Assignment 3

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Question 1

(a)

A 'over-smoothed' kernel density estimator, $\hat{f}(y)$, with $h = cT^{-k}, k < 1/5$, does not satisfy the required condition for having a limiting normal sampling distribution.

Demonstration:

$$\hat{f}(y) = \frac{1}{Th}\sum_{t=1}^{T} K(\frac{y - y_t}{h})$$

The bias is given by

$$E[\hat{f}(y)] - f(y) = \frac{h^2}{2} f^{(2)}(y)\mu_2 + O(h^4)$$

If $k < 0$, $h \nrightarrow 0$ as $T \to \infty$, the bias will not approch to zero. In other words, the consistency doesn't hold. Thus, $\hat{f}(y)$ will not have a limiting normal distribution.

If $0 < k < 1/5$, $h \to 0$ as $T \to \infty$, the consistency holds. Using a genearl version of the CLT, we know

$$\sqrt{Th}(\hat{f}(y) - E[\hat{f}(y)]) \xrightarrow{d} N(0, f(y)\int_{-\infty}^{\infty} K^2(z)dz)$$

$$\begin{aligned}
\sqrt{Th}(\hat{f}(y) - E[\hat{f}(y)]) &= \sqrt{Th}(\hat{f}(y) - f(y) + f(y) - E[\hat{f}(y)]) \\
&= \sqrt{Th}(\hat{f}(y) - f(y) - \frac{h^2}{2} f^{(2)}(y)\mu_2 - O(h^4)) \\
&= \sqrt{Th}(\hat{f}(y) - f(y)) - \sqrt{Th}\frac{h^2}{2} f^{(2)}(y)\mu_2 - \sqrt{Th}O(h^4) \\
&= \sqrt{Th}(\hat{f}(y) - f(y)) - \sqrt{Th^5}\frac{1}{2} f^{(2)}(y)\mu_2 - O(\sqrt{Th^9}) \\
&= \sqrt{Th}(\hat{f}(y) - f(y)) - \sqrt{Th^5}\frac{1}{2} f^{(2)}(y)\mu_2 - o(\sqrt{Th^5})
\end{aligned}$$

In oreder to let

$$\sqrt{Th}(\hat{f}(y) - f(y)) \xrightarrow{d} N(0, f(y)\int_{-\infty}^{\infty} K^2(z)dz)$$

We need

$$\sqrt{Th^5}\frac{1}{2}f^{(2)}(y)\mu_2 - o(\sqrt{Th^5}) \to 0$$

as $T \to \infty$

However, when $0 < k < 1/5$, $\sqrt{Th^5} \nrightarrow 0$ as $T \to \infty$.

Thus, $k < 1/5$ does not satisfy the required condition for having a limiting normal sampling distribution.

(b)

The condition on the bandwidth that is required for the limiting normality result is

$$h = cT^{-k}, \text{where } 1/5 < k < 1$$

Since this condition is satisfied, the limiting normality of $\hat{f}(y)$ is

$$\sqrt{Th}(\hat{f}(y) - f(y)) \xrightarrow{d} N(0, f(y)\int_{-\infty}^{\infty} K^2(z)dz)$$

Since $K(.)$ is the Gaussian kernel, $\int_{-\infty}^{\infty} K^2(z)dz = \frac{1}{\sqrt{4\pi}}$

Under the $H_0 : f(y) = 3$,

$$\sqrt{Th}(\hat{f}(y) - f(y)) \xrightarrow{d} N(0, \frac{3}{\sqrt{4\pi}})$$

Thus, the 95% confidence interval of the sampling distribution is

$$[-1.96 * \sqrt{\frac{3}{2}}\pi^{-1/4}, 1.96 * \sqrt{\frac{3}{2}}\pi^{-1/4}]$$

We can then check if $\hat{f}(y)$ falls into this interval. If it does, we do not reject $H_0 : f(y) = 3$. Otherwise, we reject $H_0 : f(y) = 3$.

(c)

Question 2

Question 3