Counting/Combinatorics

Chapter 6

The Basics of Counting

Section 6.1

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

CA: a: 7, b: 7^2, c: 2^7 d: IDK

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Example: How many bit strings of length seven are there?

CA: a: 7, b: 7^2, c: 2^7 d: IDK

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

CA:

a) $3 \cdot 3$

b) $3 \cdot 26 \cdot 10$

c) $26^3 \cdot 10^3$

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ different possible license plates.

$$\underbrace{\_\_\ \ \_\_\ \ \_\_}_{\text{26 choices for each letter}}\ \ \underbrace{\_\_\ \ \_\_\ \ \_\_}_{\text{10 choices for each digit}}$$

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: A set *S* contains *n* elements. Compute the number of different subsets of *S*.

CA:

a) n
b) n^2
c) 2^n
d) IDK

Counting Subsets of a Finite Set

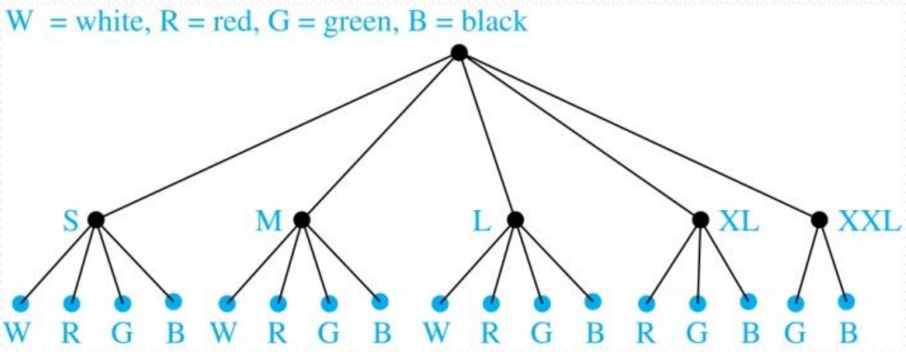
Counting Subsets of a Finite Set: A set *S* contains n elements. Compute the number of different subsets of *S*.

Solution: When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of *S* and bit strings of length n. When the *i*th element is in the subset, the bit string has a 1 in the *i*th position and a 0 otherwise.

By the product rule, there are 2^n such bit strings, and therefore 2^n subsets.

Tree Diagrams

- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL.
- Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black.
- What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?
- **Solution**: Draw the tree diagram.



- **Answer:**
- The store must stock 17 T-shirts.

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The CS department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the CS faculty and 83 CS majors and no one is both a faculty member and a student.

CA:

a) $37 \cdot 83$
b) $37 + 83$
c) IDK

Basic Counting Principles: The Sum Rule

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Example: The CS department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the CS faculty and 83 CS majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick a representative.

The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \emptyset$ for all i, j.

- The case where the sets have elements in common will be discussed when we consider the subtraction rule.

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

CA:

a) $26+10$
b) $26 \cdot 10$
c) $26 + 26 \cdot 10$
d) $26 \cdot (26 + 10)$

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the product rule.

$$26 + 26 \cdot 10 = 286$$

Counting Passwords

- Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution:

- Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.
- By the sum rule $P = P_6 + P_7 + P_8$.
- How to find each of P_6, P_7, and P_8 ?
- Find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters.

- $P_6 = 36^6 - 26^6 = 1,867,866,560.$
- $P_7 = 36^7 - 26^7 = 70,332,353,920.$
- $P_8 = 36^8 - 26^8 = 2,612,282,842,880.$

- $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

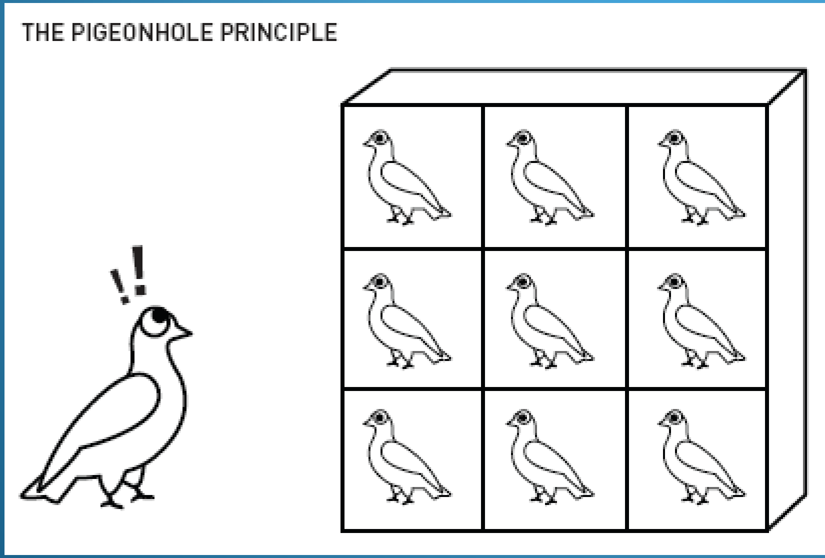
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Example: A computer company receives 350 applications from the graduating students, 220 of these applicants majored in CS, 147 – in business, 51 majored both in CS and in business.

1. How many majored in either CS or in business?
   a) 367
   b) 350
   c) 316
   d) IDK
2. How many applicants majored in neither CS or in business?
   a) 17
   b) 34
   c) 51
   d) IDK

The Pigeonhole Principle

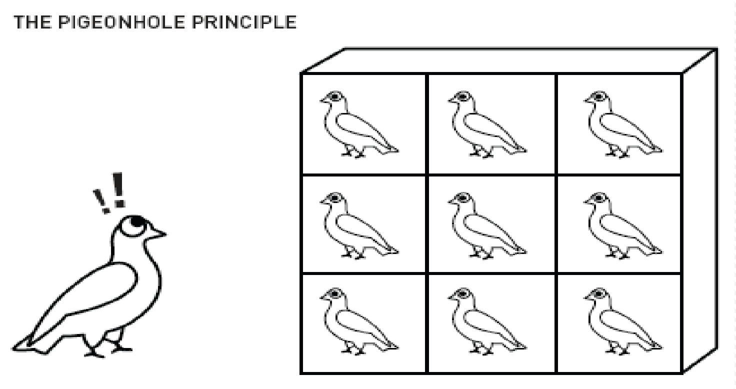


Section 6.2

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle



Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects. ◀

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Pigeonhole Principle

Example: Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution: Let n be a positive integer.

- Consider the $n + 1$ integers 1, 11, 111, …., 11…1 (the last has $n + 1$ ones).
- There are n possible remainders when an integer is divided by n. By the pigeonhole principle, when each of the $n + 1$ integers is divided by n, at least two must have the same remainder.
- Subtract the smaller from the larger having the same remainder. The result is a multiple of n that has only 0s and 1s in its decimal expansion.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < (N/k + 1)$ has been used. This is a contradiction because there are a total of n objects. ◀

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle

Example: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution:

- We assume four boxes; one for each suit. If we select N cards then (by the generalized pigeonhole principle) at least one box contains at least $\lceil N/4 \rceil$ cards.
- At least 3 cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.
- Find the smallest integer N such that $\lceil N/4 \rceil \geq 3$.
- If N/4 is an integer and $\lceil N/4 \rceil \geq 3 \Leftrightarrow (N/4 \geq 3) \Leftrightarrow (\mathrm{N} \geq 12)$
- N/4 is not an integer and $\lceil N/4 \rceil \geq 3 \Leftrightarrow (N/4 > 2) \Leftrightarrow (\mathrm{N} > 8)$
- The smallest such integer: N= 9.

How many cards must be selected from a standard deck of 52 cards to guarantee that at least 5 cards of the same suit are chosen?

a) 20
b) 17
c) 16
d) IDK