Value representation: From table to function

Problems of tabular representation of state and action:

Difficult to handle large or continuous state or action spaces.

So we can consider cur fitting.

> For example, suppose one-dimensional states $\mathcal{S}$ and their values $\{v_\pi(s_i)\}$. Since $|\mathcal{S}|$ is large, we can use a simple curve to approximate these dots to save. The simplest curve is a straight line, which can be described as
>
> $$\hat{v}(s,w) = as + b = \underbrace{[s,1]}_{\phi^T(s)}\underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{w} = \phi^T(s)w$$

TD learning of state values based on function approximation

Objective function

Let $v_\pi(s)$ be true state value, $\hat{v}(s,w)$ be approximated state value. Then the problem to be solved is to find an optimal w so that $\hat{v}(s,w)$ can best approximate $v_\pi(s)$ for every s. The objective function is

$$J(w) = \mathbb{E}\big[\big(v_\pi(S) - \hat{v}(s,w)\big)^2\big]$$

where the expectation is calculated with respect to the random variable $S \in \mathcal{S}$.

There are several ways to find probability distribution of S:

- The first way is to use a $uniform\ distribution$. i.e.

$$J(w) = \frac{1}{n}\sum_{s\in\mathcal{S}}(v_\pi(s) - \hat{v}(s,w))^2$$

- The second way is to use the $stationary\ distribution$. More specifically, after the agent executes a given policy for a sufficiently long period, the probability of the agent being located at any state can be described by this stationary distribution. Let $d_\pi(s)$ denote the probability for the agent visiting s after a long period of time. Then

$$J(w) = \sum_{s\in\mathcal{S}} d_\pi(s)(v_\pi(s) - \hat{v}(s,w))^2$$

> Let $n_\pi(s)$ denote the number of times that s has been visited in a very long episode generated by π.
>
> Then $d_\pi(s)$ can be approximated by
>
> $$d_\pi(s) \approx \frac{n_\pi(s)}{\sum_{s_k\in S} n_\pi(s_k)}$$

Optimization algorithms

To minimize the objective function $J(w)$, we can use gradient descent:

$$w_{k+1} = w_k - \alpha_k \nabla_w J(w_k)$$

where

$$\begin{aligned} \nabla_w J(w_k) &= \mathbb{E}\big[\big(v_\pi(S) - \hat{v}(s, w_k)\big)\big] \\ &= -2\mathbb{E}\big[\big(v_\pi(S) - \hat{v}(s, w_k)\big)\nabla_w \hat{v}(s, w_k)\big] \end{aligned}$$

Therefore, the gradient descent algorithm is

$$w_{k+1} = w_k + 2\alpha_k \mathbb{E}[(v_\pi(S) - \hat{v}(S, w_k))\nabla_w \hat{v}(S, w_k)]$$

where the coefficient 2 before α_k can be merged into α_k. The algorithm requires calculating the expectation. We can replace the true gradient with a stochastic gradient:

$$w_{t+1} = w_t + \alpha_t\big(v_\pi(s_t) - \hat{v}(s_t, w_t)\big)\nabla_w \hat{v}(s_t, w_t)$$

It requires the true state value v_π, which is unknown and must be estimated. We can replace it with an approximation to make the algorithm implementable. There are two ways to do so:

- MC: Let g_t be discounted return starting from s_t. Then g_t can be an approximation of $v_\pi(s_t)$.
- TD learning: $r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t)$ can be viewed as an approximation of $v_\pi(s_t)$.

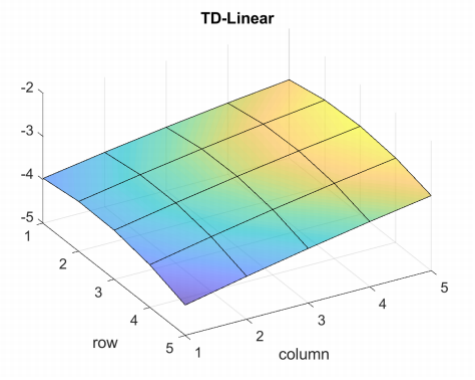
Selection of function approximators

How to select the function $\hat{v}(s, w)$?

- First approach is simply use a linear function $\hat{v}(s, w) = \phi^T(s)w$.

  It is difficult to select appropriate feature vectors. But it

  - can be better understood
  - is still powerful

  > Take 5×5 grid world as an example,
  >
  > $\phi(s) = [1, x, y, x^2, y^2, xy]^T \in \mathbb{R}^6$. The result to fit $\hat{v}$ may be
  >
  > 
  >

- The second is is to use an artificial neural network as a nonlinear function approximator.

Analysis

The algorithm minimize the following function:

- True value error:

$$J_E(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S,w))^2] = \|\hat{v}(w) - v_\pi\|_D^2$$

- Bellman error:

$$J_{BE}(w) = \|\hat{v}(w) - (r_\pi + \gamma P_\pi \hat{v}(w))\|_D^2 \doteq \|\hat{v}(w) - T_\pi(\hat{v}(w))\|_D^2$$

- Projected Bellman error:

$$J_{PBE}(w) = \|\hat{v}(w) - MT_\pi(\hat{v}(w))\|_D^2$$

  where M is a projection matrix.

TD learning of action values based on function approximation

Sarsa with function approximation

Suppose $\hat{q}(s,a,w)$ is an approximation of $q(s,a)$. Then

$$w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \hat{q}(s_{t+1}, a_{t+1}, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t)$$

If use linear functions, we have

$$\hat{q}(s,a,w) = \phi^T(s,a)w$$

Pseudocode:

Initialization: Initial parameter w_0. Initial policy π_0. $\alpha_t = \alpha > 0$ for all t. $\epsilon \in (0,1)$.
Goal: Learn an optimal policy that can lead the agent to the target state from an initial state s_0.
For each episode, do
  Generate a_0 at s_0 following $\pi_0(s_0)$
  If s_t $(t = 0, 1, 2, \dots)$ is not the target state, do
    Collect the experience sample $(r_{t+1}, s_{t+1}, a_{t+1})$ given (s_t, a_t): generate r_{t+1}, s_{t+1} by interacting with the environment; generate a_{t+1} following $\pi_t(s_{t+1})$.
    Update q-value:
    $w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \hat{q}(s_{t+1}, a_{t+1}, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t)$
    Update policy:
    $\pi_{t+1}(a|s_t) = 1 - \frac{\varepsilon}{|\mathcal{A}(s_t)|}(|\mathcal{A}(s_t)| - 1)$ if $a = \arg\max_{a \in \mathcal{A}(s_t)} \hat{q}(s_t, a, w_{t+1})$
    $\pi_{t+1}(a|s_t) = \frac{\epsilon}{|\mathcal{A}(s_t)|}$ otherwise
    $s_t \leftarrow s_{t+1}$, $a_t \leftarrow a_{t+1}$

Q-learning with function approximation

The update rule is

$$w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \max_{a \in \mathcal{A}(s_{t+1})} \hat{q}(s_{t+1}, a, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t)$$

Pseudocode:

Initialization: Initial parameter w_0. Initial policy π_0. $\alpha_t = \alpha > 0$ for all t. $\epsilon \in (0, 1)$.
Goal: Learn an optimal path that can lead the agent to the target state from an initial state s_0.

For each episode, do
  If s_t $(t = 0, 1, 2, \dots)$ is not the target state, do
    Collect the experience sample (a_t, r_{t+1}, s_{t+1}) given s_t: generate a_t following $\pi_t(s_t)$; generate r_{t+1}, s_{t+1} by interacting with the environment.
    Update q-value:
    $w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \max_{a \in \mathcal{A}(s_{t+1})} \hat{q}(s_{t+1}, a, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t)$
    Update policy:
    $\pi_{t+1}(a|s_t) = 1 - \frac{\varepsilon}{|\mathcal{A}(s_t)|}(|\mathcal{A}(s_t)| - 1)$ if $a = \arg\max_{a \in \mathcal{A}(s_t)} \hat{q}(s_t, a, w_{t+1})$
    $\pi_{t+1}(a|s_t) = \frac{\varepsilon}{|\mathcal{A}(s_t)|}$ otherwise

Deep Q-learning

We can integrate deep neural networks into Q-learning to obtain an approach called deep Q-learning or **deep Q-network** (DQN).

Algorithm description

The algorithm aims to minimize the following objective function:

$$J(w) = \mathbb{E}\left[\left(R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w) - \hat{q}(S, A, w) \right)^2 \right]$$

It is not easy to get $\nabla_w \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w)$. So let $y = R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w)$, and we assume that w in y is fixed when we calculate the gradient.

We introduce two networks:

- **Main network** $\hat{q}(s, a, w)$
- **Target network** $\hat{q}(s, a, w_T)$

Then

$$J = \mathbb{E}\left[\left(R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w_T) - \hat{q}(S, A, w) \right)^2 \right]$$

Implement details:

- In every iteration, we draw a mini-batch of samples $\{(s, a, r, s')\}$ from the replay buffer.

- Inputs of networks include state s and action a. The target output is $y_T = r + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w_T)$. Then we directly minimize the TD error or called loss function $(y_T - \hat{q}(s, a, w))^2$ over the mini-batch $\{(s, a, y_T)\}$.

Experience replay

What is Experience replay?

- Store the experience samples in a set, called replay buffer $\mathcal{B} = \{(s, a, r, s')\}$
- Every time we train the neural network, we can draw a mini-batch of random samples from the replay buffer.
- The draw of samples, or called **experience replay**, should follow a **uniform distribution**.

Why does the replay must follow a uniform distribution?

- The distribution of the state-action pair (S, A) is assumed to be uniform. (If you have prior knowledge, you can set the weight)

Pseudocode:

Initialization: A main network and a target network with the same initial parameter.
Goal: Learn an optimal target network to approximate the *optimal* action values from the experience samples generated by a given behavior policy π_b.

Store the experience samples generated by π_b in a replay buffer $\mathcal{B} = \{(s, a, r, s')\}$
  For each iteration, do
    Uniformly draw a mini-batch of samples from $\mathcal{B}$
    For each sample (s, a, r, s'), calculate the target value as $y_T = r + \gamma \max_{a \in \mathcal{A}(s')} \hat{q}(s', a, w_T)$, where w_T is the parameter of the target network
    Update the main network to minimize $(y_T - \hat{q}(s, a, w))^2$ using the mini-batch of samples
  Set $w_T = w$ every C iterations