1. a. Ω_1 is convex

$\|x-a\|_2 \le \|x-b\|_2 \Leftrightarrow (x-a)^T(x-a) \le (x-b)^T(x-b) \Leftrightarrow -2a^Tx+a^Ta \le -2b^Tx+b^Tb \Leftrightarrow 2(b-a)^Tx+a^Ta-b^Tb \le 0$

Pick $x_1, x_2 \in \Omega_1$

we have $\begin{cases} 2(b-a)^Tx_1+a^Ta-b^Tb \le 0 \\ 2(b-a)^Tx_2+a^Ta-b^Tb \le 0 \end{cases}$

Need to show $2(b-a)^T(\alpha x_1+(1-\alpha)x_2)+a^Ta-b^Tb \le 0 \qquad 0 \le \alpha \le 1$

It $\Leftrightarrow \alpha[2(b-a)^Tx_1+a^Ta-b^Tb]+(1-\alpha)[2(b-a)^Tx_2+a^Ta-b^Tb] \le 0$

Since $\begin{cases} 2(b-a)^Tx_1+a^Ta-b^Tb \le 0 \\ 2(b-a)^Tx_2+a^Ta-b^Tb \le 0 \end{cases}$

Then $\begin{cases} \alpha[2(b-a)^Tx_1+a^Ta-b^Tb] \le 0 & ① \\ (1-\alpha)[2(b-a)^Tx_2+a^Ta-b^Tb] \le 0 & ② \end{cases}$

①+② : $\alpha[2(b-a)^Tx_1+a^Ta-b^Tb]+(1-\alpha)[2(b-a)^Tx_2+a^Ta-b^Tb] \le 0$

So $\alpha x_1+(1-\alpha)x_2 \in \Omega_1$, Ω_1 is convex.

Ω_2 is convex.

Let $f(x) := x^Tx-t$

$f(x) \le 0 \Leftrightarrow x^Tx \le t$

we only need to show $f(x)$ is convex.

$\nabla^2 f(x) = 2\begin{pmatrix} I_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{pmatrix}_{(n+1)\times(n+1)}$ is PSD since the eigenvalues which are 2 or 0 ≥ 0

So $f(x)$ is convex.

Ω_2 is convex

b. Since $x \in \mathbb{R}_+^2$

Then $x_1 \ge 0 \quad x_2 \ge 0$

Since $x_1x_2 \ge 1$

Then $x_1 \ne 0 \quad x_2 \ne 0$

Then $x_1 > 0 \quad x_2 > 0$

$x_1x_2 \ge 1 \Leftrightarrow x_1 \ge \frac{1}{x_2} \Leftrightarrow \frac{1}{x_2}-x_1 \le 0$

$\nabla^2\left(\frac{1}{x_2}-x_1\right) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{x_2^3} \end{pmatrix}$ is PSD

Then $\frac{1}{x_2}-x_1$ is convex.

$\{x \in \mathbb{R}_+^2 : \frac{1}{x_2}-x_1 \le 0\}$ is convex

So $\{x \in \mathbb{R}_+^2 : x_1x_2 \ge 1\}$ is convex

c. (1) false. $\{x \in \mathbb{R} \mid 1 \le x \le 2 \text{ or } 3 \le x \le 4\}$ is a non-convex set

$\{x \in \mathbb{R} \mid 2 \le x \le 3 \text{ or } 4 \le x \le 5\}$ is a non-convex set

But $\{x \in \mathbb{R} \mid 1 \le x \le 2 \text{ or } 3 \le x \le 4\} \cup \{x \in \mathbb{R} \mid 2 \le x \le 3 \text{ or } 4 \le x \le 5\} = \{x \in \mathbb{R} \mid 1 \le x \le 5\}$ is convex.

(2) true. Since $S = \{(x,t) \in \Omega \times \mathbb{R} : f(x) \le t\}$ is convex.

Let $(x_1, f(x_1)) \in S \quad (x_2, f(x_2)) \in S$

Then $0 \le \alpha \le 1 \quad \alpha(x_1, f(x_1))+(1-\alpha)(x_2, f(x_2)) \in S$

Since Ω is convex

Then $f(\alpha x_1+(1-\alpha)x_2) \le \alpha f(x_1)+(1-\alpha)f(x_2)$

Then $f:\Omega \Rightarrow \mathbb{R}$ is a convex function.

2. a. False. Let $f(x)=-x$ $g(x)=x^Tx$ $\Rightarrow f\circ g(x)=-x^Tx$ $\Rightarrow \nabla^2 f\circ g(x)=-2I_{n\times n}$ is negative definite. $f\circ g(x)$ isn't convex.

b. True. $f\circ g:\Omega \to \mathbb{R}$ Ω is convex.

$0\le\alpha\le1$ $x_1, x_2 \in \Omega$ $(f\circ g)(\alpha x_1+(1-\alpha)x_2) \le f(\alpha g(x_1)+(1-\alpha)g(x_2))$ since f is nondecreasing and g is convex.

$I \supseteq g(\Omega) \Rightarrow g(x_1), g(x_2) \in \Omega \Rightarrow f(\alpha g(x_1)+(1-\alpha)g(x_2)) \le \alpha f(g(x_1))+(1-\alpha)f(g(x_2))$ since f is convex.

So $f\circ g(\alpha x_1+(1-\alpha)x_2) \le \alpha f\circ g(x_1)+(1-\alpha)f\circ g(x_2)$

$(f\circ g)(x)$ is convex.

c. False Let $f(x):=\frac{1}{1+e^{x^2}}$ It's easy to know $\frac{1}{1+e^{x^2}}$ is increasing and non-negative

$$\frac{d^2}{dx^2}(xf(x))=\frac{d}{dx}\left(\frac{1}{1+e^{x^2}}+\frac{2x^2e^{-x^2}}{(1+e^{x^2})^2}\right)$$

$$=\frac{-2xe^{x^2}((2x^2-3)e^{x^2}-2x^2-3)}{(e^{x^2}+1)^3}$$

when $x=2$, $\frac{d^2}{dx^2}(xf(x))=\frac{-4e^4(5e^4-11)}{(e^4+1)^3} < 0$ since $5e^4 \approx 273 > 11$

So $xf(x)$ isn't a convex function on $\mathbb{R}_+$

3. a) 1. $\frac{d^2}{dx^2}f(x)=\frac{d}{dx}\frac{-1}{x^2\sqrt{1+x^2}}=\frac{1}{x^4(1+x^2)}\left(2x\sqrt{1+x^2}+\frac{x^3}{\sqrt{1+x^2}}\right)>0$ since $x>0$ and $\mathbb{R}_{++}$ is convex.

f is convex.

2. since x^2 is convex, $x\in\mathbb{R}$., $\|x\|^2=\sum_{j=1}^n x_j^2$

Then $\|x\|^2$ is convex, $x\in\mathbb{R}^n$

Then $\frac{1}{2}\|Ax-b\|^2$ is convex, $x\in\mathbb{R}^n$

$\|\alpha x_1+(1-\alpha)x_2\|\le\|\alpha x_1\|+\|(1-\alpha)x_2\|=\alpha\|x_1\|+(1-\alpha)\|x_2\|$ for $x_1, x_2\in\mathbb{R}^n$, $0\le\alpha\le1$

Then $\|x\|_1$ is convex, $x\in\mathbb{R}^n$

Then $M\|Lx\|_1$ is convex, $x\in\mathbb{R}^n$, $M>0$

So $f(x)=\frac{1}{2}\|Ax-b\|^2+M\|Lx\|_1$ is convex.

3. $\|\lambda x_1+(1-\lambda)x_2\|\le\|\lambda x_1\|+\|(1-\lambda)x_2\|=\lambda\|x_1\|+(1-\lambda)\|x_2\|$ for $0\le\lambda\le1$, $x_1, x_2\in\mathbb{R}^n$

Then $\|x\|$ is convex $x\in\mathbb{R}^n$

Then $\frac{\lambda}{2}\|x\|$ is convex $x\in\mathbb{R}^n$ $\lambda>0$

For $\ln(1+\exp(-b_i(a_i^Tx+y)))$

Since $-b_i(a_i^Tx+y)$ is linear for x, y

To show $\ln(1+\exp(-b_i(a_i^Tx+y)))$ is convex, we only need to show $\ln(1+\exp(z))$ is convex.

$$\frac{d^2}{dz^2}\ln(1+\exp(z))=\frac{d}{dz}\frac{\exp(z)}{1+\exp(z)}=\frac{\exp(z)}{(1+\exp(z))^2}\ge0$$ So $\ln(1+\exp(z))$ is convex.

So $f(x,y)$ is convex.

b) Since $||x||_q$ is a norm.

$0 \le \alpha \le 1$, $x_1, x_2 \in \mathbb{R}$ $\quad ||\alpha x_1 + (1-\alpha) x_2||_q \le ||\alpha x_1||_q + ||(1-\alpha) x_2||_q = \alpha ||x_1||_q + (1-\alpha)||x_2||_q$

So $r(\alpha x_1 + (1-\alpha) x_2) \le \alpha r(x_1) + (1-\alpha) r(x_2)$

r is convex.

4. a) The feasible region is convex

$\frac{\partial^2}{\partial w^2} \delta^2 \cdot 1^T w = 0_{m \times m} \succeq 0 \Rightarrow \delta \cdot 1^T w$ is convex.

Let $c = \begin{bmatrix} x \\ w_i \end{bmatrix}$

$$\frac{d^2}{dc^2} \frac{(a_i^T x - b_i)^2}{1 + w_i} = \frac{d}{dc} \begin{bmatrix} \frac{2(a_i^T x - b_i) a_i}{1 + w_i} \\ -\frac{(a_i^T x - b_i)^2}{(1 + w_i)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2 a_i a_i^T}{1 + w_i} & \frac{-2(a_i^T x - b_i) a_i}{(1 + w_i)^2} \\ -\frac{2(a_i^T x - b_i) a_i^T}{(1 + w_i)^2} & \frac{2(a_i^T x - b_i)^2}{(1 + w_i)^3} \end{bmatrix} \text{ denoted by } H$$

Let $y \in \mathbb{R}^n$ $z \in \mathbb{R}$

$$[y^T, z] H \begin{bmatrix} y \\ z \end{bmatrix} = \frac{2 y^T a_i a_i^T y}{1 + w_i} - \frac{2z(a_i^T x - b_i) y^T a_i}{(1 + w_i)^2} - \frac{2z(a_i^T x - b_i) a_i^T y}{(1 + w_i)^2} + \frac{2 z^2 (a_i^T x - b_i)^2}{(1 + w_i)^3}$$

$$= \frac{2}{1 + w_i} \left[(a_i^T y)^2 - 2 \cdot \frac{z(a_i^T x - b_i)}{1 + w_i} \cdot a_i^T y + \left(\frac{z(a_i^T x - b_i)}{1 + w_i} \right)^2 \right]$$

$$= \frac{2}{1 + w_i} \left[(a_i^T y) - \frac{z(a_i^T x - b_i)}{1 + w_i} \right]^2 \ge 0$$

Then H is PSD

So $\frac{(a_i^T x - b_i)^2}{1 + w_i}$ is convex

Then $\sum_{i=1}^m \frac{(a_i^T x - b_i)^2}{1 + w_i}$ is convex

So $\sum_{i=1}^m \frac{(a_i^T x - b_i)^2}{1 + w_i} + \delta^2 \cdot 1^T w$ is convex

problem (1) is a convex optimization problem.

b)

$$\frac{\partial}{\partial w_i}\left(\sum_{i=1}^{m}\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 1^T w\right) = -\frac{(a_i^T x-b_i)^2}{(1+w_i)^2}+\delta^2 \qquad i=1,2,\cdots,m$$

when $|a_i^T x-b_i|\le\delta$

$$\frac{\partial}{\partial w_i}\left(\sum_{i=1}^{m}\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 1^T w\right) = -\frac{(a_i^T x-b_i)^2}{(1+w_i)^2}+\delta^2 \geqslant -\frac{\delta^2}{(1+w_i)^2}+\delta^2 = \frac{\delta^2 w_i(w_i+2)}{(1+w_i)^2}\geqslant 0 \quad \text{since } w\geqslant 0$$

$\sum_{i=1}^{m}\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 1^T w$ is increasing with w_i.

Since we minimize $\sum_{i=1}^{m}\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 1^T w$

Then w_i should be 0

$$\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 w_i = (a_i^T x-b_i)^2 = \varphi_\delta(a_i^T x-b_i)$$

when $|a_i^T x-b_i|>\delta$

Let $\frac{\partial}{\partial w_i}\left(\sum_{i=1}^{m}\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 1^T w\right) = -\frac{(a_i^T x-b_i)^2}{(1+w_i)^2}+\delta^2 = 0$

$$1+w_i = \frac{|a_i^T x-b_i|}{\delta}$$

$$\frac{(a_i^T x-b_i)^2}{1+w_i} = \delta|a_i^T x-b_i|$$

$$\delta^2 w_i = \delta^2\left(\frac{|a_i^T x-b_i|}{\delta}-1\right) = \delta(|a_i^T x-b_i|-\delta)$$

$$\frac{(a_i^T x-b_i)^2}{1+w_i} + \delta^2 w_i = \delta(2|a_i^T x-b_i|-\delta) = \varphi_\delta(a_i^T x-b_i)$$

So

$$\min_{\substack{x,w\\ w\geqslant 0}}\sum_{i=1}^{m}\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2\cdot 1^T w = \min_{\substack{x,w\\ w\geqslant 0}}\sum_{i=1}^{m}\left[\frac{(a_i^T x-b_i)^2}{1+w_i}+\delta^2 w_i\right]$$

$$= \min_{x}\sum_{i=1}^{m}\varphi_\delta(a_i^T x-b_i)$$