Paul's Online Notes

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Section 1.1 : Integer Exponents

We will start off this chapter by looking at integer exponents. In fact, we will initially assume that the exponents are positive as well. We will look at zero and negative exponents in a bit.

Let's first recall the definition of exponentiation with positive integer exponents. If a is any number and n is a positive integer then,

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n\ \text{times}}$$

So, for example,

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$$

We should also use this opportunity to remind ourselves about parenthesis and conventions that we have in regard to exponentiation and parenthesis. This will be particularly important when dealing with negative numbers. Consider the following two cases.

$$(-2)^4 \qquad \text{and} \qquad -2^4$$

These will have different values once we evaluate them. When performing exponentiation remember that it is only the quantity that is immediately to the left of the exponent that gets the power.

In the first case there is a parenthesis immediately to the left so that means that everything in the parenthesis gets the power. So, in this case we get,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

In the second case however, the 2 is immediately to the left of the exponent and so it is only the 2 that gets the power. The minus sign will stay out in front and will NOT get the power. In this case we have the following,

$$-2^4 = -\left(2^4\right) = -(2 \cdot 2 \cdot 2 \cdot 2) = -(16) = -16$$

We put in some extra parenthesis to help illustrate this case. In general, they aren't included and we would write instead,

$$-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$$

The point of this discussion is to make sure that you pay attention to parenthesis. They are important and ignoring parenthesis or putting in a set of parenthesis where they don't belong can completely change the answer to a problem. Be careful. Also, this warning about parenthesis is not just intended for exponents. We will need to be careful with parenthesis throughout this course.

Now, let's take care of zero exponents and negative integer exponents. In the case of zero exponents we have,

$$a^0 = 1 \qquad \text{provided } a \neq 0$$

Notice that it is required that a not be zero. This is important since 0^0 is not defined. Here is a quick example of this property.

$$(-1268)^0 = 1$$

We have the following definition for negative exponents. If a is any non-zero number and n is a positive integer (yes, positive) then,

$$a^{-n} = \frac{1}{a^n}$$

Can you see why we required that a not be zero? Remember that division by zero is not defined and if we had allowed a to be zero we would have gotten division by zero. Here are a couple of quick examples for this definition,

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25} \qquad (-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{-64} = -\frac{1}{64}$$

Here are some of the main properties of integer exponents. Accompanying each property will be a quick example to illustrate its use. We will be looking at more complicated examples after the properties.

Properties

1. $a^n a^m = a^{n+m}$ Example : $a^{-9}a^4 = a^{-9+4} = a^{-5}$

2. $\left(a^n\right)^m = a^{nm}$ — Example : $\left(a^7\right)^3 = a^{(7)(3)} = a^{21}$

3. $\dfrac{a^n}{a^m} = \begin{cases} a^{n-m} \\ \dfrac{1}{a^{m-n}} \end{cases}, \quad a \neq 0$ — Example : $\dfrac{a^4}{a^{11}} = a^{4-11} = a^{-7}$, $\dfrac{a^4}{a^{11}} = \dfrac{1}{a^{11-4}} = \dfrac{1}{a^7} = a^{-7}$

4. $\left(ab\right)^n = a^n b^n$ — Example : $\left(ab\right)^{-4} = a^{-4}b^{-4}$

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$ — Example : $\left(\dfrac{a}{b}\right)^8 = \dfrac{a^8}{b^8}$

6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$ — Example : $\left(\dfrac{a}{b}\right)^{-10} = \left(\dfrac{b}{a}\right)^{10} = \dfrac{b^{10}}{a^{10}}$

7. $\left(ab\right)^{-n} = \dfrac{1}{\left(ab\right)^n}$ — Example : $\left(ab\right)^{-20} = \dfrac{1}{\left(ab\right)^{20}}$

8. $\dfrac{1}{a^{-n}} = a^n$ — Example : $\dfrac{1}{a^{-2}} = a^2$

9. $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$ — Example : $\dfrac{a^{-6}}{b^{-17}} = \dfrac{b^{17}}{a^6}$

10. $\left(a^n b^m\right)^k = a^{nk} b^{mk}$ — Example : $\left(a^4 b^{-9}\right)^3 = a^{(4)(3)} b^{(-9)(3)} = a^{12} b^{-27}$

11. $\left(\dfrac{a^n}{b^m}\right)^k = \dfrac{a^{nk}}{b^{mk}}$ — Example : $\left(\dfrac{a^6}{b^5}\right)^2 = \dfrac{a^{(6)(2)}}{b^{(5)(2)}} = \dfrac{a^{12}}{b^{10}}$

Notice that there are two possible forms for the third property. Which form you use is usually dependent upon the form you want the answer to be in.

Note as well that many of these properties were given with only two terms/factors but they can be extended out to as many terms/factors as we need. For example, property 4 can be extended as follows.

$$\left(abcd\right)^n = a^n b^n c^n d^n$$

We only used four factors here, but hopefully you get the point. Property 4 (and most of the other properties) can be extended out to meet the number of factors that we have in a given problem.

There are several common mistakes that students make with these properties the first time they see them. Let's take a look at a couple of them.

Consider the following case.

$$\begin{aligned}\text{Correct :} \qquad & ab^{-2} = a\frac{1}{b^2} = \frac{a}{b^2} \\ \text{Incorrect :} \qquad & ab^{-2} \neq \frac{1}{ab^2}\end{aligned}$$

In this case only the b gets the exponent since it is immediately off to the left of the exponent and so only this term moves to the denominator. Do NOT carry the a down to the denominator with the b. Contrast this with the following case.

$$(ab)^{-2} = \frac{1}{(ab)^2}$$

In this case the exponent is on the set of parenthesis and so we can just use property 7 on it and so both the a and the b move down to the denominator. Again, note the importance of parenthesis and how they can change an answer!

Here is another common mistake.

$$\begin{aligned}\text{Correct :} \qquad & \frac{1}{3a^{-5}} = \frac{1}{3}\frac{1}{a^{-5}} = \frac{1}{3}a^5 \\ \text{Incorrect :} \qquad & \frac{1}{3a^{-5}} \neq 3a^5\end{aligned}$$

In this case the exponent is only on the a and so to use property 8 on this we would have to break up the fraction as shown and then use property 8 only on the second term. To bring the 3 up with the a we would have needed the following.

$$\frac{1}{(3a)^{-5}} = (3a)^5$$

Once again, notice this common mistake comes down to being careful with parenthesis. This will be a constant refrain throughout these notes. We must always be careful with parenthesis. Misusing them can lead to incorrect answers.

Let's take a look at some more complicated examples now.

Example 1 Simplify each of the following and write the answers with only positive exponents.

(a) $\left(4x^{-4}y^5\right)^3$

(b) $\left(-10z^2y^{-4}\right)^2\left(z^3y\right)^{-5}$

(c) $\dfrac{n^{-2}m}{7m^{-4}n^{-3}}$

(d) $\dfrac{5x^{-1}y^{-4}}{\left(3y^5\right)^{-2}x^9}$

(e) $\left(\dfrac{z^{-5}}{z^{-2}x^{-1}}\right)^6$

(f) $\left(\dfrac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}$

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Note that when we say "simplify" in the problem statement we mean that we will need to use all the properties that we can to get the answer into the required form. Also, a "simplified" answer will have as few terms as possible and each term should have no more than a single exponent on it.

There are many different paths that we can take to get to the final answer for each of these. In the end the answer will be the same regardless of the path that you used to get the answer. All that this means for you is that as long as you used the properties you can take the path that you find the easiest. The path that others find to be the easiest may not be the path that you find to be the easiest. That is okay.

Also, we won't put quite as much detail in using some of these properties as we did in the examples given with each property. For instance, we won't show the actual multiplications anymore, we will just give the result of the multiplication.

(a) $\left(4x^{-4}y^5\right)^3$ ***Hide Solution***

For this one we will use property 10 first.

$$\left(4x^{-4}y^5\right)^3 = 4^3x^{-12}y^{15}$$

Don't forget to put the exponent on the constant in this problem. That is one of the more common mistakes that students make with these simplification problems.

At this point we need to evaluate the first term and eliminate the negative exponent on the second term. The evaluation of the first term isn't too bad and all we need to do to eliminate the negative exponent on the second term is use the definition we gave for negative exponents.

$$\left(4x^{-4}y^{5}\right)^{3} = 64\left(\frac{1}{x^{12}}\right)y^{15} = \frac{64y^{15}}{x^{12}}$$

We further simplified our answer by combining everything up into a single fraction. This should always be done.

The middle step in this part is usually skipped. All the definition of negative exponents tells us to do is move the term to the denominator and drop the minus sign in the exponent. So, from this point on, that is what we will do without writing in the middle step.

(b) $\left(-10z^{2}y^{-4}\right)^{2}\left(z^{3}y\right)^{-5}$ ***Hide Solution*** ▾

In this case we will first use property 10 on both terms and then we will combine the terms using property 1. Finally, we will eliminate the negative exponents using the definition of negative exponents.

$$\left(-10z^{2}y^{-4}\right)^{2}\left(z^{3}y\right)^{-5} = (-10)^{2}z^{4}y^{-8}z^{-15}y^{-5} = 100z^{-11}y^{-13} = \frac{100}{z^{11}y^{13}}$$

There are a couple of things to be careful with in this problem. First, when using the property 10 on the first term, make sure that you square the "-10" and not just the 10 (*i.e.* don't forget the minus sign…). Second, in the final step, the 100 stays in the numerator since there is no negative exponent on it. The exponent of "-11" is only on the z and so only the z moves to the denominator.

(c) $\dfrac{n^{-2}m}{7m^{-4}n^{-3}}$ ***Hide Solution*** ▾

This one isn't too bad. We will use the definition of negative exponents to move all terms with negative exponents in them to the denominator. Also, property 8 simply says that if there is a term with a negative exponent in the denominator then we will just move it to the numerator and drop the minus sign.

So, let's take care of the negative exponents first.

$$\frac{n^{-2}m}{7m^{-4}n^{-3}} = \frac{m^4n^3m}{7n^2}$$

Now simplify. We will use property 1 to combine the m's in the numerator. We will use property 3 to combine the n's and since we are looking for positive exponents we will use the first form of this property since that will put a positive exponent up in the numerator.

$$\frac{n^{-2}m}{7m^{-4}n^{-3}} = \frac{m^5n}{7}$$

Again, the 7 will stay in the denominator since there isn't a negative exponent on it. It will NOT move up to the numerator with the m. Do not get excited if all the terms move up to the numerator or if all the terms move down to the denominator. That will happen on occasion.

(d) $\dfrac{5x^{-1}y^{-4}}{(3y^5)^{-2}x^9}$ ***Hide Solution*** ▾

This example is similar to the previous one except there is a little more going on with this one. The first step will be to again, get rid of the negative exponents as we did in the previous example. Any terms in the numerator with negative exponents will get moved to the denominator and we'll drop the minus sign in the exponent. Likewise, any terms in the denominator with negative exponents will move to the numerator and we'll drop the minus sign in the exponent. Notice this time, unlike the previous part, there is a term with a set of parenthesis in the denominator. Because of the parenthesis that whole term, including the 3, will move to the numerator.

Here is the work for this part.

$$\frac{5x^{-1}y^{-4}}{(3y^5)^{-2}x^9} = \frac{5(3y^5)^2}{xy^4x^9} = \frac{5\,(9)\,y^{10}}{xy^4x^9} = \frac{45y^6}{x^{10}}$$

(e) $\left(\dfrac{z^{-5}}{z^{-2}x^{-1}}\right)^6$ ***Hide Solution*** ▾

There are several first steps that we can take with this one. The first step that we're pretty much always going to take with these kinds of problems is to first simplify the fraction inside the parenthesis as much as possible. After we do that we will use property 5 to deal with the exponent that is on the parenthesis.

$$\left(\frac{z^{-5}}{z^{-2}x^{-1}}\right)^6 = \left(\frac{z^2x^1}{z^5}\right)^6 = \left(\frac{x}{z^3}\right)^6 = \frac{x^6}{z^{18}}$$

In this case we used the second form of property 3 to simplify the z's since this put a positive exponent in the denominator. Also note that we almost never write an exponent of "1". When we have exponents of 1 we will drop them.

(f) $\left(\dfrac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}$ ***Hide Solution*** ▾

This one is very similar to the previous part. The main difference is negative on the outer exponent. We will deal with that once we've simplified the fraction inside the parenthesis.

$$\left(\frac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2} = \left(\frac{4a^3a^5}{b^8b}\right)^{-2} = \left(\frac{4a^8}{b^9}\right)^{-2}$$

Now at this point we can use property 6 to deal with the exponent on the parenthesis. Doing this gives us,

$$\left(\frac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2} = \left(\frac{b^9}{4a^8}\right)^2 = \frac{b^{18}}{16a^{16}}$$

Before leaving this section we need to talk briefly about the requirement of positive only exponents in the above set of examples. This was done only so there would be a consistent final answer. In many cases negative exponents are okay and in some cases they are required. In fact, if you are on a track that will take you into calculus there are a fair number of problems in a calculus class in which negative exponents are the preferred, if not required, form.

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