* XXZ model

$$H=J_\perp\sum_i(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+J_z\sum_i S_i^zS_{i+1}^z$$

有 $[S_i^\alpha,S_j^\beta]=i\hbar\delta_{ij}\varepsilon_{\alpha\beta\gamma}S_i^\gamma \Rightarrow \underline{U_i(\vec n,\theta)=e^{i\theta\vec n\cdot\vec S_i}} \Rightarrow T=\prod_{n:odd}U_n(e_z,\pi)$

$$\Rightarrow\ =\frac{J_\perp}{2}\sum_i(S_i^+S_{i+1}^-+S_i^-S_{i+1}^+)+J_z\sum_i S_i^zS_{i+1}^z$$

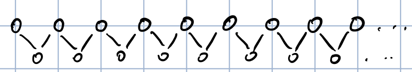
有 $THT^\dagger$ 中 $J_\perp\to -J_\perp,\ J_z\to J_z$

$\Rightarrow J_\perp$ 的符号不重要.

$$\begin{cases} S_i^+S_{i+1}^- \xrightarrow{JW} C_i^\dagger C_{i+1}\sim p \\ S_i^z\sim x\end{cases}$$

* $J_\perp=0$ Ising model (classical) $|\uparrow\uparrow\uparrow\uparrow\underline{\uparrow\downarrow}\downarrow\downarrow\downarrow\rangle$ kink

自旋关联函数 $F(r)=\langle S_i^zS_{i+r}^z\rangle\sim e^{-r/\xi}$



$$H=J_z\sum_i S_i^zS_{i+1}^z \quad (\pm\tfrac14)$$
$$=J_z\sum_i^{N-1}L_{i,i+1}$$
$$=J_z\sum_i^{N-1}L_{i+1/2} \qquad \frac{i+(i+1)}{2}=i+1/2$$

$$\Rightarrow Z=\sum_{\{L\}}\prod_i e^{-\beta J_z\sum^{N-1}L_{i+1/2}}$$
$$=\prod_i\sum_{L=\pm\frac14}e^{-\beta J_zL_{i+1/2}}=\Big(\sum_{L=\pm\frac14}e^{-\beta J_zL_{i+1/2}}\Big)^N$$

$$f=-\frac1\beta\lim_{N\to\infty}\frac1N\ln Z=-\frac1\beta\ln\Big[\sum_{L=\pm\frac14}e^{-\beta J_zL_{i+1/2}}\Big]=-\frac1\beta\ln\Big(2\cosh\frac{\beta J_z}{4}\Big).$$

$$4F(r)=\frac{\sum_{\{L\}}4L\,e^{-\beta H}}{\sum_{\{L\}}e^{-\beta H}}=\frac{\prod_{j=1}^r\Big(\sum_{L=\pm\frac14}4L\,e^{-\beta J_zL}\Big)}{\prod_{j=1}^r\Big(\sum_{L=\pm\frac14}e^{-\beta J_zL}\Big)}=\Big[\tanh\Big(\frac{-\beta J_z}{4}\Big)\Big]^r=(-\mathrm{sgn}J_z)^r e^{[r\ln\tanh(\frac{\beta|J_z|}{4})]}\Rightarrow \xi=\frac{-1}{\ln\tanh\Big(\frac{\beta|J_z|}{4}\Big)}$$

$T=0K$, 有 $F(r)=\frac14(-\mathrm{sgn}J_z)^r$

在 $\beta|J_z|\gg1$ 下, 有:

$$\xi^{-1}=2e^{-\beta|J_z|/2}$$

形成kink需要 $\Delta E=|J_z|/2$, 两个kink的平均距离为 ξ

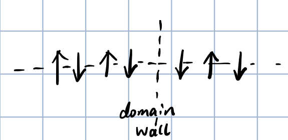
* Quantum:

def: $S_{tot}=\sum_i S_i$ 有 $[S_{tot}^2,H]=[S_{tot}^z,H]=0$.

$J_z<0$: FM: $|\uparrow\rangle$ 与 $|\downarrow\rangle$ 为本征态

$J_z>0$: AFM: $S_{staggered}=\sum_i(-1)^iS_i$ 的本征态.

$$(H-E_{Neel})\psi_n=\frac{J_z}{2}\psi_n+\frac{J_\perp}{2}(\psi_{n+2}+\psi_{n-2})$$



$|\downarrow\uparrow\downarrow\uparrow\underline{\downarrow\downarrow}\uparrow\downarrow\uparrow\downarrow\rangle$
$\wr$
$|\downarrow\uparrow\underline{\downarrow\downarrow}\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$

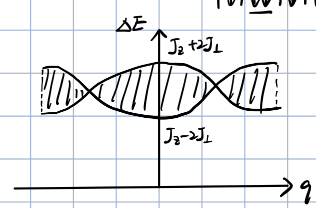
设 $\psi(k)=\frac{1}{\sqrt N}\sum_n e^{ikn}\psi_n$ 有:

$$(H-E_{Neel})\psi(k)=\Big(\frac{J_z}{2}+J_\perp\cos 2k\Big)\psi(k)$$

故激发态能谱为: $\frac{J_z}{2}+J_\perp\cos(2k)$

$$\begin{cases} q=k_1+k_2 \\ \Delta E=\varepsilon_{DW}(k_1)+\varepsilon_{DW}(k_2)\end{cases}$$

$\Rightarrow \Delta E_{LB}=J_z-2J_\perp|\cos q|$



* 利用J-W得到:

$$H=-\frac{J_\perp}{2}\sum_{i=1}^N(f_i^\dagger f_{i+1}+f_{i+1}^\dagger f_i)+J_z\sum_{i=1}^N(f_i^\dagger f_i-\tfrac12)(f_{i+1}^\dagger f_{i+1}-\tfrac12)$$

当 $J_z=0$, 用FT有:

$$H_{XY}=\sum_k(-J_\perp\cos k)f_k^\dagger f_k$$

* The Bethe Ansatz

$\Phi_{n_1,\cdots,n_r}$ r个自旋为↑, 则有 $N-r$ 个自旋为↓.

$H\Phi_0=\frac{J_z}{4}N\Phi_0=E_0\Phi_0$ (Φ_0表示所有自旋朝下)

① 仅考虑一个向上的自旋: $\Psi=\sum_{n=1}^N C_n\Phi_n$ ($C_{n+N}=C_n$)

$$\Rightarrow H\Psi=\frac{J_z}{4}(N-4)\Psi+\frac{J_\perp}{2}\sum_{n=1}^N C_n(\Phi_{n+1}+\Phi_{n-1})$$
$$=\frac{J_z}{4}(N-4)\Psi+\frac{J_\perp}{2}\sum_{n=1}^N(C_{n+1}+C_{n-1})\Phi_n$$
$$=E\Psi \Rightarrow \boxed{EC_n=(E_0-J_z)C_n+\frac{J_\perp}{2}(C_{n+1}+C_{n-1})}$$

代入 $C_n=\frac{1}{\sqrt N}e^{ikn}$

$$\Rightarrow E_k = E_0 - J_z + J_\perp \cos k \quad \Rightarrow \quad \varepsilon_k = |J_z| + J_\perp \cos k.$$

②考虑两个 Spin up 的 case:

$$\Psi = \sum_{n_1<n_2} C_{n_1,n_2} \Phi_{n_1 n_2}$$

有: $$H\Phi_{n_1 n_2} = (E_0 - \frac{J_z}{2}\cdot 4)\Phi_{n_1 n_2} + \frac{J_\perp}{2}(\Phi_{n_1-1,n_2} + \Phi_{n_1+1,n_2} + \Phi_{n_1,n_2-1} + \Phi_{n_1,n_2+1}) \quad (n_2 > n_1+1)$$

$$H\Phi_{n_1,n_1+1} = (E_0 - \frac{J_z}{2}\cdot 2)\Phi_{n_1,n_1+1} + \frac{J_\perp}{2}(\Phi_{n_1-1,n_1+1} + \Phi_{n_1,n_1+2}). \quad (n_2 = n_1+1)$$

故: $$\begin{cases} \varepsilon C_{n_1 n_2} = -2J_z C_{n_1 n_2} + \frac{J_\perp}{2}(C_{n_1+1,n_2} + C_{n_1-1,n_2} + C_{n_1,n_2+1} + C_{n_1,n_2-1}) & (n_2 > n_1+1) \\ \varepsilon C_{n_1,n_1+1} = -J_z C_{n_1,n_1+1} + \frac{J_\perp}{2}(C_{n_1-1,n_1+1} + C_{n_1,n_1+2}). & (n_2 = n_1+1) \end{cases} \quad (n_2 = n_1+1)$$

$$C_{n_1 n_2} = C_1 e^{i(k_1 n_1 + k_2 n_2)} + C_2 e^{i(k_2 n_1 + k_1 n_2)}$$

$$\underline{-J_z C_{n_1,n_1+1} + \frac{J_\perp}{2}(C_{n_1,n_1} + C_{n_1+1,n_1+1}) = 0} \qquad \varepsilon = \varepsilon_{k_1} + \varepsilon_{k_2}$$

$$-J_z(C_1 e^{ik_2} + C_2 e^{ik_1}) + J_\perp(C_1 + C_2)(1 + e^{i(k_1+k_2)}) = 0$$

$$\Rightarrow \frac{C_1}{C_2} = -\frac{J_z e^{i\frac{k_1-k_2}{2}} - J_\perp \cos\frac{k_1+k_2}{2}}{J_z e^{-i\frac{k_1-k_2}{2}} - J_\perp \cos\frac{k_1+k_2}{2}} \qquad \left|\frac{C_1}{C_2}\right| = 1$$

相移中写为:

$$C_1 = e^{i\phi/2} \qquad C_2 = e^{-i\phi/2}$$

$$\cot\frac{\phi}{2} = \frac{J_z \sin\frac{k_1-k_2}{2}}{J_\perp \cos\frac{k_1+k_2}{2} - J_z \cos\frac{k_1-k_2}{2}}$$

$$\Rightarrow \varepsilon(K,q) = \varepsilon_{\frac{K+q}{2}} + \varepsilon_{\frac{K-q}{2}} = 2|J_z|\left(1 - \cos\frac{K}{2}\cos\frac{q}{2}\right)$$

*对于 r 个 Spin up, 且 $r = \frac{N}{2}$

$$\Psi = \sum_{n_1<n_2<\cdots<n_r} C_{n_1 n_2 \cdots n_r} \Phi_{n_1 n_2 \cdots n_r} \quad 其中 \quad \underbrace{C_{n_1 n_2 \cdots n_r} = \sum_P \exp\left[i\left(\sum_{j=1}^r k_{P_j} n_j + \frac{1}{2}\sum_{j<l} \phi_{P_j,P_l}\right)\right]}$$

多体散射 → 两体散射的乘积

$$E = -\frac{NJ}{4} + J\sum_{j=1}^r \cos k_j = \frac{NJ}{4} - J\sum_{j=1}^r (1 - \cos k_j)$$

$$\Rightarrow \Delta E = \frac{\pi J}{2}|\sin q|$$