The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 6
Due: 12:00 (noon, not midnight), April 12

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 Optimality Conditions for Unconstrained Problem — I (20 pts).
Consider the function

$$f(x) = x_1^4 + 2(x_1 - x_2)x_1^2 + 4x_2^2$$

Use the first-order necessary condtion (FONC), second order necessary condition (SONC) and second order sufficient condition (SOSC) to find (i) saddle points and (ii) strict local minimizers.
Note: You can use Matlab to compute eigen values

Solution 1.

$$\nabla f = \begin{pmatrix} 4x_1^3 + 6x_1^2 - 4x_1x_2 \\ -2x_1^2 + 8x_2 \end{pmatrix}, \nabla^2 f = \begin{pmatrix} 12x_1^2 + 12x_1 - 4x_2 & -4x_1 \\ -4x_1 & 8 \end{pmatrix}$$

Step 1: Calculate all stationary poinrs of f by solving $\nabla f = 0$:

$$\nabla f(x) = 0 \Leftrightarrow 4x_2 = x_1^2 \text{ and } 3x_1^2(x_1 + 2) = 0$$

Thus, the stationary points are: $x_1^* = (0; 0), x_2^* = (-2; 1)$.

Step 2: Determine the definitenesss of the Hessian $\nabla^2 f(x^*)$ to decide whether the stationary points x^* are local minima, maxima or saddle points:

$$\nabla^2 f(x_1^*) = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix} \text{ and } \nabla^2 f(x_2^*) = \begin{pmatrix} 20 & 8 \\ 8 & 8 \end{pmatrix}$$

Due to $tr(\nabla^2 f(x_2^*)) = 28 > 0$ and $det(\nabla^2 f(x_2^*)) = 160 - 64 > 0$, the Hessian is positive definite and x_2^* is a strict local minimizer. The Hessian $\nabla^2 f(x_1^*)$ has the eigenvalues 0 and 8 and hence it satisfies SONC. We consider the function f directly around x_1^* along axis x_1.

$$f(\pm|t|, 0) = t^4 + (2 \pm |t|)t^2 = |t|^3(|t| \pm 2)$$

Consequently, f is increasing and decreasing round (0,0) and x_1^* has to be a (degenerate) saddle point. An easier way is to use Matlab to draw this function and one can visually see x_1^* is a degenerate saddle point.

Problem 2 Optimality Conditions for Unconstrained Problem — II (20 pts).
Consider the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^2} \ f(x) = \frac{1}{12}x_1^3 - x_1\left(2 + x_2^2\right) + x_2^4.$$

(a) Compute the gradient and Hessian of f and calculate all stationary points.

(b) For each stationary point, investigate whether it is a local maximizer, local minimizer, or saddle point and explain your answer.

Solution 2.

(a) The gradient and Hessian of f are given by

$$\nabla f(x) = \begin{pmatrix} \frac{1}{4}x_1^2 - x_2^2 - 2 \\ -2x_1x_2 + 4x_2^3 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x) = \begin{pmatrix} \frac{1}{2}x_1 & -2x_2 \\ -2x_2 & -2x_1 + 12x_2^2 \end{pmatrix}.$$

Moreover, $\nabla f(x) = 0$ implies $2x_2(2x_2^2 - x_1) = 0$. Let us first consider the case $x_2 = 0$. Then, it follows $x_1 = \pm 2\sqrt{2}$. Otherwise, we have $x_1 = 2x_2^2$ which implies $x_2^4 - x_2^2 - 2 = 0$, i.e.,

$$x_2^2 = 2 \text{ or } -1$$

This yields $x_2 = \pm\sqrt{2}$ and $x_1 = 4$. In total, f has the following four stationary points:

$$\bar{x}_1 = \left(2\sqrt{2}, 0\right), \quad \bar{x}_2 = \left(-2\sqrt{2}, 0\right), \quad \bar{x}_3 = \left(4, \sqrt{2}\right), \quad \bar{x}_4 = \left(4, -\sqrt{2}\right).$$

(b) We have

$$\nabla^2 f(\bar{x}_1) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -4\sqrt{2} \end{pmatrix} \quad \text{and} \quad \nabla^2 f(\bar{x}_2) = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix}.$$

These two Hessians are diagonal matrices with eigenvalues $\sqrt{2}, -4\sqrt{2}$ and $-\sqrt{2}, 4\sqrt{2}$ respectively and, hence $\nabla^2 f(\bar{x}_1)$ and $\nabla^2 f(\bar{x}_2)$ are indefinite and the stationary points $\bar{x}_1$ and $\bar{x}_2$ are saddle points. Furthermore, it holds that

$$\nabla^2 f(\bar{x}_3) = \begin{pmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 16 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(\bar{x}_4) = \begin{pmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 16 \end{pmatrix}$$

and $\mathrm{tr}(\nabla^2 f(\bar{x}_3)) = \mathrm{tr}(\nabla^2 f(\bar{x}_4)) = 18 > 0$ and $\det(\nabla^2 f(\bar{x}_3)) = \det(\nabla^2 f(\bar{x}_4)) = 32 - 8 > 0$. This shows that $\nabla^2 f(\bar{x}_3)$ and $\nabla^2 f(\bar{x}_4)$ are positive definite. Thus, by the second order sufficient conditions, $\bar{x}_3$ and $\bar{x}_4$ are strict local minimizer.

Problem 3 KKT Conditions for Constrained Problem — I (20 pts).
We consider the nonlinear program

$$\min_{x\in\mathbb{R}^2} \ f(x) := x_1^3 + x_1(2-2x_2^2) + 6x_2^2 \quad \text{subject to} \quad g(x) \le 0, \tag{1}$$

where the constraint function $g : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$g_1(x) := x_1 - 1, \quad g_2(x) := -x_2, \quad g_3(x) := 1 - (x_1^2 + x_2^2).$$

Let us further set $\bar{x} := (0, 1)$.

(a) Determine the active set $\mathcal{A}(\bar{x})$ and show that the LICQ is satisfied at $\bar{x}$.

(b) Investigate whether $\bar{x}$ is a KKT point of problem (1) and calculate a corresponding Lagrange multiplier $\bar{\lambda} \in \mathbb{R}^3$.

Solution 3.

(a) It holds

$$\nabla g_1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nabla g_2(x) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \nabla g_3(x) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix},$$

and

$$g_1(\bar{x}) = g_2(\bar{x}) = -1 < 0, \quad g_3(\bar{x}) = 0.$$

Consequently, it follows $\mathcal{A}(\bar{x}) = \{3\}$. Since the vector $\nabla g_3(\bar{x}) = (0, -2)^\top$ is not zero, the LICQ is satisfied at $\bar{x}$.

(b) As shown, $\bar{x}$ is a feasible point. Moreover, due to $g_1(\bar{x}), g_2(\bar{x}) < 0$, it follows $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$. Hence, $\bar{\lambda}_3$ must satisfy:

$$\nabla f(\bar{x}) + \nabla g_3(\bar{x})\bar{\lambda}_3 = \begin{pmatrix} 3\bar{x}_1^2 + 2 - 2\bar{x}_2^2 \\ -4\bar{x}_1\bar{x}_2 + 12\bar{x}_2 \end{pmatrix} + \bar{\lambda}_3 \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 12 - 2\bar{\lambda}_3 \end{pmatrix} = 0,$$

and we immediately obtain that $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$ and $\bar{\lambda}_3 = 6$ are valid candidates for the Lagrange multipliers. Obviously, the complementarity conditions are also satisfied. Hence, $(\bar{x}, \bar{\lambda})$ is a KKT triple and $\bar{x}$ is a KKT point.

Problem 4 KKT Conditions for Constrained Problem — II (20 pts).
Consider the optimization problem:

$$\begin{array}{rl} \text{minimize} & x_1 + 2x_2 + 4x_3 \\ \text{subject to} & \frac{4}{x_1} + \frac{2}{x_2} + \frac{1}{x_3} \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

(a) Write down the KKT conditions for this problem.

(b) Find the KKT points.

Note: This problem is actually convex and any KKT points must be globally optimal (we will study convex optimization soon).

Solution 4.

(a) First we associate a Lagrangian multiplier λ for the constraint and write down the Lagrangian function:

$$L(x_1, x_2, x_3, \lambda) = x_1 + 2x_2 + 4x_3 + \lambda \cdot (4/x_1 + 2/x_2 + 1/x_3 - 1).$$

We have the following KKT conditions:

- Main conditions
$$1 - \frac{4\lambda}{x_1^2} \geq 0; \quad 2 - \frac{2\lambda}{x_2^2} \geq 0; \quad 4 - \frac{\lambda}{x_3^2} \geq 0$$
- Dual feasibility condition: $\lambda \geq 0$;
- Complementarity conditions:
$$x_1 \cdot (1 - \frac{4\lambda}{x_1^2}) = 0; \quad x_2 \cdot (2 - \frac{2\lambda}{x_2^2}) = 0; \quad x_3 \cdot (4 - \frac{\lambda}{x_3^2}) = 0; \quad \lambda \cdot (\frac{4}{x_1} + \frac{2}{x_2} + \frac{1}{x_3} - 1) = 0$$
- Primal feasibility conditions:
$$\frac{4}{x_1} + \frac{2}{x_2} + \frac{1}{x_3} \leq 1; \quad x_1, x_2, x_3 \geq 0$$

(b) From the primal feasibility condition, we can tell that $x_1 \neq 0$, $x_2 \neq 0$, $x_3 \neq 0$. Therefore, from the complementarity conditions, we must have

$$x_1^2 = 4\lambda; \quad x_2^2 = \lambda; \quad x_3^2 = \lambda/4.$$

Thus $x_1 = 2\sqrt{\lambda}$, $x_2 = \sqrt{\lambda}$, $x_3 = \frac{\sqrt{\lambda}}{2}$. Then because $x_1 \neq 0$, we know that $\lambda \neq 0$. Thus, from the complementarity conditions, we must have

$$\frac{4}{x_1} + \frac{2}{x_2} + \frac{1}{x_3} = 1.$$

Plugging $x_1 = 2\sqrt{\lambda}$, $x_2 = \sqrt{\lambda}$, $x_3 = \frac{\sqrt{\lambda}}{2}$ into the above equation, we have $\lambda = 36$.

Therefore, the only solution to the KKT condition is $x_1 = 12$, $x_2 = 6$, $x_3 = 3$.

Problem 5 KKT Conditions for Constrained Problem — III (20 pts).
Consider the following spectrum management optimization problem

$$\begin{array}{ll} \text{maximize} & f(x) = \sum_{i=1}^n \ln(1 + \frac{x_i}{\sigma_i}) \\ \text{subject to} & \sum_{i=1}^n x_i \leq P \\ & x_i \geq 0, i = 1, 2, ..., n \end{array}$$

where $\sigma_i > 0, i = 1, 2, ..., n$, and $P > 0$.

(a) Derive the KKT conditions for this problem.

(b) Suppose $n = 3$ and $\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 1, P = 2$, show that $(\frac{1}{2}, 0, \frac{3}{2})$ is KKT point to this optimization problem.

Note: Again, this problem is convex and a KKT point is sufficient to be a global maximizer.

Solution 5.

(a) For the constraints $\sum_{i=1}^{n} x_i \leq P$, we introduce dual variable $\lambda_0 \geq 0$. For the non-negative constraints, we introduce $\lambda_i \geq 0$ for each constraint. Therefore, the KKT condition is

Main condition	$-\frac{1}{\sigma_i + x_i} + \lambda_0 - \lambda_i = 0, i = 1, 2, ..., n$
Dual feasibility	$\lambda_0 \geq 0, \lambda_i \geq 0$
Complementarity condition	$\lambda_0(\sum_{i=1}^{n} x_i - P) = 0, \lambda_i x_i = 0$
Primal feasibility	$\sum_{i=1}^{n} x_i \leq P, x_i \geq 0$

(b) To verify if a point is a KKT point, we only need to insert the solution back to the KKT condition and verify if there exists feasible dual variable λ.

Since $x_1, x_3 > 0$, by the complementarity condition we have $\lambda_1 = \lambda_3 = 0$. By main condition, we have $\lambda_0 = \frac{1}{\sigma_1 + x_1} = \frac{2}{5}$. Therefore, $\lambda_2 = -\frac{1}{\sigma_2} + \lambda_0 = \frac{1}{15}$.

Now we have $\lambda_0 = \frac{2}{5}, \lambda_1 = 0, \lambda_2 = \frac{1}{15}, \lambda_3 = 0$ to be the candidate for feasibility of KKT condition. We insert the solution x and the dual variable λ back to KKT condition. We can see all conditions are satisfied and hence $(\frac{1}{2}, 0, \frac{3}{2})$ is a KKT point.