1. Let $\nabla f = \begin{pmatrix} 4x_1^3+6x_1^2-4x_2x_1 \\ -2x_1^2+8x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

We have $\begin{matrix} 4x_1^3+6x_1^2-4x_2x_1=0 \\ x_1^2=4x_2 \end{matrix}$ $\Rightarrow 4x_1^3+6x_1^2-(x_1^2)x_1 = 3x_1^3+6x_1^2 = 3x_1^2(x_1+2)=0 \Rightarrow x_1=0$ or $-2 \Rightarrow \begin{cases} x_1=0 \\ x_2=0 \end{cases}$ or $\begin{cases} x_1=-2 \\ x_2=1 \end{cases}$

$\nabla^2 f(x) = \begin{pmatrix} 12x_1^2+12x_1-4x_2 & -4x_1 \\ -4x_1 & 8 \end{pmatrix}$

$\nabla^2 f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$ $\quad |\nabla^2 f(0,0)-\lambda I| = -\lambda(8-\lambda)=0 \Rightarrow \lambda=0$ or $8 \geq 0$

$\nabla^2 f(-2,1) = \begin{pmatrix} 20 & 8 \\ 8 & 8 \end{pmatrix}$ $\quad |\nabla^2 f(-2,1)-\lambda I| = (20-\lambda)(8-\lambda)-64 = (\lambda-4)(\lambda-24)=0 \Rightarrow \lambda=4$ or $24>0$

All eigenvalues of $\nabla^2 f(-2,1)$ are greater than 0.

$(-2,1)$ is strict minimizer.

$(0,0)$ is saddle point.

proof: $(0,0)$ is a saddle point

Consider the definition of local minimum: $\forall \bar{\alpha}\ \exists \alpha \in [0,\bar{\alpha}]\ \forall d \in \mathbb{R}^2\ f(x_0+\alpha d) \geq f(x_0)$, x_0 is the point of local min.

$\forall \bar{\alpha}>0, \exists \alpha \in [0,\bar{\alpha}]$, $f((0,0)+\alpha(-1,0)) = \alpha^4-2\alpha^3 = \alpha^3(\alpha-2)$ we let $\alpha = \begin{cases} \bar{\alpha}, & 0<\bar{\alpha}<2 \\ 1, & \bar{\alpha}\geq 2 \end{cases}$ $\Rightarrow f((0,0)+\alpha(-1,0)) < 0 = f(0,0)$

$(0,0)$ isn't local minimizer

One of the eigenvalues is 8 which >0.

$(0,0)$ isn't local maximizer.

2. a. $\nabla f(x) = \begin{pmatrix} \frac{1}{4}x_1^2 - x_2^2 - 2 \\ -2x_1x_2+4x_2^3 \end{pmatrix}$

Hessian of f: $\nabla^2 f(x) = \begin{pmatrix} \frac{1}{2}x_1 & -2x_2 \\ -2x_2 & -2x_1+12x_2^2 \end{pmatrix}$

Let $\nabla f(x)=0 \Rightarrow \begin{cases} \frac{1}{4}x_1^2-x_2^2-2=0 \\ 2x_2(2x_2^2-x_1)=0 \end{cases} \Rightarrow x_2=0$ or $x_1=2x_2^2$

if $x_2=0$: $\frac{1}{4}x_1^2-2=0 \Rightarrow x_1=\pm 2\sqrt{2}$

if $x_1=2x_2^2$: $\frac{1}{4}(2x_2^2)^2-x_2^2-2=0 \Rightarrow (x_2^2-2)(x_2^2+1)=0 \Rightarrow x_2=\pm\sqrt{2}$, $x_1=4$

Stationary points: $(2\sqrt{2},0)$ $(-2\sqrt{2},0)$ $(4,\sqrt{2})$ $(4,-\sqrt{2})$

b. ① $\nabla^2 f(2\sqrt{2},0) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -4\sqrt{2} \end{pmatrix}$ $\quad |\nabla^2 f(2\sqrt{2},0)-\lambda I| = (\sqrt{2}-\lambda)(-4\sqrt{2}-\lambda)=0 \Rightarrow \lambda_1=\sqrt{2}>0 \quad \lambda_2=-4\sqrt{2}<0$

$\Rightarrow$ it's a saddle point.

② $\nabla^2 f(-2\sqrt{2},0) = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix}$ $\quad |\nabla^2 f(-2\sqrt{2},0)-\lambda I| = (-\sqrt{2}-\lambda)(4\sqrt{2}-\lambda)=0 \Rightarrow \lambda_1=-\sqrt{2}<0 \quad \lambda_2=4\sqrt{2}>0 \Rightarrow$ it's a saddle point.

③ $\nabla^2 f(4,\sqrt{2}) = \begin{pmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 16 \end{pmatrix}$ $\quad |\nabla^2 f(4,\sqrt{2})-\lambda I| = (2-\lambda)(16-\lambda)-8=0 \Rightarrow \lambda_1=9+\sqrt{57}>0 \quad \lambda_2=9-\sqrt{57}>0 \Rightarrow$ it's a local minimizer.

④ $\nabla^2 f(4,-\sqrt{2}) = \begin{pmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 16 \end{pmatrix}$ $\quad |\nabla^2 f(4,\sqrt{2})-\lambda I| = (2-\lambda)(16-\lambda)-8=0 \Rightarrow \lambda_1=9+\sqrt{57}>0 \quad \lambda_2=9-\sqrt{57}>0 \Rightarrow$ it's a local minimizer.

3. (a) $g_1(\bar{x}) = -1$ $\quad g_2(\bar{x}) = -1$ $\quad g_3(\bar{x}) = 0$

$\mathcal{A}(\bar{x}) = \{ g_3(x) \le 0 \}$

$\nabla g_1(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \nabla g_2(\bar{x}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\quad \nabla g_3(\bar{x}) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$(\nabla g_3(\bar{x})) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ is linear independent

$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ has full rank

(b) KKT condition: $L(x,\lambda) = f(x) + g(x)^T\lambda$ $\quad \lambda := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$

$\nabla L(x,\lambda) = \nabla f(x) + \nabla g(x)\lambda = \begin{pmatrix} 3x_1^2 + 2 - 2x_2^2 \\ -4x_1x_2 + 12x_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 - 2x_1\lambda_3 \\ -\lambda_2 - 2x_2\lambda_3 \end{pmatrix} = 0$ ①

$\lambda \ge 0$ ②

$g(x) = \begin{pmatrix} x_1 - 1 \\ -x_2 \\ 1 - (x_1^2 + x_2^2) \end{pmatrix} \le 0$ ③

$\lambda_1(x_1 - 1) = 0$ ④

$\lambda_2(-x_2) = 0$ ⑤

$\lambda_3[1 - (x_1^2 + x_2^2)] = 0$ ⑥

$x_1(3x_1^2 + 2 - 2x_2^2 + \lambda_1 - 2x_1\lambda_3) = 0$ ⑦

$x_2(-4x_1x_2 + 12x_2 - \lambda_2 - 2x_2\lambda_3) = 0$ ⑧

For ④ $\lambda_1(0-1) = 0 \Rightarrow \lambda_1 = 0$

For ⑤ $\lambda_2(-1) = 0 \Rightarrow \lambda_2 = 0$

For ⑥ $\lambda_3[1 - (0^2 + 1^2)] = \lambda_3 \cdot 0 = 0$

For ⑦ $0(3\cdot 0^2 + 2 - 2\cdot 1^2 + 0 - 2\cdot 0\cdot \lambda_3) = 0$

For ⑧ $1\cdot(-4\cdot 0\cdot 1 + 12\cdot 1 - 0 - 2\cdot 1\cdot \lambda_3) = 12 - 2\lambda_3 = 0 \Rightarrow \lambda_3 = 6$

For ① $\begin{pmatrix} 0 + 2 - 2 \\ 0 + 12 \end{pmatrix} + \begin{pmatrix} 0 - 0 \\ 0 - 2\cdot 1\cdot 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ge 0$

For ② $\lambda_1 = 0 \ge 0$ $\quad \lambda_2 = 0 \ge 0$ $\quad \lambda_3 = 6 \ge 0$

For ③ $\begin{pmatrix} 0 - 1 \\ -1 \\ 1 - (0^2 + 1^2) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \le 0$

$\bar{\lambda} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ All conditions are satisfied.

4.(a) $\mathcal{L}(x,\eta_1) = x_1+2x_2+4x_3+\eta_1(\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}-1)$

$$\nabla_x \mathcal{L}(x,\eta_1) = \begin{pmatrix} 1-\frac{4\eta_1}{x_1^2} \\ 2-\frac{2\eta_1}{x_2^2} \\ 4-\frac{\eta_1}{x_3^2} \end{pmatrix} \geq 0$$

$\eta_1 \geq 0$

$\eta_1(\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}-1)=0$

$x_1(1-\frac{4\eta_1}{x_1^2})=0$

$x_2(2-\frac{2\eta_1}{x_2^2})=0$

$x_3(4-\frac{\eta_1}{x_3^2})=0$

$\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3} \leq 1$

$x \geq 0$

(b) $\because x_1, x_2, x_3 \neq 0$

$\therefore 1-\frac{4\eta_1}{x_1^2}=0 \Rightarrow x_1^2=4\eta_1$

$2-\frac{2\eta_1}{x_2^2}=0 \Rightarrow x_2^2=\eta_1$

$4-\frac{\eta_1}{x_3^2}=0 \Rightarrow 4x_3^2=\eta_1$

$\Rightarrow \eta_1>0 \Rightarrow \frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}-1=0$

$\Downarrow$

$$\frac{4}{\sqrt{4\eta_1}}+\frac{2}{\sqrt{\eta_1}}+\frac{1}{\sqrt{\frac{\eta_1}{4}}}=1$$

$\Downarrow$

$\sqrt{\eta_1}=2+2+2=6$

$\eta_1=36$

$\Downarrow$

$x_1=12$

$x_2=6$

$x_3=3$

KKT point: $(x^T,\eta_1)=(12,6,3,36)$

$$x=\begin{pmatrix}12\\6\\3\end{pmatrix} \geq 0$$

5.(a) consider $\min \; -f(x) = -\sum_{i=1}^{n} \ln(1+\frac{x_i}{\sigma_i})$

s.t. $\sum_{i=1}^{n} x_i \le P$

$x_i \geq 0 \quad i=1,2,\cdots n$

$$L(x,\eta_1) = -\sum_{i=1}^{n} \ln(1+\frac{x_i}{\sigma_i}) + \eta_1(\sum_{i=1}^{n} x_i - P)$$

$$\nabla L(x,\eta_1) = \begin{pmatrix} \vdots \\ -\frac{1}{\sigma_i + x_i} \\ \vdots \end{pmatrix} + \eta_1 \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \geq 0$$

$$\eta_1 \geq 0$$

$$x_i(\eta_1 - \frac{1}{\sigma_i + x_i}) = 0 \quad i=1,2,\cdots n$$

$$\eta_1(\sum_{i=1}^{n} x_i - p) = 0$$

$$\sum_{i=1}^{n} x_i \le p$$

$$x_i \geq 0 \quad i=1,2,\cdots,n$$

(b)

$$\begin{pmatrix} -\frac{1}{2+x_1} \\ -\frac{1}{3+x_2} \\ -\frac{1}{1+x_3} \end{pmatrix} + \eta_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \geq 0 \quad ①$$

$$\eta_1 \geq 0 \quad ②$$

$$x_1(\eta_1 - \frac{1}{2+x_1}) = 0 \quad ③$$

$$x_2(\eta_1 - \frac{1}{3+x_2}) = 0 \quad ④$$

$$x_3(\eta_1 - \frac{1}{1+x_3}) = 0 \quad ⑤$$

$$\eta_1(x_1+x_2+x_3-2) = 0 \quad ⑥$$

$$x_1+x_2+x_3 \le 2 \quad ⑦$$

$$x_i \geq 0 \quad i=1,2,3 \quad ⑧$$

For ③, $\frac{1}{2}(\eta_1 - \frac{1}{2+\frac{1}{2}}) = 0 \Rightarrow \eta_1 = \frac{2}{5}$

For ④, $0(\frac{2}{5} - \frac{1}{3+0}) = 0$

For ⑤, $\frac{3}{2}(\frac{2}{5} - \frac{1}{1+\frac{3}{2}}) = 0$

For ⑥ $\frac{2}{5}(\frac{1}{2}+0+\frac{3}{2}-2) = 0$

For ⑦ $\frac{1}{2}+0+\frac{3}{2} = 2 \le 2$

For ⑧ $\frac{1}{2} \geq 0 \quad 0 \geq 0 \quad \frac{3}{2} \geq 0$

For ① $\begin{pmatrix} -\frac{1}{2+\frac{1}{2}} \\ -\frac{1}{3+0} \\ -\frac{1}{1+\frac{3}{2}} \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{15} \\ 0 \end{pmatrix} \geq 0$ For ②, $\frac{2}{5} \geq 0$