Math Induction

based on How to Prove It (chapter 6)

- **To prove a goal of the form $\forall\, n \in \mathbb{N}\; P(n)$:**
- First prove the base case:
  - $P(n_0)$ is true
- then prove the induction step:
  - $\forall\, n \in \mathbb{N}\; n > n_0\; (P(n) \rightarrow P(n+1))$.

- Example 6.1.1. Prove that for every natural number n, $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$.
- Base case: $2^0 = 2^{0+1} - 1 = 2 - 1 = 1$ is true
- Induction case:

Givens	*Goal*
$n \in \mathbb{N}$	$2^0 + 2^1 + \cdots + 2^{n+1} = 2^{n+2} - 1$
$2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$	

$$2^0 + 2^1 + \cdots + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1.$$

- Example 6.1.2. Prove that ∀ n ∈ℕ (3 | (n³ – n)).
- Base case: ?
- Induction case:

Givens	*Goal*
$n \in \mathbb{N}$	$\exists j \in \mathbb{Z}(3j = (n+1)^3 - (n+1))$
$\exists k \in \mathbb{Z}(3k = n^3 - n)$	

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= (n^3 - n) + 3n^2 + 3n \\ &= 3k + 3n^2 + 3n \\ &= 3(k + n^2 + n).\end{aligned}$$

Example 6.1.3. Prove that $\forall\, n \geq 5(\, 2^n > n^2)$.

- Base case: $n = 5$.
- Inductive case: assume $2^n > n^2$, and try to prove that $2^{n+1} > (n+1)^2$

Givens	Goal
$2^n > n^2$	$2^{n+1} > (n+1)^2$

- We want to prove the goal: $2^{n+1} > n^2+2n+1$.
- From givens: $2^{n+1} = 2\times 2^n > 2n^2 = n^2+n^2$
- We will prove the goal if we prove that $n^2 > 2n+1$.
- Since $n \geq 5$, it follows that $n^2 \geq 5n = 2n + 3n > 2n + 1$.
- So, we proved the goal!

Strong Induction

- In the induction step of a proof by mathematical induction, we prove that **a natural number has some property based on the assumption that the previous number has the same property.**
- In some cases this assumption isn't strong enough to make the proof work, and we need to assume that **all smaller natural numbers have the property.**

- **To prove a goal of the form $\forall\, n \in \mathbb{N}\; P(n)$:**
  - Prove base step: $P(n_0)$ is true
  - Prove induction step: $\forall\, n \in \mathbb{N},\, n > n_0\, [\,(\forall k < n,\, k \geq n_0, k \in \mathbb{N}\; P(k)) \rightarrow P(n)]$
  - induction step: if every natural number smaller than *n* has the property P, then *n* has the property P.

Theorem 6.4.2. Every integer $n > 1$ is either prime or a product of primes.

Proof (by strong induction):

- base case:
  - $n=2$ is prime
- inductive hypothesis:
  - $\forall n \in \mathbb{N}, k \in \mathbb{N}, k < n[\ k > 1 \rightarrow$ (k is prime $\vee$ k is a product of primes)]
- goal:
  - $\forall n \in \mathbb{N}, n > 1 \rightarrow$ (n is prime $\vee$ n is a product of primes)

- Take arbitrary natural n. If n is prime then the goal is true.
- Suppose n is not prime.
- It means $\exists\, a \in \mathbb{N}\ \exists\, b \in \mathbb{N}\ ((n = ab) \wedge (a < n) \wedge (b < n))$.

- $\exists\, a \in \mathbb{N}\ \exists\, b \in \mathbb{N}(\, n = ab \wedge a < n \wedge b < n)$.
- Since $a < n = ab$, it follows that $b > 1$.
- Similarly $b < n = ab$, it follows that $a > 1$.
- Since $(a < n) \wedge (b < n)$, by inductive hypothesis, each of a and b is either prime or a product of primes. But then since $n = ab$, n is a product of primes.

Theorem 6.4.4. (Well-ordering principle) **Every nonempty set of natural numbers has a smallest element.**

Proof:

- **Goal:** $\forall$ S $\subseteq \mathbb{N}$ (S $\neq \emptyset \rightarrow$ S has a smallest element).
- After letting S be an arbitrary subset of $\mathbb{N}$, we'll prove the contrapositive of the conditional statement: **if S has no smallest element then S is empty (meaning $\forall$n $\in \mathbb{N}$ (n$\notin$ S)).**
- Use strong induction.
- Base case: 0$\notin$ S (if 0 $\in$S, then it is the smallest element of S, since 0 is the smallest natural number).
- Strong induction hypothesis: $\forall$ k < n(k $\notin$ S).
- Goal: n $\notin$ S.
- Proof of the goal: if n$\in$S then n would be the smallest element of S. Since S has no smallest element n$\notin$ S.
- Thus, assuming that S has no smallest element we obtained that S is empty. Therefore if S is not empty, S has a smallest element.

Theorem 6.4.5. $\sqrt{2}$ **is irrational.**

- Proof is based on well-ordering principle.
- (See textbook How To Prove It, section 6.4)