1 Answers Tutorial 4

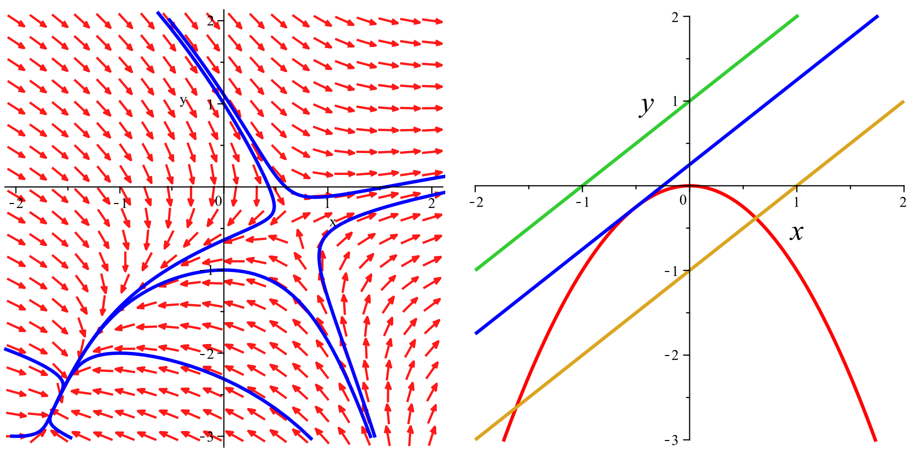
1. The equilibria are $(\theta, \theta') = (0, 0)$ with eigenvalues $\pm i$ (center) and $(\theta, \theta') = (\pm\pi, 0)$ with eigenvalues ± 1 (saddle). You may also read §10.1 from the book on the type of equilibria.

2. (a) The equilibria are $x_0 = -\frac{1}{2}\left(1 \pm \sqrt{(1-4a}\right)$ and $y_0 = a + x = a - \frac{1}{2}\left(1 \pm \sqrt{(1-4a}\right)$

   (b)
$$J(x, y) = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$$

   For $a = -1$ the eigenvalues are $\lambda_1 = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ and $\lambda_2 = -\frac{5}{2} \pm \frac{\sqrt{5}}{2}$. So the right point is a saddle, while the left one is a asymptotically stable node.

   (c) For $a = -1$ there are two equilibria, and none for $a = 1$. This change occurs for the critical value $a = 1/4$.



   The nullclines in the xy-plane show that as a increases the saddle and the node approach each other. For the critical value $a = 1/4$ the single equilibrium $(x, y) = (-1/2, -1/4)$ has eigenvalues -2 and 0. As the equilibria coalesce, this scenario is called a saddle-node bifurcation.

3. (a) Note we choose coordinates such that $x_1, x_2 = 0$ corresponds to the spring being at rest.

   (b) This is a linear system so we have $y' = Ay$ with $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2) & 0 & k_2 & 0 \\ 0 & 0 & 0 & 1 \\ k_2 & 0 & -(k_1+k_2) & 0 \end{pmatrix}$.

   (c) Solving the characteristic equation yields the eigenvalues:
$$\det(A - \lambda I) = \lambda^4 + 2(k_1+k_2)\lambda^2 + k_1(k_1+2k_2) = 0.$$

   We employ the insight of Tutorial 2 that the second component is the time derivative of the first, hence the eigenvectors may be written down using the eigenvalues.
   $v_{1,\pm} = (1, \pm\sqrt{-k_1}, 1, \sqrt{-k_1})^T$ for $\lambda_{1,\pm} = \pm\sqrt{-k_1} = \pm i\omega_1$
   $v_{2,\pm} = (-1, \mp\sqrt{-k_1-2k_2}, 1, \pm\sqrt{-k_1-2k_2})^T$ for $\lambda_{2,\pm} = \pm\sqrt{-k_1-2k_2} = \pm i\omega_2$

(d) We can now write down two complex fundamental solutions $v_{1,2+}e^{\lambda_{1,2+}t}$ and split them into real and imaginary parts.

$$x_1 = \begin{pmatrix} \cos\omega_1 t \\ -\omega_1 \sin\omega_1 t \\ \cos\omega_1 t \\ -\omega_1 \sin\omega_1 t \end{pmatrix}, \quad x_2 = \begin{pmatrix} \sin\omega_1 t \\ \omega_1 \cos\omega_1 t \\ \sin\omega_1 t \\ \omega_1 \cos\omega_1 t \end{pmatrix}, \quad x_3 = \begin{pmatrix} -\cos\omega_2 t \\ \omega_2 \sin\omega_2 t \\ \cos\omega_2 t \\ -\omega_2 \sin\omega_2 t \end{pmatrix}, \quad x_4 = \begin{pmatrix} -\sin\omega_2 t \\ -\omega_2 \cos\omega_2 t \\ \sin\omega_2 t \\ \omega_2 \cos\omega_2 t \end{pmatrix}$$

4. (a) The equilibria are $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. The latter only exist if $r > 1$.

(b) The symmetry is $S(x, y, z) = (-x, -y, z)$. You can verify that $(S(x, y, z))' = f(S(x, y, z))$ holds.

(c) The linearisation is

$$J(x, y) = \begin{pmatrix} -a & a & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

For the origin we have the eigenvalues $\lambda_1 = -b$ and $\lambda_{2,3} = \frac{1}{2}\left(-(a-1) \pm \sqrt{(a-1)^2 + 4ar}\right)$. This equilibrium is asymptotically stable if $r < 1$.

(d) The coefficients of the characteristic polynomial are $p_2 = 1+b+a$, $p_1 = b(r+a)$ and $p_0 = 2ab(r-1)$. Because r, b, a are positive and $r \geq 1$ for the nontrivial equilibria, we only need to check the condition $p_1 p_2 = p_0$. From this we can solve the critical value for r explicitly.

(e) We have the following factorisation $x^3 + ax^2 + bx + ab = (x^2 + b)(x + a)$. So for $r = \dfrac{a(3+b+a)}{a-b-1}$ the eigenvalues are $\lambda_1 = -1-a-b$ and $\lambda_{2,3} = \pm\frac{\sqrt{2ab(a+1)(1-a-b)}}{a-b-1}$.