ODE Challenge 1

1. Consider the following initial value problem (IVP)

$$x'(t) = \begin{cases} -\frac{\sqrt{1-x^2}}{x}, & 0 < x \leq 1, \\ 0, & x > 1, \end{cases} \qquad \text{with } x(0) = 1. \tag{1}$$

   (a) Sketch the direction field.

   (b) Determine a solution valid for all $t \in \mathbb{R}$.

   (c) Find a second, nontrivial solution for this IVP. What is the maximal interval of existence?

   (d) Glue these two solutions together to have a function defined on $(-\infty, 1)$.
   Show that this new function is continuous and differentiable at $t = 0$.
   Hint: use the difference quotient as on the slides of the first lecture and determine the limit.

   (e) Now change the initial condition to $x(t_0) = 1$. Adapt the previous function such that you have a nontrivial solution for $(-\infty, t_1)$. What is the value of t_1? Argue that this new function is also continuous and differentiable (without computing the limit!).

   (f) When are the conditions for existence and uniqueness violated for this ODE? What happens at those two occasions?

2. The size of a fish population in some lake is given by the logistic equation

$$P' = 0.1P(1 - P/10),$$

   where $P(t)$ denotes the number of fish in units of thousand fish and time t is measured in days. This models exponential growth for small populations, but populations cannot grow larger as resources such as food are limited. At some point in time, folks around the lake decided to catch $h * 1000$ a day.

   (a) Adapt the logistic equation to include the effect of fishery on the population.

   (b) Find and classify the equilibria of your new model in case $h = 0.1$.

   (c) Sketch the phase line for $h = 0.1$ to analyse all solutions by sketching a few characteristic ones. That is, what happens if, when fishing starts, there would be 1000 fish, and what if there were 2000 in the lake?

   (d) Show using the phase line that the population will definitely go extinct if $h = 1$.

   (e) Determine the equilibria for all values of h. Show there is a critical value of $h = h_0$ such that the population may survive when $h < h_0$, whereas it will go extinct when $h > h_0$.

   (f) Sketch characteristic solutions for $h < h_0, h = h_0$ and $h > h_0$, i.e., P as a function of time.