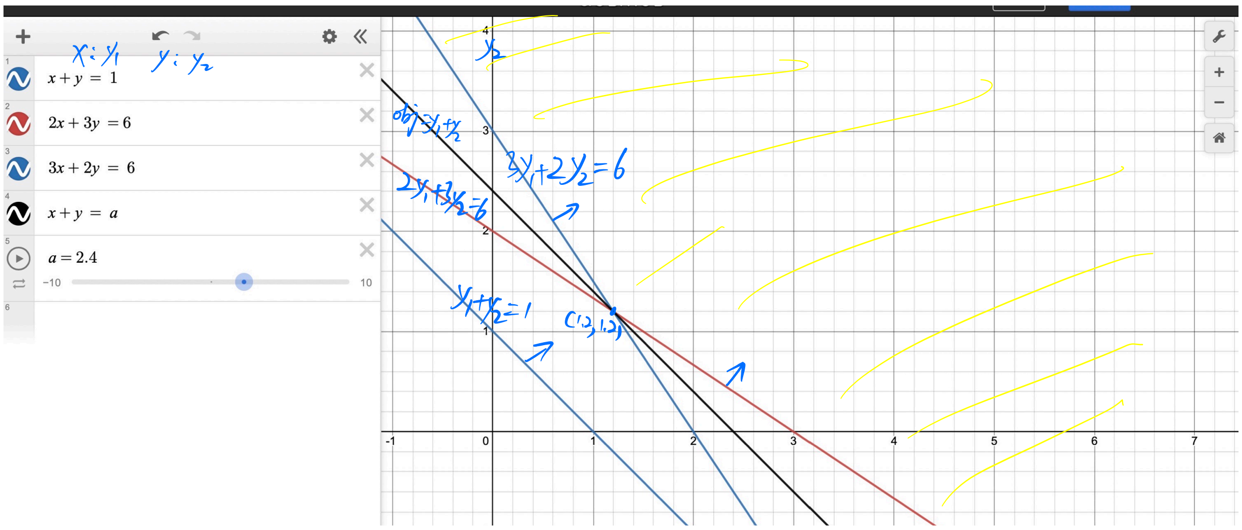
1. (1)

$$\min \ y_1 + y_2$$
$$y_1 + y_2 \geq 1$$
$$2y_1 + 3y_2 \geq 6$$
$$3y_1 + 2y_2 \geq 6$$
$$y_1, y_2 \geq 0$$

(2)



The optimal value is 2.4

The optimal solution is $(1.2, 1.2)$

(3)

$$1.2(1 - x_1 - 2x_2 - 3x_3) = 0$$
$$1.2(1 - x_1 - 3x_2 - 2x_3) = 0$$
$$x_1(1 - 1.2 - 1.2) = 0$$
$$x_2(6 - 2\times 1.2 - 3\times 1.2) = 0$$
$$x_3(6 - 3\times 1.2 - 2\times 1.2) = 0$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0.2 \\ x_3 = 0.2 \end{cases}$$

The optimal solution is $(0, 0.2, 0.2)$

The optimal value is $0 + 6\times 0.2 + 6\times 0.2 = 2.4$

2. (a)

$$\min_{\substack{x \geq 0 \\ Mx \geq -c}} c^T x = \min_{x \geq 0} c^T x \max_{y \geq 0} y^T(-c - Mx) = \max_{y \geq 0} \min_{x \geq 0} y^T(-c) + (c^T - y^T M)x = \max_{y \geq 0} y^T(-c) \min_{x \geq 0} (c^T - y^T M)x$$

The dual problem is

$$\max_y \ (-c)^T y \quad \text{s.t.} \ M^T y \leq c, \ y \geq 0 \iff \min \ c^T y \quad \text{s.t.} \ -M^T y \geq -c, \ y \geq 0 \iff \min \ c^T y \quad \text{s.t.} \ My \geq -c, \ y \geq 0$$

(b) There is a feasible solution $\Rightarrow$ The problem has optimal solution:

Since the dual problem is equivalent to the primal problem

And the primal problem is feasible

Then the dual problem is feasible, too.

So the primal prolem is bounded

Then the primal problem has optimal solution

The problem has optimal solution $\Rightarrow$ there is a feasible solution.

Since the problem has optimal solution

there must be at least one solution for the problem

So the problem is feasible

3. $\min_x c^T x + \max_{y \le 0, z} y^T(b - Ax) + z^T(d - Cx)$ $\quad y \in \mathbb{R}^m \quad z \in \mathbb{R}^p$

$= \max_{y \le 0, z} y^T b + z^T d + \min_x (c^T - y^T A - z^T C)x$

$= \max_{y, z} y^T b + z^T d \quad s.t.\ c^T - y^T A - z^T C = 0,\ y \le 0$

$= \max_{y, z} b^T y + d^T z \quad s.t.\ c = A^T y + C^T z,\ y \le 0$ which is the dual problem. denoted by LP_1

The dual problem of LP_1 is $\min_x c^T x$ s.t. $Ax \le b$ $Cx = d$

which is the same as problem (1).

4. The two LP has the same objective,

So we only need to show that they have the same feasible region, namely,

$\{x \mid \max_{a \in P_i} a^T x \le b_i \ ,i = 1, \cdots, m\} = \{x \mid d_i^T z_i \le b_i \ , C_i^T z_i = x \ , z_i \ge 0 \ , i = 1, \cdots, m\}$

First, we show that $\forall x_0 \in \{x \mid \max_{a \in P_i} a^T x \le b_i \ , i = 1, \cdots, m\}$

$\Rightarrow x \in \{x \mid d_i^T z_i \le b_i \ , C_i^T z_i = x \ , z_i \ge 0 \ , i = 1, \cdots, m\}$

consider LP: $\max_a \ a^T x_0 \le b_i \qquad i = 1, \cdots, m$

s.t. $C_i a \le d_i$

The LP is feasible since polyhedra P_i are all nonempty

The LP is bounded since $\max_a a^T x_0 \le b_i$

So the LP has optimal solution

By weak and strong theorem, the dual problem of LP is feasible and has the same optimal value.

The dual problem is $\min_{z_i} d_i^T z_i$ s.t. $C_i^T z_i = x$, $z_i \geq 0$, $i=1,\cdots,m$ whose optimal value $\leq b_i$

So x_0 is also in $\{x \mid \min_{z_i} d_i^T z_i \leq b_i$ s.t. $C_i^T z_i = x$, $z_i \geq 0$, $i=1,\cdots,m\}$

Then denote the optimal solution of $\min_{z_i} d_i^T z_i \leq b_i$ s.t. $C_i^T z_i = x_0$, $z_i \geq 0$ $i=1,\cdots,m$ by z_i^*

So we find x_0, z_i^* s.t. $d_i^T z_i^* \leq b_i$, $C_i^T z_i^* = x_0$, $z_i^* \geq 0$, $i=1,\cdots,m$

Then $x_0 \in \{x \mid d_i^T z_i \leq b_i, C_i^T z_i = x, z_i \geq 0, i=1,\cdots,m\}$

Second, we show that $\forall x_1 \in \{x \mid d_i^T z_i \leq b_i, C_i^T z_i = x, z_i \geq 0, i=1,\cdots,m\}$
$\Rightarrow x_1 \in \{\max_{a \in P_i} a^T x \leq b_i, i=1,\cdots,m\}$

Since $\exists z_{i*}$ s.t. $d_i^T z_{i*} \leq b_i$, $C_i^T z_{i*} = x_1$, $z_{i*} \geq 0$ $i=1,\cdots,m$

Then the optimal value of "$\min_{z_i} d_i^T z_i$ s.t. $C_i^T z_i = x_1$, $z_i \geq 0$, $i=1,\cdots,m$" must be less than or equal to b_i. (we will show it has optimal solution later)

So $x_1 \in \{x \mid \min_{z_i} d_i^T z_i \leq b_i$ s.t. $C_i^T z_i = x$, $z_i \geq 0$, $i=1,\cdots,m\}$

Since the dual problem of "$\min_{z_i} d_i^T z_i \leq b_i$ s.t. $C_i^T z_i = x$, $z_i \geq 0$, $i=1,\cdots,m$" is "$\max_a a^T x_0$ s.t. $C_i a \leq d_i$, $i=1,\cdots,m$" which is feasible.

Then "$\min_{z_i} d_i^T z_i \leq b_i$ s.t. $C_i^T z_i = x$, $z_i \geq 0$, $i=1,\cdots,m$" is bounded and has optimal value which is the same as the dual problem by weak and strong duality theorem.

So x_1 is in $\{x \mid \min_{z_i} d_i^T z_i \leq b_i$ s.t. $C_i^T z_i = x$, $z_i \geq 0$, $i=1,\cdots,m\}$

5. consider:

Primal: $\min\ 0^T x$

s.t. $Ax \geq 1$

Dual: $\max\ 1^T y$

$A^T y = 0$

$y \geq 0$

If Primal is feasible:

"$Ax > 0$ has a solution" holds, the optimal value of primal problem is 0.

and by Strong Duality Theorem, the optimal value of Dual must be 0, namely, $y=0$ is the only solution for Dual,

Since if $\exists\, y_0 \gneq 0$, $1^T y_0 > 0$ which is contradictory to the optimal value is 0.

So "$A^T y = 0$, $y \gneq 0$" has a solution" doesn't hold.

If Primal is infeasible:

Assume "$Ax > 0$ has a solution", denote the solution by x_0.

$Ax_0 = \begin{bmatrix} a_1 x_0 \\ a_2 x_0 \\ \vdots \end{bmatrix}$ Let $\lambda = \min\{a_1 x_0, a_2 x_0, \cdots\}$ and $\lambda > 0$ since $Ax_0 > 0$

Since Primal is infeasible, there exists $a_i x_0 < 1 \Rightarrow \lambda < 1$

since $\frac{1}{\lambda} a_i x_0 \geq \frac{1}{\lambda} \min\{a_1 x_0, a_2 x_0 \cdots\} = \frac{1}{\lambda}\lambda = 1$, $A(\frac{1}{\lambda} x_0) \geq 1$ which is contradicted to Primal is infeasible.

So $Ax > 0$ has no solution.

we know $y=0$ must be in the feasible region of Dual, namely, the dual problem must be feasible. But Primal is infeasible.

Then Dual is unbounded. Dual must have other feasible solutions except $y=0$.

$\Rightarrow$ "$A^T y = 0$, $y \gneq 0$ has a solution" holds