公式集

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目次

1 ベクトル解析（微分なし）

1.1 三重積など

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})\equiv[ABC] \tag{1.1}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \tag{1.2}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})+\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A})+\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B})=0 \tag{1.3}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{B}\cdot\boldsymbol{C})(\boldsymbol{A}\cdot\boldsymbol{D}) \tag{1.4}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=\boldsymbol{A}\cdot(\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{D})) \tag{1.5}$$

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})&=[ABD]\boldsymbol{C}-[ABC]\boldsymbol{D}\\&=[CDA]\boldsymbol{B}-[CDB]\boldsymbol{A}\end{aligned} \tag{1.6}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{B}\times\boldsymbol{C})\times(\boldsymbol{C}\times\boldsymbol{A})=[ABC]^2 \tag{1.7}$$

1.2　証明

■式 (1.2)

$$
\begin{aligned}
\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})_x &= a_y(\boldsymbol{B} \times \boldsymbol{C})_z - a_z(\boldsymbol{B} \times \boldsymbol{C})_y \\
&= a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) \\
&= (a_y b_x c_y + a_z b_x c_z) - (a_y b_y c_x + a_z b_z c_x) \\
&= b_x(a_x c_x + a_y c_y + a_z c_z) - c_x(a_x b_x + a_y b_y + a_z b_z)
\end{aligned}
$$

2　ベクトル解析（微分あり）

2.1　2 階微分

$$\nabla \times (\nabla \phi) = 0 \tag{2.1}$$

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0 \tag{2.2}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \tag{2.3}$$

$$\nabla \cdot (\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi \tag{2.4}$$

$$\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi \tag{2.5}$$

$$\nabla^2 (\phi \psi) = \phi \nabla^2 \psi + 2 \nabla \phi \cdot \nabla \psi + \psi \nabla^2 \phi \tag{2.6}$$

2.2 分配法則

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \tag{2.7}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \tag{2.8}$$

$$\nabla \times (\boldsymbol{A} + \boldsymbol{B}) = \nabla \times \boldsymbol{A} + \nabla \times \boldsymbol{B} \tag{2.9}$$

$$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) \tag{2.10}$$

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \tag{2.11}$$

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) \tag{2.12}$$

$$\nabla \times (\phi\boldsymbol{A}) = -\boldsymbol{A} \times \nabla\phi + \phi\nabla \times \boldsymbol{A} \tag{2.13}$$

$$\nabla \cdot (\phi\boldsymbol{A}) = \boldsymbol{A} \cdot \nabla\phi + \phi\nabla\boldsymbol{A} \tag{2.14}$$

$$\nabla^2(\phi\psi) = \psi\nabla^2\phi + \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi \tag{2.15}$$

$$(\boldsymbol{A} \cdot \nabla)\phi = \boldsymbol{A} \cdot (\nabla\phi) \tag{2.16}$$

$$(\boldsymbol{A} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{A} \cdot \nabla B_x)\boldsymbol{e}_x + (\boldsymbol{A} \cdot \nabla B_y)\boldsymbol{e}_y + (\boldsymbol{A} \cdot \nabla B_z)\boldsymbol{e}_z \tag{2.17}$$

$$(\boldsymbol{A} \cdot \nabla)\phi\boldsymbol{B} = \phi(\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B}(\boldsymbol{A} \cdot \nabla\phi) \tag{2.18}$$

2.3 その他

$$(\nabla \times \boldsymbol{A}) \times \boldsymbol{A} = (\boldsymbol{A} \cdot \nabla)\boldsymbol{A} - \frac{1}{2}\nabla(\boldsymbol{A} \cdot \boldsymbol{A}) \tag{2.19}$$

2.4 証明

■式 (2.1)

$$\nabla \times (\nabla\phi) = (\nabla \times \nabla)\phi = 0$$

■式 (2.2)　式 (1.1) より、∇ を $\boldsymbol{A}$ の左に置くことに注意する。

■式 (2.3)　式 (1.2) より、∇ を $\boldsymbol{A}$ の左に置くことに注意する。

■式 (2.4)

$$\begin{aligned} LHS &= \nabla_\phi \cdot (\phi\nabla\psi) + \nabla_\psi \cdot (\phi\nabla\psi) \\ &= (\nabla_\phi \cdot \phi) \cdot \nabla\psi + \phi(\nabla_\psi \cdot (\nabla\psi)) = RHS \end{aligned}$$

■式 (2.6)

$$
\begin{aligned}
LHS &= \nabla_\psi \cdot (\nabla(\phi\psi)) + \nabla_\phi \cdot (\phi\psi) \\
&= \nabla_\psi \cdot (\phi\nabla\psi + \psi\nabla\phi) + \nabla_\phi \cdot (\phi\nabla\psi + \psi\nabla\phi) \\
&= \phi\nabla^2\psi + \nabla\psi \cdot \nabla\phi + \nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi \\
&= \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi
\end{aligned}
$$

■式 (2.10)

$$
LHS = \nabla_A(\boldsymbol{A} \cdot \boldsymbol{B}) + \nabla_B(\boldsymbol{A} \cdot \boldsymbol{B})
$$

ここで、式 (1.2) から、

$$
\begin{aligned}
\boldsymbol{B}(\nabla_A \times \boldsymbol{A}) &= \nabla_A(\boldsymbol{A} \cdot \boldsymbol{B}) - (\boldsymbol{B} \cdot \nabla_A)\boldsymbol{A} \\
\boldsymbol{A}(\nabla_B \times \boldsymbol{B}) &= \nabla_B(\boldsymbol{A} \cdot \boldsymbol{B}) - (\boldsymbol{A} \cdot \nabla_B)\boldsymbol{B}
\end{aligned}
$$

が成り立つので、これらの辺々を足して整理すると、

$$
\begin{aligned}
&\nabla_A(\boldsymbol{A} \cdot \boldsymbol{B}) + \nabla_B(\boldsymbol{A} \cdot \boldsymbol{B}) \\
&\quad = \boldsymbol{B} \times (\nabla_A \times \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla_A)\boldsymbol{A} + \boldsymbol{A} \times (\nabla_B \times \boldsymbol{B}) + (\boldsymbol{A} \cdot \nabla_B)\boldsymbol{B} \\
&\quad = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})
\end{aligned}
$$

■式 (2.11)

$$
\begin{aligned}
LHS &= \nabla_A \cdot (\boldsymbol{A} \times \boldsymbol{B}) + \nabla_B \cdot (\boldsymbol{A} \times \boldsymbol{B}) \qquad \leftarrow \text{式}\ (1.1) \\
&= \boldsymbol{B} \cdot (\nabla_A \times \boldsymbol{A}) - \nabla_B(\boldsymbol{B} \times \boldsymbol{A}) \\
&= \boldsymbol{B} \cdot (\nabla_A \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla_B \times \boldsymbol{B}) \\
&= \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})
\end{aligned}
$$

■式 (2.12)

$$
\begin{aligned}
LHS &= \nabla_A \times (\boldsymbol{A} \times \boldsymbol{B}) + \nabla_B \times (\boldsymbol{A} \times \boldsymbol{B}) \qquad \leftarrow \text{式}\ (1.2) \\
&= (\nabla_A \cdot \boldsymbol{B})\boldsymbol{A} - (\nabla_A \cdot \boldsymbol{A})\boldsymbol{B} + (\nabla_B \cdot \boldsymbol{B})\boldsymbol{A} - (\nabla_B \cdot \boldsymbol{A})\boldsymbol{B} \\
&= (\boldsymbol{B} \cdot \nabla_A)\boldsymbol{A} - \boldsymbol{B}(\nabla_A \cdot \boldsymbol{A}) + \boldsymbol{A}(\nabla_B \cdot\ ) - (\boldsymbol{A} \cdot \nabla_B)\boldsymbol{B} \\
&= (\ \cdot \nabla)A - \boldsymbol{B}(\nabla\boldsymbol{A}) + \boldsymbol{A}(\nabla\boldsymbol{B}) - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}
\end{aligned}
$$

■式 (2.13)

$$
\begin{aligned}
LHS &= \nabla_\phi \times (\phi\boldsymbol{A}) + \nabla_A \times (\phi\boldsymbol{A}) \\
&= \nabla_\phi\phi \times \boldsymbol{A} + \phi\nabla_A \times \boldsymbol{A} \\
&= \nabla\phi \times \boldsymbol{A} + \phi\nabla \times \boldsymbol{A} \\
&= -\boldsymbol{A} \times \nabla\phi + \phi\nabla \times \boldsymbol{A}
\end{aligned}
$$

■式 (2.14)

$$
\begin{aligned}
LHS &= \nabla_\phi \cdot (\phi\boldsymbol{A}) + \nabla_A \times (\phi\boldsymbol{A}) \\
&= \nabla_\phi\phi \cdot \boldsymbol{A} + \phi\nabla_A \cdot \boldsymbol{A} \\
&= \boldsymbol{A} \cdot \nabla\phi + \phi\nabla\boldsymbol{A}
\end{aligned}
$$

■**式** (2.19)　式 (2.10) より。