CS 112 – Introduction to Computing II

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Lecture

Refinements to Adversary Search: Alpha-Beta Pruning;

[Same Reading]

Computer Science

Min Max Trees: From Last Time.....

The Algorithm:

(1) Generate the tree (more on this later!);

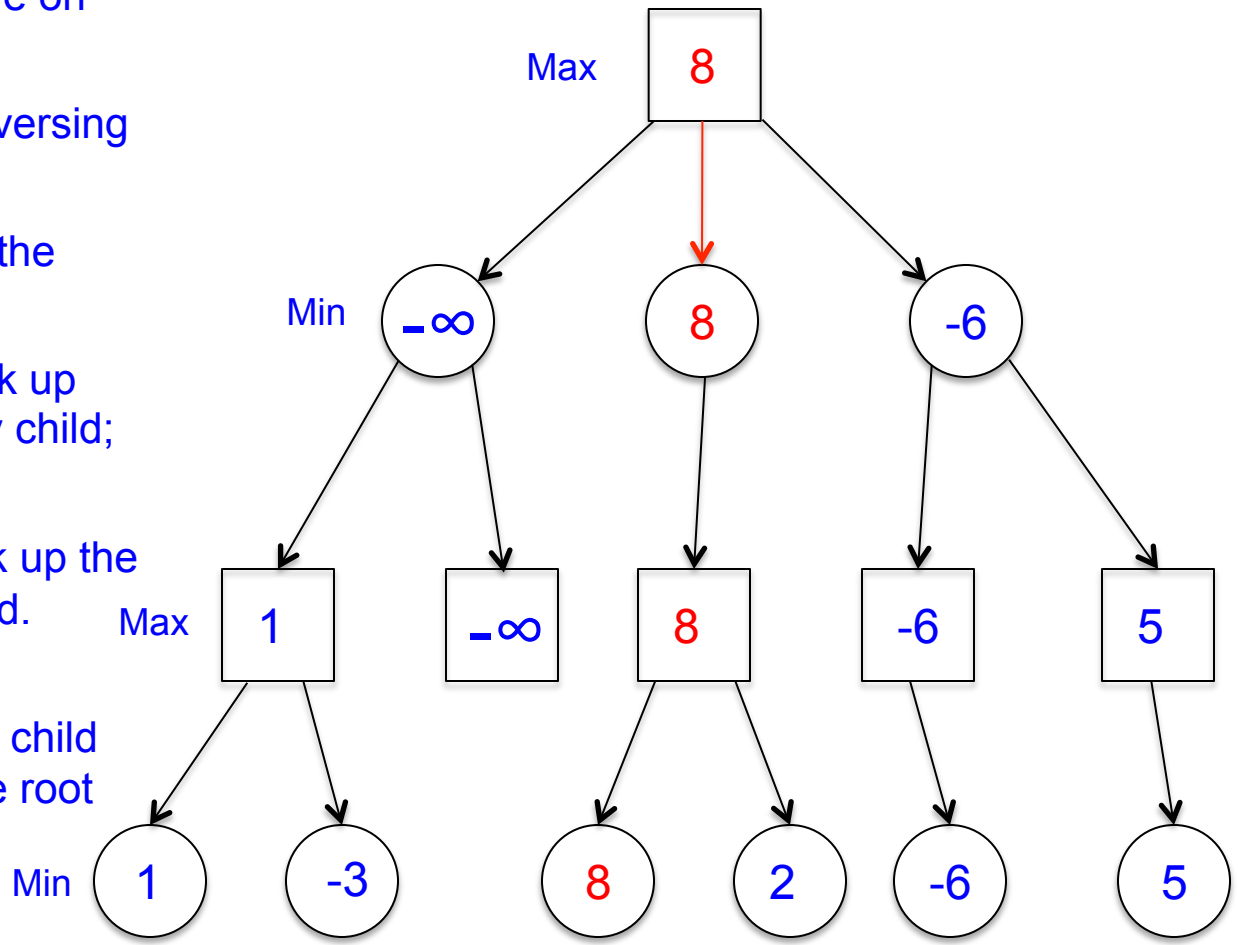
(2) Label the nodes by traversing the tree post-order and:

(A) At leaf nodes, use the eval() function;

(B) At Max nodes, back up the maximum value of any child; and

(C) At Min nodes, back up the minimum value of any child.

(3) Choose the move that corresponds to the largest child of the root (which gave the root its value).

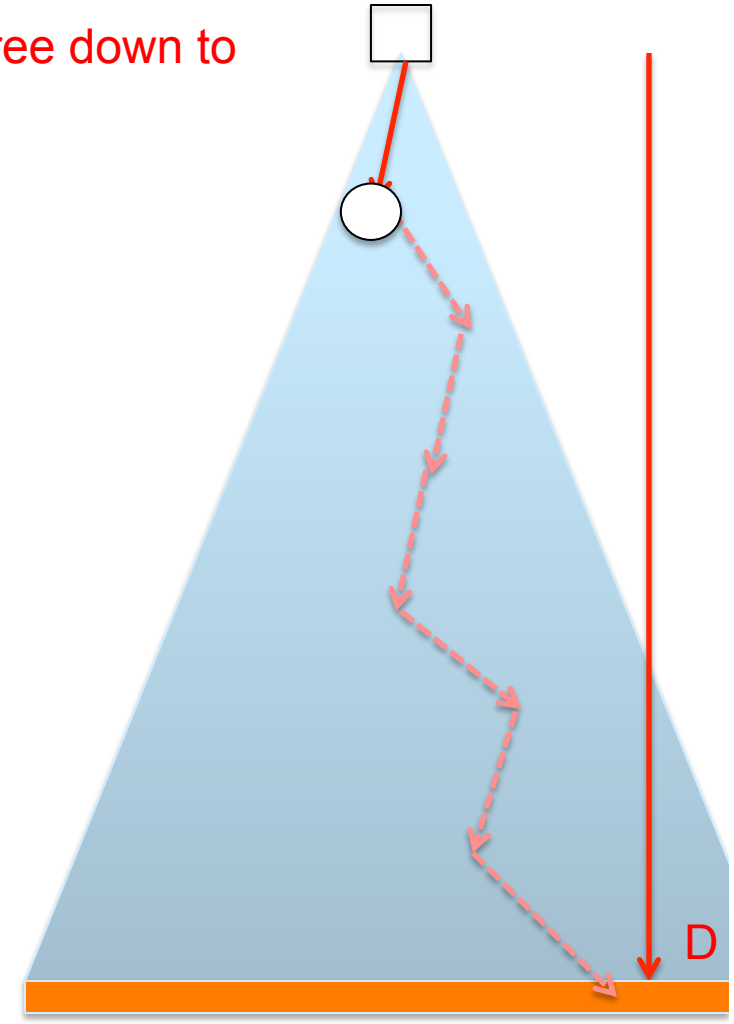


Min-Max Trees: From Last Time.....

Depth-bounded post-order traversal of a min-max tree down to some fixed depth D:

```
Move chooseMove(Node t) {
     int max = -Inf;      Move best;
     for(each move m to a child c of t) {
        int val = minMax( c, 1 );
        if(val > max) {
           best = m; max = val;
     }  }
     return best;
}

int minMax(Node t, int depth) {
  if( t is a leaf node || depth == D)
    return eval(t);
  else if( t is max node ) {
     int val = -Inf;
     for(each child c of t)
       val = max(val, minMax( c, depth+1 );
     return val;
  } else {                        // is a min node
     int val = Inf;
     for(each child c of t)
       val = min(val, minMax( c, depth+1 );
     return val;
}  }
```



What are my next two questions?

One: What's wrong with this (if anything)?

Two: How can we make it more efficient?

How can we "prune" the search space so that we find the best paths faster and eliminate useless nodes?

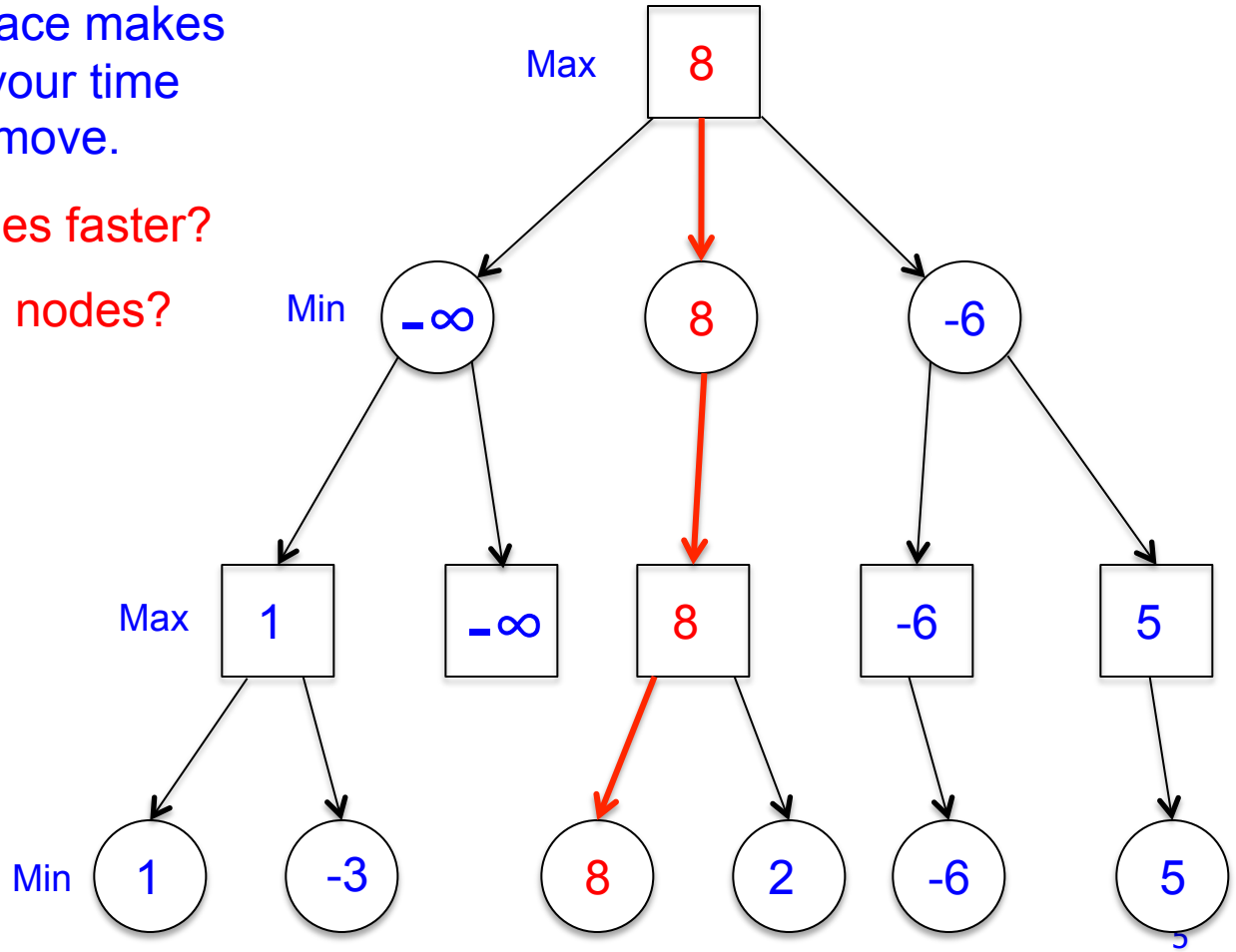
Refinements to Min-Max Search

Main Problem:

You have only a limited amount of time to search, and the combinatorial explosion of the search space makes it imperative that you use your time effectively to find the best move.

How can we find good nodes faster?

How can we avoid useless nodes?



Refinements to Min-Max Search

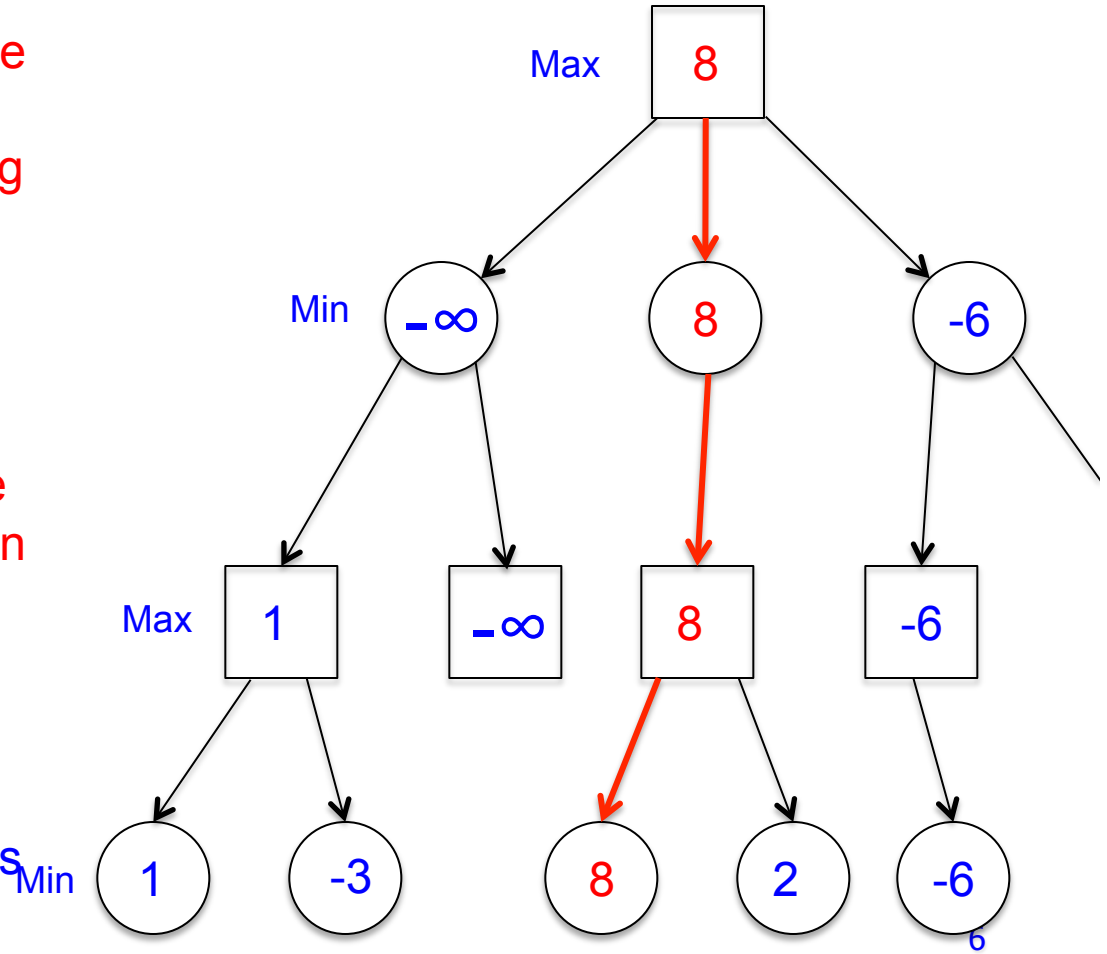
How to Find Good Nodes Faster

Basic Idea: We can apply best-first search to the Min-Max tree:

Instead of searching children in some random order, apply eval() to each child, and search them in descending order of value.

So: generate all children, sort (descending order at Max nodes, ascending order at Min nodes), then search in order. You could cut off the search after some number of children (e.g., just search half, or use a threshold value).

Note: Since we applying eval() to all nodes, perhaps we want to create a simpler, more efficient version for this part of the search.

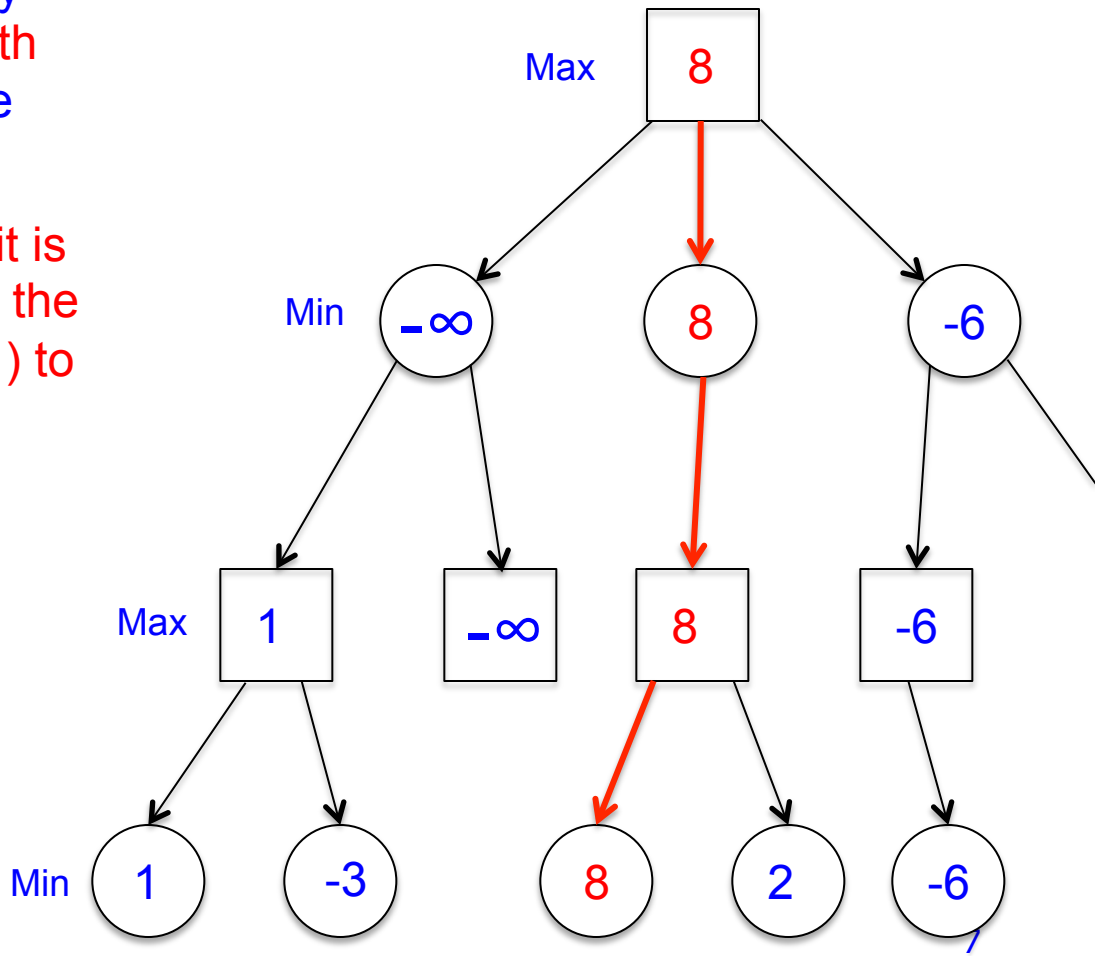


Refinements to Min-Max Search

How can we avoid useless nodes?

Some of the nodes you examine might be completely useless – they will NEVER be on the intended path which determines the choice of the next move.

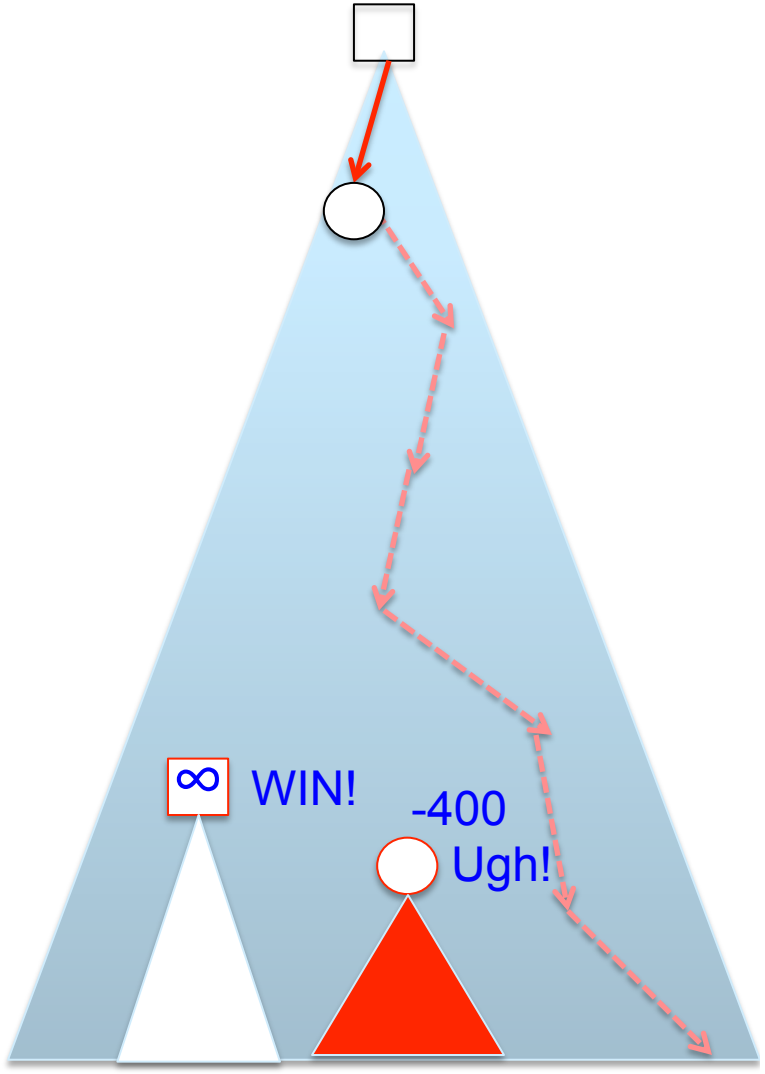
The intended path is easy to see: it is the path the root's value took from the leaf (where it was create by eval()) to the top of the tree.



How can we avoid useless nodes?

Some of the nodes you examine might be completely useless – they will NEVER be on the intended path which determines the choice of the best next move. Why?

a) Clearly bad for you (e.g., you sacrifice your Queen to capture a Pawn!) or a WIN for you



Refinements to Min-Max Search: Pruning the search space

Computer Science

How can we avoid useless nodes?

Some of the nodes you examine might be completely useless – they will NEVER be on the intended path which determines the choice of the best next move.

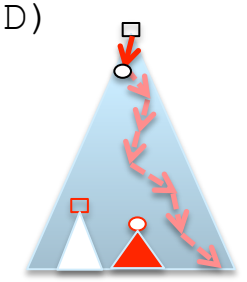
a) Clearly bad for you (e.g., you lose your Queen) or a WIN for you.

Solution: Apply Eval() during search and set thresholds at which you stop searching below that node.

Need separate thresholds for Min and Max, or just reverse them by multiplying by -1.

```
final int MAX_THRESHOLD = Infinity;
final int MIN_THRESHOLD = -300;

int minMax(Node t, int depth) {
  int e = eval(t);
  if( t is a leaf node (no moves)
   || depth == D)
     return e;
else if( t is max node ) {
   if(   e >= MAX_THRESHOLD
      || e <= MIN_THRESHOLD)
      return e;
 ....
}
else if( t is min node ) {
   if(   e <= -MAX_THRESHOLD
      || e >= -MIN_THRESHOLD)
      return e;
 ....
}
```



How can we avoid useless nodes?

Some of the nodes you examine might be completely useless – they will NEVER be on the intended path which determines the choice of the best next move.

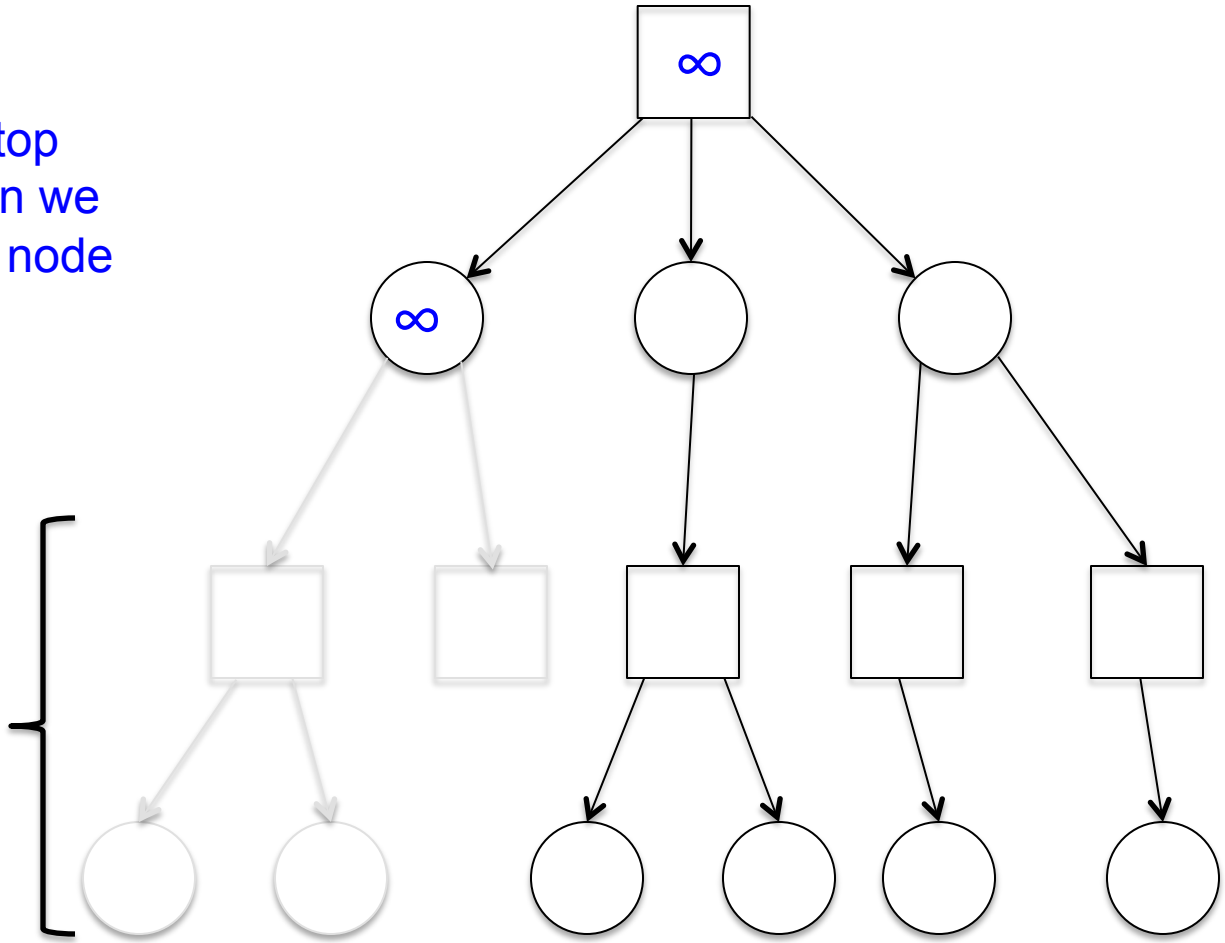
a) Clearly bad for you (e.g., you lose your Queen) or a WIN for you. Solution: Use eval to filter out "useless nodes."

b) Some nodes are useless because we have already found "good enough" nodes in other parts of the tree. This is called **α-β-Pruning**

α-β-Pruning: Examples

We have seen that we can stop searching below a node when we have found a "good enough" node such as a Win:

Why search these nodes?



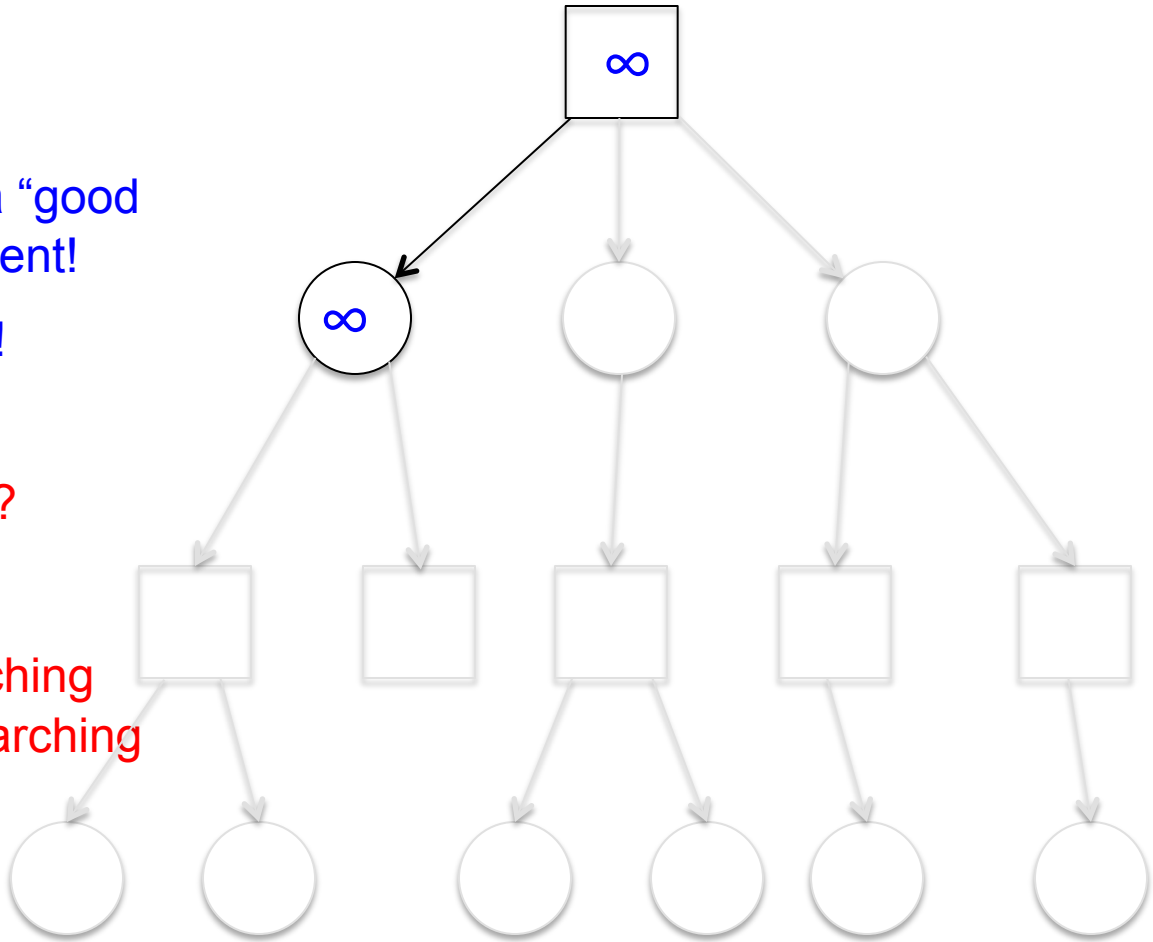
α-β-Pruning: Examples

But then why search any of the siblings? You've already found a "good enough" node to pass to the parent!

The siblings are useless as well!

Why search any of these nodes?

So, if you find a Win when searching the children left to right, stop searching and return Infinity!

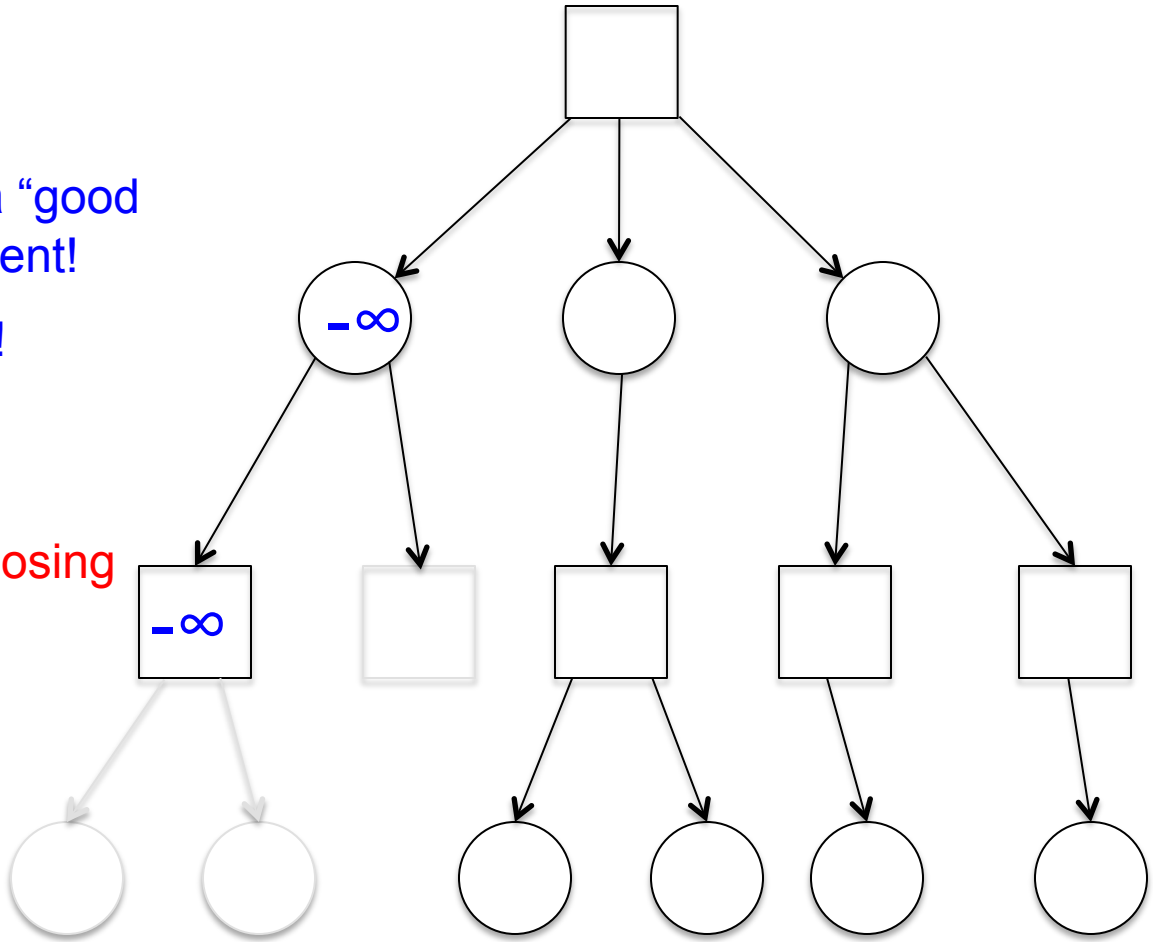


α-β-Pruning: Examples

But then why search any of the siblings? You've already found a "good enough" node to pass to the parent!

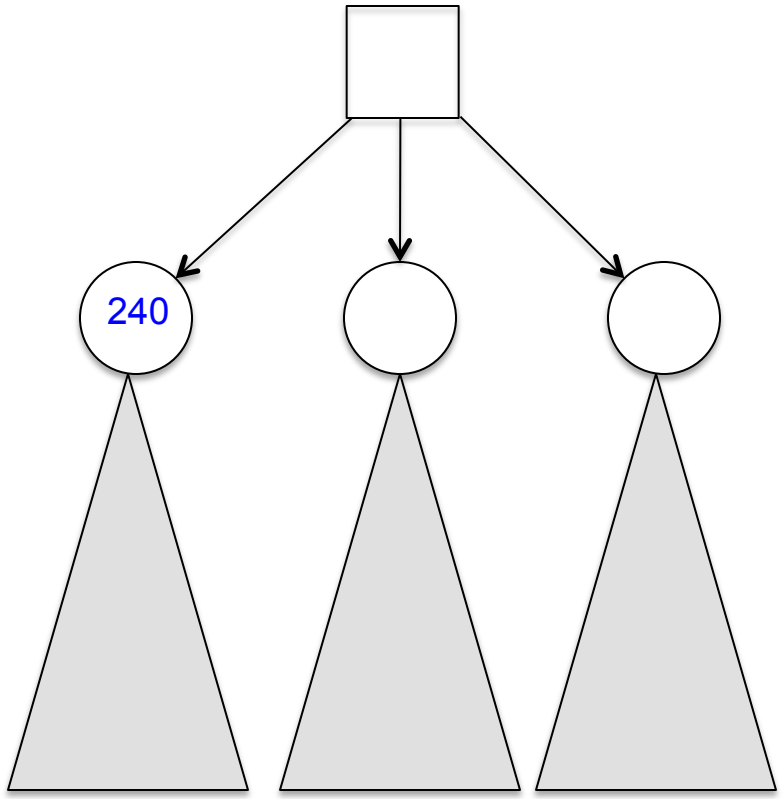
The siblings are useless as well!

A similar argument works at Minimizing nodes: if you find a Losing node, stop (that's the one your opponent will choose!)



α-β-Pruning: Examples

The general idea is: If you can not possible improve on the best value you've found so far going post-order through the children, STOP!



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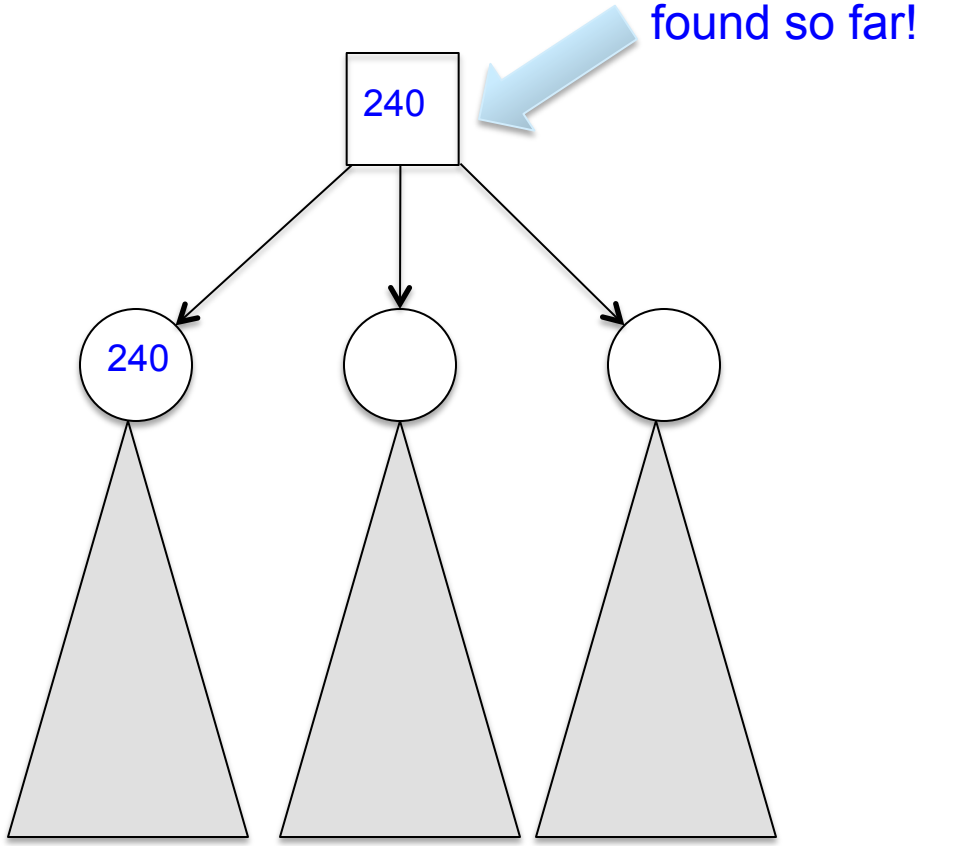

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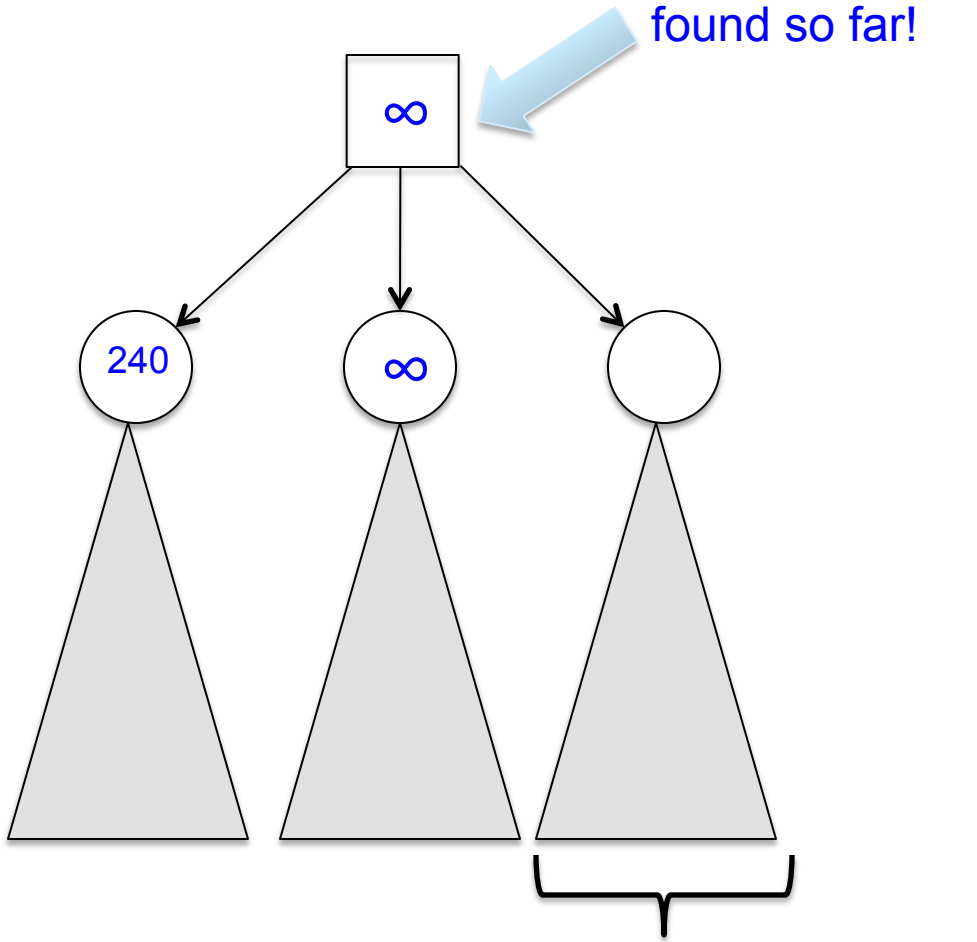

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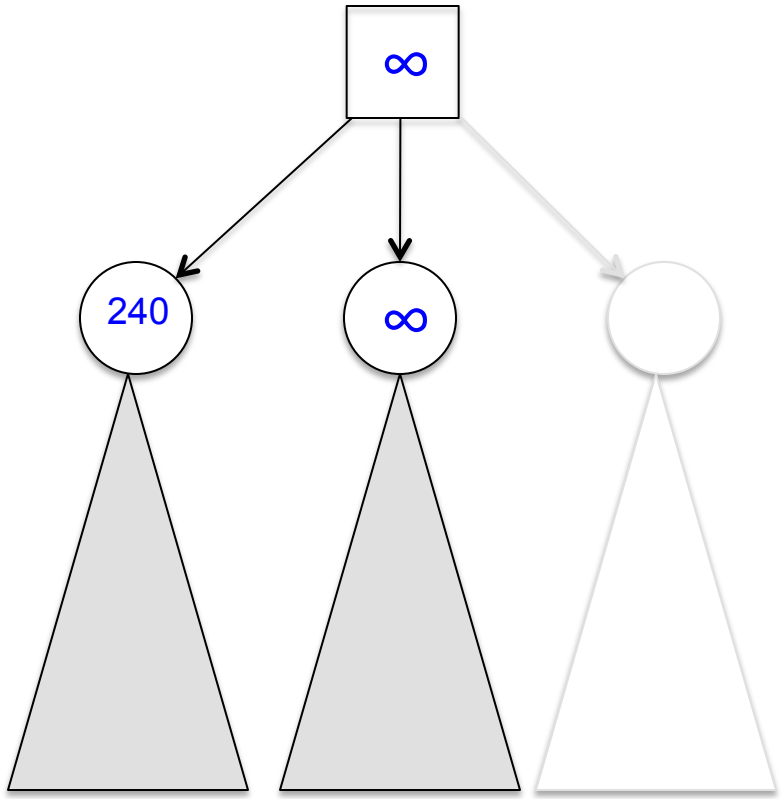


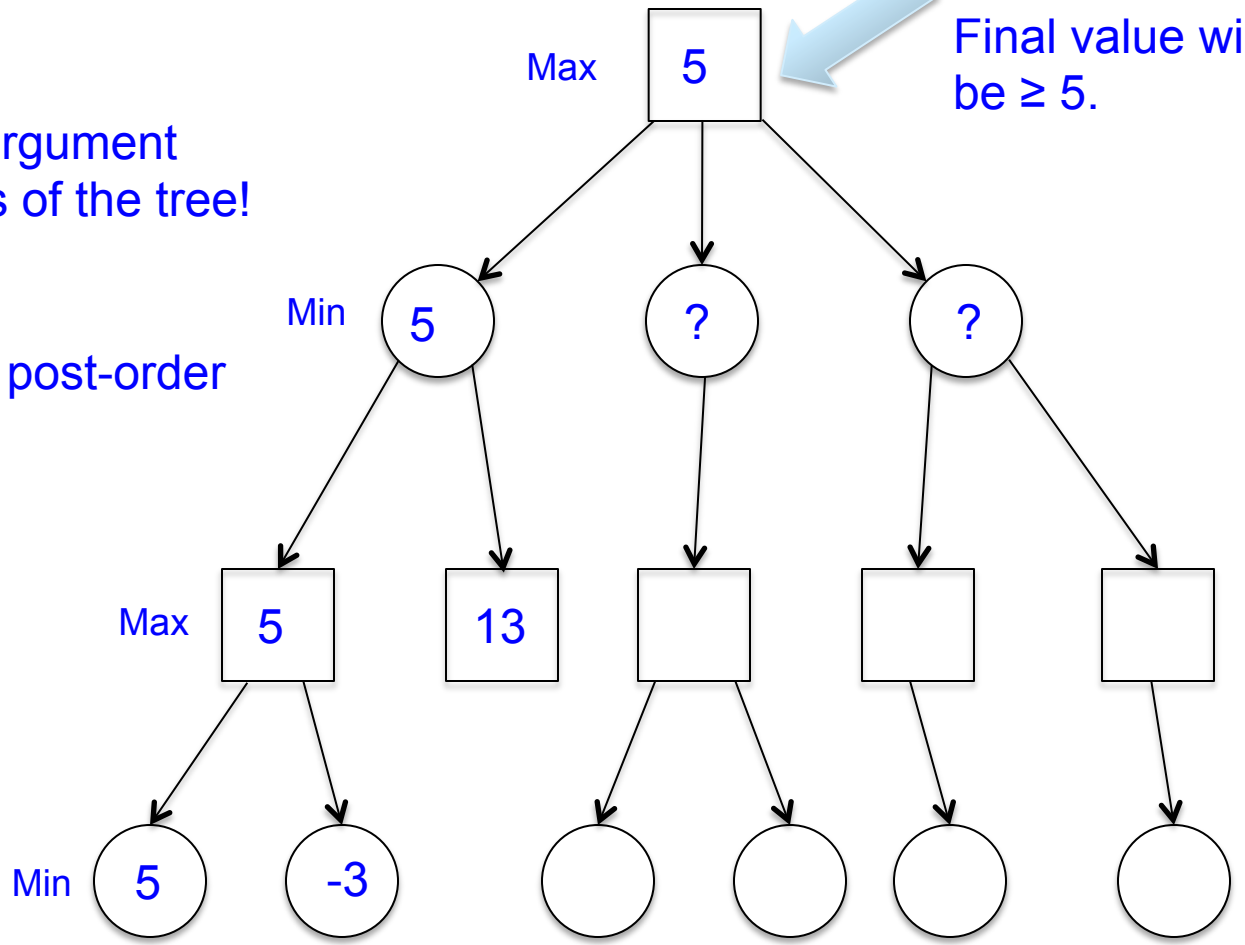
α-β-Pruning: Examples

The best value found so far! Final value will be ≥ 5.

But sometimes this kind of argument works among different levels of the tree!

Suppose we start searching post-order in this tree...

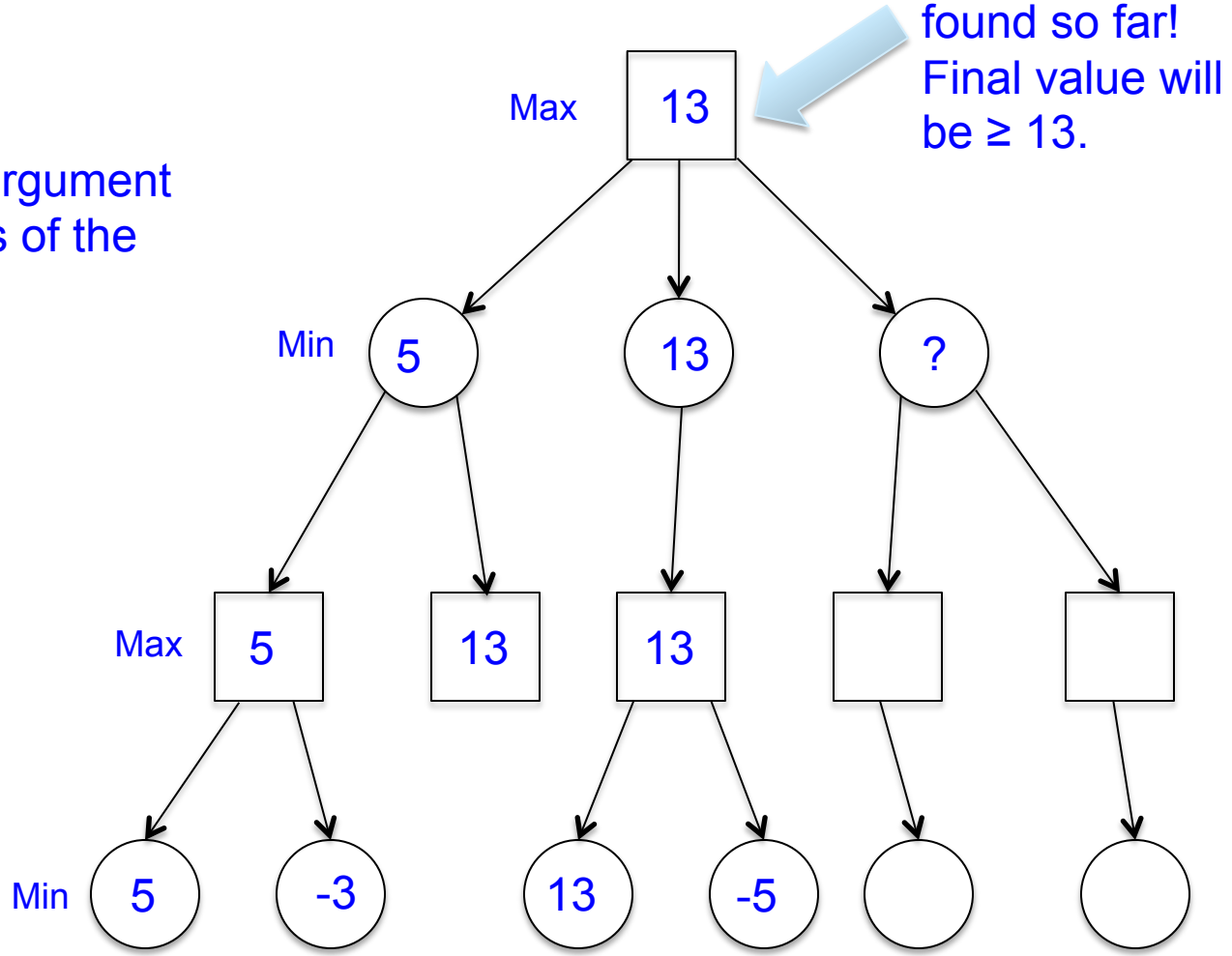
The root "knows" he can get at least a 5 from the left-most child, but what if a value > 5 can be found in the rest of the children? Keep searching!



α-β-Pruning: Examples

But sometimes this kind of argument works among different levels of the tree!

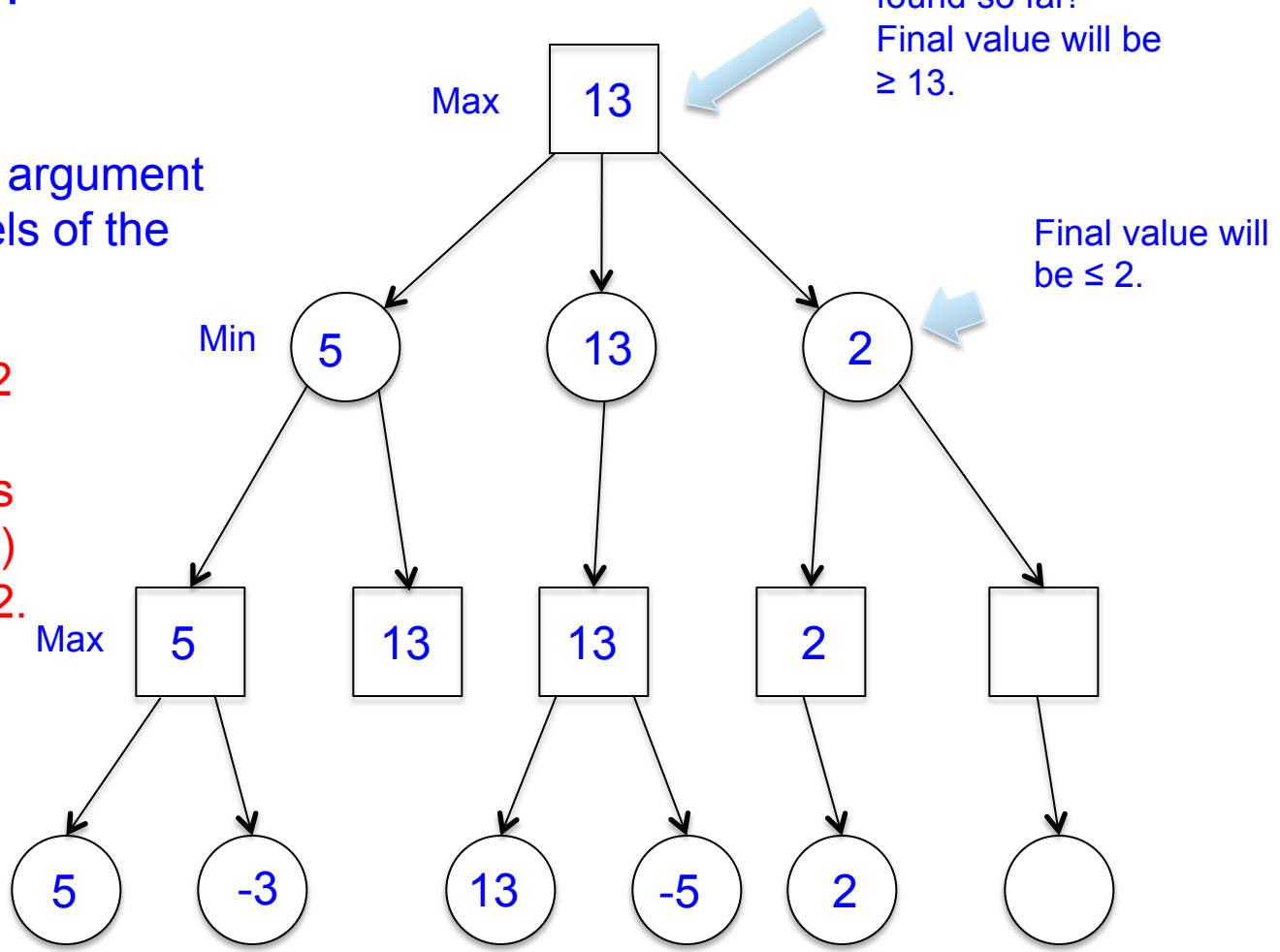
YES! Found a 13. Can a value > 13 be found in the remaining child? Keep searching.....



α-β-Pruning: Examples

But sometimes this kind of argument works among different levels of the tree!

At the next level down, a 2 has been backed up to a "grandparent". This means the right-most (minimizing) child knows he can get ≤ 2.

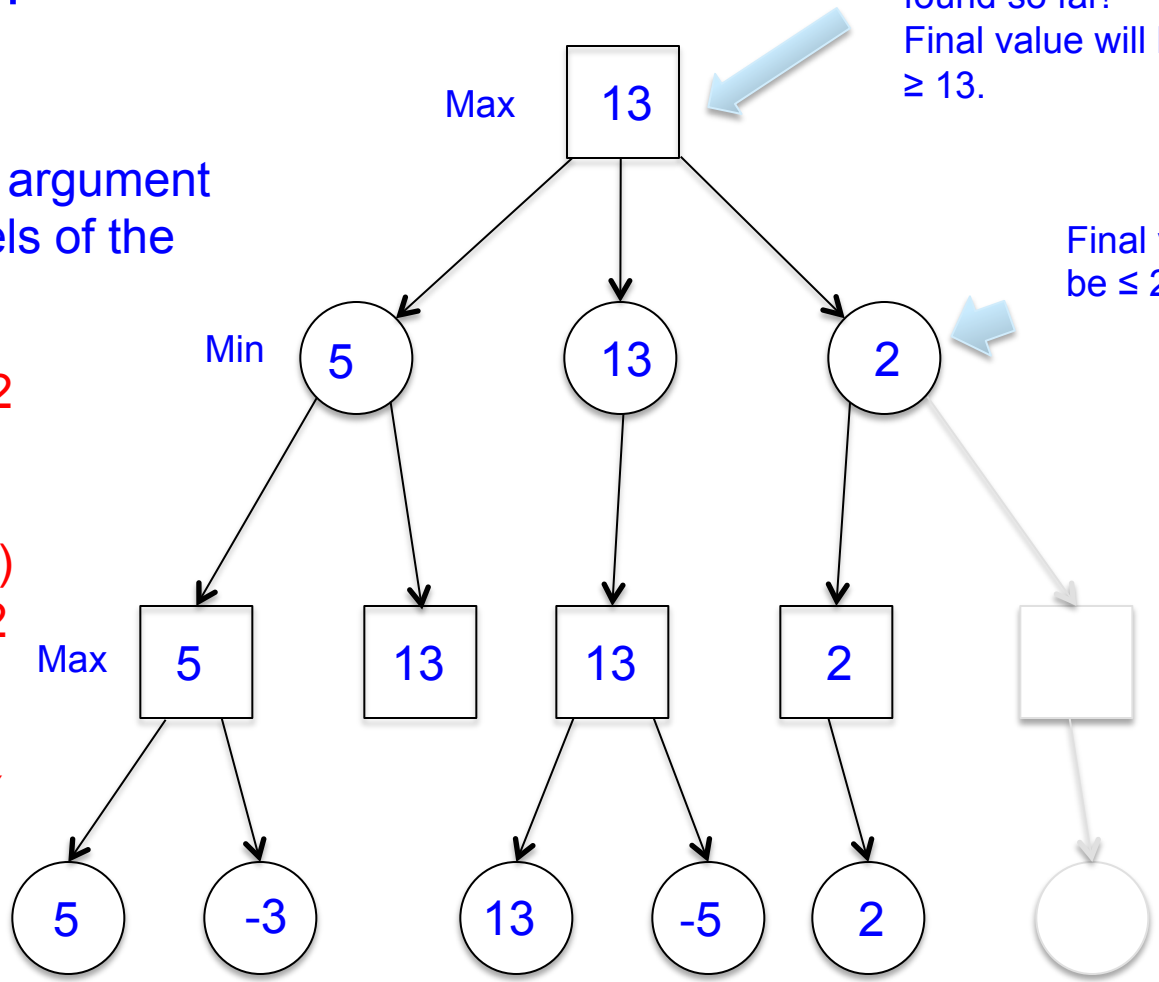


α-β-Pruning: Examples

But sometimes this kind of argument works among different levels of the tree!

At the next level down, a 2 has been backed up to a "grandchild". This means the right most (minimizing) child "knows" he can get 2 or less.

But then there is NO WAY any value from the right-most child can "beat" the 13. STOP!



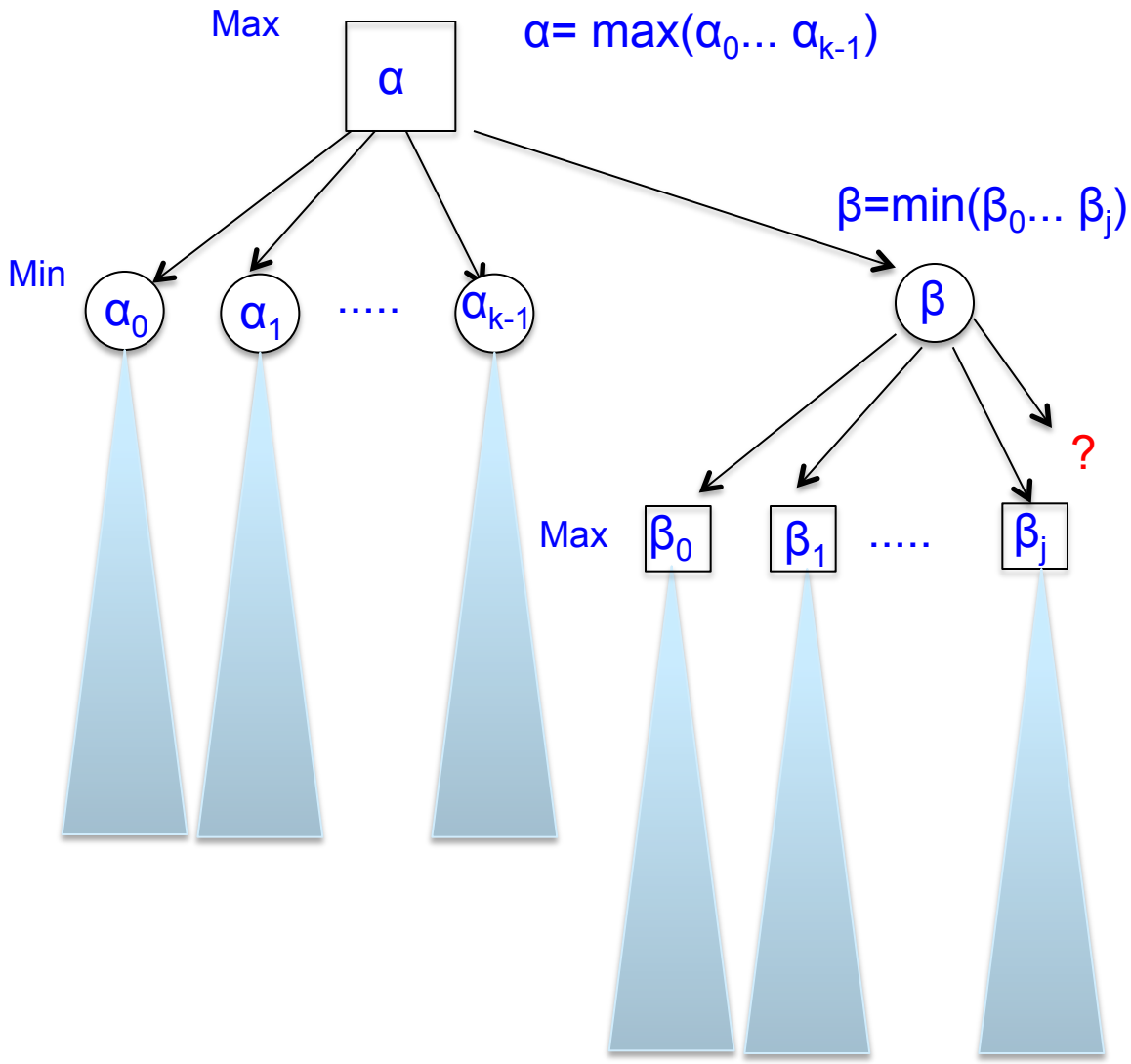
α-β-Pruning: Basic Principles

Among adjacent levels, then, we could generalize this as follows for post-order traversal below a Max node:

Keep track of the max value α among the children of all Max nodes, and the min value β among the children of all Min nodes. These are the "best-so-far" values.

If the best-so-far value of a grandchild is less than a Max nodes best-so-far value, i.e.,

if(β < α) STOP!



α-β-Pruning: Basic Principles

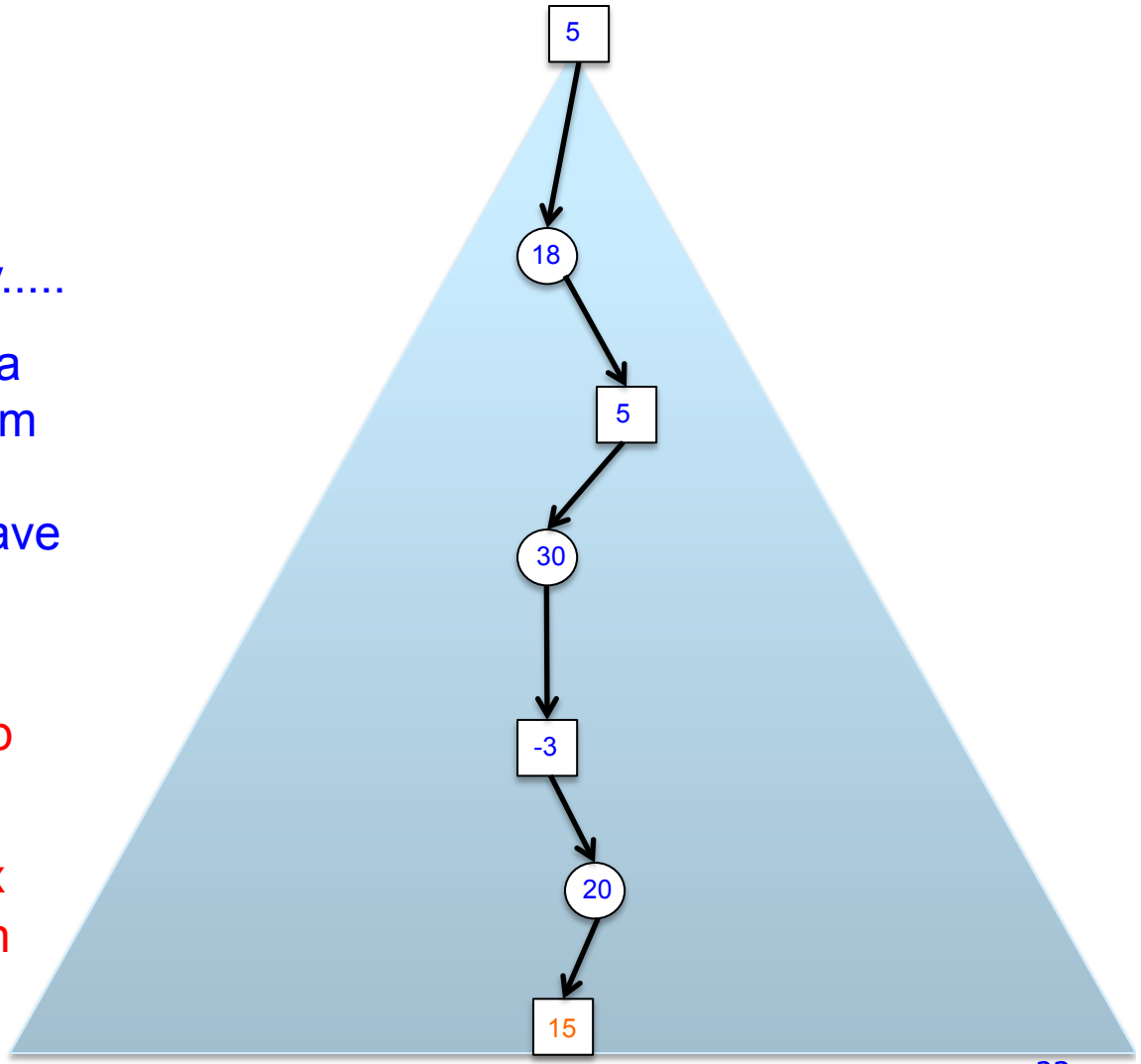
Way Complicated, right?

Not really..... let's look at it this way.....

If we could prove that a value K in a node can NEVER make its way from the leaf (where it got generated by Eval()) to the root, then we don't have to evaluate that node or any nodes below it.

How does a value K survive the trip root-ward?

It has to be the largest child at Max nodes and the smallest child at Min nodes.



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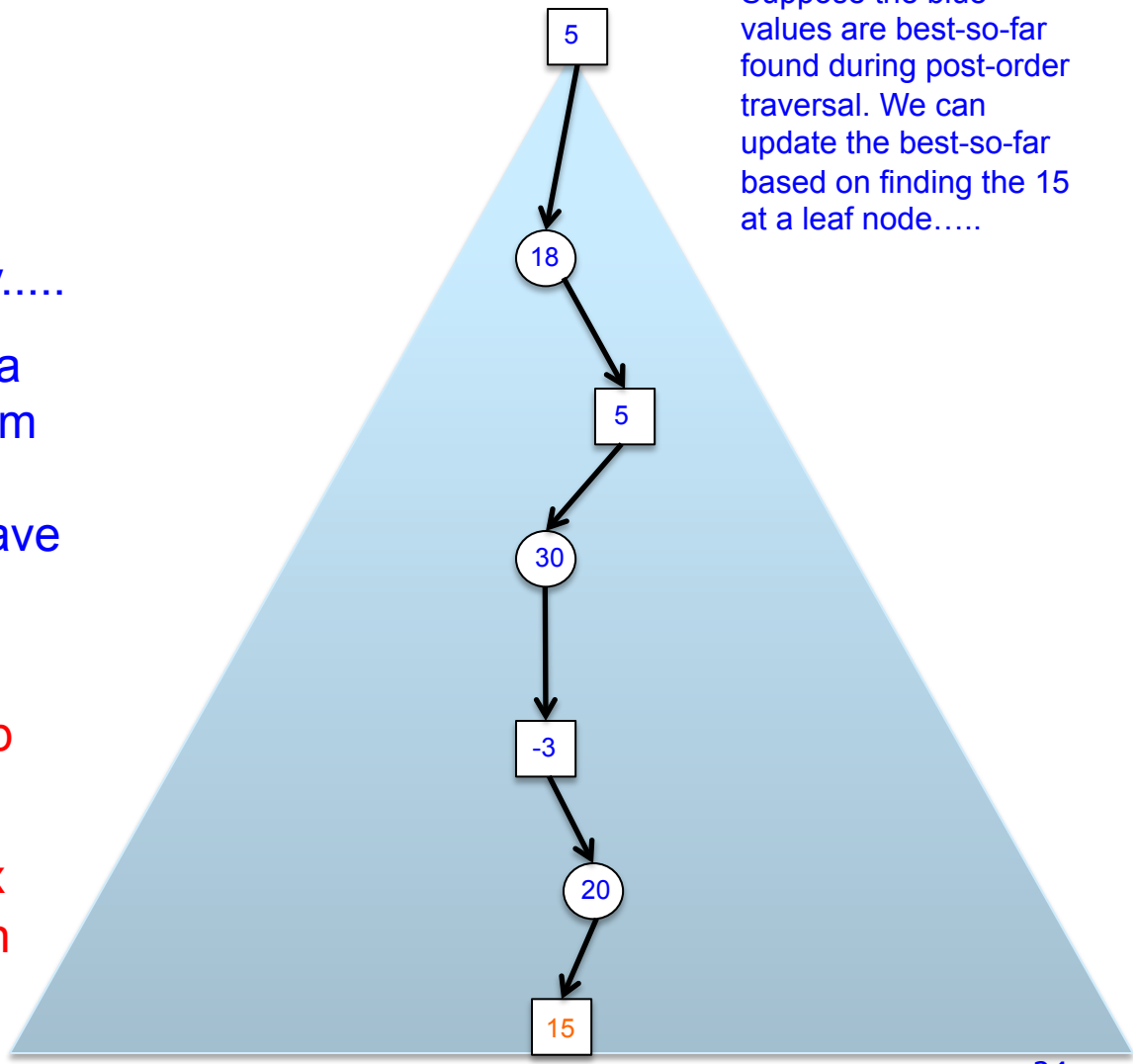

Suppose the blue values are best-so-far found during post-order traversal. We can update the best-so-far based on finding the 15 at a leaf node…..

α-β-Pruning: Basic Principles

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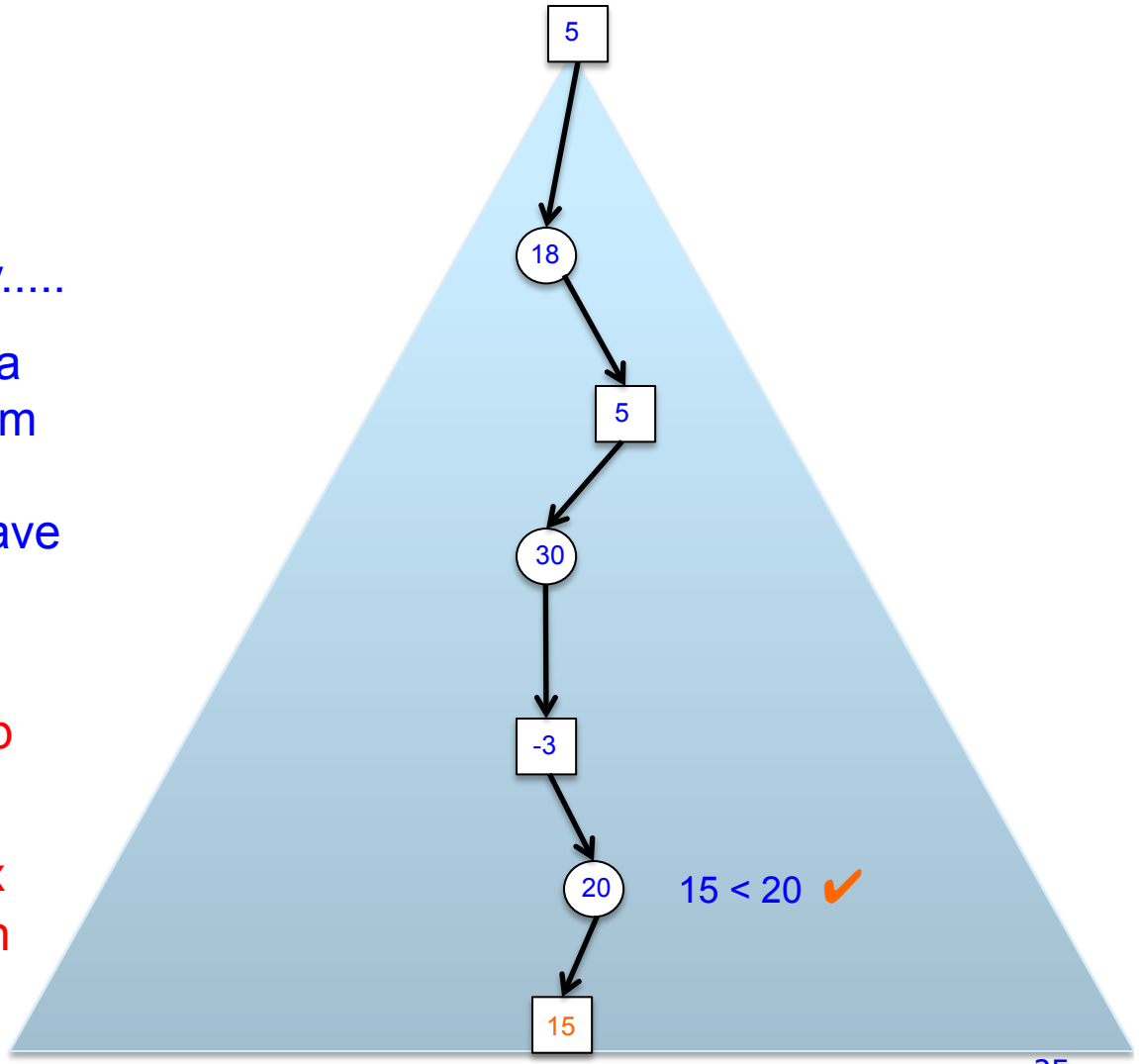

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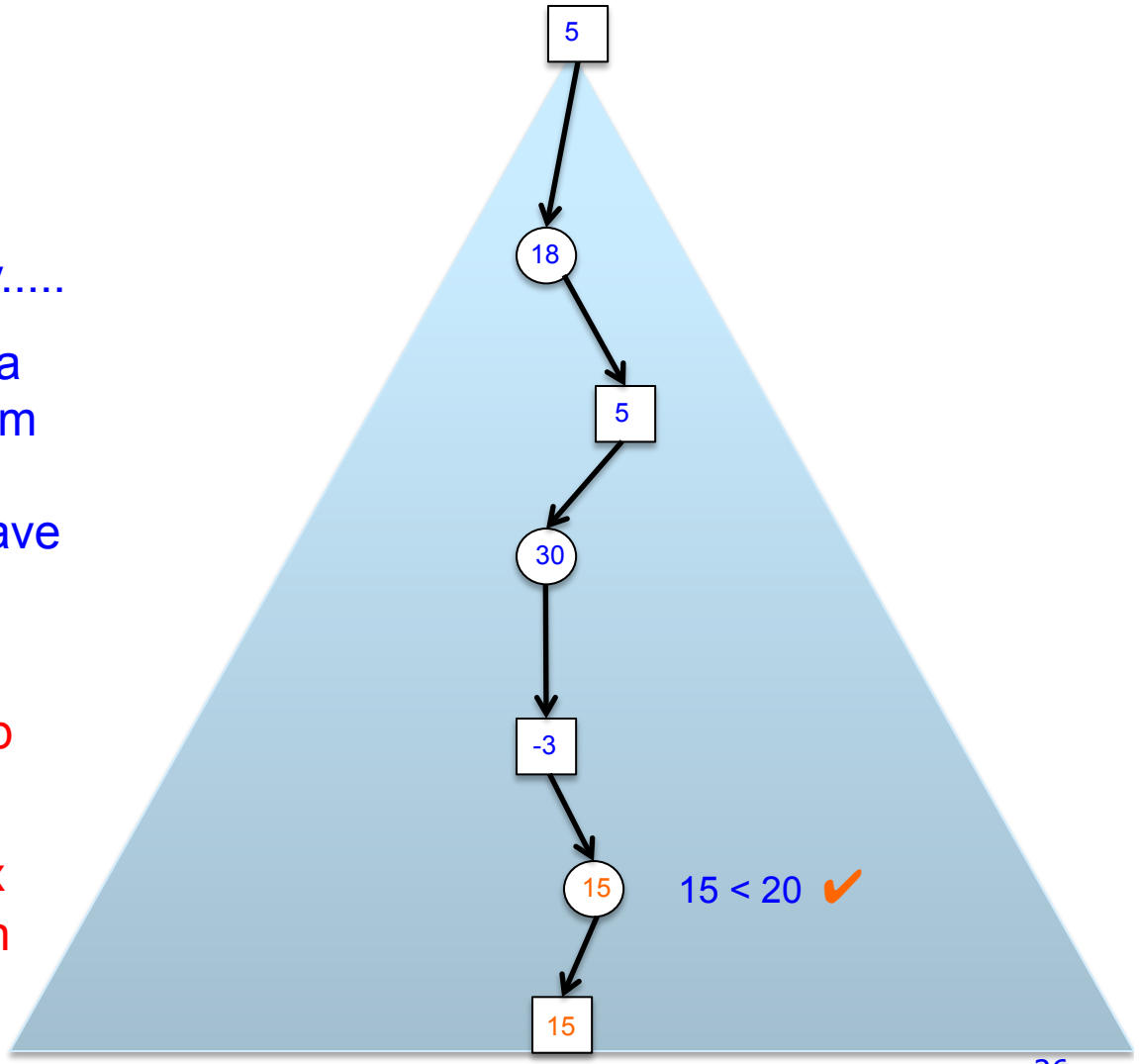

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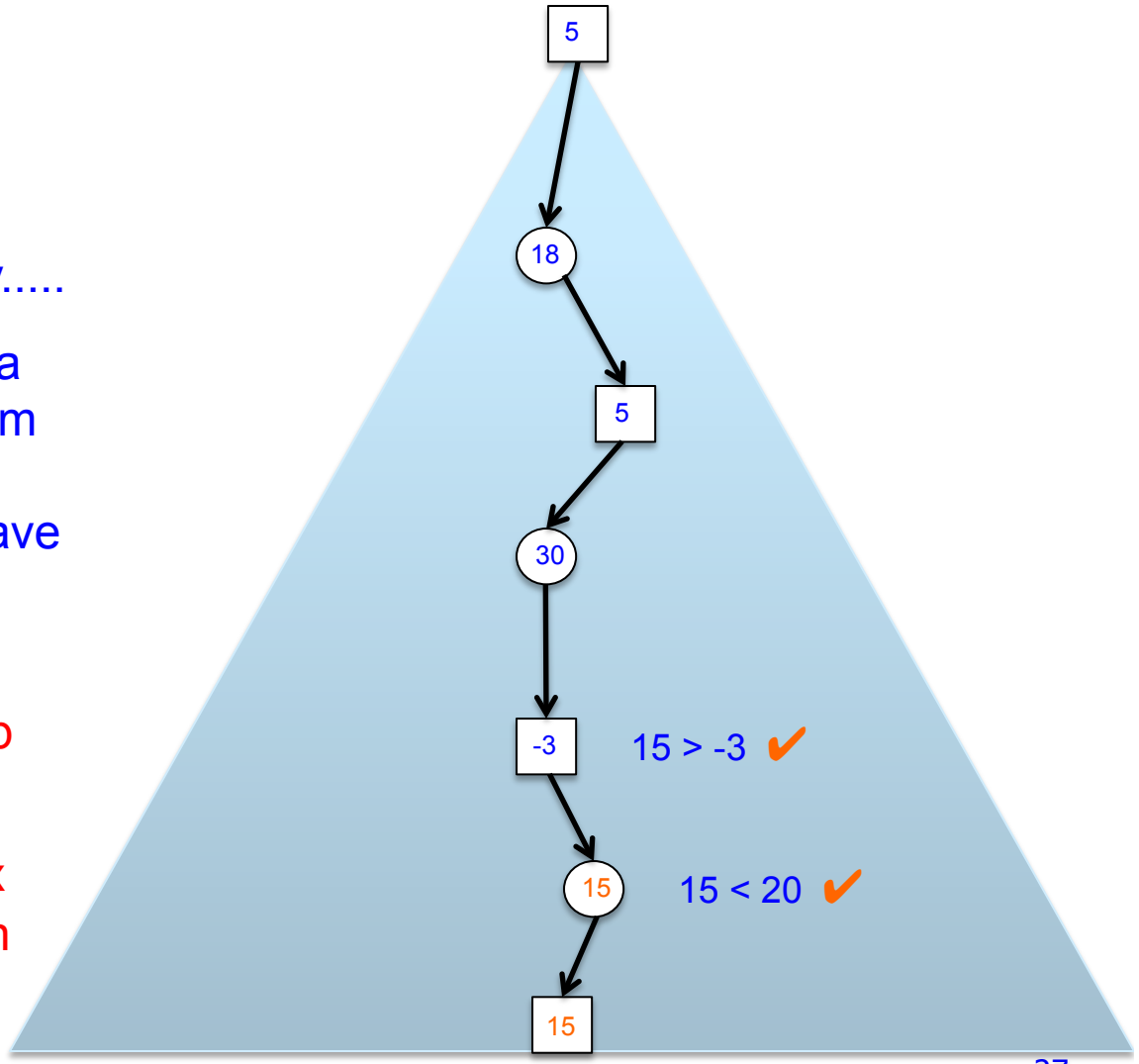

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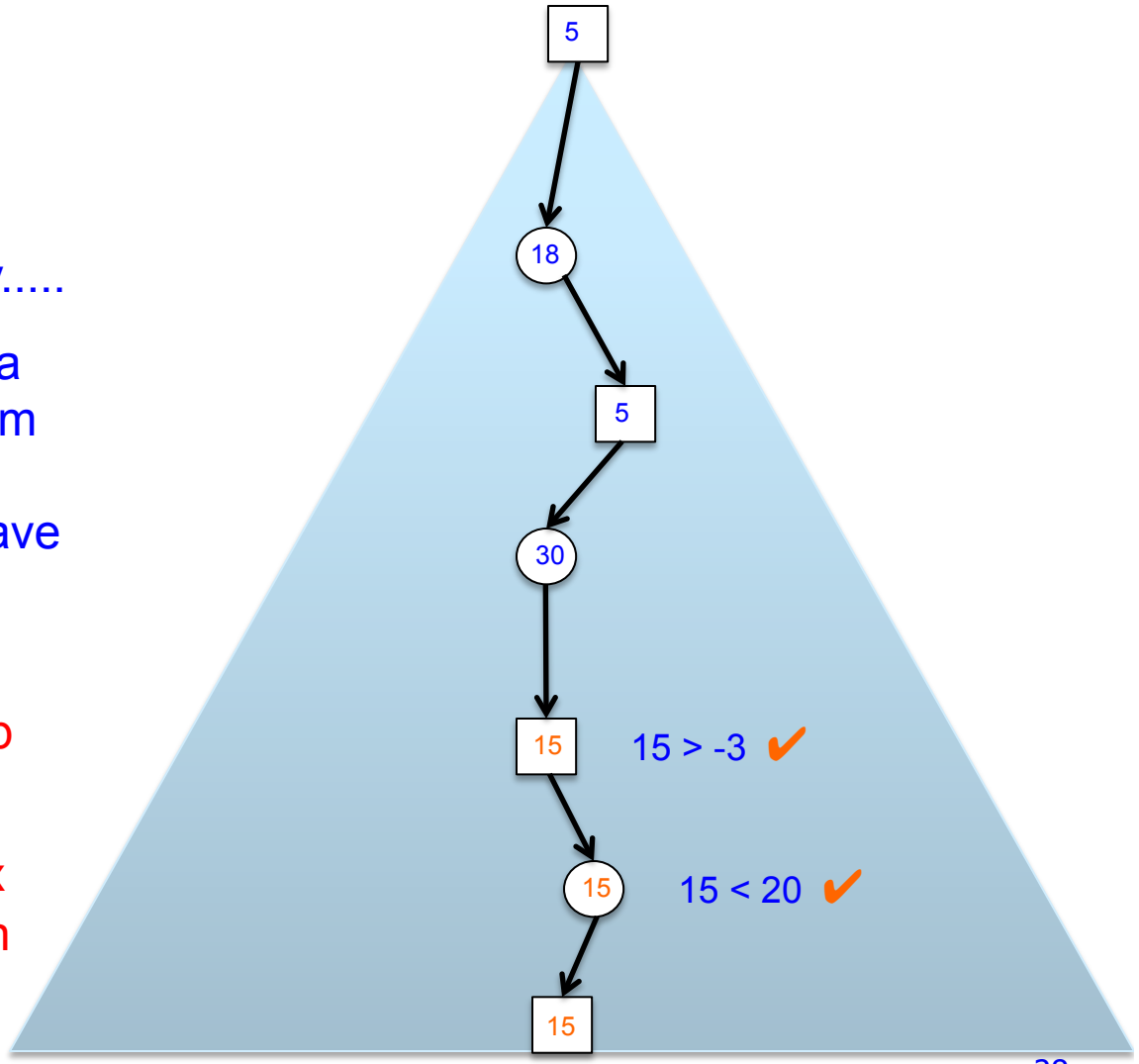

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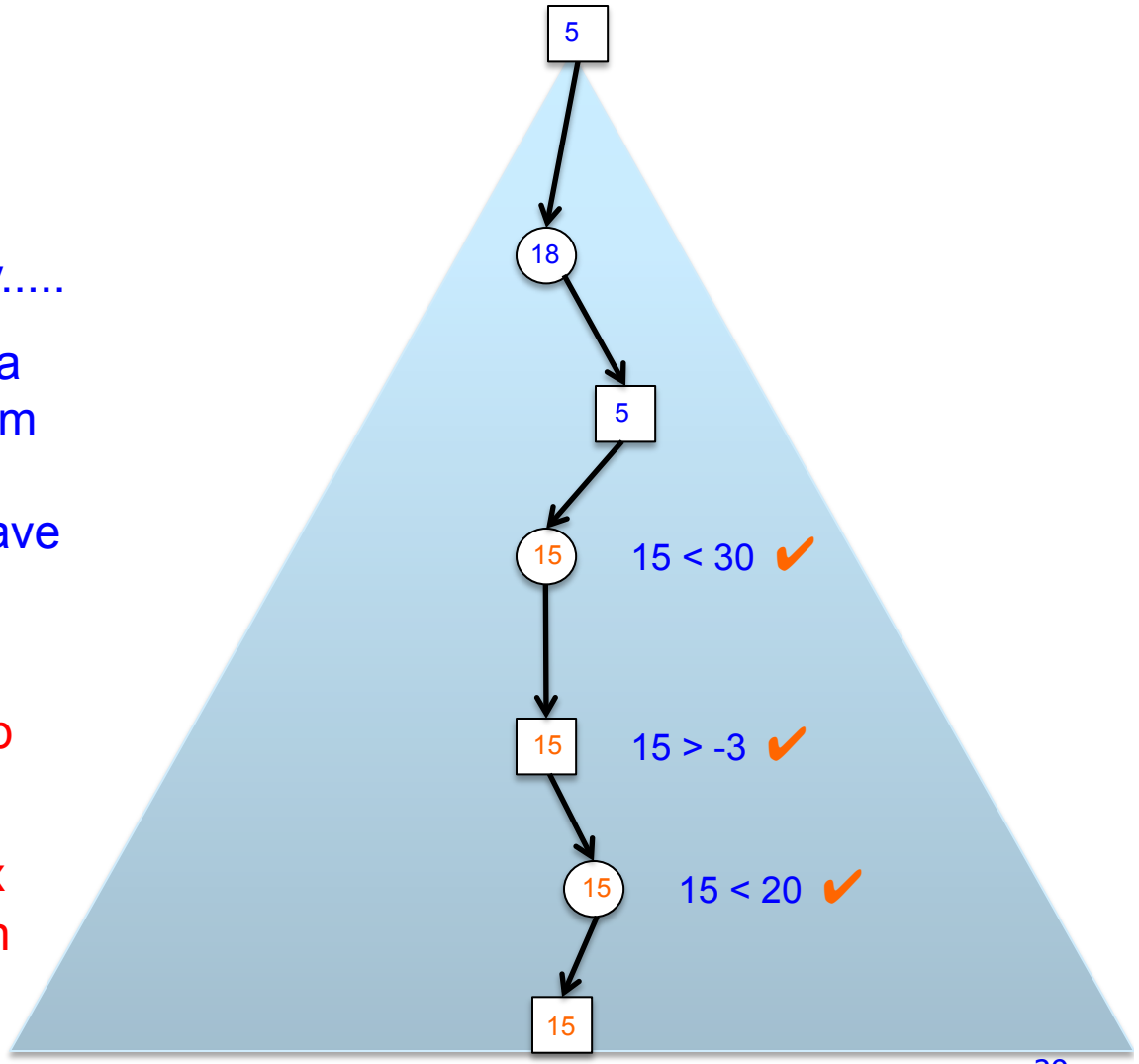

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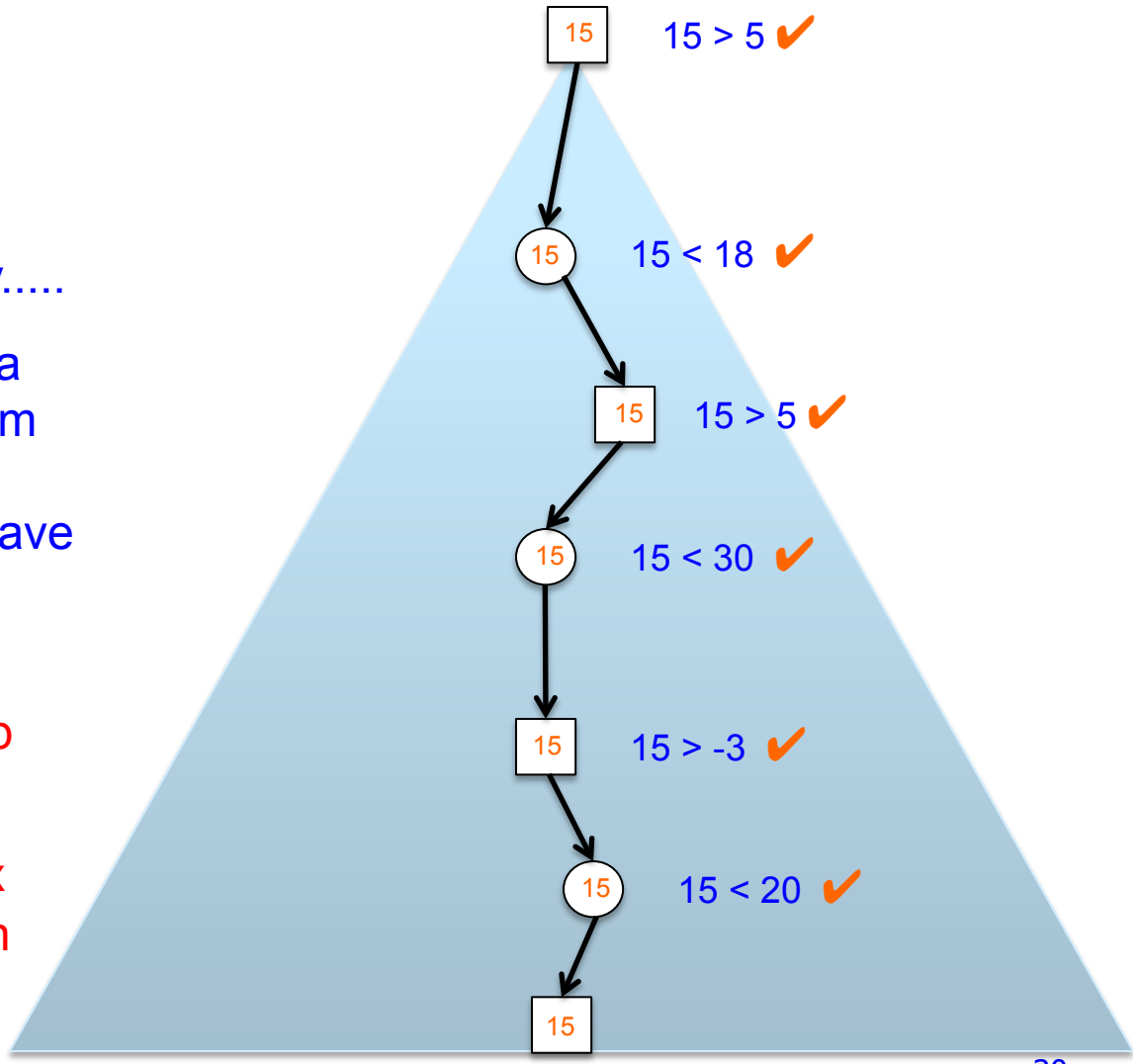


α-β-Pruning: Basic Principles

So, here is the simpler way to look at it:

15 > every value at Max nodes

is same as

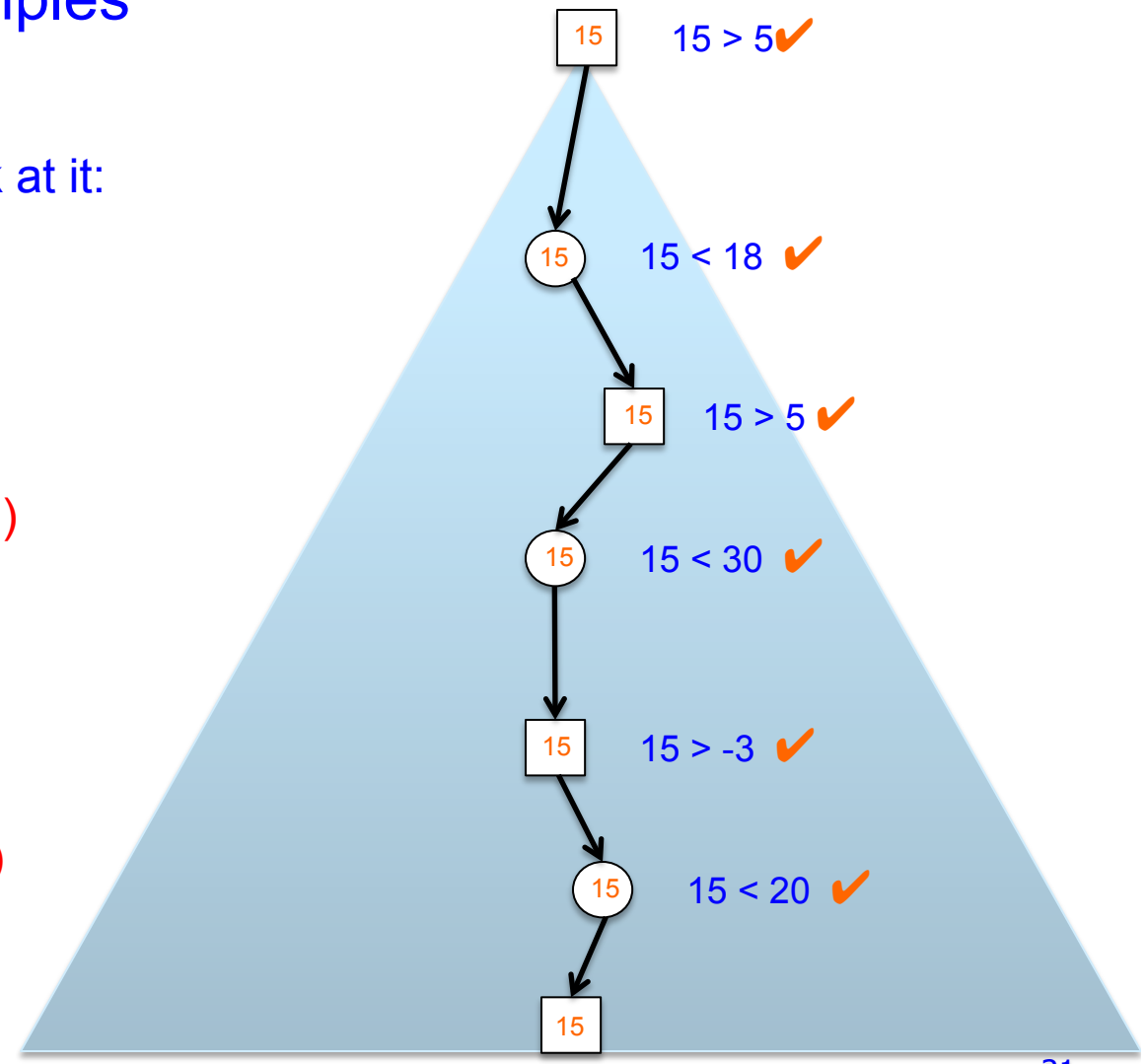
15 > max(all values at Max nodes)

and

15 < every value at Min nodes

is same as

15 < min(all values at Min nodes)

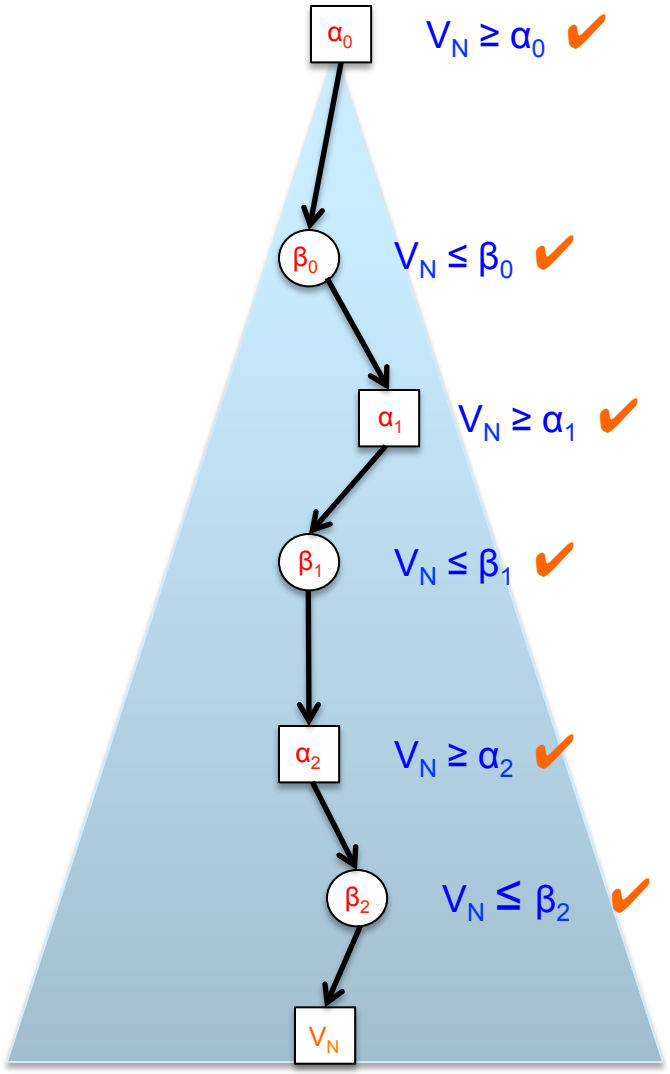


α-β-Pruning: Basic Principles

So, here is the simpler way to look at it:

Suppose

- N is a node we are visiting, with a value calculated as V_N;
- $\alpha_0 \ldots \alpha_k$ is the set of values calculated at Max nodes on the path from N to root;
- $\beta_0 \ldots \beta_j$ is the set of values calculated at Min nodes on the path from N to root;



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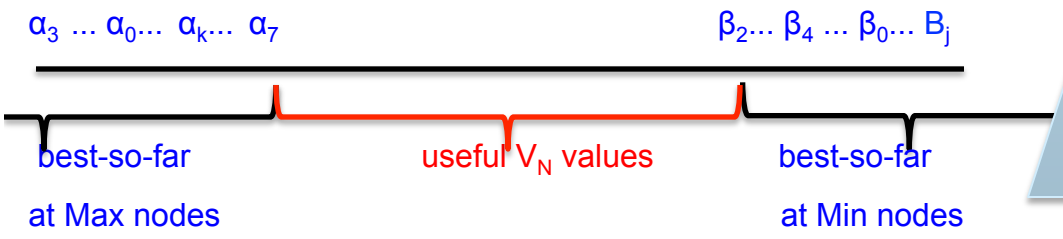
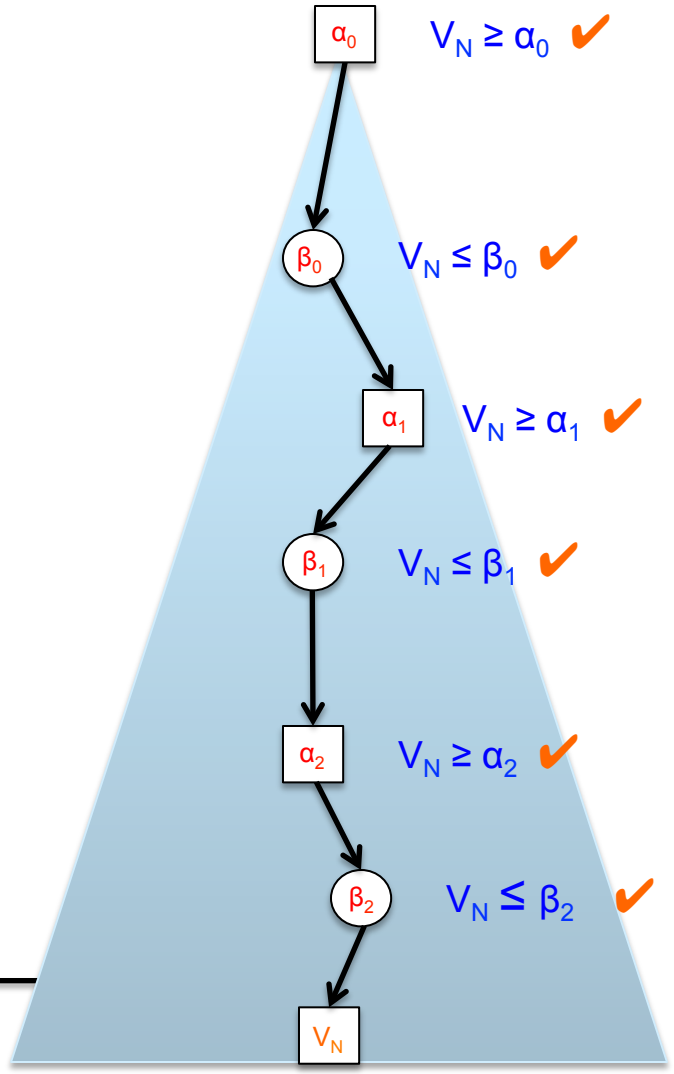
Then N is only **useful** if

$\max(\alpha_0 \ldots \alpha_k) \le V_N \le \min(\beta_0 \ldots \beta_j)$





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- $\alpha_0 \ldots \alpha_k$ is the set of values calculated at Max nodes on the path from N to root;
- $\beta_0 \ldots \beta_j$ is the set of values calculated at Min nodes on the path from N to root;

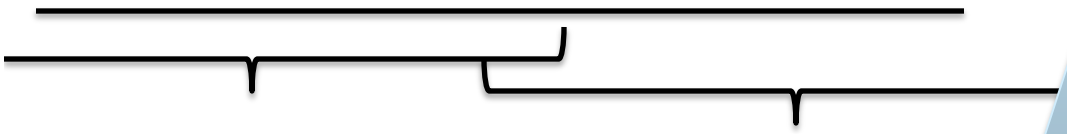
Then N is only **useful** if

$\max(\alpha_0 \ldots \alpha_k) \leq V_N \leq \min(\beta_0 \ldots \beta_j)$

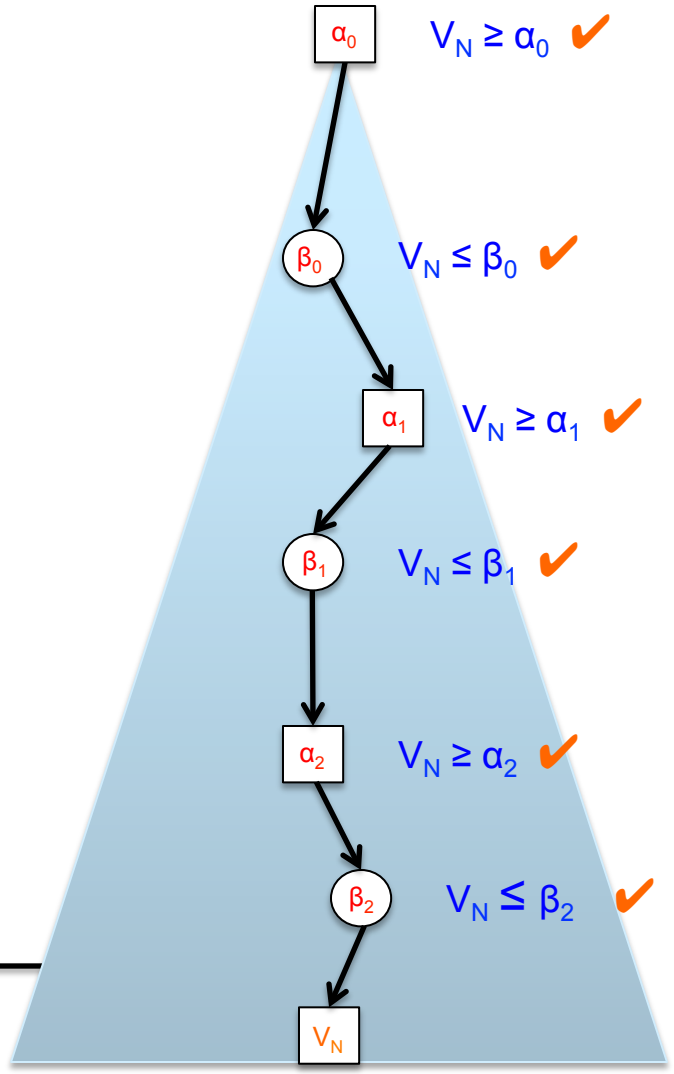
Punchline: If $\max(\alpha_0 \ldots \alpha_k) > \min(\beta_0 \ldots \beta_j)$ then V_N can NEVER be useful. STOP!

$\alpha_3 \ldots \alpha_0 \ldots \alpha_k \ldots \alpha_7$

$\beta_4 \ldots \beta_2 \ldots \beta_0 \ldots B_j$



No possible V_N values!



α-β-Pruning: Basic Principles

How can we make this into an algorithm?

- Keep track of the max value α found so far at Max nodes;
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- If ever β < α, STOP!

```
int final Inf = 1000000

Move chooseMove(Node t) {
     int max = -Inf;      Move best;
     for(each move m to a child c of t) {
        int val = minMax( c, 1, -Inf, Inf );
        if(val > max) { best = m; max = val };
     }
     return best;    }

int minMax(Node t, int depth, int alpha, int beta ) {
  if(   t is a leaf node (no moves) || depth == D)
     return eval(t);           // stop searching and return eval
  else if( t is max node ) {
     int val = -Inf;
     for(each child c of t) {
       alpha = max(alpha, val);    // update alpha with max so far
       if(beta < alpha) break;     // terminate loop
       val = max(val, minMax( c, depth+1, alpha, beta ));
     }
     return val;
  } else {                         // is a min node
     int val = Inf;
     for(each child c of t) {
       beta = min(beta, val);    // update beta with min so far
       if(beta < alpha) break;     // terminate loop
       val = min(val, minMax( c, depth+1, alpha, beta ) );
     }
     return val;
} }
```

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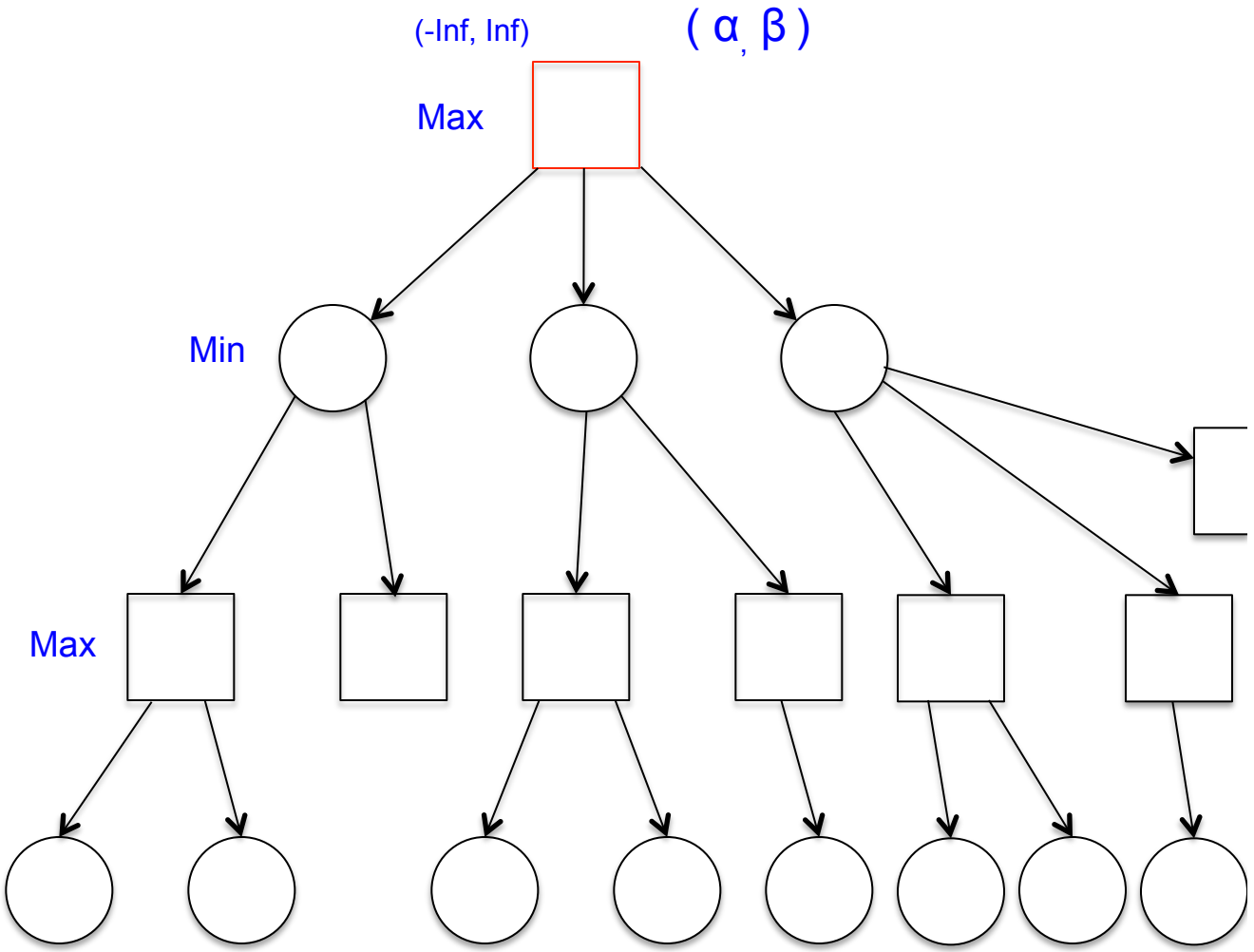

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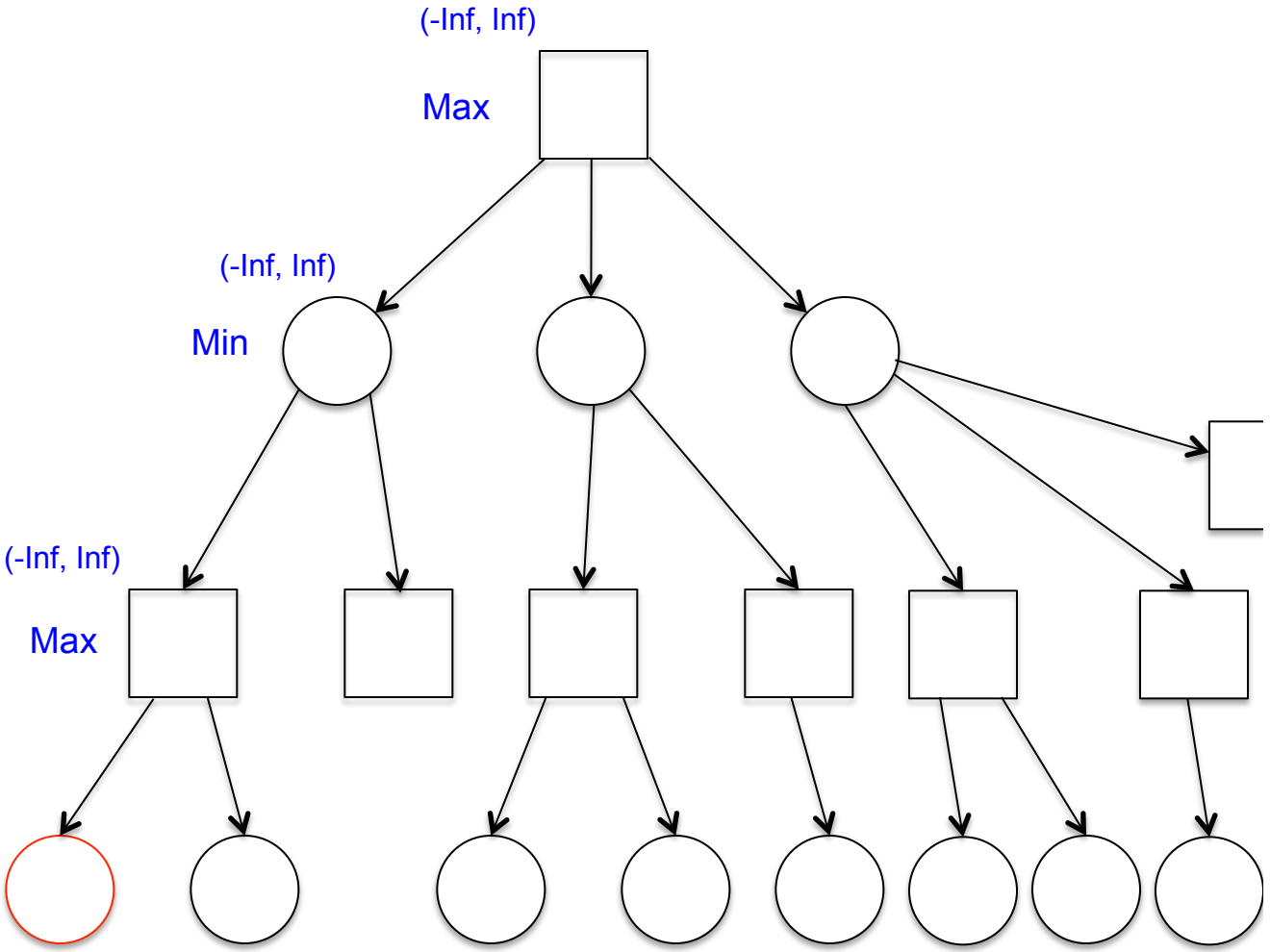

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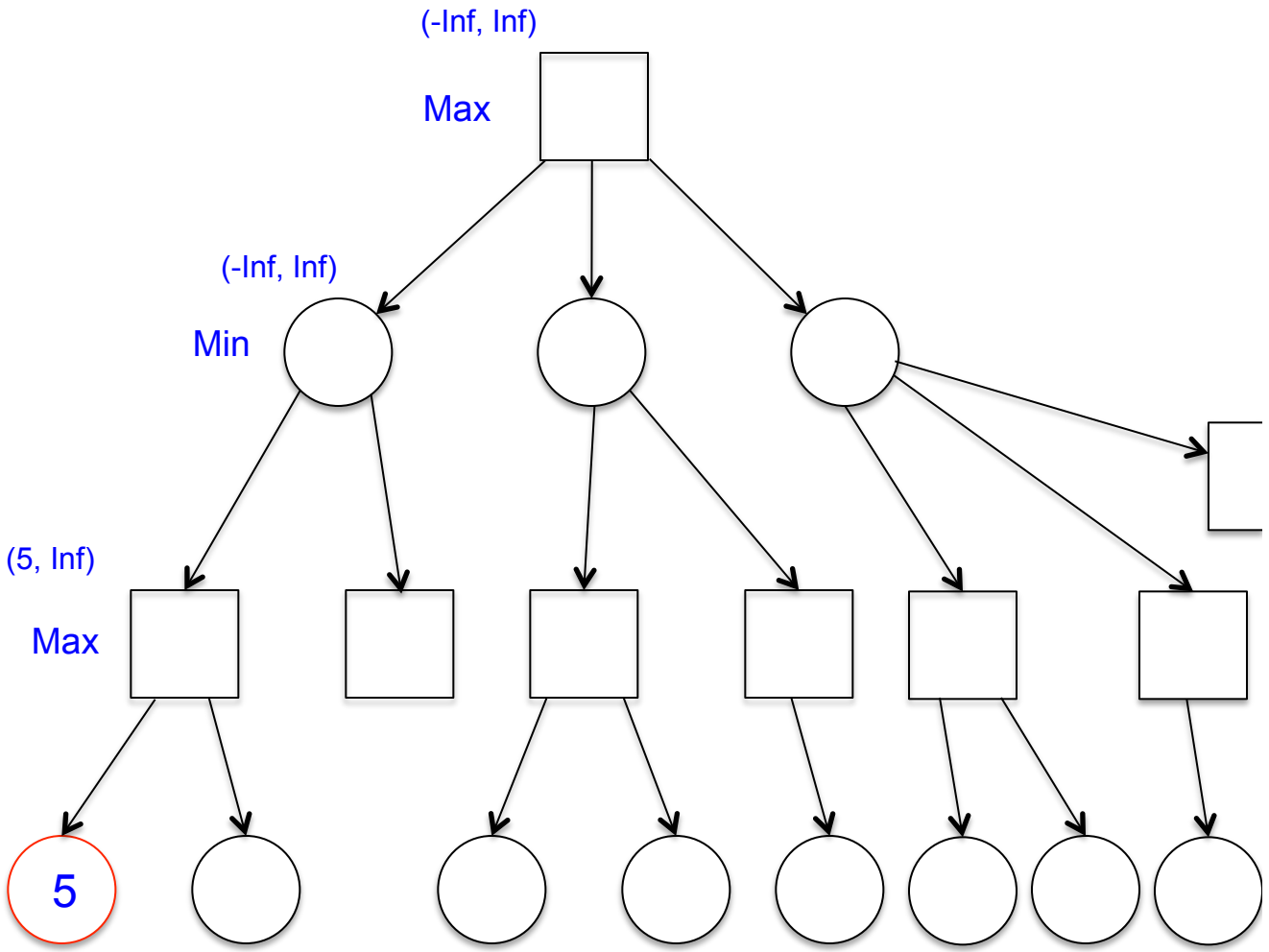

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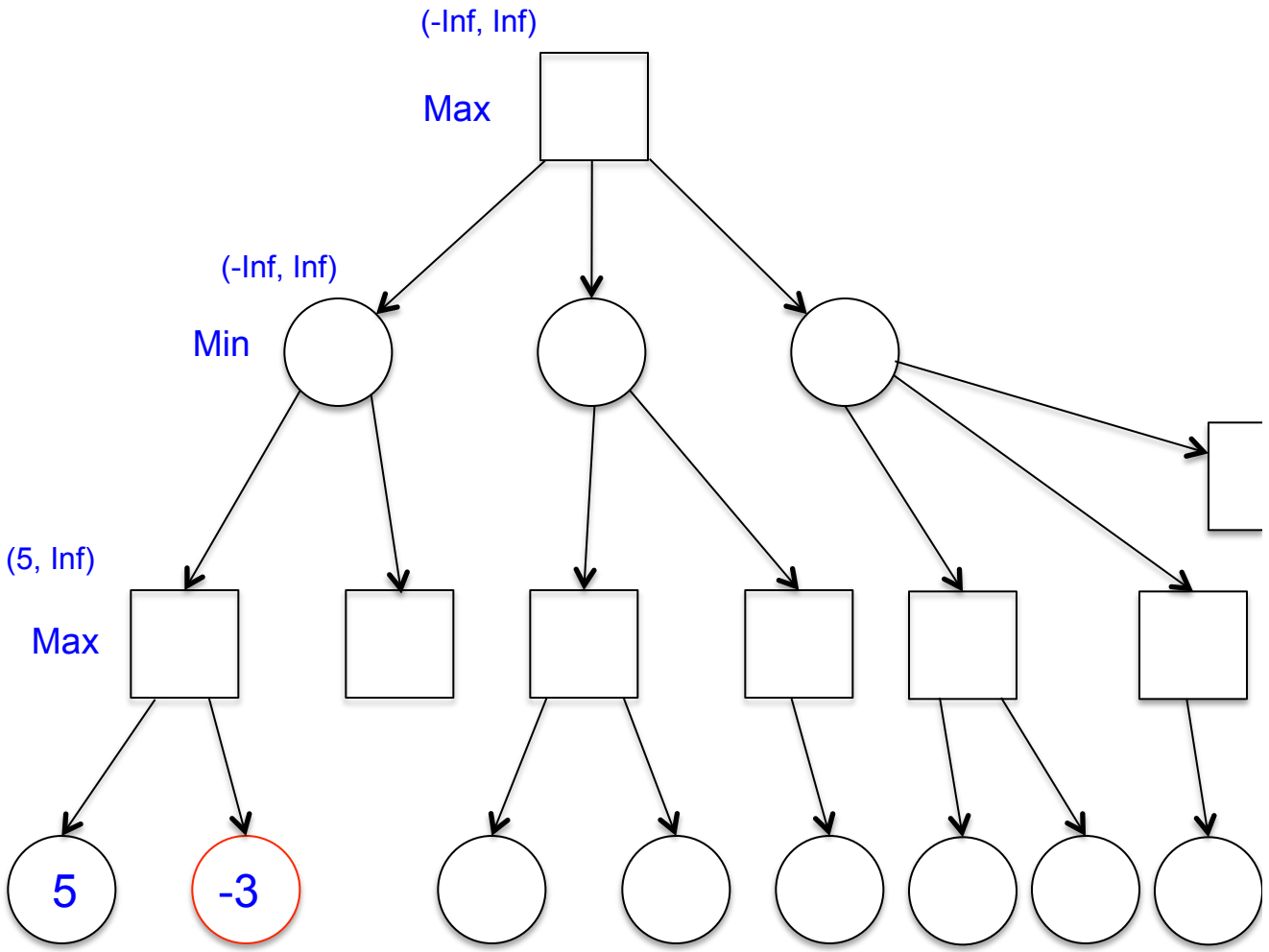

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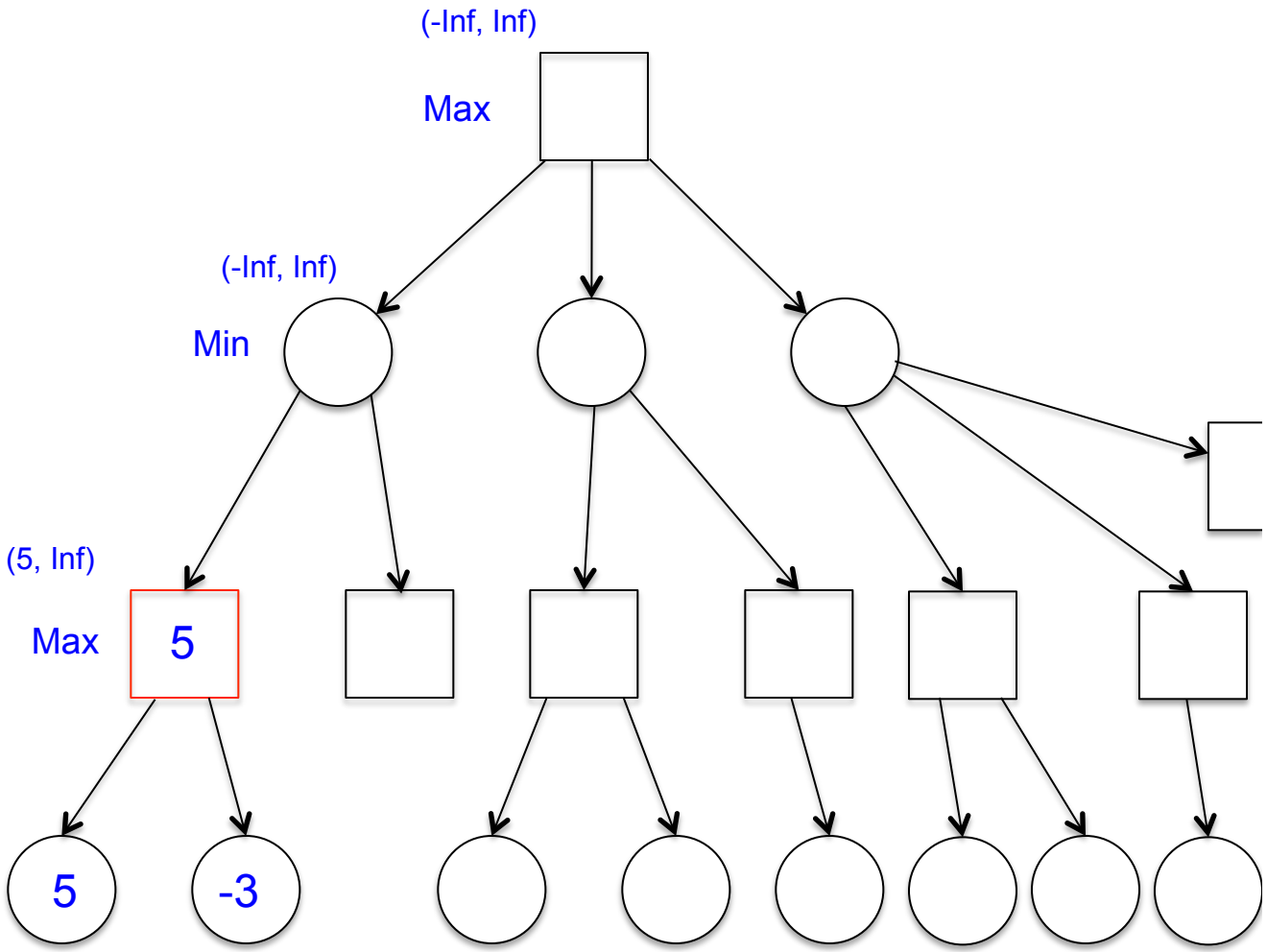
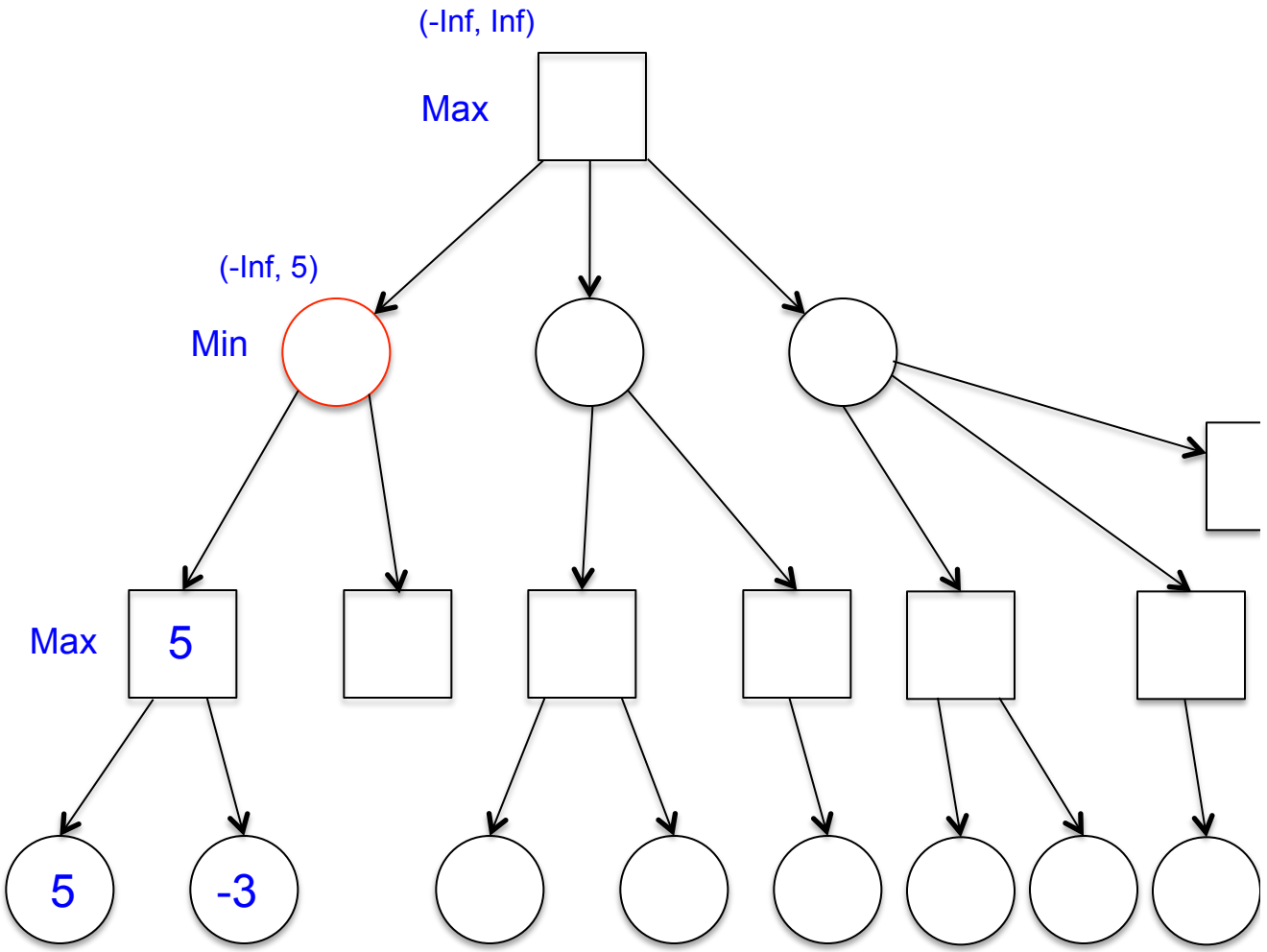


α-β-Pruning: Basic Principles

How can we make this into an algorithm?

- Keep track of the max value α found so far at Max nodes above or at the current node;
- Keep track of the min value β found so far at Min nodes above or at the current node;
- If ever β < α, STOP!

LET'S TRY IT…..



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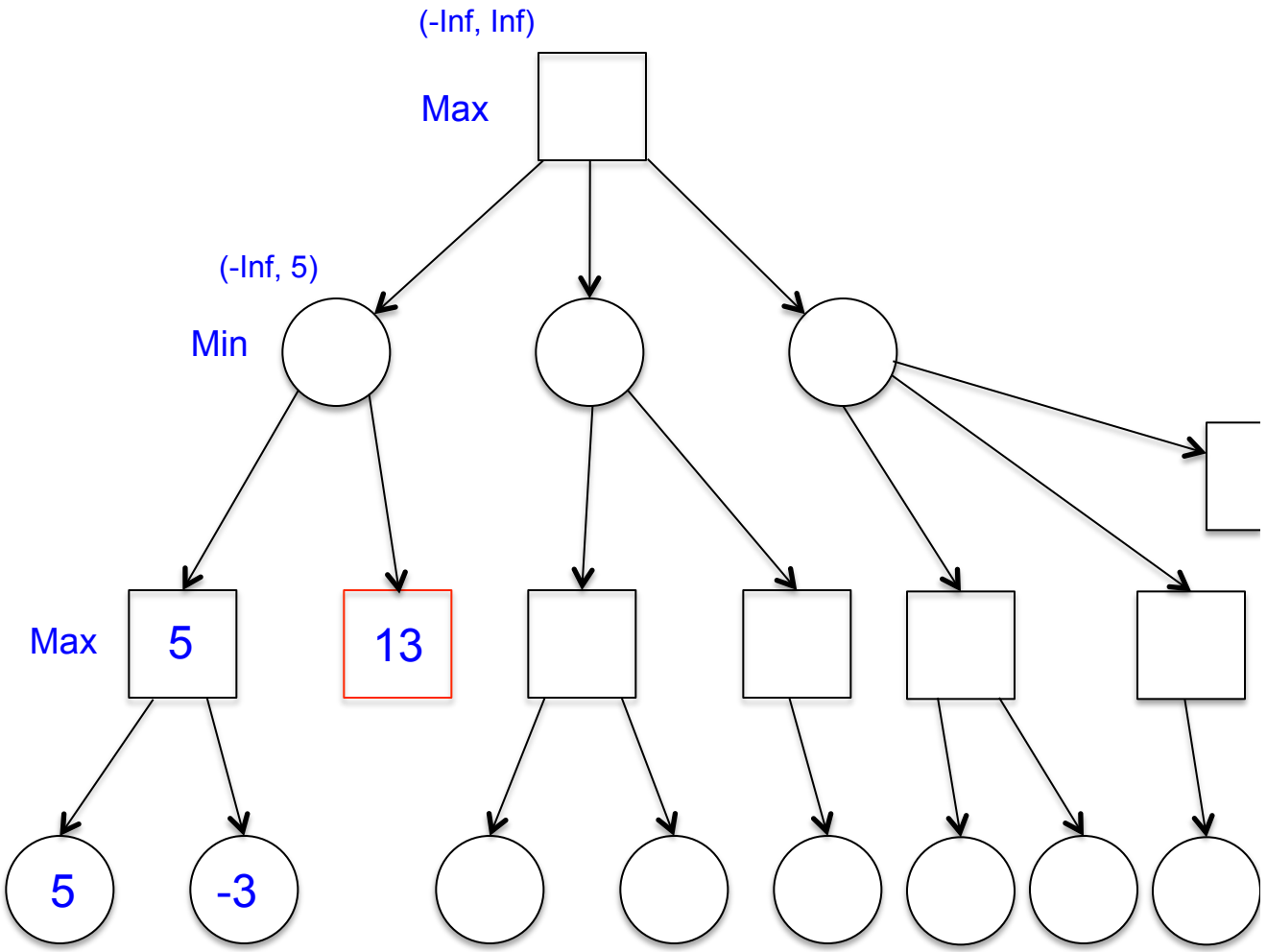

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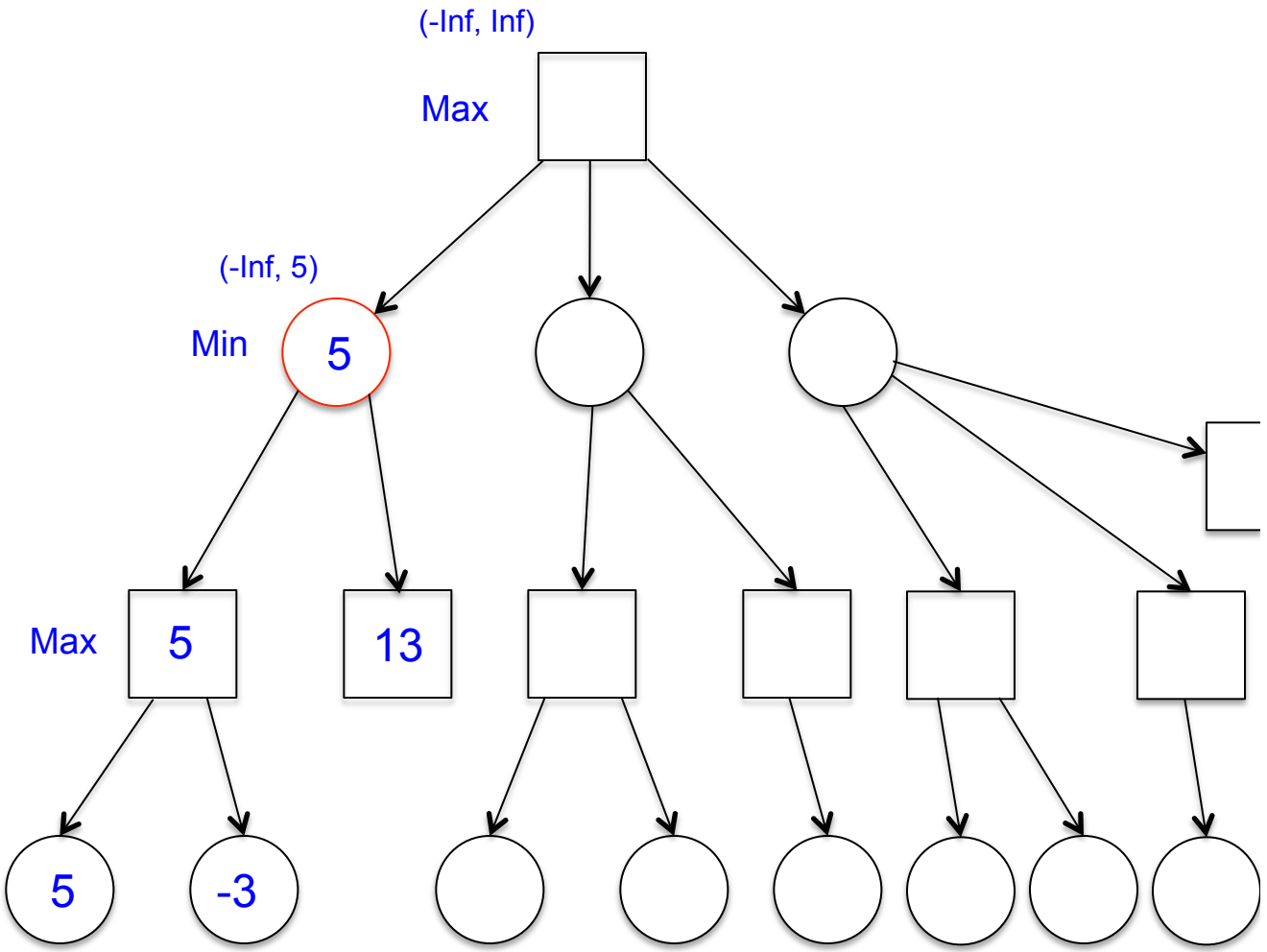

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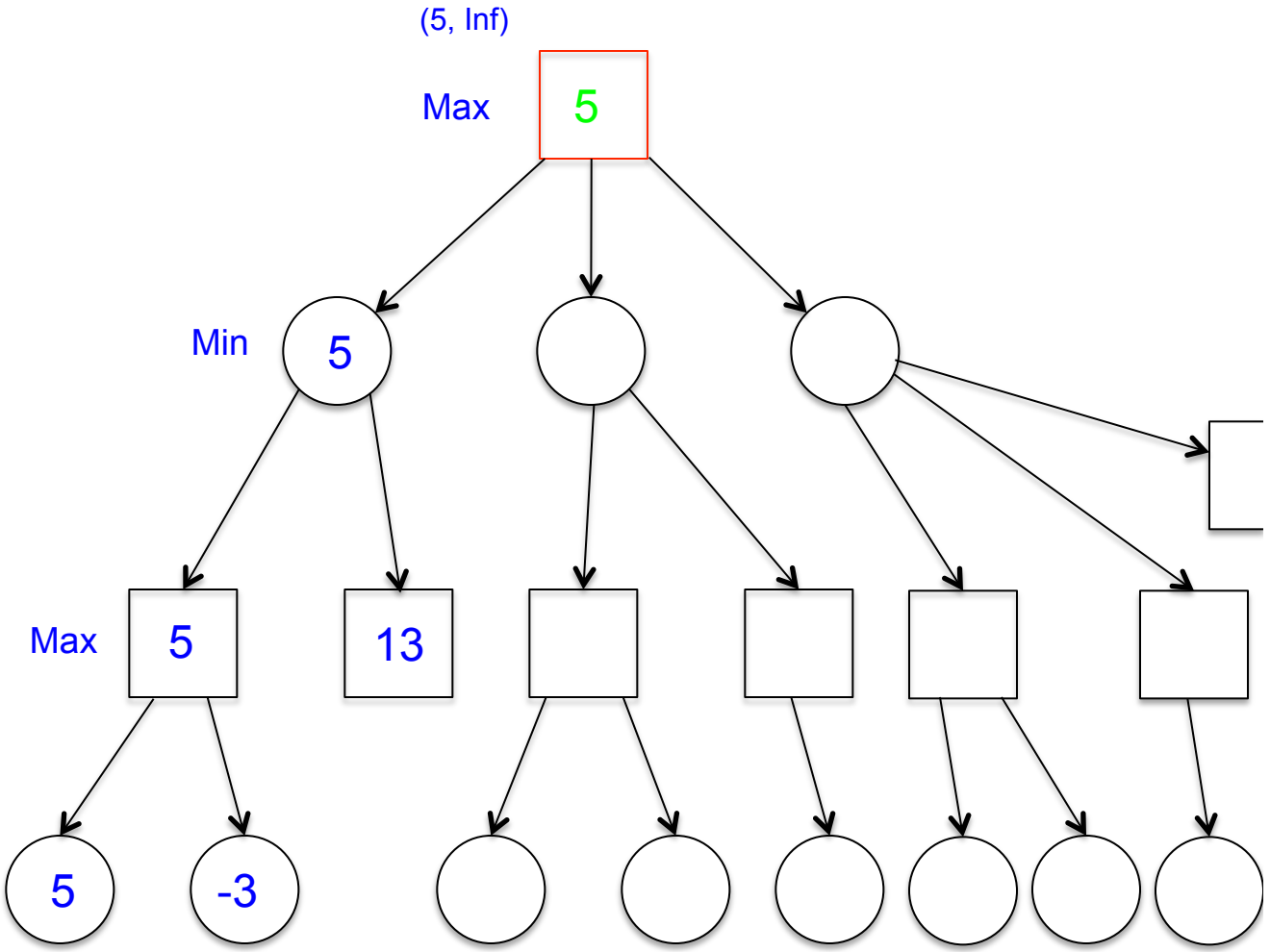

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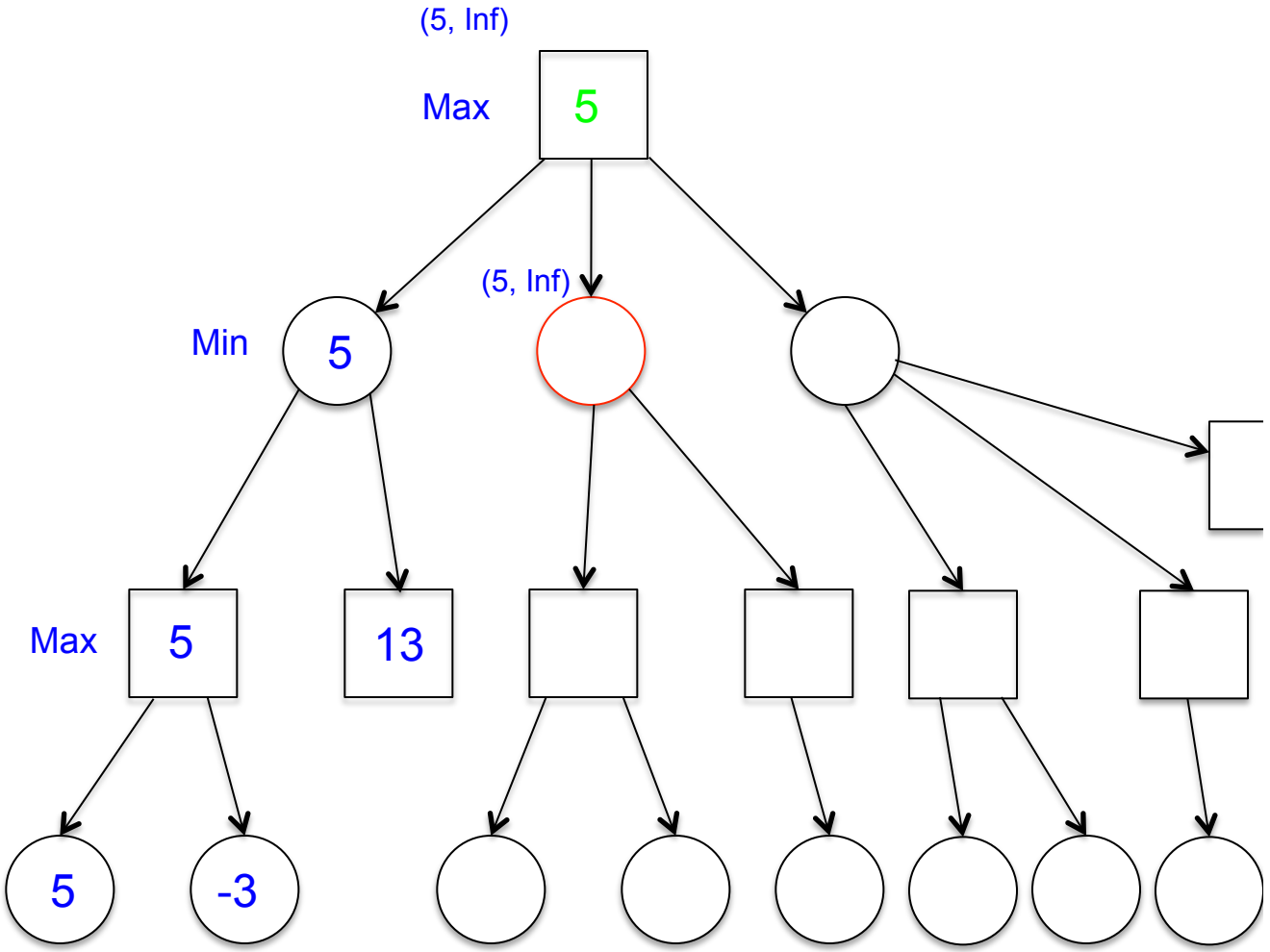

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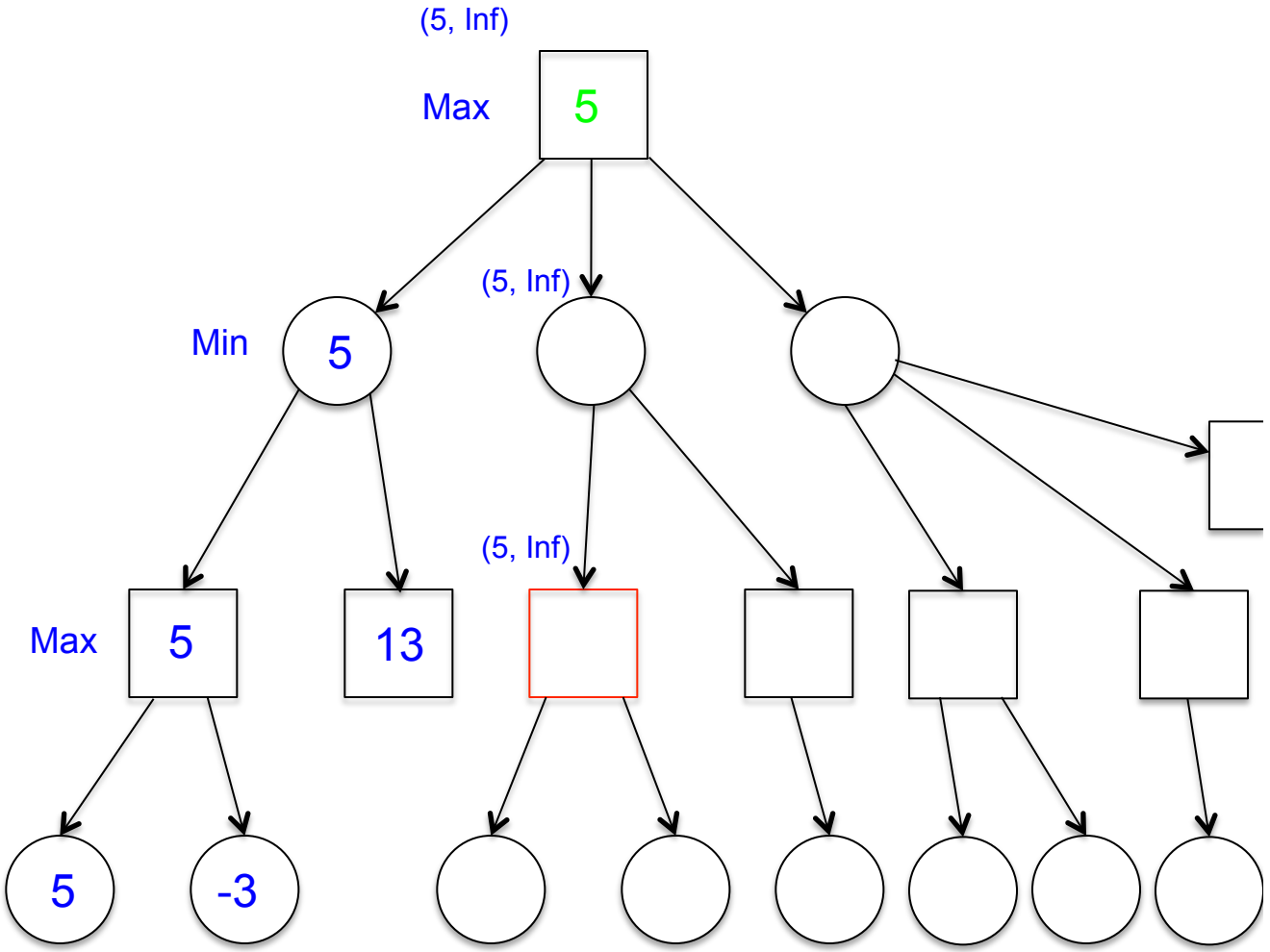

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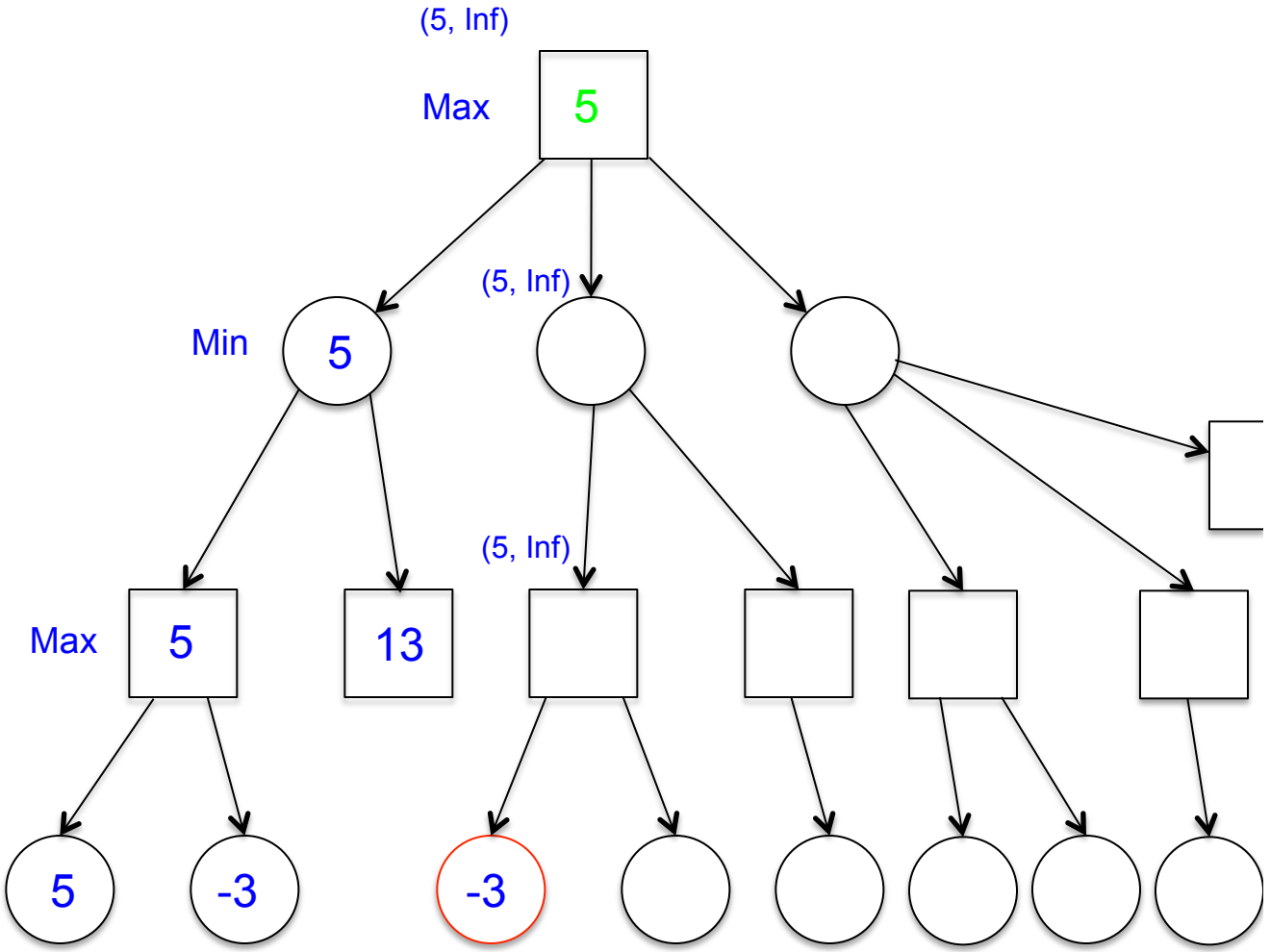

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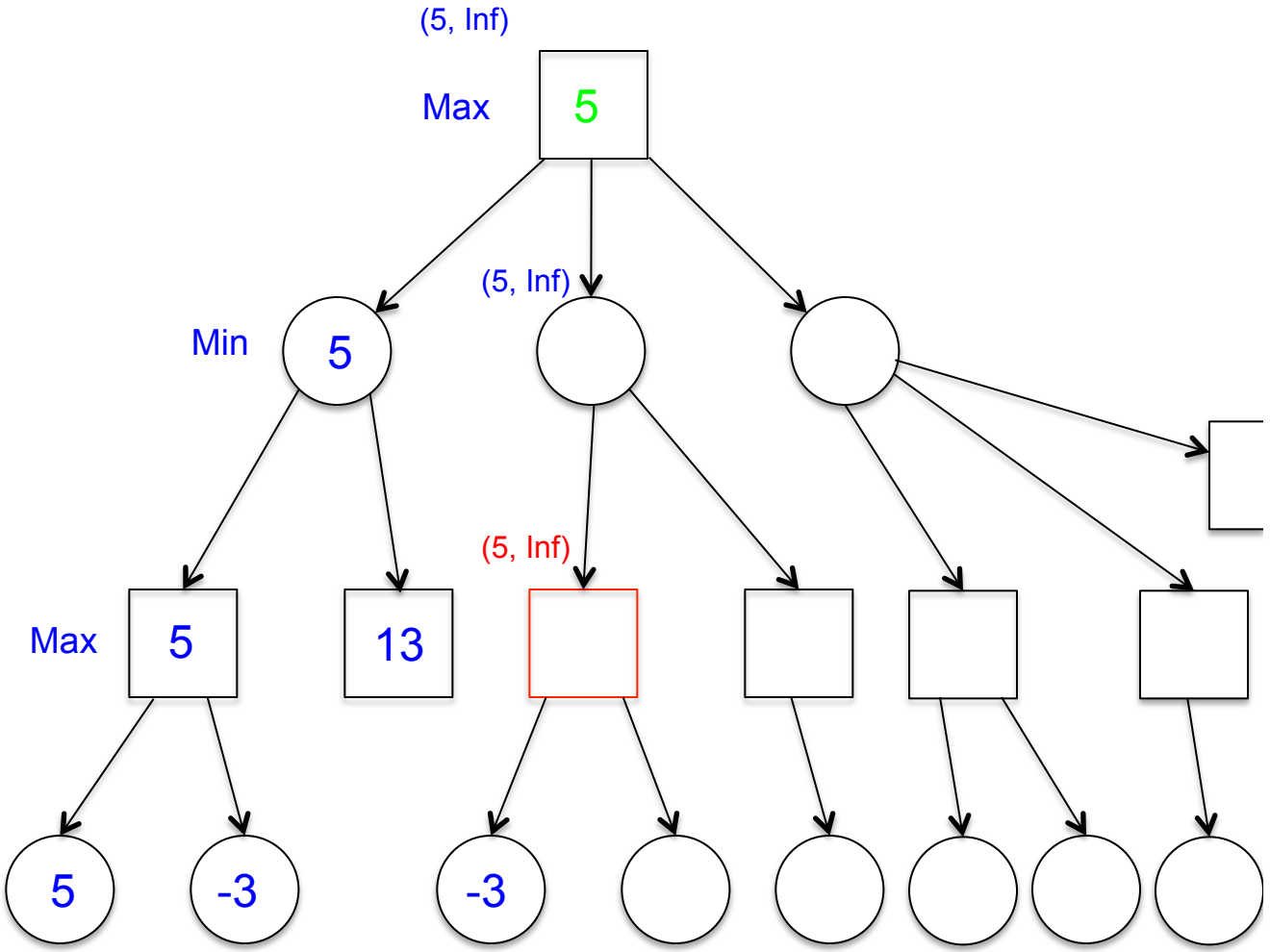

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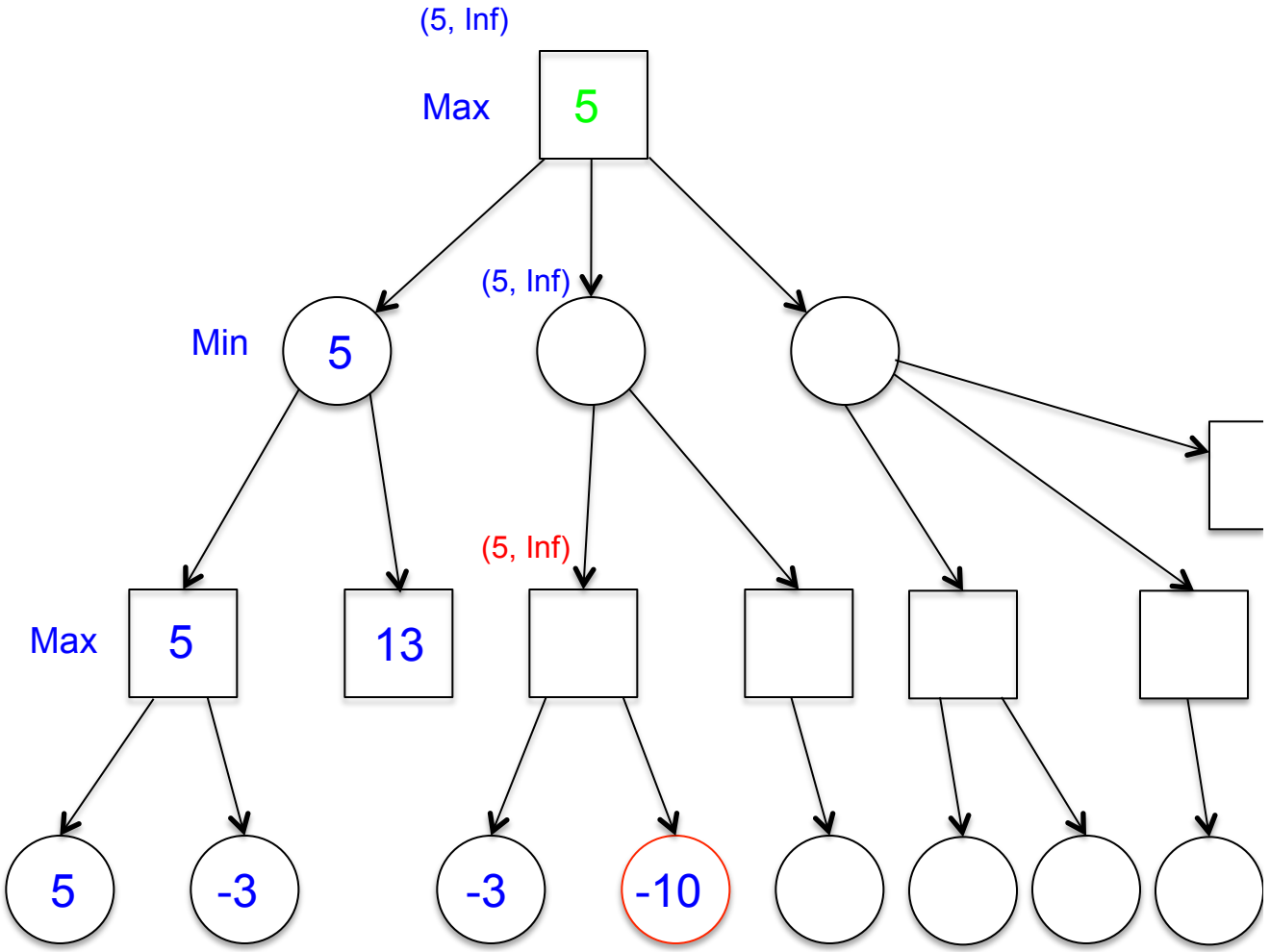

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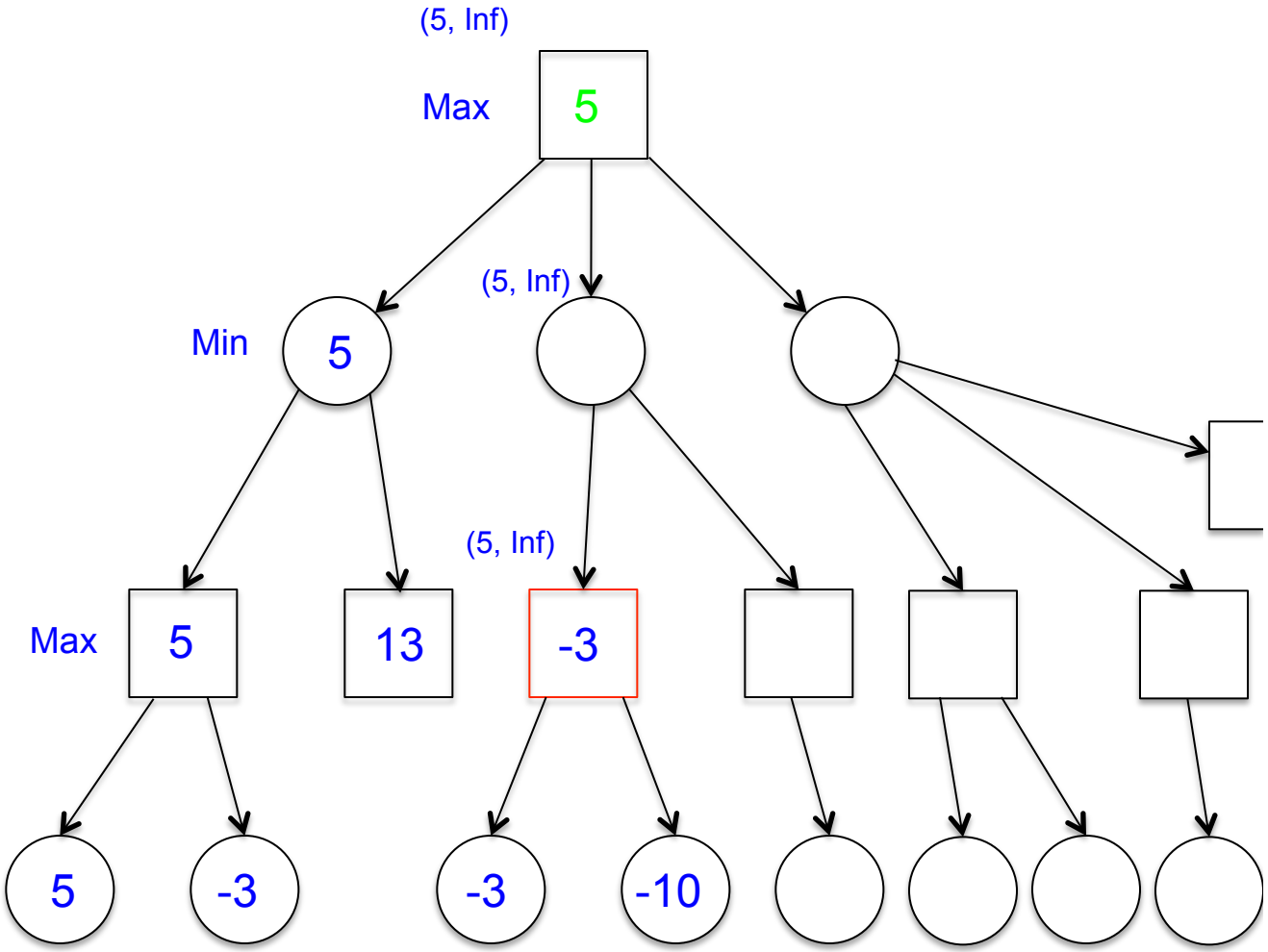

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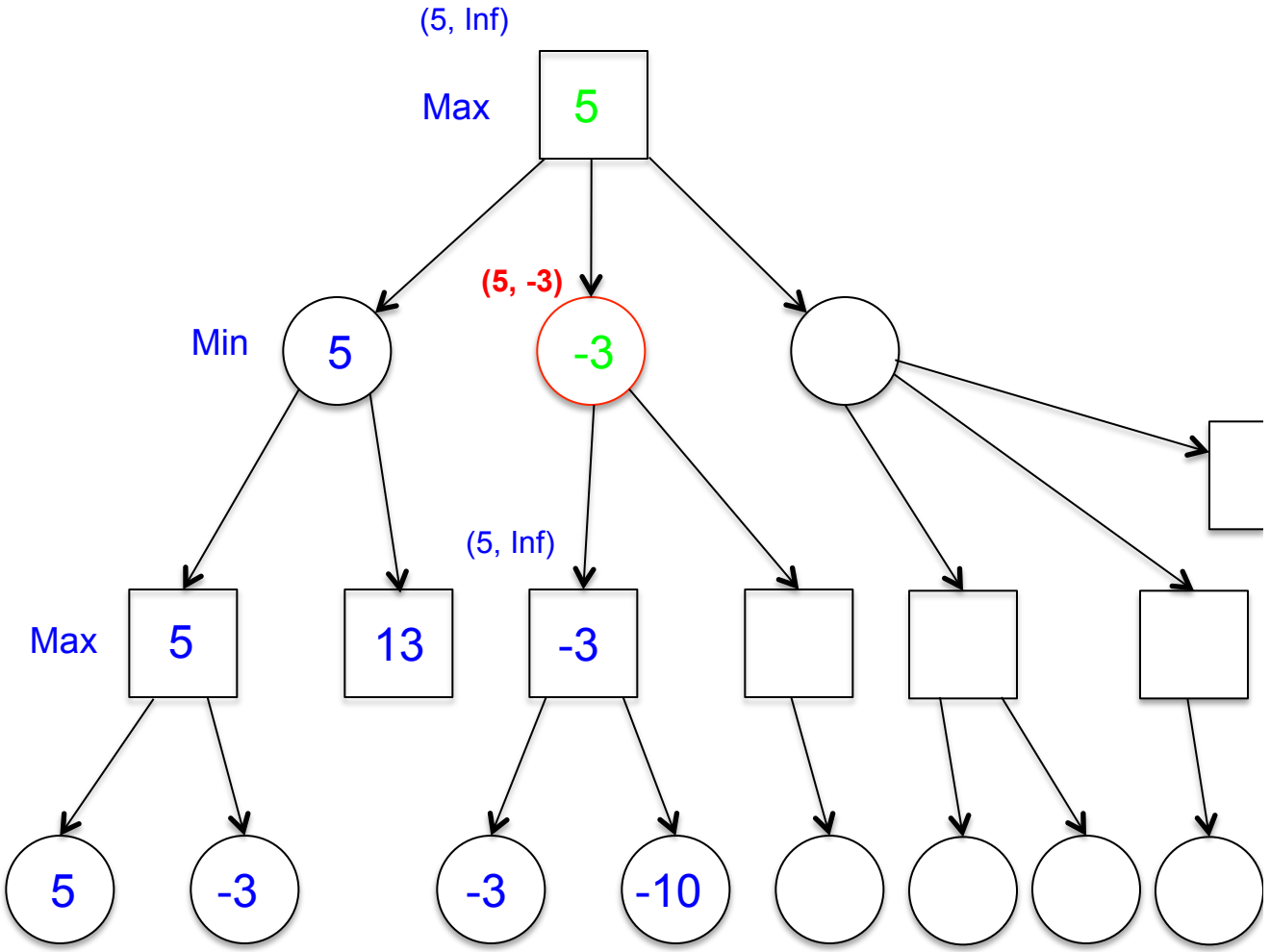

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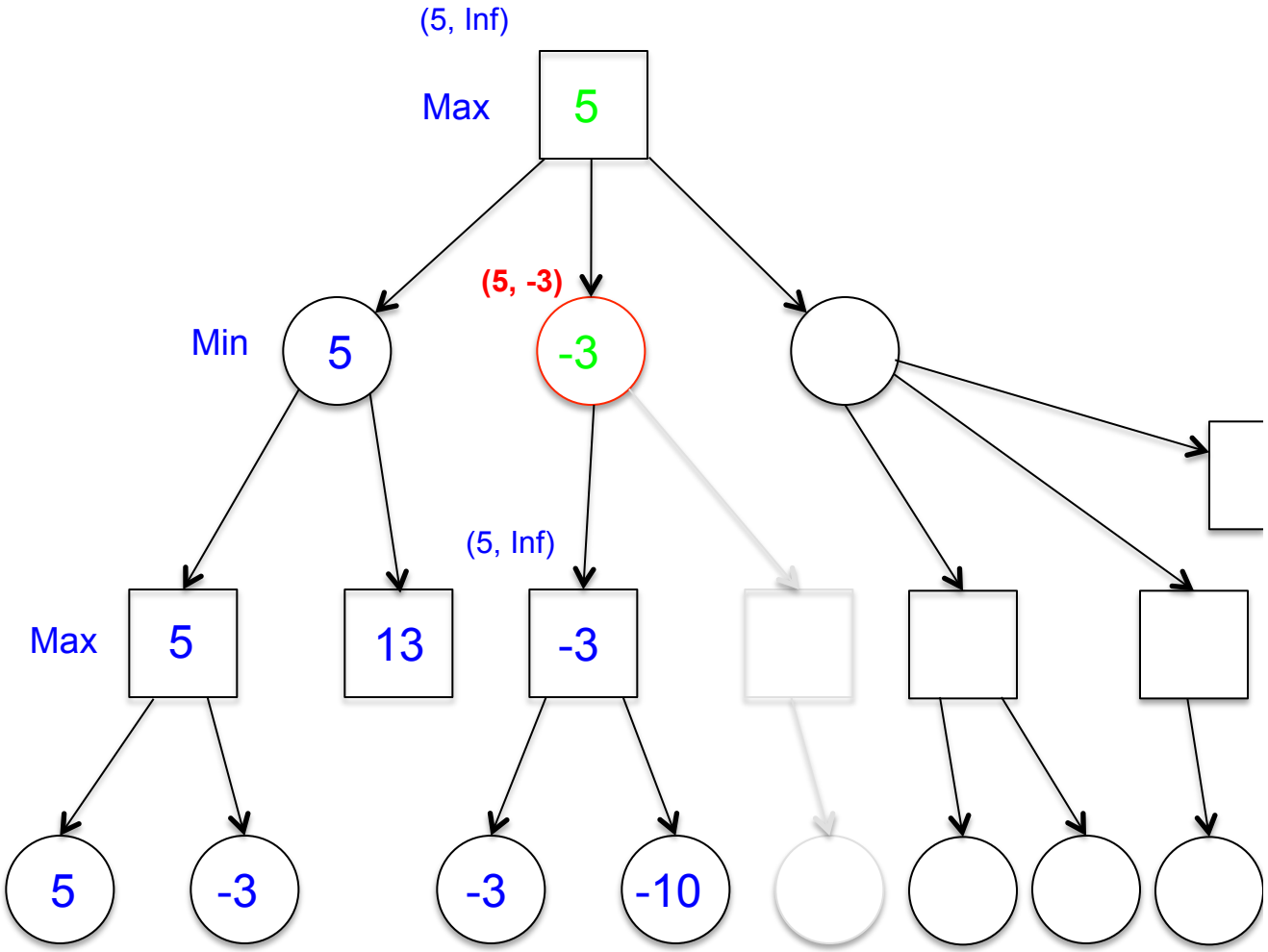

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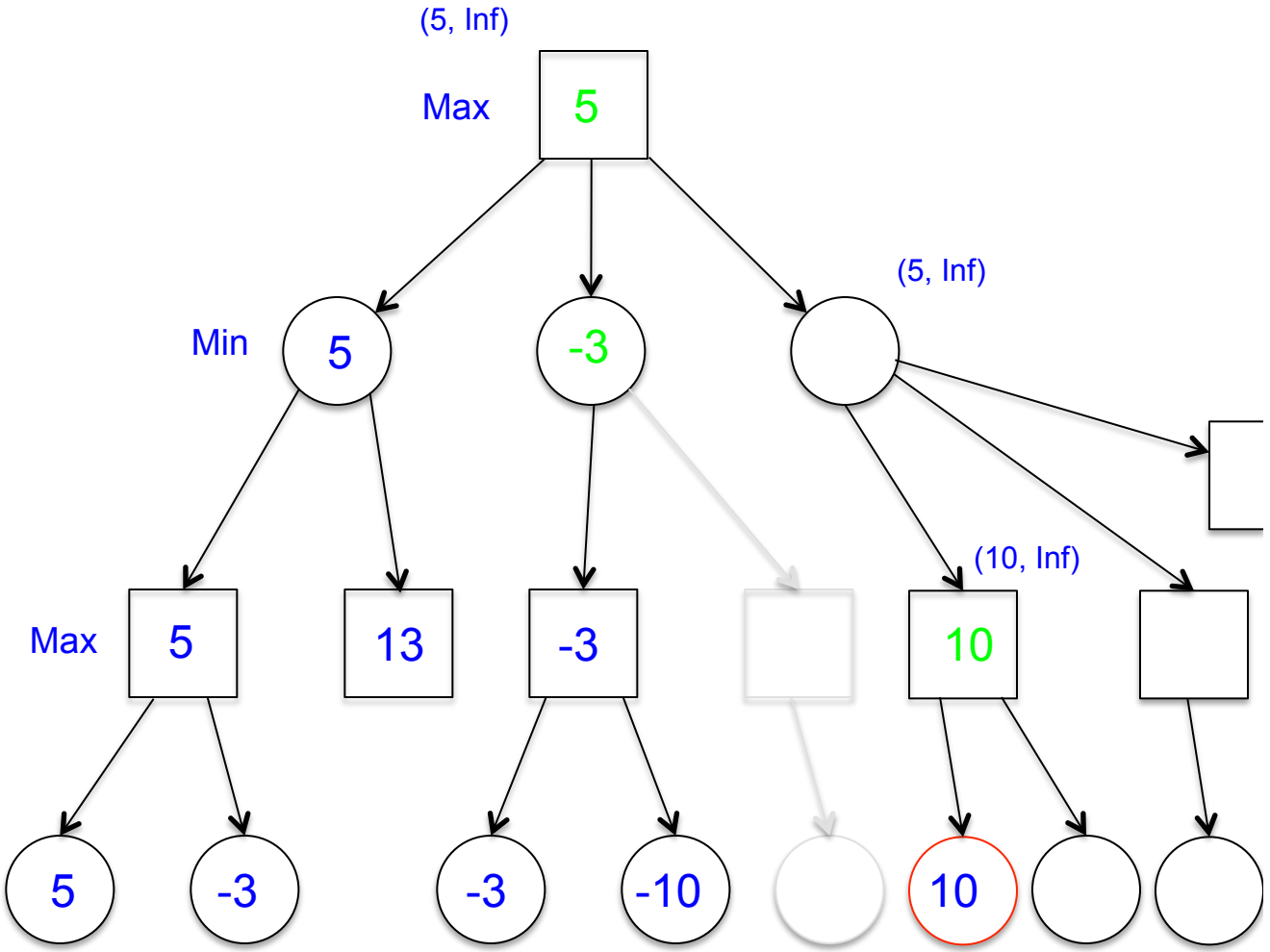

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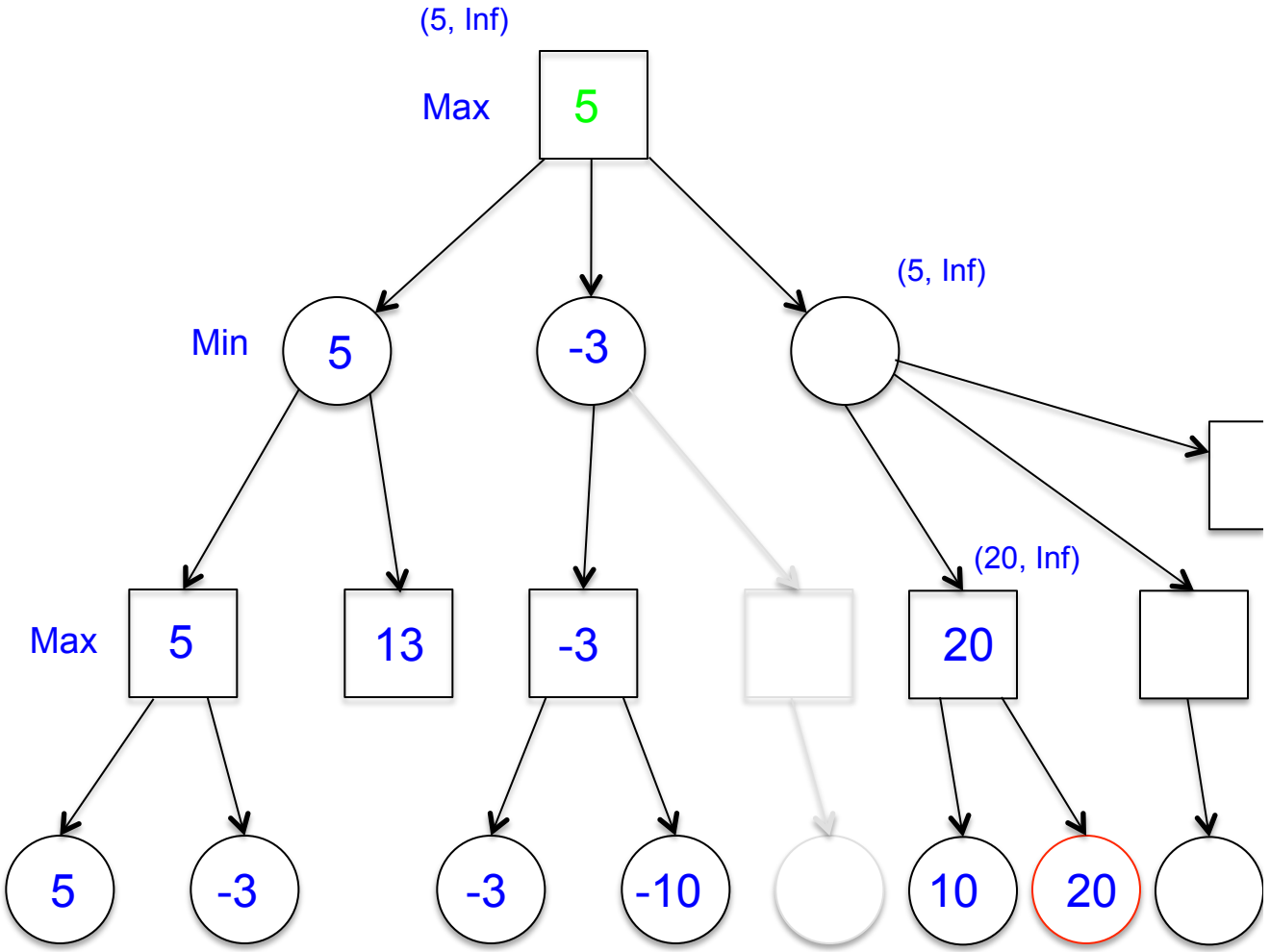

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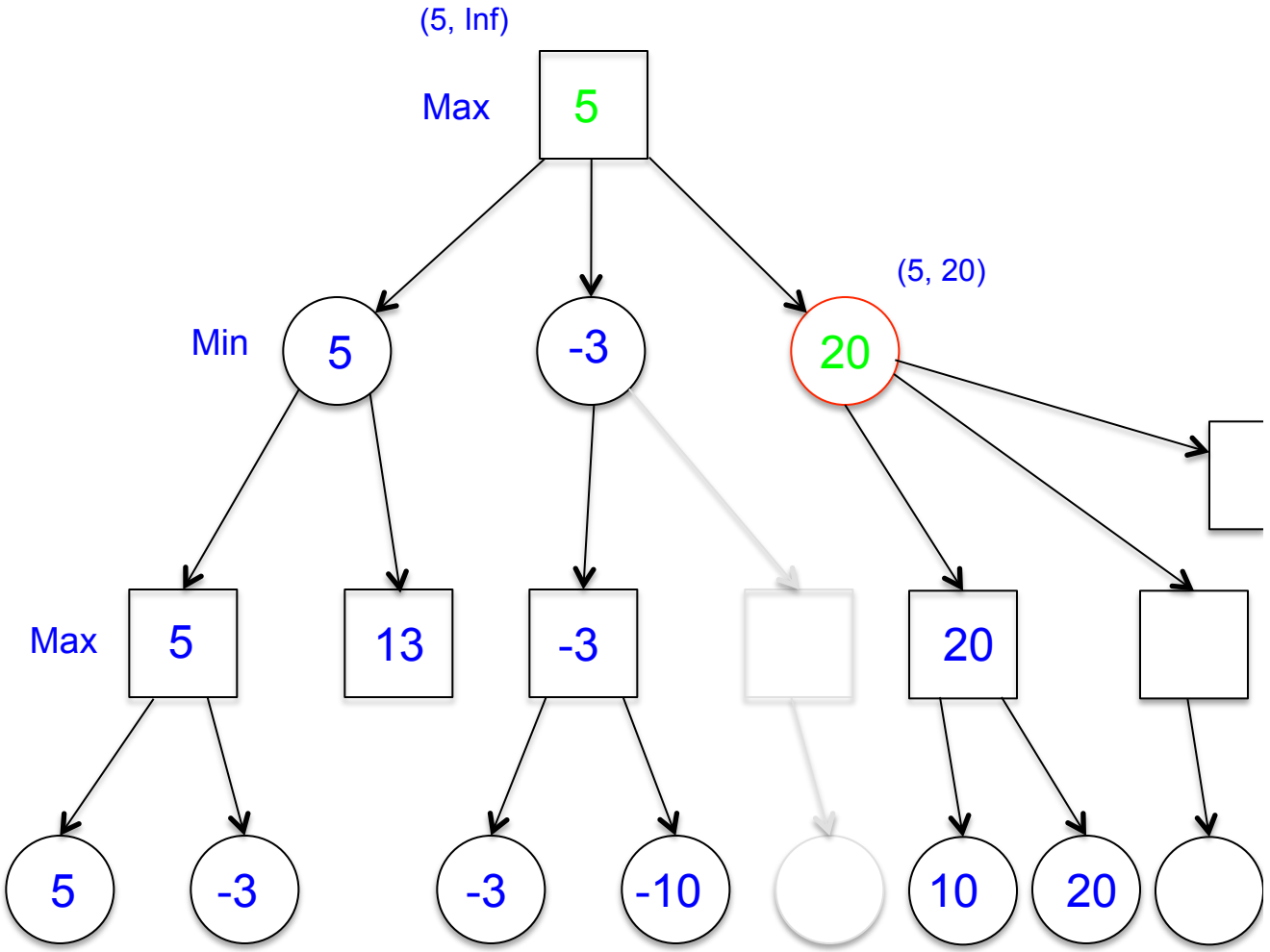

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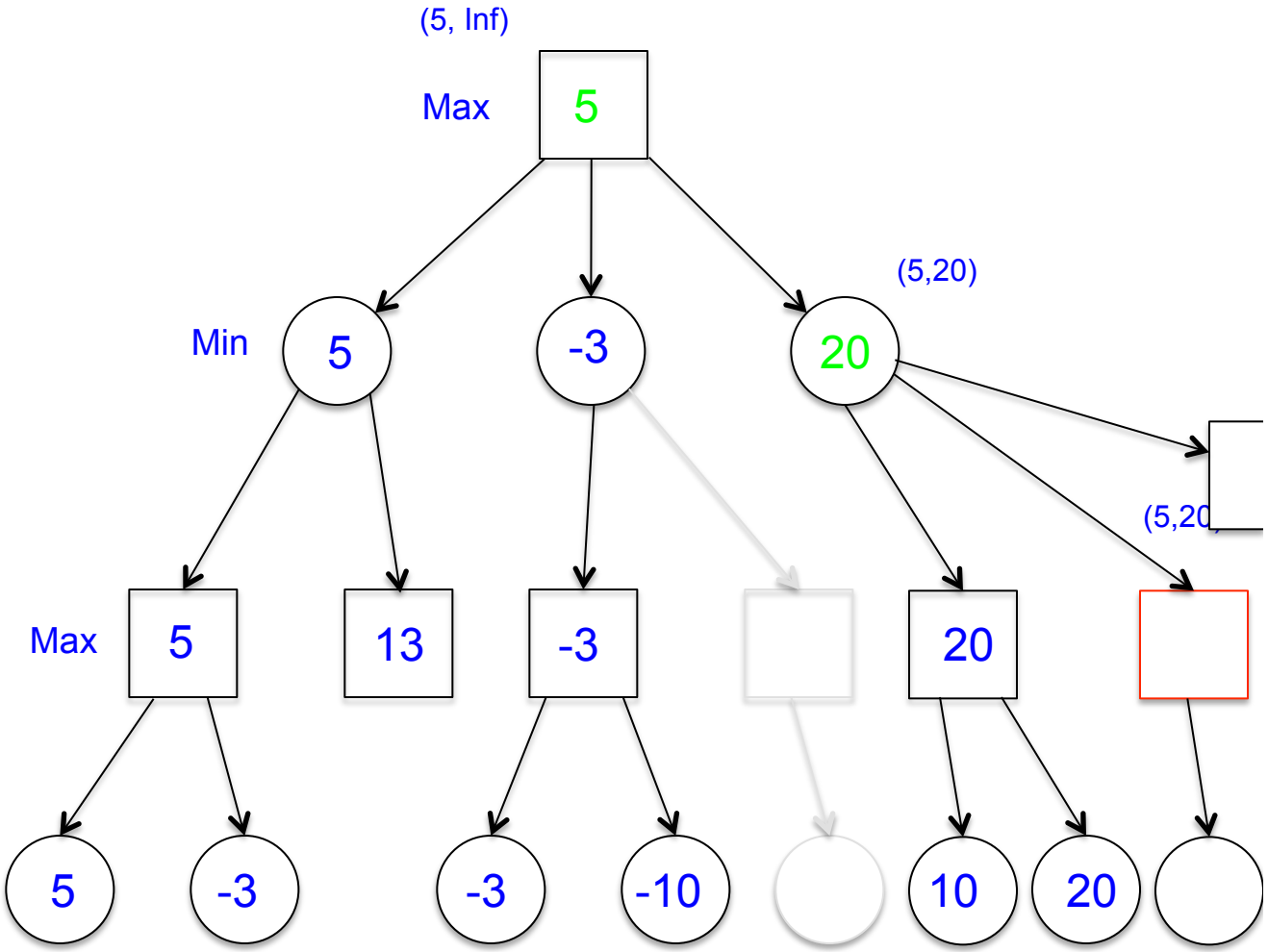

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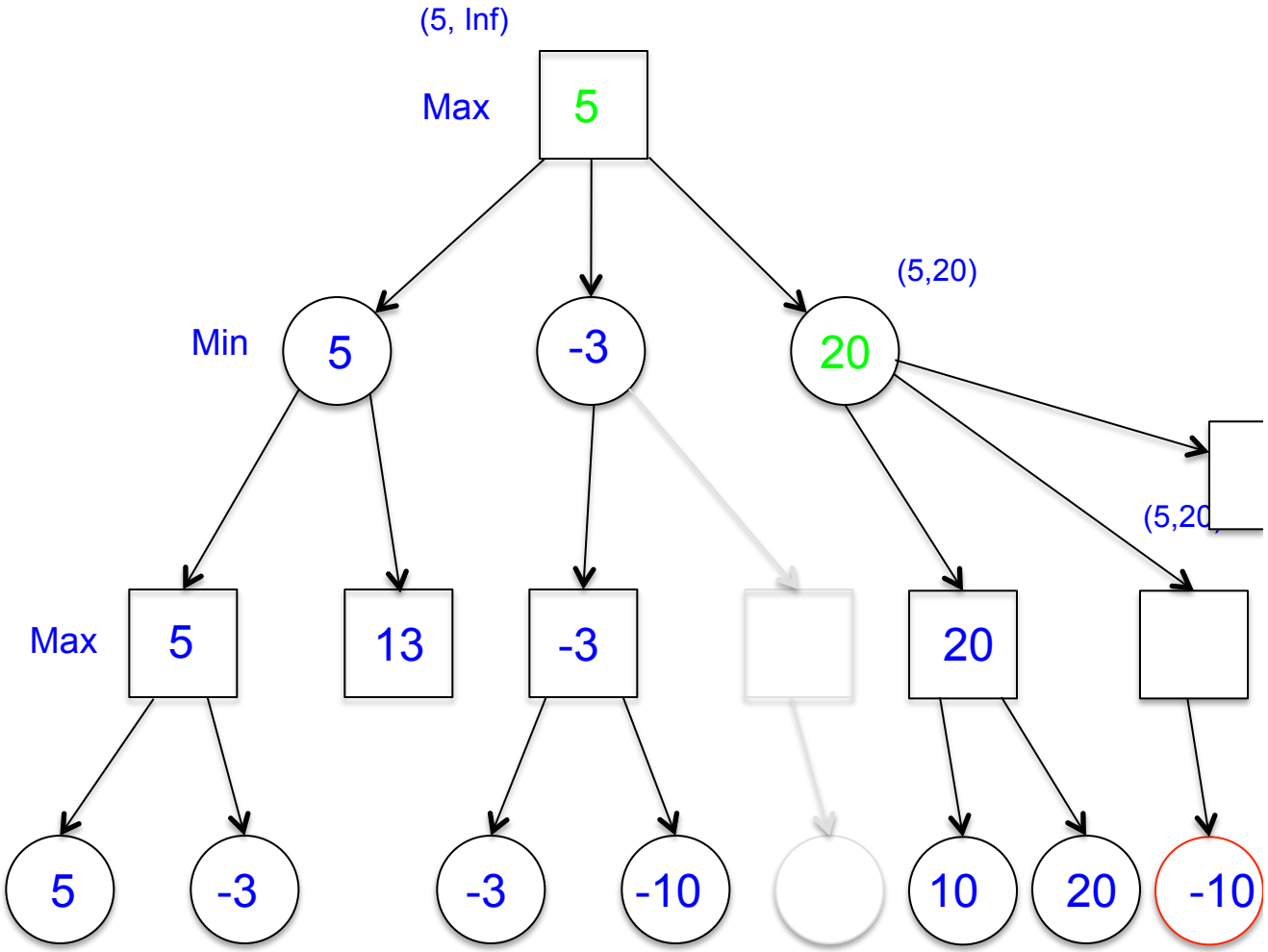

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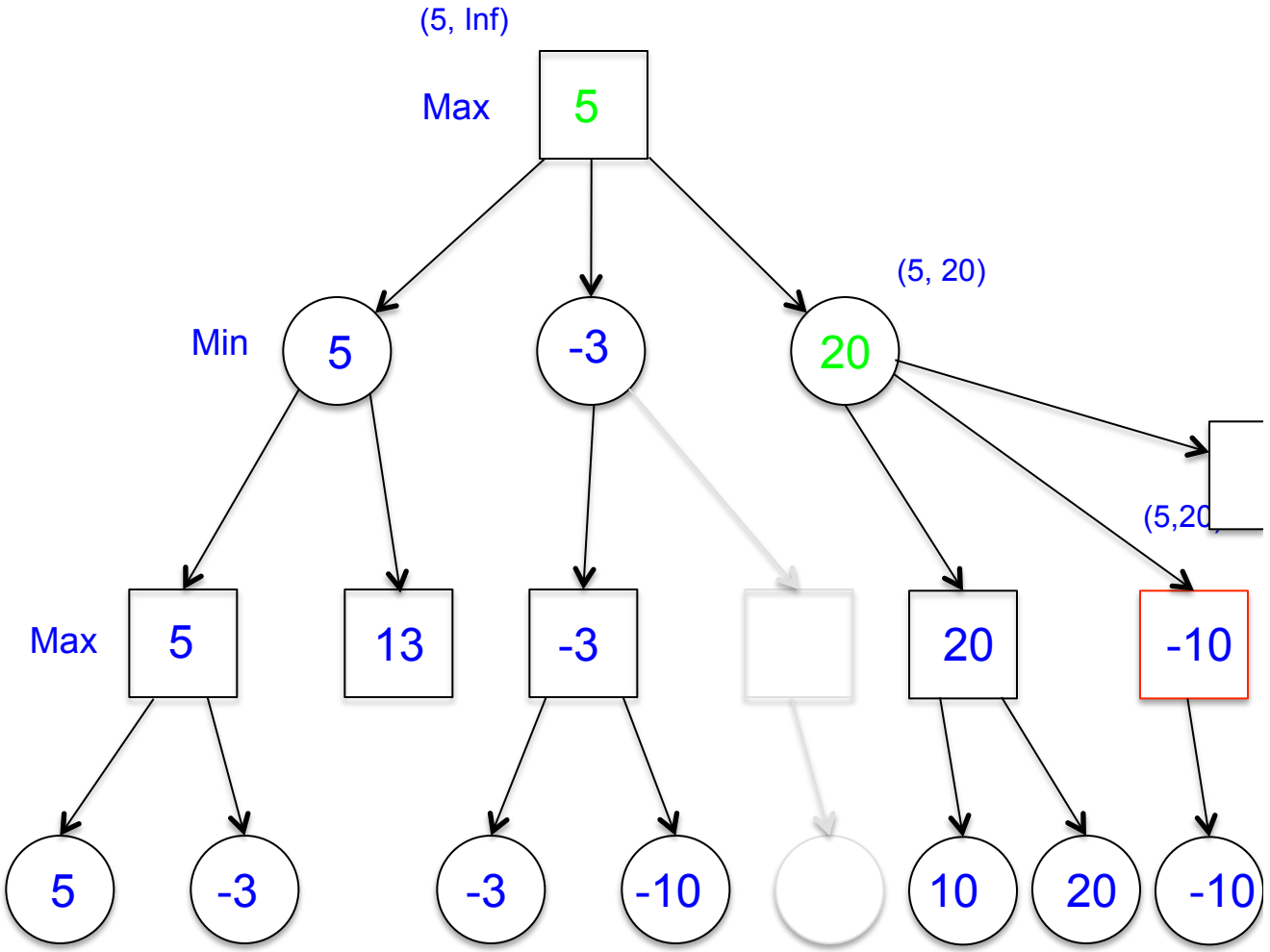

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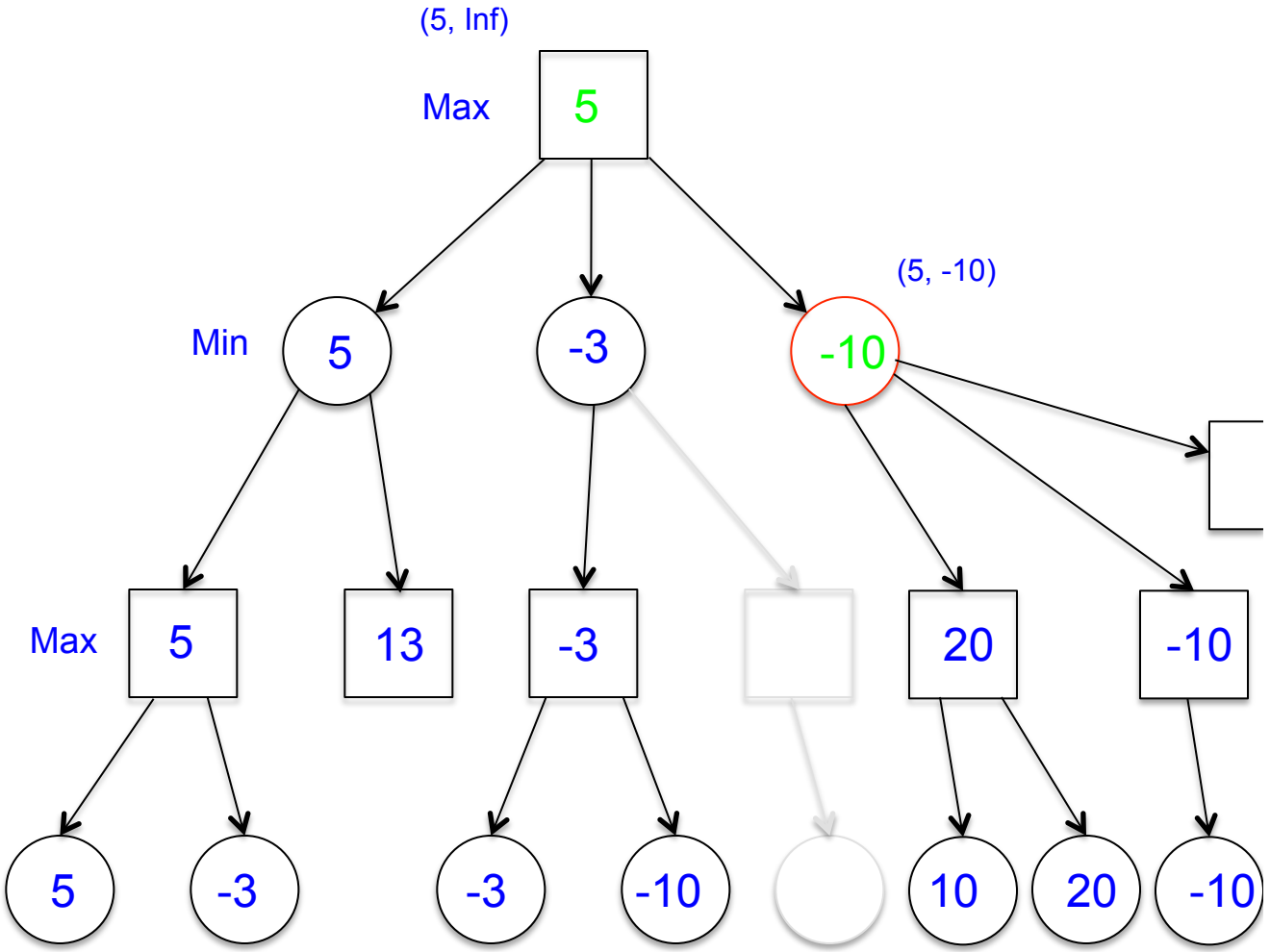

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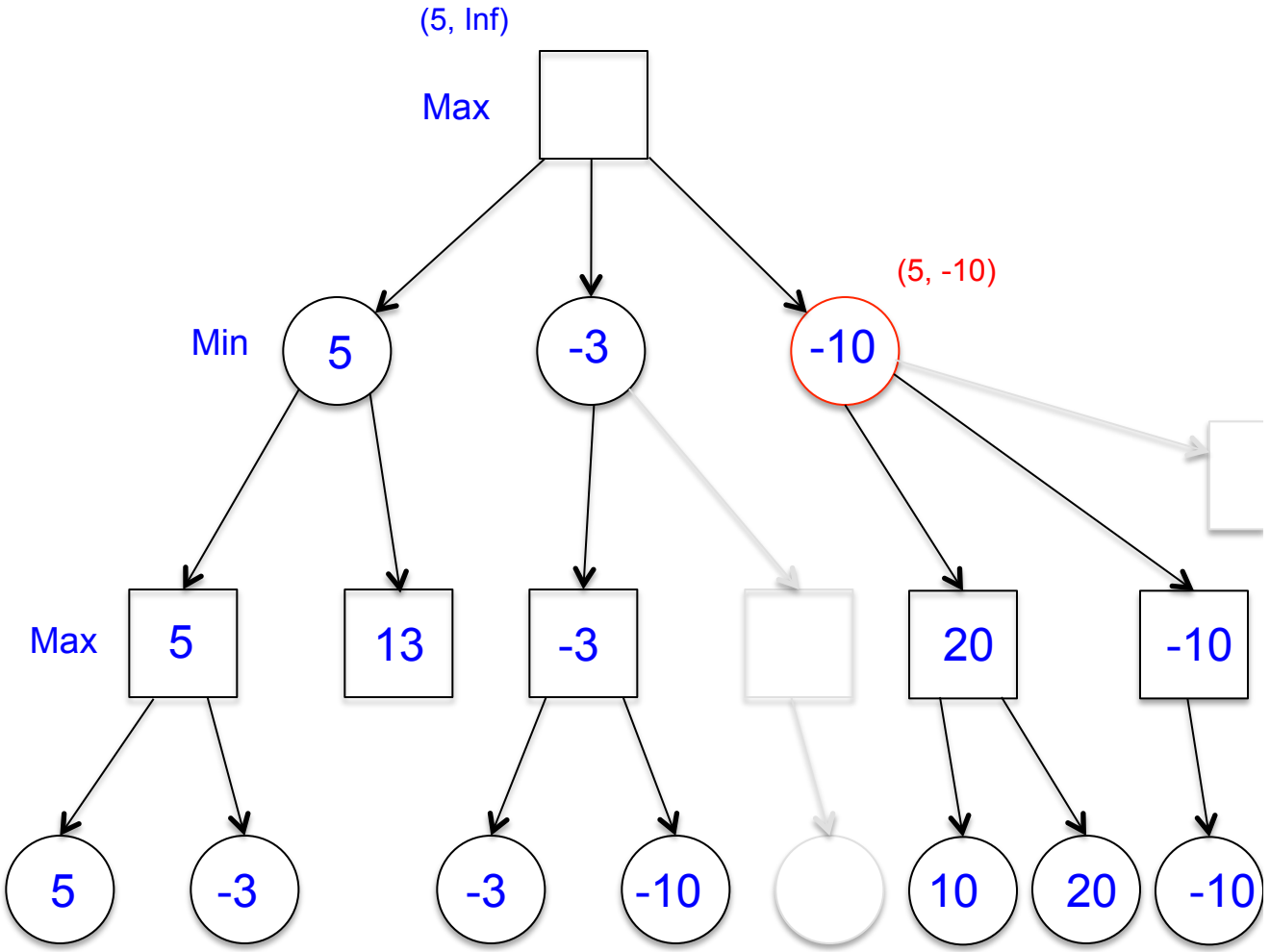

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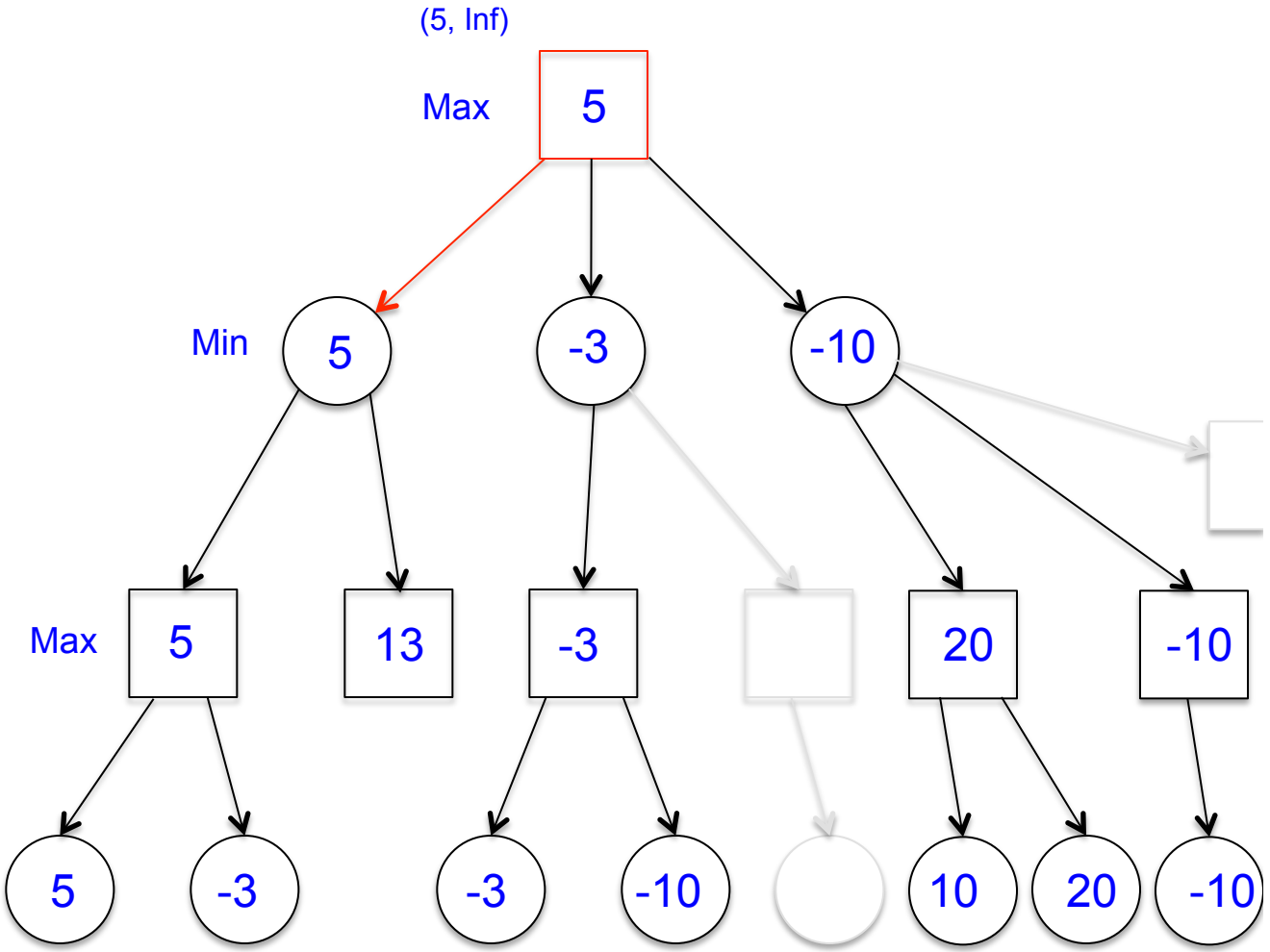

Improving the Min-Max Algorithm

How effective is Alpha-Beta Pruning?

Using a vanilla implementation of the my Connect4 program, I counted how many total board positions were examined, with and without AB Pruning. Here are the results, expressed as the percentage of boards examined under AB Pruning compared with no pruning:

Depth	Without AB	With AB	With/Without	$0.67^{(depth-2)}$
1	8	8	100.00%	
2	72	72	100.00%	
3	584	386	66.10%	0.6700
4	4,209	2,364	56.16%	0.4489
5	33,380	12,197	36.54%	0.3008
6	255,247	48,912	19.16%	0.2015
7	2,322,941	312,565	13.46%	0.1350

The improvement is about $(2/3)^{(depth - 2)}$, which is an exponential improvement; roughly, this means that using AB Pruning, you can go two layers deeper than you could otherwise.

The conclusion is obvious: Use AB Pruning!!