Paul's Online Notes

Section 1.2 : Rational Exponents

Now that we have looked at integer exponents we need to start looking at more complicated exponents. In this section we are going to be looking at rational exponents. That is exponents in the form

$$b^{\frac{m}{n}}$$

where both m and n are integers.

We will start simple by looking at the following special case,

$$b^{\frac{1}{n}}$$

where n is an integer. Once we have this figured out the more general case given above will actually be pretty easy to deal with.

Let's first define just what we mean by exponents of this form.

$$a = b^{\frac{1}{n}} \qquad \text{is equivalent to} \qquad a^n = b$$

In other words, when evaluating $b^{\frac{1}{n}}$ we are really asking what number (in this case a) did we raise to the n to get b. Often $b^{\frac{1}{n}}$ is called the n**th root of b**.

Let's do a couple of evaluations.

Example 1 Evaluate each of the following.

(a) $25^{\frac{1}{2}}$

(b) $32^{\frac{1}{5}}$

(c) $81^{\frac{1}{4}}$

(d) $(-8)^{\frac{1}{3}}$

(e) $(-16)^{\frac{1}{4}}$

(f) $-16^{\frac{1}{4}}$

Hide Discussion

When doing these evaluations, we will not actually do them directly. When first confronted with these kinds of evaluations doing them directly is often very difficult. In order to evaluate these we will remember the equivalence given in the definition and use that instead.

We will work the first one in detail and then not put as much detail into the rest of the problems.

(a) $25^{\frac{1}{2}}$ ***Hide Solution***

So, here is what we are asking in this problem.

$$25^{\frac{1}{2}} = ?$$

Using the equivalence from the definition we can rewrite this as,

$$?^{\,2} = 25$$

So, all that we are really asking here is what number did we square to get 25. In this case that is (hopefully) easy to get. We square 5 to get 25. Therefore,

$$25^{\frac{1}{2}} = 5$$

(b) $32^{\frac{1}{5}}$ ***Hide Solution***

So what we are asking here is what number did we raise to the 5$^{\text{th}}$ power to get 32?

$$32^{\frac{1}{5}} = 2 \qquad \text{because} \qquad 2^5 = 32$$

(c) $81^{\frac{1}{4}}$ ***Hide Solution***

What number did we raise to the 4$^{\text{th}}$ power to get 81?

$$81^{\frac{1}{4}} = 3 \qquad \text{because} \qquad 3^4 = 81$$

(d) $(-8)^{\frac{1}{3}}$ ***Hide Solution***

We need to be a little careful with minus signs here, but other than that it works the same way as the previous parts. What number did we raise to the 3rd power (*i.e.* cube) to get -8?

$$(-8)^{\frac{1}{3}} = -2 \qquad \text{because} \qquad (-2)^3 = -8$$

(e) $(-16)^{\frac{1}{4}}$ ***Hide Solution***

This part does not have an answer. It is here to make a point. In this case we are asking what number do we raise to the 4th power to get -16. However, we also know that raising any number (positive or negative) to an even power will be positive. In other words, there is no real number that we can raise to the 4th power to get -16.

Note that this is different from the previous part. If we raise a negative number to an odd power we will get a negative number so we could do the evaluation in the previous part.

As this part has shown, we can't always do these evaluations.

(f) $-16^{\frac{1}{4}}$ ***Hide Solution***

Again, this part is here to make a point more than anything. Unlike the previous part this one has an answer. Recall from the previous section that if there aren't any parentheses then only the part immediately to the left of the exponent gets the exponent. So, this part is really asking us to evaluate the following term.

$$-16^{\frac{1}{4}} = -\left(16^{\frac{1}{4}}\right)$$

So, we need to determine what number raised to the 4th power will give us 16. This is 2 and so in this case the answer is,

$$-16^{\frac{1}{4}} = -\left(16^{\frac{1}{4}}\right) = -(2) = -2$$

As the last two parts of the previous example has once again shown, we really need to be careful with parenthesis. In this case parenthesis makes the difference between being able to get an answer or not.

Also, don't be worried if you didn't know some of these powers off the top of your head. They are usually fairly simple to determine if you don't know them right away. For instance, in the part b we needed to determine what number raised to the 5 will give 32. If you can't see the power right off the top of your head simply start taking powers until you find the correct one. In other words compute 2^5, 3^5, 4^5 until you reach the correct value. Of course, in this case we wouldn't need to go past the first computation.

The next thing that we should acknowledge is that all of the **properties for exponents** that we gave in the previous section are still valid for all rational exponents. This includes the more general rational exponent that we haven't looked at yet.

Now that we know that the properties are still valid we can see how to deal with the more general rational exponent. There are in fact two different ways of dealing with them as we'll see. Both methods involve using property 2 from the previous section. For reference purposes this property is,

$$(a^n)^m = a^{nm}$$

So, let's see how to deal with a general rational exponent. We will first rewrite the exponent as follows.

$$b^{\frac{m}{n}} = b^{\left(\frac{1}{n}\right)(m)}$$

In other words, we can think of the exponent as a product of two numbers. Now we will use the exponent property shown above. However, we will be using it in the opposite direction than what we did in the previous section. Also, there are two ways to do it. Here they are,

$$b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m \qquad \text{OR} \qquad b^{\frac{m}{n}} = (b^m)^{\frac{1}{n}}$$

Using either of these forms we can now evaluate some more complicated expressions

Example 2 Evaluate each of the following.

(a) $8^{\frac{2}{3}}$

(b) $625^{\frac{3}{4}}$

(c) $\left(\dfrac{243}{32}\right)^{\frac{4}{5}}$

Hide Discussion

We can use either form to do the evaluations. However, it is usually more convenient to use the first form as we will see.

(a) $8^{\frac{2}{3}}$ ***Hide Solution***

Let's use both forms here since neither one is too bad in this case. Let's take a look at the first form.

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = (2)^2 = 4 \qquad\qquad 8^{\frac{1}{3}} = 2 \text{ because } 2^3 = 8$$

Now, let's take a look at the second form.

$$8^{\frac{2}{3}} = \left(8^2\right)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4 \qquad\qquad 64^{\frac{1}{3}} = 4 \text{ because } 4^3 = 64$$

So, we get the same answer regardless of the form. Notice however that when we used the second form we ended up taking the 3rd root of a much larger number which can cause problems on occasion.

(b) $625^{\frac{3}{4}}$ ***Hide Solution***

Again, let's use both forms to compute this one.

$$625^{\frac{3}{4}} = \left(625^{\frac{1}{4}}\right)^3 = (5)^3 = 125 \qquad\qquad 625^{\frac{1}{4}} = 5 \text{ because } 5^4 = 625$$

$$625^{\frac{3}{4}} = \left(625^3\right)^{\frac{1}{4}} = (244140625)^{\frac{1}{4}} = 125 \quad \text{because } \ 125^4 = 244140625$$

As this part has shown the second form can be quite difficult to use in computations. The root in this case was not an obvious root and not particularly easy to get if you didn't know it right off the top of your head.

(c) $\left(\dfrac{243}{32}\right)^{\frac{4}{5}}$ ***Hide Solution*** ▾

In this case we'll only use the first form. However, before doing that we'll need to first use **property 5** of our exponent properties to get the exponent onto the numerator and denominator.

$$\left(\frac{243}{32}\right)^{\frac{4}{5}} = \frac{243^{\frac{4}{5}}}{32^{\frac{4}{5}}} = \frac{\left(243^{\frac{1}{5}}\right)^4}{\left(32^{\frac{1}{5}}\right)^4} = \frac{(3)^4}{(2)^4} = \frac{81}{16}$$

We can also do some of the simplification type problems with rational exponents that we saw in the previous section.

Example 3 Simplify each of the following and write the answers with only positive exponents.

(a) $\left(\dfrac{w^{-2}}{16v^{\frac{1}{2}}}\right)^{\frac{1}{4}}$

(b) $\left(\dfrac{x^2y^{-\frac{2}{3}}}{x^{-\frac{1}{2}}y^{-3}}\right)^{-\frac{1}{7}}$

(a) $\left(\dfrac{w^{-2}}{16v^{\frac{1}{2}}}\right)^{\frac{1}{4}}$ ***Hide Solution*** ▾

For this problem we will first move the exponent into the parenthesis then we will eliminate the negative exponent as we did in the previous section. We will then move the term to the denominator and drop the minus sign.

$$\frac{w^{-2\left(\frac{1}{4}\right)}}{16^{\frac{1}{4}}v^{\frac{1}{2}\left(\frac{1}{4}\right)}} = \frac{w^{-\frac{1}{2}}}{2v^{\frac{1}{8}}} = \frac{1}{2v^{\frac{1}{8}}w^{\frac{1}{2}}}$$

(b) $\left(\dfrac{x^2y^{-\frac{2}{3}}}{x^{-\frac{1}{2}}y^{-3}}\right)^{-\frac{1}{7}}$ ***Hide Solution*** ▾

In this case we will first simplify the expression inside the parenthesis.

$$\left(\frac{x^2y^{-\frac{2}{3}}}{x^{-\frac{1}{2}}y^{-3}}\right)^{-\frac{1}{7}} = \left(\frac{x^2x^{\frac{1}{2}}y^3}{y^{\frac{2}{3}}}\right)^{-\frac{1}{7}} = \left(\frac{x^{2+\frac{1}{2}}y^{3-\frac{2}{3}}}{1}\right)^{-\frac{1}{7}} = \left(x^{\frac{5}{2}}y^{\frac{7}{3}}\right)^{-\frac{1}{7}}$$

Don't worry if, after simplification, we don't have a fraction anymore. That will happen on occasion. Now we will eliminate the negative in the exponent using **property 7** and then we'll use **property 4** to finish the problem up.

$$\left(\frac{x^2y^{-\frac{2}{3}}}{x^{-\frac{1}{2}}y^{-3}}\right)^{-\frac{1}{7}} = \frac{1}{\left(x^{\frac{5}{2}}y^{\frac{7}{3}}\right)^{\frac{1}{7}}} = \frac{1}{x^{\frac{5}{14}}y^{\frac{1}{3}}}$$

We will leave this section with a warning about a common mistake that students make in regard to negative exponents and rational exponents. Be careful not to confuse the two as they are totally separate topics.

In other words,

$$b^{-n} = \frac{1}{b^n}$$

and NOT

$$b^{-n} \neq b^{\frac{1}{n}}$$

This is a very common mistake when students first learn exponent rules.

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