1 Selfstudy for Tutorial 6

Polking **§10.5:** 1; **§10.6:** 6, 22, 23; **§10.7:** 22, 26

2 Exercises for Tutorial 6

1. For all of the system below

   - Determine a conserved quantity E.
   - Show that E is constant indeed, by checking $\frac{dE(x(t),y(t))}{dt} = 0$.

$$\begin{cases} x' = y \\ y' = 2x - x^2 \end{cases} \qquad\qquad \begin{cases} x' = \frac{1}{1+y^2} \\ y' = e^x \end{cases}$$

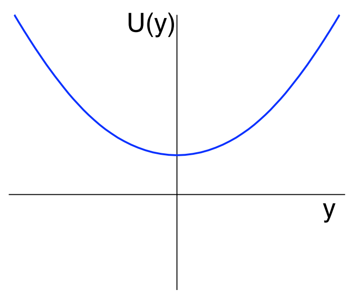
2. Given a potential $U(y) = -\int f(y)\, dy$, then the system

$$y' = v$$
$$v' = f(y)$$

   is Hamiltonian.

   Determine the corresponding Hamiltonian H, i.e. the conserved quantity.

   (b) Consider the potential $U(y)$ as in the figure below. Sketch it on your own paper, and below it, sketch a (y, v)-phase plane such that the y-axis matches. Now sketch the phase portrait, i.e. add equilibria, orbits and arrows, using the potential.



   (c) Consider the system

$$y' = v$$
$$v' = f(y) = y^2 - 9y$$

   i. Find the coordinates of the equilibria.
   ii. Determine a potential $U(y)$ and a Hamiltonian.
   iii. Sketch the phase portrait using the potential as in the previous exercise, i.e. add equilibria, orbits and arrows. Pay special attention to the level set to which the origin belongs.

3. Suppose $S(x, y)$ is a negative definite function with a maximum for (x_0, y_0).

(a) Show $V(x, y) = -S(x, y)$ is a Lyapunov function for the system defined by

$$x' = \frac{\partial S}{\partial x}, \qquad y' = \frac{\partial S}{\partial y}. \tag{1}$$

(b) Find a Lyapunov function for the system

$$x' = 2y - 2x, \qquad y' = 2x - 2y - 4y^3.$$

A system defined by (1) is known as a *gradient stelsel* system.

4. Consider the sytem

$$\begin{aligned} x' &= -x - y^2 \\ y' &= 2xy - y^3 \end{aligned}$$

(a) Investigate the stability of the origin using the linearisation.

(b) Compute $\dot{V}$ with V given by $V = \alpha x^2 + \beta y^2$, where α and β are constants yet to be determined.

(c) Determine numerical values for α and β such that $\dot{V} < 0$ for $(x, y) \neq 0$.

(d) Show the origin is asymptotically stable.