The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 4

Due: 12:00 (**noon, not midnight**), March 15

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (20pts). Consider the following linear program:

$$\begin{array}{rl} \text{maximize} & x_1 + 6x_2 + 6x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \leq 1 \\ & x_1 + 3x_2 + 2x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. What is the corresponding dual problem?
2. Solve the dual problem graphically.
3. Use complementarity slackness to solve the primal problem.

Problem 2 (20pts). Suppose M is a square matrix such that $M = -M^T$, for example,

$$M = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}.$$

Consider the following optimization problem:

$$\begin{array}{rl} \text{minimize}_x & c^T x \\ \text{subject to} & Mx \geq -c \\ & x \geq 0 \end{array}$$

(a) Show that the dual problem of it is equivalent to the primal problem.

(b) Argue that the problem has optimal solution if and only if there is a feasible solution,

Problem 3 (20pts). We consider the general linear optimization problem:

$$\min_{x} \; c^\top x \quad \text{subject to} \quad Ax \leq b, \quad Cx = d, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}, C \in \mathbb{R}^{p\times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $d \in \mathbb{R}^p$ are given. Derive the dual of problem (1) and show that the dual of the dual is equivalent to problem (1).

Problem 4 (20pts). We consider the following robust linear program

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \max_{a\in\mathcal{P}_i} \; a^\top x \leq b_i \quad i = 1, ..., m, \end{array}$$

with variable $x \in \mathbb{R}^n$ and $\mathcal{P}_i := \{a : C_i a \leq d_i\}$. The problem data are $c \in \mathbb{R}^n$, $C_i \in \mathbb{R}^{m_i\times n}$, $d_i \in \mathbb{R}^{m_i}$, and $b \in \mathbb{R}^m$. We assume that the polyhedra $\mathcal{P}_i$ are all nonempty.
Show that this problem is equivalent to the linear optimization problem

$$\begin{array}{lll} \text{minimize}_{x,z_1,\ldots,z_m} & c^\top x & \\ \text{subject to} & d_i^\top z_i \leq b_i & i = 1, ..., m \\ & C_i^\top z_i = x & i = 1, ..., m \\ & z_i \geq 0 & i = 1, ..., m. \end{array}$$

Problem 5 (20pts). Use the strong duality theorem to prove Gordan's theorem: Either $Ax > 0$ has a solution, or $A^\top y = 0, y \gneqq 0$ has a solution. (Let x, y be two vectors. Denote $x \gneqq y$ to be $x \geq y$ and $x \neq y$. Denote $x > y$ to be $x_i > y_i, \forall i = 1, \ldots, n$.)