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SDS · School of Data Science
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MAT 3007 — Optimization

Exercise Sheet 7

Problem 1 (Convex Sets): (approx. *25* pts)
In this exercise, we study convexity of various sets.

a) Verify whether the following sets are convex or not and explain your answer!

$$\Omega_1 = \{x \in \mathbb{R}^n : \|x - a\|_2 \leq \|x - b\|_2\}, \quad a, b \in \mathbb{R}^n,\ a \neq b,$$
$$\Omega_2 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x^\top x \leq t\}.$$

b) Show that the hyperbolic set $\{x \in \mathbb{R}^2_+ : x_1 x_2 \geq 1\}$ is convex, where $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x \geq 0\}$.

Hint: Rewrite the condition "$x_1 x_2 \geq 1$" in a suitable way.

c) Decide whether the following statements are true or false. Explain your answer and either present a proof / verification or a counter-example.

- The union of two non-convex sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ is always a non-convex set.
- Let $\Omega \subset \mathbb{R}^n$ be a convex set and suppose that the set $S := \{(x,t) \in \Omega \times \mathbb{R} : f(x) \leq t\} \subset \mathbb{R}^n \times \mathbb{R}$ is convex. Then, $f : \Omega \to \mathbb{R}$ is a convex function.

Problem 2 (Convex Compositions): (approx. *25* pts)
Either prove or find a counterexample for each of the following statements (you can assume that all functions are twice continuously differentiable if needed):

a) If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are convex, then the composition $f \circ g : \mathbb{R}^n \to \mathbb{R}$, $(f \circ g)(x) = f(g(x))$ is convex.

b) Let $\Omega \subset \mathbb{R}^n$ be a convex set and suppose that $g : \Omega \to \mathbb{R}$ is convex and $f : I \to \mathbb{R}$ is convex and nondecreasing where $I \supseteq g(\Omega)$ is an interval containing $g(\Omega)$. Then, $f \circ g$ is convex.

c) If $f : \mathbb{R} \to \mathbb{R}$ is increasing and non-negative, then $x \mapsto x f(x)$ is a convex function on $\mathbb{R}_+$.

Problem 3 (Convex Functions): (approx. *25* pts)
In this exercise, convexity properties of different functions are investigated.

a) Verify that the following functions are convex over the specified domain:

- $f : \mathbb{R}_{++} \to \mathbb{R}$, $f(x) := \sqrt{1 + x^{-2}}$, where $\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$.
- $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) := \frac{1}{2}\|Ax - b\|^2 + \mu \|Lx\|_1$, where $A \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^m$, and $\mu > 0$ are given and $\|y\|_1 := \sum_{i=1}^p |y_i|$, $y \in \mathbb{R}^p$.
- $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x,y) := \frac{\lambda}{2}\|x\|^2 + \sum_{i=1}^m \ln(1 + \exp(-b_i(a_i^\top x + y)))$, where $a_i \in \mathbb{R}^n$ and $b_i \in \{-1, 1\}$ are given data points for all $i = 1, \ldots, m$ and $\lambda > 0$ is a parameter.

b) Let $r(x) := \|x\|_q = (\sum_{i=1}^n |x_i|^q)^{1/q}$ be the ℓ_q-norm on $\mathbb{R}^n$ with $q \in [1, \infty)$. Show that r is a convex function.

Hint: As a norm, the mapping r satisfies certain properties that might be useful here.

Problem 4 (Weighted Least-Squares Problem): (approx. *25* pts)
We consider the following least squares-type problem with variable weights:

$$\min_{x,w} \sum_{i=1}^{m} \frac{(a_i^\top x - b_i)^2}{1 + w_i} + \delta^2 \cdot \mathbb{1}^\top w \quad \text{s.t.} \quad w \geq 0, \tag{1}$$

where $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $i = 1, \ldots, m$ are given data points, $\delta > 0$ is a parameter, and $\mathbb{1} \in \mathbb{R}^m$ is the vector of all-ones.

a) Show that problem (1) is a convex optimization problem.

b) Show that the optimization problem (1) can be simplified to the equivalent problem:

$$\min_{x} \sum_{i=1}^{m} \varphi_\delta(a_i^\top x - b_i) \quad \text{where} \quad \varphi_\delta(y) := \begin{cases} y^2 & \text{if } |y| \leq \delta \\ \delta(2|y| - \delta) & \text{if } |y| > \delta. \end{cases} \tag{2}$$

Hint: Optimize over w in (1) assuming that x is fixed and establish a connection to problem (2). Explain your steps and derivations!

Sheet 7 is due on **Apr, 19th**. Submit your solutions before **Apr, 19th**, **12:00 pm (noon)**.