1 Selfstudy for Tutorial 2

Polking **§2.9:**17, 19 **§8.1:** 7 **§8.4:** 17 **§8.5:** 9

2 Exercises for Tutorial 2

1. Consider the autonomous differential equation $x' = f(x)+a$ with $f(x) = (x+1)^2(1-x)$ with a a free parameter. Solutions cannot be obtained explicitly. Hence, we will turn to a geometric approach to study this ODE. We shall see that as a varies, the number of equilibria changes and so the dynamics change accordingly. Such a qualitative change is referred to as a *bifurcation*.

   (a) Sketch the graph of f.

   (b) Sketch the phase line of $x' = f(x) + a$ for $a = -2, -1, 0, 1$.
   Hint: Do not compute the intersections exactly, but produce a qualitatively correct figure.

   (c) Determine the values of a such that there are three equilibria. For this consider minima and maxima of f.

2. To practice phase planes and time-series, complete exercises 8.2.21-25 from *Polking*.

3. We will study the relationship between the general nth-order differential equation with constant coefficients

$$\frac{d^{(n)}x}{dt^{(n)}} + a_{n-1}\frac{d^{(n-1)}x}{dt^{(n-1)}} + \cdots + a_0 x = 0, \tag{1}$$

   and the corresponding n-dimensional first-order system

$$y' = Ay, \tag{2}$$

   where $y = (x, x', x'', ..., x^{(n-1)})$. The particular goal of this exercise is to determine the structure of the eigenvector for a given eigenvalue.

   (a) Write down the ODE as a first order system $y' = Ay$ for three variables, i.e., $n = 3$.

   (b) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$ for $n = 3$.

   (c) Show that if $x = e^{\lambda t}$ is a solution of (1) then $y = e^{\lambda t}v$ is a solution of (2), where $v = (v_0, v_1, \cdots, v_n)$ is the eigenvector for the eigenvalue $p(\lambda) = 0$. Show that for each element of v we have $v_i = \lambda^{i-1}v_1$, so we may set $v = (1, \lambda, \lambda^2, \cdots, \lambda^{n-1})$.

4. In this exercise we want to illustrate how the eigenvectors collide as we change ϵ from positive to negative. And for $\epsilon = 0$ we have a double eigenvalue b. For this we use (1) with $n = 2$ and $a_1 = -2b$ and $a_0 = b^2 - \epsilon$, i.e., the second order ODE

$$x'' - 2bx' + (b^2 - \epsilon)x = 0.$$

   (a) Determine two solutions of the form $e^{\lambda t}$ for $\epsilon \neq 0$. That is, determine the two roots $\lambda_\pm$ of $p(\lambda)$.

(b) For $\epsilon = 0$, we have only such exponential as a solution. Let us construct another. As the ODE is linear, the linear combination $x_\epsilon = \dfrac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}$ is a solution too. Note this is the difference quotient. Compute the limit $\lim_{\epsilon \to 0} x_\epsilon(t)$ to obtain the second solution.

(c) In the previous exercise we have shown the eigenvectors are given by $v_\pm = (1, \lambda_\pm)$. Draw these eigenvectors in the plane for $\epsilon > 0$ and explain what happens as $\epsilon > 0$ approaches 0.

4. In this exercise we look at a system of two coupled tanks of 360 litre completely filled with salty water. Tank A has a supply of 5 l/min pure water and 4 l/min from tank B, while 9/l/min is transported to tank B. So Tank B has a supply of 9 l/min from tank A, while 4 l/min is transported to tank A and there is a drain of 5 l/min. So the total amount of water in both tanks is constant. In the beginning tank A contains 60 kilo of salt, while tank B is clean. We denote with $x(t), y(t)$ the mass of salt and $c_\mathrm{A}(t), c_\mathrm{B}(t)$ the concentration of salt for each tank. To obtain a differential equation we can set up a mass balance for each tank, i.e. how much the mass changes during a time Δt. For instance, for tank A we find

$$x(t + \Delta t) = x(t) + \Delta t \left(\underbrace{5 * 0}_{clean} + \underbrace{4c_B}_{from\ B} - \underbrace{9c_A}_{to\ B} \right)$$

(a) How is the amount of salt in tank A related to the concentration?

(b) Write down the mass balance for tank B. Then use this to determine a system of two coupled differential equations for $(c_\mathrm{A}, c_\mathrm{B})$. Write this system in matrix form.

(c) Compute the general solution of this system first, and next also determine the constants to satisfy the initial values.

(d) Sketch the phase plane including the orbit of the solution of (c).