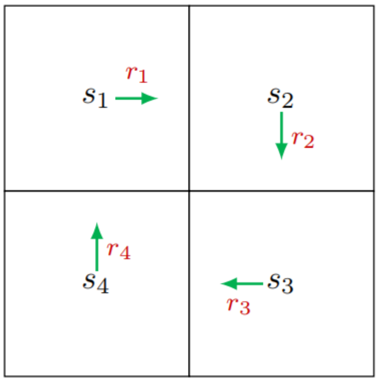
How to calculate return

e.g.



- By definition.

  To calculate v_1:

$$v_1 = r_1 + \gamma r_2 + \gamma^2 r_3 + \dots$$

  Similarly, v_2, v_3, v_4 can be obtained in the same manner.

- Bootstrapping.

$$v_1 = r_1 + \gamma v_2$$

  Similarly, v_2, v_3, v_4 can be obtained in the same manner. And get a linear matrix-vector equation:

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} + \begin{bmatrix} \gamma v_2 \\ \gamma v_3 \\ \gamma v_4 \\ \gamma v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{\mathbf{r}} + \gamma \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}}$$

  i.e.

$$\mathbf{v} = \mathbf{r} + \gamma P \mathbf{v}$$

  And get $\mathbf{v}$ by $\mathbf{v} = (I - \gamma P)^{-1} \mathbf{r}$

State value

Some notations

Consider the single-step process:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1}$$

- S_t: State at time t
- A_t: Action taken at state S_t

- R_{t+1}: Reward after taking A_t
- S_{t+1}: State transited to after taking A_t

This step is governed by the following probability distributions:

- $S_t \rightarrow A_t$ is governed by $\pi(A_t = a|S_t = s)$
- $S_t, A_t \rightarrow R_{t+1}$ is governed by $p(R_{t+1} = r|S_t = s, A_t = a)$
- $S_t, A_t \rightarrow S_{t+1}$ is governed by $p(S_{t+1} = s'|S_t = s, A_t = a)$

Multi-step trajectory:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1} \xrightarrow{A_{t+1}} R_{t+2}, S_{t+2} \ldots$$

The discounted return is:

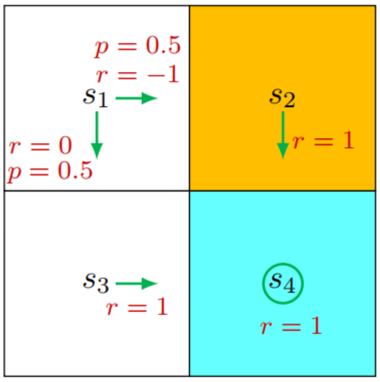
$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots$$

The expectation of G_t is defined as state-value function or simply state value:

$$v_\pi(s) = \mathbb{E}[G_t|S_t = s]$$

$v_\pi(s)$ depends on policy π. The value of a state may vary with different policies.

> State value is expectation of all possible returns.



Here is the example of policy π, and

$$v_\pi(s_1) = 0.5 \times (-1 + 1 \times \gamma + 1 \times \gamma^2 + \ldots) + 0.5 \times (0 + 1 \times \gamma + 1 \times \gamma^2 + \ldots) = -0.5 + \frac{\gamma}{\gamma - 1}$$

Deriving the Bellman equation

The return G_t of a trajectory can be written as:

$$G_t = R_{t+1} + \gamma G_{t+1}$$

So

$$v_\pi(s) = \mathbb{E}(G_t|S_t = s) = \mathbb{E}(R_{t+1}|S_t = s) + \gamma \mathbb{E}(G_{t+1}|S_t = s)$$

For $\mathbb{E}(R_{t+1}|S_t = s)$:

$$
\begin{aligned}
\mathbb{E}(R_{t+1}|S_t = s) &= \sum_a \pi(a|s)\mathbb{E}(R_{t+1}|S_t = s, A_t = a) \\
&= \sum_a \pi(a|s) \sum_r p(r|s,a)r
\end{aligned}
$$

For $\mathbb{E}(G_{t+1}|S_t = s)$:

$$
\begin{aligned}
\mathbb{E}(G_{t+1}|S_t = s) &= \sum_{s'} \mathbb{E}(G_{t+1}|S_t = s, S_{t+1} = s')p(s'|s) \\
&= \sum_{s'} \mathbb{E}(G_{t+1}|S_{t+1} = s')p(s'|s) \\
&= \sum_{s'} v_\pi(s')p(s'|s) \\
&= \sum_{s'} v_\pi(s') \sum_a p(s'|s,a)\pi(a|s)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v_\pi(s) &= \sum_a \pi(a|s) \sum_r p(r|s,a)r + \gamma \sum_{s'} v_\pi(s') \sum_a p(s'|s,a)\pi(a|s) \\
&= \sum_a \pi(a|s) \sum_r p(r|s,a)r + \gamma \sum_a \pi(a|s) \sum_{s'} v_\pi(s')p(s'|s,a) \\
&= \sum_a \pi(a|s) \big[\sum_r p(r|s,a)r + \gamma \sum_{s'} v_\pi(s')p(s'|s,a) \big]
\end{aligned}
$$

Notice that this equation is about $v_\pi(s)$ and $v_\pi(s')$, so we can use bootstrapping to solve the **equations** (like equations of all states) of states.

Matrix vector form of the Bellman equation

Rewrite the Bellman equation $v_\pi(s) = \sum_a \pi(a|s) \big[\sum_r p(r|s,a)r + \gamma \sum_{s'} v_\pi(s')p(s'|s,a) \big]$ as:

$$
v_\pi(s) = r_\pi(s) + \gamma \sum_{s'} v_\pi(s')p_\pi(s'|s)
$$

Thus, the **equations** for all states can be represented in matrix-vector form:

$$
v_\pi = r_\pi + \gamma P_\pi v_\pi
$$

where

- $v_\pi = [v_\pi(s_1), \dots, v_\pi(s_n)]^T$
- $r_\pi = [r_\pi(s_1), \dots, r_\pi(s_n)]^T$
- State transition matrix $P_\pi \in \mathbb{R}^{n \times n}$, where $[P_\pi]_{ij} = p_\pi(s_j|s_i)$

Solve state values

Why to solve state values?

- Foundation to find better policies.

The Bellman equation matrix-vector form is:

$$
v_\pi = r_\pi + \gamma P_\pi v_\pi
$$

- The closed-form solution is:

$$v_\pi = (I - \gamma P_\pi)^{-1} r_\pi$$

- To avoid the matrix inverse operation, use iterative solution:

$$v_{k+1} = r_\pi + \gamma P_\pi v_k$$

  We can show that

$$v_k \to v_\pi = r_\pi + \gamma P_\pi v_\pi, \quad k \to \infty$$

Action value

- State value: the average return of starting from a state.
- Action value: the average return of starting from a state and taking an action.

Definition:

$$q_\pi(s,a) = \mathbb{E}[G_t | S_t = s, A_t = a]$$

And,

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a) \tag{1}$$

Recall that state value is given by:

$$v_\pi(s) = \sum_a \pi(a|s) \big[\sum_r p(r|s,a) r + \gamma \sum_{s'} v_\pi(s') p(s'|s,a) \big] \tag{2}$$

By comparing (1) and (2):

$$q_\pi(s,a) = \sum_r p(r|s,a) r + \gamma \sum_{s'} v_\pi(s') p(s'|s,a) \tag{3}$$

- (1) shows how to obtain state values from action values.
- (3) shows how to obtain action values from state values.