Assignment 3

Weihao Li

2020/11/06

Question 1

(a)

A 'over-smoothed' kernel density estimator, $\hat{f}(y)$, with $h = cT^{-k}, k < 1/5$, does not satisfy the required condition for having a limiting normal sampling distribution.

Demonstration:

$$\hat{f}(y) = \frac{1}{Th}\sum_{t=1}^{T} K(\frac{y - y_t}{h})$$

The bias is given by

$$E[\hat{f}(y)] - f(y) = \frac{h^2}{2} f^{(2)}(y)\mu_2 + O(h^4)$$

If $k < 0$, $h \nrightarrow 0$ as $T \to \infty$, the bias will not approch to zero. In other words, the consistency doesn't hold. Thus, $\hat{f}(y)$ will not have a limiting normal distribution.

If $0 < k < 1/5$, $h \to 0$ as $T \to \infty$, the consistency holds. Using a genearl version of the CLT, we know

$$\sqrt{Th}(\hat{f}(y) - E[\hat{f}(y)]) \xrightarrow{d} N(0, f(y)\int_{-\infty}^{\infty} K^2(z)dz)$$

$$\begin{aligned}
\sqrt{Th}(\hat{f}(y) - E[\hat{f}(y)]) &= \sqrt{Th}(\hat{f}(y) - f(y) + f(y) - E[\hat{f}(y)]) \\
&= \sqrt{Th}(\hat{f}(y) - f(y) - \frac{h^2}{2} f^{(2)}(y)\mu_2 - O(h^4)) \\
&= \sqrt{Th}(\hat{f}(y) - f(y)) - \sqrt{Th}\frac{h^2}{2} f^{(2)}(y)\mu_2 - \sqrt{Th}O(h^4) \\
&= \sqrt{Th}(\hat{f}(y) - f(y)) - \sqrt{Th^5}\frac{1}{2} f^{(2)}(y)\mu_2 - O(\sqrt{Th^9}) \\
&= \sqrt{Th}(\hat{f}(y) - f(y)) - \sqrt{Th^5}\frac{1}{2} f^{(2)}(y)\mu_2 - o(\sqrt{Th^5})
\end{aligned}$$

In oreder to let

$$\sqrt{Th}(\hat{f}(y) - f(y)) \xrightarrow{d} N(0, f(y)\int_{-\infty}^{\infty} K^2(z)dz)$$

We need

$$\sqrt{Th^5}\frac{1}{2}f^{(2)}(y)\mu_2 - o(\sqrt{Th^5}) \to 0$$

as $T \to \infty$

However, when $0 < k < 1/5$, $\sqrt{Th^5} \nrightarrow 0$ as $T \to \infty$.

Thus, $k < 1/5$ does not satisfy the required condition for having a limiting normal sampling distribution.

(b)

The condition on the bandwidth that is required for the limiting normality result is

$$h = cT^{-k}, \text{where } 1/5 < k < 1$$

Since this condition is satisfied, the limiting normality of $\hat{f}(y)$ is

$$\sqrt{Th}(\hat{f}(y) - f(y)) \xrightarrow{d} N(0, f(y)\int_{-\infty}^{\infty} K^2(z)dz)$$

Since $K(.)$ is the Gaussian kernel, $\int_{-\infty}^{\infty} K^2(z)dz = \frac{1}{\sqrt{4\pi}}$

Under the $H_0 : f(y) = 3$,

$$\sqrt{Th}(\hat{f}(y) - f(y)) \xrightarrow{d} N(0, \frac{3}{\sqrt{4\pi}})$$

Thus, the 95% confidence interval of the sampling distribution of the test statistic is

$$[-1.96 \times \sqrt{\frac{3}{2}\pi^{-1/4}}, 1.96 \times \sqrt{\frac{3}{2}\pi^{-1/4}}]$$

We can then check if $\sqrt{Th}(\hat{f}(y) - 3)$ falls into this interval. If it does, we do not reject $H_0 : f(y) = 3$. Otherwise, we reject $H_0 : f(y) = 3$.

(c)

If we produce two curves by constructing 95% point-wise confidence interval for each $f(y)$ across the support of the Y_t, these two curves can not be interpreted as 'bounding' density functions. A proper probability density function $f(x)$ needs to satisfies two conditions: (1) $f(x) \geq 0$ and (2) $\int_{-\infty}^{\infty} f(x)dx = 1$.

The 95% confidence interval around $f(y)$ can not guarantee it will not include negative values. This does not satisfy the condition (1). Besides, neither the upper bound curve and the lower bound curve can be integrated to 1. This does not satisfy the condition (2).

Question 2

In the proof of the consistency of a nonparametric estimator, $\hat{f}(y)$ of $f(y)$, we only need to assume $f(y)$ is twice continuously differentiable, $K(.)$ satisfies some conditions as a kernel function and $h = cT^{-k}, 0 < k < 1$. The essential of the proof is built on the derivation of the bias and variance of the estimator using Taylor series. And then shows how the bias and variance will eventually approach 0 when $T \to \infty$ and $h \to 0$.

Alternatively, in the proof of the consistency of MLE, we need to assume a lot more at the beginning. These include the parametric form, the existence of the $E[lnf(Y|\theta)]$ for any θ, $Pr[f(Y|\theta^{\#}) \neq f(Y|\theta_0)] > 0$ for any

$\theta^{\#} \neq \theta_0$ and the true θ_0 is not on the boundary of Θ. The first step is to prove the expectation of the log likelihood function is maximized at θ_0. This is the most important step of the proof. And then it requires the use of weak law of large number and continuous mapping theorem to prove $\hat{\theta}_{MLE}$ is a consistent estimator of θ_0. Finally, if needed, continuous mapping theorem can be used again to prove $f(y|\hat{\theta}_{MLE})$ is a consistent estimator of $f(y|\theta_0)$. If the proof of the consistency of MLE uses a similar method as the nonparametric case, which is analysing the bias and variance. The proof could become much more complicated.

Question 3