The Chinese University of Hong Kong, Shenzhen
School of Data Science

MAT3007 · Homework 4
Due: 12:00 (**noon, not midnight**), March 15

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (20pts). Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 + 6x_2 + 6x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 \leq 1 \\ & x_1 + 3x_2 + 2x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

1. What is the corresponding dual problem?
2. Solve the dual problem graphically.
3. Use complementarity slackness to solve the primal problem.

Solution.

1.

$$c = \begin{bmatrix} 1 & 6 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

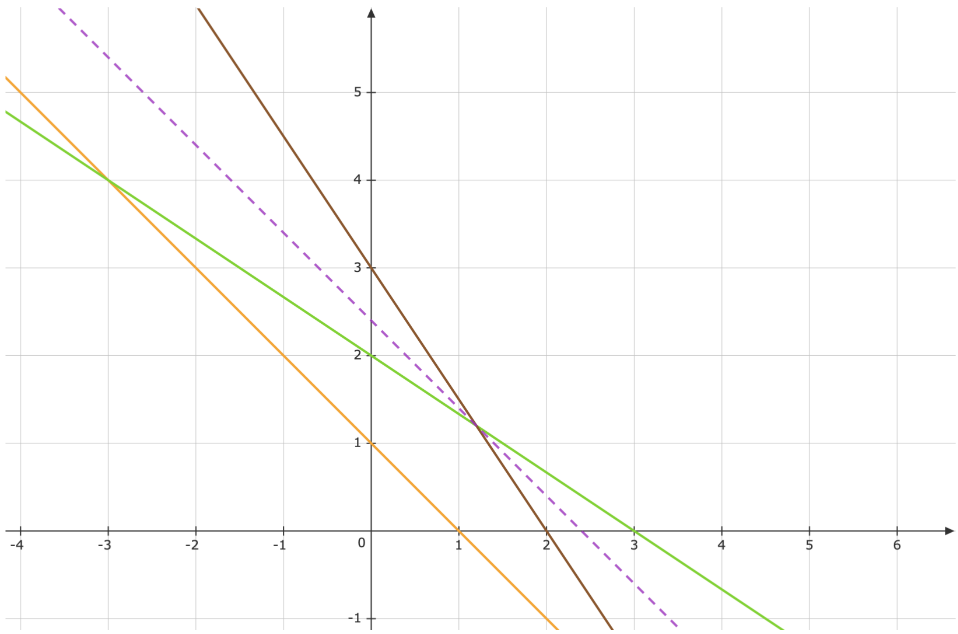
The dual problem is

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

or namely,

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 \\ \text{subject to} \quad & y_1 + y_2 \geq 1 \\ & 2y_1 + 3y_2 \geq 6 \\ & 3y_1 + 2y_2 \geq 6 \\ & y_1, y_2 \geq 0 \end{aligned}$$

2.



$$\begin{cases} y_2 \geq 1 - y_1 \\ y_2 \geq 2 - \frac{2}{3}y_1 \\ y_2 \geq 3 - \frac{3}{2}y_1 \end{cases}$$

The optimal solution is $(\frac{6}{5}, \frac{6}{5})$, and the optimal value is $\frac{12}{5}$.

3. By complementarity condition:

$$(y_1 + y_2 - 1)x_1 = 0, \tag{1}$$

$$(2y_1 + 3y_2 - 6)x_2 = 0,$$

$$(3y_1 + 2y_2 - 6)x_3 = 0,$$

$$(x_1 + 2x_2 + 3x_3 - 1)y_1 = 0, \tag{2}$$

$$(x_1 + 3x_2 + 2x_3 - 1)y_2 = 0. \tag{3}$$

Since $y_1 + y_2 - 1 \neq 0$, by (1) we have $x_1 = 0$.
Since $y_1, y_2 \neq 0$, by (2), (3) we have

$$2x_2 + 3x_3 = 1,$$

$$3x_2 + 2x_3 = 1,$$

which implies

$$x_1 = 0, x_2 = x_3 = \frac{1}{5}. \tag{4}$$

Problem 2 (20pts). Suppose M is a square matrix such that $M = -M^T$, for example,

$$M = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}.$$

Consider the following optimization problem:

$$\begin{aligned} \text{minimize}_x \quad & c^T x \\ \text{subject to} \quad & Mx \geq -c \\ & x \geq 0 \end{aligned}$$

(a) Show that the dual problem of it is equivalent to the primal problem.

(b) Argue that the problem has optimal solution if and only if there is a feasible solution,

Solution.

(a) The dual is as follows:

$$\begin{aligned} \max \quad & -c^T y \\ s.t. \quad & M^T y \leq c \\ & y \geq 0. \end{aligned}$$

Since $M = -M^T$, $M^T y \leq c$ is equivalent to $My \geq -c$. Therefore, we can reformulate the dual problem as a minimization problem as follows:

$$\begin{aligned} \min \quad & c^T y \\ s.t. \quad & My \geq -c \\ & y \geq 0 \end{aligned}$$

Therefore, the dual problem is equivalent to the primal problem.

(b) First, it is obvious that if the problem has optimal solution, then it must have a feasible solution. Now we prove the other direction. If the problem has a feasible solution x, then $y = x$ is also feasible to the dual problem. Therefore, both the primal and dual problems are feasible. According to the weak duality theorem, they both have finite optimal solution.

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Problem 3 (20pts). We consider the general linear optimization problem:

$$\min_x \; c^\top x \quad \text{subject to} \quad Ax \leq b, \quad Cx = d, \tag{5}$$

where $A \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{p \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $d \in \mathbb{R}^p$ are given. Derive the dual of problem (5) and show that the dual of the dual is equivalent to problem (5).

Solution.
As shown in the lecture the dual problem is constructed by discussing the min-max formulation

$$\min_{x}\ c^\top x + \max_{u\le 0, v\in\mathbb{R}^p}\ u^\top(b-Ax) + v^\top(d-Cx).$$

This problem is equivalent to the primal problem (5) since the inner maximum term will be $+\infty$ at any infeasible point. Exchanging the order of minimization and maximization, we obtain

$$\max_{u\le 0, v\in\mathbb{R}^p}\ b^\top u + d^\top v + \min_{x}\ (c - A^\top u - C^\top v)^\top x$$

which is equivalent to the dual problem:

$$\max_{u,v}\ b^\top u + d^\top v \quad \text{subject to} \quad u\le 0, \quad v\in\mathbb{R}^p, \quad A^\top u + C^\top v = c.$$

We can further rewrite this problem as

$$-\min_{u,v}\ (-b^\top, -d^\top)\begin{pmatrix} u \\ v\end{pmatrix} \quad \text{subject to} \quad \begin{pmatrix} I & 0\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix} \le 0 \quad \begin{pmatrix} A^\top & C^\top\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix} = c.$$

As this problem has the same format as (5), we can apply the same process with $x \equiv (u^\top, v^\top)^\top$, $c \equiv (-b^\top, -d^\top)^\top$, $A \equiv \begin{pmatrix} I & 0\end{pmatrix}$, $b = 0$, $C \equiv \begin{pmatrix} A^\top & C^\top\end{pmatrix}$, and $d \equiv c$. This leads to the following dual problem:

$$-\max_{y,z}\ c^\top z \quad \text{subject to} \quad y \le 0, \quad z\in\mathbb{R}^n, \quad \begin{pmatrix} I \\ 0\end{pmatrix} y + \begin{pmatrix} A \\ C\end{pmatrix} z = \begin{pmatrix} -b \\ -d\end{pmatrix}.$$

Setting $x = -z$, this problem is equivalent to:

$$\min_{x,y}\ c^\top x \quad \text{subject to} \quad y\le 0, \quad Ax - y = b, \quad Cx = d$$

Finally, since $y \le 0$ is only a slack variable, we can remove it by adjusting the constraint $Ax - y = b$ to $Ax \le b$. The resulting problem then coincides with (5). This shows that the dual of the dual problem is again equivalent to the primal problem (5).

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Problem 4 (20pts). We consider the following robust linear program

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & \max_{a\in\mathcal{P}_i}\ a^\top x \le b_i \quad i = 1, ..., m, \end{array}$$

with variable $x\in\mathbb{R}^n$ and $\mathcal{P}_i := \{a : C_i a \le d_i\}$. The problem data are $c\in\mathbb{R}^n$, $C_i \in \mathbb{R}^{m_i\times n}$, $d_i\in\mathbb{R}^{m_i}$, and $b\in\mathbb{R}^m$. We assume that the polyhedra $\mathcal{P}_i$ are all nonempty.
Show that this problem is equivalent to the linear optimization problem

$$\begin{array}{lll} \text{minimize}_{x,z_1,\ldots,z_m} & c^\top x & \\ \text{subject to} & d_i^\top z_i \le b_i & i = 1, ..., m \\ & C_i^\top z_i = x & i = 1, ..., m \\ & z_i \ge 0 & i = 1, ..., m. \end{array}$$

Solution.
We first interpret the constraint $[\max_{a\in\mathcal{P}_i} a^\top x] \le b_i$ as an optimization problem, i.e., we consider the linear program $\max_{a\in\mathcal{P}_i} a^\top x$ independently. The dual of this problem is given by

$$\min_{z_i} \ d_i^\top z_i \quad \text{subject to} \quad z_i \ge 0, \quad C_i^\top z_i = x.$$

If the primal problem $\max_{a\in\mathcal{P}_i} a^\top x$ is unbounded, then the feasible set of the original problem as well as the feasible set of the dual problem is empty (the condition $+\infty \le b_i$ can not be satisfied). Otherwise, since $\mathcal{P}_i$ is nonempty, there exists an optimal solution and strong duality needs to hold:

$$\max_{a\in\mathcal{P}_i} a^\top x = \min_{z_i\ge 0,\, C_i^\top z_i = x} d_i^\top z_i.$$

Furthermore, the condition $[\min_{z_i\ge 0,\, C_i^\top z_i = x} d_i^\top z_i] \le b_i$ is satisfied if and only if there exists z_i with $z_i \ge 0$, $C_i^\top z_i = x$, and $d_i^\top z_i \le b_i$. Repeating this procedure for the other constraints, we then obtain the equivalent problem:

$$\begin{array}{lll} \text{minimize}_{x,z_1,\dots,z_m} & c^\top x & \\ \text{subject to} & d_i^\top z_i \le b_i & i = 1,\dots,m \\ & C_i^\top z_i = x & i = 1,\dots,m \\ & z_i \ge 0 & i = 1,\dots,m. \end{array}$$

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Problem 5 (20pts). Use the strong duality theorem to prove Gordan's theorem: Either $Ax > 0$ has a solution, or $A^\top y = 0, y \gneqq 0$ has a solution. (Let x, y be two vectors. Denote $x \gneqq y$ to be $x \ge y$ and $x \ne y$. Denote $x > y$ to be $x_i > y_i, \forall i = 1, \dots, n$.)

Solution.
Consider the following problem:

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & Ax \ge \mathbf{1} \end{array} \tag{P}$$

, here $\mathbf{1}$ denotes the vector whose entries are all 1. Its dual is

$$\begin{array}{ll} \max & \mathbf{1}^\top y \\ \text{s.t.} & A^\top y = 0 \\ & y \ge 0 \end{array} \tag{D}$$

So we have

$Ax > 0$ has a solution
$\Longleftrightarrow Ax \ge \mathbf{1}$ has a solution
$\Longleftrightarrow$(P) is feasible
$\Longleftrightarrow$(P) has an optimial solution with optimal value 0
$\Longleftrightarrow$(D) has an optimial solution with optimal value 0
$\Longleftrightarrow A^\top y = 0, y \gneqq 0$ has no solution (otherwise, suppose it has a solution y_0, the objective value of (D) at y_0 is greater than 0)

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