero as $N\to\infty$. In such cases one says that the convergence of $s_N(t)$ to $f(t)$ is *uniform* across t.

3.7 Applications

In this section we discuss some application of the Fourier series.

Filters described by differential equations

We start with the following recapitulation of a result in Appendix A.3.

Consider the differential equation

$$\begin{aligned} p_n y^{(n)}(t) &+ p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\ &= q_m u^{(m)}(t) + q_{m-1} u^{(m-1)}(t) + \cdots + q_1 u^{(1)}(t) + q_0 u(t), \end{aligned} \tag{3.17}$$

with $p_0, p_1, \dots, p_n, q_0, \dots, q_m \in \mathbb{R}$, and $m \le n$. We assume that $u(t)$ is given, and that the problem is to determine $y(t)$. In this context $u(t)$ is usually referred to as the *input* and $y(t)$ the *output*.

Let $s\in\mathbb{C}$. For exponential inputs

$$u(t) = \mathrm{e}^{st},$$

the differential equation simplifies to

$$\begin{aligned} p_n y^{(n)}(t) &+ p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\ &= q_m s^m \mathrm{e}^{st} + q_{m-1} s^{m-1} \mathrm{e}^{st} + \cdots + q_1 s \mathrm{e}^{st} + q_0 \mathrm{e}^{st} \\ &= \left(q_m s^m + q_{m-1} s^{m-1} + \cdots + q_1 s + q_0\right) \mathrm{e}^{st}. \end{aligned} \tag{3.18}$$

[3] It can be shown that $s_N\big(t+\frac{T}{2N+1}\big) - f(t^+)$ converges to $\big(\int_0^1 \operatorname{sinc}(\pi t)\,\mathrm{d}t - \frac{1}{2}\big)(f(t^+)-f(t^-))$ as $N\to\infty$.

Then (3.18) has a particular solution of the same exponential form

$$y(t) = A\mathrm{e}^{st},$$

for some as yet unknown constant A. Substituting this form into (3.18), we find

$$\begin{aligned} p_n A s^n \mathrm{e}^{st} + p_{n-1} A s^{n-1} \mathrm{e}^{st} + \cdots + p_1 A s \mathrm{e}^{st} + p_0 A \mathrm{e}^{st} \\ = \left(q_m s^m + q_{m-1} s^{m-1} + \cdots + q_1 s + q_0\right) \mathrm{e}^{st}. \end{aligned} \tag{3.19}$$

Both sides of the equation has the invertible factor e^{st}, which hence may be canceled. This way we find

$$\left(p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0\right) A = q_m s^m + q_{m-1} s^{m-1} + \cdots + q_1 s + q_0. \tag{3.20}$$

The term in between the brackets we recognize as the characteristic polynomial $P(s)$ of the differential equation. Likewise let $Q(s)$ be the polynomial of the right-hand side of the equation. We thus have that

$$A = \frac{Q(s)}{P(s)} = \frac{q_m s^m + q_{m-1} s^{m-1} + \cdots + q_1 s + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0},$$

provided $P(s) \neq 0$, i.e. provided s is not a zero of the characteristic polynomial. We summarize the above result in the following theorem. It also defines the notion of “transfer function”:

Theorem 3.7.1 (Transfer function). Consider (3.17), and let $s \in \mathbb{C}$ be such that $p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0 \neq 0$. One particular solution $y(t)$ corresponding to input $u(t) = \mathrm{e}^{st}$ is $y_{\text{part}}(t) = H(s)\mathrm{e}^{st}$, where $H(s)$ is the *transfer function* of the system given by

$$H(s) = \frac{q_m s^m + q_{m-1} s^{m-1} + \cdots + q_1 s + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0}. \tag{3.21}$$

□

Now assume that $u(t)$ is periodic and consider its Fourier series,

$$u(t) = \sum_{k=-\infty}^{\infty} u_k \mathrm{e}^{\mathrm{i}k\omega_0 t}.$$

According to the above theorem to each exponential input $\mathrm{e}^{\mathrm{i}k\omega_0 t}$ there corresponds an exponential particular output $H(\mathrm{i}k\omega_0)\mathrm{e}^{\mathrm{i}k\omega_0 t}$. Also, by linearity (see Exercise A.3.1) we have that a particular solution for $\alpha u_1(t)+\beta u_2(t)$ is $\alpha y_1(t)+\beta y_2(t)$, where y_k is a particular solution for $u_k, k=1,2$. Repeating this argument we see that a particular solution corresponding to $\sum_{k=-N}^{N} u_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$ is given by $y_{\text{part}}(t)=\sum_{k=-N}^{N} H(\mathrm{i}k\omega_0)u_k \mathrm{e}^{\mathrm{i}\omega_0 kt}$. Letting N go to infinity and we find the following:

Theorem 3.7.2 (Response to periodic inputs). Let $u(t)$ be a T-periodic signal with Fourier coefficients u_k. Then a particular solution of differential equation (3.17) for this input is given by

$$y(t)=\sum_{k=-\infty}^{\infty} H(\mathrm{i}k\omega_0)u_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$$

where $H(s)$ is the transfer function (3.21) of the differential equation. This particular solution is also T-periodic and its Fourier coefficients are

$$y_k = H(\mathrm{i}k\omega_0)u_k. \tag{3.22}$$

□

We apply this theorem on two examples. In the first example we show that certain RC-networks can be seen as low-pass filters.

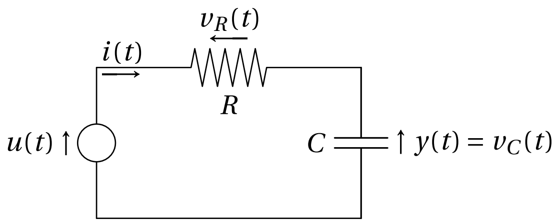


FIGURE 3.9: An RC-network (Example 3.7.3)

Example 3.7.3 (RC-network). Consider the RC-network shown in Figure 3.9. We interpret the RC-network as a system with the voltage delivered by the voltage source as the input, $u(t)$, of the system, and the voltage across the capacitor as the output, $y(t)$.

The input and output are related by a differential equation that may be obtained using Kirchhoff's voltage law and the voltage-current relations of resistors and capacitors. Kirchhoff's voltage law gives that

$$u(t) = v_R(t) + y(t) = Ri(t) + y(t). \tag{3.23}$$

The voltage across the capacitor equals

$$y(t) = \frac{q(t)}{C} = \{q(t) \text{ is the charge, } q(t) = \int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau\} = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau. \tag{3.24}$$

Differentiation with respect to time gives

$$Cy^{(1)}(t) = i(t).$$

Substituting this expression for $i(t)$ in (3.23), we get a differential equation in the input and output

$$y^{(1)}(t) + \alpha y(t) = \alpha u(t), \tag{3.25}$$

in which $\alpha = \frac{1}{RC}$. This is a first-order ordinary linear differential equation of the type (3.17) with $n = 1$, $p_1 = 1$, $p_0 = \frac{1}{RC}$, $m = 0$, and $q_0 = \frac{1}{RC}$. Hence by Theorem 3.7.2 we have that the transfer function is given by

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{1}{RCs + 1}. \tag{3.26}$$

Let us take as input $u(t)$ the sawtooth of Example 3.2.2. For this input we computed the Fourier coefficients, $u_k = \frac{\mathrm{i}}{k\omega_0}$, $k \neq 0$, and $u_0 = 0$. The Fourier coefficients of the output $y(t)$ then follow as

$$y_k = H(\mathrm{i}k\omega_0)u_k = \begin{cases} \frac{1}{\mathrm{i}k\omega_0 RC+1} \times \frac{\mathrm{i}}{k\omega_0} & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases} \tag{3.27}$$

Figure 3.10 shows the plot of the absolute value of these Fourier coefficients for different values of RC. Furthermore, we plot the corresponding time signal $y(t)$. In these plots we

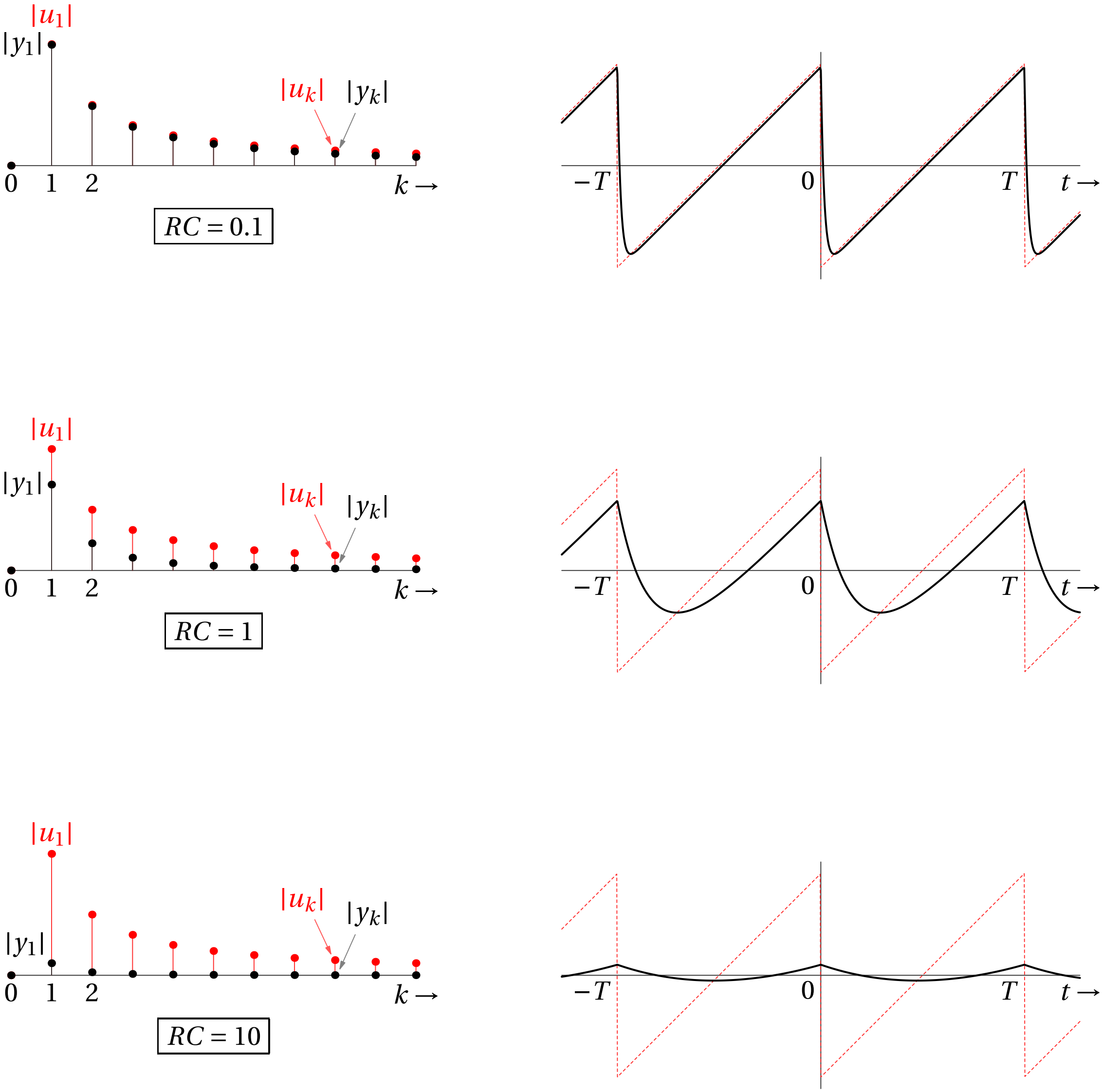


FIGURE 3.10: Left: Magnitude $|y_k|$ of the Fourier coefficients of the output $y(t)$ of the RC-circuit for $\omega_0 = 1$ and for $RC = 0.1$, 1 and 10. (The magnitudes $|u_k|$ of the input are shown in red). Right: corresponding output $y(t)$ (and the sawtooth input $u(t)$ in red). See Example 3.7.3

took $T = 2\pi$. In all figures we notice that the magnitude of the Fourier coefficients for high index k are much smaller for the output than they are for the input. Since these indices correspond to a high frequency harmonics, we conclude that this RC-network filters out high frequencies.

We can make it even more apparent when we consider the input $u(t) = \cos(t) + 0.1\cos(50t)$, see Figure 3.11. The Fourier coefficients of this signal are

$$u_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ \frac{1}{10} & k = \pm 50 \\ 0 & \text{elsewhere.} \end{cases}$$

If we choose $RC = 1$, then the Fourier coefficients of the output approximately are

$$y_k = \begin{cases} 0.5 - 0.5\mathrm{i} & k = 1 \\ 0.5 + 0.5\mathrm{i} & k = -1 \\ 0.0004 - 0.002\mathrm{i} & k = 50 \\ 0.0004 + 0.002\mathrm{i} & k = -50 \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly, when compared to the input, the values y_{-50} and y_{50} of the output are practically zero, meaning that frequency $\omega = 50$ is practically absent in $y(t)$, only frequency $\omega = 1$ remains. Its amplitude and phase differs from that of $u(t)$. This amplitude and phase change is best seen from a plot of amplitude and phase of $H(\mathrm{i}\omega)$ as a function of ω. This $H(\mathrm{i}\omega)$ seen as a function of ω is known as the *frequency response*. More on this in the next example (and, later, in § 4.6). □

Example 3.7.4 (Hypnotist). The standard act of a hypnotist is to swing a watch on a cord in front of a candidate from the audience so as to hypnotize this volunteer, see Figure 3.12. The horizontal position of his hand is denoted by u. The angle of the cord with the vertical axis is denoted by y. Without proof we state that a model describing the relation between u and y, for y not too large, is given by

$$m\ell y^{(2)}(t) + k\ell y^{(1)}(t) + mgy(t) = -mu^{(2)}(t) - ku^{(1)}(t). \tag{3.28}$$

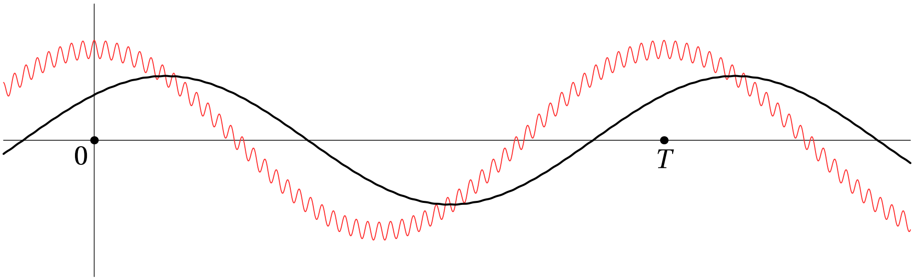


FIGURE 3.11: Input $u(t) = \cos(t) + 0.1\cos(50t)$ (in red) and its response $y(t)$ (in black) of the RC-network with $RC = 1$, see Example 3.7.3

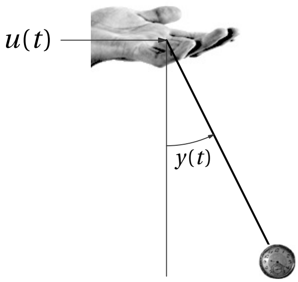


FIGURE 3.12: The hypnotist

Here ℓ is the length of the cord, g denotes the gravitational constant and k is a friction coefficient. The transfer function of (3.28) is given by

$$H(s) = \frac{-ms^2 - ks}{m\ell s^2 + k\ell s + mg}.$$

In practice k is a small positive constant, but if it would have been zero, then $H(\pm i\sqrt{g/\ell})$ would not exist. Therefore for small positive k, the denominator of $H(s)$ is small for $s = \pm i\sqrt{g/\ell}$ and $H(s)$ is large in magnitude at these points. Figure 3.13 presents a numerical example.

For the constants $m = 0.1$ [kg], $k = 0.05$ [kg/s], $\ell = 0.3$ [m], and $g = 9.81$ [kg m/s^2], we see that the maximum of $|H(i\omega)|$ is at approximately $\omega = 5$. If the hypnotist moves his hand as

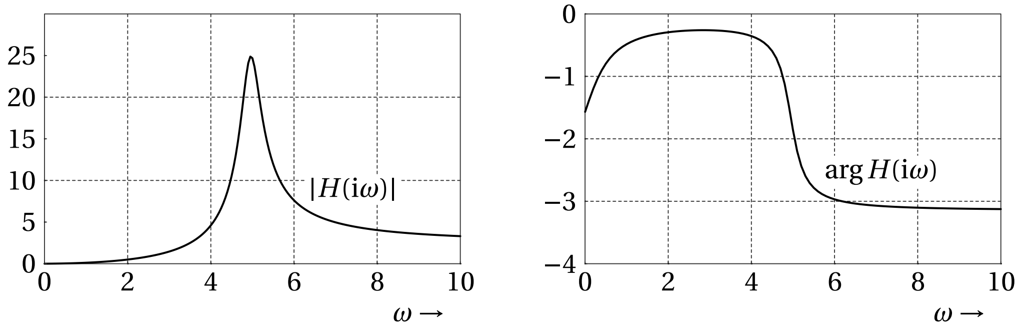


FIGURE 3.13: Amplitude and phase of $H(\mathrm{i}\omega)$ (where $m = 0.1, k = 0.05, \ell = 0.4, g = 10$), see Example 3.7.4

$\epsilon\cos(5t)$ with ϵ small, we can see that the movement of the pendulum is relatively large. Let us explicitly determine this movement. The Fourier series expansion of $\epsilon\cos(5t)$ is

$$\epsilon\cos(5t) = \frac{\epsilon}{2}\mathrm{e}^{\mathrm{i}5t} + \frac{\epsilon}{2}\mathrm{e}^{-\mathrm{i}5t}.$$

Note that this is an application of Euler's formula. The corresponding output is given by

$$y(t) = \frac{\epsilon}{2}H(\mathrm{i}5)\mathrm{e}^{\mathrm{i}5t} + \frac{\epsilon}{2}H(-\mathrm{i}5)\mathrm{e}^{-\mathrm{i}5t}.$$

Writing $H(\mathrm{i}5)$ as $|H(\mathrm{i}5)|\mathrm{e}^{\mathrm{i}\arg(H(\mathrm{i}5))}$, and using the fact that $H(-\mathrm{i}5)$ is the complex conjugate of $H(\mathrm{i}5)$, we find that

$$\begin{aligned}
y(t) &= \frac{\epsilon}{2}H(\mathrm{i}5)\mathrm{e}^{\mathrm{i}5t} + \frac{\epsilon}{2}H(-\mathrm{i}5)\mathrm{e}^{-\mathrm{i}5t} \\
&= \frac{\epsilon}{2}|H(\mathrm{i}5)|\mathrm{e}^{\mathrm{i}\arg(H(\mathrm{i}5))}\mathrm{e}^{\mathrm{i}5t} + \frac{\epsilon}{2}|H(\mathrm{i}5)|\mathrm{e}^{-\mathrm{i}\arg(H(\mathrm{i}5))}\mathrm{e}^{-\mathrm{i}5t} \\
&= |H(\mathrm{i}5)|\left[\frac{\epsilon}{2}\mathrm{e}^{\mathrm{i}(5t+\arg(H(\mathrm{i}5))} + \frac{\epsilon}{2}\mathrm{e}^{-\mathrm{i}(5t+\arg(H(\mathrm{i}5)))}\right] \\
&= |H(\mathrm{i}5)|\epsilon\cos(5t+\arg(H(\mathrm{i}5))) \\
&\approx 25\epsilon\cos(5t-2).
\end{aligned}$$

The final equality we infer from the plot (Fig. 3.13). Notice the negative phase of about -2. This means that the movement of the watch lags behind that of the hand. A modest periodic movement of the hand, say a cosine with amplitude of merely $2\,\mathrm{cm} = 0.02\,\mathrm{m}$, makes the angle of the cord behave as a cosine as well with an amplitude of approximately $0.02 \times 25 = 0.5\,\mathrm{rad}$.

Other cases are explored in Exercise 3.24. □

3.8 Exercises

3.1 Suppose $f(t)$ is 2-periodic and that

$$f(t) = 2t \quad \text{for } -1 \le t < 1.$$

(a) Determine the Fourier coefficients f_k.

(b) For which $t \in \mathbb{R}$ are the Fourier series and $f(t)$ the same?

(c) Use the previous results to compute $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.
Hint: Evaluate the Fourier series at $t = \frac{1}{2}$.

3.2 Let $f(t)$ be the π-periodic signal that is given by

$$f(t) = \cos(t) \quad 0 \le t < \pi$$

(a) Sketch the graph of $f(t)$.

(b) Determine the Fourier coefficients f_k.

3.3 Let $f(t)$ be the 2-periodic signal defined as

$$f(t) = \mathrm{e}^{-|t|}, \qquad \text{for } -1 \le t < 1.$$

(a) Determine the complex Fourier coefficients f_k.

(b) Determine the real Fourier coefficients a_k, b_k.

3.4 Express $\sin^2(\omega_0 t + \pi/3)$ as a superposition of complex harmonic signals and as superposition of sinusoids.

3.5 Suppose a T-periodic signal $f(t)$ is such that its Fourier coefficients f_k satisfy $f_{-k} = -f_k^*$ for all integers k. Show that $f(t)$ is imaginary-valued (that is, that $\mathrm{i}f(t)$ is real-valued).

3.6 Given is a T-periodic signal $f(t)$. Suppose, in addition that $f(t)$ is real and even. Show that f_k is real and that $f_{-k} = f_k$ for any integer k.

3.7 Let $f(t)$ be a T-periodic signal that on period $[0, T]$ is given by $f(t) = \text{rect}_{T/2}(t - T/2)$.

(a) Sketch the graph of $f(t)$.

(b) Determine the Fourier coefficients f_k of $f(t)$.

(c) Sketch the amplitude and phase spectrum of $f(t)$.

(d) Determine the real Fourier series of $f(t)$.

3.8 Suppose $f(t)$ is a 2π-periodic signal with Fourier coefficients $f_k = 1/(k^2 + 1)$.

(a) Show that $f(t)$ is real and even.

(b) Determine the Fourier coefficients of $f(t)\cos^2(\omega_0 t)$.

(c) Determine the Fourier coefficients of $f(2t)$.

(d) Determine the phase spectrum of $f(2t - T/2)$.

3.9 Let $f(t)$ be the 2π-periodic signal such that

$$f(t) = t^2 \quad (-\pi \le t \le \pi).$$

(a) Determine the complex Fourier coefficients of $f(t)$ and write down the Fourier series of $f(t)$.

(b) Determine the real Fourier series of $f(t)$.

(c) What is the third harmonic of $f(t)$?

(d) Calculate $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$.

3.10 Let $f(t)$ be the 2π-periodic signal such that

$$f(t) = (t - 3)^2 \quad (3 - \pi \le t \le 3 + \pi).$$

Determine the Fourier coefficients of $f(t)$.

3.11 Let $f(t)$ be the 2π-periodic signal such that

$$f(t) = \mathrm{e}^{4\mathrm{i}t}\, t^2 \quad (-\pi \le t \le \pi).$$

Determine the Fourier coefficients of $f(t)$.

3.12 Let $f(t)$ be the 2π-periodic signal such that

$$f(t) = \mathrm{e}^{2\pi\mathrm{i}t}\, t^2 \quad (-\pi < t \le \pi).$$

Determine the Fourier coefficients of $f(t)$.

3.13 Let $f(t)$ be the π-periodic signal given by $f(t) = \sin^2(t)$. Determine the second and third harmonic of $f(t)$.

3.14 Let $f(t)$ be the T-periodic signal such that

$$f(t) = \mathrm{rect}_{T/2}(t) \quad (-T/2 \le t \le T/2).$$

Determine the Fourier coefficients of $f(t)$.

3.15 Given is the T-periodic signal $f(t)$ that on one interval $[0, T]$ equals $f(t) = |t - T/2|$.

(a) Show that $f(t)$ has a real-valued Fourier coefficients.

(b) Calculate the power of the first harmonic of $f(t)$.

3.16 Let $f(t)$ be the T-periodic signal with Fourier coefficients f_k and let $\omega_0 = 2\pi/T$.

(a) Determine the Fourier coefficients of $f(t)\cos^2(\omega_0 t)$.

(b) Show that $g(t) = f(t)\mathrm{e}^{\mathrm{i}\omega_0 t/2}$ is periodic with period $2T$.

(c) Determine the Fourier coefficients and power of $g(t)$.

3.17 Let $f(t)$ be a T-periodic signal and let $g(t)$ be the signal given by

$$g(t) = \frac{1}{a}\int_{t-a}^{t} f(u)\,\mathrm{d}u.$$

Here we assume that $0 < a < T$.

(a) Show that $g(t)$ is T-periodic.

(b) Determine the Fourier coefficients of $g(t)$.

(c) What can you tell about $g(t)$ for the case that $a = T$?

3.18 Determine the power of the following signals.

(a) $f(t) = \cos(\omega_0 t) + 2\sin(\omega_0 t)$.

(b) $f(t) = |\sin(\omega_0 t)|$.

3.19 Let $f(t)$ be a real T-periodic signal with real Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\tfrac{2\pi}{T}t) + b_k \sin(k\tfrac{2\pi}{T}t)\right).$$

Which coefficients a_k, b_k are guaranteed to be zero if

(a) $f(t)$ is even, that is, if $f(t) = f(-t)$ for all t,

(b) $f(t)$ is odd, that is, if $f(t) = -f(-t)$ for all t,

(c) $f(t)$ has period $T/2$. (Explain.)

3.20 Use Example 3.6.1 to show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$.

3.21 Consider a real-valued signal $f(t)$ and its real Fourier series (Theorem 3.3.1). Show Parseval's theorem for the real Fourier series: $P_f = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$.

3.22 Determine the complex Fourier series expansion of

(a) $f(t) = \cos(t + \theta)$

(b) $f(t) = \cos(2t + \theta)$

(c) $f(t) = \sin(t) + \cos(t)$

(d) $f(t) = \sin(2t) + \cos(3t)$

3.23 Consider the differential equation (3.17). Assume further that the input is given as $\cos(\omega_0 t)$.

(a) Determine the complex Fourier series expansion of $u(t)$.

(b) Determine the complex Fourier series expansion of the corresponding output $y(t)$.

(c) Show that this output is given by

$$y(t) = |H(\mathrm{i}\omega_0)|\cos(\omega_0 t + \arg(H(\mathrm{i}\omega_0))).$$

3.24 Consider the hypnotist of Example 3.7.4. As input we take $u(t) = \cos(\omega_0 t)$. Furthermore, we assume that the constants are the same as in the example.

(a) Calculate an output if $\omega_0 = 0$. Explain your answer also physically.

(b) What happens with the output if $\omega_0 \to \infty$?

(c) Determine the frequency ω_0 for which $y(t)$ has maximal amplitude? What is this maximum?

3.25 Suppose $f(t)$ and $g(t)$ are two T-periodic continuous signals, and suppose their Fourier coefficients are the same: $f_k = g_k \ \forall k \in \mathbb{Z}$. Do we necessarily have that $f(t) = g(t)$ for all t?

More involved problems

3.26 Suppose a given piecewise smooth signal $f(t)$ is such that $f(t+T/2) = -f(t)$ for all t and a certain fixed $T > 0$. Show that $f(t)$ is periodic and that $f_{2k} = 0$ for every integer k.

3.27 Determine the period and Fourier coefficients of the periodic signal

$$f(t) = \frac{\sin(2t) + \sin(3t)}{\sin(t)}.$$

3.28 Use Euler's formula to determine a simple closed expression for the function

$$\sum_{k=0}^{\infty} \frac{\cos(kt)}{2^k}.$$

3.29 Determine $\sum_{n=1}^{\infty} \frac{1}{n^4}$. (Hint: Use Exercise 3.9.)

3.30 Lemma 3.2.5 implies that for each piecewise smooth function $f(t)$ a constant A exists such that $|f_k| < A/|k|$ for all k. Show that if $f(t)$ is n times continuously differentiable (i.e. $f^{(n)}(t)$ exists and is continuous), then $|f_k| < A/|k|^n$ for some A.

MATLAB problems

3.31 In Example 3.3.4 we found the real Fourier coefficients of $f(t) = |\sin(\pi t)|$,

$$a_k = \frac{4}{\pi}\frac{1}{1-4k^2}, \quad b_k = 0.$$

To calculate in MATLAB the partial sums

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(k w_0 t) \tag{3.29}$$

we open a file with name, say, `mysum.m` and enter the following code

```
function sn = mysum(t,N)
w0=2*pi;
sn=2/pi;
for k=1:N,
  sn=sn+ (4/pi)*(1/(1-4*k^2))*cos(k*w0*t);
end
```

Then the sum (3.29) can be computed by typing the following commands at the MATLAB prompt.

```
t=0:0.01:1;                  % Discretized time
N=5;                         %
sn=mysum(t,N);               % Calculate partial sum
plot(t,sn)                   % Plot it
hold on                      % Keep this plot
plot(t,abs(sin(pi*t)),'red') % Add a plot of f(t)
hold off                     %
```

Try this MATLAB code and then similarly plot the sum of the first N harmonics for $N = 2, 5, 10$ of the Fourier series of the π-periodic function $f(t)$ defined as

$$f(t) = t(\pi - t), \qquad (0 \leq t \leq \pi).$$

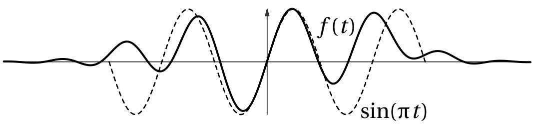
TABLE 3.2: The Gibbs phenomenon. Peak value of the $s_{2N-1}(t)$ of Example 3.6.1

N	$s_{2N-1}(\frac{\pi}{2N})$
4	1.180284
8	1.179305
16	1.179061
32	1.179000
64	1.178985
128	1.178981
256	1.178980

Chapter 4

Fourier Transform

The Fourier series expansion of the previous chapter applies to periodic signals. But what if $f(t)$ is not periodic, such as



While not periodic, we still feel that this signal $f(t)$ in some way contains a sinusoid $\sin(\pi t)$. This chapter is about a version of Fourier type expansions for aperiodic signals. Under mild assumptions, aperiodic signals $f(t)$ can be seen as a "continuous sum" of harmonic signals, that is to say, as an integral of weighted harmonic signals

$$f(t) = \int_{-\infty}^{\infty} c(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

Compare this with (3.2). Loosely speaking, integration is the same as summation, so the above integral says that $f(t)$ is a sum (integral) of weighted harmonic signals $\mathrm{e}^{\mathrm{i}\omega t}$. This integral expression of $f(t)$ has similar applications as the Fourier series, and we will encounter more applications in this chapter.

We assume throughout this chapter that the signals $f(t)$ are piecewise smooth, and that

$$f(t) = \frac{f(t^+) + f(t^-)}{2} \tag{4.1}$$

at every t. Equation (4.1) may always be achieved by redefining $f(t)$ at its points of discontinuity, if necessary.

4.1 The Fourier integral theorem

The proof of the Fourier series theorem of the previous chapter (Theorem 3.2.3) as listed in Appendix A.1 relies on the Riemann-Lebesgue Lemma (Lemma 3.2.5). A strengthened version that we need for the results in this chapter is as follows.

Lemma 4.1.1 (Riemann-Lebesgue). Suppose $f(t)$ is absolutely integrable and piecewise smooth. Then

$$\lim_{|\omega|\to\infty}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = 0. \tag{4.2}$$

□

This result is the basis of the following peculiar result that we soon need.

Lemma 4.1.2. Suppose $f(t)$ is absolutely integrable and piecewise smooth. Then

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} f(t)\,a\operatorname{sinc}(at)\,\mathrm{d}t = \pi\frac{f(0^-)+f(0^+)}{2}. \tag{4.3}$$

***Proof*.** Essentially the same as that of Lemma A.1.1 in Appendix A.1. ■

In the two lemmas we assume that $f(t)$ is absolutely integrable, which is something we have not yet defined. A signal $f(t)$ is said to be *absolutely integrable* if

$$\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t$$

is convergent (i.e. is finite). Roughly speaking this means that $f(t)$ should go to zero fast enough as $t\to\pm\infty$. This condition is needed to guarantee convergence of the integrals in the following theorem. Be aware that the next theorem assumes $f(t)=\frac{f(t^+)+f(t^-)}{2}$. With that out of the way we can prove the famous result:

Theorem 4.1.3 (The Fourier integral theorem). Suppose $f(t)$ is absolutely integrable, piecewise smooth, and that it satisfies (4.1). Then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{4.4}$$

where $\hat{f}(\omega)$ is the *Fourier transform* of $f(t)$, defined as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{4.5}$$

Proof. Substituting the integral expression (4.5) of $\hat{f}(\omega)$ in the right-hand side of (4.4) gives

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega &= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a}\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{\mathrm{i}\omega(t-\tau)}\,\mathrm{d}\tau\,\mathrm{d}\omega \\
&= \{\text{change order of integration}\} \\
&= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\int_{-a}^{a} \mathrm{e}^{\mathrm{i}\omega(t-\tau)}\,\mathrm{d}\omega\,\mathrm{d}\tau \\
&= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\frac{\mathrm{e}^{\mathrm{i}a(t-\tau)} - \mathrm{e}^{-\mathrm{i}a(t-\tau)}}{\mathrm{i}(t-\tau)}\,\mathrm{d}\tau \\
&= \lim_{a\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\, a\,\mathrm{sinc}(a(t-\tau))\,\mathrm{d}\tau.
\end{aligned}$$

That the order of integration may be changed is due to the fact that $f(t)$ is absolutely integrable. Since

$$\int_{-\infty}^{\infty} f(\tau)\, a\,\mathrm{sinc}(a(t-\tau))\,\mathrm{d}\tau = \{v = t-\tau\} = \int_{-\infty}^{\infty} f(t-v)\, a\,\mathrm{sinc}(av)\,\mathrm{d}v,$$

we see that

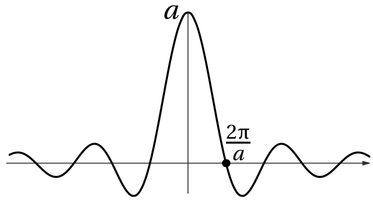
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \lim_{a\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t-v)\, a\,\mathrm{sinc}(av)\,\mathrm{d}v = f(t).$$

The last identity follows from Lemma 4.1.2. ∎

Note the striking symmetry between the expressions for $f(t)$ and $\hat{f}(\omega)$. As it turns out, absolute integrability of $f(t)$ is enough to ensure the Fourier integral theorem to be valid, i.e., we need not impose something similar on $\hat{f}(\omega)$.

Example 4.1.4. The rectangular pulse $\operatorname{rect}_a(t)$ as defined in Definition 2.5.1 is bounded and of finite duration, so it is absolutely integrable. Its Fourier transform $\hat{f}(\omega)$ equals

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} \operatorname{rect}_a(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_{-a/2}^{a/2} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega}\Big|_{t=-a/2}^{t=a/2} = \frac{\mathrm{e}^{\mathrm{i}a\omega/2} - \mathrm{e}^{-\mathrm{i}a\omega/2}}{\mathrm{i}\omega} \\ &= \frac{\sin(a\omega/2)}{\omega/2} = a\operatorname{sinc}(a\omega/2).\end{aligned}$$



□

The function $\hat{f}(\omega)$ is known under a variety of names. It is called the *Fourier transform* of $f(t)$, and sometimes it is referred to as the *spectrum* or *frequency spectrum* of $f(t)$. Also *plots* of $\hat{f}(\omega)$ as a function of ω are called spectrum or frequency spectrum. Since $\hat{f}(\omega)$ is generally complex-valued, a plot of $\hat{f}(\omega)$ consists generally of two parts, one of its amplitude versus frequency, and one of its phase versus frequency. This amplitude $A(\omega)$ and phase $\phi(\omega)$ follow from the polar form

$$\hat{f}(\omega) = A(\omega)\mathrm{e}^{\mathrm{i}\phi(\omega)}$$

in which $A(\omega)$ is real and nonnegative, and $\phi(\omega)$ also real, often restricted to the interval $[-\pi, \pi]$. We call $A(\omega)$ the *amplitude spectrum* and $\phi(\omega)$ the *phase spectrum.*

The Fourier transform $\hat{f}(\omega)$ is said to describe the function in the *frequency!domain* or the ω*-domain.* In the previous chapter we found that T-periodic signals are built up from a discrete set of frequencies, namely the multiples of the fundamental frequency. Aperiodic signals as we see now are built up from a *continuum* of frequencies: $\omega \in \mathbb{R}$.

The Fourier transform can reveal properties of the signal $f(t)$ that may not be apparent from $f(t)$ itself. Consider Example 4.1.4, where we computed the Fourier transform of the rectangular pulse $\operatorname{rect}_a(t)$. Figure 4.1 shows for three values of a the corresponding $\operatorname{rect}_a(t)$ and its Fourier transform. What we notice is that for small values of a the Fourier transform is smeared out over a wide frequency range (Figure 4.1a,b). More important for our understanding of the Fourier transform is to see what happens if a is large, such as shown in Figure 4.1(e,f). In that case $\operatorname{rect}_a(t)$ is constant equal to 1 for a long time. As we see from Figure 4.1(f), this apparently implies that the Fourier transform is practically

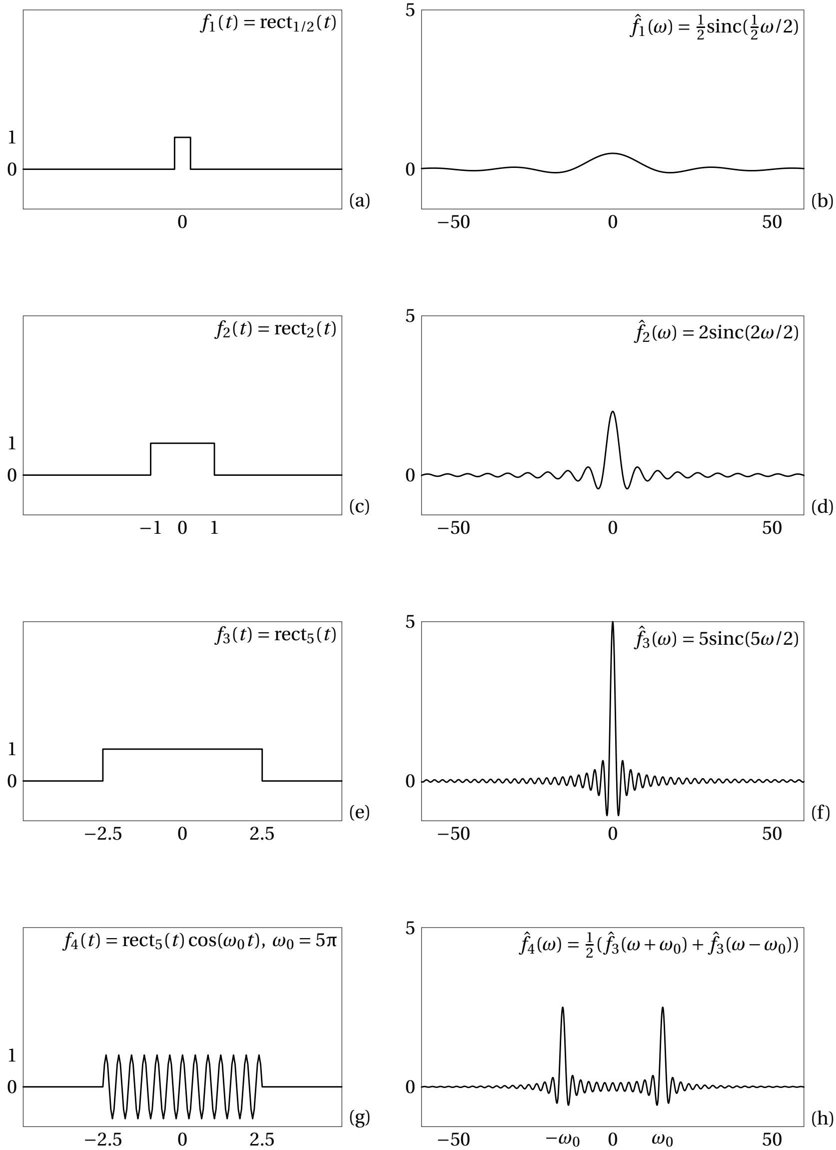


FIGURE 4.1: Examples of signals $f_i(t)$ and their Fourier transforms $\hat{f}_i(\omega)$

built up from the single frequency $\omega = 0$ only; for all other frequencies the Fourier transform $\hat{f}(\omega)$ is very small. Stated differently, the signal $\operatorname{rect}_a(t)$ for large a has its "frequency content" concentrated around $\omega = 0$. This, in hindsight, is actually not surprising, since $\hat{f}(0)$ being relatively large means that the zero frequency dominates, and zero frequency means constant signal. Slightly more concrete: if $\hat{f}(\omega)$ is relatively large on some small interval $[-\epsilon, \epsilon]$ and relatively small outside this interval, then

$$f(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega$$

approximately equals

$$\frac{1}{2\pi} \int_{-\epsilon}^{+\epsilon} \hat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega \approx \frac{2\epsilon}{2\pi} \hat{f}(0) \mathrm{e}^{0t}$$

which is constant as a function of time. For similar reasons it is to be expected that a signal $f(t)$ such as

$$f(t) = \operatorname{rect}_a(t) \cos(\omega_0 t)$$

has its frequency content concentrated around frequency $\omega = \pm\omega_0$, that is, has a Fourier transform $\hat{f}(\omega)$ with spikes near $\omega = \pm\omega_0$. Indeed, if we do the computation of $\hat{f}(\omega)$ then we get what is shown in Figure 4.1(h).

Example 4.1.5 (Low tide and high tide). Near the city of Vlissingen the water level $f(t)$ of the sea is measured every ten minutes. Figure 4.2 depicts the water level for a time span of three days and sixty days. The first measurement in both plots is from 1 September 1989 at ten minutes past 11am. With numerical recipes it is possible to compute with high accuracy the Fourier transform $\hat{f}(\omega)$ of $f(t)$. The two plots in the bottom half of the figure show $|\hat{f}(\omega)|$ over two ranges of frequencies. Note the huge spike of $|\hat{f}(\omega)|$ just to the left of $\frac{\omega}{2\pi} = 2$. This tells us that $f(t)$ is close to periodic, with period $T \approx 1/2$, which means half a day. It represents the (first harmonic of the) fluctuation of the water level due to the moons gravitational pull. Also note the little humps in $|\hat{f}(\omega)|$ at about $\frac{\omega}{2\pi} = 4$ and $\frac{\omega}{2\pi} = 6$. (We have something more to say about this in § 4.5.) Can you explain the little spike of $|\hat{f}(\omega)|$ at *precisely* $\frac{\omega}{2\pi} = 1$? □

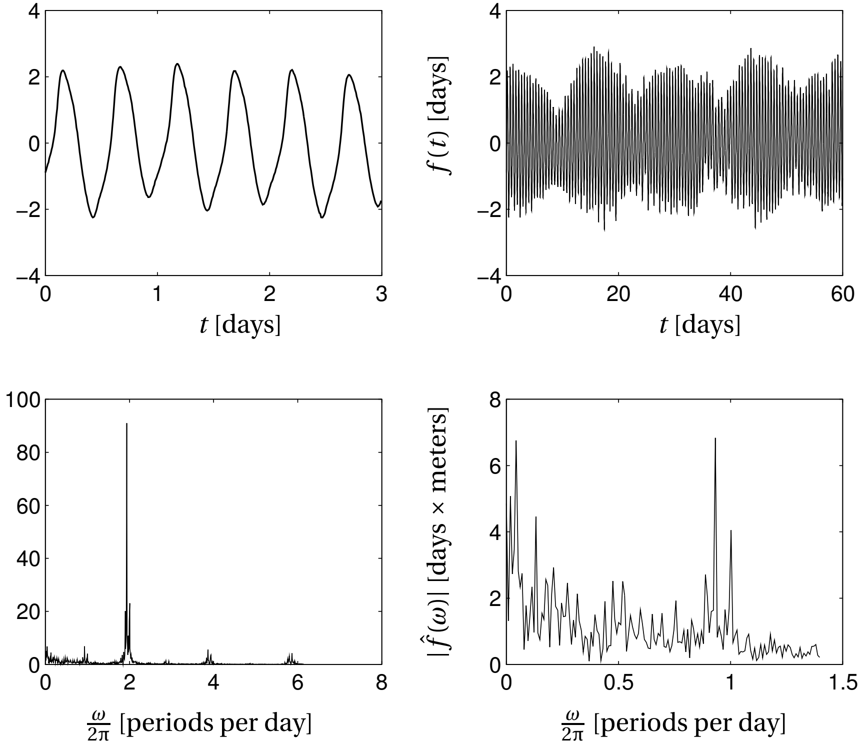
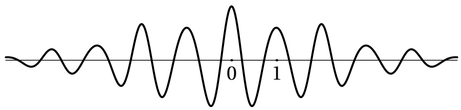


FIGURE 4.2: Vlissingen seawater level $f(t)$ and its $|\hat{f}(\omega)|$

Example 4.1.6 (Almost hidden frequencies). Visual inspection of this superposition of two damped harmonic signals

$$f(t) = [\cos(2\pi t) + \frac{1}{10}\cos(3\pi t)]\mathrm{e}^{-t^2/10}$$



reveals only one of the two frequencies, $\omega = 2\pi$. The other frequency, $\omega = 3\pi$, is hidden due to its small amplitude. Both frequencies do show up in the Fourier transform:

$\hat{f}(\omega) =$

Nice, the Fourier transform has the ability to reveal even the tiniest of harmonic signals.

□

4.2 Fourier transform properties

The Fourier transform of $f(t)$ is defined as the frequency domain *function* $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$. Often "Fourier transform" is also used for the *mapping* $\mathfrak{F}$ that sends $f(t)$ to $\hat{f}(\omega)$:

$$\mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Likewise, the *inverse Fourier transform* refers either to the mapping $\mathfrak{F}^{-1}$ that sends $\hat{f}(\omega)$ to $f(t)$ or refers to $f(t)$ itself, seen as the result of a given $\hat{f}(\omega)$.

The connection between $f(t)$ and $\hat{f}(\omega)$ is conveniently expressed as a *transform pair*

$$f(t) \xleftrightarrow{\mathfrak{F}} \hat{f}(\omega).$$

Several properties and rules of calculus are collected in Table 4.1. They are:

Linearity. This property says that $a_1 f_1(t) + a_2 f_2(t) \xleftrightarrow{\mathfrak{F}} a_1\hat{f}_1(\omega) + a_2\hat{f}_2(\omega)$ for every two complex numbers a_1 and a_2. In words: the Fourier transform $\mathfrak{F}$ is a linear mapping.

Reciprocity. This rule says $\hat{f}(t) \overset{\mathfrak{F}}{\longleftrightarrow} 2\pi f(-\omega)$. It is a curious one because $\hat{f}$ is now considered a function of time! The proof exploits the similarity between (4.4) and (4.5). Proof: first in Eqn. (4.5) interchange ω and t:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega,$$

and then substitute $w = -\omega$:

$$\hat{f}(t) = \int_{w=+\infty}^{w=-\infty} f(-w)\mathrm{e}^{\mathrm{i}wt}\,\mathrm{d}(-w).$$

Swapping the integration boundaries swaps the sign of the integral, hence

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(-w)\mathrm{e}^{\mathrm{i}wt}\,\mathrm{d}w.$$

Finally we replace w by ω,

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi f(-\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

This says that $2\pi f(-\omega)$ is the Fourier transform of $\hat{f}(t)$.

Conjugation. The conjugation rule $f^*(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}^*(-\omega)$ is easily proved:

$$\mathfrak{F}\{f^*(t)\} = \int_{-\infty}^{\infty} f^*(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \left(\int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\right)^* = \hat{f}^*(-\omega).$$

Time-scaling. This rule claims that $f(at) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{|a|}\hat{f}(\frac{\omega}{a})$, provided $a \in \mathbb{R}, a \neq 0$. In particular it says that $f(-t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}(-\omega)$. First the proof for $a > 0$. Then

$$\mathfrak{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \{\text{substitute } \tau = at\} = \frac{1}{a}\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau/a}\,\mathrm{d}\tau = \frac{1}{a}\hat{f}(\frac{\omega}{a}).$$

If $a < 0$ then the integral gains a minus sign since the boundaries of integration $-\infty$ and ∞ swap. This explains the absolute value in the general formula.

Time-shift. The rule $f(t-\tau) \xleftrightarrow{\mathfrak{F}} \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}$ follows directly

$$\mathfrak{F}\{f(t-\tau)\} = \int_{-\infty}^{\infty} f(t-\tau)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$
$$= \{\text{substitute } v = t-\tau\} = \int_{-\infty}^{\infty} f(v)\mathrm{e}^{-\mathrm{i}\omega(v+\tau)}\,\mathrm{d}v = \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}.$$

Frequency-shift. Dual to the time-shift rule is the frequency-shift rule $f(t)\mathrm{e}^{\mathrm{i}\omega_0 t} \xleftrightarrow{\mathfrak{F}} \hat{f}(\omega-\omega_0)$. Proof:

$$\mathfrak{F}\{f(t)\mathrm{e}^{\mathrm{i}\omega_0 t}\} = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t}\,\mathrm{d}t = \hat{f}(\omega-\omega_0).$$

Application of this and Euler's formula readily gives the *modulation theorem*:

$$f(t)\cos(\omega_0 t) = \frac{1}{2}(f(t)\mathrm{e}^{\mathrm{i}\omega_0 t} + f(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}) \xleftrightarrow{\mathfrak{F}} \frac{1}{2}(\hat{f}(\omega-\omega_0) + \hat{f}(\omega+\omega_0)).$$

Differentiation with respect to time. This is going to be very useful rule when we consider differential equations. The rule is $f'(t) \xleftrightarrow{\mathfrak{F}} \mathrm{i}\omega\hat{f}(\omega)$. Proof:

$$\mathfrak{F}\{f'(t)\} = \int_{-\infty}^{\infty} f'(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \Big[f(t)\mathrm{e}^{-\mathrm{i}\omega t}\Big]_{-\infty}^{\infty} + \mathrm{i}\omega\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{i}\omega\hat{f}(\omega),$$

provided $\lim_{t\to\pm\infty} f(t) = 0$, which is usually the case if $f(t)$ is absolutely integrable.

Integration with respect to time. The converse of the differentiation rule is $\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau \xleftrightarrow{\mathfrak{F}} \frac{\hat{f}(\omega)}{\mathrm{i}\omega}$, provided $\hat{f}(0) = 0$. Proof: let $g(t) = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$, then $g'(t) = f(t)$ and

$$\lim_{t\to\infty} g(t) = \int_{-\infty}^{\infty} f(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-\mathrm{i}0\tau}\,\mathrm{d}\tau = \hat{f}(0) = 0,$$

and, also, $g(t) \to 0$ as $t \to -\infty$ because $f(t)$ is assumed absolutely integrable. Using integration by parts and $g'(t) = f(t)$ we get that

$$\mathfrak{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \underbrace{\left[g(t)\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega}\right]_{-\infty}^{\infty}}_{0} + \frac{1}{\mathrm{i}\omega}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \frac{\hat{f}(\omega)}{\mathrm{i}\omega}.$$

Differentiation with respect to frequency. This rule states $-\mathrm{i}tf(t) \xleftrightarrow{\mathfrak{F}} \hat{f}'(\omega)$. It follows from integration by parts,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}'(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\left[\hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right]_{-\infty}^{\infty} - \frac{\mathrm{i}t}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = -\mathrm{i}tf(t),$$

provided $\lim_{\omega\to\pm\infty}\hat{f}(\omega) = 0$, but this is always the case for absolutely integrable $f(t)$ (see Lemma 4.1.1).

Note the symmetry between the time-shift and frequency-shift rules, and between time- and frequency differentiation rules. Table 4.1 collects the various properties. Table 4.2 brings together some of the more standard Fourier transform pairs. In the derivation of these transform pairs extensive use is made of the above properties.

4.3 Examples

Let us start with a very important example.

Example 4.3.1 (Rectangular pulse in frequency domain). In Example 4.1.4 we established the pair

$$\mathrm{rect}_a(t) \xleftrightarrow{\mathfrak{F}} a\,\mathrm{sinc}(a\omega/2).$$

Application of the reciprocity rule then gives us

$$a\,\mathrm{sinc}(at/2) \xleftrightarrow{\mathfrak{F}} 2\pi\,\mathrm{rect}_a(\omega). \tag{4.6}$$

The Fourier transform of the signal $a\,\mathrm{sinc}(at/2)$ apparently is $2\pi\,\mathrm{rect}_a(\omega)$ even though the sinc is not absolutely integrable. The formulas of the Fourier integral theorem remain valid in this case. □

The interpretation is that sinc functions contain all “low enough” frequencies with equal power, and that all “high enough” frequencies are completely absent. They play an important role in signal processing as we will see later.

TABLE 4.1: Some standard Fourier transform properties

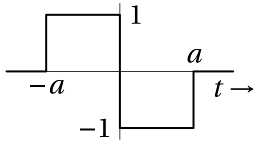
Property	Time domain	Freq. domain	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$	
Duality	$\hat{f}(t)$	$2\pi f(-\omega)$	
Conjugation	$f^*(t)$	$\hat{f}^*(-\omega)$	
Time-scaling	$f(at)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$	$a \in \mathbb{R}, a \neq 0$
Time-shift	$f(t-\tau)$	$\hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}$	
Frequency-shift	$f(t)\mathrm{e}^{\mathrm{i}\omega_0 t}$	$\hat{f}(\omega-\omega_0)$	
Modulation Thm.	$f(t)\cos(\omega_0 t)$	$\dfrac{\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)}{2}$	
Differentiation (time)	$f^{(n)}(t)$	$(\mathrm{i}\omega)^n \hat{f}(\omega)$	$\lim_{t\to\pm\infty} f(t) = 0$
Integration (time)	$\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$	$\dfrac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\hat{f}(0) = 0$
Differentiation (freq.)	$-\mathrm{i}tf(t)$	$\hat{f}'(\omega)$	

TABLE 4.2: Some standard Fourier transform pairs

$f(t)$	$\hat{f}(\omega)$	Condition
$\mathrm{rect}_a(t)$	$a\,\mathrm{sinc}(a\omega/2)$	$a>0$
$\mathrm{trian}_a(t)$	$a\,\mathrm{sinc}^2(a\omega/2)$	$a\in\mathbb{R},\ a>0$
$\mathrm{e}^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$\mathrm{Re}\,a>0$
$\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(t)$	$\dfrac{1}{(a+\mathrm{i}\omega)^{n+1}}$	$\mathrm{Re}\,a>0;\ n\in\mathbb{N}$
$-\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(-t)$	$\dfrac{1}{(a+\mathrm{i}\omega)^{n+1}}$	$\mathrm{Re}\,a<0;\ n\in\mathbb{N}$
e^{-t^2}	$\sqrt{\pi}\mathrm{e}^{-(\omega/2)^2}$	$a\in\mathbb{R}$
$a\,\mathrm{sinc}(at/2)$	$2\pi\,\mathrm{rect}_a(\omega)$	$a\in\mathbb{R},\ a>0$

Example 4.3.2 (Rectangular and triangular pulse). Recall the triangular pulse $\operatorname{trian}_a(t)$ as defined in Definition 2.5.1. Now let $f(t)$ equal

$$f(t) = \operatorname{rect}_a(t + a/2) - \operatorname{rect}_a(t - a/2)$$



Note that $\hat{f}(0) = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}t = 0$ and that

$$\operatorname{trian}_a(t) = \frac{1}{a}\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau.$$

Recall that $\operatorname{rect}_a(t) \xleftrightarrow{\mathfrak{F}} 2\sin(a\omega/2)/\omega$. Hence based on integration in time and time shift we get that

$$\operatorname{trian}_a(t) \xleftrightarrow{\mathfrak{F}} \frac{1}{\mathrm{i}\omega}\,\frac{2\sin(a\omega/2)}{a\omega}\left(\mathrm{e}^{a\mathrm{i}\omega/2} - \mathrm{e}^{-a\mathrm{i}\omega/2}\right) = \frac{4\sin^2(a\omega/2)}{a\omega^2} = a\operatorname{sinc}^2(a\omega/2).$$

Interestingly both $f(t)$ and $\hat{f}(\omega)$ are real-valued. The pair as a picture is

□

Example 4.3.3 (Towards all rational Fourier transforms). In Example 2.3.1 we saw that for $\operatorname{Re}(a) > 0$ there holds that $\int_0^{\infty} \mathrm{e}^{-at}\,\mathrm{d}t = 1/a$. An immediate consequence is the Fourier transform of $f(t) = \mathrm{e}^{-at}\,\mathbb{1}(t)$. Since $\operatorname{Re}(a + \mathrm{i}\omega) = \operatorname{Re}(a) > 0$ we have that

$$\mathrm{e}^{-at}\,\mathbb{1}(t) \xleftrightarrow{\mathfrak{F}} \int_{-\infty}^{\infty} \mathrm{e}^{-(a+\mathrm{i}\omega)t}\,\mathbb{1}(t)\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-(a+\mathrm{i}\omega)t}\,\mathrm{d}t = \frac{1}{a + \mathrm{i}\omega}.$$

Differentiating with respect to frequency n times gives

$$(-\mathrm{i}t)^n \mathrm{e}^{-at}\,\mathbb{1}(t) \xleftrightarrow{\mathfrak{F}} \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\right)^n \frac{1}{a + \mathrm{i}\omega} = (-1)^n \mathrm{i}^n n! \frac{1}{(a + \mathrm{i}\omega)^{n+1}}, \qquad (\operatorname{Re}(a) > 0).$$

Therefore

$$\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{(a+\mathrm{i}\omega)^{n+1}}, \qquad (\mathrm{Re}(a)>0). \tag{4.7}$$

Note that the Fourier transform is rational in ω.

These transform pairs are for the cases where $\mathrm{Re}(a)>0$. If $\mathrm{Re}\,a<0$ then similarly it may be shown that

$$-\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(-t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{(a+\mathrm{i}\omega)^{n+1}}, \qquad (\mathrm{Re}(a)<0).$$

The inverse Fourier transform of $1/(a+\mathrm{i}\omega)^{n+1}$ hence depends rather dramatically on a. If $\mathrm{Re}(a)>0$ then the inverse Fourier transform is $\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(t)$ which is zero for all negative time, but if $\mathrm{Re}(a)<0$ then the inverse Fourier transform is $-\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(-t)$ which is zero for positive time. □

Example 4.3.4 (More rational Fourier transforms). Suppose $\mathrm{Re}(a)>0$ and consider

$$\mathrm{e}^{-a|t|} = \mathrm{e}^{at}\,\mathbb{1}(-t) + \mathrm{e}^{-at}\,\mathbb{1}(t).$$

By linearity,

$$\mathrm{e}^{-a|t|} \overset{\mathfrak{F}}{\longleftrightarrow} \frac{-1}{-a+\mathrm{i}\omega} + \frac{1}{a+\mathrm{i}\omega} = \frac{-(a+\mathrm{i}\omega)+(-a+\mathrm{i}\omega)}{(-a+\mathrm{i}\omega)(a+\mathrm{i}\omega)} = \frac{-2a}{-a^2-\omega^2} = \frac{2a}{a^2+\omega^2}.$$

□

Example 4.3.5 (Gaussian Bell). We determine the Fourier transform of the Gaussian function

$$f(t) = \mathrm{e}^{-t^2}.$$

This is special derivation, and, interestingly, along the way we also derive the value of the famous integral

$$\beta := \int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\,\mathrm{d}t. \tag{4.8}$$

Since e^{-t^2} is an even function, its Fourier transform equals

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-t^2} \cos(\omega t)\, dt.$$

Now differentiate this expression with respect to ω and then use integration by parts

$$\hat{f}'(\omega) = \int_{-\infty}^{\infty} (-t e^{-t^2}) \sin(\omega t)\, dt = \underbrace{\left[\tfrac{1}{2} e^{-t^2} \sin(\omega t)\right]_{-\infty}^{\infty}}_{0} - \frac{\omega}{2} \int_{-\infty}^{\infty} e^{-t^2} \cos(\omega t)\, dt = -\frac{\omega}{2} \hat{f}(\omega).$$

This is a common first order differential equation: $\hat{f}'(\omega) = -\frac{\omega}{2a} \hat{f}(\omega)$. Next separate the variables,

$$\frac{\hat{f}'(\omega)}{\hat{f}(\omega)} = -\frac{\omega}{2}.$$

Integrating both sides from $\omega = 0$, we find that $\ln|\hat{f}(\omega)| - \ln|\hat{f}(0)| = -\omega^2/4$, or,

$$\hat{f}(\omega) = \hat{f}(0) e^{-(\omega/2)^2}.$$

It is interesting to see that the Fourier transform of the Gaussian function is again a Gaussian function. But what about the value of $\hat{f}(0)$? Realize that Fourier theory says that

$$\hat{f}(0) = \int_{-\infty}^{\infty} e^{-t^2} e^{0}\, dt = \beta,$$

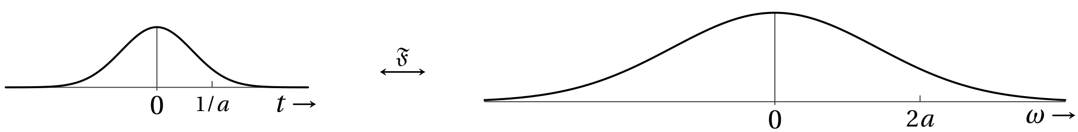
but it also says that

$$1 = f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \beta e^{-(\omega/2)^2}\, d\omega = \frac{\beta}{\pi} \int_{-\infty}^{\infty} e^{-(\omega/2)^2}\, d(\omega/2) = \frac{\beta^2}{\pi}.$$

Hence $\beta = \sqrt{\pi}$. Nice. Application of the scaling property finally gives us

$$e^{-(at)^2} \quad \overset{\mathfrak{F}}{\longleftrightarrow} \quad \frac{\sqrt{\pi}}{|a|} e^{-(\omega/(2a))^2}.$$

Graphically:



□

More examples

Often the Fourier transform $\hat{f}(\omega)$ can be found through a combination of the rules of Table 4.2.

Example 4.3.6 (Application of Fourier transform properties). The Fourier transform of $\operatorname{rect}_a(t)\cos(\omega_0 t)$ and $\mathrm{e}^{-at}\cos(\omega_0 t)\,\mathbb{1}(t)$ may be obtained using the modulation theorem:

$$\operatorname{rect}_a(t)\cos(\omega_0 t) \xleftrightarrow{\mathfrak{F}} \frac{a}{2}\operatorname{sinc}(a(\omega-\omega_0)/2) + \frac{a}{2}\operatorname{sinc}(a(\omega+\omega_0)/2), \qquad (a>0).$$

(See Figure 4.1(h).) Likewise we find that

$$\mathrm{e}^{-at}\cos(\omega_0 t)\,\mathbb{1}(t) \xleftrightarrow{\mathfrak{F}} \frac{a+\mathrm{i}\omega}{(a+\mathrm{i}\omega)^2+\omega_0^2}, \qquad (\operatorname{Re}(a)>0).$$

□

Example 4.3.7 (Another Application of Fourier transform properties). With help of the reciprocity rule it follows that

$$\frac{1}{a^2+t^2} \xleftrightarrow{\mathfrak{F}} \frac{\pi}{a}\mathrm{e}^{-a|\omega|}, \qquad (\operatorname{Re}(a)>0),$$

and the modulation theorem then gives

$$\frac{\cos(\omega_0 t)}{a^2+t^2} \xleftrightarrow{\mathfrak{F}} \frac{\pi}{2a}(\mathrm{e}^{-a|\omega-\omega_0|}+\mathrm{e}^{-a|\omega+\omega_0|}), \qquad (\operatorname{Re}(a)>0).$$

To find the Fourier transform of

$$f(t) = \frac{\sin^2(at)}{t^2}.$$

we use the reciprocity rule: since

$$\operatorname{trian}_{2a}(t) \xleftrightarrow{\mathfrak{F}} \frac{4\sin^2(a\omega)}{2a\omega^2} = \frac{2}{a}f(\omega),$$

we have by the reciprocity rule that

$$\frac{2}{a} f(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} 2\pi \operatorname{trian}_{2a}(-\omega) = 2\pi \operatorname{trian}_{2a}(\omega).$$

Hence

$$\frac{\sin^2(at)}{t^2} \stackrel{\mathfrak{F}}{\longleftrightarrow} \pi a \operatorname{trian}_{2a}(\omega).$$

□

The most important family of Fourier transforms are the rational functions of frequency.

Example 4.3.8 (Rational functions in frequency domain). In this example we calculate the inverse Fourier transform of a rational function of the form

$$\hat{f}(\omega) = \frac{Q(\mathrm{i}\omega)}{P(\mathrm{i}\omega)} := \frac{q_m(\mathrm{i}\omega)^m + q_{m-1}(\mathrm{i}\omega)^{m-1} + \cdots + q_1(\mathrm{i}\omega) + q_0}{p_n(\mathrm{i}\omega)^n + p_{n-1}(\mathrm{i}\omega)^{n-1} + \cdots + p_1(\mathrm{i}\omega) + p_0}.$$

The coefficients p_i and q_i are assumed real. We shall further assume that the rational function is *strictly proper* which means that the degree of the numerator Q is less than that of the denominator P. Additionally we assume that P has no zeros on the imaginary axis, i.e., that $P(\mathrm{i}\omega) \neq 0$ for all $\omega \in \mathbb{R}$.

For rational functions there is a straightforward algorithm that always leads to an explicit form of the inverse Fourier transform $f(t)$. Here we illustrate it by an example. Suppose that

$$\hat{f}(\omega) = \frac{6\mathrm{i}\omega}{(\mathrm{i}\omega + 1)(4 + \omega^2)}. \tag{4.9}$$

Now substitute $s = \mathrm{i}\omega$ and perform a partial fraction expansion (see Section A.4)

$$\frac{6s}{(s+1)(4-s^2)} = \frac{3}{s+2} - \frac{2}{s+1} - \frac{1}{s-2}.$$

Hence

$$\hat{f}(\omega) = \frac{3}{\mathrm{i}\omega + 2} - \frac{2}{\mathrm{i}\omega + 1} - \frac{1}{\mathrm{i}\omega - 2}.$$

For each of the three terms in this sum the inverse Fourier transform has already been determined, see Table 4.2. By linearity then the inverse Fourier transform of the sum is the sum of the inverse Fourier transforms and it equals

$$f(t) = (3\mathrm{e}^{-2t} - 2\mathrm{e}^{-t})\,\mathbb{1}(t) + \mathrm{e}^{2t}\,\mathbb{1}(-t).$$

□

4.4 Convolution and correlation

Next we formulate and prove the convolution theorem for the Fourier integral. In fact we consider two versions of the convolution theorem, one for convolution in the time domain and one for convolution in the frequency domain. In the proofs we shall silently assume that changing the order of integration is allowed. It is allowed but we do not prove it.

Theorem 4.4.1 (Convolution theorem in the time domain). Suppose that f, g are two absolutely integrable functions. Then $f * g$ is also absolutely integrable and

$$(f * g)(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}(\omega)\hat{g}(\omega).$$

Proof. Determine the Fourier transform of $(f * g)(t)$ as follows:

$$\begin{aligned}
\mathfrak{F}\{(f * g)(t)\} &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\Big(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(\tau)\Big(\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} g(t-\tau)\,\mathrm{d}t\Big)\,\mathrm{d}\tau
\end{aligned}$$

now use the rule $g(t-\tau) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{g}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}$

$$= \int_{-\infty}^{\infty} f(\tau)\hat{g}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau = \hat{f}(\omega)\hat{g}(\omega).$$

(We skip the proof that $f * g$ is absolutely integrable, although the proof is not hard.) ■

Theorem 4.4.2 (Convolution theorem in the frequency domain). Suppose that $f(t) \xleftrightarrow{\mathfrak{F}} \hat{f}(\omega)$ and $g(t) \xleftrightarrow{\mathfrak{F}} \hat{g}(\omega)$. Then

$$f(t)g(t) \xleftrightarrow{\mathfrak{F}} \frac{1}{2\pi}(\hat{f} * \hat{g})(\omega).$$

Proof. In the Fourier transform

$$\mathfrak{F}\{f(t)g(t)\} = \int_{-\infty}^{\infty} f(t)g(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$

we substitute $f(t)$ for its Fourier integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(u)\mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u.$$

Then,

$$\begin{aligned}
\mathfrak{F}\{f(t)g(t)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(u)\mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u\; g(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(u)\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}(\omega-u)t}\,\mathrm{d}t\,\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(u)\hat{g}(\omega-u)\,\mathrm{d}u = \frac{1}{2\pi}(\hat{f} * \hat{g})(\omega).
\end{aligned} \tag{4.10}$$

■

Example 4.4.3. Exercise 4.7 (page 174) claims that $(\mathrm{rect}_a * \mathrm{rect}_a)(t) = a\,\mathrm{trian}_a(t)$. Application of the convolution theorem gives

$$\mathfrak{F}\{\mathrm{trian}_a(t)\} = \frac{1}{a}(\mathfrak{F}\{\mathrm{rect}_a(t)\})^2 = a\,\mathrm{sinc}^2(a\omega/2).$$

This is in accordance with Table 4.2. □

Example 4.4.4. Given that $f(t) = \mathrm{e}^{-at}\,\mathbb{1}(t) \xleftrightarrow{\mathfrak{F}} 1/(a+\mathrm{i}\omega)$ for all $\mathrm{Re}(a) > 0$, it follows by the convolution theorem that

$$\mathfrak{F}^{-1}\left\{\frac{1}{(a+\mathrm{i}\omega)^2}\right\} = (f * f)(t) = \left(\int_0^t \mathrm{e}^{-a\tau}\mathrm{e}^{-a(t-\tau)}\,\mathrm{d}\tau\right)\mathbb{1}(t) = t\mathrm{e}^{-at}\,\mathbb{1}(t).$$

□

In the case of periodic signals we found a way to express the power of a periodic signal in terms of its Fourier coefficients. The result was called Parseval's theorem. Similarly there is a Parseval's theorem for aperiodic signals that expresses the energy of a aperiodic signal in terms of its Fourier transform. The energy of a signal $f(t)$ is defined as

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t.$$

Theorem 4.4.5 (Parseval). Let $f(t)$ be a signal with $E_f < \infty$. Then

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \,\mathrm{d}\omega.$$

Proof. The rule $f(t)g(t) \xleftrightarrow{\mathfrak{F}} \frac{1}{2\pi}(\hat{f} * \hat{g})(\omega)$ when written out becomes

$$\int_{-\infty}^{\infty} f(t)g(t)\mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(v)\hat{g}(\omega - v) \,\mathrm{d}v.$$

Now take $\omega = 0$,

$$\int_{-\infty}^{\infty} f(t)g(t) \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(v)\hat{g}(-v) \,\mathrm{d}v.$$

For a more symmetrical version, replace $g(t)$ with $g^*(t)$ and the corresponding Fourier transform $\hat{g}(\omega)$ with $\hat{g}^*(-\omega)$. Then we get

$$\int_{-\infty}^{\infty} f(t)g^*(t) \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(v)\hat{g}^*(v) \,\mathrm{d}v. \tag{4.11}$$

This is an important equality. The result follows if we take $g(t) = f(t)$. ■

Example 4.4.6. In Example 4.3.7 (for $a = 1$) we derived the Fourier transform pair $\frac{\sin^2(t)}{t^2} \xleftrightarrow{\mathfrak{F}} \pi\,\mathrm{trian}_2(\omega)$. With the help of Parseval we then get

$$\int_{-\infty}^{\infty} \frac{\sin^4(t)}{t^4} \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\pi\,\mathrm{trian}_2(\omega))^2 \,\mathrm{d}\omega = \pi\int_{0}^{2} (1 - \frac{1}{2}\omega)^2 \,\mathrm{d}\omega = \frac{2}{3}\pi.$$

□

An operation closely related to the convolution product is the cross correlation of two signals.

Definition 4.4.7 (Cross correlation). Let $f_1(t)$ and $f_2(t)$ be two signals with $E_{f_1} < \infty$ and $E_{f_2} < \infty$. The *cross correlation* $\rho_{1,2}(t)$ of $f_1(t)$ and $f_2(t)$ is defined as

$$\rho_{1,2}(t) = \int_{-\infty}^{\infty} f_1(t+\tau) f_2^*(\tau)\,\mathrm{d}\tau.$$

□

The Fourier transform of $\rho_{1,2}(t)$ follows from the convolution theorem on noting that $\rho_{1,2}(t)$ is the convolution product of the signals $f(t) = f_1(t)$ and $g(t) = f_2^*(-t)$ with respective Fourier transforms $\hat{f}_1(\omega)$ and $\hat{f}_2^*(\omega)$. Hence

$$\rho_{1,2}(t) \xleftrightarrow{\mathfrak{F}} \hat{f}_1(\omega)\hat{f}_2^*(\omega). \tag{4.12}$$

If $f_2(t) = f_1(t) = f(t)$ then $\rho(t) = \rho_{1,1}(t)$ is called the *autocorrelation* of $f(t)$. The Fourier transform of $\rho(t)$ is therefore equal to $\hat{f}(\omega)\hat{f}^*(\omega) = |\hat{f}(\omega)|^2$, which is called the *energy spectrum* or *spectral density* of $f(t)$. The inverse Fourier transform now yields the formula

$$\rho(t) = \int_{-\infty}^{\infty} f(t+\tau) f^*(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

Substitute $t = 0$ and what follows is again Parseval's equality. Moreover it follows that

$$|\rho(t)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 |\mathrm{e}^{\mathrm{i}\omega t}|\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,\mathrm{d}\omega = \rho(0),$$

In other words, the auto correlation is maximal for $t = 0$. This means that $f(t)$ correlates best with itself. That makes sense.

Example 4.4.8 (Sliding window averaging & noise reduction, continued). In Example 2.7.2 we showed that averaging $f(t)$ over an interval of length P, like

$$f_{\mathrm{swa}}(t) = \frac{1}{P}\int_{t-P/2}^{t+P/2} f(\tau)\,\mathrm{d}\tau$$

can be seen as a convolution $f_{\text{swa}} = (f * g)$ with

$$g(t) = \frac{1}{P}\,\text{rect}_P(t).$$

In frequency domain the process of averaging hence means multiplying the Fourier transform with the Fourier transform of $g(t)$,

$$\hat{g}(\omega) = \text{sinc}(\omega P/2).$$

Therefore

$$\hat{f}_{\text{swa}}(\omega) = \hat{f}(\omega)\,\text{sinc}(\omega P/2).$$

Note that $\hat{g}(\omega)$ tends to zero as $\omega \to \infty$. The high-frequency components in $f(t)$ are therefore attenuated more than the low frequency components. This agrees with our understanding of averaging. Also, the larger the averaging interval P, the faster $\hat{g}(\omega)$ decays to zero as $\omega \to \infty$, i.e., the more the high-frequency components are attenuated. Again this agrees with our understanding of averaging. □

4.5 Fourier transforms with delta functions

It makes sense to define the Fourier transform of the delta function using the sifting property, that is,

$$\mathfrak{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = 1. \tag{4.13}$$

From Theorem 4.1.3 we know that absolutely integrable signals can be recovered from their Fourier transform through an inverse Fourier transform which is in the form of an integral. In the case of the delta function, however, this integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

diverges! In a proper setup — but that is beyond the scope of this course — the inverse Fourier transform of 1 *can* be given a meaning and *can* be shown to equal $\delta(t)$. We simply define that

$$\mathfrak{F}\{\delta(t)\} = 1.$$

Its implication that $\delta(t)$ is built up from all harmonics $\frac{1}{2\pi}\hat{\delta}(\omega)\mathrm{e}^{\mathrm{i}\omega t}$ with equal weight $\frac{1}{2\pi}\hat{\delta}(\omega) = \frac{1}{2\pi}$ is a bit difficult to interpret.

Delta functions in the frequency domain $\delta(\omega)$ have a more appealing interpretation. Consider $\hat{f}(\omega) = \delta(\omega)$ and apply it to the inverse Fourier transform (assuming this makes sense)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \{\text{sifting property}\} = \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}0t} = \frac{1}{2\pi}.$$

It is a constant signal, i.e. a signal with frequency zero, and this agrees perfectly with our understanding that $\delta(\omega)$ only contains the zero frequency. Its Fourier transform $\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{1}{2\pi}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$ is now not defined, but also in this case it is possible, in a proper setup, to give it a meaning and to show that the Fourier transform equals $\hat{f}(\omega) = \delta(\omega)$.

Summarizing, delta functions in one domain correspond to constant functions in the other.

Example 4.5.1.

- $\delta(t-\tau) \xleftrightarrow{\mathfrak{F}} \mathrm{e}^{-\mathrm{i}\omega\tau}$.

  This is a direct consequence of the time-shift rule and the fact that $\delta(t) \xleftrightarrow{\mathfrak{F}} 1$.

- $\mathrm{e}^{\mathrm{i}\omega_0 t} \xleftrightarrow{\mathfrak{F}} 2\pi\delta(\omega-\omega_0)$.

  This is a direct consequence of the frequency-shift rule and the fact that $\frac{1}{2\pi} \xleftrightarrow{\mathfrak{F}} \delta(\omega)$.

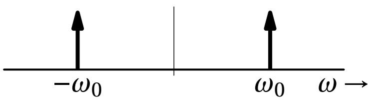
- $\cos(\omega_0 t) \xleftrightarrow{\mathfrak{F}} \pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))$.

  It follows from the Modulation theorem (Page 148).

□

That the Fourier transform of $f(t) = \cos(\omega_0 t)$ equals

$$\hat{f}(\omega) = \pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)),$$



again agrees with our understanding of what the Fourier transform $\hat{f}(\omega)$ entails. The above function $\hat{f}(\omega)$ consists of two spikes, one spike at frequency $-\omega_0$ and one at ω_0. Its frequency content is therefore concentrated at the frequencies $\pm\omega_0$ only, and does not depend on any other frequency. Indeed, $\cos(\omega_0 t)$ is like that. Table 4.3 collects some generalized Fourier transform pairs, including some that we did not treat. The rules that hold for the classical Fourier transform remain valid if we extend it with the Fourier transform pairs of Table 4.3 (proof is omitted). In those rules any derivative should now be understood to mean the generalized derivative. Even the convolution theorems remain valid.

Example 4.5.2. Let $f(t) = \mathrm{e}^{-t}\mathbb{1}(t)$. Then $f'(t) = \mathrm{e}^{-t}\delta(t) - \mathrm{e}^{-t}\mathbb{1}(t) = \delta(t) - \mathrm{e}^{-t}\mathbb{1}(t)$. The Fourier transform of $f'(t)$ equals $1 - 1/(1+\mathrm{i}\omega) = \mathrm{i}\omega/(1+\mathrm{i}\omega)$. Via the differentiation rule we get that the Fourier transform of $f'(t)$ equals the Fourier transform $\hat{f}(\omega) = 1/(1+\mathrm{i}\omega)$ of $f(t)$ multiplied with $\mathrm{i}\omega$. Indeed, this gives the same result. $\square$

TABLE 4.3: Some generalized Fourier transform pairs. (Warning: the final two pairs are not treated in this course. To understand these we would have to dig deeper into delta functions and other generalized functions.)

$f(t)$	$\hat{f}(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t-b)$	$\mathrm{e}^{-\mathrm{i}\omega b}$
$\mathrm{e}^{\mathrm{i}\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$
$\mathrm{sgn}(t)$	$\frac{2}{\mathrm{i}\omega}$
$\mathbb{1}(t)$	$\frac{1}{\mathrm{i}\omega}+\pi\delta(\omega)$

4.6 Applications

Frequency response for ODEs

In Section 4.4 we have seen that one can use the Fourier transform to calculate convolutions. In this section we show that the Fourier transform can also help to solve ordinary differential equations.

Consider the differential equation (see also (3.17))

$$\begin{aligned} p_n y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\ = q_m u^{(m)}(t) + q_{m-1} u^{(m-1)}(t) + \cdots + q_1 u^{(1)}(t) + q_0 u(t), \qquad t \in \mathbb{R}, \end{aligned} \tag{4.14}$$

with $p_0, p_1, \ldots, p_n, q_0 \ldots, q_m \in \mathbb{R}$, and $u(t)$ the input signal. Furthermore, we assume that $m \leq n$. Suppose now that both $u(t)$ and $y(t)$ possess a Fourier transform, and that $u(t), y(t)$ and its derivatives are zero at plus and minus infinity, then we may take the Fourier transform of Eqn. (4.14),

$$\begin{aligned} \mathfrak{F}\{p_n y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t)\} \\ = \mathfrak{F}\{q_m u^{(m)}(t) + q_{m-1} u^{(m-1)}(t) + \cdots + q_1 u^{(1)}(t) + q_0 u(t)\}. \end{aligned}$$

Using the linearity of the Fourier transform, this equation is equivalent to

$$\begin{aligned} p_n \mathfrak{F}\{y^{(n)}(t)\} + p_{n-1} \mathfrak{F}\{y^{(n-1)}(t)\} + \cdots + p_1 \mathfrak{F}\{y^{(1)}(t)\} + p_0 \mathfrak{F}\{y(t)\} \\ = q_m \mathfrak{F}\{u^{(m)}(t)\} + q_{m-1} \mathfrak{F}\{u^{(m-1)}(t)\} + \cdots + q_1 \mathfrak{F}\{u^{(1)}(t)\} + q_0 \mathfrak{F}\{u(t)\}. \end{aligned}$$

By the differentiation property this becomes

$$\begin{aligned} p_n (\mathrm{i}\omega)^n \hat{y}(\omega) + p_{n-1} (\mathrm{i}\omega)^{n-1} \hat{y}(\omega) + \cdots + p_1 (\mathrm{i}\omega) \hat{y}(\omega) + p_0 \hat{y}(\omega) \\ = q_m (\mathrm{i}\omega)^m \hat{u}(\omega) + q_{m-1} (\mathrm{i}\omega)^{m-1} \hat{u}(\omega) + \cdots + q_1 (\mathrm{i}\omega) \hat{u}(\omega) + q_0 \hat{u}(\omega), \end{aligned} \tag{4.15}$$

where we have used our standard notation $\hat{u}(\omega) = \mathfrak{F}\{u(t)\}, \hat{y}(\omega) = \mathfrak{F}\{y(t)\}$. The above equation we can solve for $\hat{y}(\omega)$,

$$\hat{y}(\omega) = \frac{q_m (\mathrm{i}\omega)^m + q_{m-1} (\mathrm{i}\omega)^{m-1} + \cdots + q_1 (\mathrm{i}\omega) + q_0}{p_n (\mathrm{i}\omega)^n + p_{n-1} (\mathrm{i}\omega)^{n-1} + \cdots + p_1 (\mathrm{i}\omega) + p_0} \hat{u}(\omega). \tag{4.16}$$

The function before $\hat{u}(\omega)$ is called the *frequency response*, and is denoted by $\hat{h}(\omega)$. Thus

$$\hat{h}(\omega) = \frac{q_m(\mathrm{i}\omega)^m + q_{m-1}(\mathrm{i}\omega)^{m-1} + \cdots + q_1(\mathrm{i}\omega) + q_0}{p_n(\mathrm{i}\omega)^n + p_{n-1}(\mathrm{i}\omega)^{n-1} + \cdots + p_1(\mathrm{i}\omega) + p_0}. \tag{4.17}$$

Note that the frequency response of (4.14) is the transfer function $H(s)$ of this equation evaluated at $s = \mathrm{i}\omega$, see Theorem 3.7.1:

$$\hat{h}(\omega) = H(\mathrm{i}\omega).$$

Summarizing, we see that we can write the Fourier transform of the output of (4.14) as a product of the frequency response and the Fourier transform of the input, provided both Fourier transforms exist.

Before we discuss the existence of the Fourier transform for these signals we take a closer look at (4.17). From Example 4.3.8 we know that the inverse Fourier transform of a rational function exists, provided it has no poles on the imaginary axis. Hence there exists a function $h(t)$ whose Fourier transform is given by (4.17). This function is called the *impulse response*. The name of this function can be easily understood: let $u(t)$ be the delta function $\delta(t)$. The Fourier transform of the delta function is 1, and so by (4.16) we obtain that the Fourier transform of the corresponding output is given by

$$\hat{y}(\omega) = \frac{q_m(\mathrm{i}\omega)^m + q_{m-1}(\mathrm{i}\omega)^{m-1} + \ldots + q_1(\mathrm{i}\omega) + q_0}{p_n(\mathrm{i}\omega)^n + p_{n-1}(\mathrm{i}\omega)^{n-1} + \ldots + p_1(\mathrm{i}\omega) + p_0} 1 = \hat{h}(\omega).$$

Taking the inverse Fourier transform at the left- and right-hand side, we conclude that the output (also known as the *response*) corresponding to the impulsive input signal $u(t) = \delta(t)$ is $h(t)$, hence the name impulse response.

Using the formula of (4.17), we can write Eqn. (4.16) as

$$\hat{y}(\omega) = \hat{h}(\omega)\,\hat{u}(\omega). \tag{4.18}$$

Hence the Fourier transform of the output is the product of Fourier transform of the impulse response and the Fourier transform of the input. By the Convolution Theorem 4.4.1 we obtain an expression in time domain, namely

$$y(t) = (h * u)(t) = \int_{-\infty}^{\infty} h(\tau)\,u(t-\tau)\,\mathrm{d}\tau. \tag{4.19}$$

Now we have two remaining questions. Firstly,

- do $u(t)$ and $y(t)$ possess a Fourier transform?,

and, secondly,

- why do we find here only one solution $y(t)$ even though differential equations (4.14) have many solutions?

The second question we address in Example 4.6.1. We concentrate on the first question. The existence of the Fourier transform of the input depends on the choice of the input. For example $e^{-t}\mathbb{1}(t)$ possesses a Fourier transform, whereas $e^{-t}\mathbb{1}(-t)$ does not (it is not absolutely integrable). Whether the input signal has a Fourier transform is usually easy to check. For the output $y(t)$ there are two approaches:

1. First assume that the Fourier transform of the output exists, and then solve the equation by calculating the inverse Fourier transform of (4.16). Once the output is found, check whether it has a Fourier transform. This type of formal calculus in known as *Heaviside symbolic calculus.*

2. Try to guarantee existence of the Fourier transform of $y(t)$ by exploiting properties of $u(t)$ and the differential equation or $h(t)$. For systems described by ODEs (4.14) the following can be shown to hold: if $u(t)$ is absolutely integrable and Fourier transformable, $m \leq n$ and the characteristic equation has no imaginary roots, then there always exists an absolutely integrable Fourier transformable solution $y(t)$ of the ODE.

Normally, one takes the first approach. We illustrate this on the example of the *RC*-network, see also Example 3.7.3.

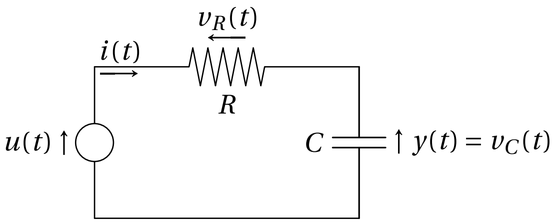


FIGURE 4.3: An *RC*-network (Example 4.6.1)

Example 4.6.1 (*RC*-network). Consider the RC-network from Example 3.7.3 as shown in Figure 4.3. In that example we derived the differential equation describing the relation between the voltage delivered by the source, $u(t)$, and the voltage across the capacitor, $y(t)$,

$$y^{(1)}(t) + \alpha y(t) = \alpha u(t), \qquad t \in \mathbb{R} \tag{4.20}$$

in which $\alpha = \frac{1}{RC}$.

From equation (4.17) we see that the frequency response of the system is given by

$$\hat{h}(\omega) = \frac{\alpha}{\mathrm{i}\omega + \alpha}$$

Using Table 4.2 we obtain that the impulse response is given by $h(t) = \alpha \mathrm{e}^{-\alpha t} \mathbb{1}(t)$.

Assume next that the input voltage is given by $u(t) = \mathrm{e}^{-2\alpha t} \mathbb{1}(t)$. This input signal has a Fourier transform, which is given by

$$\hat{u}(\omega) = \frac{1}{\mathrm{i}\omega + 2\alpha}.$$

Assuming that the output is Fourier transformable, we find that (see (4.18))

$$\hat{y}(\omega) = \frac{\alpha}{\mathrm{i}\omega + \alpha} \frac{1}{\mathrm{i}\omega + 2\alpha} = \frac{\alpha}{(\mathrm{i}\omega + \alpha)(\mathrm{i}\omega + 2\alpha)}.$$

Performing partial fraction expansion, we find

$$\hat{y}(\omega) = \frac{1}{\mathrm{i}\omega + \alpha} - \frac{1}{\mathrm{i}\omega + 2\alpha},$$

and thus

$$y(t) = \left(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-2\alpha t}\right) \mathbb{1}(t). \tag{4.21}$$

It is easy to see that this function has a Fourier transform, and so it is a solution of (4.20).

Now we can take a closer look at this solution. We know that given any particular solution, $y(t)$, we can generate another particular solution by adding any of the homogeneous solutions. The function given in (4.21) satisfies (4.20); it is *a* particular solution. Now the question is: why does the Fourier technique result in *this* particular solution and not in any of the others,

$$y(t) = \left(e^{-\alpha t} - e^{-2\alpha t}\right)\mathbb{1}(t) + y_{\text{hom}}(t). \tag{4.22}$$

Here is the reason: the homogeneous solutions satisfy $y_{\text{hom}}^{(1)}(t) + \alpha y_{\text{hom}}(t) = 0$, and it is well known that these are the functions of the form $y_{\text{hom}}(t) = c e^{-\alpha t}$ with $c \in \mathbb{C}$. The general solution of (4.20) hence is

$$\left(e^{-\alpha t} - e^{-2\alpha t}\right)\mathbb{1}(t) + c e^{-\alpha t}.$$

As the homogeneous part, $ce^{-\alpha t}$, is an exponential function defined on the whole real line, it is not absolutely integrable (unless $c = 0$). Hence it is not Fourier transformable (unless $c = 0$), see Theorem 4.1.3. Thus the only solution (4.22) that possesses a Fourier transform is the one given in (4.21). Since we have assumed from the start that our solution should possess a Fourier transform, this is the solution we find. In the next chapter we introduce a transformation that has the benefit of providing *all* solutions, with the proviso that it does so only for positive time. □

Shannon's sampling theorem

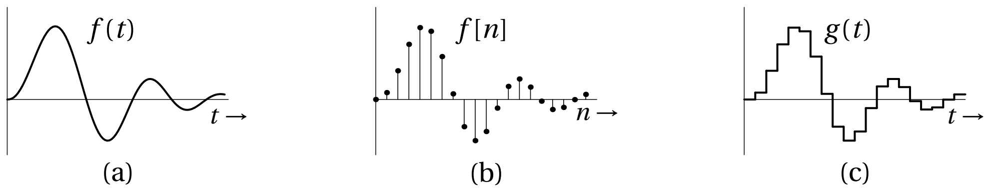


FIGURE 4.4: Continuous-time signal $f(t)$; discrete-time signal $f[n] := f(nT_s)$ derived from samples of $f(t)$ and sampling period $T_s > 0$; One possible continuous-time signal $g(t)$ derived from samples $f[n]$ and T_s

Communication between the continuous-time we live in and the discrete-time world of computers, is done through *sampling* and *holding* devices. As explained briefly on page 62, sampling is the act of taking values of a continuous-time signal $f(t)$ at multiples of a fixed sampling period T_s, resulting in a discrete-time signal $f[n] := f(nT_s)$, $(n \in \mathbb{Z})$. A holding device is any device that takes a discrete-time signal $f[n]$ and produces a continuous-time signal. The most obvious holding device is the *zero order hold,* which produces the piecewise constant continuous-time signal $g(t)$ such that $g(nT_s + t) = f[n]$ for every $t \in [0, T_s)$ and $n \in \mathbb{Z}$. Figure 4.4 illustrates the idea.

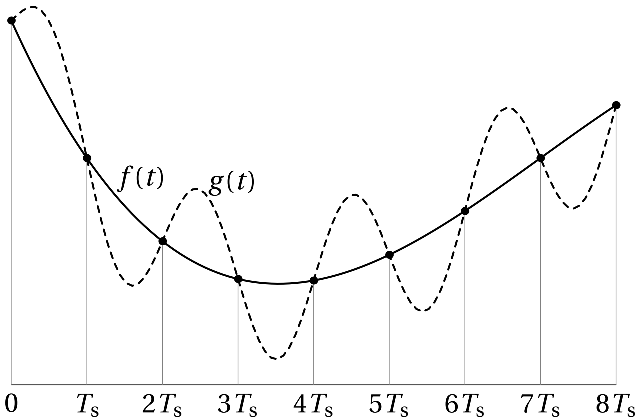


FIGURE 4.5: The signals $f(t)$ and $g(t) = f(t) + \sin(\frac{\pi}{T_s} t)$ have identical samples

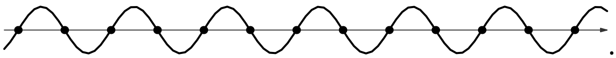
It is to be expected that with sampling some information of the original continuous-time signal $f(t)$ is lost. It is unlikely that the samples $f[n]$ are enough to reconstruct by some sort of holding device the signal $f(t)$. For example if of the signal $f(t)$ shown in Figure 4.5 we are only given its samples $f(nT_s)$, then we can not be sure that the samples come from $f(t)$ and not from $g(t) = f(t) + \sin(\frac{\pi}{T_s} t)$, because $g(t)$ and $f(t)$ are identical at the sampling instances $t = nT_s$. However, in this example the signal $g(t)$ contains a term $\sin(\frac{\pi}{T_s} t)$ which is a signal whose frequency may be unrealistically high if T_s is very small. If we know that the samples are taken from a signal that does not contain such high frequencies, then we can discard $g(t)$. In this section we derive the famous generalization of this idea, namely that *bandlimited* signals $f(t)$ can be reconstructed error-free from their samples $f(nT_s)$ provided the sampling period T_s is small enough, i.e., provided that the *sampling frequency* $\omega_s := 2\pi / T_s$ is large enough.

Definition 4.6.2 (bandlimited signals). A signal $f(t)$ is *bandlimited* if $\hat{f}(\omega) = 0$ for all $|\omega| >$

ω_b for some $\omega_\mathrm{b} \geq 0$. The smallest such value ω_b is the *bandwidth* of $f(t)$. $\square$

Bandlimited thus means that the signal $f(t)$ is not built up from unlimited high frequencies, so $f(t)$ is smooth and does not have unlimited rapid variations. The bandwidth is the highest frequency in $f(t)$.

A pathological case of sampling is when we sample a sinusoid $\sin(\omega_\mathrm{b} t)$ precisely at its zeros:



This happens when the sampling frequency $\omega_\mathrm{s} = \frac{2\pi}{T_\mathrm{s}}$ satisfies

$$\omega_\mathrm{s} = 2\omega_\mathrm{b}. \tag{4.23}$$

The value $2\omega_\mathrm{b}$ is known as the *Nyquist rate*. To allow for reconstruction of a signal $f(t)$ with bandwidth ω_b it suffices to take the sampling frequency higher than $2\omega_\mathrm{b}$:

Theorem 4.6.3 (Shannon's sampling theorem). A signal $f(t)$ with bandwidth ω_b can be reconstructed error-free from its samples $f[n] := f(nT_\mathrm{s}),\ n \in \mathbb{Z}$, iff

$$\omega_\mathrm{s} := \frac{2\pi}{T_\mathrm{s}} > 2\omega_\mathrm{b}.$$

In that case the continuous-time $f(t)$ is uniquely determined by its discrete-time samples $f[n]$ via

$$f(t) = \sum_{n=-\infty}^{\infty} f[n] \operatorname{sinc}(\pi(\tfrac{t}{T_\mathrm{s}} - n)). \tag{4.24}$$

Proof. As $\hat{f}(\omega) = 0$ for all $|\omega| > \omega_\mathrm{s}/2 > \omega_\mathrm{b}$ we have that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_\mathrm{s}/2}^{\omega_\mathrm{s}/2} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

On the interval $[-\omega_\mathrm{s}/2, \omega_\mathrm{s}/2]$ we express $\hat{f}(\omega)$ as a Fourier series with period ω_s,

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} F_n \mathrm{e}^{\mathrm{i}n\omega T_\mathrm{s}}, \qquad \omega \in (-\omega_\mathrm{s}/2, \omega_\mathrm{s}/2),$$

in which $T_\mathrm{s} = 2\pi/\omega_\mathrm{s}$. Note that T_s here is precisely the sampling period. Since $\hat{f}(\omega)$ has its support on $(-\omega_\mathrm{s}/2, \omega_\mathrm{s}/2)$ we may multiply with the rectangular pulse $\mathrm{rect}_{\omega_\mathrm{s}}(\omega)$ without changing the result,

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} F_n \mathrm{e}^{\mathrm{i}n\omega T_\mathrm{s}} \mathrm{rect}_{\omega_\mathrm{s}}(\omega), \qquad \forall \omega \in \mathbb{R}. \tag{4.25}$$

The Fourier coefficients F_n can be expressed as

$$\begin{aligned}
F_n &= \frac{1}{\omega_\mathrm{s}} \int_{-\omega_\mathrm{s}/2}^{\omega_\mathrm{s}/2} \hat{f}(\omega) \mathrm{e}^{-\mathrm{i}n\omega T_\mathrm{s}} \,\mathrm{d}\omega \\
&= \{\hat{f}(\omega) = 0 \text{ for } |\omega| > \omega_\mathrm{s}/2\} = \frac{1}{\omega_\mathrm{s}} \int_{-\infty}^{\infty} \hat{f}(\omega) \mathrm{e}^{-\mathrm{i}n\omega T_\mathrm{s}} \,\mathrm{d}\omega \\
&= \frac{2\pi}{\omega_\mathrm{s}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \mathrm{e}^{-\mathrm{i}n\omega T_\mathrm{s}} \,\mathrm{d}\omega \\
&= \{\text{inverse Fourier transform of } f(t)\} = \frac{2\pi}{\omega_\mathrm{s}} f(-nT_\mathrm{s}) = T_\mathrm{s} f[-n].
\end{aligned}$$

Now replace n by $-n$ and we see that (4.25) becomes

$$\hat{f}(\omega) = T_\mathrm{s} \sum_{n=-\infty}^{\infty} f[n] \mathrm{e}^{-\mathrm{i}n\omega T_\mathrm{s}} \mathrm{rect}_{\omega_\mathrm{s}}(\omega). \tag{4.26}$$

From Table 4.2 we know that

$$\frac{\omega_\mathrm{s}}{2\pi} \mathrm{sinc}(\omega_\mathrm{s} t/2) \overset{\mathfrak{F}}{\longleftrightarrow} \mathrm{rect}_{\omega_\mathrm{s}}(\omega),$$

and with the help of the time-shift rule we then get

$$\frac{1}{T_\mathrm{s}} \mathrm{sinc}(\omega_\mathrm{s}(t - nT_\mathrm{s})/2) \overset{\mathfrak{F}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{i}n\omega T_\mathrm{s}} \mathrm{rect}_{\omega_\mathrm{s}}(\omega).$$

With all this we can apply the inverse Fourier transform term by term to (4.26) and we get what we wanted to show,

$$f(t) = T_\mathrm{s} \sum_{n=-\infty}^{\infty} f[n] \frac{1}{T_\mathrm{s}} \mathrm{sinc}(\omega_\mathrm{s}(t - nT_\mathrm{s})/2) = \sum_{n=-\infty}^{\infty} f[n] \,\mathrm{sinc}(\pi(t/T_\mathrm{s} - n)).$$

This completes the proof. ∎

Compact discs store sampled signals that are sampled with a frequency of 44.1×10^3Hz. Knowing Shannon's sampling theorem it should be no surprise that a frequency of 44.1×10^3Hz is about twice as much as what human hearing can detect. It is in fact a bit more than twice the bandwidth of the human ear, but then again, signals stored on CDs are not actually reconstructed by the reconstruction formula (4.24), but by something more realistic. The ideal reconstruction formula is not practical since its reconstructed $f(t)$ as given by (4.24) depends on all future and past $f[n]$. This would mean that the CD has to be read in its entirety first before sound is produced! Not very practical. Shannon's sampling theorem constitutes a fundamental limitation of signal reconstruction through sampling: no one in the galaxy will be able to reconstruct signals perfectly if all that is given is that $\omega_s \le 2\omega_b$, while on the other hand if $\omega_s > 2\omega_b$ then perfect reconstruction *is* possible and (4.24) is the unique answer.

Remark: Viewing it from a different angle, the signal $f(t)$ defined by the reconstruction formula (4.24) is the signal of smallest bandwidth that has the given samples $f(nT_s)$, $n \in \mathbb{Z}$.

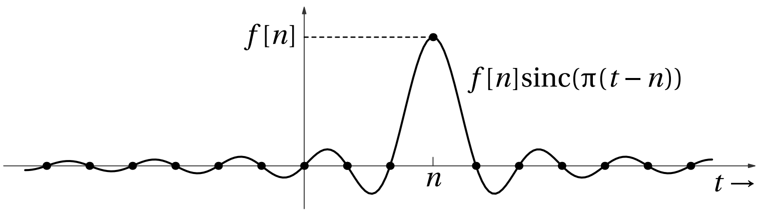


FIGURE 4.6: Graph of $f[n]\,\mathrm{sinc}(\pi(t-n))$

Example 4.6.4. For $T_s = 1$ the reconstruction formula (4.24) becomes

$$f(t) = \sum_{n=-\infty}^{\infty} f[n]\,\mathrm{sinc}(\pi(t-n)).$$

Each term $f[n]\,\mathrm{sinc}(\pi(t-n))$ in this sum is a function that is zero at all sampling instances $t \in \mathbb{Z}$ except at $t = n$ where it equals $f[n]$, see Figure 4.6. $\square$

4.7 Exercises

4.1 Sketch $f(t)$ and determine its Fourier transform.

(a)

$$f(t) = \begin{cases} e^t & \text{if } 5 < t < 6, \\ 0 & \text{if } t < 5 \text{ or } t > 6. \end{cases}$$

(b) $f(t) = t\,\mathrm{rect}_1(t)$.

(c) $f(t) = \mathrm{rect}_1(t-2) - \mathrm{rect}_1(t+2)$.

4.2 Let $f(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}(\omega)$ and $\omega_0 > 0$. Determine the Fourier transforms of the following signals.

(a) $f(t)\sin(\omega_0 t)$,

(b) $f(at)e^{i\omega_0 t}$, $\quad (a \neq 0)$,

(c) $\mathrm{Re}(f(t))$,

(d) $\mathrm{Im}(f(t))$.

4.3 Determine the Fourier transforms of the following signals.

(a) $\dfrac{\sin(4t)}{t}$,

(b) $\mathrm{trian}_a(2t) \quad (a > 0)$,

(c) $e^{-at}\,\mathbb{1}(t - t_0)$, $\quad (\mathrm{Re}(a) > 0)$,

(d) te^{-at^2}, $\quad (a > 0)$,

(e) $e^{-at}\sin(\omega_0 t)\,\mathbb{1}(t)$, $\quad (\mathrm{Re}(a) > 0)$,

(f) $\dfrac{1}{1+t^2}$,

(g) $\dfrac{1}{t^2+2t+2}$.

4.4 Let $f(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}(\omega)$. Determine the Fourier transforms of the following signals.

(a) $2f(3t-1)$,

(b) $\mathrm{e}^{-2\mathrm{i}t}f(t-2)$,

(c) $tf(t)$,

(d) $f(-\frac{1}{2}t)$,

(e) $f(1-t)$,

(f) $f(t)\cos^2(\omega_0 t)$.

4.5 The signal $f(t)$ is given by

$$f(t)=\begin{cases}\frac{1}{2}(1+\cos(\pi t/T)) & \text{if } |t|\le T,\\ 0 & \text{if } |t|>T.\end{cases}$$

Here $T>0$. Determine the Fourier transform of $f(t)$.

4.6 Let $f(t) \xleftrightarrow{\mathfrak{F}} \hat{f}(\omega)$. Determine $f(t)$ for the cases that $\hat{f}(\omega)$ equals

(a) $\hat{f}(\omega)=\mathrm{rect}_{2a}(\omega-\omega_0)+\mathrm{rect}_{2a}(\omega+\omega_0)$,

(b) $\hat{f}(\omega)=\dfrac{2+\mathrm{i}\omega}{4+5\mathrm{i}\omega-\omega^2}$,

(c) $\hat{f}(\omega)=\dfrac{9}{(1+\mathrm{i}\omega)^2(2+\mathrm{i}\omega)}$.

4.7 Show that $(\mathrm{rect}_a * \mathrm{rect}_a)(t)=a\,\mathrm{trian}_a(t)$.

4.8 Determine the convolution $(f*g)(t)$ for the following signals using Fourier transforms.

(a) $f(t)=\mathrm{e}^{at}\,\mathbb{1}(-t)$ and $g(t)=\mathrm{e}^{-bt}\,\mathbb{1}(t)$ with $a>0$ and $b>0$,

(b) $f(t)=\mathrm{sinc}(\alpha t)$ and $g(t)=\mathrm{sinc}(\beta t)$ with $\alpha>0$ and $\beta>0$,

4.9 A signal $f(t)$ is given whose Fourier transform is

$$\hat{f}(\omega)=\frac{1}{\mathrm{i}\omega+b}$$

with b a nonzero real constant. Determine the Fourier transform $\hat{g}(\omega)$ of the following signals $g(t)$.

(a) $g(t) = f(5t-4)$,

(b) $g(t) = t^2 f(t)$,

(c) $g(t) = \mathrm{e}^{2\mathrm{i}t} f(t)$,

(d) $g(t) = \cos(4t) f(t)$,

(e) $g(t) = f^{(2)}(t)$,

(f) $g(t) = (f * f)(t)$,

(g) $g(t) = f^2(t)$,

(h) $g(t) = \frac{1}{\mathrm{i}t-b}$.

4.10 Suppose $f(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}(\omega)$. Determine $f(t)$ for the cases that $\hat{f}(\omega)$ equals

(a) $\hat{f}(\omega) = \mathrm{i}\omega\,\mathrm{trian}_2(\omega)$,

(b) $\hat{f}(\omega) = \mathrm{e}^{-\mathrm{i}\omega t_0}\,\mathrm{rect}_8(\omega)$,

(c) $\hat{f}(\omega) = \cos(\omega)\,\mathrm{rect}_{2\pi}(\omega)$.

4.11 Consider the *RC*-network of Example 4.6.1. Calculate the output for the following input signals

(a) $u(t) = \mathrm{e}^{-\alpha t}\,\mathbb{1}(t)$,

(b) $u(t) = \mathrm{e}^{\alpha t}\,\mathbb{1}(-t)$,

(c) $u(t) = \delta(t-1)$

(d) $u(t) = \cos(t)$, see also Exercise 3.23.

4.12 Consider the hypnotist of Example 3.7.4 for which the differential equation is given by

$$\ell y^{(2)}(t) + k y^{(1)}(t) + g y(t) = -u^{(2)}(t).$$

The parameters of this equation are $\ell = 0.3\,\mathrm{m}$ (the length of the cord), $g = 9.81\,\mathrm{m/s^2}$ (the gravitational constant) and $k = 0.2\,\mathrm{m/s}$ (the friction).

(a) Find the frequency response of the above model.

(b) Determine the impulse response of this system.

(c) Determine the step response of the system (i.e., the output when $u(t) = \mathbb{1}(t)$)

(d) Calculate the output when the input is given as $\mathrm{e}^{-t}\,\mathbb{1}(t)$.

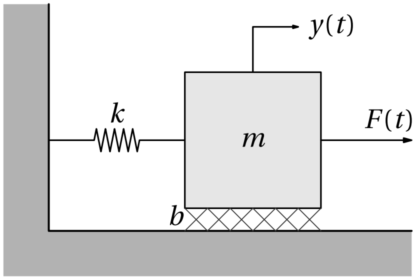


FIGURE 4.7: A mechanical system

4.13 Consider the mechanical system of Figure 4.7. The mass equals m, the spring constant is denoted by k, and the force b caused by the friction is assumed to equal 0.4 times the velocity. $F(t)$ is the external force acting on the mass, which is taken as the input to the system.

(a) Show that a model of the the mechanical system is given by

$$my^{(2)}(t) = -0.4y^{(1)}(t) - ky(t) + F(t), \tag{4.27}$$

where $y(t)$ denotes the position with respect to its equilibrium position.

(b) Calculate the expression for the frequency response.

(c) For $m = 0.1\,\mathrm{kg}$ and $k = 0.4\,\mathrm{Ns/m}$ calculate the impulse response.

(d) For the constants as given in the previous item calculate the position if the force is given as $F(t) = \mathrm{e}^{-t}\,\mathbb{1}(t)$.

4.14 Given is the bandlimited signal with Fourier transform

$$\hat{f}(\omega) = |\omega|\,\mathrm{rect}_{2\pi}(\omega).$$

(a) Is the signal $f(t)$ uniquely determined by its samples at the time instance $t = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$?
Motivate your answer.

(b) Determine $f[n]$ for $n \in \mathbb{Z}$.

(c) Determine the energy content of $f(t)$.

4.15 Given are the signals

$$f(t) = \mathrm{e}^{-|t|} \quad \text{and} \quad h(t) = \frac{\sin(at)}{t} \qquad (a > 0).$$

Let $g(t)$ be the convolution $g(t) = (f * h)(t)$. For which values of a is the convolution uniquely determined by its samples at $t = 0, \pm 1, \pm 2, \ldots$?

More involved problems

4.16 Let $f(t)$ be an absolutely integrable signal and let the signal $g(t)$ be given by

$$g(t) = \frac{1}{a} \int_{t-a/2}^{t+a/2} f(u)\,\mathrm{d}u.$$

(a) Show that $g(t)$ is absolutely integrable.

(b) Express the Fourier transform of $g(t)$ in terms of the Fourier transform of $f(t)$.

Chapter 5

Laplace Transform

FIGURE 5.1: Pierre Simon Laplace (1749–1827)

A drawback of the Fourier transform

$$\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$

is that many signals that we wish to consider do not have a Fourier transform. The unit step $\mathbb{1}(t)$, for example, only has a Fourier transform in the generalized sense (not treated in this course), and $\mathrm{e}^{t}\,\mathbb{1}(t)$ does not have a Fourier transform at all. The Laplace transform can be seen as an extension of the Fourier transform. It is an extension that allows to consider a much larger family of signals, but which still inherits most of the useful properties and insights of the Fourier transform. As it turns out, it gives rise to some useful new properties and insights as well. In accordance with the Fourier transform, the *two-sided Laplace transform* of a signal $f(t)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

In contrast to the Fourier transform, however, in the Laplace transform we allow for general complex $s \in \mathbb{C}$ and not just imaginary $s = \mathrm{i}\omega \in \mathrm{i}\mathbb{R}$. This simple extension makes it possible to take Laplace transforms of signals that hitherto were not (Fourier) transformable.

In many cases we deal with causal signals, $f(t) = 0\ \forall t < 0$. Assuming a causal signal $f(t)$ contains no delta function components, then the Laplace transform reduces to

$$\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_{0}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

The latter expression

$$\int_{0}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

is the *one-sided Laplace transform*. In this course we only consider the one-sided Laplace transform, from now on referred to as the *Laplace transform*. It is important to realize that this Laplace transform will also be used for non-causal signals! Taking the Laplace transform of a non-causal signal, means that all values $f(t),\ t < 0$ are lost in the transformation. The Laplace transform will therefore be of use only if we are not concerned with past time function values $f(t)$, $t < 0$; a situation that is often the case.

Later when functions with delta function components are allowed, we have to revise the definition of the Laplace transform a bit. In the following section we consider piecewise smooth signals.

5.1 Laplace transform

Definition 5.1.1 (Laplace transform). The *Laplace transform* $F(s)$ of a signal $f(t)$ is defined as

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{5.1}$$

for those $s \in \mathbb{C}$ for which the integral is convergent. □

Generally the Laplace transform of a given signal $f(t)$ is defined only for a subset of the complex numbers s. If $f(t)$ is *exponentially bounded* for $t > 0$, that is, if numbers C and a exist such that

$$|f(t)| \leq C\mathrm{e}^{at} \qquad \forall t > 0,$$

then the Laplace transform exists for all s with $\mathrm{Re}(s) > a$. Indeed, for such s the integral of (5.1) converges absolutely:

$$\begin{aligned}
\int_0^\infty |f(t)\mathrm{e}^{-st}|\,\mathrm{d}t &= \int_0^\infty |f(t)||\mathrm{e}^{-\mathrm{Re}(s)t}|\underbrace{|\mathrm{e}^{-\mathrm{i}\,\mathrm{Im}(s)t}|}_{1}\,\mathrm{d}t \\
&= \int_0^\infty |f(t)|\mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t \\
&\leq \int_0^\infty C\mathrm{e}^{at}\mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t \\
&= \int_0^\infty C \underbrace{\mathrm{e}^{(a-\mathrm{Re}(s))t}}_{\text{decays exponentially}}\,\mathrm{d}t < \infty.
\end{aligned}$$

All polynomials in t are exponentially bounded for $t > 0$, all exponential functions of the form e^{bt} are exponentially bounded. All piecewise smooth periodic signals are bounded, hence, exponentially bounded for $t > 0$ as well, and all products and sums of exponentially bounded signals are again exponentially bounded. The Laplace transform therefore applies to many signals, many more than can be handled with the Fourier transform.

Example 5.1.2. The Laplace transform of $f(t) = 1$ is $F(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t$. Now e^{-st} is a decaying exponential function only if $\mathrm{Re}(s) > 0$. So the integral exists iff $\mathrm{Re}(s) > 0$ in which case $F(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{\mathrm{e}^{-st}}{-s}\Big]_0^\infty = 1/s$. □

This example is instructive in that it demonstrates a fundamental feature of convergence of Laplace transforms:

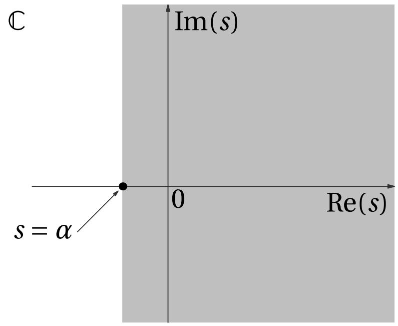


FIGURE 5.2: Region of convergence $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$

Lemma 5.1.3 (Region of convergence). For every signal $f(t)$ there is a unique $\alpha \in \mathbb{R}$, possibly $\alpha = \pm\infty$, such that $F(s)$ exists if $\mathrm{Re}(s) > \alpha$, and does not exist if $\mathrm{Re}(s) < \alpha$.

Proof. The statement is equivalent to this:

If $F(s_0)$ exists, then $F(s)$ exists $\forall\, \mathrm{Re}(s) > \mathrm{Re}(s_0)$.

This we prove. So, suppose s_0 is such that $F(s_0)$ exists. Then $\Omega(t)$ defined as $\Omega(t) = \int_0^t \mathrm{e}^{-s_0\tau} f(\tau)\,\mathrm{d}\tau$ converges to $F(s_0)$ as $t \to \infty$, which, in particular, means that $\Omega(t)$ is bounded on $[0,\infty)$. This we need soon. Now suppose that $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. Then

$$\begin{aligned} F(s) &= \lim_{M\to\infty} \int_0^M \mathrm{e}^{-st} f(t)\,\mathrm{d}t \\ &= \lim_{M\to\infty} \int_0^M \mathrm{e}^{-(s-s_0)t}\,\mathrm{e}^{-s_0 t} f(t)\,\mathrm{d}t \\ &= \lim_{M\to\infty} \int_0^M \mathrm{e}^{-(s-s_0)t}\Omega'(t)\,\mathrm{d}t \\ &= \lim_{M\to\infty} \Big(\big[\mathrm{e}^{-(s-s_0)t}\Omega(t)\big]_{t=0}^{t=M} + (s-s_0)\int_0^M \mathrm{e}^{-(s-s_0)t}\Omega(t)\,\mathrm{d}t \Big). \end{aligned} \tag{5.2}$$

Since $\Omega(t)$ is bounded, and $\mathrm{Re}(s - s_0) > 0$ we see that the limit (5.2) exists. Therefore $F(s)$ exists whenever $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. ∎

The number α here is called the *abscissa of convergence*, and we refer to the set

$$\{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}$$

as the *region of convergence* of the Laplace transform of $f(t)$, see Figure 5.2. On the region of convergence the Laplace transform exists. On the boundary $\{s \in \mathbb{C} \mid \operatorname{Re}(s) = \alpha\}$ of this region the Laplace transform may or may not exist, depending on $f(t)$ and on the value of s. In Example 5.1.2 the abscissa of convergence is $\alpha = 0$, but it may in some unlikely cases also be $-\infty$ or $+\infty$. For instance, the function

$$f(t) = \mathrm{e}^{-t^2}$$

decays to zero so incredibly fast that $F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ converges for every s. The region of convergence then is the whole complex plane and we take the abscissa of convergence to be $\alpha = -\infty$. Another function is (mind the plus sign),

$$f(t) = \mathrm{e}^{+t^2}.$$

This function grows to ∞ so unbelievably fast as $t \to \infty$ that $F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ diverges no matter what s is. Now the region of convergence is empty and then we take $\alpha = +\infty$.

Example 5.1.4 (Region of convergence).

1. The unit step $\mathbb{1}(t)$ has Laplace transform

$$\int_0^\infty \mathrm{e}^{-st}\,\mathbb{1}(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \lim_{N\to\infty} \frac{\mathrm{e}^{-st}}{-s}\Big]_{t=0}^{t=N} = \{\text{if } \operatorname{Re}(s) > 0\} = \frac{1}{s}.$$

   The above limit exists only if $\operatorname{Re}(s) > 0$. The abscissa of convergence is therefore $\alpha = 0$.

2. The causal exponential function $\mathrm{e}^{at}\,\mathbb{1}(t)$ (with a complex) has Laplace transform

$$\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\,\mathbb{1}(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a} \qquad (\operatorname{Re}(s) > \operatorname{Re}(a)).$$

   The abscissa of convergence is $\alpha = \operatorname{Re}(a)$.

3. The Laplace transform of $f(t) = \mathrm{e}^{at}\cos(bt)\,\mathbb{1}(t)$ with a complex and b real, follows similarly as above:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\cos(bt)\,\mathrm{d}t &= \frac{1}{2}\int_0^\infty \mathrm{e}^{-(s-a-\mathrm{i}b)t} + \mathrm{e}^{-(s-a+\mathrm{i}b)t}\,\mathrm{d}t \\ &= \frac{1}{2}\left(\frac{1}{s-a-\mathrm{i}b} + \frac{1}{s-a+\mathrm{i}b}\right) \qquad \text{provided } \operatorname{Re}(s-a\pm\mathrm{i}b)>0 \\ &= \frac{s-a}{(s-a)^2+b^2}.\end{aligned}$$

The abscissa of convergence is $\alpha = \operatorname{Re}(a\pm\mathrm{i}b) = \operatorname{Re}(a)$.

□

The mapping that sends a signal $f(t)$ to its Laplace transform $F(s)$ is denoted by $\mathfrak{L}$. That is,

$$\mathfrak{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t.$$

(Yes, this notation is a bit awkward, but it is quite standard.) Both $F(s)$ and the mapping $\mathfrak{L}$ are referred to as the *Laplace transform*. Without proof we state that causal piecewise smooth signals are uniquely determined by their Laplace transform, except at points of discontinuity.

5.2 Signals with delta function components

So far we assumed that $f(t)$ is a regular function. If the function has delta function components, then we have to revise the definition of the Laplace transform slightly.

We confine ourselves to signals $f(t)$ of the form

$$f(t) = g(t) + \sum_{n=0}^{N} a_n\delta(t-t_n). \tag{5.3}$$

Here $g(t)$ denotes a regular signal, the coefficients a_n are (complex) numbers and the t_n are arbitrary time instances, $t_n \in \mathbb{R}$.

The Laplace transform of the signal $f(t)$ of (5.3) is now taken to be

$$\mathfrak{L}\{f(t)\} = \mathfrak{L}\{g(t)\} + \sum_{n=0}^{N} a_n \mathfrak{L}\{\delta(t-t_n)\}.$$

The Laplace transform of $g(t)$ is the Laplace transform of a piecewise smooth signal as dealt with in the previous section. It will be no surprise that for the Laplace transform of $\delta(t-t_n)$ we shall use the sifting property of delta functions: If $t_n \neq 0$, then

$$\begin{aligned}\mathfrak{L}\{\delta(t-t_n)\} &= \int_0^{\infty} \delta(t-t_n)\mathrm{e}^{-st}\,\mathrm{d}t = \int_{-\infty}^{\infty} \delta(t-t_n)\mathrm{e}^{-st}\,\mathbb{1}(t)\,\mathrm{d}t \\ &= \{\mathbb{1}(t) \text{ is continuous at } t = t_n \neq 0\} = \begin{cases} 0 & \text{if } t_n < 0, \\ \mathrm{e}^{-st_n} & \text{if } t_n > 0. \end{cases}\end{aligned}$$

If $t_n = 0$, then we end up with the integral $\int_0^{\infty} \delta(t)\mathrm{e}^{-st}\,\mathrm{d}t$, which, as it stands, has no meaning since the delta function $\delta(t)$ has its spike precisely at $t = 0$ which is on the boundary of the interval over which is integrated. To accommodate for this problem it is customary to adjust the definition of Laplace transform by expanding slightly the interval over which is integrated. The Laplace transform will henceforth be understood as

$$F(s) = \int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t := \lim_{\epsilon \downarrow 0} \int_{-\epsilon}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{5.4}$$

Consequently, for any $t_n \in \mathbb{R}$, possibly $t_n = 0$, we have that

$$\mathfrak{L}\{\delta(t-t_n)\} = \int_{0^-}^{\infty} \delta(t-t_n)\mathrm{e}^{-st}\,\mathrm{d}t = \begin{cases} 0 & \text{if } t_n < 0, \\ \mathrm{e}^{-st_n} & \text{if } t_n \geq 0. \end{cases} \tag{5.5}$$

In particular the Laplace transform of the delta function $\delta(t)$ is equal to 1. For piecewise smooth signals $f(t)$ it makes no difference whether or not integration in (5.4) begins at 0 or 0^- or even 0^+, but for generalized functions it does make a difference, and opting for 0^- means that we want to take the effect of $\delta(t)$ fully into account.

5.3 Properties of the Laplace transform

Following is a list of properties and rules of calculus for the Laplace transform. Only those properties are proved whose derivation is substantially different from their corresponding

TABLE 5.1: Standard Laplace transform properties

Property	$f(t)$	$F(s)$	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	$\operatorname{Re} s > \max(\alpha_1, \alpha_2)$
Time-scaling	$f(at)$	$\frac{1}{a} F(\frac{s}{a})$	$a > 0,\ \operatorname{Re} s > \alpha$
Time-shift	$f(t - t_0)\,\mathbb{1}(t - t_0^-)$	$F(s)\mathrm{e}^{-st_0}$	$t_0 > 0,\ \operatorname{Re} s > \alpha$
Shift in s-domain	$f(t)\mathrm{e}^{s_0 t}$	$F(s - s_0)$	$\operatorname{Re} s > \operatorname{Re} s_0 + \alpha$
Differentiation (t)	$f^{(1)}(t)$	$sF(s) - f(0^-)$	$\operatorname{Re} s > \alpha$
	$f^{(2)}(t)$	$s^2 F(s) - s f(0^-) - f^{(1)}(0^-)$	$\operatorname{Re} s > \alpha$
Integration (t)	$\int_{0^-}^{t} f(\tau)\,\mathrm{d}\tau$	$\frac{F(s)}{s}$	$\operatorname{Re} s > \max(0, \alpha)$
Differentiation (s)	$-t f(t)$	$F'(s)$	$\operatorname{Re} s > \alpha$

Fourier transform property. In what follows, $F(s)$ denotes the Laplace transform of $f(t)$, and α denotes the abscissa of convergence of $F(s)$.

Linearity.

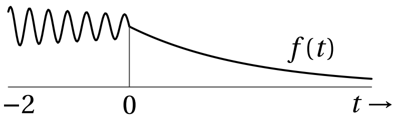
$$\mathfrak{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1\mathfrak{L}\{f_1(t)\} + a_2\mathfrak{L}\{f_2(t)\}, \qquad \mathrm{Re}(s) > \max(\alpha_1, \alpha_2).$$

Time scaling.

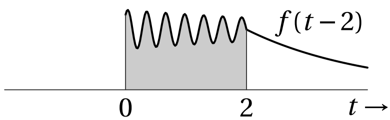
$$\mathfrak{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}), \qquad (a > 0,\ \mathrm{Re}(s) > a\alpha).$$

The only difference with Fourier is that we need $a > 0$. We challenge you to see what happens if $a < 0$.

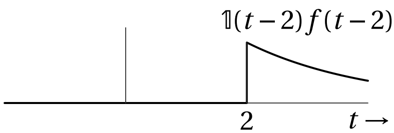
Time-shift. Based on the time-shift rule of Fourier transformation one might be tempted to conclude that $\mathfrak{L}\{f(t-t_0)\} = F(s)\mathrm{e}^{-st_0}$, but there is a catch: if $f(t)$ equals, say,



then its translation $f(t-2)$ is



This shows that we cannot expect any connection between the Laplace transforms of $f(t)$ and $f(t-2)$, because the Laplace transform only considers $t > 0$ and thus the contribution of the shaded part on the Laplace transform of $f(t-2)$ is absent in the Laplace transform of $f(t)$. The solution is to remove the initial interval $[0,2]$ by considering



The general rule is

$$\mathfrak{L}\{f(t-t_0)\,\mathbb{1}(t-t_0^-)\} = F(s)\mathrm{e}^{-st_0}, \qquad (t_0 > 0,\ \mathrm{Re}(s) > \alpha).$$

Now the proof of the corresponding Fourier transform property can be used.

Shift in the s-domain.

$$\mathfrak{L}\{f(t)\mathrm{e}^{s_0 t}\} = F(s-s_0), \qquad (\mathrm{Re}(s) > \mathrm{Re}(s_0)+\alpha).$$

Differentiation with respect to time.

$$\mathfrak{L}\{f^{(1)}(t)\} = sF(s) - f(0^-), \qquad (\mathrm{Re}(s) > \alpha). \tag{5.6}$$

Proof. We prove this only for that case that $f(t)$ is differentiable in the classical sense. We shall further assume that on the region of convergence $f(t)\mathrm{e}^{-st} \to 0$ for $t \to \infty$. This is practically always the case. Integration by parts gives that

$$\begin{aligned}
\int_{0^-}^{\infty} f^{(1)}(t)\mathrm{e}^{-st}\,\mathrm{d}t &= f(t)\mathrm{e}^{-st}\Big]_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= -f(0^-)\mathrm{e}^{-s0^-} + s\int_{0^-}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= -f(0^-) + sF(s).
\end{aligned}$$

■

If $f(t)$ is piecewise smooth, then the rule (5.6) remains valid, even if the derivative $f'(t)$ exists only in the generalized sense.

Integration with respect to time. Let $g(t) = \int_{0^-}^{t} f(\tau)\,\mathrm{d}\tau$. Then

$$\mathfrak{L}\{g(t)\} = \frac{F(s)}{s}, \qquad (\mathrm{Re}(s) > \max(0,\alpha)).$$

The derivation of this rule is postponed till we treat the convolution theorem for Laplace transforms (see Example 5.6.2).

Differentiation with respect to *s*.

$$\mathfrak{L}\{-tf(t)\} = F^{(1)}(s), \qquad (\mathrm{Re}(s) > \alpha).$$

5.4 Examples

Example 5.4.1 (Application of the differentiation rule). Let $g(t) = f'(t)$. Then by the differentiation rule we have that $G(s) = sF(s) - f(0^-)$. The Laplace transform of the second derivative can be obtained by applying the differentiation rule twice:

$$\begin{aligned}\mathfrak{L}\{f^{(2)}(t)\} &= \mathfrak{L}\{g'(t)\} = sG(s) - g(0^-) = s\big(sF(s) - f(0^-)\big) - f'(0^-) \\ &= s^2F(s) - sf(0^-) - f'(0^-).\end{aligned}$$

Repeated use (n times) of the differentiation rule will give

$$\mathfrak{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-). \tag{5.7}$$

If $f(t)$ is causal, then $f^{(k)}(0^-) = 0$, so then

$$\mathfrak{L}\{f^{(n)}(t)\} = s^nF(s).$$

$\square$

Example 5.4.2 (Differentiation rule in *s*-domain). Repeated use of differentiation rule in s gives

$$\mathfrak{L}\{(-t)^n f(t)\} = F^{(n)}(s) \qquad (n = 0, 1, \ldots). \tag{5.8}$$

$\square$

Example 5.4.3 (More applications of rules).

1. Consider the signal $f_1(t) = \mathrm{e}^{at}$ and the causal signal $f_2(t) = \mathrm{e}^{at}\mathbb{1}(t)$, and realize that they have the same Laplace transform $F_1(s) = F_2(s) = 1/(s-a)$. The derivatives of $f_1(t)$ and $f_2(t)$ are

   $$f_1^{(1)}(t) = a\mathrm{e}^{at}, \qquad f_2^{(1)}(t) = a\mathrm{e}^{at}\mathbb{1}(t) + \delta(t).$$

   The derivative of $f_2(t)$ is a generalized derivative since $f_2(t)$ is discontinuous at $t = 0$. With help of the differentiation rule we find that

   $$\begin{aligned}\mathfrak{L}\{f_1^{(1)}(t)\} &= \frac{s}{s-a} - f_1(0^-) = \frac{s}{s-a} - 1 = \frac{a}{s-a}, \\ \mathfrak{L}\{f_2^{(1)}(t)\} &= \frac{s}{s-a} - f_2(0^-) = \frac{s}{s-a}.\end{aligned}$$

   These findings may also be obtained from direct calculation of $\mathfrak{L}\{f_1'(t)\}$ and $\mathfrak{L}\{f_2'(t)\}$.

2. We know that $\mathfrak{L}\{e^{at}\} = 1/(s-a)$. Differentiate n times in the s-domain and we arrive at

$$\mathfrak{L}\{(-t)^n e^{at}\} = \frac{(-1)^n n!}{(s-a)^{n+1}},$$

hence

$$\mathfrak{L}\left\{\frac{t^n e^{at}}{n!}\right\} = \frac{1}{(s-a)^{n+1}}.$$

$\square$

Some of the more commonly used Laplace transform pairs and properties are collected in Tables 5.1 and 5.2.

TABLE 5.2: Standard Laplace transform pairs

$f(t), \quad (t > 0^-)$	$F(s)$	Region of conv.
e^{at}	$\dfrac{1}{s-a}$	$\operatorname{Re} s > \operatorname{Re} a$
$\dfrac{t^n}{n!} \quad (n = 0, 1, \ldots)$	$\dfrac{1}{s^{n+1}}$	$\operatorname{Re} s > 0$
$\dfrac{t^n}{n!} e^{at} \quad (n = 0, 1, \ldots)$	$\dfrac{1}{(s-a)^{n+1}}$	$\operatorname{Re} s > \operatorname{Re}(a)$
$\cos(bt)$	$\dfrac{s}{s^2 + b^2}$	$\operatorname{Re} s > 0$
$\sin(bt)$	$\dfrac{b}{s^2 + b^2}$	$\operatorname{Re} s > 0$
$e^{at} \cos(bt)$	$\dfrac{s-a}{(s-a)^2 + b^2}$	$\operatorname{Re} s > \operatorname{Re} a$
$e^{at} \sin(bt)$	$\dfrac{b}{(s-a)^2 + b^2}$	$\operatorname{Re} s > \operatorname{Re} a$
$\delta(t)$	1	$\forall s \in \mathbb{C}$

Example 5.4.4 (The sinc). In this example we derive that

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t = \frac{\pi}{2}. \tag{5.9}$$

This is an integral that we need in the proofs of the Fourier theorems (see Appendix A.1). It may be tempting to try to compute this integral using familiar integration techniques such as substitution and integration by parts, but this will probably fail as no closed form expression is known for the definite integral $\int_0^x \frac{\sin(t)}{t}\,\mathrm{d}t$. Instead we first use Laplace techniques to compute

$$G(s) := \mathfrak{L}\{\frac{\sin(t)}{t}\} = \int_0^\infty \frac{\sin(t)}{t}\mathrm{e}^{-st}\,\mathrm{d}t, \qquad (\mathrm{Re}\,(s) > 0) \tag{5.10}$$

and then compute $G(0)$ to determine the integral (5.9) that we want. According to the rule $-tf(t) \xleftrightarrow{\mathfrak{L}} F'(s)$ we have that

$$-\sin(t) \xleftrightarrow{\mathfrak{L}} G'(s).$$

On the other hand we have that $\mathfrak{L}\{\sin(t)\} = \frac{1}{s^2+1}$, i.e.

$$G'(s) = -\frac{1}{s^2+1}, \qquad (\mathrm{Re}\,(s) > 0).$$

Hence

$$G(s) = c - \arctan(s), \qquad (\mathrm{Re}\,(s) > 0). \tag{5.11}$$

The constant c may be determined by letting s approach ∞ (and $s \in \mathbb{R}$). Clearly the larger s the faster e^{-st} decays to zero as t increases. Therefore from (5.10) we get that $\lim_{s\to\infty} G(s) = 0$. Hence c in (5.11) equals $\arctan(\infty) = \pi/2$, and, therefore,

$$G(s) = \frac{\pi}{2} - \arctan(s), \qquad (\mathrm{Re}\,(s) > 0).$$

Now we can compute the integral that we want[1],

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t = \lim_{s\downarrow 0}\int_0^\infty \frac{\sin(t)}{t}\mathrm{e}^{-st}\,\mathrm{d}t = \lim_{s\downarrow 0} G(s) = \frac{\pi}{2}.$$

Nice. □

[1] But is the limit of the integral the same as the integral of the limit? Well, there are situations where they are *not* the same! Here, though, they are. In the course *Analysis 2* you will learn about such limits.

Example 5.4.5 (Partial fraction expansion). The inverse Laplace transform of rational functions may be determined with the help of partial fraction expansion (see Appendix A.4). The method is the same as for determining the inverse Fourier transform of rational functions. Let $F(s)$ be given as

$$F(s) = \frac{6s}{(s+1)(s^2-4)}.$$

The poles of this rational function are $s_1 = -1$, $s_2 = -2$ and $s_3 = 2$. Verify for yourself that the partial fraction expansion of $F(s)$ is

$$F(s) = \frac{2}{s+1} + \frac{1}{s-2} + \frac{-3}{s+2}.$$

Now, from Table 5.2 we can directly write down the inverse Laplace transform,

$$f(t) = 2\mathrm{e}^{-t} + \mathrm{e}^{2t} - 3\mathrm{e}^{-2t}, \qquad (t > 0^-).$$

□

5.5 Limiting behavior

From the Laplace transform $F(s)$ one can fairly easily determine the initial value $f(0)$ of the function in time domain, as well as the final value $\lim_{t\to\infty} f(t)$. Both cases use the fact that $F(s)$ is "very small" if s is positive, real and "very large". This is an intuitive result because if s is positive and "large" then e^{-st} as a function of t goes to zero rapidly, and, hence $F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ is probably going to be "small". Here is the proper statement and proof:

Lemma 5.5.1 (Behaviour of the Laplace transform of $s \to \infty$). If $f(t)$ is of exponential order, and piecewise continuous, then

$$\lim_{s\in\mathbb{R}, s\to\infty} F(s) = 0.$$

Proof. Piecewise continuity is used only to guarantee existence of integrals of the form $\int_0^M f(t)\mathrm{e}^{-st}\,\mathrm{d}t$. Since $f(t)$ is of exponential order there are $\gamma, C \in \mathbb{R}$ such that $|f(t)| < C\mathrm{e}^{\gamma t}$

for every $t > 0$. Now

$$\begin{aligned}|F(s)| &= \left|\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\right| \\ &\leq \int_0^\infty |\mathrm{e}^{-st}||f(t)|\,\mathrm{d}t \leq \int_0^\infty \mathrm{e}^{-\operatorname{Re}(s)t} C\mathrm{e}^{\gamma t}\,\mathrm{d}t = \frac{C}{\operatorname{Re}(s)-\gamma} \qquad \text{if } \operatorname{Re}(s) > \gamma.\end{aligned}$$

The upperbound goes to zero as $s \to \infty$. ■

Theorem 5.5.2 (Initial value theorem). Suppose $f(t)$ is of exponential order, and piecewise continuous, and that the *initial value* $f(0^+)$ defined as

$$f(0^+) = \lim_{t\downarrow 0} f(t)$$

exists. Then

$$\lim_{s\in\mathbb{R},\, s\to\infty} sF(s) = f(0^+).$$

Proof. Let $s \in \mathbb{R}, s > 0$. Then $sF(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}(st) = \int_0^\infty f(\tau/s)\mathrm{e}^{-\tau}\,\mathrm{d}\tau$. Notice that for every $\tau > 0$ we have $\lim_{s\to\infty} f(\tau/s) = f(0^+)$. If $f(t)$ is bounded then $\lim_{s\in\mathbb{R},\, s\to\infty} sF(s) - f(0^+) = \lim_{s\to\infty} \int_0^\infty (f(\tau/s) - f(0^+))\mathrm{e}^{-\tau}\,\mathrm{d}\tau = 0$. This proves the result for bounded $f(t)$. If $f(t)$ is not bounded then for some large enough α the function $g(t) = f(t)\mathrm{e}^{-\alpha t}$ is bounded (by assumed exponential order). Then $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} sG(s-\alpha) = \lim_{s\to\infty}(s+\alpha)G(s) = g(0^+) + \alpha G(\infty) = g(0^+) = f(0^+)$ (here we used Lemma 5.5.1). ■

To motivate the next theorem consider the function

$$f(t) = c + \mathrm{e}^{-2t}.$$

The part e^{-2t} converges to zero as $t \to \infty$ so $f(t)$ converges to c:

$$\lim_{t\to\infty} f(t) = c.$$

This limiting value is sometimes called the "final value". The final value can also be obtained through its Laplace transform. We have

$$F(s) = \frac{c}{s} + \frac{1}{s+2}$$

and we see that the final value c is the coefficient of $1/s$, or to say it differently, it is the constant part of

$$sF(s) = c + \frac{s}{s+2}.$$

The right-hand side at $s = 0$ is precisely our final value c. The property that the final value can be discerned from the Laplace transform holds under mild assumptions:

Theorem 5.5.3 (Final value theorem). Suppose that $f(t)$ is continuous on every finite interval $[0, N]$ and that the *final value* $f(\infty)$ defined as

$$f(\infty) = \lim_{t\to\infty} f(t)$$

exists. Then

$$f(\infty) = \lim_{s\downarrow 0} sF(s).$$

Proof. If the final value $f(\infty)$ exists, then the function is bounded and, hence, of exponential order 0. So $F(s)$ exists for every $\mathrm{Re}(s) > 0$. On the one hand we have that

$$\lim_{s\downarrow 0} \mathfrak{L}\{f'(t)\} = \lim_{s\downarrow 0} sF(s) - f(0^-),$$

and on the other we have that

$$\lim_{s\downarrow 0} \mathfrak{L}\{f'(t)\} = \lim_{s\downarrow 0} \int_{0^-}^{\infty} \mathrm{e}^{-st} f'(t)\,\mathrm{d}t = \int_{0^-}^{\infty} f'(t)\,\mathrm{d}t = f(\infty) - f(0^-).$$

So $\lim_{s\downarrow 0} sF(s) = f(\infty)$. We assumed here that we are allowed to interchange limit and integral. A proper technical proof is documented in the footnote[2]. ■

[2]: Let $g(t) = f(t) - f(\infty)$. Then $g(\infty) = 0$, and $|g(t)|$ over $t > 0$ is bounded by some $M > 0$. For any $N > 0$ let d_N be the supremum of $|g(t)|$ over $[N, \infty)$. Since $g(\infty) = 0$ we have $\lim_{N\to\infty} d_N = 0$. Now for any $s > 0$ we have that $|sG(s)| = |s\int g(t)\mathrm{e}^{-st}\,\mathrm{d}t|$ and this is bounded from above by $s(M\int_0^N \mathrm{e}^{-st}\,\mathrm{d}t + d_N\int_N^\infty \mathrm{e}^{-st}\,\mathrm{d}t) = M(1-\mathrm{e}^{-sN}) + d_N(\mathrm{e}^{-sN})$. Let $N = 1/\sqrt{s}$, then this upperbound is $M(1-\mathrm{e}^{-\sqrt{s}}) + d_N\mathrm{e}^{-\sqrt{s}}$. In the limit $s \downarrow 0$ we have $N \to \infty$ and the upperbound goes to 0. Hence $\lim_{s\downarrow 0} sG(s) = 0$. Finally, note that $sG(s) = sF(s) - f(\infty)$.

Example 5.5.4 (Final value). Let $f(t)$ be a signal with Laplace transform

$$F(s) = \frac{5}{s(s^2+2s+5)}.$$

To find $f(t)$ we use partial fraction expansion,

$$\frac{5}{s(s^2+2s+5)} = \frac{1}{s} - \frac{s+2}{s^2+2s+5} = \frac{1}{s} - \frac{1}{(s+1)^2+4} - \frac{s+1}{(s+1)^2+4}.$$

From Table 5.2 we copy

$$\mathfrak{L}\{1\} = \frac{1}{s}, \qquad \mathfrak{L}\{\frac{1}{2}\mathrm{e}^{-t}\sin(2t)\} = \frac{1}{(s+1)^2+4}, \qquad \mathfrak{L}\{\mathrm{e}^{-t}\cos(2t)\} = \frac{s+1}{(s+1)^2+4},$$

so that

$$f(t) = 1 - \mathrm{e}^{-t}(\sin(2t)+\cos(2t)), \qquad (t > 0^-).$$

From this the final value $f(\infty)$ can be seen to exist and it equals $\lim_{t\to\infty} f(t) = 1$. This value is indeed equal to what the final value theorem states:

$$\lim_{s\downarrow 0} sF(s) = \lim_{s\downarrow 0} \frac{5}{s^2+2s+5} = 1.$$

□

Example 5.5.5 (Final & initial value). If $f(t) = 2 + 3\mathrm{e}^{-2t}$ then $F(s) = \frac{2}{s} + \frac{3}{s+2}$ for $\mathrm{Re}(s) > 0$. In this case the final value is

$$\lim_{t\to\infty} f(t) = 2$$

and this agrees with

$$\lim_{s\downarrow 0} sF(s) = \lim_{s\downarrow 0} 2 + \frac{3s}{s+2} = 2.$$

The initial value theorem is confirmed as well, because

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} 2 + \frac{3s}{s+2} = 2+3 = 5$$

and $f(0) = 5$. □

5.6 Convolution in Laplace domain

In Section 2.7 we saw that the convolution of two causal signals $f(t)$ and $g(t)$ is again causal and that

$$(f * g)(t) = \left(\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\right)\mathbb{1}(t).$$

Since the Laplace transform only deals with the causal part of a signal (i.e., the part $f(t)$ for $t \geq 0$), it is natural to define convolutions in this respect as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau, \qquad (t > 0^-). \tag{5.12}$$

We stress that the signals $f(t)$ and $g(t)$ are allowed to be non-causal. Also, we want to allow delta components in $f(t)$ and $g(t)$, and so in (5.12) we have to extend slightly the interval $[0, t]$ over which the integration is performed,

$$(f * g)(t) = \int_{0^-}^{t^+} f(\tau)g(t-\tau)\,\mathrm{d}\tau. \tag{5.13}$$

Theorem 5.6.1 (Convolution theorem for the Laplace transform). Let $f(t)$ and $g(t)$ be signals with Laplace transforms $F(s)$ and $G(s)$ respectively. Then

$$\mathfrak{L}\{(f * g)(t)\} = F(s)G(s),$$

where $(f * g)(t)$ is the convolution product (5.13). □

Proof. Similar to that of Fourier transformation. It will be convenient to express the convolution as

$$\int_0^t f(u)g(t-u)\,\mathrm{d}u = \int_0^\infty f(u)\,\mathbb{1}(t-u)g(t-u)\,\mathrm{d}u.$$

This is correct because $\mathbb{1}(t-u)=0$ for $u>t$, and $\mathbb{1}(t-u)=1$ on $[0,t]$. Now

$$\begin{aligned}
\mathfrak{L}\{(f*g)(t)\} &= \int_0^\infty \mathrm{e}^{-st}\Big(\int_0^\infty f(u)\,\mathbb{1}(t-u)g(t-u)\,\mathrm{d}u\Big)\,\mathrm{d}t \\
&= \int_0^\infty f(u)\Big(\int_0^\infty \mathrm{e}^{-st}\,\mathbb{1}(t-u)g(t-u)\,\mathrm{d}t\Big)\mathrm{d}u \\
&= \int_0^\infty f(u)\mathfrak{L}\{\mathbb{1}(t-u)g(t-u)\}\,\mathrm{d}u \\
&= \int_0^\infty f(u)\mathrm{e}^{-su}G(s)\,\mathrm{d}u = \Big(\int_0^\infty \mathrm{e}^{-su}f(u)\,\mathrm{d}u\Big)G(s) = F(s)G(s).
\end{aligned}$$

■

For piecewise smooth signals $f(t)$ and $g(t)$ the proof of this theorem is the same as the proof of the convolution theorem for the Fourier transform. It may be shown that the result is still valid when $f(t)$ and $g(t)$ contain delta function components.

Example 5.6.2 (Convolution).

a) Consider the unit step $\mathbb{1}(t)$ and an arbitrary signal $f(t)$. Then

$$\mathfrak{L}\{(f*\mathbb{1})(t)\} = \mathfrak{L}\Big\{\int_{0^-}^t f(\tau)\,\mathrm{d}\tau\Big\} = \frac{F(s)}{s}.$$

(This also proves the integration rule on page 188.)

b) Consider the delta function $\delta(t)$ and an arbitrary signal $f(t)$. Then

$$\mathfrak{L}\{(f*\delta)(t)\} = F(s)\times 1 = F(s).$$

In other words $(f*\delta)(t) = f(t)$ for all $t>0^-$.

□

5.7 Applications

ODEs for positive time

The Laplace transform may be used in a much the same way as Fourier transforms to convert differential equations to algebraic equations. The advantage however is that with Laplace transforms

- initial conditions may be taken into account, and
- signals need only be Laplace transformable not necessarily Fourier transformable.

A minor drawback is that we have to limit attention to positive time. A typical application is that of set-point change. First an example that does not yet use Laplace transforms.

Example 5.7.1 (Set-point change). Consider again the RC-circuit of Example 3.7.3. The differential equation that relates the input voltage $u(t)$ and output voltage $y(t)$ across the capacitor was found to be

$$y^{(1)}(t) + \alpha y(t) = \alpha u(t), \qquad (\alpha = \frac{1}{RC}). \tag{5.14}$$

Suppose that since long the voltage $u(t)$ has been equal to a constant value of 1. It is easy to believe that then $y(t)$ (the voltage across the capacitor) settles to a constant value as well. Assuming that $y(t)$ is constant, gives us that $y^{(1)}(t) = 0$, and so from (5.14) we see that necessarily the output settles to a value of $y(t) = u(t) = 1$.

Now, at $t = 0$, we instantaneously increase the input voltage from 1 to 2 and, keep it constant from then on

$$u(t) = 2, \qquad \forall t > 0.$$

What will happen with $y(t)$? That is, what is $y(t)$ for $t > 0$? Intuition tells us that the response $y(t)$ is unique, but we also know that the general solution $y(t)$ of ODE (5.14) is not unique. It is readily verified — see Appendix A.3 — that the general solution for $t > 0$ is

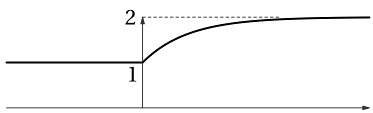
$$y(t) = 2 + \beta e^{-\alpha t}, \qquad \beta \in \mathbb{R}.$$

As argued, we have shortly before the set-point change at $t = 0$ that $y(0^-) = 1$. This gives us β:

$$1 = y(0^-) = 2 + \beta e^{-\alpha 0} = 2 + \beta.$$

So $\beta = -1$, and we found the (unique) response to the set-point change,

$$y(t) = \begin{cases} 1 & \text{if } t < 0 \\ 2 - e^{-\alpha t} & \text{if } t > 0 \end{cases}$$



In words, after that $u(t)$ is increased from 1 to 2, the output $y(t)$ grows continuously and exponentially from 1 upwards and settles to a final value of 2. □

Changing the reference temperature on your central heating system is another instance of a set-point change. Set-point changes are very common. With the Laplace transform we can redo the previous example, but now more succinctly. Indeed, initial conditions can then be taken into account right from the start and—importantly—we can solve the problem without having to find a particular solution.

Example 5.7.2 (Set-point change via Laplace). To find the solution $y(t)$ of (5.14) for $u(t) = 2$ we use Laplace transformation. Recall that

$$\mathfrak{L}\{y^{(1)}(t)\} = sY(s) - y(0^-).$$

So, taking the Laplace transform of the *Equation* (5.14) gives

$$\big(sY(s) - y(0^-)\big) + \alpha Y(s) = \alpha U(s).$$

By assumption $y(0^-) = 1$, and $u(t) = 2\ \forall t > 0$, giving $U(s) = 2/s$. Therefore

$$sY(s) - 1 + \alpha Y(s) = \alpha 2/s.$$

This is an algebraic equation, and its solution is

$$Y(s) = \frac{1 + 2\alpha/s}{s+\alpha} = \frac{s+2\alpha}{s(s+\alpha)} = \{\text{partial fraction expansion}\} = \frac{2}{s} - \frac{1}{s+\alpha}.$$

Its inverse Laplace transform yields the time-domain $y(t)$ that we are after,

$$y(t) = 2 - \mathrm{e}^{-\alpha t}, \qquad t > 0^-.$$

This agrees with what was found earlier. □

In Example 5.7.1 we found the solution by *assuming* that $y(t)$ is continuous at $t = 0$. Only then can we say that $y(0^-) = 2 + \beta \mathrm{e}^{-\alpha 0} = 2 + \beta$, which we needed to determine $y(t)$. *This assumption is not generally valid.* In certain systems $y(t)$ may jump as the result of a jump in $u(t)$, so the procedure in Example 5.7.1 is not generally applicable. The use of the Laplace transform in Example 5.7.2 does not rely on any continuity assumption. The Laplace transform *is* generally applicable. The following example demonstrates a case where $y(t)$ jumps.

Example 5.7.3 (Set-point change with jump). Consider the ODE

$$y^{(2)}(t) - 4y(t) = u^{(2)}(t) \tag{5.15}$$

and suppose that $u(t) = \mathbb{1}(t)$ and that we are given the initial conditions $y(0^-) = 0$, $\dot{y}(0^-) = 1$. To find $y(t)$ for $t > 0^-$ we apply the Laplace transform on the above equation. Using the differentiation rule (Page 186) we find that

$$\mathfrak{L}\{y^{(2)}(t)\} = s^2 Y(s) - sy(0^-) - y^{(1)}(0^-) = s^2 Y(s) - 1,$$

and

$$\mathfrak{L}\{u^{(2)}(t)\} = s^2 U(s) - su(0^-) - u^{(1)}(0^-) = s^2 U(s) = s^2 \frac{1}{s} = s.$$

Next take the Laplace transform of (5.15), and what we get is

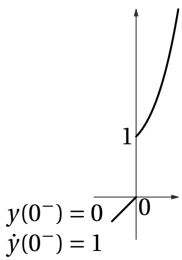
$$(s^2 Y(s) - 1) - 4Y(s) = s^2 U(s) = s.$$

This is a linear equation in $Y(s)$ with solution

$$Y(s) = \frac{s+1}{s^2-4} = \frac{s+1}{(s-2)(s+2)} = \frac{3/4}{s-2} + \frac{1/4}{s+2}.$$

The output now follows from the inverse Laplace transform,

$$y(t) = \frac{3}{4}\mathrm{e}^{2t} + \frac{1}{4}\mathrm{e}^{-2t}, \qquad (t > 0^-).$$



At $t = 0$ the output jumps from $y(0^-) = 0$ to $y(0^+) = 1$. □

Coupled ODEs

So far we have seen that the Laplace transform makes it easy to solve certain ordinary differential equations. However, sometimes it is a lot of work to derive the ODE that relates the input and output. Often, modeling results in a set of coupled ordinary differential equations. One can use this set of equations to find the output corresponding to the given input and initial conditions. This is closely related to calculating the impedance of an *RLC* circuit. Let us illustrate this by means of an example.

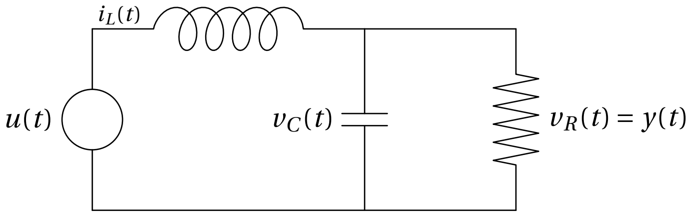


FIGURE 5.3: *LRC* circuit

Example 5.7.4. Consider the electrical circuit of Figure 5.3. We see the voltage delivered by the voltage source, $u(t)$, as the input, and the voltage $y(t)$, over the resistor as the output. We denote the current through the resistor by $i_L(t)$, and the voltage across the capacitor by $v_C(t)$. Using the basic models for the inductor and capacitor, and Kirchhoff's laws, we find

$$Li_L^{(1)}(t) = u(t) - v_C(t) \tag{5.16}$$

$$Cv_C^{(1)}(t) = i_L(t) - \frac{1}{R}v_C(t). \tag{5.17}$$

Furthermore, we have that $y(t) = v_C(t)$. We assume that the current through the inductor and the voltage across the capacitor are given for $t = 0^-$.

On both sides of both equations (5.16) and (5.17) we can apply the Laplace transform, and using the rule for derivatives, we find

$$L(sI_L(s) - i_L(0^-)) = U(s) - V_C(s) \tag{5.18}$$

$$C(sV_C(s) - v_C(0^-)) = I_L(s) - \frac{1}{R}V_C(s). \tag{5.19}$$

Putting all the unknowns variables on the left hand side of the equation and the known variables on the right-hand side, we obtain

$$LsI_L(s) + V_C(s) = U(s) + Li_L(0^-) \tag{5.20}$$

$$-I_L(s) + \left(Cs + \frac{1}{R}\right) V_C(s) = Cv_C(0^-). \tag{5.21}$$

This are two equations in two unknowns, $I_L(s), V_C(s)$, which is easy to solve. For instance, if we add Ls times (5.21) to (5.20) we get an equation in $V_C(s)$ alone:

$$V_C(s) + Ls\left(Cs + \frac{1}{R}\right) V_C(s) = U(s) + Li_L(0^-) + LCsv_C(0^-). \tag{5.22}$$

This leads to the following equation for the (Laplace transform of the) output

$$V_C(s) = \frac{U(s) + Li_L(0^-) + LCsv_C(0^-)}{LCs^2 + \frac{L}{R}s + 1}. \tag{5.23}$$

Taking the inverse Laplace transform of this function gives the desired output. □

Partial differential equations

Next is a rather spectacular example. It is a prime example of the power of Laplace transformation. The *partial* differential equation

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}$$

is the mathematical model of the temperature distribution $u(x,t)$ at position x and time t in an isolated bar of a material with thermal conductivity k. In the example discussed below the bar is between $x = 0$ and $x = 1$. The example further assumes the boundary conditions $u(0,t) = u(1,t) = 0$ and this means that the temperature is kept at 0 degrees for all time at the outer ends of the bar, and $u(x,t) = 3\sin(2\pi x)$ means that the bar has a sinusoidal distribution at initial time $t = 0$. Heat will diffuse through the bar. In the example below we show that the heat distribution is sinusoidal for all time and we derive how fast it decreases in amplitude. Fig 5.4 depicts the heat distribution over time t and space x.

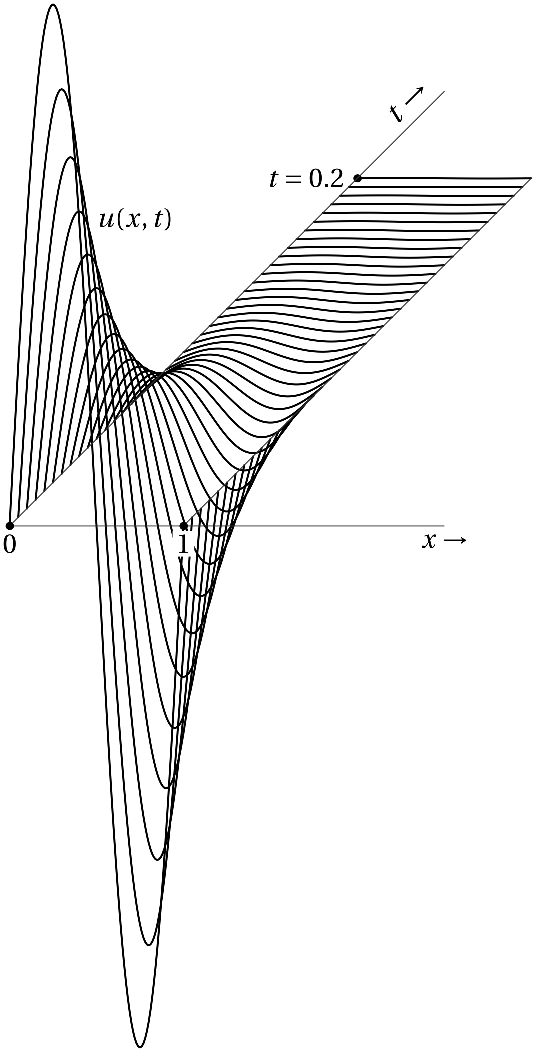


FIGURE 5.4: Temperature profile $u(x, t)$ as a function of position x and time t, see Example 5.7.5

Example 5.7.5 (Heat equation — diffusion). A function $u(x,t)$ is sought that satisfies the *partial differential equation* (PDE)

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0,$$

and which satisfies the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0 \qquad \text{for } t > 0,$$

and the initial value condition

$$u(x,0) = 3\sin(2\pi x) \qquad \text{for } 0 \le x \le 1.$$

We take the Laplace transform of both terms of the PDE with respect to the variable t. For this purpose we determine that

$$\mathfrak{L}\{\frac{\partial^2 u(x,t)}{\partial x^2}\} = \int_0^\infty \frac{\partial^2 u(x,t)}{\partial x^2} \mathrm{e}^{-st}\,\mathrm{d}t = \frac{\partial^2}{\partial x^2}\int_0^\infty u(x,t)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{\partial^2}{\partial x^2}\mathfrak{L}\{u\} = \frac{\partial^2 U(x,s)}{\partial x^2}.$$

Here we assume that exchange of differentiation of x and integration over t is permissible. The transformed PDE now becomes

$$sU(x,s) - u(x,0) = \frac{\partial^2 U(x,s)}{\partial x^2},$$

so

$$\frac{\partial^2 U(x,s)}{\partial x^2} - sU(x,s) = -3\sin(2\pi x). \tag{5.24}$$

The PDE is thus transformed into an ordinary DE. For any fixed s it is a DE with constant coefficients. We solve this DE with using characteristic polynomials. (Solving using the Laplace transform is also possible!) The characteristic equation is

$$\lambda^2 - s = 0.$$

Therefore the general solution of this homogenous solution is:

$$U_{\text{hom}}(x,s) = A(s)\mathrm{e}^{x\sqrt{s}} + B(s)\mathrm{e}^{-x\sqrt{s}}.$$

Notice that the integration constants A, B may depend on s so they are, as yet, arbitrary functions of s! For the particular solution of the inhomogeneous DE we try

$$U_{\text{part}}(x) = c\sin(2\pi x) + d\cos(2\pi x).$$

Substitution in the DE (5.24) results in $d = 0$ (verify this yourself) and

$$-4\pi^2 c\sin(2\pi x) - sc\sin(2\pi x) = -3\sin(2\pi x),$$

so

$$c = \frac{3}{s+4\pi^2}.$$

The general solution of the inhomogeneous DE hence is

$$U(x,s) = A(s)e^{x\sqrt{s}} + B(s)e^{-x\sqrt{s}} + \frac{3}{s+4\pi^2}\sin(2\pi x). \tag{5.25}$$

We will now transform the boundary conditions, which are in principle functions of t as well,

$$\begin{aligned} U(0,s) &= \mathfrak{L}\{u(0,t)\} = \mathfrak{L}\{0\} = 0, \\ U(1,s) &= \mathfrak{L}\{u(1,t)\} = \mathfrak{L}\{0\} = 0. \end{aligned}$$

Using the general form (5.25) this gives

$$\begin{aligned} A(s) + B(s) &= 0, \\ A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} &= 0. \end{aligned}$$

This results in the rather special solution $A(s) = B(s) = 0$, so (5.25) reduces to

$$U(x,s) = \frac{3}{s+4\pi^2}\sin(2\pi x).$$

This function is rational in s. Its inverse Laplace transform brings us back the time domain,

$$u(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x).$$

Done. See Fig. 5.4. The amplitude of the sinusoid decays exponentially fast. □

5.8 Exercises

5.1 Determine the Laplace transform and its domain of convergence for the following signals.

(a) $t \sin(\pi t)$

(b) $t^2 \sin(\pi t)$

(c) $\mathrm{e}^t \mathbb{1}(a-t)$, for arbitrary $a \in \mathbb{R}$.

(d) $\frac{\mathrm{d}}{\mathrm{d}t} \cos(t)$

(e) $\frac{\mathrm{d}}{\mathrm{d}t} (\cos(t) \mathbb{1}(t))$

5.2 Determine $f(t)$, $(t > 0^-)$, whose Laplace transform equals

(a) $\dfrac{1}{s-2}$

(b) $\dfrac{s}{s-2}$

(c) $\dfrac{s}{(s-2)^2}$

(d) $\dfrac{s^3}{(s-2)^3}$

(e) $\dfrac{1}{s^2+2s+2}$

(f) $\dfrac{s}{s^2+2s+2}$

(g) $\dfrac{2}{s^2+s}$

(h) $\dfrac{3s+4}{s^2+s}$

(i) $\dfrac{s}{s^2+2s-2}$

5.3 Let $f(t) = (1 + \mathbb{1}(t-1)) \cos(t)$. Verify that

$$\mathfrak{L}\{f'(t)\} = s\mathfrak{L}\{f(t)\} - f(0^-).$$

5.4 Determine the inverse Laplace transform of the signals

(a) $\dfrac{1}{(s^2+1)^2}$,

(b) $\dfrac{s^2-3s+2}{s^2-7s+12}$,

(c) $\dfrac{1+\mathrm{e}^{-s\pi}}{s^2+1}$,

(d) $\dfrac{\mathrm{e}^{-(s+a)t_0}}{s+a}$.

5.5 Suppose $f(t)$ is a periodic signal with period T and let $F(s)$ be the Laplace transform of $f(t)$. Now periodic signals do not have a final value $f(\infty)$ unless $f(t)$ is constant. Express $\lim_{s\downarrow 0} sF(s)$ in terms of the Fourier coefficients of $f(t)$.

5.6 Use the Laplace transform to determine $(f*g)(t)$ for the cases

(a) $f(t)=t^n\,\mathbb{1}(t)$ and $g(t)=t^m\,\mathbb{1}(t)$,

(b) $f(t)=g(t)=\mathrm{e}^{-t}\,\mathbb{1}(t)$,

(c) $f(t)=\mathrm{e}^{-t}\,\mathbb{1}(t)$ and $g(t)=\sin(t)\,\mathbb{1}(t)$.

5.7 Use the Laplace transform to solve the differential equation

$$y^{(2)}(t)+3y^{(1)}(t)+2y(t)=u(t)$$

for $t>0$. The input equals $u(t)=\mathbb{1}(t)$ and the initial conditions are $y(0^-)=y^{(1)}(0^-)=0$.

5.8 Solve the differential equation of Exercise 5.7 for $t>0$ if the input equals $u(t)=\mathrm{e}^{-t}$ and the initial conditions are $y(0^-)=-1$ and $y^{(1)}(0^-)=6$.

5.9 Solve the differential equation of Exercise 5.7 for $t>1$ if the input equals $u(t)=\mathrm{e}^{-t}$ and the initial conditions are $y(1^-)=-1$ and $y^{(1)}(1^-)=6$.

5.10 The following differential equation is given

$$y^{(2)}(t)+2y^{(1)}(t)+5y(t)=u^{(2)}(t)-9u(t)$$

Furthermore, the input is given by $u(t) = \mathrm{e}^{3t}\mathbb{1}(t)$.

Show that there exist initial conditions $y(0^-)$ and $y^{(1)}(0^-)$ such that the solution y of the differential equation is zero for $t > 0$.

More involved problems:

5.11 Determine the Laplace transform of the following signals

(a) $f(t) = |\sin(\omega_0 t)|$,

(b) $f(t) = \lfloor t \rfloor$.

Here $\lfloor t \rfloor$ denotes the *floor* of t defined as

$$\lfloor t \rfloor = \max_{n\in\mathbb{Z}}\{n : n \le t\}.$$

5.12 Let $\beta > 0$. Determine all $s \in \mathbb{C}$ for which the Laplace transform of the signal $f(t) = 1/(1+t^\beta)$ exists. (Hint: distinguish various cases of β.)

5.13 First define the constant $\gamma := -\int_0^\infty \ln(t)\mathrm{e}^{-t}\,\mathrm{d}t \approx 0.5772156649$. Show that the Laplace transform of

$$f(t) = \ln(t)$$

equals

$$F(s) = -\frac{\ln(s)+\gamma}{s}$$

for every real $s > 0$.

5.14 Show that the Laplace transform of

$$f(t) = \frac{1}{\sqrt{t}}$$

is

$$F(s) = \frac{\sqrt{\pi}}{\sqrt{s}}$$

whenever $\mathrm{Re}(s) > 0$. [Hint: you may want to know that $\int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{1}{2}\sqrt{\pi}$.]

Appendix A

Appendices

A.1 Proofs

This appendix collects proofs that are either too complicated for the course or not insightful enough to warrant inclusion in the main body of the lecture notes.

A.1.1 Chapter 2

Proof of Theorem 2.2.2. The modulus of a non-constant polynomial $p(s)$ grows without bound as $|s| \to \infty$ so there is an $R > 0$ such that

$$\inf_{s\in\mathbb{C}} |p(s)| = \inf_{|s|\leq R} |p(s)|.$$

The disc $\{s \in \mathbb{C} : |s| \leq R\}$ is a compact set and $|p(s)|$ is continuous, hence the above infimum over this disc is in fact a minimum,

$$\min_{s} |p(s)| = \min_{|s|\leq R} |p(s)|. \tag{A.1}$$

Let s_0 be a minimizer. If $p(s_0) = 0$ then we are done. Now, to obtain a contradiction, assume that $p(s_0) \neq 0$. Consider the (finite) Taylor series expansion of p around s_0,

$$p(s_0 + h) = p(s_0) + hp'(s_0) + \cdots + h^k \frac{p^{(k)}(s_0)}{k!} + \cdots.$$

At least one of the derivatives in this expansion is nonzero because p is non-constant. Let $p^{(k)}(s_0)$ be the first nonzero derivative, that is,

$$p(s_0+h) = p(s_0) + h^k \frac{p^{(k)}(s_0)}{k!} + o(h^k). \tag{A.2}$$

Every $h \in \mathbb{C}$ can be expressed as

$$h^k = -\epsilon^k \frac{k!\,p(s_0)}{p^{(k)}(s_0)}$$

for some $\epsilon \in \mathbb{C}$. Then (A.2) takes the form

$$p(s_0+h) = p(s_0)\big(1-\epsilon^k + o(\epsilon^k)\big).$$

For $\epsilon > 0$ but small enough the term $\big(1-\epsilon^k + o(\epsilon^k)\big)$ is less than 1 (in magnitude) so then $|p(s_0+h)| < |p(s_0)|$. This is a contraction, hence $p(s_0) = 0$. ■

A.1.2 Chapter 3

Proof of Theorem 3.1.3. Let $f(t)$ denote the Fourier series $f(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}$. We show that $f(t)$ is continuous for every time $a \in \mathbb{R}$, that is, we show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|t-a| < \delta$ implies that $|f(t) - f(a)| < \epsilon$.

Define the partial sums

$$s_N(t) = \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}.$$

Then there holds that

$$|f(t) - s_N(t)| = |\sum_{|k|>N} c_k \mathrm{e}^{\mathrm{i}k\omega_0 t}| \leq \sum_{|k|>N} |c_k|. \tag{A.3}$$

By assumption $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, so that

$$\sum_{|k|>N} |c_k| = \sum_{k=-\infty}^{\infty} |c_k| - \sum_{k=-N}^{N} |c_k| \to 0 \quad \text{as } N \to \infty.$$

Consequently there is a large enough positive integer N_1 such that for every $N > N_1$ we get $\sum_{|k|>N} |c_k| < \epsilon/3$. Considering Equation (A.3) we conclude that $|f(t) - s_{N_1}(t)| \le \sum_{|k|>N_1} |c_k| < \epsilon/3$ for every t.

For every t the partial sum $s_{N_1}(t)$ is a finite sum of continuous functions, hence is itself continuous. For the given $\epsilon > 0$, therefore, a $\delta > 0$ can be found such that $|s_{N_1}(t) - s_{N_1}(a)| < \epsilon/3$ whenever $|t-a| < \delta$. Finally then for all such t,

$$\begin{aligned}
|f(t) - f(a)| &= |f(t) - s_{N_1}(t) + s_{N_1}(t) - s_{N_1}(a) - (f(a) - s_{N_1}(a))| \\
&\le |f(t) - s_{N_1}(t)| + |f(a) - s_{N_1}(a)| + |s_{N_1}(t) - s_{N_1}(a)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

This completes the proof. ∎

Lemma A.1.1. If $f(t)$ is piecewise smooth on $[-T/2, T/2]$, then

$$\lim_{a\to\infty} \int_{-T/2}^{T/2} f(t) \frac{\sin(at)}{t} \, \mathrm{d}t = \pi \frac{f(0^+) + f(0^-)}{2}.$$

Proof. It suffices to prove that

$$\lim_{a\to\infty} \int_{0}^{T/2} f(t) \frac{\sin(at)}{t} \, \mathrm{d}t = \frac{\pi}{2} f(0^+),$$

Indeed, if the above holds then replacing t with $-t$ readily gives

$$\lim_{a\to\infty} \int_{-T/2}^{0} f(t) \frac{\sin(at)}{t} \, \mathrm{d}t = \lim_{a\to\infty} \int_{0}^{T/2} f(-t) \frac{\sin(at)}{t} \, \mathrm{d}t = \frac{\pi}{2} f(0^-).$$

Define $I(a) = \int_0^{T/2} f(t) \frac{\sin(at)}{t} \, \mathrm{d}t$ and express $I(a)$ as a sum $I(a) = I_1(a) + f(0^+) I_2(a)$ with

$$\begin{aligned}
I_1(a) &= \int_0^{T/2} \frac{f(t) - f(0^+)}{t} \sin(at) \, \mathrm{d}t, \\
I_2(a) &= \int_0^{T/2} \frac{\sin(at)}{t} \, \mathrm{d}t.
\end{aligned}$$

We will show that $\lim_{a\to\infty} I_1(a) = 0$ and that $\lim_{a\to\infty} I_2(a) = \pi/2$.

To calculate the limit of $I_2(a)$ we make use of the standard integral (see Example 5.4.4)

$$\int_0^\infty \frac{\sin(t)}{t}\,\mathrm{d}t = \frac{\pi}{2}.$$

This gives

$$\lim_{a\to\infty} I_2(a) = \lim_{a\to\infty}\int_0^{T/2} \frac{\sin(at)}{t}\,\mathrm{d}t = \{\tau = at\} = \lim_{a\to\infty}\int_0^{aT/2} \frac{\sin(\tau)}{\tau}\,\mathrm{d}\tau = \frac{\pi}{2}.$$

Now take an $\epsilon > 0$. We show that $|I_1(a)| < \epsilon$ for large enough a. Since $f'(0^+) = \lim_{t\downarrow 0}(f(t) - f(0^+))/t$ exists, the function $(f(t) - f(0^+))/t$ is bounded on $(0, T/2]$, that is,

$$|f(t) - f(0^+)| \le Mt \quad \forall\, t \in (0, T/2]$$

for some $M > 0$. Choose $\delta > 0$ such that $\delta < \epsilon/(2M)$ and $\delta < T/2$. Then

$$\left|\int_0^\delta \frac{f(t) - f(0^+)}{t}\sin(at)\,\mathrm{d}t\right| \le M\int_0^\delta |\sin(at)|\,\mathrm{d}t \le M\delta < \frac{\epsilon}{2}.$$

We have found that

$$\begin{aligned}|I_1(a)| &= \left|\int_0^\delta \frac{f(t) - f(0^+)}{t}\sin(at)\,\mathrm{d}t + \int_\delta^{T/2} \frac{f(t) - f(0^+)}{t}\sin(at)\,\mathrm{d}t\right| \\ &< \frac{\epsilon}{2} + \left|\int_\delta^{T/2} \frac{f(t) - f(0^+)}{t}\sin(at)\,\mathrm{d}t\right|.\end{aligned}$$

On the interval $[\delta, T/2]$ the function $(f(t) - f(0^+))/t$ is piecewise smooth since the only possible singularity is at $t = 0$ and this is not in the interval. The Riemann–Lebesgue lemma therefore applies, which gives

$$\lim_{a\to\infty}\int_\delta^{T/2} \frac{f(t) - f(0^+)}{t}\sin(at)\,\mathrm{d}t = 0.$$

So for sufficiently large a we have that

$$\left|\int_\delta^{T/2} \frac{f(t) - f(0^+)}{t}\sin(at)\,\mathrm{d}t\right| < \frac{\epsilon}{2},$$

Then, finally, $|I_1(a)| < \epsilon/2 + \epsilon/2 = \epsilon$, implying that $\lim_{a\to\infty} I_1(a) = 0$. ■

Theorem A.1.2 (Theorem 3.2.3). Let $f(t)$ be a T-periodic signal and suppose it is piecewise smooth on $[-T/2, T/2]$. Then for every $t \in \mathbb{R}$ there holds that

$$\frac{f(t^+)+f(t^-)}{2} = \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t},$$

where f_k are the Fourier coefficients of $f(t)$ defined as

$$f_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-\mathrm{i}k\omega_0 t}\,\mathrm{d}t. \tag{A.4}$$

Proof. We need to show that $(f(t^+)+f(t^-))/2 = \lim_{N\to\infty} s_N(t)$ for every t, where

$$s_N(t) = \sum_{k=-N}^{N} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t}. \tag{A.5}$$

First we derive an integral representation for $s_N(t)$ by substituting the defining integral (A.4) for f_k in (A.5). This gives

$$\begin{aligned}
s_N(t) &= \sum_{k=-N}^{N} f_k \mathrm{e}^{\mathrm{i}k\omega_0 t} \\
&= \sum_{k=-N}^{N} \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\mathrm{e}^{\mathrm{i}k\omega_0(t-\tau)}\,\mathrm{d}\tau \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}k\omega_0(t-\tau)}\,\mathrm{d}\tau,
\end{aligned}$$

using Problem 2.31 this becomes

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)(2N+1)\frac{\operatorname{sinc}((N+1/2)\omega_0(t-\tau))}{\operatorname{sinc}(\omega_0(t-\tau)/2)}\,\mathrm{d}\tau,$$

and now substitute $x = t-\tau$,

$$\begin{aligned}
&= \frac{1}{T}\int_{t-T/2}^{t+T/2} f(t-x)(2N+1)\frac{\operatorname{sinc}((N+1/2)\omega_0 x)}{\operatorname{sinc}(\omega_0 x/2)}\,\mathrm{d}x \\
&= \frac{1}{T}\int_{-T/2}^{T/2} f(t-x)(2N+1)\frac{\operatorname{sinc}((N+1/2)\omega_0 x)}{\operatorname{sinc}(\omega_0 x/2)}\,\mathrm{d}x.
\end{aligned}$$

To apply Lemma A.1.1 we rearrange this as

$$s_N(t) = \int_{-T/2}^{T/2} g(x) \times (N+1/2)\omega_0 \operatorname{sinc}((N+1/2)\omega_0 x)\,\mathrm{d}x,$$

where

$$g(x) = \frac{2f(t-x)}{T\omega_0 \operatorname{sinc}(\omega_0 x/2)} = \frac{f(t-x)}{\pi \operatorname{sinc}(\omega_0 x/2)}.$$

Therefore by Lemma A.1.1 we have

$$\lim_{N\to\infty} s_N(t) = \pi \frac{g(0^+)+g(0^-)}{2} = \pi \frac{f(t^-)/\pi + f(t^+)/\pi}{2} = \frac{f(t^+)+f(t^-)}{2}.$$

This completes the proof. ■

A.2 Bounded Linear Operators and Contractions

Definition A.2.1 (Bounded operator). Let $\mathbb{X}$ and $\mathbb{Y}$ be normed vector spaces (either both real or both complex). A linear operator $F : \mathbb{X} \to \mathbb{Y}$ is *bounded* if a $c \geq 0$ exists such that

$$\|F(x)\|_{\mathbb{Y}} \leq c\|x\|_{\mathbb{X}} \qquad \forall x \in \mathbb{X}. \tag{A.6}$$

□

If c for instance ≤ 1 then the norm of the image $F(x)$ never exceeds that of x, irrespective of the choice of x. Likewise if (A.6) holds for $c = 2$ then the norm of $F(x)$ will never be more than twice the norm of x, et cetera. The smallest possible c is what is called the *operator norm*[1]:

[1] The attentive reader will wonder why we call it operator *norm*. Doesn't this require that some set of operators F is a vector space and that on this vector space the operator norm has the property of norm? The answers are yes and yes, but we will not deal with such matters in this course, even though we are close to settling it.

Definition A.2.2 (Operator norm). Let $\mathbb{X}, \mathbb{Y}$ be normed vector spaces and $F : \mathbb{X} \to \mathbb{Y}$ a bounded linear operator. The *operator norm* $\|F\|$ of F is defined as[2]

$$\|F\| = \sup_{x \neq 0} \frac{\|F(x)\|_{\mathbb{Y}}}{\|x\|_{\mathbb{X}}}.$$

If $\mathbb{X} = \{0\}$ then we define $\|F\| = 0$. □

By definition of operator norm we have for every nontrivial vector space $\mathbb{X}$ and every $x \in \mathbb{X}$ that

$$\|F(x)\|_{\mathbb{Y}} \leq \|F\| \|x\|_{\mathbb{X}}. \tag{A.7}$$

Example A.2.3 (Bounded operator). We determine the operator norm of the linear mapping $A : \mathscr{C}([a,b];\mathbb{R}) \to \mathbb{R}$ defined as

$$A(f) = \int_a^b f(t) \,\mathrm{d}t.$$

On the domain $\mathscr{C}([a,b];\mathbb{R})$ we take the max-norm, on the codomain $\mathbb{R}$ we take as norm the absolute value. We have

$$\|A(f)\| = |A(f)| = \left| \int_a^b f(t) \,\mathrm{d}t \right| \leq \int_a^b |f(t)| \,\mathrm{d}t \leq \int_a^b \|f\|_\infty \,\mathrm{d}t = (b-a)\|f\|_\infty.$$

Therefore, for every f there holds

$$\frac{\|A(f)\|}{\|f\|_\infty} \leq b - a.$$

The operator A thus is bounded and its operator norm is at most $b - a$. For the constant function $f(t) = 1$, the above is an equality,

$$\frac{|A(1)|}{\|1\|_\infty} = \frac{\left| \int_a^b 1 \,\mathrm{d}t \right|}{1} = b - a.$$

So the upperbound $b - a$ is achieved for some functions f. The operator norm hence *equals* $b - a$. □

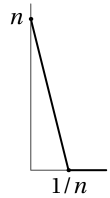
[2] supremum means least upperbound.

Example A.2.4 (Unbounded operator). Consider $\mathscr{C}([0,1];\mathbb{R})$ with the 1-norm. On this space the operator $\Delta : \mathscr{C}([0,1];\mathbb{R}) \to \mathbb{R}$ defined as

$$\Delta(f) = f(0)$$

is unbounded. To see this take for instance the sequence of functions

$$f_n(t) = \begin{cases} n(1-nt) & 0 \leq t \leq \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}.$$



The 1-norm of each f_n is $1/2$ while $|\Delta(f)| = n$. The ratio $|\Delta(f)|/\|f\|_1 = 2n$ is unbounded. This shows that Δ is an unbounded operator. □

Given normed vector spaces $\mathbb{X}, \mathbb{Y}$ we say that a mapping $A : \mathbb{X} \to \mathbb{Y}$ (not necessarily linear) is *continuous at* $x_0 \in \mathbb{X}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|x - x_0\|_{\mathbb{X}} < \delta \implies \|A(x) - A(x_0)\|_{\mathbb{Y}} < \epsilon. \tag{A.8}$$

If the mapping is continuous at x_0 for every $x_0 \in \mathbb{X}$, then A is said to be *continuous.*

Lemma A.2.5 (Norms are continuous). Every norm $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ is continuous.

Proof. Let $A = \|\cdot\|$ and on $\mathbb{R}$ use the standard norm (absolute value). The reverse triangle inequality says that $|A(x) - A(x_0)| = |\|x\| - \|x_0\|| \leq \|x - x_0\|$. Hence (A.8) holds for $\delta = \epsilon$. ■

Notice that norms are not linear mappings. For *linear* mappings there is a very neat characterization of continuity:

Theorem A.2.6 (Bounded = continuous for linear maps). Let $\mathbb{X}$ and $\mathbb{Y}$ be two normed vector spaces (either both real, or both complex). For a linear operator $A : \mathbb{X} \to \mathbb{Y}$ the following three statements are equivalent.

1. A is continuous,

2. A is continuous at $x_0 = 0$,

3. A is bounded.

Proof. (1. $\implies$ 2.) is trivial.

Now (2. $\implies$ 3.): If A is continuous at $x_0 = 0$ then $\forall \epsilon > 0$ there exists $\delta > 0$ such that $\|x\|_{\mathbb{X}} < \delta$ implies $\|A(x)\|_{\mathbb{Y}} < \epsilon$. Every vector $z \in \mathbb{X}$ can be written as a scaled vector $z = \alpha x$ such that $\|x\|_{\mathbb{X}} < \delta$. Just take α big enough, for instance $\alpha = 2\|z\|_{\mathbb{Y}}/\delta$. Then by linearity we have $\|A(z)\|_{\mathbb{Y}} = \|\alpha A(x)\|_{\mathbb{Y}} \leq |\alpha|\epsilon = \epsilon 2\|z\|_{\mathbb{X}}/\delta$. Hence $\|A\| \leq 2\epsilon/\delta < \infty$.

Remains to prove (3. $\implies$ 1.): If A is bounded then $\|A(x) - A(x_0)\|_{\mathbb{Y}} = \|A(x - x_0)\|_{\mathbb{X}} \leq \|A\|\|x - x_0\|_{\mathbb{X}}$ for some finite $\|A\|$. Take $\delta = \epsilon/\|A\|$. ∎

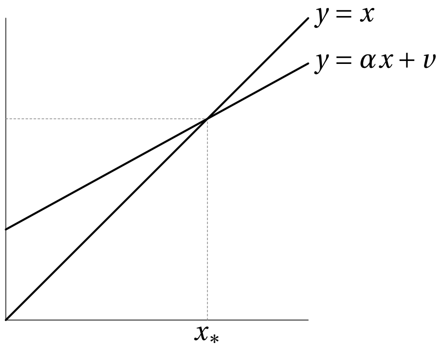
Now something special. Suppose we have to determine a solution x of the equation

$$x = A(x) + v,$$

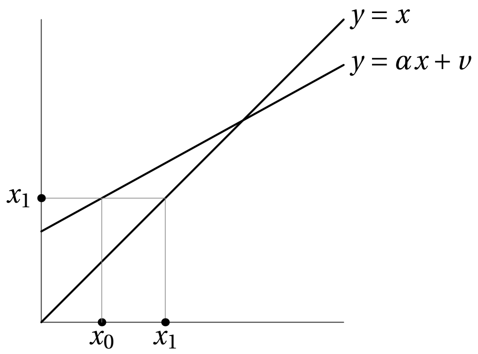
with v a given vector and A some mapping, possibly some horribly complicated mapping. In general this is daunting problem because the inverse of $I - A$ in the solution

$$x = (I - A)^{-1} v$$

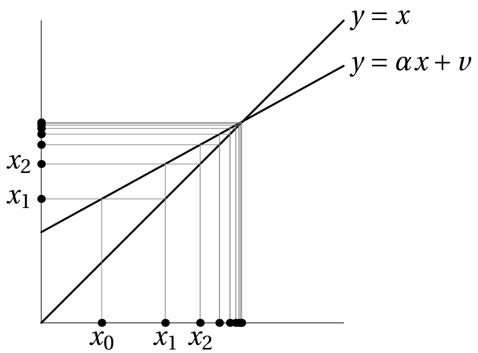
(assuming this inverse exists at all) may be very difficult to get a handle on. However there is a very important set of operators — still including really tricky ones — for which we can solve the problem and solve it constructively. To get an idea of the problem we start with real numbers x and v and with $A(x) = \alpha x$ a scalar multiplication. Then the point x where $x = \alpha x + v$, is the intersection of the two lines as indicated in this figure,



Instead of solving the intersection point x_* directly we propose to iteratively determine x_*. Just start wherever you like, x_0, and from that compute $x_1 := \alpha x_0 + v$,



and repeat this process. So now that you have x_1, compute $x_2 := \alpha x_1 + v$ and then compute $x_3 := \alpha x_2 + v$, et cetera,



As the figure suggests, the x_k converges to x_* which is the solution we are after. For this procedure to work we need the slope α of the line to satisfy $|\alpha| < 1$ (we challenge you to see what happens if $|\alpha| > 1$).

And now the big leap: this idea works for every linear mapping $A : \mathbb{X} \to \mathbb{X}$, provided that $\mathbb{X}$ is a Banach space and A is a contraction. This we shall prove.

Definition A.2.7 (Contraction). A linear operator $A : \mathbb{X} \to \mathbb{X}$ is a *contraction* if $\|A\| < 1$. □

For every contraction A on Banach space the inverse of $I - A$ exists and we have efficient ways of approximating it:

Theorem A.2.8 (Neumann series). Suppose A is a contraction on a Banach space $\mathbb{X}$. Then $I - A$ is invertible and bounded on $\mathbb{X}$ and we have

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and its operator norm is bounded from above as

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Proof. The operator norm satisfies the submultiplicative property $\|A^k\| \leq \|A\|^k$ (Exercise A.2.4). For every $x \in \mathbb{X}$ the $(I + A + \cdots + A^n)(x)$ is a Cauchy sequence, because for every $m \geq n \geq N$ we have

$$\begin{aligned}
&\|(I + \cdots + A^m)(x) - (I + \cdots + A^n)(x)\| \\
&= \|(A^{n+1} + \cdots + A^m)(x)\| \\
&\leq \|A^{n+1}(x)\| + \cdots + \|A^m(x)\| \\
&\leq (\|A^{n+1}\| + \cdots + \|A^m\|)\|x\| \\
&\leq \left(\sum_{k=N+1}^{\infty} \|A\|^k\right)\|x\| \\
&= \frac{\|A\|^{N+1}}{1 - \|A\|}\|x\| \to 0 \text{ as } N \to \infty, \text{ because } \|A\| < 1.
\end{aligned}$$

Since $\mathbb{X}$ is a Banach space, the above Cauchy property implies that $\lim_{n\to\infty}(I + \cdots + A^n)(x)$ exists for every x. Denote the limit by $B(x)$. The so defined B is in fact bounded because, using continuity of norm, the norm of the limit is the limit of the norm:

$$\begin{aligned}
\|B(x)\| &= \lim_{n\to\infty} \|(I + A + \cdots + A^n)(x)\| \\
&\leq \lim_{n\to\infty} \|x\| + \|A(x)\| + \cdots + \|A^n(x)\| \\
&\leq \lim_{n\to\infty} (1 + \|A\| + \cdots + \|A\|^n)\|x\| \\
&= \frac{1}{1 - \|A\|}\|x\|.
\end{aligned}$$

Remains the show that $B = (I - A)^{-1}$, i.e. that it satisfies $(I - A)B = I = B(I - A)$. It does:

$$\begin{aligned}(I-A)B(x) &= (I-A)\lim_{n\to\infty}(I+\cdots+A^n)(x)\\ &= \lim_{n\to\infty}(I-A)(I+\cdots+A^n)(x)\\ &= \lim_{n\to\infty}(I-A^{n+1})(x) = x.\end{aligned}$$

Here we used continuity of $(I - A)$ (because it is bounded), and that $\|A^{n+1}(x)\| \leq \|A\|^{n+1}\|x\| \to 0$ as $n \to \infty$. Entirely similarly it can be shown that also $B(I - A)(x) = x$. ■

The above theorem is just one rendition of the famous and exceptionally useful set of contraction mapping theorems. The notion of "contraction" exists for nonlinear mappings as well and the contraction theorems are then possibly even more spectacular. Contraction arguments are used to prove existence and construct solutions of many differential equations, and it is at the heart of a great many numerical routines.

Example A.2.9 (Fredholm integral operator). Consider the integral equation in $f \in \mathscr{L}^2([-1,1];\mathbb{R})$,

$$f(x) = x^4 + \int_{-1}^{1} K(x,y) f(y)\,\mathrm{d}y, \tag{A.9}$$

with $K(x,y) = x^3y^2$. The integral in this equation is linear in f. Now denote this integral as $A(f)$. Then the problem becomes finding an f in Banach space $\mathscr{L}^2([-1,1];\mathbb{R})$ for which

$$f = g + A(f) \qquad \text{with } g(x) = x^4.$$

The previous theorem guarantees a unique solution if A is a contraction (i.e. $\|A\| < 1$). Using inner products, it can be shown that

$$\|A\| \leq \sqrt{\int_{-1}^{1}\int_{-1}^{1} |K(x,y)|^2\,\mathrm{d}x\mathrm{d}y}.$$

For our function $K(x,y) = x^3y^2$ we have that

$$\int_{-1}^{1}\int_{-1}^{1} K(x,y)^2\,\mathrm{d}y\mathrm{d}x = \int_{-1}^{1}\int_{-1}^{1} x^6y^4\,\mathrm{d}y\mathrm{d}x = \frac{2}{7}\frac{2}{5} = \frac{4}{35}.$$

So $\|A\| \le \sqrt{4/35} < 1$ and hence A is a contraction. The Neumann series now says that there is a unique solution $f_* \in \mathscr{L}^2([-1,1]\mathbb{R})$ of the integral equation and that it equals

$$f_* = \sum_{k=0}^{\infty} A^k(g) = g + A(g) + A^2(g) + \cdots.$$

We are lucky in this example because it turns out that

$$\begin{aligned}
(Ag)(x) &= \int_{-1}^{1} x^3 y^2 y^4 \,\mathrm{d}y = \tfrac{2}{7}x^3, \\
(A^2 g)(x) = A(\tfrac{2}{7}x^3) &= \int_{-1}^{1} x^3 y^2 \tfrac{2}{7} y^3 \,\mathrm{d}y = 0
\end{aligned}$$

and consequently $(A^k g)(x) = 0$ for every $k \ge 2$. The unique solution $f_* \in \mathscr{L}^2([-1,1];\mathbb{R})$ of (A.9) hence is

$$f_*(x) = g(x) + (Ag)(x) = x^4 + \tfrac{2}{7}x^3.$$

□

A.2.1 Exercises

A.2.1 Consider the linear mapping $A : \mathbb{C}^2 \to \mathbb{C}^2$ defined as

$$A(x) = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} x.$$

Take the standard 2-norm on $\mathbb{C}^2$. Determine $\|A\|$.

A.2.2 Suppose $A : \mathbb{V} \to \mathbb{V}$ has eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$. Show that $\|A\| \ge \max_i |\lambda_i|$.

A.2.3 We consider three linear mappings from ℓ^2 to ℓ^2,

$$\begin{aligned}
K(u_1, u_2, \ldots) &= (0, u_1, u_2, \ldots) \\
L(u_1, u_2, \ldots) &= (u_2, u_3, \ldots) \\
M(u_1, u_2, \ldots) &= ((2 - \tfrac{1}{1})u_1, (2 - \tfrac{1}{2})u_2, \ldots)
\end{aligned}$$

On ℓ^2 we take the standard 2-norm.

(a) Determine $\|K\|$

(b) Determine $\|L\|$

(c) Determine $\|M\|$

A.2.4 Prove the submultiplicative property of operator norm: $\|AB\| \leq \|A\|\|B\|$.

A.2.5 Consider the linear mapping $A : \ell^2 \to \ell^2$,

$$A(u_1, u_2, \ldots) = (u_1, \tfrac{1}{2}u_2, \tfrac{1}{3}u_3, \ldots)$$

On ℓ^2 we take the standard 2-norm.

(a) Show that A is bounded

(b) Determine $\|A\|$

(c) Determine $\ker(A)$

A.2.6 Consider the mapping $A : \mathscr{L}^2(\mathbb{R};\mathbb{R}) \to \mathscr{L}^2(\mathbb{R};\mathbb{R})$ defined as

$$(Af)(x) = f(x) + f(-x).$$

On $\mathscr{L}^2(\mathbb{R};\mathbb{R})$ we take the standard 2-norm.

Determine $\|A\|$

A.2.7 Consider $\mathscr{L}^2([2,3];\mathbb{R})$ with standard 2-norm. Let $A : \mathscr{L}^2([2,3];\mathbb{R}) \to \mathscr{L}^2([2,3];\mathbb{R})$ be the linear mapping

$$(Af)(x) = xf(x).$$

(a) Show that $\|A\| \leq 3$

(b) Let

$$f_n(x) = \begin{cases} \sqrt{n} & 3 - 1/n \leq x \leq 3 \\ 0 & 2 \leq x < 3 - 1/n \end{cases}$$

Determine $\|f_n\|_2$ and show that $\|A(f_n)\|_2 \geq 3 - 1/n$

(c) Determine $\|A\|$

A.2.8 Suppose $A: \mathbb{X} \to \mathbb{X}$ is a contraction and that $\mathbb{X}$ is Banach. Let $f_0, g \in \mathbb{X}$. Does the sequence $\{f_n\}_{n\in\mathbb{N}}$ defined as $f_{n+1} = g + A(f_n)$ converge?

A.2.9 Let $A: \mathbb{X} \to \mathbb{Y}$ be a linear operator. Show that $\|A\| = \sup_{\|x\|\leq 1} \|A(x)\|$.

A.3 Ordinary Differential Equations (ODEs)

In this appendix we demonstrate how to obtain solutions $y(t)$ for ordinary linear differential equations with constant coefficients,

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = f(t) \tag{A.10}$$

for a given function $f(t)$. It is customary to associate with such differential equations (A.10) a *homogeneous equation* and a *characteristic equation*. The homogeneous equation is (A.10) in which $f(t)$ is taken equal to zero,

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) = 0,$$

and the associated characteristic equation is the polynomial equation

$$\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0, \qquad (\lambda \in \mathbb{C}). \tag{A.11}$$

The fundamental theorem of algebra states that a polynomial equation of degree n has exactly n roots in $\mathbb{C}$, counting multiplicities. Now if λ_1 is a root of the characteristic equation (A.11), then $y(t) = \mathrm{e}^{\lambda_1 t}$ is a solution of the homogeneous equation. Indeed, if $y(t) = \mathrm{e}^{\lambda_1 t}$, then

$$\begin{aligned}
& y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_1 y^{(1)}(t) + p_0 y(t) \\
={}& \lambda_1^n \mathrm{e}^{\lambda_1 t} + p_{n-1}\lambda_1^{n-1}\mathrm{e}^{\lambda_1 t} + \cdots + p_1\lambda_1\mathrm{e}^{\lambda_1 t} + p_0\mathrm{e}^{\lambda_1 t} \\
={}& \left(\lambda_1^n + p_{n-1}\lambda_1^{n-1} + \cdots + p_1\lambda_1 + p_0\right)\mathrm{e}^{\lambda_1 t} \\
={}& 0.
\end{aligned}$$

Example A.3.1.

1. The characteristic equation of

$$y^{(2)}(t) - 10y(t) = 0$$

is $\lambda^2 - 10 = 0$. Its roots are $\lambda_1 = \sqrt{10}$ and $\lambda_2 = -\sqrt{10}$. Hence $y_1(t) = \mathrm{e}^{\sqrt{10}t}$ and $y_2(t) = \mathrm{e}^{-\sqrt{10}t}$ are solutions of $y^{(2)}(t) - 10y(t) = 0$. By linearity, then,

$$y(t) = \alpha_1 \mathrm{e}^{\sqrt{10}t} + \alpha_2 \mathrm{e}^{-\sqrt{10}t},$$

is a solution of $y^{(2)}(t) - 10y(t) = 0$ for any $\alpha_1, \alpha_2 \in \mathbb{C}$.

2. The characteristic equation of

$$y^{(3)}(t) - 3y^{(2)}(t) + 2y^{(1)}(t) = 0$$

is $\lambda^3 - 3\lambda^2 + 2\lambda = 0$. Since

$$\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2)$$

we see that $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 2$ are the characteristic roots. Then $\mathrm{e}^{0t} = 1$, and e^{t} and e^{2t} are three solutions of the homogeneous equation, and then by linearity every $y(t)$ of the form

$$y(t) = \alpha_1 + \alpha_2 \mathrm{e}^{t} + \alpha_3 \mathrm{e}^{2t}, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C},$$

is a solution of the homogeneous equation.

3. The characteristic equation of

$$y^{(3)}(t) = 0$$

is $\lambda^3 = 0$. This has one root $\lambda_1 = 0$ with multiplicity 3. Now $y_1(t) = \mathrm{e}^{\lambda_1 t} = \mathrm{e}^{0t} = 1$ is obviously a solution of the homogeneous equation, but so are $y(t) = t$ and $y(t) = t^2$. Apparently not every solution is a linear combination of exponential functions.

□

In the last of the three examples we saw that not every solution of the homogeneous equation is a sum of exponential functions. This has something to do with the fact that the multiplicity of the characteristic root λ_1 in that example is more than 1. The general result we state without proof:

Theorem A.3.2. To each characteristic root λ_i of multiplicity m_i, the m_i functions

$$y_{i,k}(t) = t^{k-1}\mathrm{e}^{\lambda_i t}, \quad (k = 1, \dots, m_i)$$

are solutions of the homogeneous equation. These solutions $y_{i,k}(t)$ are called the *basis solutions.*

Furthermore, $y(t)$ is a solution of the homogeneous equation if and only if it is a linear combination of the basis solutions,

$$y(t) = \sum_{i,k} \alpha_{i,k} y_{i,k}(t), \quad \alpha_{i,k} \in \mathbb{C}. \tag{A.12}$$

Example A.3.3.

1. The characteristic equation of

   $$y^{(n)}(t) = 0$$

   is $\lambda^n = 0$. It has one root $\lambda_1 = 0$ with multiplicity n. The basis solutions hence are

   $$y_{1,1}(t) = 1, \quad y_{1,2}(t) = t, \quad y_{1,3}(t) = t^2, \quad \dots, \quad y_{1,n}(t) = t^{n-1}.$$

   The general solution of $y^{(n)}(t) = 0$ is therefore $y(t) = \alpha_{1,1} + \alpha_{1,2}t + \cdots + \alpha_{1,n}t^{n-1}$, that is, the solutions are the polynomials in t of degree $n-1$ or less.

2. The characteristic equation of

   $$y^{(3)}(t) - 4y^{(2)}(t) + 5y^{(1)}(t) - 2y(t) = 0 \tag{A.13}$$

   is $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$. Since

   $$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda - 1)^2(\lambda - 2)$$

we obtain as basis solutions

$$y_{1,1}(t) = \mathrm{e}^{t}, \quad y_{1,2}(t) = t\mathrm{e}^{t}, \quad y_{2,1}(t) = \mathrm{e}^{2t}.$$

The general solution of (A.13) then is

$$y(t) = \alpha_{1,1}\mathrm{e}^{t} + \alpha_{1,2}t\mathrm{e}^{t} + \alpha_{2,1}\mathrm{e}^{2t}, \quad \alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1} \in \mathbb{C}.$$

□

A.3.1 Particular solutions

Up to now we considered only homogeneous equations, that is, the case that $f(t)$ in (A.10) is zero. In this section we consider ODE (A.10) for the case that $f(t)$ is non-zero.

Suppose that for a given $f(t)$ we found *one* solution $y_{\mathrm{part}}(t)$ of the ODE (A.10). How does the general solution then look like?

Lemma A.3.4. Suppose $f(t)$ is given and let $y_{\mathrm{part}}(t)$ be one solution of the ODE (A.10). Then the general solution $y(t)$ of (A.10) is

$$y(t) = y_{\mathrm{part}}(t) + y_{\mathrm{hom}}(t)$$

where $y_{\mathrm{hom}}(t)$ is any solution of the associated homogeneous equation.

Proof (sketch). If $y_{\mathrm{p1}}(t)$ and $y_{\mathrm{p2}}(t)$ are two solutions then by linearity the difference $y_{\mathrm{h}}(t) := y_{\mathrm{p1}}(t) - y_{\mathrm{p2}}(t)$ satisfies the homogenous equation. Conversely if $y_{\mathrm{h}}(t)$ satisfies the homogeneous equation then for any solution $y_{\mathrm{p1}}(t)$ of the ODE also $y_{\mathrm{p1}}(t) + y_{\mathrm{h}}(t)$ is a solution. Therefore given any solution $y_{\mathrm{p1}}(t)$ the function $y_{\mathrm{p2}}(t)$ is also a solution if and only if $y_{\mathrm{p2}}(t) = y_{\mathrm{p1}}(t) + y_{\mathrm{h}}(t)$ for some homogenous solution $y_{\mathrm{h}}(t)$. ■

In our quest for the general solution it therefore suffices to find *one* solution of the ODE. All others then follow by adding the general solution of the homogeneous equation. One solution $y_{\mathrm{part}}(t)$ of the ODE is commonly called a *particular solution.* Generally it is difficult to find a particular solution. For certain signals $f(t)$ it is however possible to make an educated guess. The following three examples demonstrate three such cases.

Example A.3.5 (Constant right-hand side). If the right-hand side of the ODE is constant

$$f(t) = c,$$

then we may contemplate a constant particular solution $y_{\text{part}}(t)$. As all derivatives of a constant signal are zero, the ODE (A.10) for constant $f(t)$ and $y(t)$ reduce to

$$p_0 y(t) = c.$$

If $p_0 \neq 0$ then clearly

$$y_{\text{part}}(t) = \frac{c}{p_0}$$

is a constant particular solution.

Often we are only concerned with solutions for positive time. Consider the ODE

$$y^{(2)}(t) - 4y(t) = f(t)$$

and suppose that $f(t) = \mathbb{1}(t)$. Since we are interested in the signals for positive time, we may consider the $f(t) = \mathbb{1}(t)$ to be a constant 1. A particular solution follows as $y_{\text{part}}(t) = 1/(-4) = -1/4$. Hence *for positive time* the general solution $y(t)$ is

$$y(t) = -\frac{1}{4} + \alpha_{1,1}\mathrm{e}^{2t} + \alpha_{2,1}\mathrm{e}^{-2t}, \quad \alpha_{1,1}, \alpha_{2,1} \in \mathbb{C}.$$

□

Example A.3.6 (Exponential right-hand side). The constant signal $f(t)$ of the previous example is a degenerate case of an exponential signal $f(t) = \mathrm{e}^{s_0 t}$. For an exponential signal $f(t) = \mathrm{e}^{s_0 t}$ we contemplate a particular solution of the similar form

$$y_{\text{part}}(t) = A\mathrm{e}^{s_0 t}, \quad \text{for some } A \in \mathbb{C}.$$

The left-hand side of the ODE (A.10) then becomes

$$y_{\text{part}}^{(n)}(t) + p_{n-1} y_{\text{part}}^{(n-1)}(t) + \cdots + p_0 y_{\text{part}}(t) = A\big(s_0^n + p_{n-1} s_0^{n-1} + \cdots + p_0\big)\mathrm{e}^{s_0 t}$$

Equating this with $f(t) = e^{s_0 t}$ yields A,

$$A = \frac{1}{s_0^n + p_{n-1} s_0^{n-1} + \cdots + p_0}.$$

For A to exist we will need to assume that s_0 is not a characteristic root, otherwise the above denominator is zero. For $s_0 = 0$ we recover that case of constant $f(t)$.

Consider the ODE

$$y^{(2)}(t) - 4y(t) = f(t)$$

with $f(t) = e^{s_0 t}$. Then as long as s_0 is not a characteristic root, we obtain as particular solution

$$y_{\text{part}}(t) = \frac{1}{s_0^2 - 4} e^{s_0 t}.$$

Like in the previous example, the general solution then is

$$y(t) = \frac{1}{s_0^2 - 4} e^{s_0 t} + \alpha_{1,1} e^{2t} + \alpha_{2,1} e^{-2t}, \quad \alpha_{1,1}, \alpha_{2,1} \in \mathbb{C}.$$

If s_0 is a characteristic root, i.e. $s_0 = 2$ or $s_0 = -2$ then it can be shown that

$$y_{\text{part}}(t) = \frac{t}{2 s_0} e^{s_0 t}.$$

is a particular solution and the general solution then is:

$$y(t) = \frac{t}{2 s_0} e^{s_0 t} + \alpha_{1,1} e^{2t} + \alpha_{2,1} e^{-2t}, \quad \alpha_{1,1}, \alpha_{2,1} \in \mathbb{C}.$$

□

Example A.3.7 (Polynomial solutions). Consider the case that the right-hand side $f(t)$ of the ODE (A.10) is polynomial,

$$y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \cdots + p_0 y(t) = \sum_{k=0}^{M} \beta_k t^k, \qquad (\beta_k \in \mathbb{C}).$$

The claim is then that there is a particular solution $y(t)$ that is polynomial in t as well. The method is best demonstrated on an example. Consider the ODE

$$y^{(2)}(t)+y^{(1)}(t)+2y(t)=t^2+2t. \tag{A.14}$$

Differentiate both sides as often as needed up to the point where the right-hand side becomes constant.

$$\begin{aligned} \text{Original equation: } & y^{(2)}(t)+y^{(1)}(t)+2y(t)=2t+t^2 && \text{(A.15)}\\ \text{differentiate once: } & y^{(3)}(t)+y^{(2)}(t)+2y^{(1)}(t)=2+2t && \text{(A.16)}\\ \text{differentiate once again: } & y^{(4)}(t)+y^{(3)}(t)+2y^{(2)}(t)=2. && \text{(A.17)} \end{aligned}$$

The last equation (A.17) has a solution $y^{(2)}(t)=1$. Now we use that in the preceding equation (A.16) to solve for $y^{(1)}(t)$. Since $y^{(3)}(t)=0$ we obtain from (A.16) that

$$y^{(1)}(t)=\frac{1}{2}\big((2+2t)-y^{(3)}(t)-y^{(2)}(t)\big)=t+\frac{1}{2}.$$

Now that $y^{(1)}(t)$ is determined we return to Eqn. (A.15) and solve that for $y(t)$,

$$y(t)=\frac{1}{2}\big((2t+t^2)-y^{(2)}(t)-y^{(1)}(t)\big)=\frac{1}{2}(2t+t^2-1-(t+\frac{1}{2}))=\frac{1}{2}t^2+\frac{1}{2}t-\frac{3}{4}.$$

This is a particular solution.

The characteristic equation of (A.14) is $\lambda^2+\lambda+2=0$ and its roots are complex, $\lambda_1=-\frac{1}{2}+\mathrm{i}\frac{1}{2}\sqrt{7}$ and $\lambda_2=-\frac{1}{2}-\mathrm{i}\frac{1}{2}\sqrt{7}$. The general solution of (A.14) hence is

$$y(t)=\frac{1}{2}t^2+\frac{1}{2}t-\frac{3}{4}+\alpha_{1,1}\mathrm{e}^{(-\frac{1}{2}+\mathrm{i}\frac{1}{2}\sqrt{7})t}+\alpha_{2,1}\mathrm{e}^{(-\frac{1}{2}-\mathrm{i}\frac{1}{2}\sqrt{7})t} \qquad (\alpha_{1,1},\alpha_{2,1}\in\mathbb{C}).$$

□

A.3.2 Exercises

A.3.1 Suppose that $y_k(t)$ is a particular solution of (A.10) for $f(t)=f_k(t)$. Determine a particular solution $y(t)$ for $f(t)=\alpha f_1(t)+\beta f_2(t)$, $\alpha,\beta\in\mathbb{R}$.

A.3.2 Determine the general solution of the homogeneous equation associated with the following ODEs:

(a) $y^{(2)}(t)+2y^{(1)}(t)+2y(t)=0$,

(b) $y^{(2)}(t)-4y^{(1)}(t)+4y(t)=t^2$,

(c) $y^{(2)}(t)+7y^{(1)}(t)+12y(t)=1-t$,

(d) $y^{(4)}(t)=\mathbb{1}(t)$,

(e) $y^{(1)}(t)+\beta y(t)=2$,

(f) $y^{(1)}(t)+\beta y(t)=\mathrm{e}^{4t}+\mathrm{e}^{-t}$.

A.3.3 Determine a particular solution $y(t)$ for the ODEs of the previous problem.

A.4 Partial Fraction Expansion

The topic of this appendix is partial fraction expansion. As an example, consider the identity

$$\frac{1}{(s+1)(s+2)(s+3)}=\frac{\frac{1}{2}}{s+1}+\frac{-1}{s+2}+\frac{\frac{1}{2}}{s+3}. \tag{A.18}$$

It is easy to verify that the above identity is indeed correct: multiply both left and right-hand side by $(s+1)(s+2)(s+3)$, and the identity reduces to the polynomial identity

$$1=\frac{1}{2}(s+2)(s+3)-(s+1)(s+3)+\frac{1}{2}(s+1)(s+2),$$

whose validity is subsequently readily verified.

In this section we discuss a procedure for obtaining *partial fraction expansions.* A partial fraction expansion of a rational function is an expansion of that function as a sum of elementary rational terms of the form

$$\frac{\alpha}{(s-\beta)^k},\qquad \alpha,\beta\in\mathbb{C}$$

such as in the right-hand side of (A.18). More generally, the partial fraction expansion of a rational function

$$\frac{Q(s)}{P(s)} = \frac{q_m s^m + q_{m-1} s^{m-1} + \cdots + q_1 s + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0}$$

is an expansion of the form

$$\frac{Q(s)}{P(s)} = A_0 + \sum_{k=1}^{M} \Big(\frac{A_{k,1}}{s-s_k} + \frac{A_{k,2}}{(s-s_k)^2} + \cdots + \frac{A_{k,m_k}}{(s-s_k)^{m_k}}\Big) \tag{A.19}$$

with $A_0, A_{k,l}, s_k \in \mathbb{C}$ and $m_k, M \in \mathbb{N}$. Two properties are immediate. Firstly, the right-hand side of (A.19) is not defined for $s = s_k$, so the left-hand side, $Q(s_k)/P(s_k)$ is not defined either. Therefore the s_k are necessarily zeros of the polynomial $P(s)$. Secondly, the limit as $|s| \to \infty$ of the right-hand side of (A.19) is finite,

$$\lim_{|s|\to\infty} A_0 + \sum_{k=1}^{M} \Big(\frac{A_{k,1}}{s-s_k} + \frac{A_{k,2}}{(s-s_k)^2} + \cdots + \frac{A_{k,m_k}}{(s-s_k)^{m_k}}\Big) = A_0, \tag{A.20}$$

so the left-hand side $Q(s)/P(s)$ is also finite in the limit $|s| \to \infty$. This is the case if and only if the degree of $Q(s)$ is less than or equal to the degree of $P(s)$. Rational functions $Q(s)/P(s)$ with $\deg Q(s) \le \deg P(s)$ are called *proper* rational functions.

Theorem A.4.1. Every proper rational function $Q(s)/P(s)$ has a partial fraction expansion.

More concretely, let s_k, $(k = 1, 2, \ldots, M)$ denote the zeros of $P(s)$. Then $Q(s)/P(s)$ has a partial fraction expansion of the form

$$\frac{Q(s)}{P(s)} = A_0 + \sum_{k=1}^{M} \Big(\frac{A_{k,1}}{s-s_k} + \frac{A_{k,2}}{(s-s_k)^2} + \cdots + \frac{A_{k,m_k}}{(s-s_k)^{m_k}}\Big)$$

where M is the number of different zeros of $P(s)$, m_k is the multiplicity of zero s_k of $P(s)$, and A_0 and the $A_{k,l}$ are (complex) constants. □

If you want a partial fraction expansion with real coefficients then we can identify real zeros $s_1, \ldots s_M$ and the complex zeros can be characterized as the zeros of the second order polynomial $s^2 + a_i s + b_i$ with $a_i^2 < 4b_i$ while a_i and b_i are real-valued for $i = 1, \dots N$. In that case we obtain:

$$\begin{aligned}\frac{Q(s)}{P(s)} &= A_0 + \sum_{k=1}^{M} \Big(\frac{A_{k,1}}{s - s_k} + \frac{A_{k,2}}{(s - s_k)^2} + \cdots + \frac{A_{k,m_k}}{(s - s_k)^{m_k}}\Big) \\ &\quad + \sum_{k=1}^{N} \Big(\frac{B_{k,1} + C_{k,1} s}{s^2 + a_k s + b_k} + \frac{B_{k,2} + C_{k,2} s}{(s^2 + a_k s + b_k)^2} + \cdots + \frac{B_{k,n_i} + C_{k,n_i} s}{(s^2 + a_k s + b_k)^{n_i}}\Big).\end{aligned}$$

A.4.1 If $Q(s)/P(s)$ is strictly proper

A rational function $Q(s)/P(s)$ is *strictly proper* if the degree of $Q(s)$ is less than the degree of $P(s)$,

$$\frac{Q(s)}{P(s)} = \frac{q_{n-1} s^{n-1} + \cdots + q_1 s + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0}, \qquad (p_n \neq 0).$$

Strictly proper rational functions tend to zero as $|s| \to \infty$, so in view of (A.20), we have that

$$A_0 = 0.$$

In this subsection we demonstrate partial fraction expansion techniques for strictly proper rational functions.

Example A.4.2. Let $Q(s)/P(s) = 1/((s+1)(s+2))$. The zeros of $P(s)$ are $s_1 = -1$ and $s_2 = -2$ and they both have multiplicity one. Therefore by the above theorem there are constants $A = A_{1,1}$ and $B = A_{2,1}$ such that

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

To determine the values of A and B we may multiply left and right-hand side by $(s+1)(s+2)$ to obtain,

$$1 = (s+2)A + (s+1)B = s(A+B) + (2A+B).$$

Subtracting 1 from both sides gives

$$0 = (s+2)A + (s+1)B = s(A+B) + (2A+B-1).$$

As this has to hold for every $s \in \mathbb{C}$ we must have that the polynomial on the right-hand side is identically zero:

$$
\begin{aligned}
0 &= A + B \\
0 &= 2A + B - 1.
\end{aligned}
$$

These are two equations in two unknowns, and its solution is

$$A = 1, \qquad B = -1.$$

We found the partial fraction expansion $\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$. □

The method of the previous example is generally applicable, but if $P(s)$ has many zeros, then the method becomes unwieldy. In such cases it is often easier to work with a direct method, such as the one demonstrated on the following example. The method assumes that the zeros of $P(s)$ have multiplicity 1.

Example A.4.3. Suppose

$$F(s) = \frac{s+4}{(s+1)(s+2)(s+3)}.$$

We see that $F(s)$ is strictly proper, so $F(s)$ has the partial fraction expansion,

$$\frac{s+4}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

for some constants A, B and C. Note that A is the coefficient of $\frac{1}{s+1}$ which has a pole at $s = -1$. Now to find A we simply evaluate at this pole $s = -1$ the function $F(s)$ with the term $(s+1)$ removed:

$$A = \left.\frac{s+4}{\boxed{(s+1)}\,(s+2)(s+3)}\right|_{s=-1} = \frac{-1+4}{(-1+2)(-1+3)} = \frac{3}{2}.$$

Likewise the coefficients B and C of $\frac{1}{s+2}$ and $\frac{1}{s+3}$ may be directly determined as

$$B = \left.\frac{s+4}{(s+1)\,\boxed{(s+2)}\,(s+3)}\right|_{s=-2} = \frac{-2+4}{(-2+1)(-2+3)} = -2,$$

and

$$C = \left.\frac{s+4}{(s+1)(s+2)(s+3)}\right|_{s=-3} = \frac{-3+4}{(-3+1)(-3+2)} = \frac{1}{2}.$$

So the partial fraction expansion is

$$\frac{s+4}{(s+1)(s+2)(s+3)} = \frac{3/2}{s+1} + \frac{-2}{s+2} + \frac{1/2}{s+3}.$$

This method works for rational functions $F(s) = Q(s)/P(s)$ whose denominator $P(s)$ has zeros of multiplicity 1 only. □

The exposition in the previous example was deliberately taken rather graphical as this makes the method easier to perform by hand. Mathematically, we did nothing but compute

$$A = \lim_{s\to-1}(s+1)F(s) = \frac{3}{2},$$
$$B = \lim_{s\to-2}(s+2)F(s) = -2,$$
$$C = \lim_{s\to-3}(s+3)F(s) = \frac{1}{2}.$$

If $F(s)$ has a multiple pole then a similar result holds.

Lemma A.4.4. Suppose s_k is a zero of a polynomial $P(s)$ with multiplicity m_k. Then the coefficient A_{k,m_k} of the term of highest order

$$\frac{A_{k,m_k}}{(s-s_k)^{m_k}}$$

in the partial fraction expansion of $F(s) = Q(s)/P(s)$ equals

$$A_{k,m_k} = \lim_{s\to s_k}(s-s_k)^{m_k}F(s).$$

□

The interested reader may want to prove for herself why this is the case, and then with help of the next example a proof of Theorem A.4.1 should be within reach, well to the interested reader at least.

Example A.4.5. Suppose

$$F(s) = \frac{s+4}{(s+1)^2(s+2)}.$$

The multiplicity of the zero $s_1 = -1$ is $m_1 = 2$. So the partial fraction expansion of $F(s)$ is of the form

$$F(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}. \tag{A.21}$$

Since the multiplicity of the zero $s_1 = -1$ is 2, the coefficient B of the highest order term $\frac{1}{(s+1)^2}$ equals

$$B = \lim_{s \to -1} (s+1)^2 F(s) = \frac{s+4}{(s+1)^2 (s+2)}\bigg|_{s=-1} = \frac{-1+4}{-1+2} = 3.$$

Now that B is known we may bring it to the left-hand side of (A.21),

$$F(s) - 3\frac{1}{(s+1)^2} = \frac{A}{s+1} + \frac{C}{s+2}.$$

The left-hand side may be simplified to

$$F(s) - 3\frac{1}{(s+1)^2} = \frac{(s+4) \quad -3(s+2)}{(s+1)^2(s+2)} = \frac{-2s-2}{(s+1)^2(s+2)} = -\frac{2}{(s+1)(s+2)}.$$

We have reduced the problem to one of lower order. We leave it to the reader to verify that

$$-\frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{C}{s+2}$$

for $A = -2$ and $C = 2$. The partial fraction expansion of $F(s)$ is now determined,

$$\frac{s+4}{(s+1)^2(s+2)} = \frac{-2}{s+1} + \frac{3}{(s+1)^2} + \frac{2}{s+2}.$$

□

A.4.2 If $Q(s)/P(s)$ is proper

A rational function $Q(s)/P(s)$ is proper if the degree of $Q(s)$ is less than *or equal* to the degree of $P(s)$. So proper rational functions are of the form

$$\frac{Q(s)}{P(s)} = \frac{q_n s^n + q_{n-1} s^{n-1} + \cdots + q_1 s + q_0}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + p_0}, \qquad (p_n \neq 0). \tag{A.22}$$

Partial fraction expansion of a proper rational function can easily be reduced to that of a strictly proper rational function. Again the general procedure should be clear from an example.

Example A.4.6. Suppose

$$\frac{Q(s)}{P(s)} = \frac{s^2}{s^2+3s+2}.$$

The degree of the numerator $Q(s)$ is the same as the degree of the denominator $P(s)$. Adding and subtracting A_0 does not change the result, so

$$\frac{Q(s)}{P(s)} = A_0 + \frac{s^2 - A_0(s^2+3s+2)}{s^2+3s+2}.$$

For $A_0 = 1$ the numerator polynomial $s^2 - A_0(s^2+3s+2)$ drops degree,

$$\frac{Q(s)}{P(s)} = 1 + \frac{s^2-(s^2+3s+2)}{s^2+3s+2} = 1 + \frac{-3s-2}{s^2+3s+2}.$$

Now $\frac{-3s-2}{s^2+3s+2}$ is strictly proper and it has partial fraction expansion $\frac{1}{s+1} - 4\frac{1}{s+2}$ (verify this yourself). Then the partial fraction expansion of $Q(s)/P(s)$ follows as

$$\frac{s^2}{s^2+3s+2} = 1 + \frac{1}{s+1} - 4\frac{1}{s+2}.$$

□

A.4.3 Complex poles

The function

$$\frac{1}{s^2+4s+5}$$

has poles $-2+\mathrm{i}$ and $-2-\mathrm{i}$. The poles form a complex conjugate pair. The standard PFE thus takes the form

$$\frac{A_1}{s-(-2+\mathrm{i})}+\frac{A_2}{s-(-2-\mathrm{i})},$$

but in cases of a complex pole pair we prefer to combine the two terms into a single second order term of the form

$$\frac{Bs+C}{(s+a)^2+b^2}$$

with, now, A, B, a, b real numbers. In our example this is

$$\frac{1}{s^2+4s+5}=\frac{1}{(s+2)^2+1}.$$

That is sufficient if we need to determine the inverse Laplace transform. In general, rational functions can be decomposed into a sum of functions of the form

$$\frac{A}{(s+a)^k} \quad \text{and} \quad \frac{Bs+C}{((s+a)^2+b^2)^k}.$$

Example A.4.7. We determine the PFE of $\frac{1}{s(s^2+1)}$. The form of the PFE is

$$\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}=\frac{As^2+A+Bs^2+Cs}{s(s^2+1)}.$$

So $A+B=0, C=0, A=1$ and hence $B=-1$. This gives

$$\frac{1}{s(s^2+1)}=\frac{1}{s}-\frac{s}{s^2+1}.$$

□

A.4.4 Exercises

A.4.1 Construct the partial fraction expansion of

(a) $\dfrac{3s+4}{s^2+s}$

(b) $\dfrac{1}{s^2+5s+6}$

(c) $\dfrac{s}{s^2+5s+6}$

(d) $\dfrac{s^2}{s^2+5s+6}$

(e) $\dfrac{s^2+3s-100}{s^2+5s+6}$

(f) $\dfrac{s}{(s-2)^2}$

(g) $\dfrac{s^2}{(s-2)^2}$

(h) $\dfrac{s}{s^2+2s-2}$

(i) $\dfrac{1}{s}+\dfrac{s-2}{s(s+2)}$

(j) $\dfrac{s-\beta}{(s-1)(s+2)^2}$. Check your answer for $\beta=1$ and $\beta=-2$.

A.4.2 Construct the partial fraction expansion of

(a) $\dfrac{1}{s^2+2s+2}$

(b) $\dfrac{5}{s(s^2-2s+5)}$

(c) $\dfrac{4s^2+2s+4}{(s+0.5)(s^2-s+1.25)}$

A.5 Complex Integration

Integration as we know it for real-valued functions is easily extended to complex-valued functions. The integral of a complex function $f(t) = f_1(t) + \mathrm{i} f_2(t)$ on an interval (a, b) is defined as

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b f_1(t)\,\mathrm{d}t + \mathrm{i}\int_a^b f_2(t)\,\mathrm{d}t. \tag{A.23}$$

In effect this says that for complex-valued functions the integral exists if and only if both its real and imaginary part can be integrated. In the above, $a = -\infty$ and $b = \infty$ are allowed. From Equation (A.23) it follows that

$$\left(\int_a^b f(t)\,\mathrm{d}t\right)^* = \int_a^b f^*(t)\,\mathrm{d}t.$$

A.5.1 Three examples

Like in the real case it is often possible to obtain an explicit function description of the *primitive* of $f(t)$, also called the *antiderivative* of $f(t)$. Also the rules of integration by parts and substitution remain valid for complex-valued functions. This is illustrated in the following three examples.

Example A.5.1 (Integration). Let n be a positive integer and let $T > 0$ and $\omega_0 = 2\pi/T$. Then

$$\frac{1}{T}\int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}n\omega_0 t}\,\mathrm{d}t = \begin{cases} 0 & \text{if } n \neq 0, \\ 1 & \text{if } n = 0. \end{cases} \tag{A.24}$$

This may be seen as follows. For $n = 0$ we have that $\mathrm{e}^{\mathrm{i}n\omega_0 t} = 1$ which immediately establishes the case for $n = 0$. If $n \neq 0$ then

$$\int_{-T/2}^{T/2} \mathrm{e}^{\mathrm{i}n\omega_0 t}\,\mathrm{d}t = \frac{\mathrm{e}^{\mathrm{i}n\omega_0 t}}{\mathrm{i}n\omega_0}\bigg]_{-T/2}^{T/2} = \frac{1}{\mathrm{i}n\omega_0}(\mathrm{e}^{\mathrm{i}n\pi} - \mathrm{e}^{-\mathrm{i}n\pi}) = \frac{1}{\mathrm{i}n\omega_0}((-1)^n - (-1)^n) = 0.$$

That the answer is zero is not that strange, it simply says that the average of a harmonic signal over a period is zero. □

Example A.5.2 (Indefinite integration). Let $a \in \mathbb{C}$ and suppose that $\operatorname{Re} a > 0$. Then

$$\int_0^\infty \mathrm{e}^{-at}\,\mathrm{d}t = 1/a. \tag{A.25}$$

This is because

$$\int_0^\infty \mathrm{e}^{-at}\,\mathrm{d}t = \lim_{M\to\infty}\int_0^M \mathrm{e}^{-at}\,\mathrm{d}t = \lim_{M\to\infty} \frac{-\mathrm{e}^{-at}}{a}\Big]_0^M = \Big(\lim_{M\to\infty} \frac{-\mathrm{e}^{-aM}}{a}\Big) + \frac{1}{a} = \frac{1}{a}.$$

□

And then there is the rule of *integration by parts* which is that

$$\int_a^b h(t)g(t)\,\mathrm{d}t = h(t)G(t)|_a^b - \int_a^b h^{(1)}(t)G(t)\,\mathrm{d}t.$$

Here $h^{(1)}(t)$ is the derivative of $h(t)$ and $G(t)$ is any function whose derivative is $g(t)$. The rule is valid if all functions involved exist and are piecewise smooth.

Example A.5.3 (Integration by parts). Suppose $T > 0$, and let $\omega_0 = 2\pi/T$ and $n \in \mathbb{Z}$, $n \neq 0$. We shall establish that

$$\int_0^T t\mathrm{e}^{\mathrm{i}n\omega_0 t}\,\mathrm{d}t = \frac{T^2}{2\pi\mathrm{i}n}. \tag{A.26}$$

Integration by parts yields

$$\begin{aligned}\int_0^T t\mathrm{e}^{\mathrm{i}n\omega_0 t}\mathrm{d}t &= t\frac{\mathrm{e}^{\mathrm{i}n\omega_0 t}}{\mathrm{i}n\omega_0}\Big|_0^T - \frac{1}{\mathrm{i}n\omega_0}\int_0^T \mathrm{e}^{\mathrm{i}n\omega_0 t}\mathrm{d}t \\ &= \frac{T}{\mathrm{i}n\omega_0}\mathrm{e}^{\mathrm{i}n\omega_0 T} - \frac{1}{(\mathrm{i}n\omega_0)^2}(\mathrm{e}^{\mathrm{i}n\omega_0 T} - 1).\end{aligned}$$

Since $\omega_0 T = 2\pi$ and $\mathrm{e}^{2\pi\mathrm{i}n} = 1$ we find (A.26). □

The following inequalities are often used when only existence of integrals or bounds on integrals are needed and not so much their precise value.

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t.$$

That is, the absolute value of an integral is at most the integral of the absolute value. This is readily verified. A straightforward applications is this: If $|f(t)| \leq M$ on the interval of integration $[a, b]$, then

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t \leq \int_a^b M\,\mathrm{d}t = M(b-a).$$

In particular it then follows that the integral exists.

A.6 Matlab & Python

One interesting byproduct of Shannon's sampling theorem is that it provides a way to determine Fourier transforms $\hat{f}(\omega)$ of signals $f(t)$ on the basis of samples $f[n]$ alone. Shannon says that $f(t)$ follows uniquely from its samples as

$$f(t) = \sum_{n=-\infty}^{\infty} f[n]\,\mathrm{sinc}(\pi(\tfrac{t}{T_\mathrm{s}} - n)).$$

provided the bandwidth ω_b of $f(t)$ is less than the Nyquist frequency $\omega_\mathrm{nyq} := \omega_\mathrm{s}/2$. In the proof of Shannon's sampling theorem we actually derived an explicit expression of the Fourier transform, which we copy here:

$$\hat{f}(\omega) = T_\mathrm{s} \sum_{n=-\infty}^{\infty} f[n]\mathrm{e}^{-\mathrm{i}n\omega T_\mathrm{s}}, \qquad f \in [0, \omega_\mathrm{s}/2].$$

Example A.6.1 (Magnitude of Fourier transform via FFT). The MATLAB script below computes $|\hat{f}(\omega)|$ of

$$f(t) = \cos(2\pi \times 5t) + \frac{1}{2}\cos(2\pi \times 10t).$$

Its bandwidth is $\omega_\mathrm{b} = 2\pi \times 10$. Then Shannon says we need $2\pi/T_\mathrm{s} > 2\omega_\mathrm{b} = 2\pi \times 20$, that is, $T_\mathrm{s} < 1/20$, i.e we should take more than 20 samples per time unit.

```
Tepoch=1;                       % some epoch length
Ts=0.01;                        % sampling period < 1/20
t=0:Ts:Tepoch;
```

```
f=cos(2*pi*5*t)+cos(2*pi*10*t)/2;
N=length(f);
M=2^11;                      % take M = 2^? ≥ N
w=2*pi*(0:M/2)/Ts/M;         % gridded [0, ω_nyq]

fhat=fft(f,M)*Ts;            %
fhat=fhat(1+(0:M/2));        % only need first half
plot(w/(2*pi),abs(fhat));    % plot of |f̂(ω/(2π))|. See Fig. A.1
grid
```

The result is shown in Fig. A.1. Notice that we plotted against $\omega/(2\pi)$ and not ω. The $\omega/(2\pi)$ means "cycles per time unit". Our signal $f(t)$ is a sum of sinusoids of period $T = 1/5$ and $T = 1/10$. Not surprisingly, then, the Fourier transform has peaks at $\omega/(2\pi) = 5$ cycles per second and at $\omega/(2\pi) = 10$ cycles per second, see Fig. A.1. □

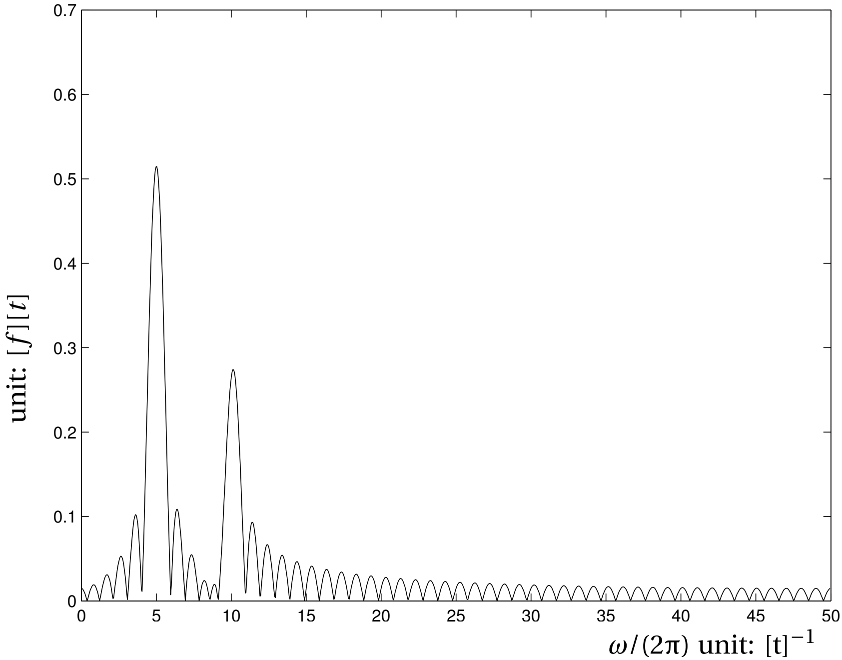


FIGURE A.1: Estimate of $|\hat{f}(\frac{\omega}{2\pi})|$, see Example A.6.1. The unit of $\hat{f}(\frac{\omega}{2\pi})$ is that of f times that of t

The same can be done in PYTHON:

```
import numpy as np

Tepoch  = 1
Ts      = 0.01
t       = np.arange(0,Tepoch+Ts/2,Ts)
f       = np.cos(2*np.pi*5*t)+np.cos(2*np.pi*10*t)/2;
N       = len(f)
M       = 2**11
w       = np.arange((M//2)+1)*2*np.pi/M/Ts

fhat    = np.fft.fft(f,M)*Ts
fhat    = fhat[0:(M//2+1)]
```

To plot the above Fourier transforms and to add labels and save the plot as pdf one can do:

```
import matplotlib.pyplot as plt

plt.plot(w/(2*np.pi), abs(fhat))

plt.rc('text', usetex=True)                                         # optional
plt.rcParams['text.latex.preamble'] = [r'\usepackage{fourier}']  # optional
plt.rc('font', family='serif', size=18)                             # optional

plt.xlabel('frequency $\omega/(2\pi)$ [t]$^{-1}$')
plt.ylabel('unit: $[f][t]$')
plt.grid()
plt.savefig('code1p1.pdf')               # save it as pdf
#plt.show()                              # (the plot is similar to Fig. A.1)
```

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