MAT3007 · Homework 6

Due: 12:00 (noon, not midnight), April 12

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 Optimality Conditions for Unconstrained Problem — I (20 pts).
Consider the function

$$f(x) = x_1^4 + 2(x_1 - x_2)x_1^2 + 4x_2^2$$

Use the first-order necessary condtion (FONC), second order necessary condition (SONC) and second order sufficient condition (SOSC) to find (i) saddle points and (ii) strict local minimizers.
Note: You can use Matlab to compute eigen values

Problem 2 Optimality Conditions for Unconstrained Problem — II (20 pts).
Consider the unconstrained optimization problem

$$\min_{x\in\mathbb{R}^2} \; f(x) = \frac{1}{12}x_1^3 - x_1\left(2 + x_2^2\right) + x_2^4.$$

(a) Compute the gradient and Hessian of f and calculate all stationary points.

(b) For each stationary point, investigate whether it is a local maximizer, local minimizer, or saddle point and explain your answer.

Problem 3 KKT Conditions for Constrained Problem — I (20 pts).
We consider the nonlinear program

$$\min_{x\in\mathbb{R}^2} \; f(x) := x_1^3 + x_1(2 - 2x_2^2) + 6x_2^2 \quad \text{subject to} \quad g(x) \le 0, \tag{1}$$

where the constraint function $g : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$g_1(x) := x_1 - 1, \quad g_2(x) := -x_2, \quad g_3(x) := 1 - (x_1^2 + x_2^2).$$

Let us further set $\bar{x} := (0, 1)$.

(a) Determine the active set $\mathcal{A}(\bar{x})$ and show that the LICQ is satisfied at $\bar{x}$.

(b) Investigate whether $\bar{x}$ is a KKT point of problem (1) and calculate a corresponding Lagrange multiplier $\bar{\lambda} \in \mathbb{R}^3$.

Problem 4 KKT Conditions for Constrained Problem — II (20 pts).
Consider the optimization problem:

$$\begin{array}{ll} \text{minimize} & x_1 + 2x_2 + 4x_3 \\ \text{subject to} & \frac{4}{x_1} + \frac{2}{x_2} + \frac{1}{x_3} \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

(a) Write down the KKT conditions for this problem.

(b) Find the KKT points.

Note: This problem is actually convex and any KKT points must be globally optimal (we will study convex optimization soon).

Problem 5 KKT Conditions for Constrained Problem — III (20 pts).
Consider the following spectrum management optimization problem

$$\begin{array}{ll} \text{maximize} & f(x) = \sum_{i=1}^{n} \ln(1 + \frac{x_i}{\sigma_i}) \\ \text{subject to} & \sum_{i=1}^{n} x_i \le P \\ & x_i \ge 0, i = 1, 2, ..., n \end{array}$$

where $\sigma_i > 0, i = 1, 2, ..., n$, and $P > 0$.

(a) Derive the KKT conditions for this problem.

(b) Suppose $n = 3$ and $\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 1, P = 2$, show that $(\frac{1}{2}, 0, \frac{3}{2})$ is KKT point to this optimization problem.

Note: Again, this problem is convex and a KKT point is sufficient to be a global maximizer.