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CS 527 HW 3;

Problem 1.

$\because Y = X + Z \pmod 5$, $X \in \{0, 1, 2, 3, 4\}$.

and Z is independent of X. $\because Z = \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

so, $C = \max_{p(x)} [H(Y) - H(Y/X)]$.

$\because H(Y/X) = H(X+Z/X) = H(Z|X)$

$\therefore H(Y/X) = H(Z)$ (Z and X is independent).

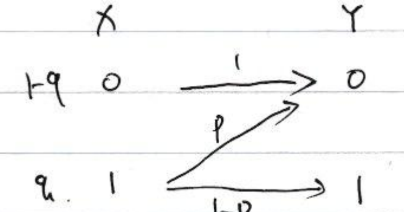
$= (1-p)\log_2 \frac{1}{(1-p)} + p \log_2 \frac{1}{p}$

and $p(x=0) = p(x=1) = p(x=2) = p(x=3) = p(x=4) = \frac{1}{5}$.

so $H(Y) = \log_2 5$. bits.

$\therefore C = \max_{(p(x)=\frac{1}{5})} \underline{\log_2 5 - H(p)}$. bits/transmission.

Problem 2: $\because Q = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ for transition matrix $p(Y/X)$



so, by the question we can get $p = \frac{1}{2}$.

$1-p = \frac{1}{2}$.

let $p(x=1) = q$, $p(x=0) = 1-q$.

since $C = \max_{p(x)} [H(Y) - H(Y/X)]$

$\Rightarrow$ ① $H(Y) = \frac{1}{2} q \log_2 \frac{2}{q} + (1 - q\cdot\frac{1}{2}) \log_2 \frac{1}{(1-\frac{1}{2}q)}$ bits.

② $H(Y/X) =$ ~~$p(x=0)$~~

$p(x=0) H(Y/x=0) + p(x=1) H(Y/x=1)$

$= p(x=1) = q$ bits.

put ① ② in to C.

$\Rightarrow \underline{C = \max_{q(x)} [\frac{1}{2} q \log_2 \frac{2}{q} + (1-\frac{1}{2}q) \log_2 \frac{1}{(1-\frac{1}{2}q)} - q]}$ --- ③

$\because \frac{dI}{dq} = 0$ so, $\Rightarrow \frac{1}{2} \log_2 \frac{(1-\frac{1}{2}q)}{\frac{1}{2}q} - 1 = 0$.

so $\frac{1}{2}\log_2\frac{(1-\frac{1}{2}q)}{\frac{1}{2}q} = 1 \Rightarrow 1-\frac{1}{2}q = 2q \Rightarrow q = \frac{2}{5}$. ···· *

let $q=\frac{2}{5}$ into ③ C

$$\Rightarrow C = \frac{1}{2}\cdot\left(\frac{2}{5}\right)\log_2\frac{2}{\frac{2}{5}} + \left(1-\frac{1}{2}\cdot\frac{2}{5}\right)\cdot\log_2\frac{1}{\left(1-\frac{1}{2}\cdot\frac{2}{5}\right)} - \frac{2}{5}$$

$$= 0.721 - \frac{2}{5} = \underline{0.3219 \text{ bits}}$$

problem 3: We ~~can know~~ need to prove $P(E) = \frac{1}{2}\left(1-(1-2p)^n\right)$

so $P(1) = \frac{1}{2}\left(1-(1-2p)\right) = p$.

for $P(n) = \frac{1}{2}\left(1-(1-2p)^n\right)$

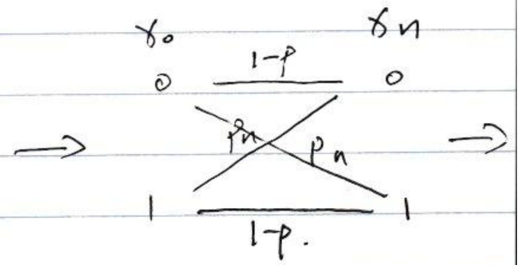
for $P(n+1) \Rightarrow P(n+1) = \frac{1}{2}\left(1-(1-2p)^{n+1}\right)$

so. $P(n+1) = P(X_{n+2}|X_n) = P(X_{n+1}=0|X_n=0)\cdot P(X_{n+2}=1|X_{n+1}=0) + P(X_{n+1}=1/X_n=0)\cdot P(X_{n+2}=1|X_{n+1}=1)$.

$$= (1-P(n))\cdot p + P(n)\cdot(1-p)$$
$$= p - 2pP(n) + P(n)$$
$$= p + (1-2p)P(n)$$
$$= p + (1-2p)\cdot\left(\frac{1}{2}\left(1-(1-2p)^n\right)\right).$$

So. $P(n+1)$ holds:

$$= p + \frac{1}{2}(1-2p) - \frac{1}{2}(1-2p)^{n+1}$$
$$= \underline{\frac{1}{2}\left(1-(1-2p)^{n+1}\right)}$$



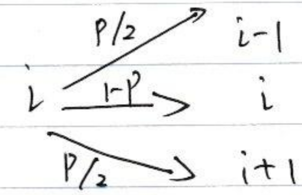
So $C = H(X_{n+2}) - H(X_{n+2}|X_n)$

$$= 1 - H\left(\frac{1}{2}\left(1-(1-2p)^n\right)\right).$$

when $n\to\infty$ $P(E) = \lim_{n\to\infty}\frac{1}{2}\left(1-(1-2p)^n\right) = \frac{1}{2}$,

So $C = 1 - H(\frac{1}{2}) = 0$.

problem 4 $\because$ by the question, we can know that :



so $C = \max_{p(x)} I(X;Y)$

$$= \max_{p(x)} [H(Y) - H(Y/X)].$$

$$H(Y/X) = -[(1-p)\log_2(1-p) + \frac{p}{2}\log_2(\frac{p}{2}) + \frac{p}{2}\log_2(\frac{p}{2})]$$

$$= -[(1-p)\log_2(1-p) + p\log_2(\frac{p}{2})]$$

$$= H(p) + p \text{ bits}.$$

$\because H(Y)$ is max when X is $p(x=0) = \cdots = p(x=15) = \frac{1}{16}$.

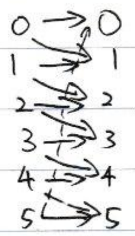
so. $\max_{p(x)} H(Y) = 16[\frac{1}{16}\log_2 16] = 4$ bits.

$$\underline{C = 4 - H(p) - p \text{ bits / transmission}}.$$

problem 5: a) let we call n: length of the codeword ; l: error ; m: symbols (levels)

so $(\lfloor \frac{m}{l+1} \rfloor)^n \le$ max number of codewords $\le (\frac{m}{l+1})^n$.

$\therefore$ mod 6:



$\therefore$ when $n=1$; $(\lfloor \frac{6}{2} \rfloor)^1 \le |C| \le (\frac{6}{2})^1 \Rightarrow |C| = 3$.

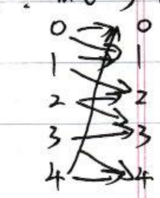
we pick $\{0, 2, 4\}$

when $n=2$; $(\lfloor \frac{6}{2} \rfloor)^2 \le |C| \le (\frac{6}{2})^2 \Rightarrow |C| = 9$.

$\therefore \underline{C = \{00, 02, 04, 20, 22, 24, 40, 42, 44\}}$.

b). so, it same as a part, we can know :

$\because$ mod 5 levels : $(\lfloor \frac{m}{l+1} \rfloor)^n \le$ max # of codewords $\le (\frac{m}{l+1})^n$.



so when $n=1$; $(\lfloor \frac{5}{2} \rfloor)^1 \le |C| \le (\frac{5}{2})^1 \therefore \Rightarrow |C| \le 2$.

$C = \{0, 2\}$ or $\{1, 3\}$.

when $n=2$; $\lfloor \frac{5}{2} \rfloor^2 \le |C| \le (\frac{5}{2})^2 \Rightarrow 4 \le |C| \le 6$

$\therefore |C| = 5.$

so we pick $\Rightarrow C = \{02, 10, 23, 31, 44\}$.