Recurrence Relations

Chapter 8

Applications of Recurrence Relations

Section 8.1

Section Summary

- Applications of Recurrence Relations
  - Fibonacci Numbers
  - The Tower of Hanoi
  - Bit Strings

Recurrence Relations
(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$.

- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect (e.g. $a_0=1$, $a_1=1$).

Rabbits and the Fibonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

The sequence $\{f_n\}$ satisfies the recurrence relation
$f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$
with the initial conditions $f_1 = 1$ and $f_2 = 1$.

f_n - n-th Fibonacci number.

Rabbits and the Fiobonacci Numbers (*cont.*)

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Modeling the Population Growth of Rabbits on an Island

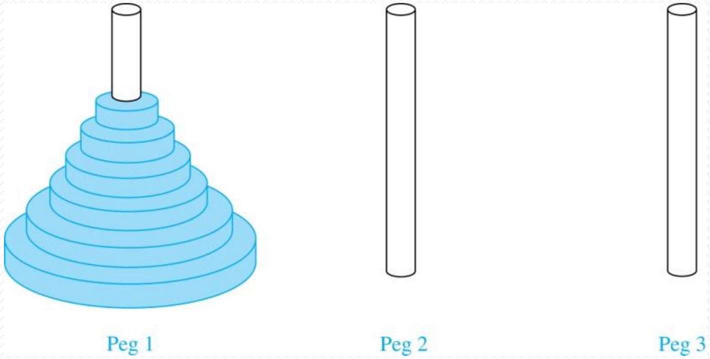
The Tower of Hanoi

In the late nineteenth century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

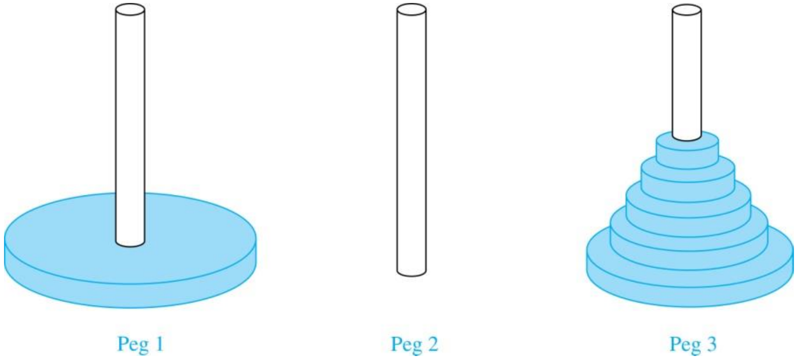
The Tower of Hanoi (*continued*)



The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi (*continued*)

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi (*continued*)

- We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

  $H_n = 2H_{n-1} + 1$

  $= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$ *using the method of backward substitutions*

  $= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$

  $\vdots$

  $= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$

  $= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1$ *because $H_1 = 1$*

  $= 2^n - 1$ *using the formula for the sum of the terms of a geometric series*

  - There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end. In how many years the world will end?
  - Using this formula for the 64 gold disks of the myth,

    $2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$ minutes,

  which is about … years.

Counting Bit Strings

Example 3:

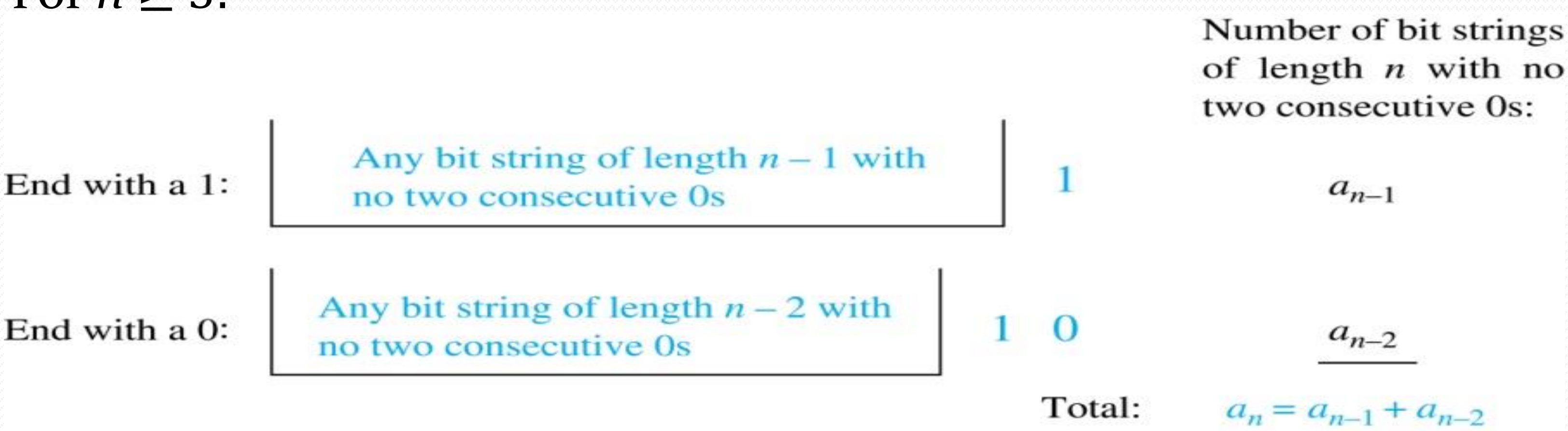
- Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s.
- How many such bit strings are there of length =5?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s.

Initial conditions:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

For $n \geq 3$:



Bit Strings (*continued*)

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $\boldsymbol{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k\, a_{n-k}}$, where $c_1, c_2, \ldots., c_k$ are real numbers, and $c_k \neq 0$

- ***linear*** because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- ***homogeneous*** because no terms occur that are not multiples of the a_js.
- with ***constant coefficients*** - each c_k is a constant.
- ***of degree k*** because a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence $\{a_n\}$ satisfying such a recurrence relation is **uniquely determined** by the recurrence relation and the ***k*** initial conditions $\boldsymbol{a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}}$.

CQ: Which Ones are Linear Homogeneous Recurrence Relations with Constant Coefficients?
(could be more than one answer)

a) $P_n = 2P_{n-1}$

b) $f_n = f_{n-1} + f_{n-2}$

c) $a_n = a_{n-1} + a_{n-2}^2$

d) $H_n = 2H_{n-1} + 1$

e) $B_n = nB_{n-1}$

Examples of Linear Homogeneous Recurrence Relations

- $P_n = 2P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ $(n=2,3,4 \ldots)$

iff $a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants, which can be found from initial conditions:

$a_o = C_o$, $a_1 = C_1$.

Proof.

- $a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n} \rightarrow a_o = C_o,\ a_1 = C_1,\ a_n = c_1 a_{n-1} + c_2 a_{n-2}$
- $a_o = C_o,\ a_1 = C_1,\ a_n = c_1 a_{n-1} + c_2 a_{n-2} \rightarrow a_n = \alpha_1 r_1^{\ n} + \alpha_2 r_2^{\ n}$

$a_n = \alpha_1 r_1{}^n + \alpha_2\, r_2{}^n \;\rightarrow a_n = c_1 a_{n-1} + c_2 a_{n-2}\, (n{=}2{,}3{,}4\ \ldots)$

Proof.

$$
\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n.
\end{aligned}
$$

$$a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} \rightarrow a_o = C_o,\ a_1 = C_1$$

Proof.

$$a_o = \alpha_1 + \alpha_2 = C_o,$$

$$a_1 = \alpha_1 r_1 + \alpha_2 r_2 = C_1,$$

Solving the system of linear equations:

$$\alpha_1 + \alpha_2 = C_o,$$

$$\alpha_1 r_1 + \alpha_2 r_2 = C_1,$$

for α_1 and α_2 we get:

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

and

$$\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

- $a_o = C_o$, $a_1 = C_1$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} \rightarrow a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$

Proof.

- We know that $\{a_n\}$ and $\{\alpha_1 r_1{}^n + \alpha_2 r_2{}^n\}$ are both solutions of the same recurrence relation with the same initial conditions.
- The solution of hom. linear rec. relation with 2 initial conditions is unique (see slide 16).
- Then $a_n = \alpha_1 r_1{}^n + \alpha_2 r_2{}^n$. ■

Using Theorem 1

Example: What is the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution:
The characteristic equation is
$r^2 - r - 2 = 0$.
Its roots are
$r = 2$ and $r = -1$.
Therefore, $\{a_n\}$ is a solution to the recurrence relation iff
$a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$$a_0 = 2 = \alpha_1 + \alpha_2 \text{ and } a_1 = 7 = \alpha_1 2 + \alpha_2(-1).$$

Solving these equations, we find that
$\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with
$a_n = 3\cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution:

The roots of the characteristic equation

$r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 \,.$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha r_0^n + \alpha_2 n r_0^n$ for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Using Theorem 2

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution:

The characteristic equation is

$r^2 - 6r + 9 = 0.$

The only root is

$r = 3.$

Therefore, $\{a_n\}$ is a solution to the recurrence relation iff

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

Find the constants α_1 and α_2:

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.

Hence, $a_n =$

$= 3^n + n3^n$.

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients **of any degree** when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}$$

iff

$$a_n = \alpha_1 r_1{}^n + \alpha_2\, r_2{}^n + \ldots + \alpha_k\, r_k{}^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$
$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$$
$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \text{ iff}$$

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

$$(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2) 2^n + (\alpha_{2,0} + \alpha_{2,1} n) 5^n + \alpha_{3,0} 9^n.$$

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

$$r^3 + 3r^2 + 3r + 1 = (r + 1)^3,$$

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$a_0 = 1 = \alpha_{1,0},$$
$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},$$
$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.$$

$$\alpha_{1,0} = 1, \ \alpha_{1,1} = 3, \text{ and } \alpha_{1,2} = -2.$$

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

Match the rec. relations and their solutions

Recurrence relations	Solutions
1. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, char equation $r^2 - c_1 r - c_2 = 0$ has one repeated root	a) $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n$
2. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ char equation $r^2 - c_1 r - c_2 = 0$ has two distinct roots	b) $a_n = \alpha_1 r^n + \alpha_2 n r^n$
3. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$, char. eq-n $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has k distinct roots	c) $a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$ $+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$ $+ \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$
4. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$, char. eq-n $r^k - c_1 r^{k-1} - \cdots - c_k = 0$ has t $(t<k)$ distinct roots with multiplicities $m_1, m_2, \ldots, m_t$	d) $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n + \ldots + \alpha_k {r_k}^n$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear* ***nonhomogeneous recurrence relation*** *with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k} + \boldsymbol{F(n)},$$

where c_1, c_2, ….,c_k are real numbers, and $\boldsymbol{F(n)}$ **is a function not identically zero depending only on *n*.**

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k},$$

is called the associated **homogeneous recurrence relation**.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{\boldsymbol{a_n}^{(p)}\}$ **is a particular solution** of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution of the recurrence relation is of the form $\{\boldsymbol{a_n}^{(p)} + \boldsymbol{a_n}^{(h)}\}$, where $\{\boldsymbol{a_n}^{(h)}\}$ is a solution of the associated **homogeneous** recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Theorem 6 (on particular solution of nonhomogenous rec. relation)

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution:

- The associated **linear homogeneous rec. relation** is
- *A)* $r^2 = 3r + 2$ *B)* $r^2 = 3r$ *C)* $a_n = 3a_{n-1}$ *D)* $a_n = 3a_{n-1} + 2$
- What is its degree?
  - A) 1 B) 2 C)3
- What is its characteristic equation?
  - A) r=3 B) $r^2=3r$
- What are its solutions:
  - A) $a_n^{(h)} = \alpha_1 3^n + \alpha_2 n 3^n$ B) $a_n^{(h)} = \alpha 3^n$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

- **Solution (cont.)**:
- *Let's find a particular solution.*
- $F(n)= 2n$,
- 1 is not a root of char. equation
- *According to Theorem 6:* $a_n^{(p)} = p_1 n + p_0$
- Plug it into rec. relation: $a_n = 3a_{n-1} + 2n$ and get:

  $p_1 n + p_0 = 3(p_1(n-1) + p_0)+ 2n.$

  $(2 + 2p_1)n + (2p_0 - 3p_1) = 0.$

  $2 + 2p_1 = 0$ and $2p_0 - 3p_1 = 0.$

  $p_1 = -1$ and $p_0 = -3/2.$
- Consequently, $\boldsymbol{a_n^{(p)} = -n - 3/2}$ **is a particular solution.**

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.

What is the solution with $a_1 = 3$?

Solution (ending):

- By Theorem 5, all solutions are of the form:

  $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.
- To find the solution with $a_1 = 3$, let $n = 1$ in the above formula.
- Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$.
- Hence, the solution is $\boldsymbol{a_n = -n - 3/2 + (11/6)3^n}$.

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

with $a_0=1$, $a_1=2$

Solution.

- the solution of homogeneous recurrence relation: $$a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n,$$
- particular solution of nonhom. recurrence relation: $$a_n^{(p)} = C \cdot 7^n,$$
- find C $$C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n.$$

$49C = 35C - 6C + 49$, which implies that $20C = 49$,

- solution of nonhom. recurrence relation: $$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$
- find α_1 and α_2
  - from $a_0=1$, $a_1=2$

Match the rec. relations and their solutions

Recurrence relations	Solutions
$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots.. + c_k\, a_{n-k} + F(n)$	a) $a_n = \alpha_1 {r_1}^n + \alpha_2\, {r_2}^n + \ldots + + \alpha_k {r_k}^n$
$a_n = c_1a_{n-1} + c_2a_{n-2}$, char equation $r^2 - c_1r - c_2 = 0$ has one repeated root	b) $a_n = \alpha_1 {r_1}^n + \alpha_2\, {r_2}^n$
$a_n = c_1a_{n-1} + c_2a_{n-2}$ char equation $r^2 - c_1r - c_2 = 0$ has two distinct roots	c) $a_n = \alpha_1 r^n + \alpha_2\, nr^n$
$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots.. + c_k\, a_{n-k}$, char. eq-n $r^k - c_1r^{k-1} - \cdots - c_k = 0$ has k distinct roots	d) $a_n = (\alpha_{1,0} + \alpha_{1,1}n + \cdots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + (\alpha_{2,0} + \alpha_{2,1}n + \cdots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1}n + \cdots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n$
$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots.. + c_k\, a_{n-k}$, char. eq-n $r^k - c_1r^{k-1} - \cdots - c_k = 0$ has t *($t<k$)* distinct roots with multiplicities $m_1, m_2, \ldots, m_t$	e) ${a_n}^{(p)} + {a_n}^{(h)}$

Divide-and-Conquer Algorithms and Recurrence Relations

Section 8.3

Section Summary

- Divide-and-Conquer Algorithms
  - Definition
  - Examples
- Divide-and-Conquer Recurrence Relations
  - Form
  - Master Theorem (important tool for solving rec. relations)

Divide-and-Conquer Algorithmic Paradigm

Definition: A *divide-and-conquer algorithm* works by first *dividing* a problem recursively into smaller size problems and then *conquering* the original problem using the solutions of the smaller problems.

Examples:

- Binary search
- Merge sort

Divide-and-Conquer Recurrence Relations

- Suppose that a recursive algorithm divides a problem of size n into a subproblems.
- Assume each subproblem is of size n/b.
- Suppose $g(n)$ extra operations are needed in the conquer step.
- Then $f(n)$ represents the number of operations to solve a problem of size n satisisfies the following recurrence relation:

$$f(n) = af(n/b) + g(n)$$

- This is called a *divide-and-conquer recurrence relation*.

Example: Binary Search

- Binary search reduces the search for an element (key) in a sequence of size n to the search in a sequence of size $n/2$.
- 5 comparisons are needed to implement this reduction (see slide 43 of Lecture 10).
- Hence, if $f(n)$ is the number of comparisons required to search for an element in a sequence of size n, then

$$f(n) = f(n/2) + 3.$$

Example: Merge Sort

- The merge sort algorithm splits a list of n (assuming n is even) items to be sorted into two lists with $n/2$ items. It uses fewer than n comparisons to merge the two sorted lists.
- Hence, the number of comparisons required to sort a sequence of size n, is no more than than $M(n)$ where

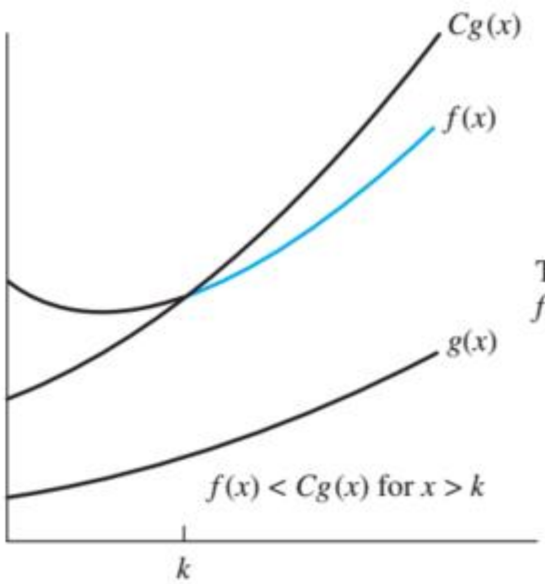
$$M(n) = 2M(n/2) + n.$$

Big-O Notation

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]



The part of the graph of $f(x)$ that satisfies $f(x) < Cg(x)$ is shown in color.

FIGURE 2 **The Function $f(x)$ is $O(g(x))$.**

The term big-O was coined by a German mathematician Paul Bachmann in 1892, and later used by Edmund Landau for number theory, and Donald Knuth for complexity of algorithms.

Match

Functions

1) $f(n)=C$
2) $f(n)= 5n+10$
3) $f(n)= an^2 +bn+c$
4) $f(n)=n^3 + n^2$
5) $f(n)= \log_2 n +3$
6) $f(n)= (n+3) \ln n$
7) $f(n)=2^n + n^3 + n^2$

Sets of functions

a) $O(1)$
b) $O(\log n)$
c) $O(n)$
d) $O(n \log n)$
e) $O(n^2)$
f) $O(n^3)$
g) $O(2^n)$

Estimating the Size of Divide-and-Conquer Functions

Theorem 1: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number.

Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if} \quad a > 1 \\ O(\log n) & \text{if} \quad a = 1. \end{cases}$$

Furthermore, when $a > 1$ and $n = b^k$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2$$

where $C_1 = f(1) + c/(a-1)$ and $C_1 = -c/(a-1)$.

Complexity of Binary Search

Binary Search Example: Give a big-*O* estimate for the number of comparisons used by a binary search.

a) $O(n^{\log_1 2})$

b) $O(n^{\log_2 1})$

c) $O(\log n)$

Complexity of Binary Search

Solution: Since the number of comparisons used by binary search is

$f(n) = f(n/2) + 5$ where n is even,

by Theorem 1, it follows that

$f(n)$ is $O(\log n)$.

Complexity of Merge Sort

Merge Sort Example: Give a big-O estimate for the number of comparisons used by merge sort.

Estimating the Size of Divide-and-conquer Functions (*continued*)

Theorem 2. Master Theorem: Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if} \quad a < b^d, \\ O(n^d \log n) & \text{if} \quad a = b^d, \\ O(n^{\log_b a}) & \text{if} \quad a > b^d. \end{cases}$$

Complexity of Merge Sort

Solution: Since the number of comparisons used by merge sort to sort a list of n elements is determined by

$M(n) = 2M(n/2) + n$,

by the master theorem $M(n)$ is

a) $O(n)$.
b) $O(n \log n)$
c) $O(n^2)$