1 ODE Challenge 2

On Canvas you find the MATLAB tool `pplane9`. Download it and start it using `pplane9` from the Matlab command line in the folder where you placed it. Enter the differential equations as you would enter normal Matlab expressions. Press **Proceed** and a window with a phase space will appear. Clicking on a particular point will produce a simulation, both in forward and backward time. Use **Stop** if you want to stop the simulation. In the menu there are options for finding equilibria and periodic orbits as well as an option to plot nullclines and (un)stable orbits/manifolds of saddles.

1. In this exercise we investigate *Kepler's problem*. The variable q describes the radius of planet or comet in an orbit around the sun. Keplers second law states that the angular momentum $L = q^2\dot{\theta} = L$ is constant. That relation allows one to derive the following differential equation for q

$$\begin{cases} q' &= p, \\ p' &= \frac{2C}{q^3} - \frac{1}{q^2}. \end{cases}$$

   Here $q' = \frac{dq}{d\theta}$, so the $'$ denotes the derivative w.r.t. to the angle θ in the plane around the sun. Meanwhile we write $\dot{\theta}$ for the time derivative of θ.

   (a) Determine a conserved quantity.

   (b) Draw the phase portrait in the (q,p)-plane. You may use `pplane9` to verify your result.

   (c) Describe the orbits of these bodies geometrically? That is, how does the radius q vary as a planet makes one rotation around the sun. Sketch how several different orbits in the (q,p)-plane correspond to orbits in the (x,y)-plane around the sun. Note $x = q(\theta)\cos(\theta)$ and $y = q(\theta)\sin(\theta)$, with θ changing monotonically.

2. Consider the system

$$\begin{cases} x' &= -2y - x^3 + xy^2 + x^5, \\ y' &= x - x^2 y. \end{cases}$$

   We will use Lasalle's invariance principle to show that the origin is asymptotically stable. In addition, we carefully construct the regions Ω_β and B_δ that were used in the proof for Lyapunov's theorem. With that we determine the basin of attraction.

   (a) Choose a such that $\dot{V}$ for the function $V(x,y) = x^2 + ay^2$ is negative semi-definite in a neighbourhood of the origin.

   (b) Show that the origin is asymptotically stable.

   (c) What basin of attraction for the origin do we find based on this Lyapunov function? In particular determine the sets Ω_β and $B_{r,\delta}$ as in the proof.

   (d) Sketch the phase portrait, and use `pplane9`!. What is the obstruction for a larger basin of attraction?