The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 5

Due: 12:00 (noon, not midnight), March 22

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (50pts). Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & 3x_1 + 4x_2 + 3x_3 + 6x_4 & \\ \text{subject to} & 2x_1 + x_2 - x_3 + x_4 & \geq 12 \\ & x_1 + x_2 + x_3 + x_4 & = 8 \\ & -x_2 + 2x_3 + x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array} \tag{1}$$

After transforming the problem into standard form and apply Simplex method, we obtain the final tableau as follow:

B	0	2	9	0	3	0	36
1	1	0	-2	0	-1	0	4
4	0	1	3	1	1	0	4
6	0	-2	-1	0	-1	1	6

a) Derive the dual problem of the linear program (1) and calculate a dual solution based on complementarity conditions. Given that the optimal solution to the primal solution is unique, investigate whether the dual solution is unique.

b) Do the optimal primal solution and the objective function value change if we

- decrease the objective function coefficient for x_3 to 1?
- increase the objective function coefficient for x_3 to 12?
- decrease the objective function coefficient for x_1 to 1?
- increase the objective function coefficient for x_1 to 7?

e) Find the possible range for adjusting the coefficient 8 of the second constraint such that the current basis is kept optimal.

Solution 1.

a) The dual of problem (1) is given by

$$\begin{array}{lll} \text{minimize} & 12y_1 + 8y_2 + 12y_3 & \\ \text{subject to} & y_1 \le 0, y_2 \text{ free}, y_3 \ge 0 & \\ & 2y_1 + y_2 & \ge 3 \\ & y_1 + y_2 - y_3 & \ge 4 \\ & -y_1 + y_2 + 2y_3 & \ge 3 \\ & y_1 + y_2 + y_3 & \ge 6. \end{array}$$

Using the complementarity conditions, we can infer $2y_1 + y_2 - 3 = 0$, $y_1 + y_2 + y_3 - 6 = 0$, and $y_3(-x_2^* + 2x_3^* + x_4^* - 10) = -6y_3 = 0$. (Since the optimal slack variables are not all zero, the corresponding primal constraints needs to be inactive). This yields $y_3 = 0$ and:

$$2y_1 + y_2 = 3, \quad y_1 + y_2 = 6 \quad \Longrightarrow \quad y_1 = -3, \quad y_2 = 9.$$

Since the primal solution is unique and the complementarity conditions fully characterize the dual solution $y^* = (-3, 9, 0)^\top$, the dual problem has a unique solution as well.

b) Using the final simplex tableau, we obtain:

$$A_B^{-1} A_N = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{pmatrix} \quad \text{and we have} \quad A_B^{-1} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

We now discuss the different questions step by step:

- Decreasing $c_3 = 3$ to 1 means that the new costs (in standard form) are given by: $\tilde{c} = (-3, -4, -1, -6, 0, 0)^\top$. Due to $\{3\} \notin B$, we can simply check:

  $$r_N^\top + \begin{pmatrix} 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 11 & 3 \end{pmatrix} \ge 0.$$

  In this case, both the optimal value as well as the optimal solution do not change.

- Increasing $c_3 = 3$ to 12 means that the new costs (in standard form) are given by: $\tilde{c} = (-3, -4, -12, -6, 0, 0)^\top$. Due to $\{3\} \notin B$, we can simply check:

  $$r_N^\top + \begin{pmatrix} 0 & -9 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix} \ge 0.$$

  In this case, both the optimal value as well as the optimal solution do not change. (However, the problem might have multiple optimal solutions - as demonstrated by running CVX).

- Decreasing $c_1 = 3$ to 1 means that the new costs (in standard form) are given by: $\tilde{c} = (-1, -4, -3, -6, 0, 0)^\top$. We then have:

  $$\tilde{c}_N^\top - \tilde{c}_B^\top A_B^{-1} A_N = \begin{pmatrix} -4 & -3 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -6 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 13 & 5 \end{pmatrix}.$$

  Thus, $x^* = (4, 0, 0, 4, 0, 6)^\top$ will remain optimal solution with new optimal value 28.

- Increasing $c_1 = 3$ to 7 means that the new costs (in standard form) are given by: $\tilde{c} = (-7, -4, -3, -6, 0, 0)^\top$. We then have:

$$\tilde{c}_N^\top - \tilde{c}_B^\top A_B^{-1} A_N = \begin{pmatrix} -4 & -3 & 0 \end{pmatrix} - \begin{pmatrix} -7 & -6 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & -1 \\ 1 & 3 & 1 \\ -2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}.$$

  The optimal solution will change in this case. (Calling CVX, we can find $x^* = (8, 0, 0, 0, 4, 10)^\top$ with optimal value 56).

c) We need to find the range of λ such that $x_B^* + \lambda A_B^{-1} e_2$ is nonnegative, i.e.,

$$\begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 - \lambda \\ 4 + 2\lambda \\ 6 - 2\lambda \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields $-2 \leq \lambda \leq 3$ and thus, the coefficient 8 can be chosen from the interval $[6, 11]$ without changing the current optimal basis.

Problem 2 (50pts).
Consider the following linear program:

$$\begin{array}{lccccccccl} \text{minimize} & x_1 & + & x_2 & + & 2x_3 & + & x_4 & & \\ \text{subject to} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 2 \\ & 2x_1 & & & + & x_3 & - & x_4 & \leq & 2 \\ & -x_1 & & & - & 2x_3 & + & x_4 & \geq & 1 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

Denote $x = (x_1, x_2, x_3, x_4, s_1, s_2)$ as the decision variable to the standard form of the above problem, where s_1, s_2 are the slack variables corresponding to the second and third constraints. The following table gives the final simplex tableau when solving the standard form of the above problem:

B	1	0	$\frac{7}{2}$	0	0	$\frac{1}{2}$	$-\frac{3}{2}$
2	1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
4	-1	0	-2	1	0	-1	1
5	1	0	-1	0	1	-1	3

a) In what range can we change the first objective coefficient $c_1 = 1$ so that the current optimal basis still remains optimal? If we change $c_1 = 1$ to $c_1 = 100$, what will be the new primal optimal solution and optimal value?

b) In what range can we change the second objective coefficient $c_2 = 1$ so that the current optimal basis still remains optimal?

c) In what range can we change the coefficient of the third constraint $b_3 = 1$ (the one appearing in the constraint $-x_1 - 2x_3 + x_4 \geq 1$) so that the current optimal basis still remains optimal?

d) What will be the new optimal primal and dual solutions when we change $b_3 = 1$ to $b_3 = \frac{3}{2}$?

Solution 2.

a) Since $j \in N$, the condition is

$$r_N + \lambda e_j \geq 0,$$

where

$$e_j = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From the simplex tableau, we can read that

$$r_N = \left[1, \tfrac{7}{2}, \tfrac{1}{2}\right]^\top.$$

Thus, the condition on λ is

$$\left[1, \tfrac{7}{2}, \tfrac{1}{2}\right]^\top + \lambda \left[1, 0, 0\right]^\top \geq 0$$

which gives $\lambda \geq -1$. Thus, we can choose $c_1 \geq 0$.

If $c_1 = 100$, the optimal basis will remain the same and changing c does not affect x_B^*. Thus, we have the new primal optimal solution and the optimal value keep unchanged. From the simplex tableau, we have $x^* = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$ and optimal value $V^* = \frac{3}{2}$.

b) Since $j \in B$, the condition is

$$r_N^\top - \lambda e_j^\top A_B^{-1} A_N \geq 0,$$

where

$$e_j = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From the simplex tableau, we can read that

$$A_B^{-1} A_N = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Thus, the condition on λ is

$$\left[1, \tfrac{7}{2}, \tfrac{1}{2}\right] - \lambda \left[1, 0, 0\right] \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix} \geq 0,$$

which gives $\lambda \leq 1$. Thus, we can choose $c_2 \leq 2$.

c) The condition is

$$x_B^* + \lambda A_B^{-1} e_3 \geq 0,$$

where

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & -2 & 1 & 0 & -1 \end{bmatrix}.$$

Let

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

be a diagonal matrix formed by the 2th, 5th, and 6th columns of matrix A. Then, we have (from the simplex tableau) that

$$S = A_B^{-1} D = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Since

$$D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Thus,

$$A_B^{-1} = SD^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, the condition on λ is

$$\begin{bmatrix} \frac{1}{2} \\ 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix} \geq 0,$$

which gives $-1 \leq \lambda \leq 1$. Overall, we can choose $b_3 \in [0, 2]$.

More comments on reading A_B^{-1}: What if there are no columns that can form an identity matrix in A? In this case, if there are columns that forms a diagonal matrix, we can still use the final tableau to get A_B^{-1} without explicitly computing the inverse, like the way we used in this question. For instance, let us assume

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 & 0 \\ 5 & 5 & 0 & 0 & 3 \end{pmatrix}$$

We can see $[A_3, A_4, A_5]$ forms a diagonal matrix. Define $D := [A_3, A_4, A_5]$. Suppose we read $S := A_B^{-1} D$ from the the corresponding columns of the final tableau. Then, we can calculate A_B^{-1} as

$$A_B^{-1} = SD^{-1}$$

This approach relies on the fact that the inverse of the diagonal matrix D is directly computable.

d) The basic part of the new optimal primal solution is

$$
\begin{aligned}
\widetilde{x}_B &= A_B^{-1}(b + \Delta b) = x_B^* + A_B^{-1}\Delta b \\
&= \begin{bmatrix} \frac{1}{2} \\ 1 \\ 3 \end{bmatrix} + A_B^{-1} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{4} \\ \frac{3}{2} \\ \frac{7}{2} \end{bmatrix}.
\end{aligned}
$$

Thus, the new optimal primal solution is $\widetilde{x} = (0, \frac{1}{4}, 0, \frac{3}{2})$.

The new optimal dual solution is

$$
y^* = (A_B)^{-\top} c_B = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right).
$$

You can also compute y^* by the complementary conditions.