Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an **ordered arrangement** of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

- The permutations of S are: 1,2,3; 2,1,3; 3,2,1, …
- The 2-permutations of S are: 1,2; 1,3; 2,1; 2,3; 3,1; 3,2.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer then number of permutations of n distinct elements $P(n) = n!$

If n, r are integers with $1 \leq r \leq n$, then number of r-permutations of a set with n distinct elements.

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = \frac{n!}{(n-r)!}$$

Note: $P(n,n) = P(n)$

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

a) 8
b) 8^8
c) 8!
d) 7!

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

a) 8!
b) **6!**
c) 5!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Cl.A:

a) 100^3

b) **$100 \cdot 99 \cdot 98$**

c) 100

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Combinations

Definition: An ***r*-combination** of elements of a set is an **unordered selection** of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

- **Example**: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is the same 3-combination as $\{d, c, a\}$, but different from $\{a, b, c\}$ and $\{b, c, d\}$.

- The **number of *r*-combinations** of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

1) What is $C(n, n)$? *$C(n,n)=1$*
2) What is bigger – number of combinations $C(n, r)$ or permutations $P(n, r)$? $P(n, r) > C(n, r)$

Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: To list all r-permutations of n elements, we should list all *r*-combinations, and then do all permutations of each *r*-combination.

By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$.

Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Possible Answers:

a) 50
b) 250
c) **252**
d) 10·9 ·8 ·7 ·6

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards?

a) $= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$

b) $= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$

a) None of the above

Answer: b

- Is this number equal to number of ways to select 47 cards from 52?
- Yes.

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} .$$

Hence, $C(n, r) = C(n, n - r)$. ◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$.
  - Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$.
  - Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◀

Binomial Coefficients and Identities

Section 6.4

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$.

The number of terms $x^{n-j}y^j$ equals to number of ways of picking y for j (x+y) terms (out of n) and picking x for other (x+y) terms. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{j}$.

◀

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j =$ 13.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

◀

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is

$$\sum_{k=0}^{n} \binom{n}{k}.$$

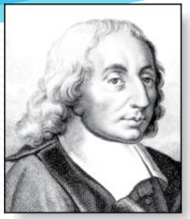
Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

◀

Pascal's Identity

Blaise Pascal
(1623-1662)



Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's Triangle

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
1 \\
1\quad 1 \\
1\quad 2\quad 1 \\
1\quad 3\quad 3\quad 1 \\
1\quad 4\quad 6\quad 4\quad 1 \\
1\quad 5\quad 10\quad 10\quad 5\quad 1 \\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1 \\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1 \\
1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1 \\
\cdots
\end{array}$$

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Generalized Permutations and Combinations

Section 6.5

Section Summary

- Permutations with Repetition
- Permutations with Indistinguishable Objects
- Combinations with Repetition

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition. ◀

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Permutations with Repetition

a) 26^r
b) $P(26, r) = 26!/r!$
c) $C(26, r) = 26!/(r!(26-r)!)$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\ C(n - n_1, n_2)\ \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.
- Applying product rule we get:

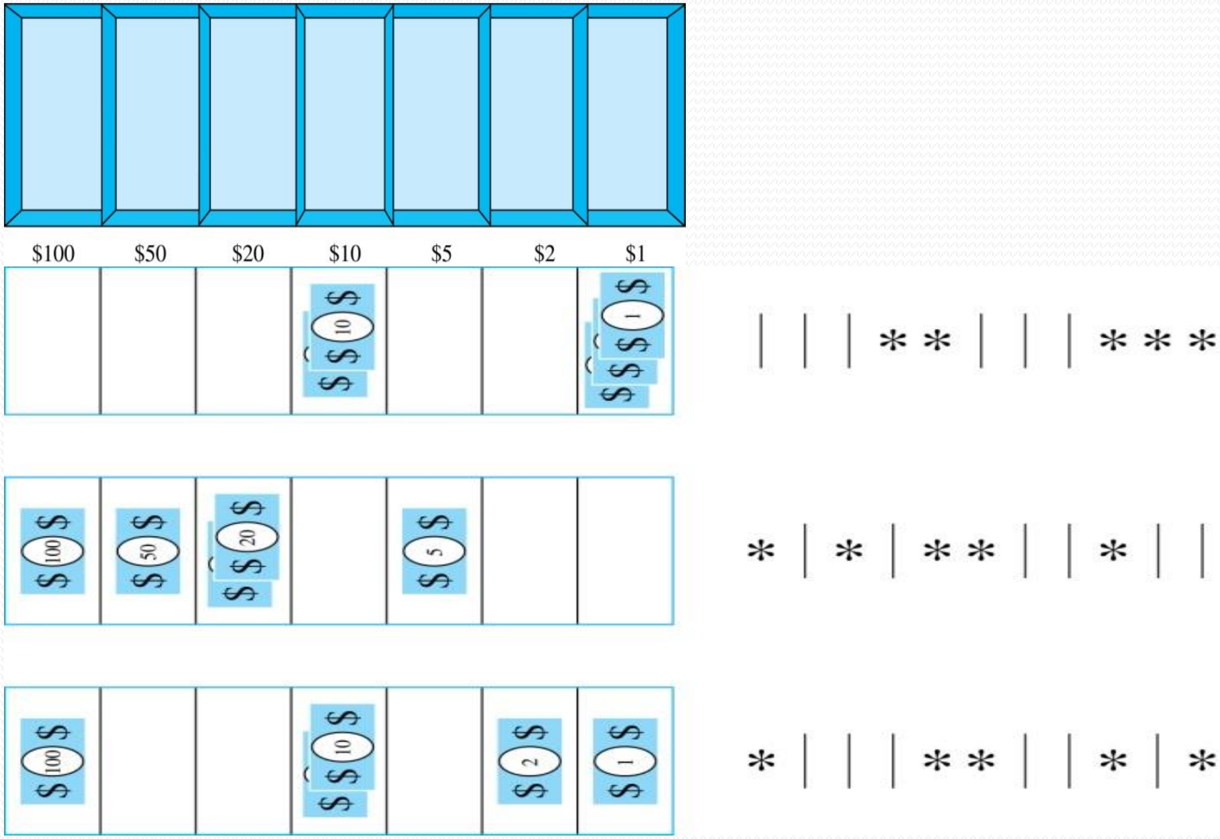
$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Combinations with Repetition

Example: How many ways are there to select five bills from a box containing at least five of each of the following 7 denominations: $1, $2, $5, $10, $20, $50, and $100?

Combinations with Repetition

Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



Combinations with Repetition

- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

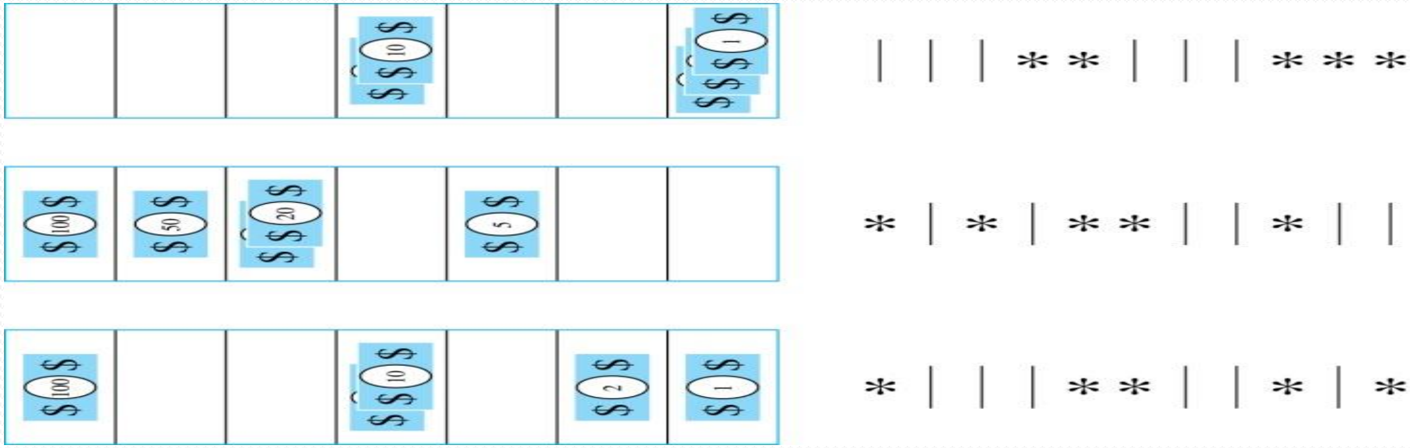
ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of ***r*-combinations** from a set with ***n*** **types of elements** when **repetition of elements is allowed** is

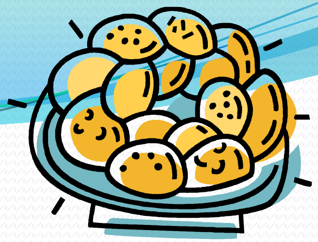
$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Combinations with Repetition



Proof:

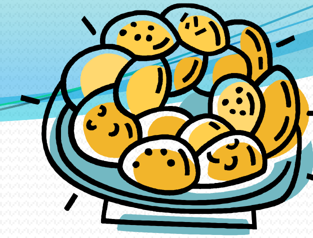
- Each r-combination of a set with n elements with repetition allowed can be represented by a list of n –1 bars and r stars.
- n –1 bars mark the n cells separating different elements,
- r stars mark r elements; stars located in the same cell are identical elements, stars in different cells are different elements.
- The number of such lists is $C(n + r – 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r – 1$ positions to place the stars and the bars.
- This is also equal to $C(n + r – 1, n –1)$, which is the number of ways to place the n –1 bars.



Combinations with Repetition

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

a) C(4,6)
b) C(6,4)
c) C(9,6)
d) C(10,6)



Combinations with Repetition

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

permutations of n objects with indistinguishable n_1 objects of type 1, n_2 – of type 2, … $\frac{n!}{n_1!n_2!\cdots n_k!}$.

Inclusion-Exclusion

Section 8.5

Principle of Inclusion-Exclusion

- In Section 2.2, we developed the following formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- We will generalize this formula to finite sets of any size.

Two Finite Sets

Example: In a discrete mathematics class every student is a major in CS or Math or both.

The number of students with CS major is 25; the number of students with Math major is 13; double-major is allowed. The number of students majoring in both CS and Math is 8.

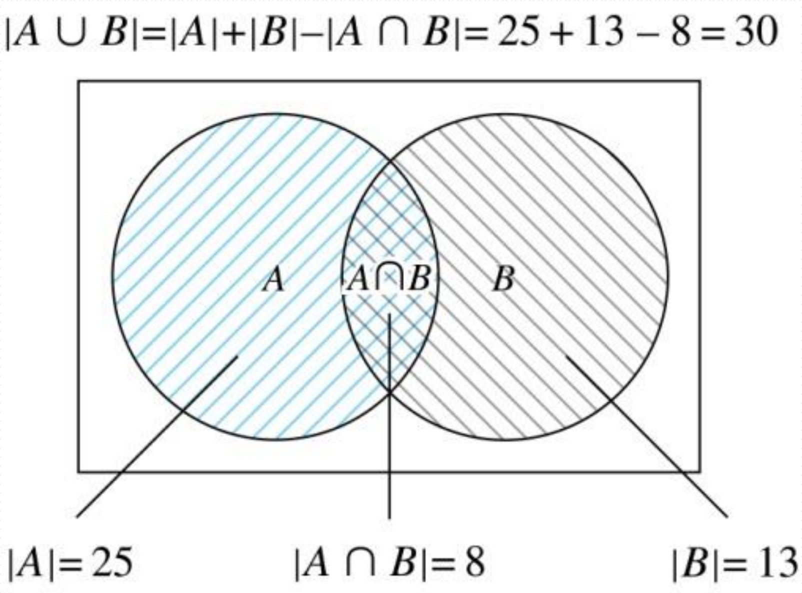
How many students are in the class?

Cl.A:

a) 46
b) 38
c) 30
d) IDK

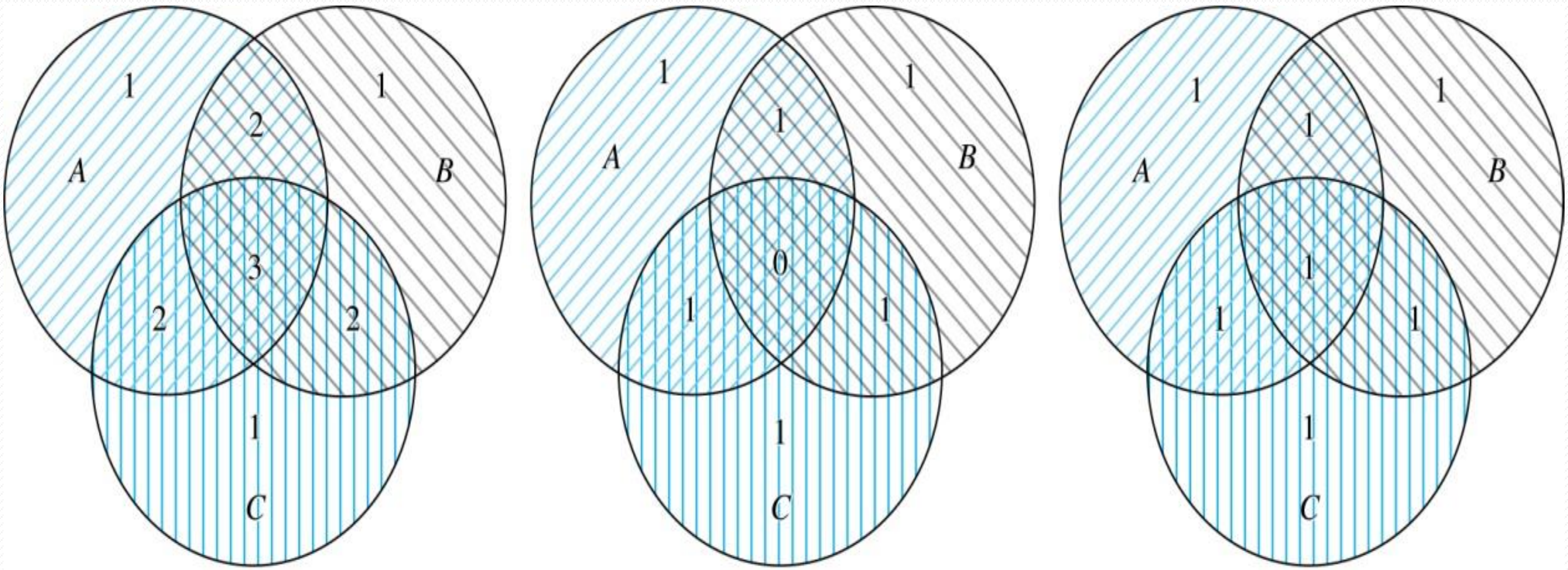
Two Finite Sets

Solution: $|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$



Three Finite Sets $|A \cup B \cup C| =$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

Three Finite Sets Continued

Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian, how many students have taken a course in all 3 languages.

Cl.A:

a) 2,225
b) 2,085
c) 133
d) 7
e) IDK

Three Finite Sets Continued

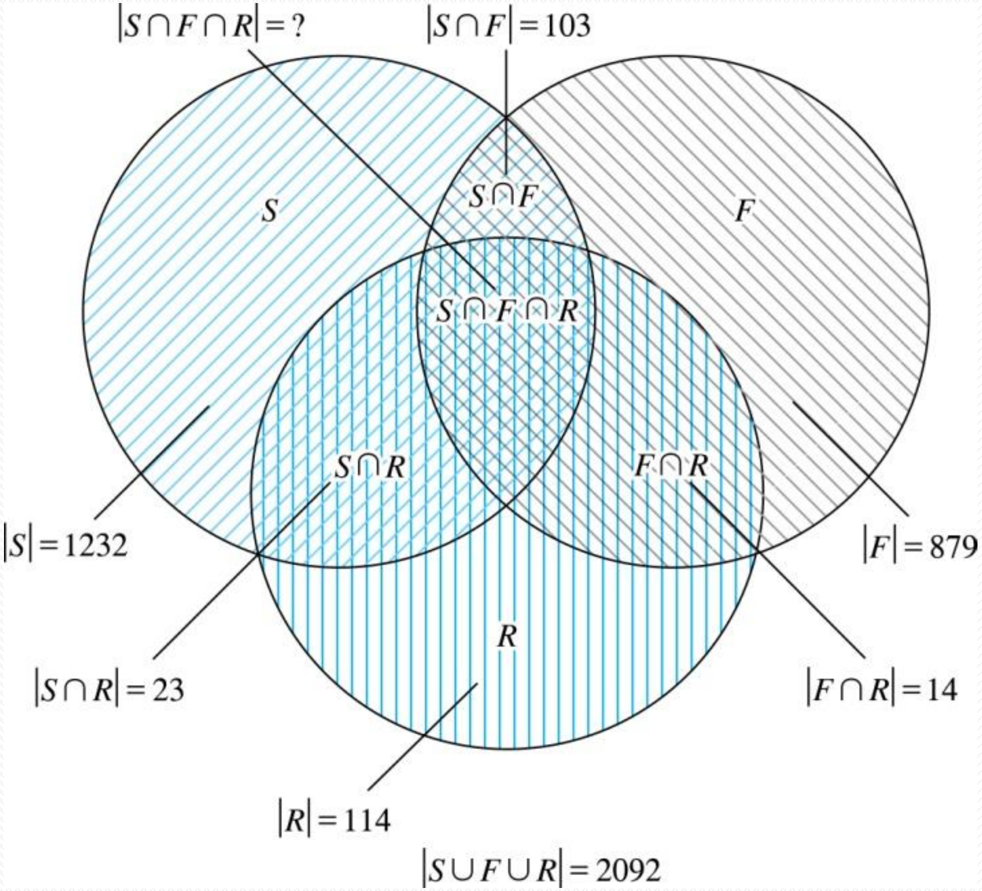
Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken a course in at least one of Spanish French and Russian. How many students have taken a course in all 3 languages.

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian.

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 2092$.

$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$,
$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.
$2092 = 2085 + |S \cap F \cap R|$.
$|S \cap F \cap R| = 7$

Illustration of Three Finite Set Example



The Principle of Inclusion-Exclusion

Theorem 1. The Principle of Inclusion-Exclusion:
Let $A_1, A_2, \ldots, A_n$ be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| =$$

$$\sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i \leq j \leq n} |A_i \cap A_j| +$$

$$\sum_{1 \leq i \leq j \leq k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

The Principle of Inclusion-Exclusion

Example. How many elements are in the union of 5 sets, if

- the set contains 10,000 elements each,
- each pair of sets has 1000 common elements,
- each triple of sets has 100 common elements,
- every four of the sets have 10 common elements,
- and there is 1 element in all five sets.

Solution: There are

- 5 sets
- C(5,2) pairs of sets,
- C(5,3) combinations of three sets,
- C(5,4) combinations of four sets,
- 1 way to combine all five sets

The union of 5 sets contains

$5\times10{,}000 - C(5,2)\times1000 + C(5,3)\times100 - C(5,4)\times10 + 1\times1 =$

$= 5\times10{,}000 - 10\times1000 + 10\times100 - 5\times10 + 1 =$

$= 40{,}951$ elements.

Proof of inclusion-exclusion principle by math induction

The base case is $n=2$, for which we already know the formula to be valid. Assume that the formula is true for n sets. Look at a situation with $n+1$ sets, and temporarily consider $A_n \cup A_{n+1}$ as one set. Then by the inductive hypothesis we have

$$|A_1 \cup \cdots \cup A_{n+1}| = \sum_{i<n} |A_i| + |A_n \cup A_{n+1}| - \sum_{i<j<n} |A_i \cap A_j|$$

$$- \sum_{i<n} |A_i \cap (A_n \cup A_{n+1})| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_{n-1} \cap (A_n \cup A_{n+1})| .$$

Next we apply the distributive law to each term on the right involving $A_n \cup A_{n+1}$, giving us

$$\sum |(A_{i_1} \cap \cdots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \sum |(A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1})| .$$

Now we apply the basis step to rewrite each of these terms as

$$\sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n| + \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1}| - \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n \cap A_{n+1}| ,$$

which gives us precisely the summation we want.

Combinatorial Proof of Inclusion-Exclusion

Consider an element a that is a member of r of the sets $A_1, \ldots, A_n$ where $1 \leq r \leq n$.

- It is counted exactly once in $|A_1 \cup A_2 \cup \cdots \cup A_n| =$
- It is counted $r = C(r,1)$ times by $\sum |A_i|$
- It is counted $C(r,2)$ times by $\sum |A_i \cap A_j|$
- In general, it is counted $C(r,m)$ times by counting elements in all the intersections of m sets A_i.

Combinatorial Proof of Inclusion-Exclusion

By Corollary 2 of Section 6.4, we have

$C(r,0) - C(r,1) + C(r,2) - \cdots + (-1)^r C(r,r) = 0.$

$1 = C(r,0),$

so $1 = C(r,1) - C(r,2) + \cdots + (-1)^{r+1} C(r,r).$

Thus, each element ***a*** in $A_1 \cup A_2 \cup \ldots \cup A_n$ **,** that is a member of r of the sets $A_1, \ldots, A_n$ is counted exactly **once**

by $\sum |A_i| - \sum |A_i \cap A_j| + \ldots + (-1)^{r+1} \sum |A_{i_1} \cap A_{i_2} \cap \ldots A_{i_r}|$

So exactly once by both left and right hand side of inclusion-exclusion principle on slide 47