Mathematical and Computational Physics
Lecture Notes

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Contents

Preface

This book collects informal lecture notes to accompany the videos and problem sets. When the course was originally given on Georgetown's campus in 2016, these notes formed the basis for the in-class lectures. As such, they contain some extra materials that did not make it onto the flipped course format (and also the MOOC); I have decided to keep that extra material in these notes. This is not intended to be a formal book by any means. It is more of a supplement to complement the other resources in the course. If you like, *it is a learning guide*. It synthesizes many different materials including those you will find in the readings.

There are many people who have helped with different aspects of the class. Geoffrey Fitzgerald carefully went through all of the on-line materials and checked them for accuracy. Mike Rushka typeset the problem sets and solutions and Matthew Werner and Paul Heyden converted them to on-line assignments using the MITx grading system. Matthew Werner and Gabby Olshan-Cantin typeset these notes into book form. David Wolf carefully reviewed all problems and Jolyon Bloomfield provided technical assistance with MIT's grading extensions to the edX platform. Chris Cothran carefully set up the laboratory demonstrations. Leanne Doughty also taught the flipped version of the class at Georgetown. Anna Kruse, Zhuquing Ding and Linda Lemus provided assistance from CNDLS. Joe King and Xiaoke Ding shot some of the demonstration videos and helped with the course trailer. Eddie Maloney paid the bills. We also benefited from financial support from NASA through the DC area space grant.

The main goal is for you to make a transition from a *technician* to a *practitioner*. Practitioners understand not just the mechanical steps of how to use math, but they understand the how and why as well. Empowered in this fashion, you too will be able to move seamlessly through the physics curriculum. Enjoy the ride and work hard. It will payoff well!

Finally, I thank all of the students who took this class, became practitioners of math, and helped improve the class and myself during the journey. And, of course, any errors, omissions, typos, or mistakes are on me. Tell me about them and I will fix them as soon as I can!

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Devious Math Tricks

0.1 Introduction

Before jumping into the main text, we start with a primer on mathematical tools that you can use to make your mathematical work easier, faster, and more accurate. I hope you enjoy these ideas and have a chance to implement them through the course. They take some time to master, but are well worth it.

Remember that kid from algebra class who sat across from you and seemed to know the answers to every question faster and more accurately than you? I am going to let you in on a secret—most likely that "curve buster" was not a genius or born to do math—instead he or she simply learned a number of devious tricks to get through the material faster and with fewer errors. After finishing this chapter, you will be able to do so too!

The biggest myth to dispel is that geniuses in algebra have photographic memories and can recall the relevant formulas and apply them at will. Such people may exist but the brainiac across the row from you was not one of these. The reality is you need to only know something like ten important equations, and by using them as I am about to describe to you, you will find that you too will just know them when you need them. You too can become an algebra master.

0.2 Devious trick number 1 (Add zero)

$x + 0 = x$. This is called the "add zero" trick. It comes up so many times you cannot get through most problems without encountering it. The problem is no one has told you about it, so you often stumble onto it and fail to recognize its importance. We will see and use the add zero trick again and again. Right now, let me illustrate it for you. Consider the following problem: factor $x^2 - 25$.

Go ahead and give it a try, but do not just write down the answer if you have memorized it, see if you can derive it. Here is how the add zero trick works. An educated guess says we should add a number times x as one of the terms, the 25, suggests. We add 5x- 5x to get:

$$x^2 + 5x - 5x - 25 = (x^2 + 5x) - (5x + 25) \tag{1}$$

$$= x(x + 5) - 5(x + 5) \tag{2}$$

$$= (x - 5)(x + 5). \tag{3}$$

All we did was add zero! The creativity was to recognize what zero to add. Indeed this is how you apply the technique. You learn pretty quickly what ideas might work for what to add by practicing the method a number of times.

0.3 Devious trick number 2 (Multiply by 1)

$x \times 1 = x$. Just like the "add zero" trick, the "multiply by one" trick can be used to simplify many different expressions. It often is seen in cases where we want to rationalize a denominator, such as the following:
Rationalize the denominator of $\frac{1-\sqrt{3}}{1+2\sqrt{3}}$.

The strategy is to multiply the numerator and denominator by the same factor such that the denominator no longer has a square root. Recalling that $(a + b)(a - b) = a^2 - b^2$, our strategy is to use $1 = \frac{1-2\sqrt{3}}{1-2\sqrt{3}}$ as the term we multiply by. Now we compute:

$$\frac{1-\sqrt{3}}{1+2\sqrt{3}} \times \frac{1-2\sqrt{3}}{1-2\sqrt{3}} = \frac{1-\sqrt{3}-2\sqrt{3}+6}{1-12} = -\frac{7-3\sqrt{3}}{11}. \tag{4}$$

This "multiply by one" trick can also be used to simplify products of integers via factorials. For example:

$$(n+1)(n+2)...(2n-1)(2n) = \frac{1 \times 2 \times 3...n}{1 \times 2 \times 3...n}(n+1)(n+2)...(2n-1)(2n) = \frac{(2n)!}{n!} \tag{5}$$

0.4 Devious trick number 3 (Manipulate identities)

Remember simple identities and use them to make complicated identities. As I mentioned before, there are only a handful of identities worth remembering. Nearly everything else can be re-derived from these few. As an example, think about your trigonometry identities. There are loads upon loads of them. But they all follow from two main results—remembering the definitions of the trig functions in terms of sin and cos and using the fact that $\sin^2 + \cos^2 = 1$. Let's illustrate how. Suppose I want to express $\sec^2(\theta)$ in terms of $\tan(\theta)$. I simply recall that $\sec(\theta) = \frac{1}{\cos(\theta)}$. Then I use our "multiply by one" trick using $1 = \cos^2(\theta) + \sin^2(\theta)$, so

$$\begin{aligned} \sec^2(\theta) &= \frac{1}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}\left(\cos^2(\theta) + \sin^2(\theta)\right) = \frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} \\ &= 1 + \tan^2(\theta). \end{aligned} \tag{6}$$

0.5 Devious trick number 4 (Recognize abstracted identities)

Recognize simple identities when expressed in a more complex form. This one takes a bit more practice, as it requires you to use abstraction to recognize the simple identity. Here is an example. We already saw that $x^2 - 25 = (x+5)(x-5)$. We need to recognize it in other contexts. So

$$\frac{\sin^2(\theta) - 25}{\cos^2(\theta)} = \tan^2(\theta) - 25\sec^2(\theta) = (\tan(\theta) + 5\sec(\theta))(\tan(\theta) - 5\sec(\theta)) \tag{7}$$

will also hold. A more complicated example is to factor $(x^4 + 2x^2 - 24)$. We use the add zero trick first:

$$\begin{aligned} \left(x^4 + 2x^2 + (1-1) - 24\right) &= \left((x^2+1)^2 - 25\right) = (x^2+1+5)(x^2+1-5) \\ &= (x^2+6)(x^2-4). \end{aligned} \tag{8}$$

Note how we had to recognize $x^4 + 2x^2 + 1 = (x^2+1)^2$ was the square in the expression.

0.6 Devious trick number 5 (subtract identities)

Use information you know and change expressions to those you and know and their difference. This one is a little more difficult to recognize but is quite important. We will illustrate by example. Consider evaluating the following:

$$x^2 + x^4 + x^6 + ... = \sum_{n=1}^{\infty} x^{2n}. \tag{9}$$

This looks like a geometric series, but for x^2, not x. In other words,

$$\sum_{n=1}^{\infty} x^{2n} = \sum_{n=1}^{\infty} (x^2)^n. \tag{10}$$

This would be the geometric series except it is missing the first term. Hence,

$$\sum_{n=1}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n - 1 = \frac{1}{1-x^2} - 1 = \frac{x^2}{1-x^2}. \tag{11}$$

We also could find it another way, by recognizing that

$$x^2 + x^4 + x^6 + ... = x^2(1 + x^2 + x^4 + ...) = x^2 \sum_{n=0}^{\infty} (x^2)^n = x^2 \frac{1}{1-x^2} = \frac{x^2}{1-x^2}, \tag{12}$$

as before.

0.7 Devious trick number 6 (index gymnastics)

We end by discussing one other skill, which involves shifting indices in summations. Students often struggle with this skill. Yet it is fairly easy to master. Let's look at the last example:

$$x^2 + x^4 + x^6 + ... = \sum_{n=1}^{\infty} x^{2n}. \tag{13}$$

We know the geometric series starts with $n = 0$, not $n = 1$. So let's shift $n \to n' + 1$. Then n' runs from 0 to ∞. We obtain:

$$\sum_{n=1}^{\infty} x^{2n} = \sum_{n'=0}^{\infty} x^{2(n'+1)} = \sum_{n'=0}^{\infty} x^{2n'} x^2 = \frac{1}{1-x^2} x^2 = \frac{x^2}{1-x^2}, \qquad (14)$$

just like before. We will see that this index shifting skill becomes critically important in mastering many of the identities we develop later. I will help you recognize and develop this skill as we move forward.

0.8 Important identities

We end this section with the important identities you should know already (some others will be developed in this chapter). The main ones are the following:
1) $ax^2 + bx + c$ has two roots given by $r_\pm = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$
2) $\cos^2(\theta) + \sin^2(\theta) = 1$
3) $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$, $\sec(\theta) = \frac{1}{\cos(\theta)}$, and $\csc(\theta) = \frac{1}{\sin(\theta)}$
4) $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$
5) $x^2 - a^2 = (x + a)(x - a)$
6)$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$
7) $\sum_{n=0}^{N} \binom{N}{n} x^{N-n} y^n = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} x^{N-n} y^n = (x + y)^N$ (the binomial theorem)
8) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Ok, now you are armed with the skills to tackle some tough algebra problems. Let's give it a try.

0.9 Problems

These problems are optional. But I encourage you to try them to hone your devious algebra trickery...

1. Factor $(x^2 + 2xy + y^2 - 9)$
2. Use the double angle formula, in the form $\cos(\theta) = \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2})$ to find the relation
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta) + 1}{2}}$$

Hint: $\cos^2(\frac{\theta}{2})+\sin^2(\frac{\theta}{2}) = 1$ will help. How do you pick the correct sign?

3. Use the same double angle formula to find $\sin(\frac{\theta}{2})$. Verify your results for 2) and 3) satisfy $\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}) = 1$

4. Show that $\sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n = \frac{e^x - 1}{x}$.
   Hint: Use formula (8) above.

5. This is a challenging one. Show that

$$\sqrt{3 + 2\sqrt{2}} = \pm(1 + \sqrt{2})$$

   by using the add zero trick to establish that the argument of the square root on the left hand side is a perfect square.

6. Use the double angle formulas for sine and cosine to show

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

7. Use multiplication by $(1 - x)$ (trick used in identity (6) above for the appropriate x) to show that:

$$1 + \sqrt[4]{2} + \sqrt{2} + \sqrt[4]{8} = \frac{1}{\sqrt[4]{2} - 1}$$

8. Show that

$$\frac{(2n)!}{2^n n!} = 1 \times 3 \times 5 \times 7 \cdots (2n-3) \times (2n-1) = (2n-1)!!$$

   where the last equality defines the double factorial !! (the double factorial skips integers, so it is a product of all even or all odd integers only).

9. Show that

$$\sum_{n=1}^{\infty} (\cos(\theta))^{2n} = \frac{1}{\tan^2(\theta)}$$

   Hint: Use the geometric series.

Chapter 1

Irrational Numbers and Ratios

1.1 Introduction and Course Goals

The course will cover the following:

- Review of calculus, focusing on key concepts and ideas likely missed the first time through the material
- Multivariable calculus, Div, Grad, Curl, and multivariable integral theorems
- Complex variables and the calculus of residues
- Linear algebra, solving linear equations, eigenvectors and eigenvalues
- First and second order differential equations
- Fourier Series

Throughout the course we stress ideas in addition to learning the mechanics of how to do something. We also apply this to problems with a physics background.

1.2 Irrational Numbers

We begin with the proof of the existence of an irrational number. An irrational number is one that cannot be written as the ratio of two integers—$\frac{p}{q}$ with p and q relatively prime (have no common factors, or in "lowest-terms").

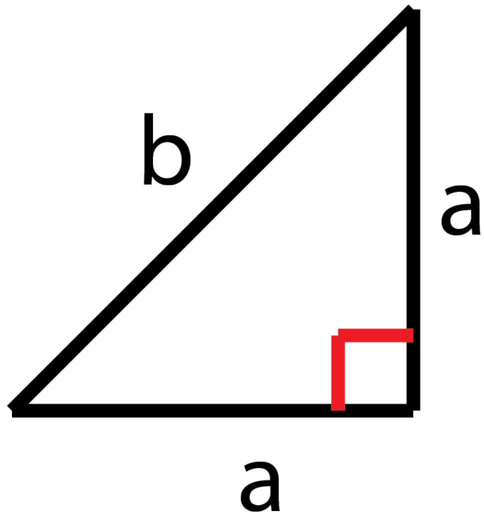

Figure 1.1: Right triangle employed in the proof. The length of each edge is a and the hypotenuse is $b = \sqrt{2}a$.

The argument is as follows: Consider the hypotenuse of an isosceles right triangle. Pythagoras says $a^2 + a^2 = b^2$ or $b^2 = 2a^2$.

Suppose we have an even integer $2n$. When we square it, it becomes $4n^2$, which is even and divisible by 4. Squaring the odd integer $2n + 1$ gives $(2n + 1)^2 = 4n^2 + 4n + 1$ which is an odd integer.

Now suppose $p \times e = b$ and $q \times e = a$ so that $\frac{a}{b} = \frac{q}{p}$ with p and q relatively prime and e the common factor in both a and b. Then $p^2 = 2q^2$, so p^2 is even. This means it must be the square of an even integer, so $p = 2n$. But then $2q^2$ is a multiple of 4, so $q = 2m$ for some m. Then p and q have a common factor of 2, which is not allowed, because we have set up the problem such that p and q have no common factors. Therefore the ratio $\frac{a}{b}$ cannot be written as a rational number ... it is irrational!

Irrational numbers can have odd properties. For example, if we take an irrational number and raise it to an irrational power, we can get a rational number. Here is a proof:

Consider $x = \sqrt{3}^{\sqrt{2}}$, which is an irrational raised to an irrational power. Now consider

$$x^{\sqrt{2}} = \left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{3}\right)^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{3}\right)^{2} = 3 \tag{1.1}$$

which is rational. Therefore either x is rational or $x^{\sqrt{2}}$ is rational. So an irrational number raised to an irrational power can yield a rational number! Note that this argument does not tell us whether the irrational is $\sqrt{3}$ or $x = \left(\sqrt{3}\right)^{\sqrt{2}}$, which when raised to an irrational power produces a rational number. (There are techniques that do answer this question, but we will not discuss further here.)

1.3 Zeno's Paradox

The book by Toeplitz discusses Zeno's Paradox — that if every step I take is half as large as the previous one, then I can never get from point A to point B because it would take an infinite number of steps.

The resolution is simple. I do need an infinite number of steps, but the total sum of all of these steps is *finite*, which is why we can arrive at point B. Next lecture, we will actually prove that the sum of all of those steps is just twice the length of the first one. This is because

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2. \tag{1.2}$$

1.4 Ratios

What is the definition of π? Think about this before I reveal the answer.

Be sure to give it a try.

Really. Think about it.

Don't peek yet

It is the ratio of the circumference of a circle to its diameter or the ratio of the area of a circle to the square of its radius. The idea of defining π as a *ratio* comes from the Greeks. Many math concepts have hidden definitions in terms of ratios. Examples include area, sine, cosine, tangent, hyperbolic tangent, and so on.

Next we will show amazing things you can do with ratios. We will assume a well known (and in many respects self-evident) fact that the ratio of the areas of any two pie slices of the same angle of two circles is proportional to the ratio of the squares of the radii of the two circles. Consider the circle with radius 1:

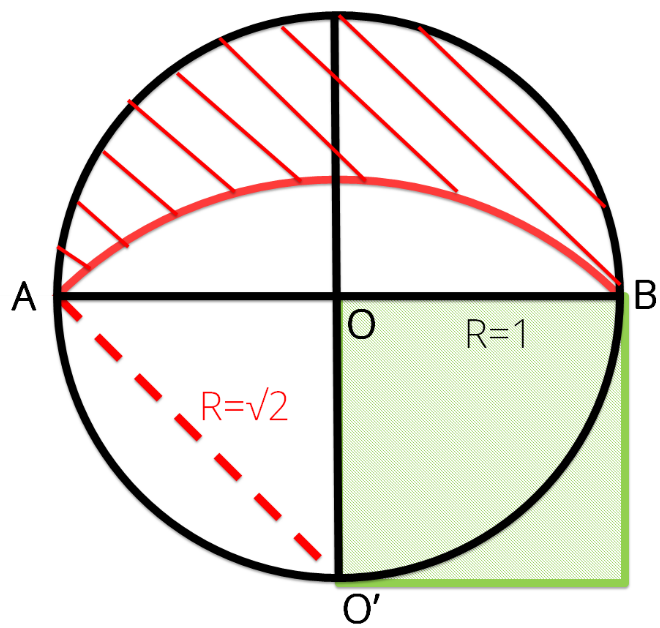

Figure 1.2: This figure has a circle of radius 1 (black) and the chord of a circle of radius $\sqrt{2}$ (red) drawn on it. The proof we will make is that the area between the two circles (hatched red) is equal to the area of the square (green).

We will prove that the red area and the green area are the same. This is close to the problem known as "squaring the circle," which has been proven to be impossible 2000 years after the Greeks gave up on their attempts. But this one can be proven, and perhaps inspired them on the bigger goal.

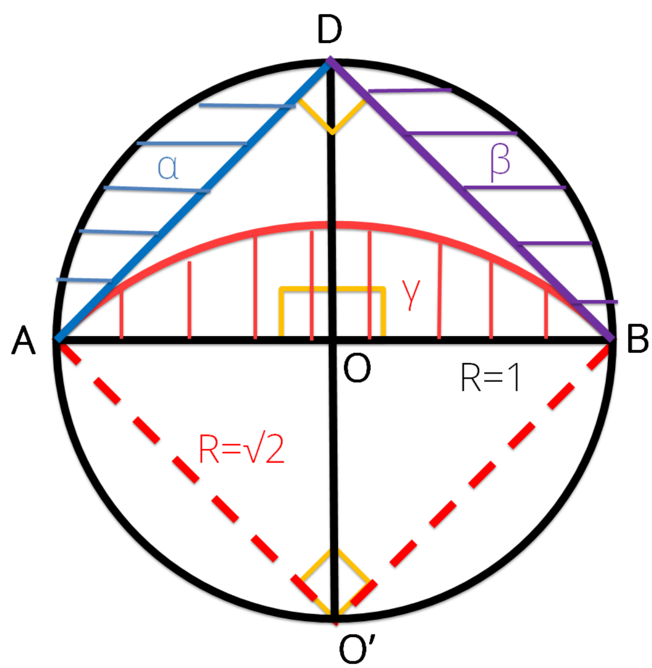

Figure 1.3: We start at the midpoint D of the upper half of the circle. Draw lines from D to A and B. This defines the hatched areas α (blue) and β (purple). We also have the hatched red area γ.

Draw straight lines from midpoint D to A and B (blue and purple). The angles AOD, DOB, and $AO'B$ are $90°$, so the the assumption we stated above implies that the ratio

$$\frac{\text{Area}\,\alpha}{\text{Area}\,\gamma} = \left(\frac{\text{rad}\,\alpha}{\text{rad}\,\gamma}\right)^2 = \frac{1}{2}. \tag{1.3}$$

A similar argument can be done for β, so we can immediately conclude that the areas satisfy $\alpha+\beta = \gamma$. Now consider $\triangle ADB$. Its area is $\frac{1}{2}\times\sqrt{2}\times\sqrt{2} = 1$. So $1 + \alpha + \beta = \frac{\pi}{2}$, which is the area of the semicircle with radius 1. Hence,

$$\frac{\pi}{2} - \gamma = 1 = \text{area of original red section in Figure 1.2} \tag{1.4}$$

But the area of the square is also 1 (since its side has length 1), so the area of the crescent and the square are the same!

I don't know about you, but I think this is *really cool*.

1.5 Archimedes

Archimedes is arguably one of the best mathematicians of all time. He discovered many important things. One was determining the size of π. Archimedes calculated π by the exhaustion method (now called the "squeeze method" by some for the methodology used).

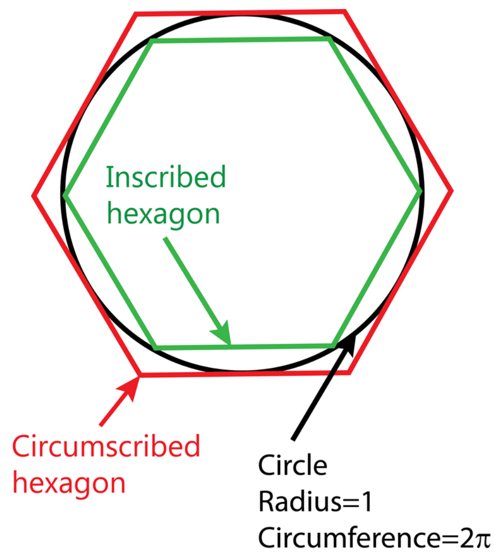

Figure 1.4: Schematic of the Archimedes exhaustion principle. The value of 2π lies in between the perimeter of the circumscribed polygon and the inscribed polygon. As the number of sides is made larger and larger the two perimeters approach each other, and the value of 2π.

Here is how the argument goes. Take a circle of unit radius. We will calculate the perimeter of an n-gon and compare it to the perimeter of a $2n$-gon both *inscribed inside* the circle.

The sides of the $2n$-gon and n-gon (see Fig. 1.5) can be written $s_{2n} = BD$, and $s_n = BC$. We can see that $\triangle ACP$, $\triangle ADB$, and $\triangle BDP$ are similar (this is because they are all right triangles with an acute angle of θ). One verifies this by comparing angles: we have $\angle ABC = 90^{\circ} - 2\theta = \angle ABP$, $\angle ABD = 90^{\circ} - \theta$, and $\angle PBD = \theta$. This follows because for any point on a circle the angle created from two diameter endpoints to the point is always $90°$, (sums of squares of chords)^2 = (diameter)^2. The proof of this is a homework problem.

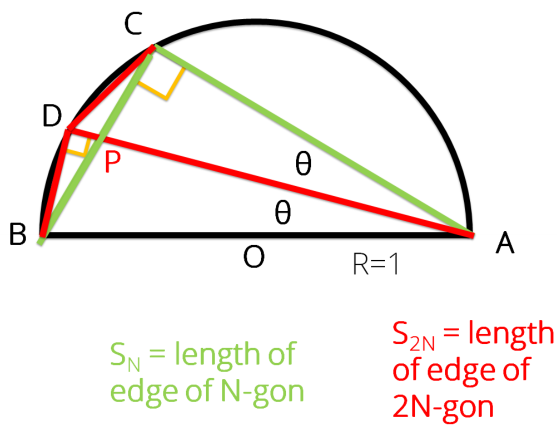

Figure 1.5: Triangles used to relate the side of one edge of the n-gon, given by $s_n = BC$ and the side of the $2n$-gon, given by $s_{2n} = BD$. The strategy is to first identify the similar triangles $\triangle ACP$, $\triangle ADB$, and $\triangle BDP$.

Observing these similar triangles (see also Fig. 1.6), we see

$$\frac{AB}{AD} = \frac{BP}{BD} \tag{1.5}$$

and hence

$$\frac{AC}{PC} = \frac{AD}{BD} \implies \frac{AC}{AD} = \frac{PC}{BD}. \tag{1.6}$$

Adding the top equation to the right part of the second equation yields

$$\frac{AB + AC}{AD} = \frac{BP + PC}{BD} = \frac{BC}{BD}. \tag{1.7}$$

Cross multiply

$$\frac{AB + AC}{BC} = \frac{AD}{BD}. \tag{1.8}$$

Squaring this equation gives

$$\frac{(AB)^2 + 2(AB)(AC) + (AC)^2}{(BC)^2} = \frac{(AD)^2}{(BD)^2} \tag{1.9}$$

Now we add 1 to both sides:

$$\frac{(AB)^2 + 2(AB)(AC) + (\mathbf{AC})^{\mathbf{2}} + (\mathbf{BC})^{\mathbf{2}}}{(BC)^2} = \frac{(\mathbf{AD})^{\mathbf{2}} + (\mathbf{BD})^{\mathbf{2}}}{(BD)^2} \tag{1.10}$$

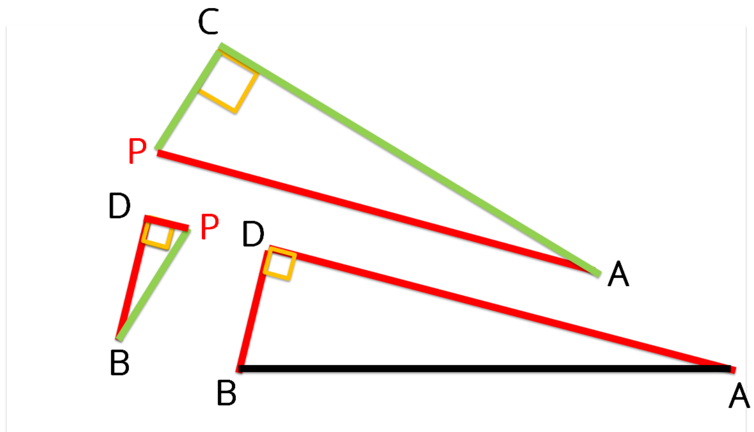

Figure 1.6: Closeup of the three triangles used in the proof. Matching the ratios of the lengths similar sides is used to finish the proof.

Note that each of the bold face terms are equal to $(AB)^2$. Substituting that result into the equation (in two places), then yields

$$\frac{2(AB)(AB + AC)}{(BC)^2} = \frac{(AB)^2}{(BD)^2} \tag{1.11}$$

We remember that $AB = 2 =$ diameter, $BD = s_{2n}$, and $BC = s_n$. We have

$$(AC)^2 = (AB)^2 - s_n^2 = 4 - s_n^2 \tag{1.12}$$

So

$$\frac{4(2 + \sqrt{4 - s_n^2})}{s_n^2} = \frac{4}{s_{2n}^2} \tag{1.13}$$

$$s_{2n}^2 = \frac{s_n^2}{2 + \sqrt{4 - s_n^2}}. \tag{1.14}$$

To start the recursion, We consider a hexagon, $n = 6$. The hexagon is made of six equilateral triangles, each with an edge equal to the radius of the circle, which is 1. So the initial perimeter is 6. (Be sure to draw picture and verify this is so.)

We solve the recursion numerically for the circumference $n \times s_n \to 2\pi r = 2\pi$ as n gets large. The results are given in the table (for the perimeter divided by 2).

N	S_N	Pi	T_N	Error width
6	3.0	3.14159265359	3.46410161514	0.464101615138
12	3.10582854123	3.14159265359	3.21539030917	0.109561767943
24	3.13262861328	3.14159265359	3.1596599421	0.0270313288163
48	3.13935020305	3.14159265359	3.14608621513	0.00673601208453
96	3.14103195089	3.14159265359	3.14271459965	0.0016826487548
192	3.14145247229	3.14159265359	3.14187304998	0.000420577694665
24576	3.14159264503	3.14159265359	3.14159267174	2.67075961347e-08

Figure 1.7: Table of the recursion results for increasing n. One can clearly see the convergence to π. This table is showing the perimeter divided by 2 for the inscribed and circumscribed polygons.

Archimedes also found the outer polygons to bound the value of π. The relationship between the n-gon and the $2n$-gon is

$$t_{2n} = \frac{2\sqrt{4+t_n^2}-4}{t_n} \tag{1.15}$$

Proving this is true will also be a homework problem.

Chapter 2

Everything you want to know about series but were afraid to ask

2.1 The geometric series

The book by Toeplitz discusses Zeno's paradox — that one can never go from point A to point B because it takes an infinite number of steps, if one takes half as big a step each time. Since one needs an infinite number of steps, one can never make it. But the flaw in Zeno's argument is that the sum of all of the steps is *finite*, which is why we can do it. We will prove

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 \tag{2.1}$$

next.

We perform the proof in full generality, using an abstract x in the *geometric series*. Our goal is to show that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \tag{2.2}$$

for $|x| < 1$. The strategy is to use the multiply by one trick. We have

$$\begin{aligned} &\left(1 + x + x^2 + x^3 + x^4 + \cdots + x^N\right) \frac{(1-x)}{(1-x)} \\ &\quad = \left(1 - x + x - x^2 + x^2 - \cdots - x^N + x^N - x^{N+1}\right) \frac{1}{(1-x)}. \end{aligned} \tag{2.3}$$

Note how all the terms except the first and the last have pairs appearing—one with a plus sign (black) and one with a minus sign (red)—so they will cancel. This shows that

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}. \tag{2.4}$$

If $|x| < 1$, then as $N \to \infty$, one will have $|x|^{N+1} \to 0$ (be sure you understand why).

So we take the limit $N \to \infty$ to obtain

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \tag{2.5}$$

for $|x| < 1$. This sum is called the geometric series. It is an important result to remember (both the final answer and the methodology used to derive it). It will come up again many times.

2.2 The (dreaded) Taylor series

Finally, we talk about a MacLaurin and Taylor series. These are essentially the same thing. But to be completely precise, we note that the MacLaurin series is a Taylor series expanded about the point $x_0 = 0$. The Taylor series can be expanded about any point x_0. It is important to remember this, because most examples are MacLaurin series, but there are times when a Taylor series about a different point is needed.

Suppose we want a polynomial that has the same function value and same first n derivatives of a function at the origin. How do we make such a thing? Obviously the first term is the value of the function at the origin, or f(0). The second term must have the correct slope, so the coefficient of x is $\frac{df(0)}{dx}$, which we denote as $f^{(1)}(0)$, with the superscript indicating the number of derivatives. The third term is proportional to x^2. Since the second derivative of x^2 is 2, we must have its term be $f^{(2)}(0)$ multiplied by $\frac{x^2}{2}$. Hence, we have already found that the quadratic polynomial that satisfies this criterion is

$$f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(0)x^2. \tag{2.6}$$

To get the full series, we just continue in the same fashion, recalling that

$$\frac{d^n}{dx^n}x^n = n! \tag{2.7}$$

yields

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}, \tag{2.8}$$

which, one can see, includes the quadratic polynomial we derived above.

As a check in your understanding, you should be able to take the nth derivative of the series, and evaluate it at $x = 0$, and show it is equal to $f^{(n)}(0)$. You must assume that you an interchange the order of taking a derivative and performing the sum when you carry out this calculation. Note that the result in Eq. (2.7) holds at every point x_0! This is why the form for the Taylor series will be exactly the same as the MacLaurin series (with the only change being $x \to x - x_0$).

Now we are ready for a worked-out example. Evaluate the MacLaurin series of $\sqrt{1+x}$. We need to calculate a number of different derivatives and evaluate them at $x = 0$.

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2}\frac{1}{\sqrt{1+x}}\bigg|_{x=0} = \frac{1}{2} \tag{2.9}$$

$$\frac{d^2}{dx^2}\sqrt{1+x} = \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{(1+x)^{\frac{3}{2}}}\bigg|_{x=0} = -\frac{1}{4} \tag{2.10}$$

$$\frac{d^3}{dx^3}\sqrt{1+x} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{(1+x)^{\frac{5}{2}}}\bigg|_{x=0} = \frac{3}{8} \tag{2.11}$$

At this stage, it is common for students, who are "technicians," to try to recognize the pattern and simply "hope" it continues that way forever. But it is better to carefully verify (via an induction argument), as a "practitioner" would, that it really holds. We will not go through these details here. Instead, we just show what we explicitly derived:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots \tag{2.12}$$

which follows from

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\left(\frac{x^2}{2}\right) + \frac{3}{8}\left(\frac{x^3}{6}\right) + \dots \tag{2.13}$$

You should try to do the full inductive argument to determine the general series on your own if you have never done this before.

2.3 Hyperbolic functions

This material does not exactly fit here, but it is useful for you to have a quick review of hyperbolic functions, which I have seen cause more than their fair share of consternation.

We begin with some basic definitions. The hyperbolic cosine is defined as

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \tag{2.14}$$

and it is an *even* function of x. The hyperbolic sine is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{2.15}$$

and it is an *odd* function of x. The derivatives follow immediately from the definitions:

$$\frac{d\cosh(x)}{dx} = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x) \tag{2.16}$$

and

$$\frac{d\sinh(x)}{dx} = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x). \tag{2.17}$$

Recall how similar this is to the conventional trig functions except for some signs. Indeed, this is common with hyperbolics. Another example is the following identity

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1. \tag{2.18}$$

In summary, we have shown that

$$\frac{d\cosh(x)}{dx} = \sinh(x), \tag{2.19}$$

$$\frac{d\sinh(x)}{dx} = \cosh(x), \tag{2.20}$$

and

$$\cosh^2(x) = 1 + \sinh^2(x). \tag{2.21}$$

Plots of the hyperbolic functions are shown in Fig. 2.1.

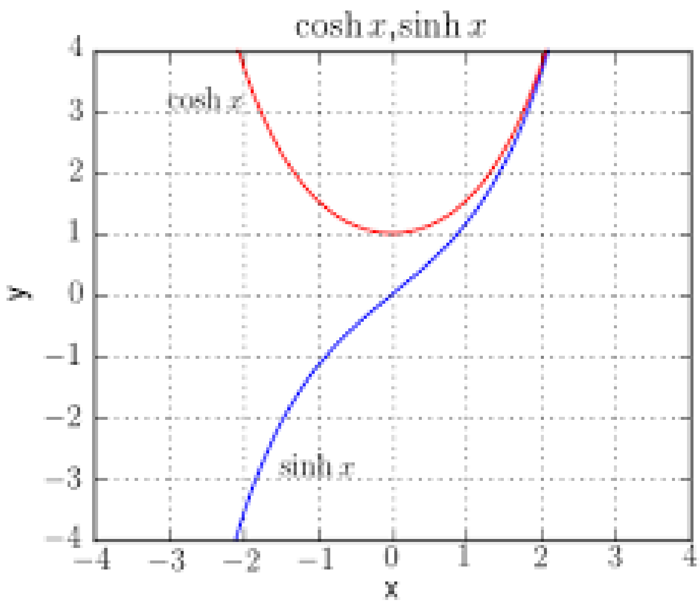

Figure 2.1: Plot of hyperbolic cosine (red) and sine (blue). One can immediately see their even and odd characters. The hyperbolic cosine is always larger than 1 and always large than the hyperbolic sine.

Chapter 3

Integrals and Limits

3.1 Origins of the concept of integration

We begin with the early work by the pioneers Archimedes and Cavalieri who showed how to integrate x^2 and x^k respectively. In modern formulas, we want to show how to integrate $\int_0^1 x^k$. The strategy is to relate these integrals to sums of powers of integers. We start by taking the interval of 0 to 1 and divide into n intervals running from $\frac{0}{n}$ to $\frac{n}{n}$.

The red rectangles have an area less than the integral, while the green rectangles cover a larger area. This sounds like we will be invoking the "squeeze principle" again. Indeed, we will. Let t_n denote the sum of the green rectangles and s_n the sum of the blue rectangles.

$$t_n = \left(\frac{1}{n}\right)^k \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^k \cdot \frac{1}{n} + \left(\frac{3}{n}\right)^k \cdot \frac{1}{n} + ... + \left(\frac{n}{n}\right)^k \cdot \frac{1}{n} \tag{3.1}$$

$$s_n = \left(\frac{0}{n}\right)^k \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^k \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^k \cdot \frac{1}{n} + ... + \left(\frac{n-1}{n}\right)^k \cdot \frac{1}{n} \tag{3.2}$$

So

$$s_n \le \int_0^1 x^k \le t_n \tag{3.3}$$

But

$$t_n = \sum_{j=1}^{n} \frac{j^k}{n^{k+1}}, \tag{3.4}$$

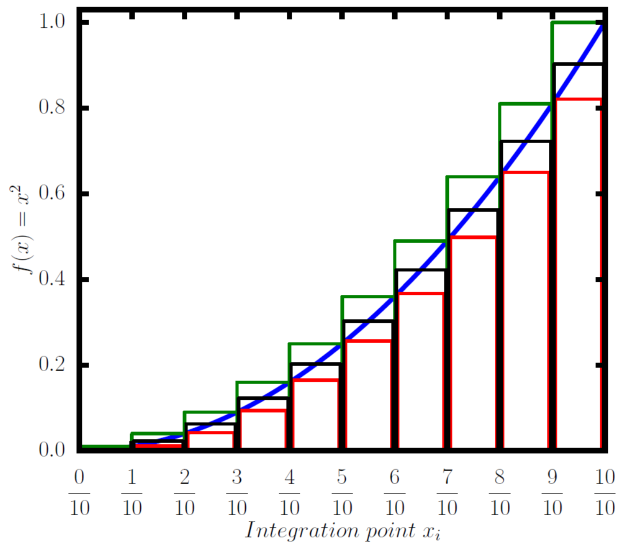

Figure 3.1: Three different rectangular integration routines for $\int_0^1 x^k\,dx$ with $k = 2$. The function is plotted in blue, the three different numerical integration schemes are left point (red), midpoint (black) and right point (green). One can clearly see the integral is bounded to lie in between the sum of the red and green rectangular areas.

and

$$s_n = \sum_{j=0}^{n-1} \frac{j^k}{n^{k+1}} = \sum_{j=1}^{n-1} \frac{j^k}{n^{k+1}}, \tag{3.5}$$

where in the second equality in the s_n equation, we note that we can drop the $j = 0$ term.

Now we invoke the critical piece of the squeeze argument. To begin, note that $t_n - s_n = \frac{1}{n}$ so as $n \to \infty$, $t_n - s_n \to 0$ or $t_n \to s_n$ and this limit is the integral. How do we know this result, that $t_n - s_n = \frac{1}{n}$? Each term in s_n also appears in t_n. But t_n has one more term—its last term. Hence, the difference is precisely that last term $\frac{n^k}{n^{k+1}} = \frac{1}{n}$. Then the rest of the argument follows as above. To determine the final result, we need to now evaluate these finite sums of powers of integers.

3.2 Sums of powers of integers

The skill to learn how to sum powers of integers is a useful one. You might think it is an odd thing to do, as we cannot extend these sums to infinity (unlike some popular youtube videos would say), because such sums always *diverge*. But we can actually get closed-form expressions for *finite summations*. And this is a rather marvelous result. We show you how to do this next.

Define

$$\mathrm{sum}_k(n) = \sum_{j=1}^{n} j^k = \text{ sum of first n integers raised to the k power.} \tag{3.6}$$

The simplest case is $k = 1$.

$$\mathrm{sum}_1(n) = 1 + 2 + 3 + ... + n. \tag{3.7}$$

We can write $\mathrm{sum}_1(n)$ in reverse order underneath, and add down the columns:

$$\begin{aligned} \mathrm{sum}_1(n) &= 1 + \quad 2 \quad + \quad 3 \quad + \ldots + n \\ \mathrm{sum}_1(n) &= n + (n-1) + (n-2) + \ldots + 1 \end{aligned} \tag{3.8}$$

$$2 \times \mathrm{sum}_1(n) = \underbrace{(n+1) + (n+1) + (n+1) + ... + (n+1)}_{n \text{ terms}} \tag{3.9}$$

There are n terms of $(n + 1)$, so

$$\mathrm{sum}_1(n) = \frac{n(n+1)}{2}. \tag{3.10}$$

This result is very direct, simple and neat.

Unfortunately, this approach is not easily generalized to other k powers. So we have to proceed a different way. We do so by computing differences of finite sums in two different ways. Consider

$$\sum_{j=1}^{n} \left[(j+1)^2 - j^2\right] = (n+1)^2 - 1 = n^2 + 2n, \tag{3.11}$$

which follows since many of the terms (all except the first term of the second sum and the last term of the first sum) cancel when the sums are subtracted from ach other. Now, we compute a second way, by expanding the terms in the sums (this is mathematically fine because all sums are *finite*). We find

$$\sum_{j=1}^{n}\left[(j+1)^2-j^2\right]=\sum_{j=1}^{n}\left(\cancel{j^2}+2j+1-\cancel{j^2}\right)=\sum_{j=1}^{n}2j+\sum_{j=1}^{n}1 \tag{3.12}$$

We realize that

$$\sum_{j=1}^{n}2j=2\sum_{j=1}^{n}1 \tag{3.13}$$

and

$$\sum_{j=1}^{n}1=n. \tag{3.14}$$

So we immediately discover (after equating the two ways to evaluate the summations) that

$$2\,\mathrm{sum}_1(n)=n^2+2n-n=n(n+1) \tag{3.15}$$

so

$$\mathrm{sum}_1(n)=\frac{n(n+1)}{2} \tag{3.16}$$

as before.

But this strategy can be generalized to higher powers of k.

We can check the next case where we work with cubes instead of squares. We have

$$\sum_{j=1}^{n}\left[(j+1)^3-j^3\right]=(n+1)^3-1=n^3+3n^2+3n \tag{3.17}$$

and

$$\begin{aligned}\sum_{j=1}^{n}\left[(j+1)^3-j^3\right]&=\sum_{j=1}^{n}\left(\cancel{j^3}+3j^2+3j+1-\cancel{j^3}\right) && (3.18)\\ &=3\,\mathrm{sum}_2(n)+3\,\mathrm{sum}_1(n)+\mathrm{sum}_0(n) && (3.19)\\ &=n^3+3n^2+3n. && (3.20)\end{aligned}$$

So

$$\begin{aligned} \mathrm{sum}_2(n) &= \frac{1}{3}\left(\underbrace{n^3+3n^2+3n}_{\text{sum first way}}\underbrace{-\frac{3}{2}n^2-\frac{3}{2}n}_{-3\,\mathrm{sum}_1(n)}\underbrace{-n}_{-\mathrm{sum}_0(n)}\right) && (3.21)\\ &= \frac{1}{3}\left(n^3+n^2+\frac{1}{2}n\right) && (3.22)\\ &= \frac{1}{6}n\left(2n^2+3n+1\right) && (3.23)\\ &= \frac{1}{6}n\left(2n+1\right)\left(n+1\right). && (3.24)\end{aligned}$$

Hence, we have found that

$$\sum_{j=1}^{n} j^2 = \frac{1}{6}n(2n+1)(n+1). \tag{3.25}$$

For the $k = 2$ case, the integral of the parabola then gives

$$\int_0^1 x^2\,dx = \lim_{n\to\infty}\frac{1}{6}n(2n+1)(n+1)\frac{1}{n^3} = \frac{1}{3} \tag{3.26}$$

as expected. One can use this technique to calculate higher integer powers as well. Cavalieri extended this to all the way to $k = 10$. The strategy is to extend the expression for the difference of the sums of powers to higher k values. The highest-order terms, proportional to j^k cancel in the two summations. Then the procedure continues as we did above. We need to use the results for all previous powers to obtain the final answer. Note that there does not seem to be any simply pattern to these final answers, so we do not try to establish them by induction. But if you like, you could determine the result for the highest power of n (proportional to n^{k+1}), which is what is needed for the calculation of the integral.

Fermat used a slightly different method, with an infinite number of steps and the identity

$$\lim_{x\to1}\frac{1-x^{k+1}}{1-x} = \lim_{x\to1}\left(1+x+x^2+x^3+\ldots+x^k\right) = k+1 \tag{3.27}$$

to prove the integral result for all integer k.

3.3 Issues with defining an integral

The book discusses issues with defining an integral in greater depth, but the main results are

- The integral is well defined for piecewise monotonic (strictly increasing or decreasing) functions
- Strange functions (like the one in section 17) show that one has to be careful and precise in defining the integral. This often does not play a role in integrals that arise in physics, but occasionally is relevant in some areas like Cantor sets or strange attractors. The field of real analysis spends significant time sorting out all possible subtleties in how one defines an integral.
- *The definite integral is not the area under the curve.* It is often that one can think of it as the *signed* area under the curve, where negative values correspond to negative areas.

3.4 Numerical integration

We do not go into detail into numerical techniques in this class, but it is important for you to know some of the basics. Below is a crash course on numerical quadrature.

While the definite integral is defined in the limit where the maximal step size approaches 0, we must work with a finite step size to calculate a numerical approximation to the integral. The simplest quadrature rule is a left point, midpoint, or right point integration rule. These are demonstrated in Fig. 3.2.

We call the coordinate where the function is evaluated x_i, and the value of the function used $f(x_i)$. Then the numerical approximation to the integral replaces the integral with a finite sum of terms

$$\int_a^b f(x)\,dx = \Delta x \sum_{i=1}^{n} f(x_i) \tag{3.28}$$

where the $\{x_i\}$ are chosen as described in Fig. 3.2, corresponding to the specific integration rule that is being used. For example, in left-point integration, we choose $\Delta x = (b-a)/n$ and $x_i^{\text{left}} = (i-1)\Delta x$. The right point rule is

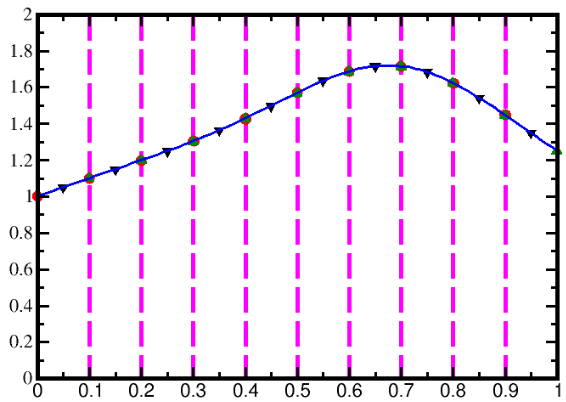

Figure 3.2: Different rectangular integration rules. In this example, we have ten rectangles whose area we sum to approximate the integral $\int_0^1 f(x)\,dx$ ($f(x)$ in blue). The left-point rule, takes the values of the functions at the red circles for each rectangle. The right point rule takes the values with the green triangles, while the midpoint rule takes the black triangular values. In the limit as the step size goes to zero, all rules will give the same answer, but their results differ for $\Delta x \neq 0$.

$x_i^{\text{right}} = i\Delta x$. The midpoint rule, averages the two and is $x_i^{\text{mid}} = \left(i - \frac{1}{2}\right)\Delta x$. In all cases $1 \leq i \leq n$ ($n = 10$ in the figure). These first rules are rectangular integration rules. Images most effectively illustrate the different methods, as can be seen in the corresponding Figs. 3.3-3.5.

The trapezoidal rule uses trapezoids to fit the curve (red trapezoids). This result is exactly equivalent to a different set of rectangles centered at the gridpoints; in this case, for the first and last points, only one-half of the corresponding rectangle contributes.

The next to consider is the so-called Simpson's rule, which approximates the integral via summing with weights that exactly integrate constant, linear, and quadratic functions. It can also be thought of as approximating the function with a quadratic for every sequence of three adjacent points on the grid. We will not draw a figure for this, but you can certainly imagine what it looks like. As with the trapezoidal rule, the construction simplifies, and can be thought of as integration over rectangles with the weights alternating

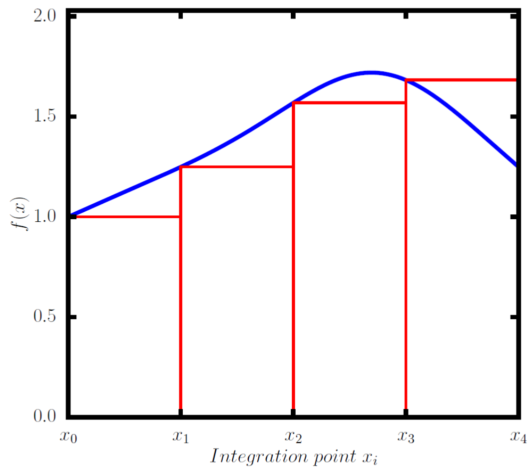

Figure 3.3: Left-hand integration rule with four points, given by $\int_{x_0}^{x_4} f(x)dx \approx \Delta x\,[f(x_0) + f(x_1) + f(x_2) + f(x_3)]$.

from 4/3 to 2/3 with the endpoints given by 1/3 again:

$$\int_{x_0}^{x_5} f(x)dx \approx \Delta x \left[\frac{1}{3}f(x_0) + \frac{4}{3}f(x_1) + \frac{2}{3}f(x_2) + \frac{4}{3}f(x_3) + \frac{1}{3}f(x_4)\right] \quad (3.29)$$

The sum alternates $\frac{4}{3}$ weight and $\frac{2}{3}$ weight, with $\frac{1}{3}$ weight on the endpoints. The generalization of Simpson's rule to higher polynomials is called Romberg integration. Many prefer this type of extrapolation technique to other methods. One can think of it as a technique that tries to extrapolate all the way to $\Delta x \to 0$.

There is one other technique for integration called Gaussian integration where the abscissae and weights are determined from a prescribed formula that exactly integrates a weight function times a polynomial of some degree. In this case, the spacing of the grid points is not uniform and it changes for every point as the number of points in the sum changes. This can make it inconvenient for computation and determining accuracy when one computes for different numbers of grid points. The reason why, is for grids that are evenly spaced, we simply double the number of points. Then half of the old grid points are on the new grid and we do not need to recalculate the function at those points. This never holds with Gaussian integration techniques. They are used, however, because if you have an integral of the form of a Gaussian

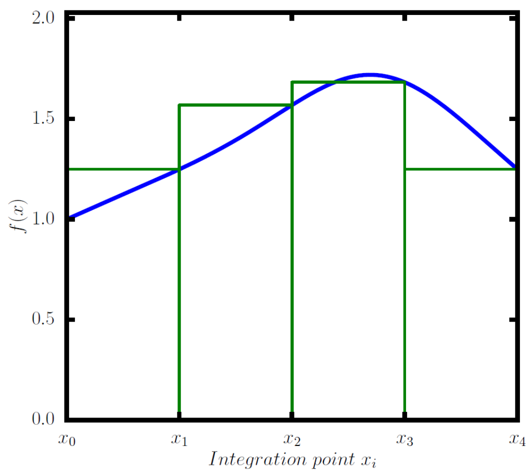

Figure 3.4: Right-hand integration rule with four points, given by $\int_{x_0}^{x_4} f(x)dx \approx \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4)]$.

integration weight function, this tailor-made approach is likely to be superior. Calculating the weights and the gridpoints, however, is complicated.

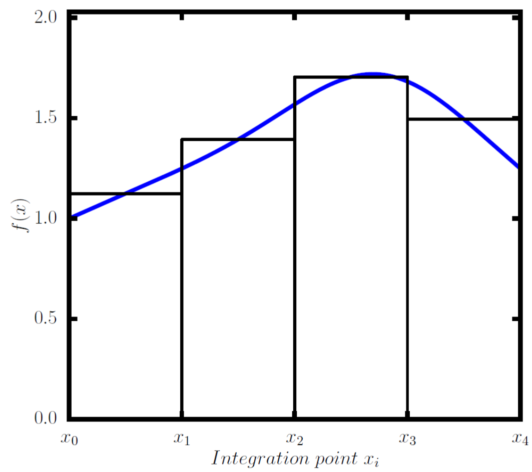

Figure 3.5: Midpoint integration rule with four points, given by $\int_{x_0}^{x_4} f(x)dx \approx \Delta x \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right)\right]$.

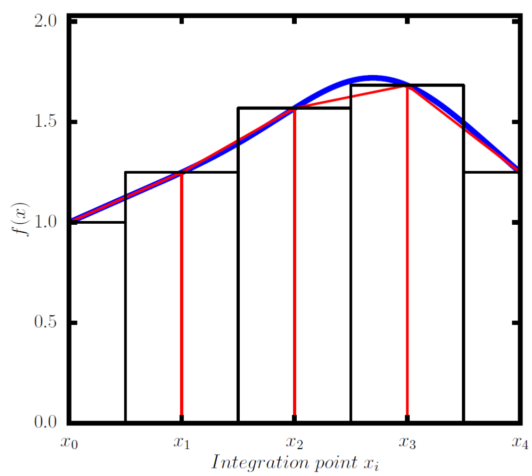

Figure 3.6: Trapezoidal integration rule with four points. The red trapezoids show the integration following the direct rule. It is equivalent to the black rectangles with the first and last counted half. The rule is given by $\int_{x_0}^{x_4} f(x)dx \approx \Delta x \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right]$.

Chapter 4

Tangents and Logarithms

4.1 Optimizations with constraints

The book by Toeplitz describes how a significant effort was expended by mathematicians on finding tangents to curves and how calculus makes it much easier to do. We will examine, as an example, one such derivative problem—for the set of rectangles of fixed perimeter p, what shape has the largest area?

Recall that area = length×width, and perimeter = 2×(length + width).

So if we let w denote the width and p denote the perimeter, then we have

$$l = \text{length} = \frac{p}{2} - w. \tag{4.1}$$

The area satisfies

$$A = l \times w = \left(\frac{p}{2} - w\right) \times w = \left(\frac{p}{2}\right) w - w^2. \tag{4.2}$$

Differentiating to determine the maximum gives

$$\frac{dA}{dw} = \frac{p}{2} - 2w = 0 \rightarrow w = \frac{p}{4}. \tag{4.3}$$

Solving for the length, then yields

$$l = \frac{p}{2} - \frac{p}{4} = \frac{p}{4} = w \rightarrow l = w, \tag{4.4}$$

which means the shape is a square (because the length is equal to the width)!

Some people would like to solve this problem in a much simpler way by saying such a rectangle must be a special rectangle. But the only special rectangle we know is a square. So it must be square. Sometimes such arguments are deep and meaningful because the arguments are based on symmetry principles. Other times, it is just good luck that it gives the right answer. It is safer, at this stage of your career, to err on the side of producing a rigorous argument to support such statements, rather than falling back on a "symmetry" or "unique" argument.

4.2 Trigonometric tables

I want to take the remainder of our time discussing sine and logarithm tables. Back in the period from 200 BC to the early to mid 1900's, nearly all calculations were done with tables or instruments that acted as tables, such as slide rules.

The Greeks constructed sine tables, while logarithms didn't come about until the early 1600's. The accuracy needed was one part in ten million, or 7 digits, in order to perform astronomical calculations accurately enough. In other words, this is what Kepler needed to establish his laws of planetary motion. Trying to develop such tables was very demanding but it was a key to technological advance at the time.

The generation of a sine table was known to the Greeks. A 30–60–90 triangle gives $\sin(30°)$. The construction of a regular pentagon gives $\sin(36°)$. Archimedes formulas, which can be employed to produce $\sin\left(\frac{x}{2}\right)$ from $\sin(x)$, give us $\sin(15°)$ and $\sin(18°)$. The Greeks also knew sine addition formulas such as

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta). \tag{4.5}$$

So using this, one obtains $\sin(18° - 15°) = \sin(3°)$. Archimedes again gives $\sin(1\frac{1}{2}°)$ and $\sin(\frac{3}{4}°)$. Then, they use the identity

$$\frac{\sin(x)}{\sin(y)} < \frac{x}{y} \text{ for } 0 < y < x < 90°, \tag{4.6}$$

but recall that the angles must be expressed in radians for all of these calculations.

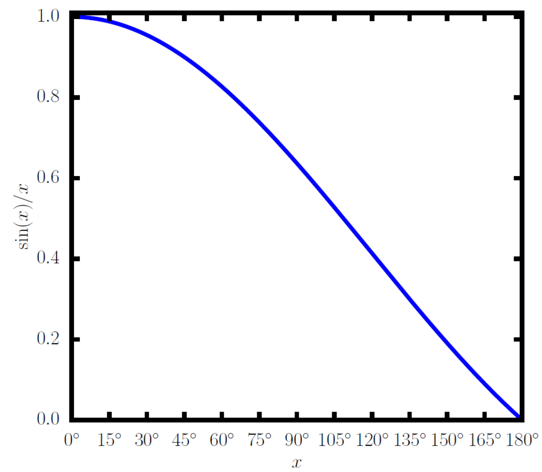

Figure 4.1: The function $\sin(x)/x$ for the range from 0 to π. Note how it is always a decreasing function of x. Such a function is called monotonic.

The identity follows from $\frac{\sin(x)}{x} < 1$ being a monotonic decreasing function of x (see Fig. 4.1) so

$$\frac{\sin(x)}{x} < \frac{\sin(y)}{y} \text{ for } y < x. \tag{4.7}$$

Cross multiplying gives us

$$\frac{\sin(x)}{\sin(y)} < \frac{x}{y} \text{ for } y < x. \tag{4.8}$$

This general result implies that

$$\frac{\sin(1\frac{1}{2}^\circ)}{\sin(1^\circ)} < \frac{3}{2} \text{ or } \frac{2}{3}\sin\left(1\frac{1}{2}^\circ\right) < \sin(1^\circ) \tag{4.9}$$

and

$$\frac{\sin(1^\circ)}{\sin(\frac{3}{4}^\circ)} < \frac{4}{3} \text{ or } \sin(1^\circ) < \frac{4}{3}\sin\left(\frac{3}{4}^\circ\right). \tag{4.10}$$

Using the values for $\sin(1\frac{1}{2}^\circ)$ and $\sin(\frac{3}{4}^\circ)$ pins (or better squeezes) the value of $\sin(1^\circ)$ to 4-5 digits of accuracy. They then computed $\sin(\frac{1}{2}^\circ)$ from Archimedes and finally used the addition formula to generate sine tables for every $\frac{1}{2}^\circ$.

4.3 Tables of logarithms

Log tables were even more valuable. Multiplying two 7-digit numbers to keep 7 digits of accuracy was tedious. But using a logarithm table reduced multiplication to addition because $\ln(xy) = \ln(x)+\ln(y)$, or calculating roots to division because $\ln(\sqrt[n]{x}) = \frac{1}{n}\ln(x)$.

We start with an example. Compute $\sqrt[3]{36000}$. A log table with a specific step size tells us that

12809	35996.4763
12810	36000.0759

where the first entry is the step (or exponent) and the second is $(1+x)^{step}$, where x is the value (step size) used in generating the table.

To solve this problem, we first interpolate to find $\log(3600) = 12809.98$. We then divide by 3 to get 4299.99. We next go to the table and find the step associated with this number (not shown here) to find the cube root (ans: 33.019272). So, one could compute complex things with these tables.

Generating these tables was mind numbing because it had to be done by hand. A four digit accuracy table required 2,300 steps. So, here are the first few steps of an example table with $x = 0.0001$.

step	$10,000 \times (1+\frac{1}{10,000})^{step}$
0	10,000.0000
1	10,001.0000
2	10,002.0001
3	10,003.0003

The key trick to making the table is that one never uses multiplication. They are formed instead *by addition* using the following result:

$$a \times \left(1 + \frac{1}{10,000}\right) = a + \frac{a}{10,000}. \tag{4.11}$$

The addition is done in a simple fashion. Write down the previous number a. Take the same number and shift it four digits to the right. Then add them together, truncating terms that are beyond the desired accuracy. Then one repeats. Again. And again. And again.

Generating these tables in base 10 seemed to be most convenient for calculations, but to get 7 digits of accuracy requires 23,000,000 steps and no

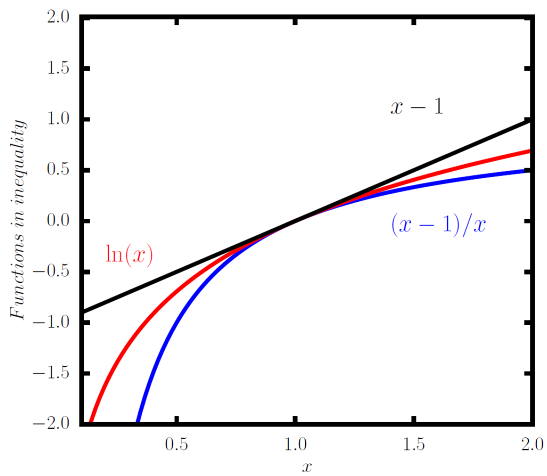

Figure 4.2: Functions used in the inequality. Blue is $\frac{x-1}{x}$, red is $\ln(x)$, and black is $x-1$. Note how they remain ordered even though they become equal at $x = 1$.

one could take up such work. This was further confounded by the fact that any error made at one step, invalidates all further entries in the table.

John Napier figured out some simplifications, that allowed one to actually construct such a table.

1. Compute a table with entries $(1 + \frac{1}{100})^{step}$ for each integer step.

2. Since steps in different tables are related by fixed ratios, compute every 100^{th} entry of a $(1 + \frac{1}{10000})^{step}$ table by multiplying the entries in the lower-precision table by that specific factor. Then fill in the first 100 entries of the higher precision table, and by simple addition, one finds all subsequent entries.

3. The remaining problem was the ratio between the low-precision and the high-precision table, which was not clear how to compute. Napier sought for an *absolute* logarithm table to resolve this issue.

4. To do so, Napier examined how the geometric table entries (called $f(x) = (1 + x)^{step}$) varied with respect to the table parameter x and found that
$$\lim_{x_1 \to x} \frac{f(x) - f(x_1)}{x - x_1} = \frac{c}{x} = f'(x). \tag{4.12}$$

So the simplest table would have $c = 1$, which defines the "absolute" logarithm table.

5. Napier further notices that if we compare the curves

$$\frac{x-1}{x}, \qquad \ln(x), \qquad x-1 \tag{4.13}$$

all three vanish at $x = 1$ (and hence meet each other) and the slopes are

$$\frac{1}{x^2}, \qquad \frac{1}{x}, \qquad 1 \tag{4.14}$$

respectively. Since $\frac{1}{x^2} < \frac{1}{x} < 1$, for $x > 1$ (and the opposite for $x < 1$), we find the curves remain strictly ordered with respect each other, as shown in Fig. 4.2. Hence,

$$\frac{x-1}{x}, \qquad \ln(x), \qquad x-1 \tag{4.15}$$

becomes, after dividing by $x - 1$,

$$\frac{1}{x} < \frac{\ln(x)}{x-1} < 1, \tag{4.16}$$

for all x. Now, let $x = \frac{a}{b} \to \frac{1}{a} < \frac{\ln(a)-\ln(b)}{a-b} < \frac{1}{b}$ for $a > b$. This identity allows the factor between different tables to be found and further allows one to compute the table entries more easily for the higher-precision tables.

What base did Napier use? Let $x = \frac{n+1}{n}$ in the inequality in Eq. (4.16). Then we find that

$$\frac{n}{n+1} < \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} < 1 \tag{4.17}$$

or

$$\frac{n}{n+1} < \ln\left(1+\frac{1}{n}\right)^n < 1. \tag{4.18}$$

now, as we take $n \to \infty$, we find that $(1+\frac{1}{n})^n \to e$, so $\ln(e) = 1 \to \text{base} = e$.

They then used very clever tricks, described at the end of section 22 of the Toeplitz book, to compute tables with seven digits of accuracy. These tables were extremely influential for hundreds of years and were needed for many different calculations. The first important problem was allowing Kepler to develop his rules of planetary motion.

Note that we hardly use such tables anymore. But it is important that how they were constructed not become a lost art. There is much insight to be found from understanding how one constructed these tables.

Chapter 5

Fundamental Theorem of Calculus and Manipulation of Integrals

5.1 Fundamental theorem of calculus

We start with the fundamental theorem of calculus: If $F(t) = \int_a^t f(x)\,dx$ with $a < t < b$ and if $f(x)$ is continuous and monotonic for $a \leq x \leq b$, then $F'(t) = f(t)$. (Barrow)

To prove the fundamental theorem, we just compute the derivative

$$\lim_{t_1 \to t} \frac{F(t_1) - F(t)}{t_1 - t} = \lim_{t_1 \to t} \frac{\int_0^{t_1} f(x)\,dx - \int_0^t f(x)\,dx}{t_1 - t} = \lim_{t_1 \to t} \frac{\int_t^{t_1} f(x)\,dx}{t_1 - t} \tag{5.1}$$

Since $f(t) < f(x) < f(t_1)$ for $t < x < t_1$ because f is monotonic, we have

$$(t_1 - t)f(t) < \int_t^{t_1} f(x)dx < (t_1 - t)f(t_1) \tag{5.2}$$

$$\implies f(t) < \frac{F(t_1) - F(t)}{t_1 - t} < f(t_1) \tag{5.3}$$

$$\implies \lim_{t_1 \to t} \frac{F(t_1) - F(t)}{t_1 - t} = f(t) \tag{5.4}$$

since f is continuous.

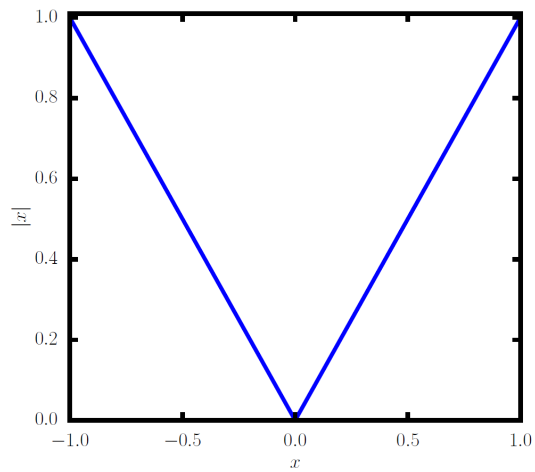

Figure 5.1: Absolute value function. Note how the derivative is -1 for $x < 1$ and 1 for $x > 1$. The limit as $x \to 0$ does not exists for the derivative because the left limit is not equal to the right limit.

The fundamental theorem of calculus has an obvious corollary as well. It is, in fact, the most common use of the theorem.

Corollary: If $\phi'(x) = f(x)$, $\phi(x) = F(x) + c$ where c is a constant.

If f is continuous at x, then the left and right limits match: $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$, but continuity also requires such a limit to exist in the first place (see figure 81 in the Toeplitz book for an example of a function with no limit). Note, however, that continuity does not imply that f is differentiable. $f(x) = |x|$ is a classic example: the function is continuous but is not differentiable at $x = 0$.

The converse is, however, true. If a function is differentiable at x, then it must also be continuous at x.

5.2 Product Rule

The product rule, sometimes called the Leibnitz rule, shows how one can calculate the derivative of a product of functions. The formula is well known to any student in a Calculus I class.

$$w(x) = u(x)v(x) \implies w'(x) = u'(x)v(x) + u(x)v'(x) \tag{5.5}$$

Proof:

$$\begin{aligned}
w'(x) &= \lim_{x_1 \to x} \frac{u(x_1)v(x_1) - u(x)v(x)}{x_1 - x} && (5.6)\\
&= \lim_{x_1 \to x} \frac{u(x_1)v(x_1) - u(x)v(x_1) + u(x)v(x_1) - u(x)v(x)}{x_1 - x} && (5.7)\\
&= \lim_{x_1 \to x} \frac{u(x_1) - u(x)}{x_1 - x} v(x_1) + u(x) \frac{v(x_1) - v(x)}{x_1 - x} && (5.8)\\
&= u'(x)v(x) + u(x)v'(x) && (5.9)
\end{aligned}$$

Note the use of the "add zero" trick in the second line (terms in red). Now, since v is differentiable, it is also continuous, so we can replace $v(x_1)$ by $v(x)$ in the last line.

We will make use of the product rule to derive the important result called integration by parts.

5.3 Integration by Parts

Integating the Leibnitz rule (via the fundamental theorem of calculus) shows us that

$$u(x)v(x) = \int u'(y)v(y)\, dy + \int u(y)v'(y)\, dy. \tag{5.10}$$

Rearranging, we find

$$\int u'(y)v(y)\, dy = u(y)v(y) - \int u(y)v'(y)\, dx \tag{5.11}$$

Integration by parts is usually used for a definite integral, as follows:

$$\int_a^b u'(x)v(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u(x)v'(x)dx. \tag{5.12}$$

Integration by parts is an extremely useful technique for evaluating integrals. For example, consider for $n \neq -1$: $\int x^n \ln(x)\, dx$. We take $u' = x^n$ and $v = \ln(x)$.

$$\int x^n \ln(x)\, dx = \frac{x^{n+1}}{n+1} \ln(x) - \int \frac{x^{n+1}}{n+1} \frac{1}{x}\, dx = \frac{x^{n+1}}{n+1} \ln(x) - \frac{x^{n+1}}{(n+1)^2} \tag{5.13}$$

In particular, for $n = 0$, we have $\int \ln(x)dx = x\ln(x) - x$ which would have been difficult to guess by any other means.

5.4 Inverse Chain Rule

The so-called chain rule for the derivatives of functions of functions is the following: if $g(u)$ is a differentiable function of u and $u(x)$ is a differentiable function of x, then

$$\frac{dg(u)}{dx} = \frac{dg(u)}{du}\frac{du}{dx}. \tag{5.14}$$

This is one of the most useful relations of differential calculus. It can simplify derivatives if you are clever. For example, consider $\frac{d}{d\theta}\left(\sin^2\theta + \frac{1}{\sin\theta}\right)$. It is easy to calculate as

$$\frac{d}{d\sin\theta}\left(\sin^2\theta + \frac{1}{\sin\theta}\right)\frac{d\sin\theta}{d\theta} = \left(2\sin\theta - \frac{1}{\sin^2\theta}\right)\cos\theta \tag{5.15}$$

In fact, we may be doing this subconsciously as we are following the rules for derivatives, but there are many situations where using the chain rule in this fashion can make calculations much easier to finish.

Related to this idea is the "inverse" of the chain rule

$$\int g(u)u'(x)\,dx = \int g(u)\frac{du}{dx}dx = \int g(u)du, \tag{5.16}$$

which is another valuable tool for integration. An example is $\int \frac{\ln x}{x}dx$. We have $g(u) = \ln x$ and $u'(x) = \frac{1}{x}$, so that

$$\int \frac{\ln x}{x}dx = \int uu'dx = \int udu = \frac{u^2}{2} = \frac{1}{2}(\ln x)^2. \tag{5.17}$$

Another example is

$$\int \frac{1}{x\ln x}dx = \int \frac{u'}{u}dx = \int \frac{du}{u} = \ln u = \ln(\ln x) \tag{5.18}$$

and so on.

5.5 Inverse Functions

Inverse functions are important throughout physics and math. One thing you must remember is that a function relates *one value* to each argument

x, which makes it single-valued: for each x, there is only one $f(x)$. If we want to compute the inverse of $f(x)$, for each y we must find the x such that $f(x) = y$, or $f^{-1}(y) = x$. We must have only one x value for each y value. This is, in general, where the complexity occurs, because the definition of a single-valued function does not guarantee that the inverse function is also single-valued. Indeed, many are not, which means we must restrict the domain for the inverse function to have it be single valued.

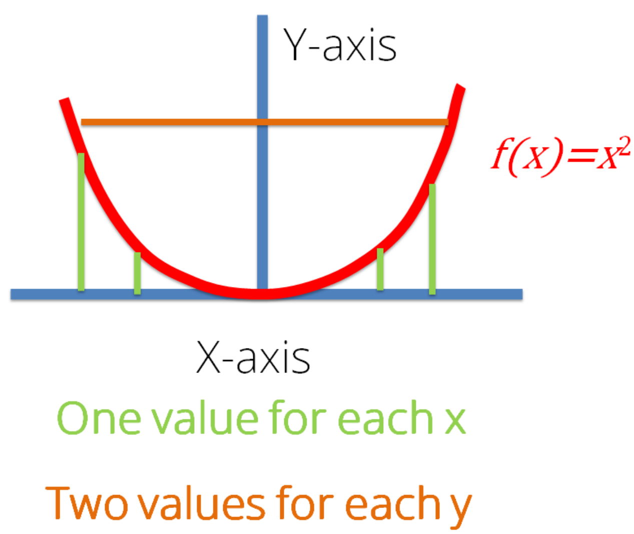

Figure 5.2: Schematic of functions and inverse functions for $y = x^2$.

Take a simple function like $f(x) = x^2$ for $-\infty < x < \infty$. Note how for every x, there is one and only one $y = x^2$ value (green lines in Fig. 5.2), so f is a single valued function. But for the inverse, if we set a value of y, there are two roots, one positive and one negative (red horizontal line in Fig.'5.2), so there are two inverse functions defined on different domains: $f^{-1}(y) = \sqrt{y}$ gives answers in the range 0 to ∞ and $\tilde{f}^{-1}(y) = -\sqrt{y}$ gives answers in the range $-\infty$ to 0. Both are valid inverse functions. For general functions that are not strictly monotonic, the inverse functions will be defined on different ranges. This is particularly true for trigonometric functions which we will treat in the next lecture.

5.6 Examples

We end the chapter with some examples.
1. Differentiate a^x.

Solution: $a^x = e^{x \ln a}$

$$\frac{d}{dx}a^x = e^{x\ln a}\frac{d}{dx}(x\ln a) = e^{x\ln a}\ln a = a^x \ln a. \tag{5.19}$$

So $\frac{d}{dx}a^x = a^x \ln a$. Note how $\frac{d}{dx}e^x = e^x$ since $\ln e = 1$.

2. Compute the following integral:

$$\int \frac{x^4 - a^4}{x^2 + a^2} dx \tag{5.20}$$

Solution: It looks impossible to get a simple answer at first, but recall that

$$x^4 - a^4 = \left(x^2 - a^2\right)\left(x^2 + a^2\right) \tag{5.21}$$

So the integral is

$$\int \left(x^2 - a^2\right) dx = \frac{x^3}{3} - xa^2 \tag{5.22}$$

3. Compute the following integral:

$$\frac{d}{dx}\ln\left(\frac{x^2+1}{x^2-1}\right) \tag{5.23}$$

Solution:

$$\frac{d}{dx}\ln\left(\frac{x^2+1}{x^2-1}\right) = \frac{d}{dx}\ln\left(x^2+1\right) - \frac{d}{dx}\ln\left(x^2-1\right) = \frac{2x}{x^2+1} - \frac{2x}{x^2-1} = \frac{-4x}{x^4-1}. \tag{5.24}$$

Could you show or recognize that

$$\int \frac{x}{1-x^4}dx = \frac{1}{4}\ln\left(\frac{x^2+1}{x^2-1}\right)? \tag{5.25}$$

To do this one would first expand by partial fractions, but one would then have to recognize that integrals of the form $\int \frac{2x}{x^2+1}dx$ are also of the form $\int \frac{du}{dx}dx = \int du = u$ for $u = \ln(x^2+1)$, which is often difficult to remember or recognize.

Chapter 6

How to integrate

6.1 Elementary examples of integration

The Toeplitz book starts by showing how to determine the derivatives of trigonometric functions, which I will assume you all know: $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, $\frac{d}{dx}\tan(x) = \frac{1}{[\cos(x)]^2}$ and so on.

It also describes the inverse trigonometric functions, which are defined, typically, on the range from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, although this can vary. The restricted range is to ensure that the inverse trigonometric functions are single-valued, which we discussed in the last lecture (pictures of the inverse functions can be found in the Toeplitz book).

The critically important idea that we will use here and in the lab is the question of the derivatives of the inverse functions. If $y = \sin(x)$ so $\arcsin(y) = x$, then $\frac{dx}{dy} = \frac{1}{cos(x)} = \frac{1}{\sqrt{1-y^2}}$ so $\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}$. Note that we can replace $\cos(x)$ by $\sqrt{1-y^2}$, because $\sin(x) = y$. One might be worried about whether we choose the plus root or the minus root. This is determined by looking at the angles being considered for the given problem. Similarly, $\frac{d}{dy}\arccos(y) = \frac{-1}{\sqrt{1-y^2}}$ and $\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}$. These identities become important in evaluating integrals as we now show.

So, we want to generate a general set of methods for how to integrate different functions. We already know how to integrate polynomials, since

$$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} & n \neq -1 \\ \ln(x) & n = -1 \end{cases} \tag{6.1}$$

$$\implies \int (ax+b)^n dx = \begin{cases} \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} & n \neq -1 \\ \frac{1}{a}\ln(ax+b) & n = -1 \end{cases} \tag{6.2}$$

If we let $P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ denote an n^{th} degree polynomial, then we know immediately how to integrate the polynomial. What about a more complicated integral, like a polynomial divided by a linear function? We can use the above formulas to integrate $\int \frac{P_n(x)}{x-a}dx$ via

$$\int \frac{P_n(x)}{x-a}dx = \int \frac{P_n(x) - P_n(a)}{x-a}dx + \int \frac{P_n(a)}{x-a}dx \tag{6.3}$$

$$= \int \frac{a_1(x-a) + a_2(x^2-a^2) + a_3(x^3-a^3) + ... + a_n x^n}{x-a}dx + P_n(a)\ln(x-a) \tag{6.4}$$

$$\int dx[a_1 + a_2(x+a) + a_3(x^2+ax+a^2) + ... + a_n(x^{n-1} + ax^{n-2} + ... + a^{n-1})] + P_n(a)\ln(x-a) \tag{6.5}$$

which can be integrated. Note how we used the relation in deriving the geometric series to go from the second to the third line—in other words, all of these integrals can be evaluated analytically, because we can factor the polynomial appropriately after using first the add zero trick and then replacing the remaining polynomial by finite-sum expansions.

What about even more complicated integrals, like $\int \frac{P_n(x)}{(x-a)^m}dx$? This can be written (after using the add-zero trick) as

$$\int \frac{P_n(x) - P_n(a)}{(x-a)} \frac{dx}{(x-a)^{m-1}} + \int \frac{P_n(a)}{(x-a)^m}dx \tag{6.6}$$

with the second term being able to be integrated, and $[P_n(x) - P_n(a)]/(x-a)$ being a polynomial of degree $n-1$. So we can reduce $\int \frac{P_n(x)}{(x-a)^m}dx$ to known integrals plus an integral of the form $\int \frac{P_{n-1}(x)}{(x-a)^{m-1}}dx$. Continue iterating this way until the remaining polynomial becomes a constant, or the exponent on the denominator becomes 1. At this point, all integrals are known how to do by our previous work. Hence, this implies that all integrals of the form

$$\int \frac{P_n(x)}{(x-a)^m}dx \tag{6.7}$$

can also be integrated. Note, that it does not say working out the results for individual cases will be easy. In general, it will not be simple to do so. But the path to doing so is completely known.

Note: old-schoolers, like myself, use integral tables to find these integrals, because the procedure is very tedious to do by hand. If you know mathematica, or wolfram alpha, then this is a very useful exercise that these symbolic manipulation packages can do for you.

What about more complex denominators? We know $\int \frac{dx}{x^2+1} = \arctan(x)$ and

$$\int \frac{dx}{(x-a)(x-b)} = \int dx \left(\frac{1}{x-a} - \frac{1}{x-b} \right) \frac{1}{(a-b)} = \frac{1}{a-b} \ln \left(\frac{x-a}{x-b} \right). \tag{6.8}$$

The Toeplitz book shows how you can generalize these results to $\int \frac{dx}{Ax^2+Bx+C}$. In fact, similar to rational functions, one can integrate any integral of the form $\int \frac{P_n(x)}{(Ax^2+Bx+C)^m} dx$, as well.

If we have an integral of a rational function of $\cos(x)$ and $\sin(x)$, that can be integrated to let $t = \tan(\frac{x}{2})$, so that $\frac{dx}{dt} = \frac{2}{1+t^2}$ but one can show that $\frac{2t}{1+t^2} = \sin(x)$, and $\frac{1-t^2}{1+t^2} = \cos(x)$. So any integral of a rational function of $\sin(x)$ and $\cos(x)$ can be written as an integral of a rational function of t.

An example is:

$$\int \frac{dx}{[\sin(x)]^m} = \int \left(\frac{1+t^2}{2t} \right)^m \frac{2}{1+t^2} \, dt = \int \frac{(1+t^2)^{(m-1)}}{2^{m-1} t^m} \, dt, \tag{6.9}$$

which can be integrated, but the result won't be written out explicitly here.

We next consider integrals that include terms that behave like square roots—particularly $\sqrt{a^2 - x^2}$ for $-a \leq x \leq a$. Any rational function of x and $\sqrt{a^2-x^2}$ can be integrated as follows: Let $x = a\cos(\phi)$, so that $dx = -a\sin(\phi)d\phi$, and$\sqrt{a^2-x^2} = a\sin(\phi)$. A rational function of x and $\sqrt{a^2-x^2}$ (multiplied byb dx) becomes a rational function of $\cos(\phi)$ and $\sin(\phi)$ (multiplied by $\sin(\phi)$), which remains a rational function of $\cos(\phi)$ and $\sin(\phi)$. We just explained how to integrate such quantities. So this approach can be further extended to all rational functions of x and $\sqrt{Ax^2+Bx+C}$, but they require figuring out carefully all the different possible cases (and really requires using integral tables or wolfram alpha in order to ensure the final result is correct).

This approach goes no further. If we have a rational function of x and $\sqrt{Ax^4 + Bx^3 + Cx^2 + Dx + E}$, one cannot integrate this without introducing what are called elliptic functions. Elliptic functions are similar to trigonometric functions but are defined on ellipses, not circles. This topic is beyond what we will cover here in this class, but don't be afraid if you encounter these functions in the future. They can be handled just like sines and cosines. They satisfy lots of similar identities and so forth.

6.2 Gaussian and Frullani integrals

I want to end with showing two more integrals. The first is the Gaussian integral:

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}. \tag{6.10}$$

You may wonder, where in the heck does $\sqrt{\pi}$ come from? It turns out to be very simple and follows from a really neat trick that everyone should know.

We begin by letting $I = \int_{-\infty}^{\infty} dx e^{-x^2}$. Then, we have $I^2 = \int_{-\infty}^{\infty} dx e^{-x^2} \times \int_{-\infty}^{\infty} dy e^{-y^2}$ This is an integral is over the entire $x - y$ plane. Now convert from Cartesian coordinates to polar coordinates:

$$I^2 = \int_0^{\infty} dr \int_0^{2\pi} r e^{-r^2}\, d\theta, \tag{6.11}$$

since $x^2 + y^2 = r^2$. The integral over θ gives 2π, resulting in

$$I^2 = 2\pi \int_0^{\infty} dr\, r e^{-r^2}. \tag{6.12}$$

Let $r^2 = u$, so that $2r\, dr = du$. This then gives

$$I^2 = \pi \int_0^{\infty} du\, e^{-u} = \pi \left(-e^{-u}\right)\Big|_0^{\infty} = \pi(-0 - (-1)) = \pi. \tag{6.13}$$

Taking the square root then yields $I = \sqrt{\pi}$!

The last thing I want to show is Frullani's theorem. This is a theorem that is little known, but extremely powerful if you encounter a function to integrate that satisfies its conditions. The Frullani integral is

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = [f(0) - f(\infty)] \ln\left(\frac{b}{a}\right). \tag{6.14}$$

The functions must decay fast enough, as discussed below, for the integral to make sense.

Proof: Examine the double integral $\int_a^b dx \int_0^\infty dy\,(-f'(xy))$, by integrating over y first:

$$\int_a^b dx\left(\frac{-f(xy)}{x}\right)\Big|_{y=0}^{y=\infty} = \int_a^b dx[f(0)-f(\infty)]\frac{1}{x} = \ln\left(\frac{b}{a}\right)[f(0)-f(\infty)]. \tag{6.15}$$

Now integrate instead over x first:

$$\int_0^\infty dy\left(\frac{-f(xy)}{y}\right)\Big|_{x=a}^{x=b} = \int_0^\infty dy\frac{f(ay)-f(by)}{y}. \tag{6.16}$$

So, we have established that

$$\int_0^\infty dx\frac{f(ax)-f(bx)}{x} = [f(0)-f(\infty)]\ln\left(\frac{b}{a}\right), \tag{6.17}$$

which is the Frullani result.

Example:

$$\int_0^\infty dx\,\frac{e^{-ax}-e^{-bx}}{x} = \text{a Frullani integral with } f(x)=e^{-x} \tag{6.18}$$

$$= \ln\left(\frac{b}{a}\right)$$

$$\int_0^\infty dx\frac{b\sin(ax)-a\sin(bx)}{abx^2} = \text{ Frullani with } f(x)=\frac{\sin(x)}{x} \tag{6.19}$$

$$= \ln\left(\frac{b}{a}\right)$$

and so on. This is an interesting integral to remember.

Chapter 7

Multivariable Integration: Cubic, Cylindrical, and Spherical Coordinates

7.1 Arc Length and Area

We start this lecture by discussing how to calculate the arc-length of a curve. Suppose we have a curve $y = f(x)$. The arc length along the curve (see Fig. 7.1) is

$$ds = \sqrt{dx^2 + dy^2}, \tag{7.1}$$

which arises from computing the length of the hypotenuse of a small right triangle with displacements dx and dy on the two legs. Factoring out a dx, then gives us that

$$ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\,\sqrt{1 + [f'(x)]^2}. \tag{7.2}$$

The arc-length of the curve S is found by adding all of these little ds lengths up, as follows:

$$S = \int_{x_{min}}^{x_{max}} ds = \int_{x_{min}}^{x_{max}} dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \int_{x_{min}}^{x_{max}} dx\,\sqrt{1 + [f'(x)]^2}. \tag{7.3}$$

Our first example will be an arc-length of a circle. We know the full circumference is $2\pi r$, but when we compute the arc-length, we can break the

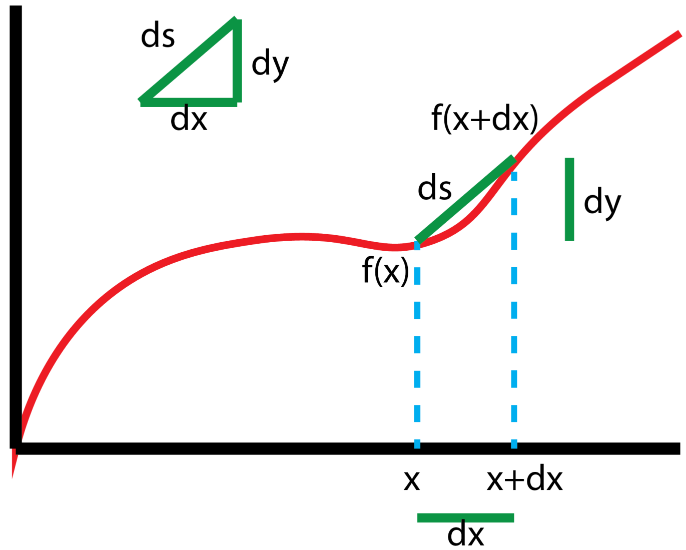

Figure 7.1: Geometry needed to construct the arc-length of a curve $y = f(x)$. One can see that the infinitesimal arc-length along the curve ds is found from computing the hypotenuse of the right triangle with legs dx and $dy = f'(x)dx$.

circle into four pieces. Then we have

$$y(x) = \sqrt{r^2 - x^2} \tag{7.4}$$

in the first quadrant. Taking the derivative, gives us

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}}, \tag{7.5}$$

and then the term in the square-root of the infinitesimal arc-length becomes

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}. \tag{7.6}$$

So

$$\text{arclength} = \frac{1}{4}\text{perimeter} = \int_0^r dx\sqrt{\frac{r^2}{r^2 - x^2}}. \tag{7.7}$$

To compute this integral, we recall the rules from Chapter 6 and let $x = r\sin\phi$, with $dx = r\cos\phi\, d\phi$. The arc-length becomes

$$\frac{1}{4}\text{perimeter} = \int_0^{\frac{\pi}{2}} r\cos\phi\, d\phi\left(\frac{r}{r\cos\phi}\right) = r\int_0^{\frac{\pi}{2}} d\phi = \frac{\pi r}{2}. \tag{7.8}$$

This then implies that the perimeter is $2\pi r$, which we already know to be true. But it is always nice to re-derive something familiar with our new and powerful calculus machinery.

Lets try it now with an ellipse. The ellipse satisfies

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \tag{7.9}$$

with $a \neq b$. We can solve for y in the first quadrant again, to learn that

$$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}. \tag{7.10}$$

Computing the derivative is next

$$\frac{dy}{dx} = \frac{b}{2}\left(\frac{-\frac{2x}{a^2}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right) = \frac{b}{a}\left(\frac{x}{\sqrt{a^2-x^2}}\right), \tag{7.11}$$

followed by squaring it and adding 1. This then gets put into the formula for the arc-length, which finally is

$$\frac{1}{4}\text{perimeter} = \int_0^a dx\sqrt{1+\frac{b^2}{a^2}\left(\frac{x^2}{a^2-x^2}\right)} \tag{7.12}$$

$$= \int_0^a dx\frac{1}{a}\sqrt{\frac{a^4+(b^2-a^2)x^2}{a^2-x^2}} \tag{7.13}$$

$$= \int_0^a dx\frac{1}{a}\left(\frac{a^4+(b^2-a^2)x^2}{\sqrt{(a^2-x^2)(a^4+(b^2-a^2)x^2)}}\right). \tag{7.14}$$

This has a square root of a quartic polynomial in the denominator. As we learned in Chapter 6, to integrate this requires introducing elliptic functions. We will not do that here.

Remarkably though, the areas of the circle and ellipse can be calculated analytically. For the ellipse we have the geometry given in Fig. 7.2.

$$\text{area} = 4\int_0^a dx\, y \tag{7.15}$$

$$= 4\int_0^a dx\, b\sqrt{1-\left(\frac{x}{a}\right)^2} \tag{7.16}$$

$$= 4\,\frac{b}{a}\int_0^1 dx\,\sqrt{a^2-x^2} \tag{7.17}$$

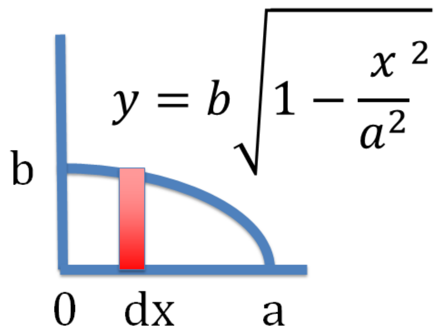

Figure 7.2: Set-up for an integral of the area of an ellipse. We add up the little rectangles of width dx and height $y(x)$.

Let $x = a \sin\phi$ and $dx = a \cos\phi \, d\phi$. Then $\sqrt{a^2 - x^2} = a\cos\phi$. Substituting in gives us

$$\text{area} = 4\,\frac{b}{a}\int_0^{\frac{\pi}{2}} a\cos\phi\, d\phi\, a\cos\phi \tag{7.18}$$

$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\phi\, d\phi. \tag{7.19}$$

But $\cos^2\phi = \frac{\cos(2\phi)+1}{2}$. So

$$\text{area} = \frac{4ab}{2}\int_0^{\frac{\pi}{2}} (\cos 2\phi + 1)\, d\phi \tag{7.20}$$

$$= 2ab\left(\left.\frac{\sin 2\phi}{2}\right|_0^{\frac{\pi}{2}} + \frac{\pi}{2}\right) \tag{7.21}$$

$$= \pi ab. \tag{7.22}$$

For a circle, $a = b = r$, and area $= \pi r^2$ as we already know.

7.2 Differential volume elements

We now move on to integration in three dimensions. We start with volumes and the different three-dimensional coordinate systems (Cartesian, cylindrical, or spherical). Volume is the ratio of the amount of an object to the amount in a unit cube. In cubic coordinates, the volume "interval" is $dx\,dy\,dz$.

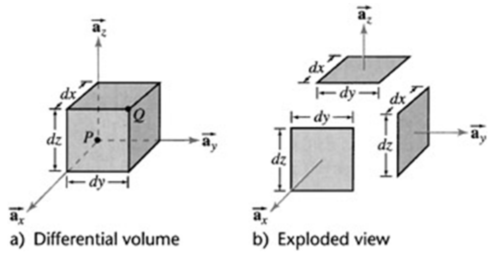

Figure 7.3: The Cartesian differential volume element, of size $dx \times dy \times dz$.

For cylindrical coordinates, we use polar coordinates for the plane and z for the third dimension, corresponding to the height. This yields a volume element given by $dr\,rd\theta\,dz$.

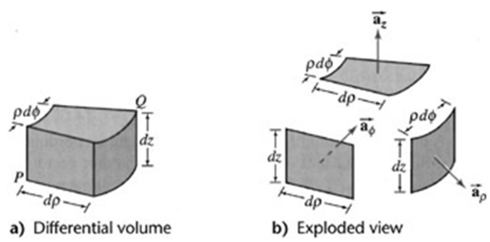

Figure 7.4: The differential volume element in cylindrical coordinates, given by $dr \times rd\phi \times dz$. (Note, we usually use r instead of ρ for the radial coordinate, but both are common.)

For spherical coordinates, the volume element is $dr\,rd\theta\,r\sin\theta\,d\phi = r^2\,dr\,d\cos\theta\,d\phi$. Learning how to evaluate integrals in this second form, integrating with respect to $d\cos\theta$, is the difference between a technician and a practitioner!

7.3 Examples of volume and surface area integrals

Our first example is the volume of a sphere of radius a. The reason why we use spherical coordinates is because the equation of a sphere is so simple in

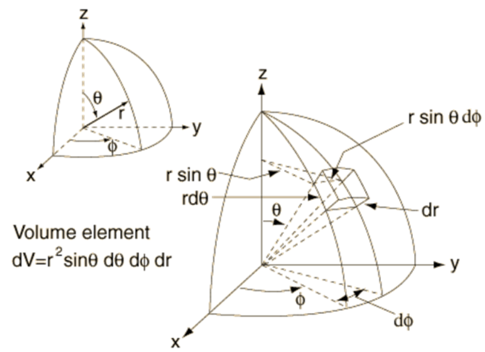

Figure 7.5: The differential volume element in spherical coordinates. Note that the angle θ is the rotation down from the z-axis and the angle ϕ is the angle in the $x-y$ plane. This convention is the opposite to what mathematicians use. But it is the one adopted by physicists.

these coordinates. It is just $r = a$. So

$$\text{Volume} = \int_0^a r^2\,dr \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi \tag{7.23}$$

$$= 2\pi \int_0^a r^2\,dr \int_{-1}^1 d\cos\theta \tag{7.24}$$

$$= 2 \times 2\pi \int_0^a r^2\,dr \tag{7.25}$$

$$= 4\pi \left(\frac{r^3}{3}\right)\Big|_0^a \tag{7.26}$$

$$= \frac{4}{3}\pi a^3 \tag{7.27}$$

which we know to be the volume of a sphere.

Our second example is the volume of a cylinder of radius a and height h. The cylinder lives in $0 \leq z \leq h$ and $r \leq a$. We compute the volume using cylindrical coordinates:

$$\text{Volume} = \int_0^h dz \int_0^a dr\, r \int_0^{2\pi} d\theta \tag{7.28}$$

$$= h \times \frac{a^2}{2} \times 2\pi \tag{7.29}$$

$$= \pi a^2 h \tag{7.30}$$

as expected.

Now we move on to surface areas, starting with the surface area of a cylinder. We know from a previous calculation that the surface area of each of the circular endcaps is πa^2. We add the area of the two endcaps to the area of the sides of the cylinder, which is the perimeter of the circle integrated over the height of the cylinder.

$$\text{Area} = 2\pi a^2 + \int_0^h dz 2\pi a \tag{7.31}$$

$$= 2\pi a^2 + 2\pi a h. \tag{7.32}$$

The final simple example is the surface area of a sphere. The area element is $r d\theta \, r \sin\theta d\phi$ with $r = a$.

$$\text{Area} = a^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi = 4\pi a^2. \tag{7.33}$$

Our final topic is surfaces of revolution. It is critical that you understand surfaces of revolution to be comfortable with the ideas of this chapter. Students often have trouble with them because the curves can be revolved around the x or y axes. One needs to always pay attention.

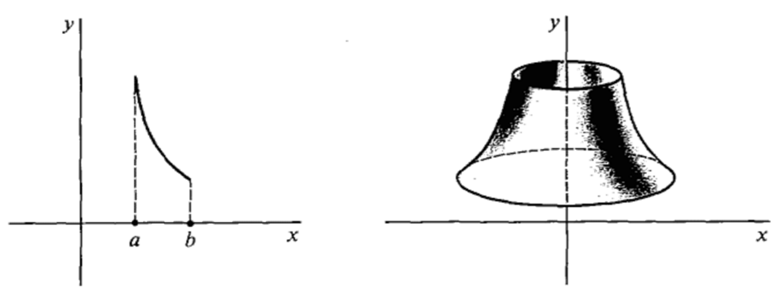

Figure 7.6: In this example, we have the surface of revolution about the y-axis.

The volume of a surface of revolution is the sum of the volumes of different disks, each with radius x.

$$\text{Volume} = \int_{y_{min}}^{y_{max}} dy \, \pi x^2 \tag{7.34}$$

We know the equation for $y(x)$, so $dy = \frac{dy}{dx}dx$. The volume integral then becomes

$$\text{Volume} = \pi \int_0^{x_{max}} dx \frac{dy}{dx} x^2. \tag{7.35}$$

Now we go onto the final topic, the surface area of a surface of revolution. Each area element was just an arc-length ds times $2\pi x$ the perimeter of the circle.

$$\text{Surface Area} = \int ds\, 2\pi x \tag{7.36}$$

$$= \int_0^{x_{max}} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, 2\pi x \tag{7.37}$$

$$= 2\pi \int_0^{x_{max}} dx\, x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{7.38}$$

You will have some examples of this on the homework.

Chapter 8

The vanishing sphere and other multidimensional integrals

8.1 Three-dimensional integrals

Recall we discussed how to integrate volumes and areas in cubic, cylindrical, and spherical coordinates in Chapter 7. Now we will discuss some other examples:

Example 1: What is the mass of a sphere of radius R with constant mass density ρ_0? To find the total mass, we add each volume element, weighted by the mass density. Since the mass density is a constant, it does not change the volume integral. Rather, it just multiplies the result, as you can see below.

$$M = \int_0^R dr \int_{-1}^1 r\, d\cos\theta \int_0^{2\pi} r\rho_0\, d\phi \tag{8.1}$$

$$= \int_0^R r^2 \rho_0 dr \times 2 \times 2\pi = \frac{4\pi}{3} R^3 \rho_0 \tag{8.2}$$

as you might have guessed.

Example 2: Now we try this again, but with a mass density that *varies with the radius*, such as $\rho(r) = \rho_0(\frac{r}{R})^\alpha$. The integral now becomes

$$M = \int_0^R dr r^2 \rho_0 (\frac{r}{R})^\alpha \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi. \tag{8.3}$$

The integral over the angles is trivial and yields a factor of 4π. One can then complete the integral as follows:

$$= \frac{4\pi\rho_0}{R^\alpha}\int_0^R dr r^{\alpha+2} = \frac{4\pi\rho_0}{R^\alpha}\frac{r^{\alpha+3}}{\alpha+3}\bigg|_0^R = \frac{4\pi\rho_0}{R^\alpha}\frac{R^{\alpha+3}}{\alpha+3} = \frac{4\pi R^3}{\alpha+3}\rho_0. \tag{8.4}$$

Note that e have a nice check of this formula. If we set $\alpha = 0$, then the same mass will be found as we calculated before in Example 1. Note further that as α increases, the mass goes down. Do you understand why this is so?

8.2 The vanishing sphere

Let's examine the phenomenon of the "vanishing sphere"—as the number of dimensions d increases, the volume of the sphere in d dimensions approaches *zero* for any finite radius.

We start by defining a sphere in d-dimensions. The position vector $\vec{x}$ is an d-tuple, given by $\vec{x} = (x_1, x_2, x_3, x_4, ...x_d)$. A sphere is defined by the volume inside the set

$$\vec{x}\cdot\vec{x} \le R^2. \tag{8.5}$$

This implies that

$$(x_1)^2 + (x_2)^2 + (x_3)^2 + ...(x_d)^2 \le R^2. \tag{8.6}$$

Now we need to determine spherical coordinates in d dimensions. We work by generalizing the spherical coordinates from $3d$, where we have r and two angles θ and ϕ, to d "hyper-spherical" coordinates given by r, θ_1, ..., θ_{d-1}. The procedure is a s follows. We start with the first Cartesian coordinate x_1, which we define in terms of the r-coordinate and the first angle θ_1

$$x_1 = r\cos(\theta_1). \tag{8.7}$$

One should think of this as the projection of the vector $\vec{x}$ on the first Cartesian axis, which defines the angle θ_1. The perpendicular component of $\vec{x}$ now has a length given by $r\sin\theta_1$. So, its projection onto the second axis is

$$x_2 = \underbrace{r\sin(\theta_1)}_{length\ of\ \perp\ component}\cos(\theta_2). \tag{8.8}$$

The procedure is then iterated. Each subsequent perpendicular component is of length of the previous perpendicular component multiplied by the sine of the latest angle. Hence, we have

$$x_3 = \underbrace{r\sin(\theta_1)\sin(\theta_2)}_{length\ of\ \perp\ component}\ \cos(\theta_3). \tag{8.9}$$

This continues until we get to the last angle. It is different. The second-to-last angle is defined via

$$x_{d-1} = \underbrace{r\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)...\sin(\theta_{d-2})}_{length\ of\ \perp\ component}\cos(\theta_{d-1}). \tag{8.10}$$

And then the last one is

$$x_d = \underbrace{r\sin(\theta_1)\sin(\theta_2)\sin(\theta_3)...\sin(\theta_{d-2})\sin(\theta_{d-1})}_{length\ of\ final\ \perp\ component} \tag{8.11}$$

where the final cosine is replaced by a sine representing the last perpendicular component.

Now to find the volume element, it is just like before in $3d$. Each dimension uses the length of the corresponding perpendicular component. One needs to be careful about the last term:

$$dr\times r d\theta_1\times r\sin(\theta_1)d\theta_2\times r\sin(\theta_1)\sin(\theta_2)d\theta_3...r\sin(\theta_1)\sin(\theta_2)...\sin(\theta_{d-2})d\theta_{d-1}. \tag{8.12}$$

Here r goes from $0\to\infty$, $\theta_1...\theta_{d-2}$ from $0\to\pi$, and θ_{d-1} from $0\to 2\pi$. So an integral for the volume in d-dimensions is:

$$V_d = \int_0^R dr r^{d-1}\int_0^\pi \sin(\theta_1)^{d-2}d\theta_1\int_0^\pi \sin(\theta_2)^{d-3}d\theta_2\cdots\int_0^\pi \sin(\theta_{d-2})d\theta_{d-2}\int_0^{2\pi} d\theta_{d-1} \tag{8.13}$$

Hence, to find the volume, we need to integrate $\int_0^\pi(\sin\theta)^\alpha d\theta = I_\alpha$, with $u=(\sin\theta)^{\alpha-1}$ and $v'=\sin(\theta)$, using integration by parts with $v=-\cos(\theta)$. This yields

$$I_\alpha = -(\sin\theta)^{\alpha-1}\cos\theta\Big|_0^\pi + \int_0^\pi(\alpha-1)(\sin\theta)^{\alpha-2}\cos\theta\cos\theta\,d\theta. \tag{8.14}$$

Simplifying, we find

$$I_\alpha = 2\delta_{\alpha=1} + (\alpha-1)\int_0^\pi(\sin\theta)^{\alpha-2}(1-\sin^2\theta)\,d\theta. \tag{8.15}$$

Using the definition of I_α, we find a recurrence relation between the integrals, given by

$$I_\alpha = 2\delta_{\alpha=1} + (\alpha - 1)\left(I_{\alpha-2} - I_\alpha\right). \tag{8.16}$$

We re-arrange the recurrence relation to its final form, which is

$$\alpha I_\alpha = (\alpha - 1) I_{\alpha-2} + 2\delta_{\alpha=1}. \tag{8.17}$$

The recurrence relation then tells us that $I_1 = 2$ (which can also be verified from integrating the definition of I_1). Hence, we have the odd integrals satisfy $I_3 = \frac{2}{3} I_1 = \frac{4}{3}$, $I_5 = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} I_1 = \frac{16}{15}$, $I_7 = \frac{6}{7} I_5 = \frac{6\times4\times2}{7\times5\times3} \times 2 = \frac{32}{35}$, and so on. One can recognize the clear pattern. We therefore have

$$I_{2n+1} = \frac{2^n n!}{(2n+1)!!} \times 2 = \frac{2^{n+1} n!}{(2n+1)!!}. \tag{8.18}$$

To be completely rigorous, one should verify this result by induction. The proof should take two to three lines and it is useful for you to try it yourself.

Now, onto the even integrals. We have

$$I_2 = \int_0^\pi (\sin\theta)^2 \, d\theta = \frac{1}{2} \int_0^\pi \left(\sin^2\theta + \cos^2\theta\right) \, d\theta = \frac{\pi}{2}. \tag{8.19}$$

Using this initial result, $I_2 = \frac{\pi}{2}$, we can then generate all other even integrals from the recurrence relation: $I_4 = \frac{3}{4} I_2 = \frac{3}{8}\pi$, $I_6 = \frac{5}{6} I_4 = \frac{5*3}{6*4}\frac{\pi}{2}$, and so on. Again, the pattern is clear, so we know that

$$I_{2n} = \frac{(2n-1)!!}{2^n n!}\pi. \tag{8.20}$$

Again, you should verify this is correct by induction.

In order to compute the volumes of the spheres, we first consider the case where d is an odd number. Evaluating the initial volume integral is now straightforward, because all angular integrals are known. We have

$$V_d = \int_0^R r^{d-1} \, dr \times 2\pi \times I_{d-2} \times I_{d-3} \times I_{d-4} \cdots I_2 \times I_1. \tag{8.21}$$

The radial integral can be done and we group the angular integrals to give

$$V_d = \frac{1}{d} R^d \times 2\pi \times (I_1 I_3 I_5 \cdots I_{d-2}) \times (I_2 I_4 I_6 \cdots I_{d-3}). \tag{8.22}$$

Now, we substitute in the results for each of the angular integrals:

$$V_d = \frac{1}{d}R^d \times 2\pi \times 2 \times \left(2 \times \frac{2}{3}\right) \times \left(2 \times \frac{2}{3} \times \frac{4}{5}\right) \times \left(2 \times \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7}\right) \cdots \left(\frac{2^{\frac{d-1}{2}}(\frac{d-3}{2})!}{(d-2)!!}\right)$$
$$\times \frac{\pi}{2} \times \left(\frac{\pi}{2} \times \frac{3}{4}\right) \times \left(\frac{\pi}{2} \times \frac{3}{4} \times \frac{5}{6}\right) \cdots \left(\frac{\pi(d-4)!!}{2^{\frac{d-3}{2}}(\frac{d-3}{2})!}\right). \quad (8.23)$$

So,

$$V_d = \frac{1}{d}R^d \times 2\pi \times \left(2 \times \frac{\pi}{2}\right) \times \left(2 \times \frac{2}{3}\right) \times \left(\frac{\pi}{2} \times \frac{3}{4}\right) \times \left(2 \times \frac{2}{3} \times \frac{4}{5}\right)$$
$$\times \left(\frac{\pi}{2} \times \frac{3}{4} \times \frac{5}{6}\right) \cdots \times \left(\frac{2\pi}{d-3}\right)\left(\frac{2 \times 2 \times 4 \cdots (d-3)}{3 \times 5 \cdots (d-2)}\right) \quad (8.24)$$

We finally simplify in a couple of steps:

$$V_d = \frac{1}{d}R^d \times (2\pi)^{\frac{d-1}{2}} \times \frac{1}{2^{\frac{d-3}{2}}\left(\frac{d-3}{2}\right)!} \times \frac{2^{\frac{d-1}{2}}\left(\frac{d-3}{2}\right)!}{(d-2)!!}, \quad (8.25)$$

$$V_d = \frac{1}{d}R^d(2\pi)^{\frac{d-1}{2}}\frac{2}{(d-2)!!}, \quad (8.26)$$

$$V_d = R^d(2\pi)^{\frac{d-1}{2}} \times \frac{2}{d!!} = (\sqrt{2\pi}R)^d\sqrt{\frac{2}{\pi}}\frac{1}{d!!}. \quad (8.27)$$

Note how, for fixed R and as d increases, we eventually have $d > 2\pi R^2$, so that $\frac{(\sqrt{2\pi}R)^d}{d!!} \to 0$ as $d \to \infty$. This implies that the Volume of a sphere vanishes as $d \to \infty$!

Why?

Recall that volumes are measured *relative to the unit cube* (which always has volume equal to 1). Note that the diagonal of a unit cube has length $\sqrt{d} \to \infty$ as $d \to \infty$, so the unit cube looks like a "porcupine" as $d \to \infty$. This implies that any finite radius sphere lies in a small volume inside the unit cube, so $V_d \to 0$ as $d \to \infty$.

In general, high dimensional integrals are hard to evaluate except by a Monte Carlo method—the Monte Carlo method picks a volume that encloses the object that is being integrated. Next, one chooses a random point uniformly in the volume. If the randomly chosen point lies inside the sphere, then we add 1 to the sum. If not, then we don't add 1. Hence, the volume is estimated to be the total number of points in sphere divided by the total number in the volume and multiplied by the volume.

Lets try with a python code for odd dimensions with a unit radius sphere. We will use $100,000,000$ randomly chosen points, which gives an accuracy of better than 3 digits for low d. Recall that

$$V_d = \left(\sqrt{2\pi}R\right)^d \sqrt{\frac{2}{\pi}}\,\frac{1}{d!!}, \tag{8.28}$$

so we have $V_3 = \frac{4}{3}\pi$, $V_5 = \frac{8}{15}\pi^2$, $V_7 = \frac{16}{105}\pi^3$, $\cdots$, $V_{19} = \frac{1024}{19!!}\pi^4$. We compare the two results in Table 8.1.

	Monte Carlo	exact
1:	2.0	2.0
3:	4.18854	4.18879
5:	5.26270	5.26379
7:	4.72316	4.72477
9:	3.29940	3.29851
11:	1.88584	1.88410
13:	0.91628	0.91063
15:	0.36438	0.38144
17:	0.14811	0.14098
19:	0.03670	0.04662

Table 8.1: Comparison of a Monte Carlo integration to the exact results for unit spheres of different dimensions.

Note how, from $d = 13$ and beyond, it can be seen from the table the the numbers become less and less accurate.

8.3 Moments of Inertia

Over the years, I have noticed that students really struggle with knowing how to set up the integral for a moment of inertia and evaluate it. We will spend an entire laboratory working on this problem, after which you should become a “practitioner.”. We introduce some of the issues here.

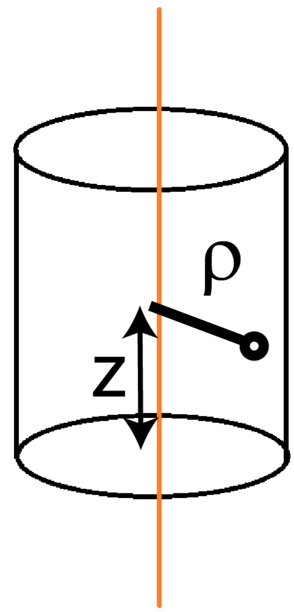

Figure 8.1: Schematic of the cylinder and the axis of rotation (orange) about which the cylinder rotates. Note how the *distance* to the axis is given by ρ and is independent of the angle and the height.

Note that a moment of inertia simply involves an integral similar to the ones used to find the mass given a particular mass density (constant or not); we looked at these in Examples 1 and 2 above. The difference is that we must add an additional factor of l^2, where l is the perpendicular distance of the particular volume element from the axis of rotation. It is this last step that often stymies students. Look carefully at the geometry and see whether you can see how this is done.

As an example, we show a cylinder in black and the axis of rotation in orange in Fig. 8.1. If you recall cylindrical coordinates, you should be able to immediately see that the perpendicular distance of some point within the cylinder to the rotation axis is just given by the radial coordinate ρ (it is independent of z or θ). Hence, we have $l = \rho$, and the additional factor needed in the integral will be a factor of $l^2 = \rho^2$. We examine this in complete detail in the lab.

Chapter 9

Feynman or Parametric Integration

9.1 Powers and Gaussians

This technique is usually called differentiating under the integral sign. Take a function that depends on α and x. Under proper converging conditions

$$\frac{d}{dx}\int f(x,\alpha)dx = \int \frac{\delta}{\delta x} f(x,\alpha)\,dx \tag{9.1}$$

The most common example of this comes with a variant of the Gaussian integral. Suppose we want to find

$$\int_0^\infty x^n e^{-x^2}\,dx. \tag{9.2}$$

Recall, that we already worked out the integral of the Gaussian without any power. It was

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}. \tag{9.3}$$

Since the integrand is even, we also know the integral from 0 to infinity is given by

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}. \tag{9.4}$$

Now, we want to investigate powers of x times Gaussians. For $n = 2m+1$ odd, we let $x^2 = u$ and $2x\,dx = du$. The integral transforms to

$$\int_0^\infty x^{2m+1} e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty u^m e^{-u}\,du. \tag{9.5}$$

The hardest part of this technique is figuring out where to introduce a parameter that we differentiate with respect to. Recall that a derivative of an exponential returns the exponential times a derivative of the exponent. This motivates us to introduce α in the following fashion:

$$\int_0^\infty x^{2m+1} e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty u^m e^{-u}\,du = \frac{1}{2}(-1)^m \int_0^\infty \frac{d^m}{d\alpha^m} e^{-\alpha u}\,du\bigg|_{\alpha=1}. \tag{9.6}$$

Convince yourself that the derivative, evaluated when $\alpha = 1$ does indeed produce the required power. Now, we complete the calculation by performing the integral first. We have

$$\begin{aligned}
\int_0^\infty x^{2m+1} e^{-x^2}\,dx &= \frac{1}{2}(-1)^n \frac{d^m}{d\alpha^m}\int_0^\infty e^{-\alpha u}\,du\bigg|_{\alpha=1} && (9.7)\\
&= \frac{1}{2}(-1)^m \frac{d^m}{d\alpha^m}\left(-\frac{1}{\alpha}e^{-\alpha u}\bigg|_0^\infty\right)\bigg|_{\alpha=1} && (9.8)\\
&= \frac{1}{2}(-1)^m \frac{d^m}{d\alpha^m}\frac{1}{\alpha}\bigg|_{\alpha=1} && (9.9)\\
&= \frac{1}{2}(-1)^m(-1)^m m!\,\frac{1}{\alpha^{n+1}}\bigg|_{\alpha=1} && (9.10)\\
&= \frac{1}{2}m!. && (9.11)
\end{aligned}$$

Be sure you can evaluate the m-fold derivative of $1/\alpha$ to be able to find this final result. To be rigorous, prove it via induction.

Now for $n = 2m$ even, we begin in the same way

$$\int_0^\infty x^{2m} e^{-x^2}\,dx = (-1)^n \int_0^\infty \frac{d^m}{d\alpha^m} e^{-\alpha x^2}\bigg|_{\alpha=1} dx \tag{9.12}$$

Next, we set $y = \sqrt{\alpha}\,x$, $dy = \sqrt{\alpha}\,dx$ to remove the α from the exponent and we do the integral and the subsequent derivatives:

$$\int_0^\infty x^{2m} e^{-x^2}\, dx = (-1)^n \frac{d^m}{d\alpha^m} \int_0^\infty e^{-y^2} \frac{1}{\sqrt{\alpha}}\, dy \bigg|_{\alpha=1} \tag{9.13}$$

$$= (-1)^m \frac{d^m}{d\alpha^m} \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\alpha}} \bigg|_{\alpha=1} \tag{9.14}$$

$$= (-1)^m (-1)^m \frac{\sqrt{\pi}}{2} \frac{1}{2} \frac{3}{2} \frac{5}{2} \cdots \frac{2m-1}{2} \frac{1}{a^{m+\frac{1}{2}}} \bigg|_{\alpha=1} \tag{9.15}$$

$$= \frac{\sqrt{\pi}}{2} \frac{(2m-1)!!}{2^m}. \tag{9.16}$$

So we have

$$\int_0^\infty x^n e^{-x^2}\, dx = \begin{cases} \frac{\sqrt{\pi}}{2} \frac{(n-1)!!}{2^{\frac{n}{2}}}, & \text{if } n \text{ even} \\ \\ \frac{1}{2} \left(\frac{n-1}{2}\right)!, & \text{if } n \text{ odd} \end{cases}. \tag{9.17}$$

Clearly, this method is powerful. But it often is quite confusing. The biggest issue is how do we put in the parameter we differentiate with respect to? One way is as we saw before—we add the parameter to an argument of a function. Another way is to add an entire function into the integrand. We show how this works next. Consider the following integral.

$$\int_0^\infty \frac{\sin x}{x}\, dx \tag{9.18}$$

Adding an α to the argument of the sin function does not help us. Instead, we introduce $e^{-\alpha x}$ with $\alpha > 0$, and choose the limit where $\alpha \to 0$. This will reproduce the original integral.

$$I(\alpha) = \int_0^\infty \frac{\sin x}{x} e^{-\alpha x}\, dx \tag{9.19}$$

But now the derivative satisfies

$$\frac{d}{d\alpha} I(\alpha) = \frac{d}{d\alpha} \int_0^\infty \frac{\sin x}{x} e^{-\alpha x}\, dx \tag{9.20}$$

$$= \int_0^\infty \frac{\sin x}{x} \frac{d}{d\alpha} e^{-\alpha x}\, dx \tag{9.21}$$

$$= \int_0^\infty \frac{\sin x}{x} (-x)\, e^{-\alpha x}\, dx \tag{9.22}$$

$$= -\int_0^\infty \sin x\, e^{-\alpha x}. \tag{9.23}$$

But $\sin x = \mathrm{Im}\,[e^{ix}]$ (recall that $e^{ix} = \cos x + i \sin x$). So, we have

$$\frac{d}{d\alpha} I(\alpha) = -\mathrm{Im} \int_0^\infty e^{-\alpha x + ix}\, dx. \tag{9.24}$$

Now we can evaluate the integral to find

$$\frac{d}{d\alpha} I(\alpha) = -\mathrm{Im} \frac{1}{-\alpha + i} e^{-\alpha x + ix} \Big|_0^\infty \tag{9.25}$$

$$= \mathrm{Im} \frac{1}{-\alpha + i}. \tag{9.26}$$

We next evaluate the imaginary part. We use the multiply by one trick, so that the imaginary part becomes simple:

$$\frac{d}{d\alpha} I(\alpha) = \mathrm{Im} \frac{-\alpha - i}{(-\alpha - i)(-\alpha + i)} \tag{9.27}$$

$$= \mathrm{Im} \frac{-\alpha - i}{\alpha^2 + 1} \tag{9.28}$$

$$= -\frac{1}{1 + \alpha^2}. \tag{9.29}$$

This is a differential equation. But do not panic. We actually know how to solve it. Recall that $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$ so

$$I(\alpha) = -\tan^{-1} x + C. \tag{9.30}$$

How do we find the constant? It is not quite so obvious at first, but with a little thought, we try to examine the limit where $\alpha \to \infty$. Then the integral $\to 0$. But $\tan^{-1}(\infty) = \frac{\pi}{2}$. So we can find the constant C via

$$-\frac{\pi}{2} + C = 0 \implies C = \frac{\pi}{2}. \tag{9.31}$$

Putting this all together tells us that

$$I(\alpha) = \frac{\pi}{2} - \tan^{-1} \alpha \tag{9.32}$$

and for $\alpha = 0$,

$$I(0) = \frac{\pi}{2}. \tag{9.33}$$

Hence
$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \tag{9.34}$$

Clearly this approach has a lot of surprises. Sometimes we convert evaluating an integral into solving a differential equation (we will cover how to solve such equations in great generality later in these notes). The key point is it takes practice to learn how to make the approach work. There are only a handful of things we can try to make it work. One simply tries them to see if Feynman integration will work for a given problem.

Overall, this is an impressive method to remember. It requires some creativity to employ the technique.

We will do one more example:

$$\int_0^1 \frac{x^2 - 1}{\ln x}\, dx \tag{9.35}$$

It is not obvious how to enter the parameter here. It turns out the trick is to replace the exponent 2 with α. Then $\frac{d}{d\alpha} x^\alpha = \frac{d}{d\alpha} e^{\alpha \ln x} = \ln x\, e^{\alpha \ln x} = \ln x\, x^\alpha$ (if you have not seen this before, be sure to remember it—you can differentiate an expression with respect to its exponent by converting it to an exponential). Using this, we rewrite the integral as

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\, dx \tag{9.36}$$

The simplification occurs when we differentiate with respect to α (which is the common theme in this approach):

$$\begin{aligned}
\frac{d}{d\alpha} I(\alpha) &= \int_0^1 \frac{d}{d\alpha} \frac{x^\alpha - 1}{\ln x}\, dx && (9.37)\\
&= \int_0^1 \frac{\ln x\, x^\alpha}{\ln x}\, dx && (9.38)\\
&= \int_0^1 x^\alpha\, dx && (9.39)\\
&= \left.\frac{x^{\alpha+1}}{\alpha + 1}\right|_0^1 && (9.40)\\
&= \frac{1}{\alpha + 1} && (9.41)
\end{aligned}$$

Now we need to integrate with respect to α to get the solution:

$$I(\alpha) = \int d\alpha \frac{1}{\alpha + 1} + C = \ln(\alpha + 1) + C. \tag{9.42}$$

If $\alpha = 0$, then $x^\alpha = 1$ and the integral vanishes $I(\alpha) = 0$. This implies that $\ln(1) + C = 0$, so $C = 0$. Finally, we have

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x} \, dx = \ln(\alpha + 1) \tag{9.43}$$

For $\alpha = 2$,

$$I(2) = \int_0^1 \frac{x^2 - 1}{\ln x} \, dx = \ln 3. \tag{9.44}$$

You might recall that we already solved the integral for $\alpha = 1$, which gives $\ln 2$.

Chapter 10

Vector-valued functions

10.1 Vector fields

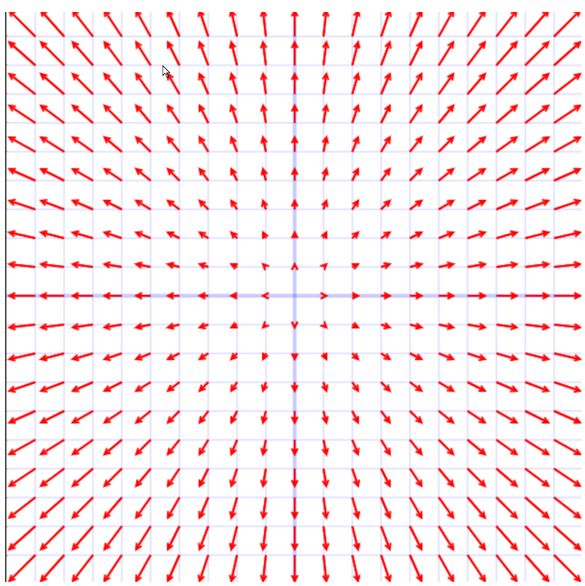

Figure 10.1: Vector field $x\hat{i} + y\hat{j}$.

You have been studying functions throughout calculus. In most cases the result of the function is a real number. Vector-valued functions, on the other hand map to a vector. The easiest way to visualize such functions is when they map a number on the plane to a vector. This is called a vector field.

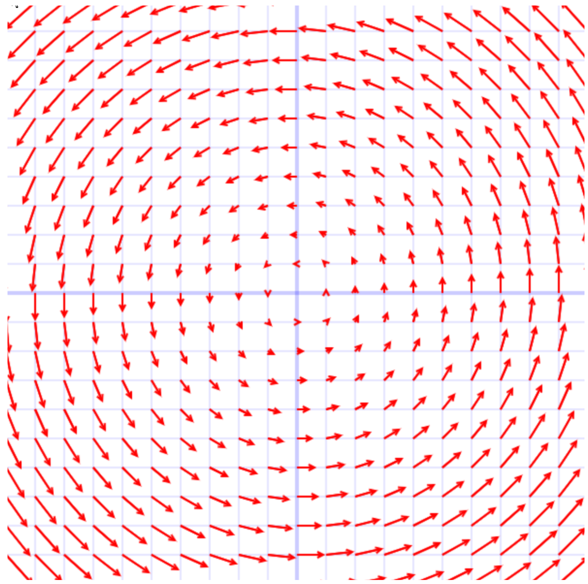

Figure 10.2: Vector field $-y\hat{i} + x\hat{j}$.

At each point, the point is mapped to a vector, which can be illustrated as a small arrow (of the size and direction of the vector value), located at the point.

There is an on-line internet tool that can be used to plot these vector fields. It is https://kevinmehall.net/p/equationexplorer/index.html .

Let's look at some examples of vector fields. These are all taken from the website above. We plot three vector fields: $x\hat{i} + y\hat{j}$ in Fig. 10.1; $-y\hat{i} + x\hat{j}$ in Fig. 10.2; and $-(x-2)\hat{i} - y\hat{j}$ in Fig. 10.3.

As we look at these different vector fields, we get sense of "motion" in them. In Fig. 10.1, we get the sense of stuff moving out from the origin. In Fig. 10.2, we get the feeling of rotation around the origin. In Fig. 10.3, we get a sense of stretching in the vertical and compression in the horizontal.

Be sure that you can describe in words exactly what a vector field is. Best if you use your own words. For me, I like to say a vector field is a little arrow that has a direction and length that I draw at each point in space.

One of the aspects of vector fields, that we saw in the three images, was that they convey different senses of "motion." There is a sense of expansion (almost like an explosion). The mathematical way to describe this is by using a concept called divergence. We also saw a sense of rotation. In

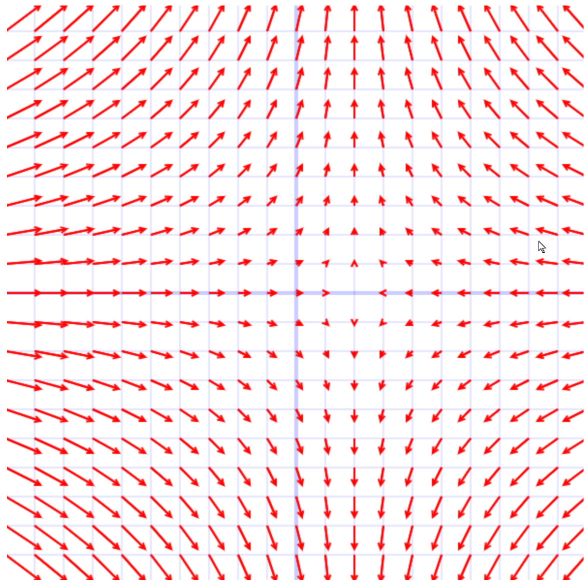

Figure 10.3: Vector field $-(x-2)\hat{\imath} + y\hat{\jmath}$.

mathematics, we use the concept of curl to describe it. We will define these concepts precisely in due course. For now, we will focus on some applications and examples of vector fields.

We begin with Coulomb's Law: $\vec{F}_E = q_0 \vec{E}(\vec{r})$, which expresses the electric force as the charge of the particle q_0 multiplied by the electric field $\vec{E}$. The electric field, in SI units, is the following for a charge q_1 sitting at the location $\vec{r}_1$: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r}-\vec{r_1}|^2}$. Hence, if we have a charge distribution $\rho_E(\vec{r}\,')$, then the electric field is found by summing (really integrating) all of these little charges described by $\rho_e(\vec{r}\,')$, or

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho_e(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\vec{e}_{\vec{r}-\vec{r}\,'}, \tag{10.1}$$

with $\vec{e}_{\vec{r}-\vec{r}\,'}$ the unit vector pointing in the direction of $\vec{r}-\vec{r}\,'$. Note that this integral is quite difficult to carry out because it is integrating *vectors*—we need to worry about where the unit vector $\vec{e}_{\vec{r}-\vec{r}\,'}$ is pointing when summing over all of the charges in the charge distribution.

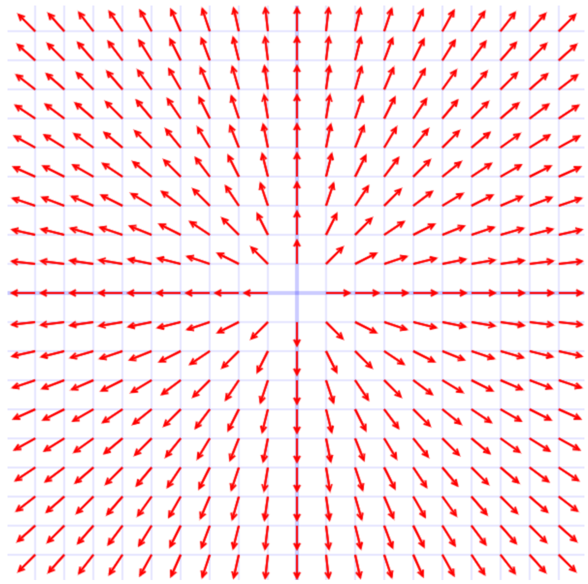

Figure 10.4: Radial vector field composed of unit vectors. Note how it is undefined at the origin, because we cannot tell what direction the vector points at when we are at the origin.

10.2 Examples of vector fields

Example 1: Find a formula for a vector field that is a unit vector in the radial direction in $2d$.

Recall that the vector $\vec{r} = x\hat{i} + y\hat{j}$. The length of the vector is $r = \sqrt{x^2 = y^2}$, so this radial unit vector field becomes

$$\vec{E}(x, y) = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}, \tag{10.2}$$

which is plotted in Fig.10.4.

What happens as $x, y \to 0$? This point is a singular point, as shown in Fig. 10.4. Note that singular points do not need to have a vector diverging to infinity. Simply having a point where the vector field is undefined because we cannot choose its direction is enough. This is a real singularity.

Example 2: Find a vector field which points at 45° to the x-axis and whose magnitude is $(x + y)^2$ at x, y.

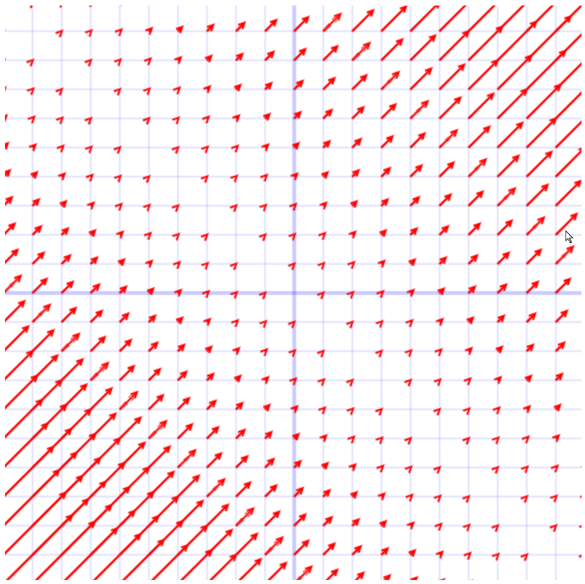

Figure 10.5: The vector field that points at 45° and has a magnitude $(x+y)^2$.

The vector field has each component the same to have them point in a 45° direction. We pick the magnitude to yield the required $(x+y)^2$. This gives us

$$\vec{E}(x,y) = \frac{(x+y)^2}{\sqrt{2}}\hat{i} + \frac{(x+y)^2}{\sqrt{2}}\hat{j}, \tag{10.3}$$

which is plotted in Fig. 10.5. Note how the vector field goes to zero along the line $x = -y$. But there is no singularity here, because a zero vector has no direction.

Example 3: Construct a vector field that points tangential to a circle and has a magnitude given by the distance from the origin.

The vector field is pointing in a tangential direction. This is given by a vector field that behaves as $-y\hat{i} + x\hat{j}$. Computing the length of the vector at that point, we find its length is r. So this is the vector field we are after. It is already plotted in Fig. 10.2.

$$\vec{E}(x,y) = -y\hat{i} + x\hat{j}. \tag{10.4}$$

We are not yet going to define the mathematical terms of divergence and of curl, but we will depict them in images. The divergence is like an "explosion". The arrows all point outwards as if the origin is a source and

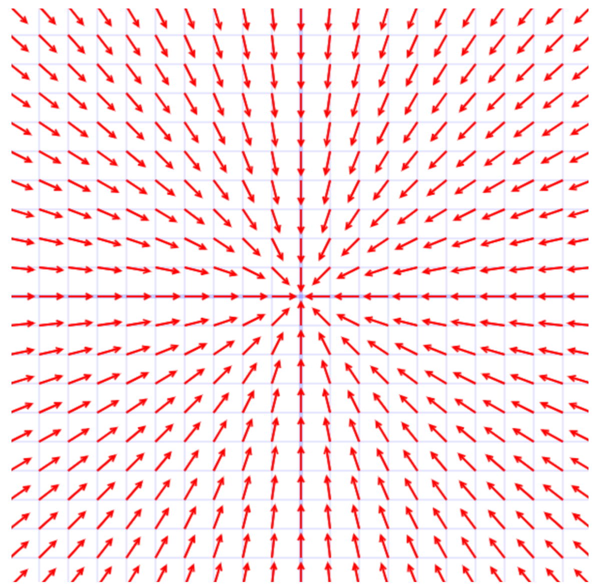

Figure 10.6: Example of a drain. This case is a singular case because the arrows all have the same magnitude. The image of water flowing into a drain at the origin is clear.

everything moves outwards away from it. We already have seen two images of this, in Fig. 10.1 and Fig. 10.4. The difference between these two is just the magnitudes of the arrows. In one case they approach zero at the origin (and there is no singularity), while in the other case they are the same magnitude as we approach the origin (and there is a singularity).

The opposite of a source is a drain. Here, everything flows into the origin, just like water flowing down a drain. We show an example in Fig. 10.6 of the singular version of this vector fields, where every term has a unit magnitude.

The concept of a curl is a bit more complicated. We have one example already in Fig. 10.2 that clearly rotates and has a curl that is nonzero. But there are some vector fields that look like they rotate, but actually have no curl. So, once we define curl, one needs to carefully use the definition to determine whether a vector field actually has a nonzero curl.

We do have another example of a curl, which may not seem obvious at all. This is shown in Fig. 10.7. While there is no clear rotation of these vectors, if we look at the horizontal row of vectors and we move down the page, we clearly see an over all rotation in a counter-clockwise direction by a total of

180°. It turns out that this vector field also has a nonzero curl.

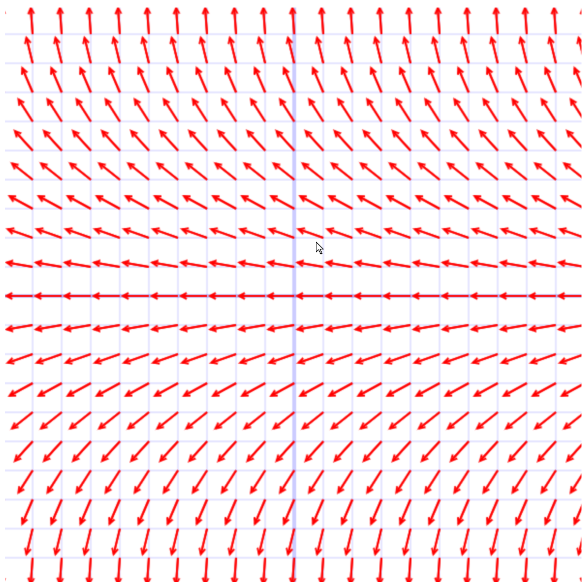

Figure 10.7: A vector field with nonzero curl but no obvious global rotation to it. The rotation is from one horizontal row to another.

We end with a simple summary of our terms:

Vector fields—a vector-valued function at each point in space.

Electric field—a force on an infinitesimal test charge divided by the magnitude of the test charge.

Electric force—the electric field times the charge.

One can compute the electric field from the charge density of a set of charged objects. But the integral is not easy to evaluate, as we mentioned above.

Chapter 11

Surface Integrals

11.1 How to construct the integral of a vector on a surface

We have already learned about vector-valued functions. Suppose $\vec{F}(x, y, z)$ is such a vector-valued function. While one could imagine integrating the individual components of $\vec{F}$ over the surface, in many applications, it is more appropriate to integrate a scalar over the surface. This requires us to construct a scalar at each point of the vector field. We do this by taking the dot product with the unit outward pointing normal on the surface. The surface integral over a surface S is then defined to be

$$\int_S \vec{F}(x, y, z) \cdot \hat{n}(x, y, z)\, dS \tag{11.1}$$

where $\hat{n}$ is the unit normal vector at the point (x, y, z) on the surface (pointing in an *outward* direction). This is illustrated in Fig. 11.1. We take the dot product (which yields a scalar number at each point) and sum over the surface. Note that this procedure, in principle, is quite straightforward to implement, but one has to be careful to ensure that it is done correctly.

The book by Schey describes in detail how one constructs the unit vector $\hat{n}(x, y, z)$ for a surface defined by $z = f(x, y)$. The result is

$$\hat{n}(x, y, z) = \frac{-\hat{i}\frac{df}{dx} - \hat{j}\frac{df}{dy} + \hat{k}}{\sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}}. \tag{11.2}$$

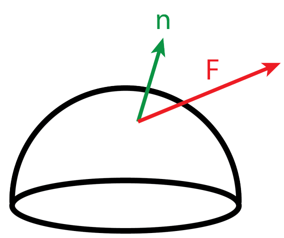

Figure 11.1: Normal vector (green) and vector field (red) on the surface of a hemisphere.

The unit vector is found by determining the vector perpendicular to the two linearly independent tangent vectors to the surface, as described in *Div, Grad, and Curl*. To perform the integration, we break the surface up into tiny surface elements, each of area ΔS_i, and compute the vector function at (x_i, y_i, z_i) in the center of each surface patch ΔS. Then we find the unit vector $\hat{n}$, take the dot product with the vector field F, multiply by ΔS_i and add up the results of all of the patches

$$\int_S \vec{F}(x,y,z)\cdot\hat{n}\,dS \sim \sum_i \Delta S_i \vec{F}(x_i,y_i,z_i)\cdot\hat{n}(x_i,y_i,z_i). \tag{11.3}$$

This is illustrated in Fig. 11.2.

Let us next evaluate the surface area of a hemisphere of radius R. The image of the hemisphere can be found in Fig. 11.1. Since the integral involves the patch area multiplied by $\hat{n}\cdot F$. If we choose $\vec{F}=\hat{n}$, then the scalar number we multiply the area patch magnitude by will be 1 and we will be calculating the surface area of the hemisphere. We will work in spherical coordinates for the hemispherical cap and polar coordinates for the circular bottom. The integral separates into two pieces as follows:

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_0^{\frac{\pi}{2}} R^2\sin\theta\,d\theta\int_0^{2\pi} d\phi\,\hat{n}\cdot\hat{n} + \int_0^R dr\int_0^{2\pi} r\,d\theta\,\hat{n}\cdot\hat{n}. \tag{11.4}$$

The area element is $R\,d\theta\,R\sin\theta\,d\phi$ for the spherical cap as worked out before, and $dr\,r\,d\theta$ for the circle. We also have $\hat{n}\cdot\hat{n}=1$ since $\hat{n}$ is a unit vector; note that in the clever way we have worked out this integral, we never need

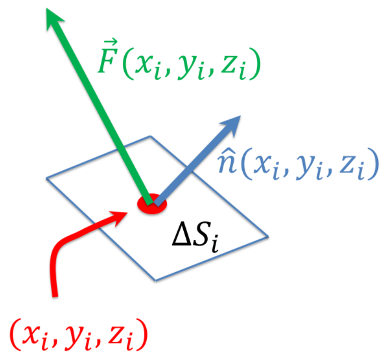

Figure 11.2: Strategy for calculating the surface integral. We first create the area patches ΔS_i. Then, using the normal vector to the patch $\hat{n}$, we take the dot product of the normal with the vector field. This is a number that we multiply by the patch area and add up to determine the total integral.

to determine $\hat{n}(x, y, z)$, because its square is always 1. So, after performing the integrals, we have

$$\int_S \vec{F} \cdot \hat{n}\, dS = R^2 \left(-\cos\theta \Big|_0^{\frac{\pi}{2}} \right) 2\pi + \frac{r^2}{2}\Big|_0^R 2\pi \tag{11.5}$$

$$= 2\pi R^2 + \pi R^2 \tag{11.6}$$

$$= 3\pi R^2. \tag{11.7}$$

Let's redo the same problem, but now with $\vec{F} = x\hat{i} = R\sin\theta\cos\phi$. In this case, we do need to compute the normal vector. On the spherical hemisphere part of the integral, we have

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \tag{11.8}$$

$$= \frac{R\sin\theta\cos\phi\hat{i} + R\sin\theta\sin\phi\hat{j} + R\cos\theta\hat{k}}{R} \tag{11.9}$$

$$= \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k} \tag{11.10}$$

and on the circular bottom, we have $\hat{n} = -\hat{k}$ (recall that the normal points outwards for these integrals). So on the hemisphere, the integral becomes

$$\vec{F} \cdot \hat{n} = R\sin^2\theta\cos^2\phi \tag{11.11}$$

and on the circle $\vec{F} \cdot \hat{n} = 0$. This yields

$$\int_S \vec{F} \cdot \hat{n}\, dS = R^2 \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi\, \sin\theta \sin^2\theta \cos^2\phi + \int_0^R dr \int_0^{2\pi} r\, d\phi \times 0 \tag{11.12}$$

$$= R^2 \int_0^1 \left(1 - \cos^2\theta\right)\, d\cos\theta \int_0^{2\pi} \cos^2\phi\, d\phi \tag{11.13}$$

$$= R^2 \left(1 - \frac{1}{3}\right)\left(2\pi\frac{1}{2}\right) \tag{11.14}$$

$$= \frac{2}{3}\pi R^2. \tag{11.15}$$

Note how we converted the integral over θ into an integral over $\cos\theta$. It usually is easier to do it this way.

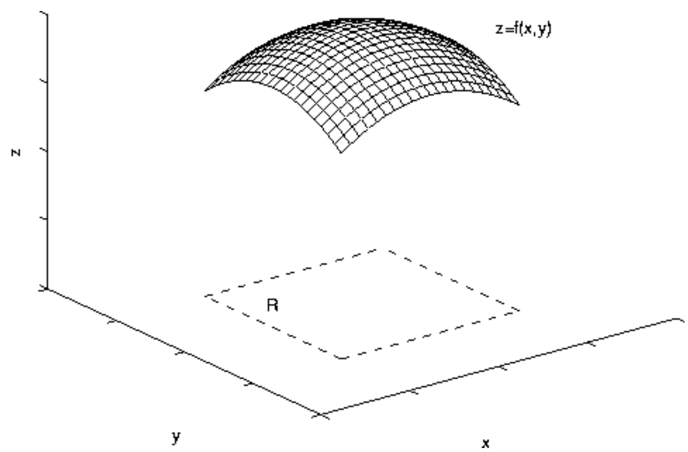

Figure 11.3: Surface integral projection onto a plane. We divide the surface into little patches on the surface and then map those little surfaces onto the plane below in order to integrate. The little patches need to have their area corrected if they lie at an angle to the plane that they are projected to below (see the next figure).

The point of these examples is that surface integrals are not frightening if you just work out all of the terms carefully and reduce them to integrals we already know how to do.

There is one other technical point we need to sort out. If we want to project the surface onto the plane, we map the dS elements onto the plane by projection.

$$dS = \frac{dx\,dy}{\cos\theta} = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} \tag{11.16}$$

See Figs. 11.3 and 11.4 to see how this works.

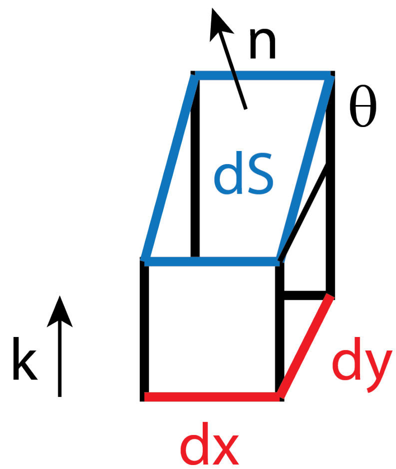

Figure 11.4: Schematic showing how the projection of the surface element dS at an angle of θ to the horizontal maps onto the element $dxdy$ in the plane. We see that $dS = dxdy/\cos\theta = dxdy/(\hat{n}\cdot\hat{k})$.

Recall for a surface $z = f(x, y)$, we had

$$\hat{n} = \frac{-\hat{i}\frac{df}{dx} - \hat{j}\frac{df}{dy} + \hat{k}}{\sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}}. \tag{11.17}$$

So we also have

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}}. \tag{11.18}$$

This then implies that (see Fig. 11.4)

$$dS = \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2}\,dx\,dy, \tag{11.19}$$

which is very similar to how we calculated arc lengths.

In summary, the general form of the integral is

$$\int_S \vec{F}\cdot\hat{n}\, dS = \int dx\, dy\, \vec{F}\,(x, y, f(x, y))\cdot\hat{n}\,(x, y, f(x, y)) \sqrt{1 + \left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2} \tag{11.20}$$

The final thing we look at in this chapter is Gauss' Law, which is

$$\int_S \vec{E}\cdot\hat{n}\, dS = \frac{q_{\text{enclosed}}}{\epsilon_0} = \int dV\,\frac{\rho}{\epsilon_0}, \tag{11.21}$$

where ρ is the charge density. We want to examine what happens "locally." We examine a small volume and a small surface area.

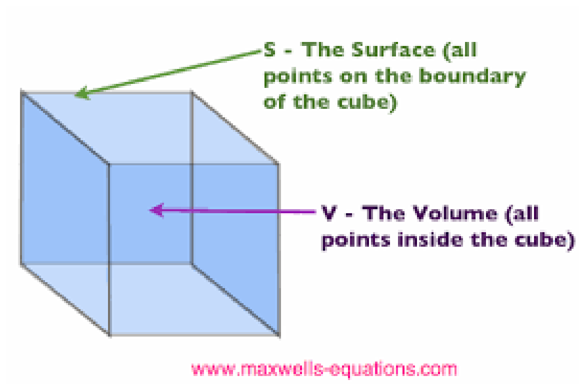

Figure 11.5: Volume and surface elements used in Gauss' law. The point (x, y, z) lies at the center of the cube.

Here, dS is the area of the prism and dV is the volume. This tells us that

$$\int_S \vec{E}\cdot\hat{n}\, dS = \int dV\,\frac{\rho}{\epsilon_0} = \int \frac{\rho(x, y, z)}{\epsilon_0}\, dV \tag{11.22}$$

We define

$$\frac{1}{\delta V}\int_{\delta S} \vec{E}\cdot\hat{n}\, dS = \text{“divergence”}. \tag{11.23}$$

We will see in the next chapter an alternative way to compute divergence, since the expression above is painful to try to evaluate.

Chapter 12

The Divergence Theorem

12.1 Defining divergence

Recall from last lecture, where we showed if we consider a small rectangular prism about the point (x_0, y_0, z_0) with widths dx, dy, dz, then:

$$\frac{1}{\delta V} \int_{\delta S} \vec{F} \cdot \hat{n}\, ds = \frac{\rho(x,y,z)}{\epsilon_0} \tag{12.1}$$

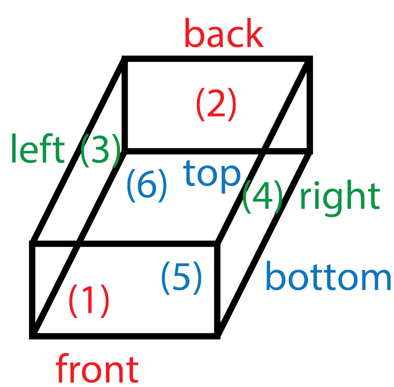

Figure 12.1: Rectangular prism employed for defining the divergence.The sides are labeled as follows: (1) front; (2) back; (3) left; (4) right; (5) bottom and (6) top.

There are 6 faces to the prism. Let's do the surface integral::

$$\int_{\delta S} \vec{F}\cdot\hat{n}\,ds = \int_{S_1} \vec{F}\cdot\hat{n}\,ds + \int_{S_2} \vec{F}\cdot\hat{n}\,ds + \int_{S_3} \vec{F}\cdot\hat{n}\,ds + \int_{S_4} \vec{F}\cdot\hat{n}\,ds + \int_{S_5} \vec{F}\cdot\hat{n}\,ds + \int_{S_6} \vec{F}\cdot\hat{n}\,ds. \tag{12.2}$$

We denoted the corresponding contributions according to the colors in Fig. 12.1.

side	normal	coordinate	area
1	$-\hat{j}$	$x\hat{i} + \left(y - \frac{dy}{2}\right)\hat{j} + z\hat{k}$	$dx\,dz$
2	$\hat{j}$	$x\hat{i} + \left(y + \frac{dy}{2}\right)\hat{j} + z\hat{k}$	$dx\,dz$
3	$-\hat{i}$	$\left(x - \frac{dx}{2}\right)\hat{i} + y\hat{j} + z\hat{k}$	$dy\,dz$
4	$\hat{i}$	$\left(x + \frac{dx}{2}\right)\hat{i} + y\hat{j} + z\hat{k}$	$dy\,dz$
5	$-\hat{k}$	$x\hat{i} + y\hat{j} + \left(z - \frac{dz}{2}\right)\hat{k}$	$dx\,dy$
6	$\hat{k}$	$x\hat{i} + y\hat{j} + \left(z + \frac{dz}{2}\right)\hat{k}$	$dx\,dy$

Using the results of the table above, we can directly compute the total surface integral for the small volume element of the rectangular prism:

$$\begin{aligned}\int_S \vec{F}\cdot\hat{n}\,ds = &\left[F_y\left(x, y+\frac{dy}{2}, z\right) - F_y\left(x, y-\frac{dy}{2}, z\right)\right] dx\,dz \\ &+ \left[F_x\left(x+\frac{dx}{2}, y, z\right) - F_x\left(x-\frac{dx}{2}, y, z\right)\right] dy\,dz \\ &+ \left[F_z\left(x, y, z+\frac{dz}{2}\right) - F_z\left(x, y, z-\frac{dz}{2}\right)\right] dx\,dy.\end{aligned} \tag{12.3}$$

Using $\delta V = dx\,dy\,dx$, we define the divergence to be

$$\left(\frac{1}{\delta V}\int_{\delta S} \vec{F}\cdot\hat{n}dS\right) = \text{ divergence}, \tag{12.4}$$

which then becomes

$$\text{divergence} = \left(\frac{\partial F_y}{\partial y} + \frac{\partial F_x}{\partial x} + \frac{\partial F_z}{\partial z}\right)\bigg|_{(x,y,z)}. \tag{12.5}$$

This can be rewritten as

$$\text{divergence} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{F}(x,y,z) = \vec{\nabla}\cdot\vec{F}(x,y,z). \tag{12.6}$$

This then leads to our final result for the divergence

$$\vec{\nabla} \cdot \vec{F}(x,y,z) = \frac{\partial}{\partial x} F_x(x,y,z) + \frac{\partial}{\partial y} F_y(x,y,z) + \frac{\partial}{\partial z} F_z(x,y,z), \tag{12.7}$$

with $\vec{\nabla}$ called the divergence operator.

Using this new divergence notation allows us to re-express Gauss' law as

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}. \tag{12.8}$$

12.2 The divergence theorem

Consider a closed surface surrounding a volume V:

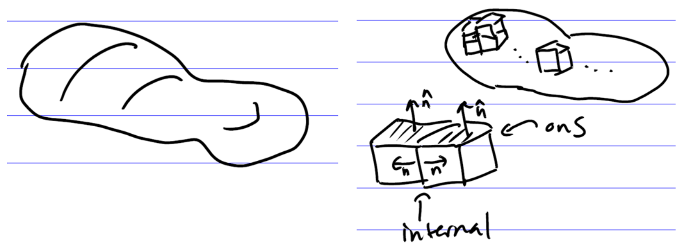

Figure 12.2: On the left, we show a general volume that will be integrated. On the right, it is broken into small volumes for carrying out the integration. Note how the outward pointing normals cancel for all internal boundaries, but are unpaired for the outer boundaries.

Any face on S has a unit normal which points out and accumulates $\vec{F} \cdot \hat{n}$ terms in the integral, but all internal faces have two unit normals $\hat{n}$, one in each direction (+ and -) on the boundary surfaces of two adjacent prisms, which *cancel* when integrated over their respective surfaces. So we learn that

$$\int_S \vec{F} \cdot \hat{n}\, dS = \sum_i \int_{S_i} \vec{F} \cdot \hat{n}\, dS, \tag{12.9}$$

where S_i is the surface of one of the cubes in the volume. But we know for each small cube:

$$\int_{S_i} \vec{F} \cdot \hat{n} dS = \int_{V_i} \vec{\nabla} \cdot \vec{F}\, dV. \tag{12.10}$$

Since $\frac{1}{V_i}\int_{S_i} \vec{F}\cdot\hat{n}\,dS = \vec{\nabla}\cdot\vec{F}$, hence

$$\int_S \vec{F}\cdot\hat{n}\,dS = \sum_i \int_{S_i} \vec{F}\cdot\hat{n}\,dS = \sum_i \int_{V_i} \vec{\nabla}\cdot\vec{F}\,dV = \int_V \vec{\nabla}\cdot\vec{F}\,dV \tag{12.11}$$

or,

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_V \vec{\nabla}\cdot\vec{F}\,dV, \tag{12.12}$$

which is called the divergence theorem.

Example: This is a cute example that allows you to compute a volume by integrating a surface area.

$$\vec{F} = \vec{r} = xi + yj + zk \tag{12.13}$$

$$\vec{\nabla}\cdot\vec{F} = 1+1+1 = 3 \tag{12.14}$$

$$\frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS = \frac{1}{3}\int_V \vec{\nabla}\cdot\vec{r}dV = \frac{1}{3}\int_V 3dV = V \tag{12.15}$$

so,

$$\frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS = V \tag{12.16}$$

Check: Volume of a sphere

$$V = \frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS \tag{12.17}$$

with $\hat{n} = \hat{r} = \vec{r}/r =$ unit vector in radial direction, so $\vec{r}\cdot\hat{n} = \vec{r}\cdot\vec{r}/r = r$ and we have

$$V = \frac{1}{3}\int_S rdS. \tag{12.18}$$

The r comes out of the integral because we are integrating over the surface, which has a fixed radius. We obtain

$$V = \frac{r}{3}\int_S dS = \frac{r}{3}\times 4\pi r^2 = \frac{4}{3}\pi r^3. \tag{12.19}$$

This checks out!

Check: Rectangular prism, given in Fig. 12.1.

Start by identifying the six different normal vectors and the areas of each of the faces. Then we assemble the results via

$$V = \frac{1}{3}\int_S \vec{r}\cdot\hat{n}\,dS = \frac{1}{3}\left[-y_1 ab + y_2 ab - x_3 ac + x_4 ac - z_5 bc + z_6 bc\right], \quad (12.20)$$

since the normals point as described in the table after Eq. (12.2). But, $y_2 - y_1 = c$, $x_4 - x_3 = b$, $z_6 - z_5 = a$ and

$$V = \frac{1}{3}\int_S \vec{r}\cdot\hat{n}dS = \frac{1}{3}abc3 = abc. \quad (12.21)$$

We end by discussing divergence in cylindrical and spherical coordinates. Cylindrical:

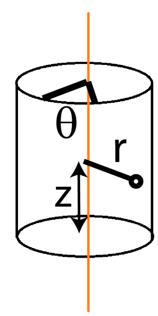

Figure 12.3: Schematic of cylindrical coordinates ρ, θ, and z.

The unit vectors are as follows:

$$\hat{e_z} = \hat{k} = \text{ unit vector in the } z \text{ direction;} \quad (12.22)$$

$$\hat{e_r} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i} + y\hat{j}}{r} = \text{ unit vector in the } r \text{ direction;} \quad (12.23)$$

and

$$\hat{e_\theta} = \frac{y\hat{i} - x\hat{j}}{r} = \text{ unit vector perpendicular to } z \text{ and } r. \quad (12.24)$$

The divergence in cylindrical coordinates is then

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}. \quad (12.25)$$

For spherical coordinates, the book *Div, Grad, Curl, and all that* interchanges θ and ϕ—this is common, math uses the notation in the book and physics uses the interchanged notation. We use our notation, not the book's, because ours is more standard for physics. Yes, it is a pain, yes it will cause confusion. But you need to learn the language of physicists as you become one.

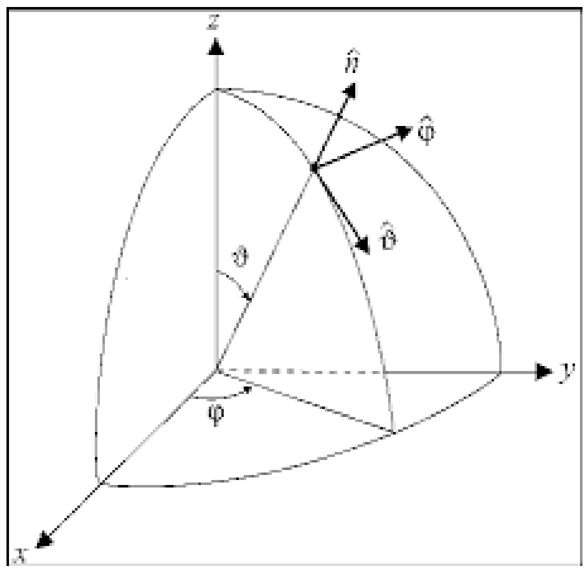

Figure 12.4: Physicist's notation for spherical coordinates (r, θ, ϕ) and the corresponding unit vectors.

The unit vectors satisfy

$$\hat{e_\phi} = \hat{e_r} \times \hat{e_\theta}. \tag{12.26}$$

The radial direction is

$$\hat{e_r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \tag{12.27}$$

and the *theta* direction is

$$\hat{e_\theta} = z\frac{(x\hat{i} + y\hat{j})}{r\sqrt{x^2+y^2}} - \hat{k}\frac{\sqrt{x^2+y^2}}{r}. \tag{12.28}$$

We need to work out the above cross product to get the ϕ direction (it lies in the x-y plane).

In spherical coordinates, the divergence then becomes

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta) F_\theta) + \frac{1}{r\sin(\theta)}\frac{\partial F_\phi}{\partial \phi} \tag{12.29}$$

which is $\theta \to \phi$, $\phi \to \theta$ from the book result.

Chapter 13

The Line Integral and the Curl

13.1 Line Integrals

We have already investigated line integrals when we looked at calculating the arc length. We found

$$S = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx \tag{13.1}$$

And the picture in Fig. 13.1 should remind you why this was this way.

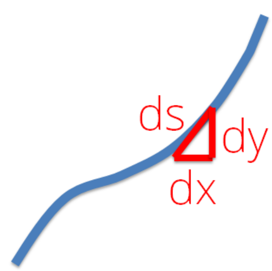

Figure 13.1: Schematic of the line integral. The blue curve is the continuous function $y(x)$ that we are seeking the arc length of. The red triangle shows how to relate the differential of the arc length to the differentials in the x and y directions.

Using the Pythagorean theorem, we immediately see that $ds^2 = dx^2 + dy^2$

or,

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{13.2}$$

when the curve is expressed in the form $y = f(x)$.

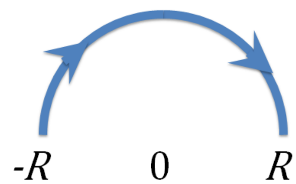

Figure 13.2: Line integral curve for the integration over a semicircle. One must separate a full circle into pieces like this, because a function is single-valued (one value $y = f(x)$ for each x); otherwise, it is not well-defined.

Let's be concrete and work out the example of the line integral over a semicircle of radius r in the clockwise direction (see Fig. 13.2). Here The y-component of the semicircle satisfies

$$y = f(x) = \sqrt{R^2 - x^2}. \tag{13.3}$$

Computing the derivative yields

$$\frac{df(x)}{dx} = -\frac{x}{\sqrt{R^2 - x^2}}. \tag{13.4}$$

Plugging into the formula for the arc length then becomes

$$S = \int_{-R}^{R} \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \int_{-R}^{R} \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx \tag{13.5}$$

$$= \int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}}\, dx. \tag{13.6}$$

This integral can be evaluated directly via a simple trigonometric substitution. Let $x = R\sin\theta$, so that $dx = R\cos\theta\, d\theta$. Then we have that the arc length becomes

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{R^2\cos\theta}{R\sqrt{1 - \sin^2\theta}} = R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi R, \tag{13.7}$$

exactly as expected (recall that the circumference of a circle of radius R is $2\pi R$).

The wrinkle we have now for the line integrals we want to consider is that we are integrating functions *that have vector dot products* as the integrand. Consider the integral over the same curve, but now the integrand is not the scalar 1, but is the vector dot product given by $\vec{F}(x) \cdot \vec{t}$ where $\vec{F}(x) = -y\hat{\imath} + x\hat{\jmath} = -\sqrt{R^2 - x^2}\,\hat{\imath} + x\hat{\jmath}$.

The unit vector $\hat{t}$ for each step dx is taken to be the tangent to the curve at the point $(x, f(x)) = (x, y)$. This tangent vector is proportional to $\hat{\imath} + \frac{df}{dx}\hat{\jmath} = \hat{\imath} - \frac{x}{\sqrt{R^2-x^2}}\hat{\jmath}$. Putting this all together gives us

$$\hat{t} = \frac{\sqrt{R^2 - x^2}\,\hat{\imath} - x\hat{\jmath}}{R}. \tag{13.8}$$

So $\vec{F}(x) \cdot \hat{t} = -\sqrt{R^2 - x^2}\frac{\sqrt{R^2-x^2}}{R} - \frac{x^2}{R} = \frac{x^2-R^2+x^2}{R} = -R$, which turns out to be independent of x. This won't always happen. So the inegral then becomes

$$\int \vec{F} \cdot \hat{t}\, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx(-R)\frac{R}{R^2 - x^2} = -R(\pi R) = -\pi R^2 \tag{13.9}$$

The key to calculating these integrals is to carefully work out each term and then you are left with an ordinary integral at the end (which you should be able to evaluate). Note that there are many steps. Just work through these problems slowly and remember all the different steps you need to assemble to finish the evaluation of the final result.

We consider another example. If $\vec{F}(\vec{x})$ is proportional to $\hat{r} = \frac{\hat{\imath}x+\hat{\jmath}y+\hat{k}z}{\sqrt{x^2+y^2+z^2}} = \hat{\imath}\frac{x}{r} + \hat{\jmath}\frac{y}{r} + \hat{k}\frac{z}{r}$ and is a function of r only, then $\int \vec{F} \cdot \vec{t}\, ds$ depends only on the initial and final points.

Proof:

$$\int_1^2 \vec{F}(r) \cdot \hat{t}\, ds = \int_1^2 \left[F_x(r)dx + F_y(r)dy + F_z(r)\right] dz. \tag{13.10}$$

This form follows because $ds = \sqrt{dx^2 + dy^2 + dz^2}$ and $\hat{t} = \hat{\imath}\frac{dx}{ds} + \hat{\jmath}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}$. Thus, we find the unit vector times the differential of the arc length becomes

$$\hat{t}\, ds = \left(\hat{\imath}\frac{dx}{ds} + \hat{\jmath}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}\right) ds = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz. \tag{13.11}$$

Taking the dot product with $\vec{F}(r)$ then gives the integral in Eq. (13.10). But, we also have that $dr = \frac{dr}{dx}dx + \frac{dr}{dy}dy + \frac{dr}{dz}dz = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$, since $r = \sqrt{x^2 + y^2 + z^2}$, so our specific vector field can also be expressed as

$$\vec{F}(r) = \left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right) f(r) \tag{13.12}$$

according to the hypothesis of the problem. Using the result in Eq. (13.11) yields the following for the dot product

$$\vec{F}(r) \cdot \hat{t}\, ds = f(r) \left(\frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz\right) = f(r)\, dr. \tag{13.13}$$

This then means that we can re-express the integral as

$$\int_1^2 \vec{F}(\vec{r}) \cdot \hat{t}\, ds = \int_1^2 f(r)\, dr, \tag{13.14}$$

which is a function of r_1 and r_2 only (by the fundamental theorem of calculus). So the integral cares only about endpoints not the paths between them. If the endpoints are the same, we must have that the integral vanishes

$$\oint \vec{F}(\vec{r}) \cdot \hat{t}\, ds = 0. \tag{13.15}$$

So a function $\vec{F}(\vec{r})$ that can be expressed in th form $f(r)\hat{r}$ has no circulation, since $\oint \vec{F}(\vec{r}) \cdot \hat{t}\, ds = 0$. The electrostatic field is an example of a vector field with this property. It has no circulation.

13.2 Curl

The curl will be defined analogous to the way we defined the divergence. Examine the circulation over a small path and divide the circulation by the area enclosed by the path. You have a problem like this that you will have to do on the homework. Here we examine a simplification of the general definition and examine the curl in the $x - y$ plane using a rectangle for the path (see Fig. 13.3).

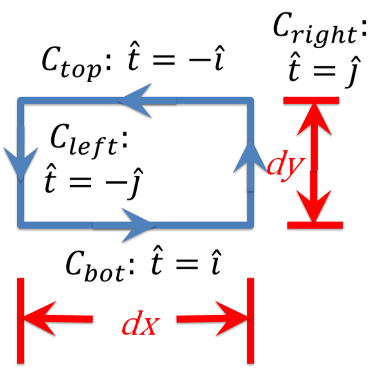

Figure 13.3: Example of how one can define the curl.

Since dx and dy are small, we immediately see that we can approximate $\oint \vec{F}(x,y,z) \cdot \hat{t}\, ds$ by

$$\oint \vec{F}(x,y,z) \cdot \hat{t}\, ds = \int_{c_{\text{top}}} \vec{F} \cdot \hat{t}\, ds + \int_{c_{\text{left}}} \vec{F} \cdot \hat{t}\, ds + \int_{c_{\text{bottom}}} \vec{F} \cdot \hat{t}\, ds + \int_{c_{\text{right}}} \vec{F} \cdot \hat{t}\, ds$$

$$= -F_x\left(x, y + \frac{\Delta y}{2}, z\right)\Delta x - F_y\left(x - \frac{\Delta x}{2}, y, z\right)\Delta y$$

$$+ F_x\left(x, y - \frac{\Delta y}{2}, z\right)\Delta x + F_y\left(x + \frac{\Delta x}{2}, y, z\right) \tag{13.16}$$

$$= \Delta x \Delta y \left[\frac{F_x\left(x, y - \frac{\Delta y}{2}, z\right) - F_x\left(x, y + \frac{\Delta y}{2}, z\right)}{\Delta y} + \right.$$

$$\left. \frac{F_y\left(x + \frac{\Delta x}{2}, y, z\right) - F_y\left(x - \frac{\Delta x}{2}, y, z\right)}{\Delta x}\right] \tag{13.17}$$

$$= \Delta x \Delta y \left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)(x,y,z). \tag{13.18}$$

Similarly, if we did this about rectangles in the $y - z$ or $z - x$ planes, we would get

$$\Delta y \Delta z \left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right)(x,y,z) \tag{13.19}$$

and

$$\Delta z \Delta x \left(\frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x}\right)(x,y,z) \tag{13.20}$$

We describe the rectangles by their unit normals that we get by curling the fingers of our right hand in the direction of the path of the integral, so the thumb points in the direction of the normal to the enclosed area. So the xy rectangle is associated with $\hat{k}$, the yz rectangle with $\hat{i}$, and the xz rectangle with $\hat{j}$. These results are summarized in Fig. 13.4.

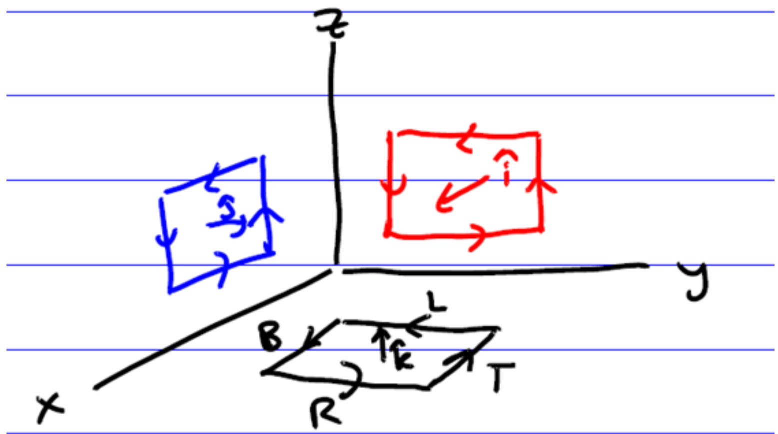

Figure 13.4: Schematic of how to determine the curl using rectangular paths in the three different Cartesian planes.

The curl of $\vec{F}$ is a vector, so we multiply our calculations by the respective normals and add. Schematically (see Fig. 13.4), we have

$$\oint \frac{\vec{F} \cdot \hat{t}\, ds}{\text{area enclosed}} \implies \hat{i}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) + \hat{j}\left(\frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x}\right) + \hat{k}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right). \tag{13.21}$$

Because the area and the path both $\to 0$, we call

$$\lim_{\to 0} \oint \frac{\vec{F} \cdot \hat{t}], ds}{\text{area enclosed}} = \text{curl}\, \vec{F} = \vec{\nabla} \times \vec{F}. \tag{13.22}$$

Since

$$\det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) + \hat{j}\left(\frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x}\right) + \hat{k}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right), \tag{13.23}$$

which is the same as the expression in Eq. (13.21). Hence, the familiar formula for the curl, given by $\nabla \times \vec{F} = \det |...|$.

Chapter 14

Stokes Theorem

14.1 Defining Stokes Theorem

Stokes theorem extends the result that holds for small surfaces given by

$$\oint \vec{F} \cdot \hat{t}\, ds = \int\int \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS \tag{14.1}$$

to large surfaces. The first integral is a line integral of the vector field dotted into the tangent vector of the curve. The second integral is an area integral of the curl of the vector field dotted into the normal unit vector perpendicular to the area.

Focus on Fig. 14.1. It shows a surface, denoted by the symbol Σ, a normal vector at one point, denoted by $\hat{n}$, and the boundary of the area, denoted $\partial\Sigma$. We consider a line integral over the closed path $\partial\Sigma$.. A capping area is defined to be any area that has the above curve as its boundary. The area given by Σ is an example of a capping area.

If you are having any trouble with this concept, think of putting a rubber sheet or a soap film with its boundary constrained to lie of $\partial\Sigma$. Any shape you can deform to is a capping area.

Stokes theorem says

$$\oint \vec{F} \cdot \hat{t}\, ds = \int\int_{\text{capping area}} \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS \tag{14.2}$$

To prove Stokes theorem, break the capping area into small pieces. For each

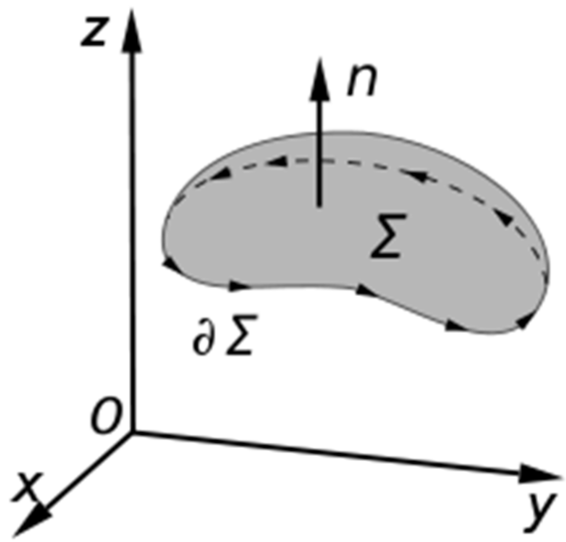

Figure 14.1: Schematic of Stokes theorem. The curve, denoted $\partial\Sigma$, is the line integral over the boundary of the area. The area, denoted Σ, can be thought of as any area that has the underlying boundary curve as its edge.

small piece, or area $\triangle S_i$, we already showed

$$\oint_{\text{piece i}} \vec{F} \cdot \hat{t}\, ds = \int_{\triangle S_i} \vec{\nabla} \times \vec{F} \cdot \hat{n}\, dS. \tag{14.3}$$

Fig. 14.2 shows that when you hook line integrals over different areas together, the integrals over the interior paths cancel (red lines in Fig. 14.2).

So, when we add all up all of the paths, we get a curve that goes around the boundary. We add all areas up and we obtain the surface integral over the full surface. This is Stokes theorem.

One of the hardest concepts to become comfortable with is given by the question What is the meaning of curl? In essence, curl means a vector field has *some kind of rotational character*, but this is subtle. We show four examples of the curl in the next section. You will clearly see that the notion of a curl is not always so obvious.

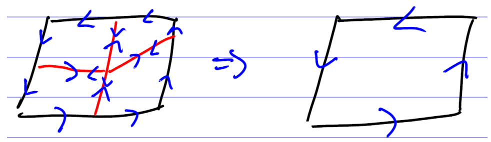

Figure 14.2: Figure illustrating the crux of Stokes theorem. As we integrate over the small line integrals, we find the internal lines (given in red) will have their integrals all cancel, because they are integrated over in both directions. This means the sum over all of those integrals is given by the line integral over the boundary of the curve only,

14.2 Examples of Stokes Theorem

We describe four examples of the curl of vector fields.

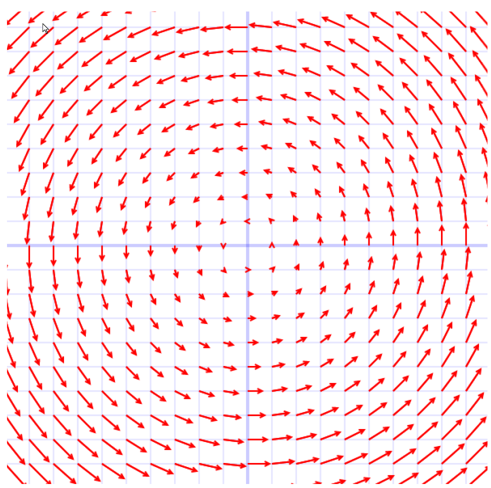

Figure 14.3: The vector field given by $\vec{F} = -y\hat{i} + x\hat{j}$.

Example 1: $\vec{F} = -y\hat{i} + x\hat{j}$ (see Fig. 14.3).

This vector field obviously rotates. Computing the curl (by using the definition in terms of derivatives) yields

$$\vec{\nabla} \times \vec{F} = \hat{k}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) = 2\hat{k}. \quad (14.4)$$

Indeed, this vector field, which clearly rotates, has a nonzero curl.

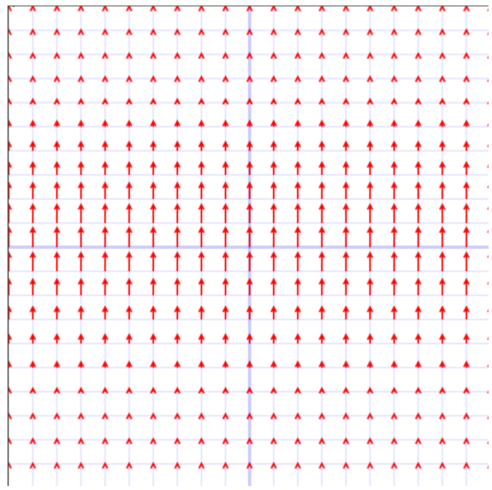

Figure 14.4: The vector field given by $\vec{F} = \hat{\jmath} e^{-0.05y^2}$.

Example 2: $\vec{F} = \hat{\jmath} e^{-0.05y^2}$ (see Fig. 14.4).
Our second example has its vector field satisfy

$$\vec{\nabla} \times \vec{F} = 0, \tag{14.5}$$

because F_y has no x or z dependence. Clearly for this vector field, we can see that there is no rotation, because the direction of the vectors never changes. So this concept of associating curl with rotation seems to work well.

Example 3: $\vec{F} = \hat{\jmath} e^{-0.05x^2}$ (see Fig. 14.5).
The curl for this vector field is given by

$$\vec{\nabla} \times \vec{F} = \hat{k}(-0.1xe^{-x^2}). \tag{14.6}$$

This curl is clearly $\neq 0$. But this "rotation" is subtle, since the direction of the vector field never changes either.

Example 4: $\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$, with $(x, y) \neq (0, 0)$ (see Fig. 14.6).
This vector field clearly looks like it has rotation, but,

$$\vec{\nabla} \times \vec{F} = \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \tag{14.7}$$

Evaluating the derivatives gives us

$$\vec{\nabla} \times \vec{F} = \hat{k}\left[\frac{1}{x^2+y^2} - \frac{x \times 2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y \times 2y}{(x^2+y^2)^2}\right]. \tag{14.8}$$

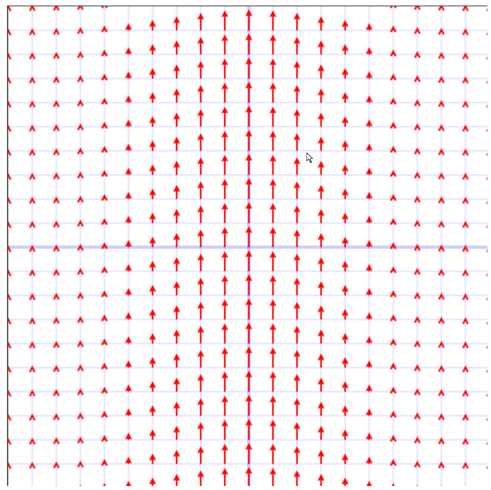

Figure 14.5: The vector field given by $\vec{F} = \hat{j}e^{-0.05x^2}$.

Simplifying, we find

$$\vec{\nabla} \times \vec{F} = \hat{k}\frac{x^2 + y^2 - 2x^2 + x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = 0. \tag{14.9}$$

This shows us that curl is different from just thinking of it as the rotation of a vector field.

The concept and comfort level of the curl is one of the harder things ti develop as you work with vector fields. Take your time, think carefully, and you will be able to master it.

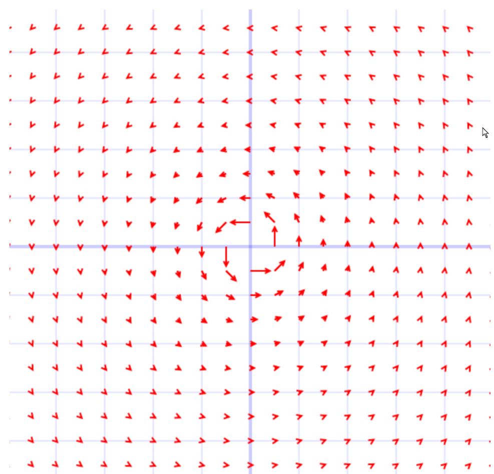

Figure 14.6: The vector field given by $\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$, with $(x, y) \neq (0, 0)$.

Chapter 15

Line Integrals and the Gradient

15.1 Path Independence of the Line Integral

Previously we saw that if $\int_a^b \vec{F}\cdot\hat{t}\,ds$ is independent of the path, then $\nabla\times F = 0$. The converse, $\nabla\times F = 0$ implies $\int_a^b \vec{F}\cdot\hat{t}\,ds$ is independent of the path required the region we were in to be "simply connected" which essentially means "has no holes." But we haven't focused on that too much because most physics problems are in simply connected regions of space.

Now we look at a third possibility. Suppose $\vec{F}(\vec{r}) = \nabla\psi(\vec{r})$ with ψ a scalar function and $\nabla = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$ the gradient operator. Note that the gradient operator converts a scalar function into a vector field, but the vector field is not arbitrary, it has constraints due to the fact that it is defined via the gradient operator. We let s be the arc length parametrization of the path $\vec{r}(s) = \hat{i}x(s) + \hat{j}y(s) + \hat{k}z(s)$. In this case, we then have that the tangent vector satisfies $\hat{t} = \hat{i}\frac{dx}{ds} + \hat{j}\frac{dy}{ds} + \hat{k}\frac{dz}{ds}$. (Note that arc length parameterization means the parameter s used to parametrically determine the curve via the three functions $(x(s), y(s), z(s))$ *is the arc length traveled along the path*; determining such a parameterization is not trivial to do, but we assume it has been done.) The scalar dot product of the vector field (defined via the gradient) with the tangent vector becomes

$$\vec{F}\cdot\hat{t} = \frac{d\psi}{dx}\frac{dx}{ds} + \frac{d\psi}{dy}\frac{dy}{ds} + \frac{d\psi}{dz}\frac{dz}{ds}. \tag{15.1}$$

It is clear, that if we think in terms of the chain rule, we have $\vec{F}\cdot\hat{t} = \frac{d\psi}{ds}$. This means we can integrate it immediately when we perform the line integral.

We find

$$\int_{(x,y,z)}^{(x',y',z')} \vec{F}\cdot\hat{t}\,ds = \int_{(x,y,z)}^{(x',y',z')} \frac{d\psi}{ds}\,ds = \psi(x',y',z') - \psi(x,y,z), \tag{15.2}$$

because it is a perfect differential. Since this depends only on the endpoints, it is *independent of the path*, as we claimed.

So if we integrate from some fixed point (x_0, y_0, z_0) to (x, y, z), we have

$$\tilde{\psi}(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F}\cdot\hat{t}\,ds \tag{15.3}$$

is a well defined function, because the integral is path independent; that is, for each point (x, y, z), the function is well defined and single valued. Now suppose we take the following specific paths: the first path c_1 is given by $(x_0, y_0, z_0) \to (x_0, y, z)$ and the secodn path c_2 is given by $(x_0, y, z) \to (x, y, z)$. The first integral path is independent of x (because $x = x_0$ throughout the path), so the derivative is nonzero only for the second path, or

$$\frac{d\tilde{\psi}}{dx} = \frac{d}{dx}\int_{c_2} \vec{F}\cdot\hat{t}\,ds = \frac{d}{dx}\int_{x_0}^{x} \vec{F}(\bar{x},y,z)\cdot\hat{\imath}\,d\bar{x} = F_x(x,y,z) \tag{15.4}$$

using similar paths for y and z, we get $\frac{d}{dy}\tilde{\psi}(x,y,z) = F_y$ and $\frac{d}{dz}\tilde{\psi}(x,y,z) = F_z$.

So this is how one can construct a scalar function ψ, whose gradient is $\vec{F}$. If $\int \vec{F}\cdot\hat{t}\,ds$ is independent of path, then we can always find a function ψ such that $\vec{F} = \vec{\nabla}\psi$. Be sure you can understand the construction of such a function.

Now, assume that the vector field is the gradient of a scalar function. We have the curl satisfies

$$\begin{aligned}
\nabla\times F &= \hat{\imath}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right) + \hat{k}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\\
&= \hat{\imath}\left(\frac{\partial^2\psi}{\partial y\partial z} - \frac{\partial^2\psi}{\partial z\partial y}\right) + \hat{\jmath}\left(\frac{\partial^2\psi}{\partial z\partial x} - \frac{\partial^2\psi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x}\right)\\
&= 0 \qquad\qquad (15.5)
\end{aligned}$$

because mixed second partial derivatives are independent of the order of the derivatives.

So we see that the three different results: (i) $\int_c \vec{F} \cdot \hat{t}\, ds$ is independent of path; (ii) $\nabla \times F = 0$ and (iii) $\vec{F} = \vec{\nabla}\psi$ are all equivalent. The issue of simple-connectedness is subtle and discussed further in *Div, Grad, and Curl.* One needs to understand it to properly apply the equivalence of these three results.

15.2 Laplace's and Poisson's Equations

We will apply this general result to Maxwell's equations and electric fields. This will also introduce us to Laplace's and Poisson's equations.

Consider a static electric field $\vec{E}$. Maxwell's equations tell us that $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \vec{E} = 0$. The fact that the curl vanishes implies that we can find a scalar potential such that $\vec{E} = -\nabla\phi$, which automatically gives $\nabla \times E = 0$. (Note that the scalar function here is defined with a minus sign. Obviously, it is a small generalization of the discussion given above.) Using the potential in the other Maxwell equation (Gauus' law) then gives $-\nabla \cdot \nabla\phi = \frac{\rho}{\epsilon_0}$ or $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$. The symbol ∇^2 is the Laplacian (often called "del squared"). We compute it just like you might think—it is the divergence of the gradient, or

$$\begin{aligned} \nabla^2\phi &= \nabla \cdot \nabla\phi && (15.6)\\ &= \nabla \cdot \left(\hat{\imath}\partial_x\,\phi + \hat{\jmath}\partial_y\,\phi + \hat{k}\partial_z\,\phi\right) && (15.7)\\ &= \partial_x^2\,\phi + \partial_y^2\,\phi + \partial_z^2\,\phi, && (15.8)\end{aligned}$$

where we used symbols $\partial_x = \frac{\partial}{\partial x}$ and the like as a shorthand notation for partial derivatives. Hence, we say that the Laplacian satisfies

$$\text{Laplacian} = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{15.9}$$

Poisson's equation is

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}, \tag{15.10}$$

and it is used in problems where one has free charges and we are trying to determine the scalar potential. But many electrostatic problems are to find the fields in a space where there are no free charges. Then we have Laplace's equation

$$\nabla^2\phi = 0. \tag{15.11}$$

In both of these cases, the idea is that we solve Poisson's or Laplace's equation for the scalar potential ϕ first, and then use the gradient to determine the electric field.

Both of these equations are not enough to determine the solution. They are subject to different boundary conditions. These boundary conditions are often given in terms of the potentials or fields at the boundaries surfaces of the problem (usually metals). It is easiest to see how this works by going through some examples.

Example 1

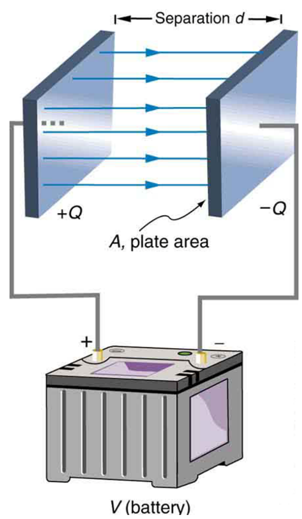

Figure 15.1: Parallel-plate capacitor set up. In a real capacitor, the plates are finite in size and of area A, as depicted in the above figure, but we will work with *infinite* plates, which are infinite parallel planes, separated by a distance d.

What is the potential and electric field of an infinite parallel plate capacitor with $\phi = 0$ on the left plane, $\phi = V_0$ on the right plane and the separation between the planes being d.

Since the planes are infinite, the problem is translationally invariant in y and z, so there is no dependence of the potential on y and z. (Be sure you understand this!) So we have

$$\frac{d^2\phi}{dx^2} = 0 \implies \phi = ax + b, \tag{15.12}$$

a linear function. Now we apply the boundary conditions: at $x = 0$, we have $\phi = 0$, which implies that $b = 0$ and at $x = d$, we have $\phi = V_0$, which implies that $a = \frac{V_0}{d}$. So

$$\phi = V_0 \frac{x}{d} = \text{a linear function.} \tag{15.13}$$

The electric field is now found by taking the gradient

$$\vec{E}(x) = -\nabla\phi(x) = \left(-\frac{V_0}{d}, 0, 0\right). \tag{15.14}$$

As expected, the electric field inside a parallel plate capacitor is a constant field, perpendicular to the plates.

Example 2: Spherical Capacitors

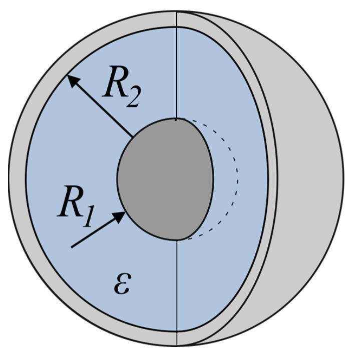

Figure 15.2: Schematic of a spherical capacitor, with inner radius R_1 and outer radius R_2. The region between the two spheres is filled with a dielectric with dielectric constant ϵ.

Because we are working in spherical coordinates, we will use the symbols Ψ instead of ϕ for the potential, in order to not confuse the potential with the angle ϕ used in spherical coordinates. The boundary conditions we have are $\Psi_{\text{inner}} = V_1$ and $\Psi_{\text{outer}} = 0$.

In spherical coordinates, The Laplacian becomes

$$\nabla^2\Psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Psi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\Psi}{d\phi^2}, \tag{15.15}$$

which you should have seen in one of your previous math or physics classes (the derivation is rather long, so if you have never seen it, go find an appropriate textbook or web resource to see the derivation). We do show another derivation in Chapter 17.

Because we have spherical symmetry, the potential is rotationally invariant, which means that there is no θ or ϕ dependence, so

$$\frac{d\Psi}{d\theta} = \frac{d\Psi}{d\phi} = 0. \tag{15.16}$$

Hence, we have

$$\nabla^2\Psi \to \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = 0. \tag{15.17}$$

This means that

$$r^2\frac{d\Psi}{dr} = c_1 \implies \Psi = \int \frac{c_1}{\bar{r}^2}\,d\bar{r} + c_2 = -\frac{c_1}{r} + c_2. \tag{15.18}$$

Now we use the boundary conditions: at $r = R_1$, we have $\Psi = V_1$

$$\implies -\frac{c_1}{R_1} + c_2 = V_1. \tag{15.19}$$

Similarly, at $r = R_1$, we have $\phi = 0$ and

$$-\frac{c_1}{R_1} + c_2 = 0 \implies c_2 = \frac{c_1}{R_2}. \tag{15.20}$$

These two equations can be combined to yield

$$-\frac{c_1}{R_1} + \frac{c_1}{R_2} = V_1. \tag{15.21}$$

Solving, we find that

$$c_1\left(\frac{R_1 - R_2}{R_1R_2}\right) = V_1 \implies c_1 = \left(\frac{R_1R_2}{R_1 - R_2}\right)V_1. \tag{15.22}$$

Putting this all together, we find that the scalar potential becomes

$$\Psi(r) = \left(\frac{R_1R_2}{R_1 - R_2}\right)\frac{V_1}{r} - \frac{R_1}{R_2 - R_1}V_1 = \frac{R_1V_1}{R_2 - R_1}\left(\frac{R_2}{r} - 1\right). \tag{15.23}$$

We use the gradient to find the field. This becomes

$$\vec{E} = \frac{R_1R_2}{R_1 - R_2}\frac{V_1}{r^2}\hat{e}_r \tag{15.24}$$

for $R_1 \leq r \leq R_2$, and $E_\theta = E_\phi = 0$.

Chapter 16

Laplace's equation

16.1 A harder Laplace's equation problem

Last time, we worked out two simple examples of solving Laplace's equation: (i) the parallel plate capacitor and (ii) the spherical capacitor. One thing we did not mention is that these solutions to Laplace's equation are unique, so however one finds a solution (including guessing), if they work, they are the solution. Hence, the technique of "guess and check if it works" is a valid method to try. It does require sophisticated guessing in some cases.

Example 3: The cylinder between two parallel planes capacitor.

We use cylindrical coordinates (the angular variable will be θ, so we can use ϕ again for the scalar potential). The one important observation to make here is that when we are far away from the cylinder, but still between the parallel plates, the electric field must be pointing in the $\hat{\imath}$ direction. Hence, we know that the electric field satisfies $\vec{E} \to E_0\hat{\imath}$ far from cylinder. The symmetry of the problem also implies that $E_z = 0$, because the problem is translationally invariant in the z-direction.

The Laplacian in cylindrical coordinates is

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{16.1}$$

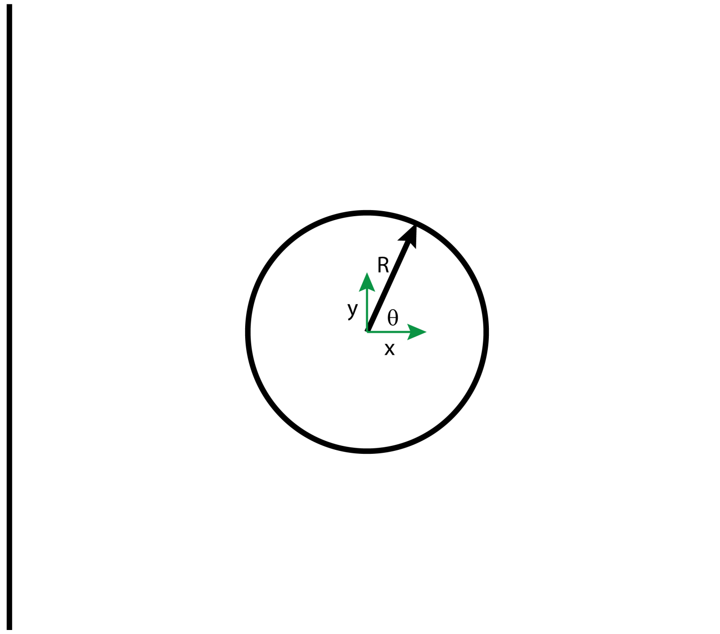

Figure 16.1: The "cylinder in the parallel plates" capacitor.

Since we have no z dependence, we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0. \tag{16.2}$$

The boundary condition at $r = R$ is $\phi(r,\theta) = 0$ for $r = R$. Far from the cylinder, we know $\vec{E} \to E_0\hat{i} \implies \phi \to -E_0 x$ or $\phi \to -E_0 r\cos(\theta)$ for $r \gg R$. We choose this as the second boundary condition and motivate a guess:

$$\phi(r,\theta) = f(r)\cos(\theta) \tag{16.3}$$

with $f(r) = 0$ at $r = R$ and $f(r) \to -E_0 r$ as $r \to \infty$. Substitute into Laplace's equation to find that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r)}{\partial r}\right)+\frac{1}{r^2}(-f(r))=0. \tag{16.4}$$

We further guess that $f(r) = r^\lambda$ is a possible solution.

$$\frac{1}{r}\frac{\partial}{\partial r}\lambda r^\lambda - r^{\lambda-2} = 0 \implies (\lambda^2-1)r^{\lambda-2} = 0. \tag{16.5}$$

This means that $\lambda = \pm 1$ or $f(r) = Ar + \frac{B}{r} \implies \phi(r,\theta) = (Ar + \frac{B}{r})\cos(\theta)$. Now, examine the boundary conditions:

$$AR + \frac{B}{R} = 0, B = -AR^2, r \to \infty, \phi \to Ar\cos(\theta) \implies A = -E_0. \tag{16.6}$$

So,

$$\phi(r,\theta) = -E_0 r\left(1 - \left(\frac{R}{r}\right)^2\right)\cos(\theta). \tag{16.7}$$

This has determined the scalar potential. The electric field is found by taking the gradient:

$$\vec{E} = -\nabla\phi = -\frac{\partial\phi}{\partial r}\hat{e_r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e_\theta}, E_z = 0. \tag{16.8}$$

Evaluating, we find that

$$E_r = E_0\cos(\theta) + E_0\frac{R^2}{r^2}\cos(\theta) = E_0\left(1 + \left(\frac{R}{r}\right)^2\right)\cos(\theta) \tag{16.9}$$

and

$$E_\theta = -E_0\left(1 - \left(\frac{R}{r}\right)^2\right)\sin(\theta). \tag{16.10}$$

Note that as $r \to \infty$, we have $\vec{E} \to E_0\left(\cos(\theta)\hat{e_r} - \sin(\theta)\hat{e_\theta}\right) = E_0\hat{i}$, as expected.

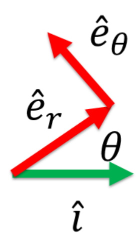

Figure 16.2: The combination of the unit vectors that we find for $r \gg R$ approach $\hat{i}$.

Fig. 16.2 clearly shows how the unit vectors add together to give $\hat{i}$ in the $r \gg R$ limit:

$$\hat{e_r}\cos(\theta) - \hat{e_\theta}\sin(\theta) = \hat{i}. \tag{16.11}$$

16.2 Relaxation method

Now we discuss a numerical method used to solve Laplace equation problems in general. It is called the relaxation method.

We focus on a two-dimensional problem and we employ Cartesian coordinates. Laplace's equation becomes

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{16.12}$$

Let's discretize this on a grid with spacing Δ in the x and y directions. Whenever we want to evaluate derivatives numerically, we must do so *approximately*, which we do by using the definition of the derivative, but evaluated at a finite "nudge" (Δ), rather than taking the limit as the "nudge" goes to zero. This procedure is called discretization.

For evaluating the Laplacian, we need to evaluate second derivatives, which require three discrete points in the x and in the y-directions. We use the so-called "cross" geometry, as shown in Fig. 16.3.

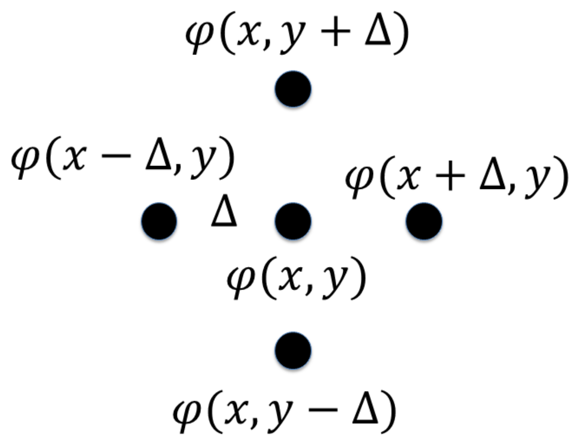

Figure 16.3: The "cross" employed in calculating approximate second derivatives in the x- and y-directions.

Following the geometry in Fig. 16.3, we find the Laplacian is approximated by

$$\nabla^2\Phi \sim \frac{\Phi(x+\Delta, y) - 2\Phi(x,y) + \Phi(x-\Delta, y)}{\Delta^2} + \frac{\Phi(x, y+\Delta) - 2\Phi(x,y) + \Phi(x, y-\Delta)}{\Delta^2}. \tag{16.13}$$

Check, by evaluating the Taylor series expansion for the potential at the "nudged" positions:

$$\Phi(x+\Delta,y)=\Phi(x,y)+\Delta\frac{\partial}{\partial x}\Phi(x,y)+\frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y)+... \tag{16.14}$$

and

$$\Phi(x-\Delta,y)=\Phi(x,y)-\Delta\frac{\partial}{\partial x}\Phi(x,y)+\frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y)+... \tag{16.15}$$

Now, substitute into the expression on the "cross"

$$\begin{aligned}\Phi(x+\Delta,y)-2\Phi(x,y)+\Phi(x-\Delta,y) &= \Phi(x,y)-2\Phi(x,y)+\Phi(x,y)\\ &\quad+\Delta\frac{\partial}{\partial x}\Phi(x,y)-\Delta\frac{\partial}{\partial x}\Phi(x,y)\\ &\quad+\frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y)+\frac{1}{2}\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y)\\ &=\Delta^2\frac{\partial^2}{\partial x^2}\Phi(x,y),\end{aligned} \tag{16.16}$$

to lowest nonvanishing order in Δ^2, which checks out! Setting $\nabla^2\Phi=0$ and solving for $\Phi(x,y)$ in terms of the surrounding points on the cross gives

$$\Phi(x,y)=\frac{1}{4}\left[\Phi(x+\Delta,y)+\Phi(x-\Delta,y)+\Phi(x,y+\Delta)+\Phi(x,y-\Delta)\right] \tag{16.17}$$

The relaxation method algorithm, then, is as follows:

1. Set up a uniformly-spaced grid in x and y with spacing Δ.
2. Set Φ equal to its values on the boundaries.
3. Guess Φ values in the interior (usually they vary linearly with respect to the boundary values).
4. Visit every point in the interior and update Φ via the above equation.
5. Repeat until Φ values stop changing.

We will explore this technique further in the lab.

16.3 Directional derivatives

In one dimension, the derivative is uniquely defined. But in higher dimensions, we can think of "nudging" in different directions, which gives rise to the so-called directional derivative. Consider a surface $z = f(x, y)$ that maps the x-y plane onto a surface in $3d$. An example is plotted in Fig. 16.4.

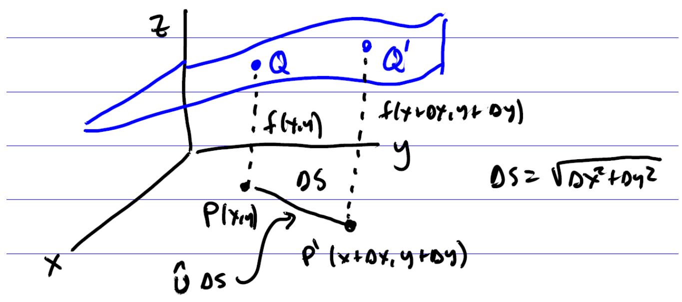

Figure 16.4: Example of a surface defined by a function $f(x, y)$ that we will use for determining directional derivatives.

The directional derivative requires us to compute the infinitesimal change in the arc length along a specific direction relative to how the function changes in the same direction (the two "nudges" Δ_x and Δ_y determine the direction of the derivative). So, we define $\Delta_s = \sqrt{\Delta_x^2 + \Delta_y^2}$ and determine Δf via a Taylor-series expansion:

$$\begin{aligned}
\Delta f &= f(x + \Delta_x, y + \Delta_y) - f(x, y) \\
&= f(x, y) + \Delta_x \frac{\partial}{\partial x} f(x, y) + \Delta_y \frac{\partial}{\partial y} f(x, y) - f(x, y) \\
&= \Delta_x \frac{\partial}{\partial x} f(x, y) + \Delta_y \frac{\partial}{\partial y} f(x, y) \\
&= \hat{u}\Delta_s \cdot \nabla f(x, y). \qquad (16.18)
\end{aligned}$$

Here, we used $\hat{u} = \frac{\Delta_x \hat{\imath} + \Delta_y \hat{\jmath}}{\Delta_s}$, and $\nabla f = \hat{\imath}\frac{\partial}{\partial x} f + \hat{\jmath}\frac{\partial}{\partial y} f$. This finally implies that

$$\frac{\Delta f}{\Delta_s} = \hat{u} \cdot \nabla f(x, y). \qquad (16.19)$$

This result is called the directional derivative of f ($\hat{u}$ is a unit vector). Note that the directional derivative is maximal when $\hat{u}$ is in the direction of ∇f, since then $\hat{u} \cdot \nabla f(x,y) = |\nabla f(x,y)|$. This says *the gradient points in the direction of maximal change to the function f*. This is useful for minimizing functions of multiple variables. Similar to Newton's method, we step in the direction of maximal change (called steepest descents).

One other property is of note. Suppose we plot iso-surfaces of f, which are given by the sets where $f(x,y)$ is equal to some constant. These are similar to contour plots on a map which plot lines of constant height. Since f does not change its value along an iso-surface, we must have the gradient is perpendicular to the iso-surface lines. This is a pictorial way of seeing the steepest descents method as a way to get to the maximum or minimum of a function. The situation is sketched in Fig. 16.5

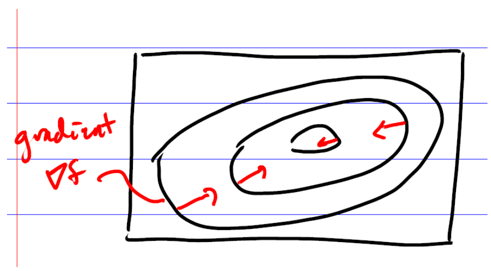

Figure 16.5: Example of a plot of different iso-surfaces of f. The gradient is perpendicular to the iso-surfaces and shows the direction of steepest descent (or ascent).

Chapter 17

The Laplacian in Cylindrical and Spherical Coordinates

17.1 Cylindrical Coordinates, the hard way

Cylindrical coordinates and their unit vectors are given in Fig. 17.1. Note that in curvilinear coordinate systems, it is common for the unit vectors to depend on position. For example, the radial unit vector is oriented along a radius at an angle θ in the $x-y$ plane. *This direction will change as the position changes*; then the unit vector changes direction as well. When we compute derivatives, *we need to take into account the change of direction of these unit vectors.* It is very common for students to have trouble understanding this aspect of unit vectors in curvilinear coordinates. Be sure that you can clearly see how this works; think about what happens to these vectors as the coordinates change.

We begin with the definitions of the different unit vectors in cylindrical coordinates. If you remember how this is done in polar coordinates, you will find it is essentially the same here. The first one is the radial unit vector, pointing in the direction of the position vector projected to the plane:

$$\hat{e}_r = \frac{\hat{\imath}x + \hat{\jmath}y}{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta. \tag{17.1}$$

The final result uses the facts that $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$. Next is $\hat{e}_\theta$, pointing in the tangential direction in the plane. It is easy to construct because there is only one vector perpendicular to $\hat{e}_r$ (recall the "easy" way to construct such a vector—we interchange the horizontal and vertical components,

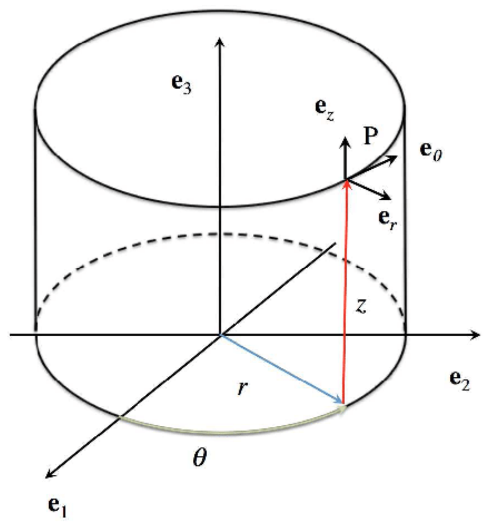

Figure 17.1: Cylindrical coordinates and unit vectors (note that sometimes ρ is used for r and ϕ for θ, but you should always be able to figure it out by the context). The cylindrical coordinates are polar coordinates for the $x-y$ plane and the Cartesian z-coordinate. The radial unit vector is in a radial direction in the $x-y$ plane, the θ unit vector is in the tangential direction to the circle in the $x-y$ plane and the z-component unit vector is along the z-axis. Note that we use the symbol $\hat{e}_\alpha$ to denote these different unit vectors (including $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ for the standard $\hat{i}$, $\hat{j}$ and $\hat{k}$ Cartesian unit vectors.)

changing the sign of one of them—one only has to decide on the overall sign to use)

$$\hat{e}_\theta = \frac{-\hat{i}y + \hat{j}x}{r} = -\hat{i}\sin\theta + \hat{j}\cos\theta. \tag{17.2}$$

These two relations can be inverted as well, so we have

$$\hat{i} = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta \tag{17.3}$$

and

$$\hat{j} = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta. \tag{17.4}$$

As r changes $\hat{e}_r$ and $\hat{e}_\theta$ don't change. As z changes $\hat{e}_r$ and $\hat{e}_\theta$ don't change. But as θ changes, we have

$$\frac{\partial\hat{e}_r}{\partial\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta = \hat{e}_\theta \tag{17.5}$$

and

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{\imath}\cos\theta - \hat{\jmath}\sin\theta = -\hat{e}_r. \tag{17.6}$$

Note that Cartesian unit vectors always have vanishing derivatives with respect to any parameter.

The derivative of a unit vector is always perpendicular to the unit vector. Do you know why? By definition, a unit vector satisfies $\hat{e}_\alpha \cdot \hat{e}_\alpha = 1$. So the derivative with respect to some parameter γ gives $(\partial_\gamma \hat{e}_\alpha) \cdot \hat{e}_\alpha + \hat{e}_\alpha \cdot (\partial_\gamma \hat{e}_\alpha) = 0$, because the length of the unit vector is a constant, independent of any parameter. Since the two terms are the same, we see that $(\partial_\gamma \hat{e}_\alpha) \cdot \hat{e}_\alpha = 0$, or the derivative of the unit vector is *perpendicular* to the unit vector. Note that this does not say that the derivative of the unit vector must also be a unit vector! It turned out this way here, but does not need to be true in general.

Our next task is to calculate the derivatives of the Cartesian coordinates with respect to the cylindrical coordinates. Since we have $x = r\cos\theta$, $y = r\sin\theta$, and z, we find

$$\frac{\partial x}{\partial r} = \cos\theta \tag{17.7}$$

and

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \tag{17.8}$$

for x. For y we have

$$\frac{\partial y}{\partial r} = \sin\theta \tag{17.9}$$

and

$$\frac{\partial y}{\partial \theta} = r\cos\theta. \tag{17.10}$$

Next, we carefully work out the chain rule for derivatives with respect to the three cylindrical coordinates. We obtain

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial}{\partial z}\frac{\partial z}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \tag{17.11}$$

for the r-coordinate. We obtain

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \tag{17.12}$$

for the θ-coordinate. We now use these results to find the gradient operator in cylindrical coordinates. We find that

$$\begin{aligned}\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} &= \hat{\imath}\cos^2\theta\frac{\partial}{\partial x}+\cancel{\hat{\jmath}\sin\theta\cos\theta\frac{\partial}{\partial x}}+\cancel{\hat{\imath}\cos\theta\sin\theta\frac{\partial}{\partial y}}+\hat{\jmath}\sin^2\theta\frac{\partial}{\partial y}\\ &+\hat{\imath}\sin^2\theta\frac{\partial}{\partial x}-\cancel{\hat{\jmath}\sin\theta\cos\theta\frac{\partial}{\partial x}}-\cancel{\hat{\imath}\sin\theta\cos\theta\frac{\partial}{\partial y}}+\hat{\jmath}\cos^2\theta\frac{\partial}{\partial y}\\ &= \hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y},\end{aligned} \tag{17.13}$$

since the cross terms cancel. So in cylindrical coordinates, we have the gradient operator is

$$\nabla=\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_z\frac{\partial}{\partial z}. \tag{17.14}$$

The Laplacian is found by properly computing the dot product of the gradient with itself. Using the derivatives we computed already above for the unit vectors, we immediately find that

$$\begin{aligned}\nabla\cdot\nabla &= \left[\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_z\frac{\partial}{\partial z}\right]\cdot\left[\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_z\frac{\partial}{\partial z}\right]\\ &= \hat{e}_r\cdot\hat{e}_r\frac{\partial^2}{\partial r^2}+\hat{e}_\theta\frac{1}{r}\cdot\left(\frac{\partial}{\partial\theta}\hat{e}_r\right)\frac{\partial}{\partial r}+\hat{e}_\theta\cdot\hat{e}_\theta\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\hat{e}_z\cdot\hat{e}_z\frac{\partial^2}{\partial x^2}\\ &= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}.\end{aligned} \tag{17.15}$$

You should check that we only included the nonzero terms in the second line. All others vanish either due to derivatives or due to unit vectors being perpendicular. One needs to do this analysis carefully. In the last line, we compressed the derivatives in the radial direction to one term instead of two.

17.2 Cylindrical Coordinates, the easy way

The easy way involves looking at how functions of the cylindrical coordinates behave. Consider a function $f(r,\theta,z)$. We start by looking at the lowest-order change in the function due to small changes in the coordinates

$$\Delta f=\frac{\partial f}{\partial r}\Delta r+\frac{\partial f}{\partial\theta}\Delta\theta+\frac{\partial f}{\partial z}\Delta z+... \tag{17.16}$$

Next, we define a vector corresponding to the shifts (measured in a length) along each unit vector direction corresponding to the changes in the coordinates

$$\Delta\vec{s} = \hat{e}_r \Delta r + \hat{e}_\theta r \Delta\theta + \hat{e}_z \Delta z \tag{17.17}$$

and recognize that the change in f is actually a directional derivative

$$\Delta f = \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z} \right) \cdot \Delta\vec{s} = \vec{\nabla} f \cdot \Delta\vec{s}. \tag{17.18}$$

From this, we can immediately find the gradient

$$\nabla f = \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z} \right). \tag{17.19}$$

To find the divergence in cylindrical coordinates, we either take derivatives like above, or proceed similar to pages 42-43 and problem II-20 of *Div, Grad, and Curl* to find the divergence. In either case, one has

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}, \tag{17.20}$$

where the subscripts on F denote the components in terms of the unit vectors in cylindrical coordinates. Using $\vec{F} = \vec{\nabla} f$ gives

$$\begin{aligned} \nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r} \right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta} \right) + \frac{\partial^2 f}{\partial z^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} f \right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned} \tag{17.21}$$

That is it for cylindrical coordinates. We now move on to spherical coordinates, which is a lot harder.

17.3 Spherical Coordinates

We begin the discussion of spherical coordinates by doing it the "fast way". Recall that we compute the change in a function f

$$\Delta f = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial \phi}\Delta\phi \tag{17.22}$$

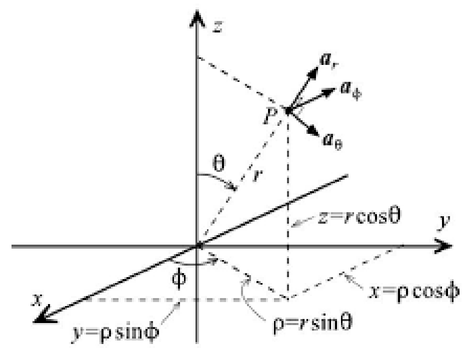

Figure 17.2: Unit vectors for spherical coordinates. Note that the physicist and mathematics conventions for θ and ϕ are reversed. It is confusing. I am sorry. Here we see the unit vectors. Just replace the label a with $\hat{e}$ for our convention.

and define a vector $\Delta\vec{s}$ that corresponds to the lengths displaced along each unit vector due to the changes in the coordinates

$$\Delta\vec{s} = \hat{e}_r\Delta r + \hat{e}_\theta r\Delta\theta + \hat{e}_\phi r\sin\theta\Delta\phi \tag{17.23}$$

and recognize that we have a directional derivative

$$\Delta f = \vec{\nabla} f \cdot \Delta\vec{s} = \left(\hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\cdot\Delta\vec{s}. \tag{17.24}$$

The gradient can then be extracted "by inspection"

$$\vec{\nabla} f = \hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}. \tag{17.25}$$

In other words, we have

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{17.26}$$

in spherical coordinates.

We can next work out the Laplacian $\vec{\nabla}\cdot\vec{\nabla} f$, which requires us to work out derivatives of the unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ with respect to changes in r, θ, ϕ or we can use the formula for the divergence in spherical coordinates which can be derived as discussed in the book. We show the former here.

To begin, let's find $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ as before. The radial unit vector is in the direction $\vec{r}/r$

$$\hat{e}_r = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{r}, \tag{17.27}$$

the $\hat{e}_\theta$ vector lies in the plane formed by the z axis and $\hat{e}_r$

$$\hat{e}_\theta = \text{unit vector in } z,\ \rho \text{ plane perpendicular to } \hat{e}_r. \tag{17.28}$$

We first find the unit vector in the $x - y$ plane along the projection of $\vec{r}$, denoted $\hat{e}_\rho$

$$\hat{e}_\rho = \frac{\hat{i}x + \hat{j}y}{r\sin\theta} \tag{17.29}$$

so

$$\hat{e}_\theta = -\cos\theta\hat{e}_\rho + \sin\theta\hat{k}. \tag{17.30}$$

Simplifying these two results yields

$$\hat{e}_r = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \tag{17.31}$$

and

$$\hat{e}_\theta = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta. \tag{17.32}$$

. Our last unit vector is in the ϕ direction. We find it by simple cross product

$$\hat{e}_\phi = \hat{e}_r \times \hat{e}_\theta = -\hat{i}\sin\phi + \hat{j}\cos\phi. \tag{17.33}$$

Now, on to the derivatives. No unit vector depends on r. For the θ and ϕ, note that

$$\frac{\partial\hat{e}_r}{\partial\theta} = \hat{e}_\theta \tag{17.34}$$

and

$$\frac{\partial\hat{e}_r}{\partial\phi} = \sin\theta\hat{e}_\phi \tag{17.35}$$

for the radial unit vector. We have

$$\frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r \tag{17.36}$$

and

$$\frac{\partial\hat{e}_\theta}{\partial\phi} = \cos\theta\hat{e}_\phi \tag{17.37}$$

for the θ direction unit vector. Finally, we have

$$\frac{\partial \hat{e}_\phi}{\partial \theta} = 0 \tag{17.38}$$

and

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\sin\theta \hat{e}_r - \cos\theta \hat{e}_\theta \tag{17.39}$$

for the ϕ direction unit vector. We now employ these results in calculating the Laplacian

$$\begin{aligned}
\nabla^2 &= \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \right) \cdot \left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \right) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{\sin\theta}\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}.
\end{aligned} \tag{17.40}$$

In the last line, we have compressed the notation for the derivatives.

If we use

$$\vec{\nabla} \cdot F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \tag{17.41}$$

on

$$F = \vec{\nabla} f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}, \tag{17.42}$$

we find

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}, \tag{17.43}$$

which also agrees.

The bottom line from all of this work is that calculating the divergence, gradient, curl and Laplacian in other coordinate systems requires patience to carry out. It is our classic "French cooking" exercise!

Chapter 18

Introduction to complex numbers

18.1 Manipulations of complex numbers

The book, *Introduction to Linear Algebra and Differential Equations*, provides a review of a number of arithmetic manipulations with complex numbers in Chapter 1. Be sure to read and master that material. Here, we review the most critical facts and clarify points that are often confusing.

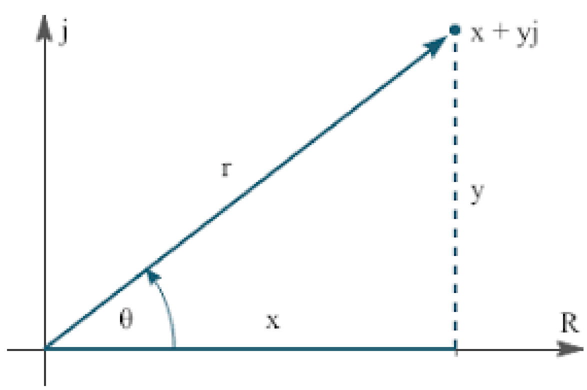

Figure 18.1: Polar representation of a complex number.

Let $z = x + iy$ be a complex number, with x and y being real. The imaginary number i satisfies $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Sometimes, we prefer to express the complex numbers in a polar representation where $x + iy = re^{i\theta}$, which implies that $x = r\cos\theta$ and $y = r\sin\theta$ (more on this later). We also have $r = \sqrt{x^2 + y^2}$ as usual. The complex arithmetic way of determining the polar radius is with the complex conjugate $\bar{z}$ or z^*. If $z = x + iy$, then $\bar{z} = x - iy$ and $z\bar{z} = |z|^2 = x^2 + y^2$ so $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = r$. Note how the complex conjugate is precisely what is needed to ensure that we get the sum of the squares of the Cartesian coordinates $[(x + iy)(x - iy) = x^2 + y^2]$.

This leads to an important exercise with complex arithmetic called *rationalizing the denominator*. Consider the inverse of a complex number. Use our "multiply by one" trick to insert the complex conjugate. This gives

$$\frac{1}{x + iy} = \frac{1}{x + iy} \times 1 = \frac{1}{x + iy}\frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2}, \tag{18.1}$$

or in complex notation: $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$. Often, people forget about this simple manipulation when working with complex numbers. Please be sure this is not you!

18.2 Complex exponentials

We will now develop De Moivre's theorem, which will ell us about roots of unity. The starting point is the so-called Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{18.2}$$

which we used when discussing the polar form of a complex number. This can be proved geometrically (see Fig. 18.1), or it can be done algebraically, as we do now. We begin with

$$e^{i\theta} = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \dots, \tag{18.3}$$

which is the standard form of the Taylor series expansion of an exponential. Note that $\operatorname{Re} z = x = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = y = \frac{z-\bar{z}}{2i}$ so $\operatorname{Re} e^{i\theta} = \frac{e^{i\theta}+e^{-i\theta}}{2}$. Let's see how this works for the Taylor series. Note that

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + ... \tag{18.4}$$

and

$$e^{-i\theta} = 1 - i\theta - \frac{1}{2}\theta^2 + \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + ... \tag{18.5}$$

So, we have

$$e^{i\theta} + e^{-i\theta} = 2 - \theta^2 + \frac{2}{4!}\theta^4 + ... \tag{18.6}$$

and therefore

$$\text{Re } e^{i\theta} = 1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + ... = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{18.7}$$

(One verifies this either by comparing with the Taylor series for the cosine or by using the definition of the cosine as a way of determining the series.)

Similarly

$$\text{Im } e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(2i\theta - \frac{2i}{3!}\theta^3 + ...\right) = \theta - \frac{1}{3!}\theta^3 + ... = \sin(\theta). \tag{18.8}$$

So, we have

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta. \tag{18.9}$$

Note that it is quite important for you to become facile in using both relations for cosine and sine—those from the power series and those in terms of the complex exponentials.

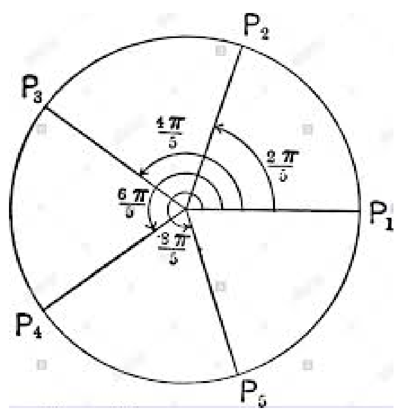

Figure 18.2: Schematic of the five roots of unity as found by de Moivre's theorem.

Now we complete de Moivre's theorem (which involves finding the complex roots of unity). So if $z = e^{i\theta}$ then $z^n = e^{in\theta}$ and we find

$$\text{Re } e^{in\theta} = \cos(n\theta) + i\sin(n\theta). \tag{18.10}$$

This allows us to find the N roots of unity $1 = e^{i2\pi} \implies 1^{\frac{1}{N}} = e^{\frac{i2\pi}{N}}$ and the N roots are given by the expression $e^{\frac{i2n\pi}{N}}$, for $0 \leq n \leq N-1$. Note how the N roots of unity are evenly distributed around a unit circle (see Fig. 18.2 for an example with $N = 5$). Note how we used the "multiply by one" trick again, replacing 1 by $e^{2i\pi}$. Remember this. It comes up often.

Since we know

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta, \tag{18.11}$$

one can determine the logarithm in a straightforward fashion. The logarithm function is often confusing in complex analysis. While it is the inverse of the exponential, the weird looking real and imaginary parts are often mishandled. Be sure to work carefully with logarithms. From the above Euler relation, we immediately have

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + \ln e^{i\theta} = \frac{1}{2}\ln z\bar{z} + i\theta. \tag{18.12}$$

This implies that the real part of the logarithm is the ordinary logarithm of the modulus of z and the imaginary part is i times the phase of z. So as we wrap around a circle, the phase changes by 2π, *and the logarithm does not return to itself*. This is one reason why we have to be very careful in working with logarithms in complex analysis.

18.3 Cauchy-Riemann equations

We end with an introduction to the theory of analytic functions—complex-valued functions of $z = x + iy$ only (no dependence on $\bar{z} = x - iy$). This restriction ends up providing lots of constraints on f, which subsequently result in lots of interesting properties, such as analytic functions being infinitely differentiable, always having Taylor series expansions, being continuous with all derivatives continuous, etc. They are *super well-behaved functions*. Let's see how some of these properties arise.

A function is analytic when it depends only on z (and not $\bar{z}$)

$$f(z) = f(x + iy) = \text{ analytic.} \tag{18.13}$$

We can evaluate derivatives with respect to the real and imaginary parts of z, respectively:

$$\frac{df}{dx} = f' \quad \text{and} \quad \frac{df}{dy} = if'. \tag{18.14}$$

So, if we think of the analytic function as a two-dimensional vector field (since the complex numbers live in a plane), we have $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$ with u and v being real-valued functions. Then, we have

$$\frac{df}{dx} = \frac{du}{dx} + i\frac{dv}{dx} \quad \text{and} \quad \frac{df}{dy} = \frac{du}{dy} + i\frac{dv}{dy}. \tag{18.15}$$

But we saw above that $\partial_x f = f'$ and $\partial_y f = if'$, so

$$-i\frac{df}{dy} = \frac{df}{dx} \implies \frac{du}{dx} + i\frac{dv}{dx} = \frac{dv}{dy} - i\frac{du}{dy}. \tag{18.16}$$

Simplifying, by equating the real and imaginary parts, we have

$$\frac{du}{dx} - \frac{dv}{dy} = 0 \quad \text{and} \quad \frac{dv}{dx} + \frac{du}{dy} = 0. \tag{18.17}$$

These two relations are called the Cauchy-Riemann equations. They resemble $\nabla \cdot f = 0$ and $\nabla \times f = 0$, except there seems to be a sign difference. It turns out these relations are closely related and there is a minus sign issue. We will discuss this and find the right way to think about this in the next lecture.

Any function that satisfied the Cauchy-Riemann equations is an analytic function.

We end by checking the Cauchy-Riemann equations for some common known functions next. First, the exponential:

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)). \tag{18.18}$$

We have

$$u = e^x \cos(y), v = e^x \sin(y). \tag{18.19}$$

The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = e^x \cos(y), \quad \frac{\partial v}{\partial y} = e^x \cos(y) \implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{18.20}$$

which checks out! In addition,

$$\frac{\partial u}{\partial y} = -e^x \sin(y), \; \frac{\partial v}{\partial x} = e^x \sin(y) \implies \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{18.21}$$

which also checks out!

Now, try the logarithm:

$$\ln z = \frac{1}{2}\ln(x^2+y^2) + i\arctan\frac{y}{x}, \tag{18.22}$$

so that

$$u = \frac{1}{2}\ln(x^2+y^2),\ v = \arctan\frac{y}{x}. \tag{18.23}$$

Next, we check the Cauchy-Riemann relations

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2},\quad \frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}. \tag{18.24}$$

Rearranging, we have

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2},\quad \frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2+y^2}, \tag{18.25}$$

so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ again.

You will find many of the “good” functions you know will also be analytic when they are extended to be defined on the complex plane. Analytic functions are not only good. You should think of them as being truly “super.”

Chapter 19

Cauchy Theorem and Introduction to Residues

19.1 Contour Integrals in the Complex Plane

Recall that we derived the Cauchy-Riemann equations

$$f(x+iy) = u(x,y) + iv(x,y) \tag{19.1}$$

$$\frac{du}{dx} = \frac{dv}{dy} \tag{19.2}$$

$$\frac{du}{dy} = -\frac{dv}{dx} \tag{19.3}$$

in the previous chapter. We will use these results to help us understand integration in the complex plane. It turns out that this is quite similar to the Stokes' theorem we worked with for line integrals of vector fields.

Consider the integral of a complex-valued function $f(z)$ over a path in the complex plane starting at a and ending at b. Converting to a more standard form using the real and imaginary parts, we have

$$\begin{aligned} \int_a^b f(z)\,dz &= \int_a^b \left(u(x,y) + iv(x,y)\right)(dx + idy) && (19.4) \\ &= \int_a^b \left[u(x,y)dx - v(x,y)dy\right] + i\left[u(x,y)dy + v(x,y)dx\right]. && (19.5) \end{aligned}$$

Let us think of the first integral as a line integral of the vector field $\vec{F}_1 = u\hat{e}_x - v\hat{e}_y$ with $\hat{t} = \frac{dx\hat{e}_x + dy\hat{e}_y}{ds}$. The second as another vector field $\vec{F}_2 = u\hat{e}_x + v\hat{e}_y$ and same $\hat{t}$. Then

$$\int_a^b f(z)\,dz = \int \vec{F}_1 \cdot \hat{t}\,ds + i\int \vec{F}_2 \cdot \hat{t}\,ds \tag{19.6}$$

with $\vec{F}_1$ and $\vec{F}_2$ real-valued vector fields. Let's find the curl of each of them. Compute

$$\nabla \times \vec{F}_1 = \hat{e}_z \left(-\frac{dv}{dx} - \frac{du}{dy} \right) = 0 \tag{19.7}$$

and

$$\nabla \times \vec{F}_2 = \hat{e}_z \left(\frac{dv}{dx} - \frac{du}{dy} \right) = 0 \tag{19.8}$$

where they both vanish by the Cauchy-Riemann equations. So since the curl is zero, these integrals are independent of the path. From this, we immediately learn the first form of Cauchy's Theorem:

$$\oint f(z)\,dz = 0 \tag{19.9}$$

for a function f that is analytic on the interior of a closed curve. This means, for example, that there are no singularities in the interior of the closed curve (which is similar to the simply connected region requirement we had with Stokes' theorem).

Since the above integral is independent of the shape of the path, we can deform the path to any shape we want as long as f remains analytic in the region of the path.

Example: *Path independence for the integral along a circle and along the axes.*

Integrate e^z from 1 to i along a quarter of the unit circle: $\int_1^i e^z\,dz$. We let $z = \gamma(t) = \cos t + i \sin t$ for $0 \le t \le \frac{\pi}{2}$. Then we have $dz = \gamma'(t) =$

Figure 19.1: Two different paths for integration of the analytic function $f = e^z$ used to show path independence of a contour integral.

$(-\sin t + i\cos t)dt$ and the integral becomes

$$\begin{aligned}
\int_1^i e^z\,dz &= \int_0^{\frac{\pi}{2}} dt\, e^{\cos t + i\sin t}(-\sin t + i\cos t) \\
&= \int_0^{\frac{\pi}{2}} dt e^{\cos t}\left(\cos(\sin t) + i\sin(\sin t)\right)(-\sin t + i\cos t) \\
&= \int_0^{\frac{\pi}{2}} dt e^{\cos t}\,[] - \cos(\sin t)\sin t - \sin(\sin t)\cos t \\
&\qquad\qquad -i\sin(\sin t)\sin t + i\cos(\sin t)\cos t] \\
&= \int_0^{\frac{\pi}{2}} dt\,\frac{d}{dt}\left[e^{\cos t}\cos(\sin t) + ie^{\cos t}\sin(\sin t)\right] \\
&= 1\cos(1) + i\sin(1) - e^1\cos(0) - ie^1\sin(0) \\
&= e^i - e^1.
\end{aligned} \tag{19.10}$$

One can perform a similar integration around the "ell" shaped path in Fig. 19.1. The details are given in the video lecture. Here, we will be even more general and simply note that $e^z = \frac{d}{dz}e^z$ so $\int_1^i \frac{d}{dz}e^z\,dz = e^z\big|_1^i = e^i - e^1$ for *any path*, which shows again that the integral is independent of the path!

Now we consider the single most important contour integral in complex analysis. It is amazing how many interesting results derive from this one simple integral. It is also amazing how many students have trouble with analyzing and understanding this. So pay attention! We consider an integral around a unit circle centered at the point z_0 of $(z - z_0)^n$:

$$\oint (z - z_0)^n\,dz. \tag{19.11}$$

In this integral, n is an integer. The integration contour is illustrated schematically in Fig. 19.2.

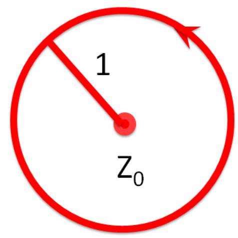

Figure 19.2: Contour around a unit circle centered at z_0 and traversed in the counterclockwise direction.

Note that for all n not equal to -1, we have that the integrand is a perfect differential, so that

$$(z - z_0)^n = \frac{d}{dz} \frac{(z - z_0)^n}{n + 1} \tag{19.12}$$

and

$$\oint (z - z_0)^n \, dz = \left. \frac{z - z_0}{n + 1} \right|_{\text{start}}^{\text{end}} = 0 \tag{19.13}$$

since the starting point and end point are the same for integration around a circle.

On the other hand, for $n = 1$, we use $\gamma(t) = z_0 + e^{it}$, $0 \leq t \leq 2\pi$. $dz = \gamma'(t) \, dt = ie^{it} \, dt$ and the integral becomes

$$\oint \frac{1}{z - z_0} \, dz = \int_0^{2\pi} dt \, ie^{it} \frac{1}{z_0 + e^{it} - z_0} = \int_0^{2\pi} dt \, i \frac{e^{it}}{e^{it}} = 2\pi i. \tag{19.14}$$

Summarizing, we have

$$\oint (z - z_0)^n \, dz = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{19.15}$$

Be sure you understand clearly why this integral is what it is and also be sure you can derive it yourself. It is that important.

19.2 Proof of Cauchy's Theorem

Now we will prove the full Cauchy's theorem. Let γ be a path that encircles z_0 once in the counterclockwise direction (the circle in Fig. 19.2 is an example of just such a path). Then, we claim that

$$f(z_0) = \int_\gamma \frac{f(z)}{z - z_0}\frac{dz}{2\pi i}. \tag{19.16}$$

To prove this, we expand $f(z)$ in a Taylor series expansion about z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(z - z_0)^n \frac{d^n}{dz^n} f(z)\bigg|_{z=z_0} \tag{19.17}$$

$$= f(z_0) + (z - z_0)f'(z_0) + \frac{1}{2}(z - z_0)^2 f''(z_0) + ... \tag{19.18}$$

which holds for all analytic functions. So

$$\int_\gamma \frac{f(z)}{z - z_0}\frac{dz}{2\pi i} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dz^n} f(z)\bigg|_{z=z_0} \int_\gamma (z - z_0)^{n-1}. \tag{19.19}$$

Note that $\frac{d^n}{dz^n} f(z)\big|_{z=z_0}$ can be removed from the integral because it is a number, not a function of z. Then, we find that only the $n = 0$ term contributes and integral becomes $2\pi i \frac{f(z_0)}{2\pi i} = f(z_0)$, which proves the result. Note, that if you are not happy with the rigor of our using a Taylor series expansion, or with our interchanging the order of the integration and the summation, congratulations, you are quite mathematically inclined. We will not go into the details as to why this is not an issue here. That is one of the major results you will learn in a complex analysis class. The goal here is just to make the result seem plausible, with a heuristic derivation and to show you the implications of it and how to use it. Rest assured, the result is perfectly rigorous.

Summarizing, we have that Cauchy's theorem says

$$f(z_0) = \int_\gamma \frac{f(z)}{z - z_0}\, dz \tag{19.20}$$

for any z_0 that lies *inside* the curve γ. Think about this for a moment. It is a truly remarkable result. It says that an analytic function is *completely* *determined in the interior of a region by its values on the boundary of that region*! This incredible result is true for all analytic functions.

We end the chapter with a discussion of a generalization of the Taylor series. Recall that the Taylor series was a best fit polynomial to a function. When you look carefully at Taylor series, you see they do not work well for functions that diverge at singular points. This is because the slope of a polynomial is always finite, but the slope at a singularity is always infinite. A Laurent series is a generalization of a Taylor series to include such singular behavior at z_0. If there are only a finite number of singular terms then we can expand

$$g(z) = \frac{1}{n!}\frac{b_{-n}}{(z-z_0)^n} + \frac{1}{(n-1)!}\frac{b_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{1}{2}\frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \frac{1}{2}a_2(z-z_0)^2 + \cdots + \frac{1}{n!}a_n(z-z_0)^n + \cdots \quad (19.21)$$

The term b_{-1} is called the residue of the function g. For example, if f is analytic, and $g = \frac{f(z)}{z-z_0}$ then $f(z_0)$ is the residue of g at z_0. In many cases finding b_{-1} is simple. In some cases, it can be difficult. But the residue is also an extremely important quantity to know about a singular function. We will see why in the next chapter.

Chapter 20

The Residue theorem

20.1 Calculating integrals via residues

Recall last time when we showed the residue of a function $g(z)$ is the coefficient of the $\frac{1}{z-z_0}$ term in the Laurent expansion for $g(z)$. Each point where $g(z)$ diverges as $\frac{b_{-1}}{z-z_0}$ is called a <u>*pole*</u> of $g(z)$ with b_{-1} the <u>*residue*</u> of the pole.

The idea of calculating real-valued integrals via residues is not at all obvious. When we reviewed the one-dimensional integral calculus earlier in the course, we covered how one can integrate a wide range of different functions. Special examples like Frullani's integral and Feynman integration produced more options. Here we expand the repertoire even more. Probably the single most useful integration technique is via residues. If you recall from our example given in Chapter 19, the only integral you need is the integral of powers of $z - z_0$ and only one of those integrals is nonzero. Couple this with a Laurent expansion about any singular point and you can almost see the methodology emerging. One issue often leads to confusion, though. Lot's of confusion. It is how to relate an integral from $-\infty$ to $+\infty$ along the real axis to a closed contour integral, where we can use the residue theorem. We employ examples below to describe how this works. Please look carefully at how this is done and try to master it. It is the key to becoming a practitioner on the calculus of residues.

Let's start with a careful description of the residue theorem. The residue theorem says if γ is a curve that encircles some poles of $g(z)$, each circled once in the counter-clockwise direction, then

$$\int_\gamma g(z)dz = 2\pi i \sum_{\text{poles inside } \gamma} (\text{residues of } g \text{ inside } \gamma) \qquad (20.1)$$

The residue theorem is primarily used to integrate many different definite integrals (usually over the full real axis, but sometimes over smaller regions). In most cases, it does not help with indefinite integrals.

20.2 Examples of the residue theorem

We illustrate with a few examples here and then cover more in the laboratories. We start with an integral that works with simply identifying how the residue theorem works. This first example will not lead to an application to one-dimensional integrals.

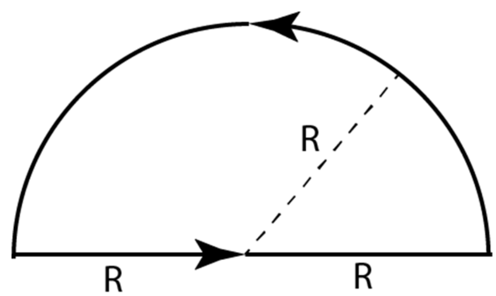

Figure 20.1: Path γ used for integrating the first residue theorem example. The path here is given for arbitrary R and our specific γ for the first example has $R = 2$.

Example: *Integrate* $\frac{1}{z^4+1}$ *over* γ.

Suppose we integrate $\frac{1}{z^4+1}$ over γ: $\int_\gamma \frac{dz}{z^4+1}$. The poles of $\frac{1}{z^4+1}$ occur when $z^4 + 1 = 0 \implies z^4 = -1$. A simple modification of the way we solved De Moivre's theorem allows us to find where the roots are. We have the four roots z_n ($n = 0$, 1, 2 and 3) are found via

$$z^4 = e^{i\pi+2in\pi} \implies z_n = e^{\frac{i\pi}{4}+\frac{in\pi}{2}}, n = 0, 1, 2, 3, \qquad (20.2)$$

because $-1 = e^{i\pi}$. Note that all of these roots lie on the unit circle. You can immediately see that two poles lie inside γ, circled in blue in Fig. 20.2.

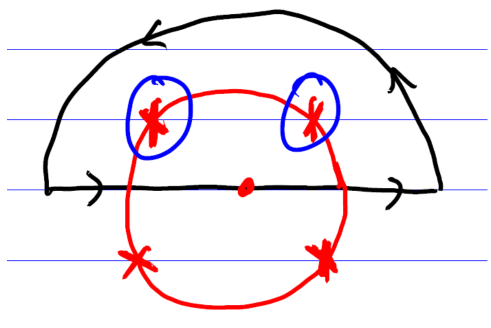

Figure 20.2: Schematic showing the two roots, circled in blue, that lie inside γ. These two points are where the two poles lie that are inside the contour.

The next step is to find the residues. The residue of $g(z) = \frac{1}{z^4+1}$ at z_n is

$$\text{Res at } z_n = \lim_{z \to z_n} \frac{z - z_n}{z^4 + 1}. \tag{20.3}$$

We know this because the pole is an isolated pole (implying all terms in the Laurent expansion with b_{-n} vanish for $n \geq 2$). This then allows us to compute the residue by multiplying by $(z - z_n)$ and taking the limit as $z \to z_n$. Because $g(z)$ has a singularity, we find this limit becomes $\frac{0}{0}$. To evaluate such an indeterminate form, we use l'Hôpital's rule and differentiate the numerator and denominator separately:

$$\text{Res at } z_n = \lim_{z \to z_n} \frac{\frac{d}{dz}(z - z_n)}{\frac{d}{dz}(z^4 + 1)} = \frac{1}{4z_n^3}. \tag{20.4}$$

So the residue for $n = 0$ becomes

$$\frac{1}{4e^{\frac{3i\pi}{4}}} = \frac{1}{4}e^{-\frac{3i\pi}{4}} = \frac{-1-i}{4\sqrt{2}} \tag{20.5}$$

and at $n = 1$, the residue becomes

$$\frac{1}{4e^{\frac{9i\pi}{4}}} = \frac{1}{4}e^{-\frac{9i\pi}{4}} = \frac{1-i}{4\sqrt{2}} \tag{20.6}$$

and the integral becomes $2\pi i$ times the sum of the two residues

$$2\pi i \left(\frac{1}{4\sqrt{2}}\right)(-1 - i + 1 - i) = \frac{\pi}{\sqrt{2}}. \tag{20.7}$$

We finally end up with the result of

$$\int_\gamma \frac{1}{z^4+1}dz = \frac{\pi}{\sqrt{2}}. \tag{20.8}$$

Our next example will help us evaluate an indefinite integral over the real axis. The key strategy we need to use here is the "add zero" trick. We need to find a contour in the complex plane, which is constructed out of two pieces: (i) one which will become the integral over the real axis and (ii) the other which will equal zero and hence add nothing to the final result, Once we have obtained such a closed contour, we can then use the residue theorem to evaluate the integral. Because all we did was add zero, the integral remains equal to the original value of the integral over the real axis. Now that is cool!

But how do we find a piece of the contour integral that will be zero? It turns out to be rather simple to do. Think of the integral over a semicircle of radius R given in Fig. 20.1. The horizontal part of the semicircle becomes the entire real axis as $R \to \infty$. What about the other piece, given by the circular part of the semicircle? The perimeter is πR. If the integrand decays faster than $1/R$ as $R \to \infty$, then the total integral over that arc is equal to zero as $R \to \infty$! And that is how we do it. Note that it is critically important to verify this for any integral you want to evaluate. Sometimes doing this shows how you "close the contour" one way or another.

Example: *Show how to use the residue theorem to evaluate an indefinite integral over the real axis.* We examine the following integral

$$\int_{-\infty}^{\infty} dx \frac{1}{1+x^{2n}}, n = 1, 2, ... \tag{20.9}$$

Note that if we close the integral on the upper half plane, as discussed above and shown in Fig. 20.1 for $R \to \infty$, then we add a term of size $\approx \frac{\pi R}{1+R^{2n}} \to 0$ as $R \to \infty$ (because $n \geq 1$), so

$$\int_{-\infty}^{\infty} dx\, \frac{1}{1+x^{2n}} = \int_\gamma dz\, \frac{1}{1+z^{2n}} = 2\pi i \sum \text{residues inside } \gamma. \tag{20.10}$$

Residues occur when $z^{2n} = -1$ (so we solve by generalizing De Moivre's theorem again), $z_k = e^{\frac{i\pi}{2n}+\frac{ik\pi}{n}}$ for $0 \leq k \leq 2n$. The residues are equally spaced on the unit circle and those with $k = 0, 1, ..., n-1$ lie inside γ (those with $n \leq k < 2n$ lie outside γ), because they are inside the upper half plane (above the real axis).

$$\text{Residue at } z_n = \lim_{z\to z_n} \frac{(z-z_n)}{1+z^{2n}} \overset{\text{l}'Hopital}{=} \frac{1}{2nz_k^{2n-1}} = -\frac{z_k}{2n}, \tag{20.11}$$

since $z_k^{2n} = -1$. Hence, we find that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}} = \sum_{k=0}^{n-1} 2\pi i \frac{1}{2n}(-e^{\frac{i\pi}{2n}+\frac{ik\pi}{n}}) \tag{20.12}$$

The sum looks like it is hard to evaluate, but if you think of our devious math tricks, maybe you will think that perhaps there might be some way to simplify it. First, we pull out the factors independent of k:

$$= -\frac{\pi i}{n} e^{\frac{i\pi}{2n}} \sum_{k=0}^{n-1} e^{i\frac{k\pi}{n}}. \tag{20.13}$$

Now we see that we have something that looks like a finite piece of the geometric series. We always treat this by multiplying by $(1-x)/(1-x)$ for $x = e^{i\pi/n}$ here:

$$= -\frac{\pi i}{n} e^{\frac{i\pi}{2n}} \frac{1-e^{i\pi}}{1-e^{\frac{i\pi}{n}}}. \tag{20.14}$$

Rationalizing the denominator then gives us

$$= -\frac{2\pi i}{n} e^{\frac{i\pi}{2n}} \frac{1-e^{-\cdot\frac{i\pi}{n}}}{(1-e^{\frac{i\pi}{n}})(1-e^{-\frac{i\pi}{n}})}. \tag{20.15}$$

Simplifying, we have

$$= -\frac{2\pi i}{n} \frac{e^{\frac{i\pi}{2n}} - e^{-\frac{i\pi}{2n}}}{1-e^{\frac{i\pi}{n}} - e^{-\frac{i\pi}{n}} + 1}. \tag{20.16}$$

Now, we recall the definition of sin in terms of exponentials:

$$= -\frac{2\pi i}{n} \frac{2i\sin\frac{\pi}{2n}}{2-2\cos\frac{\pi}{n}}, \tag{20.17}$$

and we simplify the final result y collecting all the factors

$$= \frac{2\pi}{n} \frac{\sin\frac{\pi}{2n}}{1-\cos\frac{\pi}{n}}. \tag{20.18}$$

But, we are not done yet. The following manipulations are things that you might not easily recognize. But with practice, eventually you should be able to do them too. We have

$$1-\cos\frac{\pi}{n}=1-\left(\cos^2\left(\frac{\pi}{2n}\right)-\sin^2\left(\frac{\pi}{2n}\right)\right) \tag{20.19}$$

after using the double angle formula for the cosine. Then we replace $\sin^2$ by $1-\cos^2$ and vice versa

$$=1-2\cos^2\left(\frac{\pi}{2n}\right)+1=2\left(1-\cos^2\left(\frac{\pi}{2n}\right)\right)=2\sin^2\left(\frac{\pi}{2n}\right). \tag{20.20}$$

We put this together with our final result to obtain

$$\int_{-\infty}^{\infty}dx\frac{1}{1+x^{2n}}=\frac{2\pi}{n}\frac{\sin\frac{\pi}{2n}}{2\sin^2\left(\frac{\pi}{2n}\right)}=\frac{\pi}{n}\csc\frac{\pi}{2n} \tag{20.21}$$

and, because the integrand is even, we get:

$$\int_0^{\infty}\frac{1}{1+x^{2n}}=\frac{\pi}{2n}\csc\frac{\pi}{2n}, \tag{20.22}$$

which is a result that is hard to imagine one could evaluate analytically (or that the result is so simple). So, this is pretty remarkable stuff!
We have one final example:

$$I(a,b)=\int_0^{\pi}\frac{d\theta}{a+b\cos\theta},\quad 0<b<a. \tag{20.23}$$

This integral is only over a finite subset of the real axis, so our previous ideas do not look like they can be applied here. But the integrand is a function of sines and cosines. In this case, there is a new trick we can invoke. We want to change the integral from 0 to π to an integral over the unit circle. This works better if the integral goes from 0 to 2π, so we first extend the integral to twice the interval (now going from $-\pi$ to π) by using the fact that cosine is an even function and then we note that integrating over a whole period is independent of where one starts, so we can shift the integral by π without changing the final answer

$$I(a,b)=\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}, \tag{20.24}$$

where the factor of 1/2 comes from extending the integral to $[-\pi, 0]$. We now change the variables to $z = e^{i\theta}, dz = ie^{i\theta}d\theta$, and $d\theta = \frac{dz}{iz}$. Then we also have $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and

$$I(a,b) = \frac{1}{2}\oint \frac{dz}{iz}\frac{1}{a + \frac{b}{2}(z + \frac{1}{z})} = \frac{1}{2i}\oint \frac{1}{\frac{b}{2}z^2 + az + \frac{b}{2}} \tag{20.25}$$

$$= \frac{1}{ib}\oint dz \frac{1}{z^2 + 2z\frac{a}{b} + 1} \tag{20.26}$$

The roots of the denominator are easily found because it is a quadratic equation in z. We have the two roots given by

$$z_\pm = -\frac{2a}{2b} \pm \frac{1}{2}\sqrt{\frac{4a^2}{b^2} - 4} = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}. \tag{20.27}$$

Since $z_+z_- = 1$ [check it directly from the values for $z_\pm$ or compare $(z - z_+)(z - z_-) = z^2 - (z_+ + z_-)z + z_+z_-$ to the polynomial in the denominator], there must be one root *inside* and one root *outside* the circle. But since $0 < b < a$, this implies that the $-$ root is outside the circle and the $+$ root is inside the circle. The denominator can be written as $= (z - z_+)(z - z_-)$, so the residue is

$$\frac{1}{z_+ - z_-} = \frac{1}{\frac{2}{b}\sqrt{a^2 - b^2}} = \frac{b}{2\sqrt{a^2 - b^2}}. \tag{20.28}$$

The then integral becomes

$$I(a,b) = \frac{1}{ib}2\pi i\frac{b}{2\sqrt{a^2 - b^2}} = \pi\frac{1}{\sqrt{a^2 - b^2}} \quad \text{for } 0 < b < a. \tag{20.29}$$

Hence, we have

$$\int_0^\pi \frac{d\theta}{a + b\cos\theta} = \pi\frac{1}{\sqrt{a^2 - b^2}}. \tag{20.30}$$

We can check this with a case that is trivial to evaluate, namely the case with $b = 0$. In this case, we are integrating a constant, and the result is $I(a, 0) = \pi/a$. The other case that can be exactly determined (although this is harder to immediately see) is the case with $a = b$. In this case, we find that $I(a, a) \to \infty$, and that is what our answer gives too.

Note that this last integral might seem to be a challenging one for you. It is. But as you become more sophisticated in your math abilities, you need to be able to push yourself to the limit (pun intended).

I hope you will agree this method of doing integrals is <u>*very*</u> powerful.

Chapter 21

Gaussian Elimination

21.1 Solving m linear equations in n unknowns

In physics, we often have to solve simultaneous linear equations. These can be expressed in the following schematic form: $Ax = b$, where A is an $m \times n$ matrix, x is an n vector of unknowns, and b is a vector of m values. Written out, it looks like:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m. \end{aligned} \tag{21.1}$$

If $m \neq n$, there may be no solutions or infinitely many. The same turns out to be true when $m = n$ as well. In this lecture, we will focus on the square $m = n$ case. The book by Dettman shows an example of a rectangular case. The strategy to solve this problem is called Gaussian elimination. It is simply a systematic way of solving these equations as you might guess you would do it. We reduce the number of nonzero terms in each equation by one by adding or subtracting multiples of one equation from another. At the end, we can reverse substitute from the bottom up to solve the entire problem (if a solution exists). But the way of doing it is perhaps more strategic than how you would do it. It is foolproof in the sense that if a solution exists, you will always find it and you do so in an efficient fashion. Let's see how.

Start by multiplying row 1 by a_{21}/a_{11} and subtract it from row 2, assuming ($a_{11} \neq 0$). This is set up to cancel the leftmost coefficient in the second equation. Then repeat for $a_{31}, a_{41}...a_{n1}$ to get a new matrix with $n-1$ zeroes in the left most column. Then start with the second row and zero out all 2^{nd} column elements below it; then repeat for all subsequent columns. At this stage, the matrix looks like:

$$a'_{11}x_1 + a'_{12}x_2 + ... + a'_{1n}x_n = b'_1 \tag{21.2}$$

$$0 + a'_{22}x_2 + ... + a'_{2n}x_n = b'_2 \tag{21.3}$$

$$\vdots \tag{21.4}$$

$$0 + 0 + ... + a'_{nn}x_n = b'_n \tag{21.5}$$

Now, solve first for x_{n1}, then substitute into each row above in turn to get all solutions for $(x_1, ...x_n)$.

Some subtleties - if during row reduction, one of the first entries in the row that we subtract from all lower rows is equal to zero, then we cannot do the procedure. In this case, interchange that row with a lower one and continue to go with the algorithm. It is easiest to add the b column to the last column of the matrix when doing the calculations.

21.2 Concrete example of Gaussian elimination

It is easiest to see how to do this with an example:

$$\begin{aligned} x_1 \qquad\quad + 2x_3 - \ x_4 &= 0 \\ 2x_1 + \ x_2 - \ x_3 \qquad\ &= 5 \\ -x_1 + 2x_2 + \ x_3 + 2x_4 &= 3 \\ 3x_2 - 2x_3 + 5x_4 &= 1 \end{aligned} \tag{21.6}$$

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 5 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 1 \end{bmatrix}. \tag{21.7}$$

So our augmented matrix is:

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & \mathbf{0} \\ 2 & 1 & -1 & 0 & \mathbf{5} \\ -1 & 2 & 1 & 2 & \mathbf{3} \\ 0 & 3 & -2 & 5 & \mathbf{1} \end{array}\right], \tag{21.8}$$

Step 1: subtract 2× row 1 from row 2. Step 2: add row 1 to row 3. This yields

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & \mathbf{0} \\ 0 & 1 & -5 & 2 & \mathbf{5} \\ 0 & 2 & 3 & 1 & \mathbf{3} \\ 0 & 3 & -2 & 5 & \mathbf{1} \end{array}\right]. \tag{21.9}$$

Step 3: Subtract 2× row 2 from row 3. Step 4: Subtract 3× row 2 from row 4. This gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & \mathbf{0} \\ 0 & 1 & -5 & 2 & \mathbf{5} \\ 0 & 0 & 13 & -3 & \mathbf{-7} \\ 0 & 0 & 13 & -1 & \mathbf{-14} \end{array}\right]. \tag{21.10}$$

Step 5: Subtract row 3 from row 4. This finally gives

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & \mathbf{0} \\ 0 & 1 & -5 & 2 & \mathbf{5} \\ 0 & 0 & 13 & -3 & \mathbf{-7} \\ 0 & 0 & 0 & 2 & \mathbf{-7} \end{array}\right]. \tag{21.11}$$

Now we solve by reverse substitution. Starting from row 4, we find $x_4 = -\frac{7}{2}$. Then row 3 tells us that $13x_3 - 3x_4 = -7$

$$13x_3 = -\frac{21}{2} - 7 = -\frac{35}{2} \tag{21.12}$$

or

$$x_3 = -\frac{35}{26}. \tag{21.13}$$

Next, row 2 says $x_2 - 5x_3 + 2x_4 = 5$

$$x_2 = -\frac{175}{26} + 7 + 5 = \frac{-175 + 312}{26} = \frac{137}{26}. \tag{21.14}$$

Finally, row 1 implies that $x_1 + 2x_3 - x_4 = 0$

$$x_1 = \frac{35}{13} - \frac{7}{2} = \frac{70 - 91}{26} = -\frac{21}{26}. \tag{21.15}$$

So,

$$x_1 = -\frac{21}{26}, x_2 = \frac{137}{26}, x_3 = -\frac{35}{26}, \text{ and } x_4 = -\frac{7}{2}. \tag{21.16}$$

It can be seen that this procedure can be easily automated on a computer and it is also clear that it is a tedious approach that is easy to make an error while solving. So, using a computer is often better than using pencil and paper.

21.3 General considerations about Gaussian elimination

Uniqueness: The solution $Ax = b$ is unique unless one of the rows of the reduced matrix is all zeros. In that case, if the corresponding b value is not zero, then there is no solution. If it is zero, then there often are infinite solutions.

In essence, if $Ax = 0$ has solutions that are <u>*not*</u> $x = 0$, then any solution to $Ax = b$ can have any multiple of the solution of $Ax = 0$ added to it, and there is an infinite number of these. So, $Ax = b$ will have a unique solution if $Ax = 0$ has no solutions except $x = 0$.

We will find out in the next lecture that for square matrices, the system has no nontrivial solutions to $Ax = 0$ if and only if the determinant of A is nonzero. This is often the criterion used ($\det A = 0 \implies$ nontrivial solutions to $Ax = 0$) and you should become comfortable with it.

Chapter 22

Determinants

22.1 Definition of a determinant

The determinant of a matrix A with matrix elements a_{ij} is

$$\det A = \sum_{\substack{\text{permutations} \\ P \text{ of } n \text{ objects}}} (-1)^p a_{p_1 1} a_{p_2 2} a_{p_3 3} \ldots a_{p_n n}, \tag{22.1}$$

where $(p_1, p_2, \ldots, p_n)$ is a permutation of $(1, 2, \ldots, n)$ and

$$(-1)^p = \begin{cases} \ \ 1 & \text{if } p \text{ is an even permutation} \\ -1 & \text{if } p \text{ is an odd permutation.} \end{cases} \tag{22.2}$$

We illustrate this with some examples. The permutations are all possible reorderings of the set of numbers. For example, the six permutations of 3 numbers are 123, 231, 312, 132, 213, and 321. How do we determine whether a permutation is even or odd? Consider 312. To get back to 123, we need to

$$\begin{aligned} &\text{interchange 31 so then } 312 \to 132 \\ &\text{interchange 32 so then } 132 \to 123. \end{aligned} \tag{22.3}$$

In this case, these are two pair permutations. The number of pair permutations determines whether a permutation is classified as even or odd. An even permutation involves an even number of pair permutations. This implies that the even permutations make up one half of the $n!$ permutations of n numbers and odd permutations are the other half.

While this form for the definition of the determinant is simple to use for small matrices, it rapidly becomes painful to work with. Start with a 2×2 matrix

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underbrace{a_{11}a_{22}}_{12\to 12} - \underbrace{a_{21}a_{12}}_{12\to 21}, \tag{22.4}$$

where the 12 permutation is even $(+1)$ and the 21 permutation is odd (-1). The 3×3 matrix satisfies

$$\begin{aligned} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= \underbrace{a_{11}a_{22}a_{23}}_{123\to 123} + \underbrace{a_{31}a_{12}a_{23}}_{123\to 312} + \underbrace{a_{21}a_{32}a_{13}}_{123\to 231} \\ &- \underbrace{a_{11}a_{32}a_{23}}_{123\to 132} - \underbrace{a_{31}a_{22}a_{13}}_{123\to 321} - \underbrace{a_{21}a_{12}a_{33}}_{123\to 213} \end{aligned} \tag{22.5}$$

Where the permutations $123, 312, 231$ are even and the permutations $132, 321, 213$ are odd.

22.2 Properties of determinants

We now examine properties of the determinant. It turns out that the definition of the determinant implies a number of different powerful properties. These properties allow us to calculate determinants much more easily than having to sum $n!$ products of n terms. We start with some notation by introducing the completely antisymmetric tensor ϵ. We write

$$\det A = \sum_p \epsilon_{p_1 p_2 \ldots p_n} a_{p_1 1} a_{p_2 2} \ldots a_{p_n n} \tag{22.6}$$

with the completely antisymmetric tensor satisfying

$$\epsilon_{p_1 p_2 \ldots p_n} = \begin{cases} 0 & \text{any two } p_i\text{'s are the same} \\ 1 & p_1 \ldots p_n \text{ is an even permutation} \\ -1 & p_1 \ldots p_n \text{ is an odd permutation.} \end{cases} \tag{22.7}$$

Properties of the determinant:

1) If a row is all zeros, then $\det A = 0$ (because an element from each row appears in the product of each term, so each product has a zero in it). As a

result, the determinant ends up as the sum of $n!$ terms that are all zero and the determinant is 0.

2) If a column has all zeros, then $\det A = 0$ (the argument for this is similar to the one above).

3) If a row is multiplied by a constant λ, the determinant is multiplied by λ (since each row has an element that appears in one factor of each term in the sum for the determinant). The same thing happens for an entire column multiplied by a constant.

4) If two rows of a matrix are interchanged, the sign of the determinant changes ($\det A \to -\det A$). The interchange of a row is like adding a permutation to each term in the determinant. This changes the even permutations to odd permutations, and *vice versa*—hence, the sign of the determinant changes.

5) If two rows are proportional to each other, then $\det A = 0$. Take one row and multiply by a constant so it is equal to the second row. This changes the determinant by that factor according to property 4 above. Then, interchange the two rows—the matrix doesn't change, but $\det A = -\det A$, which implies that $\det A = 0$.

6) If row i can be written as $a_{ij} = b_{ij} + c_{ij}$ for $i \le j \le n$, then $\det A = \det B + \det C$ where B is the matrix A with the i^{th} row replaced by b_i and C is the matrix A with the i^{th} row replaced by c_i.

Proof: every place a_{ix} appears in a term in the determinant (one factor for each term), we replace the a_{ix} factor with $b_{ix} + c_{ix}$. Then we group all the b terms together and all the c terms together in the summation and we immediately see that $\det A = \det B + \det C$.

7) If we take α times row i and add to row j, it does not change the determinant. This is because we think of $b_{jx} = a_{jx}$ and $c_{jx} = a_{jx}\alpha$ but C has two rows proportional to one another, so $\det C = 0$ and $\det A = \det B + \det C$ implies that the determinant is unchanged.

8) The transpose of a matrix A^T interchanges the columns of the matrix with its rows—so $(A^T)_{ij} = a_{ji} = (A)_{ji}$. The determinant of $A^T = \det A$. This follows, since we can permute a product $a_{p_1 1} a_{p_2 2} a_{p_3 3} ... a_{p_n n}$ into the product $a_{1 p_1} a_{2 p_2} a_{3 p_3} ... a_{n p_n}$ by reordering the terms in the sum. The permutations remain even or odd, so overall, the determinant stays the same.

9) The ij cofactor of a matrix is defined to be $(-1)^{i+j}$ times the det of the $(n-1) \times (n-1)$ matrix whose elements are found by removing the i^{th}

row and the j^{th} column of A. The determinant satisfies

$$\det A = \sum_{j=1}^{n} a_{ij} c_{ij} \tag{22.8}$$

and similarly, we have

$$\det A = \sum_{i=1}^{n} a_{ij} c_{ij}. \tag{22.9}$$

This is often a useful way to find determinants if a row or column has many zeros.

10) If $C = AB$, such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, then $\det C = \det A \det B$.

Proof:

$$\begin{aligned}
\det C &= \sum_{p} \epsilon_{p_1 \cdots p_n} c_{p_1 1} c_{p_2 2} \cdots c_{p_n n} && (22.10)\\
&= \sum_{p} \epsilon_{p_1 \ldots p_n} \sum_{k_1=1}^{n} a_{p_1 k_1} b_{k_1 1} \sum_{k_2=1}^{n} a_{p_2 k_2} b_{k_2 2} \cdots \sum_{k_n=1}^{n} a_{p_n k_n} b_{k_n n} && (22.11)\\
&= \sum_{p_1 \cdots p_n} \sum_{k_1 \cdots k_n} \epsilon_{p_1 \cdots p_n} a_{p_1 k_1} a_{p_2 k_2} \cdots a_{p_n k_n} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n}. && (22.12)
\end{aligned}$$

So, we end up with

$$\sum_{k_1 \cdots k_n} \det A \cdot \epsilon_{k_1 \cdots k_n} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n} \tag{22.13}$$

or

$$\det A \cdot \det B \tag{22.14}$$

because

$$\sum_{p_1 \cdots p_n} \epsilon_{p_1 \cdots p_n} a_{p_1 k_1} a_{p_2 k_2} \cdots a_{p_n k_n} = \det A \cdot \epsilon_{k_1 \cdots k_n}. \tag{22.15}$$

This last identity summarizes how the permutation $k_1 k_2 \cdots k_n$ of $12 \cdots n$ is even or odd according to whether the completely antisymmetric tensor is 1 or -1.

These properties greatly simplify how one can compute a determinant. We show next how to calculate a determinant via row reduction with a concrete example. The strategy is to change the matrix by adding multiples of one row to another in order to make the entries below the diagonal all zeros.

The determinant is unchanged by all of these transformations. Here we go. The first equation took 3× the first row and subtracted it from the second. Then we subtract the first row from the fourth row. This makes zeros for all of the entries below the diagonal in the first column. Then we add 2× the second row to the third and subtract the second from the fourth. This zeroes out the second column below the diagonal. Finally, we take 3/10 of the third row and add to the fourth, to zero out the final subdiagonal element.

$$\begin{aligned}
\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 5 & 1 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix} &= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -8 & -12 \\ 0 & 2 & 6 & 0 \\ 0 & -1 & -5 & -1 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -8 & -12 \\ 0 & 0 & -10 & -24 \\ 0 & 0 & 3 & 11 \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -8 & -12 \\ 0 & 0 & -10 & -24 \\ 0 & 0 & 0 & 3.8 \end{pmatrix}. \qquad (22.16)
\end{aligned}$$

Now, the matrix has zeroes all below the diagonal. If you think of it, only one term in the expansion of the determinant now contributes. It is the product of all of the diagonal elements. so $\det = 1 \times -1 \times -10 \times 3.8 = 38$. This is how we calculate determinants via row reduction.

The other way we can do it is via cofactors. Here we expand the cofactors running down the fourth column (in red)

$$\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 5 & 1 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix} = -4 \times \det\begin{pmatrix} 3 & 5 & 1 \\ 0 & 2 & 6 \\ 1 & 1 & -2 \end{pmatrix} + 3 \times \det\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 1 \\ 0 & 2 & 6 \end{pmatrix}. \qquad (22.17)$$

Now, we evaluate the determinant of the remaining 3×3 matrices (from the cofactors) using the definition of the determinant

$$\begin{aligned}
\det &= -4(-12 + 30 + 0 - 2 - 0 - 18) + 3(30 + 0 + 18 - 0 - 36 - 2) \\
&= -4(-2) + 3(10) = 38. \qquad (22.18)
\end{aligned}$$

Lastly, we discuss the Hilbert matrix: $H_{ij} = \frac{1}{i+j-1}$ as an interesting matrix to determine determinants for. This set of matrices is constructed in such a way that the determinants become very mall very rapidly as the size of the matrix increases.

$$H_1 = 1, \tag{22.19}$$

$$H_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}, \tag{22.20}$$

$$H_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \tag{22.21}$$

$$H_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \ldots \tag{22.22}$$

Direct computation yields

$$\det H_1 = 1, \tag{22.23}$$

$$\det H_2 = 1 \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{22.24}$$

$$\begin{aligned} \det H_3 &= 1 \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} - 1 \cdot \frac{1}{4} \cdot \frac{1}{4} - \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{15} + \frac{1}{24} + \frac{1}{24} - \frac{1}{27} - \frac{1}{16} - \frac{1}{20}, \quad \text{use lcd} = 16 \times 27 \times 5 = 2160 \\ &= \frac{144 + 90 + 90 - 80 - 135 - 108}{2160} = \frac{1}{2160}. \end{aligned} \tag{22.25}$$

As we can see, the determinants get small very fast. In fact, $\det H_4 = \frac{1}{6{,}048{,}000}$. We will work with these matrices in the lab.

Chapter 23

Inverse of a Matrix

The inverse of a matrix A^{-1} satisfies

$$A \cdot A^{-1} = 1 = A^{-1} \cdot A, \tag{23.1}$$

where we assume square $n \times n$ matrices (in other words, the 1 stands for a diagonal $n \times n$ matrix whose diagonal elements are all ones; we also denote such a matrix with I). Writing out the matrix multiplication explicitly, we have

$$\sum_k a_{ik}\, a_{kj}^{-1} = \delta_{ij}, \qquad \sum_k a_{ik}^{-1} a_{kj} \; = \delta_{ij}. \tag{23.2}$$

The Kronecker delta function δ_{ij} is 1 if $i = j$ and 0 otherwise. If A is the inverse of a matrix, it is unique. Here is why. Suppose B was a different inverse. Then, $A \cdot B = I$. We multiply both sides of the equation on the left by A^{-1} to yield

$$A^{-1} \cdot A \cdot B = A^{-1} \cdot I. \tag{23.3}$$

Next, we recognize on the left hand side that $A^{-1} \cdot A = I$. Similarly, on the right-hand side, we have $A^{-1} \cdot I = A^{-1}$. In other words, we have

$$(A^{-1} \cdot A) \cdot B = A^{-1}, \tag{23.4}$$

$$I \cdot B = A^{-1}, \tag{23.5}$$

$$B = A^{-1}. \tag{23.6}$$

This implies that A^{-1} is unique!

Consider the following object

$$q_{ij} = \sum_k a_{ik} c_{jk}, \quad \text{where} \;\; c_{jk} = \text{ the } jk \text{ cofactor of } A. \tag{23.7}$$

If $i = j$, then $q_{ij} = \sum_k a_{ik} c_{ik} = \det A$ by the cofactor expansion for the determinant. If $i \neq j$, then $q_{ij} = \det$ of a matrix with the j^{th} row replaced by a_i. But then this matrix has two rows that are the same, so its determinant is 0. Hence,

$$q_{ij} = \det A \delta_{ij} = |A| \delta_{ij}, \tag{23.8}$$

where we introduced the shorthand notation $|A| = \det A$ for the determinant of A. If we define $(A^{-1})_{ij} = \frac{c_{ji}}{|A|}$ (this is the ijth element of the inverse of A; note the change in the order of the indices in c_{ji}), then

$$(A \cdot A^{-1})_{ij} = \sum_k a_{ik} \, a^{-1}_{kj} = \sum_k \frac{a_{ik} \cdot c_{jk}}{|A|} = \frac{|A| \delta_{ij}}{|A|} = \delta_{ij}. \tag{23.9}$$

So, we have that $A \cdot A^{-1} = I$, which implies A^{-1} is the unique inverse of A. This means that *we have an explicit formula for the inverse of the matrix A.* The problem is that it requires calculating many determinants so it is not so efficient to use for large matrices; but it is reasonable for 2×2 or 3×3 matrices. This formula is called *Cramer's rule for the inverse of a matrix.*

Note that since $\det AB = \det A \cdot \det B$, we have $\det A \cdot \det A^{-1} = \det I$ or $\det A \cdot \det A^{-1} = 1$. This then implies that $|A^{-1}| = \frac{1}{|A|}$. So, we see that an inverse will exist if and only if $|A| \neq 0$.

Example: Use this method to find the inverse of

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} = A. \tag{23.10}$$

To start, we compute the determinant using the multiply along "diagonals" and add or subtract rule)

$$\det A = 1 \cdot 0 \cdot 3 + 2 \cdot 1 \cdot 1 + 1 \cdot (-1) \cdot 2 - 1 \cdot 0 \cdot 1 - 1 \cdot 1 \cdot 2 - 2 \cdot (-1) \cdot 3 = 0 + 2 - 2 - 0 - 2 + 6 = 4. \tag{23.11}$$

Next, we construct the matrix of cofactors (called the transpose of the adjugate matrix for jargon lovers), given by

$$C = \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} \\ \\ -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ \\ \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \end{pmatrix}, \tag{23.12}$$

which becomes

$$C = \begin{pmatrix} -2 & 4 & -2 \\ -4 & 2 & 0 \\ 2 & -2 & 2 \end{pmatrix}. \tag{23.13}$$

Taking the transpose, we obtain

$$C^T = \begin{pmatrix} -2 & -4 & 2 \\ 4 & 2 & -2 \\ -2 & 0 & 2 \end{pmatrix}, \tag{23.14}$$

which is the adjugate matrix. Dividing by the determinant, we find

$$A^{-1} = \frac{C^T}{|A|} = \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{23.15}$$

Now we check by matrix multiplication of $A \cdot A^{-1}$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23.16}$$

This verifies that $A \cdot A^{-1} = I$.

Because determinants are very difficult to calculate for large matrices, we now describe a more efficient way to compute the inverse via row reduction. The only change is that in this case the augmented matrix includes the identity matrix rather than just one column of b, and we row reduce to produce zeros *both above and below* the diagonal. We illustrate this methodology with the same example as before:

$$\left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \tag{23.17}$$

In the first step, we zero out the first column, as well as the second column:

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right) \tag{23.18}$$

We now normalize so that the diagonals are 1:

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right). \tag{23.19}$$

Now we remove the upper zeros. We do the third column first:

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 2 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right), \tag{23.20}$$

and then the second

$$\Longrightarrow \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right). \tag{23.21}$$

The inverse is now read off from the augmented part:

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{23.22}$$

This is the same inverse as we found before!

We can formally use the inverse found from cofactors to solve the original problem $Ax = b$ via a multiplication by A^{-1} on the left of both sides of the equation and then using the cofactor-based formula:

$$A^{-1}Ax = A^{-1}b \implies I \cdot x = A^{-1}b \implies x = A^{-1} \cdot b \tag{23.23}$$

$$A^{-1} = \frac{C^T}{|A|} \text{ so } x_i = (A^{-1} \cdot b)_i = \sum_k \frac{C_{ki}}{|A|} b_k \tag{23.24}$$

or $x_i = \frac{1}{|A|} \sum_k b_k c_{ki}$ - this is also called Cramer's Rule.

Note that this sum is the determinant of the matrix made by taking A and replacing the i^{th} column by the vector b.

This method is often not good for calculations unless a specific matrix element or a specific x_i is needed, then it can provide a fast way to directly obtain that result only.

Now for some nomenclature: An orthogonal matrix O satisfies $O^{-1} = O^T$ (its inverse is its transpose). Orthogonal matrices are made from unit vectors that are all perpendicular on the rows or the columns. A unitary matrix is a general complex matrix which satisfies the inverse is the Hermitian conjugate or $U^{-1} = (U^*)^T$ where U^* mean take the complex conjugate of all elements of the matrix U. These are formed by rows or columns of unit vectors using the complex norm which will be discussed in a future lecture.

If you know your matrix is orthogonal or unitary, then you can compute its inverse very easily. This can save huge time in calculations.

Chapter 24

Vector Spaces

24.1 Definition of a vector space

We start with describing the abstract definition of a vector space. The idea is that vectors do not need to be just the n-tuples in $\mathbb{R}^n$ that we are used to, but instead, we abstract the critical properties that vectors in $\mathbb{R}^n$ have and then any other set of objects that satisfy these properties will also form a vector space. If we can prove properties of the vector space using just these bare requirements, then anything that satisfies these properties also satisfies the theorems we prove. This is in essence how mathematicians use abstraction to generalize properties. For physics, these ideas become most important when we study quantum mechanics. Quantum mechanics works with the abstraction that functions are infinite-dimensional vectors. We won't go into extensive detail here, but this is why these ideas and the ability to use abstraction are so important to physicists.

Vector space: A vector space is a collection of objects $v \in V$ that have the arithmetic operations of addition and multiplication defined by the following postulates:

1. If $\vec{u}$ and $\vec{v}$ are in V, then so is $\vec{u} + \vec{v}$ (closure under addition)

2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (vector addition is commutative)

3. $u + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (vector addition is associative)

4. There is a zero vector such that $\vec{v} + \vec{0} = \vec{v}$ for all $v \in V$. (existence of an additive identity)

5. If $\vec{v}$ is in V, then there is a vector $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = \vec{0}$. This is called the negative of $\vec{v}$. (existence of an additive inverse)

6. If a is a scalar and $\vec{v} \in V$, then $a\vec{v} \in V$ (closure under scalar multiplication)

7. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ (distributive property)

8. $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ (distributive property)

9. $(ab)\vec{v} = a(b\vec{v})$ (scalar multiplication is associative)

10. $1\vec{v} = \vec{v}$ (existence of a scalar multiplicative identity)

If all scalars are real numbers, this is a *real vector space*. If all scalars are complex numbers, this is a *complex vector space*. Note this definition comes from the scalars in the vector space. Ordinary vectors like $\mathbb{R}^3$ are real vector spaces.

A subspace is a vector space $S \subset V$ such that all vectors in the subspace form a vector space. An example of this is that $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$. Note: The zero vector is always in a vector subspace because any vector space must have an additive inverse.

A set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n, \}$ are *dependent* if there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = 0$ and not all scalars α_i are zero. This definition can be difficult to understand. Let's look at an example: $(1, 0, 0), (1, 1, 0), (0, 1, 0)$ are dependent since $-1 \times (1, 0, 0) + 1 \times (1, 1, 0) + (-1) \times (0, 1, 0) = 0$. Note how at least one of the scalars is nonzero—one does not need all of them to be nonzero, usually at least two is sufficient.

A set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n\}$ is said to span V if every vector in V can be written as $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$. If the set of vectors is dependent, we can reduce it by removing basis vectors that can be expressed in terms of the remaining ones until we finally reach a set of independent spanning vectors. The *number* of these vectors is called the *dimension* of V. Because these vectors are independent, the scalars used to represent any vector are unique to that vector. They are called the coordinates of the vector in the given basis. While it may sound abstract, you are familiar with it. Consider the basis vectors $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, and $\hat{k} = (0, 0, 1)$ in $\mathbb{R}^3$. The normal way we write a vector as $\vec{v} = (v_1, v_2, v_3)$ is actually a shorthand for $v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$—we already know that these numbers v_1, v_2 and v_3 are a *unique* representation of the vector $\vec{v}$.

24.2 Scalar product

Let V be a complex vector space. Then the complex-valued scalar product satisfies:

1. $(\vec{u} \cdot \vec{v}) = (\vec{v} \cdot \vec{u})^*$ (note the complex conjugate!)

2. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$

3. $(a\vec{u} \cdot \vec{v}) = a(\vec{u} \cdot \vec{v})$ (Note that this means $\vec{u} \cdot (a\vec{v}) = a^* \vec{u} \cdot \vec{v}$)

4. $(\vec{u} \cdot \vec{u}) \geq 0$ for all $\vec{u} \in V$

5. $(\vec{u} \cdot \vec{u}) = 0$ implies $\vec{u} = \vec{0}$.

Example: n-tuples of complex numbers: $\vec{u} = (u_1, ...u_n)$, $\vec{v} = (v_1, ...v_n)$. So $(\vec{u} \cdot \vec{v}) = u_1 v_1^* + ... + u_n v_n^*$. You should verify that this satisfies all properties of a scalar product.

We now define a norm, which can be thought of as the length of a vector. A norm is less general than an inner product. We will see all vector spaces with inner products automatically have norms, but the converse is not always true—a normed vector space need not also have an inner product.

Norm: The norm is defined by $\|\vec{u}\| = (\vec{u} \cdot \vec{u})^{\frac{1}{2}}$

1. $\|\vec{u}\| \geq 0$

2. $\|\vec{u}\| = 0 \implies \vec{u} = \vec{0}$

3. $\|a\vec{u}\| = |a| \|\vec{u}\|$

4. $|(\vec{u} \cdot \vec{v})| \leq \|\vec{u}\| \|\vec{v}\|$, this is called the Cauchy Identity.

5. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$, this is called the Triangle Inequality.

Proof of property 4 (Cauchy inequality): If $\vec{u} = 0$ or $\vec{v} = 0$, then property 4 obviously holds, so let's assume $\vec{u} \neq 0$ and $\vec{v} \neq 0$. Also assume that $(\vec{u} \cdot \vec{v}) \neq 0$, otherwise the inequality also obviously holds. So we define

$\lambda = \frac{|(\vec{u}\cdot\vec{v})|}{(\vec{u}\cdot\vec{v})}$, and note that $|\lambda| = 1$. Then consider

$$\begin{aligned}
0 \leq \left\| \frac{\lambda\vec{u}}{\|u\|} - \frac{\vec{v}}{\|v\|} \right\|^2 &= |\lambda|^2 \frac{\|u\|^2}{\|u\|^2} - \frac{\lambda(\vec{u}\cdot\vec{v}) + \lambda^*(\vec{v}\cdot\vec{u})}{\|u\|\|v\|} + \frac{\|v\|^2}{\|v\|^2} \\
&= 1 - \frac{|(\vec{u}\cdot\vec{v})| + |(\vec{v}\cdot\vec{u})|}{\|u\|\|v\|} + 1 \\
\implies 2|(\vec{u}\cdot\vec{v})| \leq 2\|u\|\|v\| & \qquad (24.1)
\end{aligned}$$

so, $|(\vec{u}\cdot\vec{v})| \leq \|u\|\|v\|$.

Now we prove the Triangle Inequality (Property 5):

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= (\vec{u}\cdot\vec{u}) + (\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{u}) + (\vec{v}\cdot\vec{v}) \\
&= \|\vec{u}\|^2 + 2\,\mathrm{Re}\,(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2|(\vec{u}\cdot\vec{v})| + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\
&\leq (\|\vec{u}\| + \|\vec{v}\|)^2 \\
\implies \|\vec{u}+\vec{v}\| &\leq \|\vec{u}\| + \|\vec{v}\|. \qquad (24.2)
\end{aligned}$$

24.3 Polynomials and functions as vectors

Let's now try a nontrivial example. Let $V_n = \{$polynomials of degree $< n$ with complex coefficients$\}$ which is a complex vector space and let $(\vec{u}\cdot\vec{v}) = u_1 \cdot v_1^* + ... + u_n \cdot v_n^*$. We will denote the two vectors as $\vec{u} = u_1 + u_2 x + ... + u_n x^{n-1}$, and $\vec{v} = v_1 + v_2 x + ... + v_n x^{n-1}$ (note the overarrow is schematic, these are *not* n-tuples in a conventional vector space). If we think of the powers of x in the polynomial as the "placeholder" for the different elements in a conventional n-dimensional vector, then we can immediately see that this vector space is isomorphic to $\mathbb{C}^n$. We can think of the basis vectors being $1, x, x^2, \cdots, x^{n-1}$, and so on.

A more complex example is the vector space of real-valued continuous functions on the interval $x : 0 \leq x \leq 1$. The scalar product is:

$$(f\cdot g) = \int_0^1 dx f(x) g(x). \qquad (24.3)$$

On can immediately verify that all properties of a scalar product hold—the only one that really needs to be checked is

$$(f \cdot f) = \int_0^1 dx \Big(f(x)\Big)^2 \geq 0, \tag{24.4}$$

which is clear for real-valued functions.

What is a basis for this vector space? We can show that $\{\sin(2\pi x), \sin(4\pi x), \sin(6\pi x), \cdots, \sin(2n\pi x)\}$ are all independent. If they were not independent, then

$$c_1 \sin(2\pi x) + c_2 \sin(4\pi x) + c_3 \sin(6\pi x) + ... + c_n \sin(2n\pi x) = 0 \tag{24.5}$$

for some set of $c_1 ... c_n$ that are not all zero. Multiply the above by $\sin(2m\pi x)$ and integrate from 0 to 1. Note that

$$\sin^2(2m\pi x) = \frac{[]1 - \cos(4m\pi x)]}{2} \tag{24.6}$$

and

$$\sin(2m\pi x)\sin(2m'\pi x) = \frac{[\cos(2m\pi - 2m'\pi)x - \cos(2m\pi + 2m'\pi)x]}{2} \tag{24.7}$$

for $m \neq m'$. So we can integrate to find

$$\int_0^1 dx \sin^2(2m\pi x) = \int_0^1 \frac{[]1 - \cos(4m\pi x)]}{2} = \frac{1}{2} - \frac{\sin(4m\pi x)}{8m\pi}\Bigg|_0^1 = \frac{1}{2}$$

$$\int_0^1 dx \sin(2m\pi x)\sin(2m'\pi x) = \int_0^1 dx \left[\frac{\cos(2m\pi x - 2m'\pi x) - \cos(2m\pi x + 2m'\pi x)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(2m\pi x - 2m'\pi x)}{2m\pi - 2m'\pi} - \frac{\sin(2m\pi x + 2m'\pi x)}{2m\pi + 2m'\pi}\right]\Bigg|_0^1 = 0. \tag{24.8}$$

So we find that $c_m = 0$. We repeat this for all m and find the solution holds for all m. This then implies that $c_1 = c_2 = ...c_n = 0$, which implies this set of functions is independent for this vector space.

Chapter 25

Scalar Products and Orthonormal Bases

Last chapter, where we defined vector spaces, showed us how to define a scalar product and a norm and we even started looking at the dimensionality of a vector space. Today, we continue this theme and develop the so-called Gram-Schmidt orthogonalization procedure.

Supposed I have a set of four 4-tuples. How do I tell whether they span $\mathbb{R}^4$? (Spanning means any vector in $\mathbb{R}^4$ can be expressed as a linear combination of the given 4 vectors.)

Hence, if they span R^4, then any vector (x_1, x_2, x_3, x_4) can be written in terms of the 4 vectors. Let's say they are: $u_1 = (1,1,1,1), u_2 = (1,-1,1,-1), u_3 = (1,2,3,4), u_4 = (1,0,2,0)$ Then we want to solve $c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3} + c_4\vec{u_4} = \vec{x}$ for nonzero c's. This can be written in matrix form as $Mc = x$, with M being

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & -1 & 4 & 0 \end{pmatrix} \tag{25.1}$$

If $\det M \neq 0$, then these equations can be solved. So we compute the determinant. Note that one column has two zeros in it. So we expand by minors on the fourth column.

$$\det M = (-1)\cdot 1 \cdot \det \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{pmatrix} - (2)\cdot \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{pmatrix} \tag{25.2}$$

$$= -(4-3-2-2+3+4) - 2(-4+2-1+1+2-4) = -4+8 = 4 \quad (25.3)$$

Since this is nonzero, a solution exists. Hence, the vectors span R^4. But since the dimension of R^4 is 4, we need at least 4 vectors to span, so these vectors do form a basis for R^4. One can see that finding the coordinates in this basis is painful because the basis vectors are not orthogonal.

In general, having orthonormal basis vectors is convenient for doing calculations. One set for R^n is just $(1,0,0,...),(0,1,0,....),...,(0,0,1,0,...)$, the so-called Cartesian basis. But there is another basis that is easy to understand when one basis vector must be $\vec{e_1} = (1,1,1,...,1)\frac{1}{\sqrt{N}}$ then the other orthonormal vectors are $\vec{e_2} = (1,-1,0,0,...,0)\frac{1}{\sqrt{2}}$, $\vec{e_3} = (1,1,-2,0,...,0)\frac{1}{\sqrt{6}}$, ..., $\vec{e}_n = (1,1,1,...,1,-n)\frac{1}{\sqrt{n^2+n-1}}$. One can check $\vec{e_i}\cdot\vec{e_j} = 0$ directly for $i \neq j$ and $\vec{e_i}\cdot\vec{e_i} = 1$ by construction.

So is there a direct way to construct an orthonormal basis from a set of independent spanning vectors? The answer is yes and the procedure is called *Gram-Schmidt orthogonalization.*

Let $\vec{u_1}, \vec{u_2}, ..., \vec{u_n}$ = spanning, independent set. Then let $\vec{e_1} = \frac{\vec{u_1}}{\sqrt{(\vec{u_1}\cdot\vec{u_1})}}$ so $\vec{e_1}$ has a unit norm. Then let $\vec{e_2} = \frac{\vec{u_2}-\vec{e_1}(\vec{e_1}\cdot\vec{u_2})}{\text{normalization}}$ and then normalize. This projects $\vec{u_2}$ onto $\vec{e_1}$ and removes that projection from $\vec{e_2}$ so that $\vec{e_2}\cdot\vec{e_1} = 0$. Now proceed the same way for the other cases -

$$\vec{e_3} = \frac{\vec{u_3} - \vec{e_1}(\vec{e_1}\cdot\vec{u_3}) - \vec{e_2}(\vec{e_2}\cdot\vec{u_3})}{\text{normalization}} \quad (25.4)$$

In each step, we project out the parts that are on the vectors we already have chosen. Consider our earlier example:

$$\{\vec{u_i}\} = \{(1,1,1,1),(1,-1,1,-1),(1,2,3,4),(1,0,2,0)\} \quad (25.5)$$

So $\vec{e_1} = \frac{1}{2}(1,1,1,1)$, $\vec{e_2} = \frac{(1,-1,1,-1)-0\cdot\vec{e_1}}{\text{norm}} = \frac{1}{2}(1,-1,1,-1)$

$$\vec{e_3} = \frac{(1,2,3,4) - \frac{1}{2}\cdot(1,1,1,1)\cdot 5 - \frac{1}{2}(1,-1,1,-1)\cdot(-1)}{\text{norm}} = \frac{1}{2}(-1,-1,1,1) \quad (25.6)$$

$$\vec{e_4} = \frac{(1,0,2,0) - \frac{1}{2}\cdot(1,1,1,1)\cdot\frac{3}{2} - \frac{1}{2}(1,-1,1,-1)\cdot\frac{3}{2} - \frac{1}{2}(-1,-1,1,1)\cdot\frac{1}{2}}{\text{norm}} \quad (25.7)$$

$$= (-\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4}) \implies \frac{1}{2}(-1,1,1,-1) \quad (25.8)$$

So the basis is $\frac{1}{2}(1,1,1,1), \frac{1}{2}(-1,-1,1,1)\frac{1}{2}(1,-1,1,-1)$ and $\frac{1}{2}(-1,1,1,-1)$, which is an orthonormal basis.

Suppose we now construct an orthonormal basis for the set of regular functions on $[0,1]$ with scalar product $\int_0^1 f(x)g(x)dx$. Start with the first vector being a constant: $\vec{e}_0 = 1$. Then the second is a linear function: $\vec{e}_1 = ax+b$; the third is a quadratic: $\vec{e}_2 = \alpha x^2 + \beta x + \gamma + ...$ and so on. The first vector is already normalized since $\int_0^1 |\vec{e}_0|^2 dx = 1$. Now we apply Gram-Schmidt.

$$\int \vec{e}_0\vec{e}_1 dx = 0 \implies \int_0^1 (ax+b)dx = 0 \implies \left(\frac{ax^2}{2}+bx\right)\Big|_0^1 = 0 \implies \frac{a}{2}+b=0 \tag{25.9}$$

$$\int |\vec{e}_1|^2 dx = 1 \implies \int_0^1 (ax+b)^2 dx = \int_0^1 (a^2x^2+2abx+b^2)dx = a^2\frac{1}{3}+ab+b^2 = 1 \tag{25.10}$$

So $b = \frac{-a}{2}$, and $(\frac{a^2}{3} - \frac{a^2}{2} + \frac{a^2}{4}) = 1$ so $a^2 = 12 \implies a = 2\sqrt{3}$ and $b = -\sqrt{3}$. Therefore, $\vec{e_1} = 2\sqrt{3}x - \sqrt{3}$.

For e_2, we have three integrals to compute:

$$\int \vec{e}_0\vec{e}_2 dx = 0 \implies \int_0^1 (\alpha x^2 + \beta x + \gamma) = 0 \implies \tfrac{\alpha}{3} + \tfrac{\beta}{2} + \gamma = 0; \tag{25.11}$$

$$\int \vec{e}_1\vec{e}_2 dx = 0 \implies \int_0^1 (2\sqrt{3}x - \sqrt{3})(\alpha x^2+\beta x+\gamma) = 0 \implies \tfrac{\sqrt{3}}{6}\alpha + \tfrac{\sqrt{3}}{6}\beta = 0; \tag{25.12}$$

and

$$\int |\vec{e}_2|^2 dx = 1 \implies \int_0^1 (\alpha x^2+\beta x+\gamma)^2 dx = 1 \implies \tfrac{\alpha^2}{5}+\tfrac{\alpha\beta}{2}+\tfrac{2\alpha\gamma+\beta^2}{4}+\beta\gamma+\gamma^2 = 1. \tag{25.13}$$

Solving these equations yields

$$\alpha = 6\sqrt{5}, \quad \beta = -6\sqrt{5} \quad \text{and} \quad \gamma = \sqrt{5}. \tag{25.14}$$

Hence, we have $\vec{e}_2 = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$.

One can continue this procedure indefinitely, but it is complicated and tedious; it can be completed on a computer. Such collections of polynomials are called orthogonal polynomials.

Chapter 26

Change of Bases

26.1 Linear maps

We consider linear transformations from a vector space U (dimension n) to a vector space V (dimension m) that is linear (see Fig. 26.1 for a schematic).

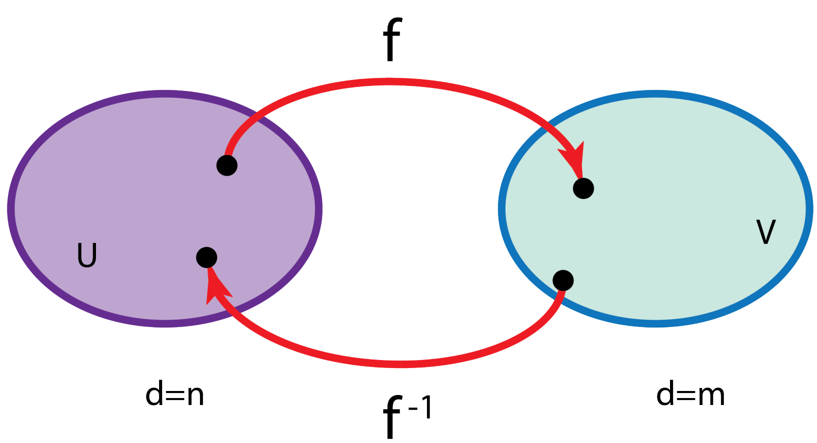

Figure 26.1: Schematic of a map f between two vector spaces U and V. The map produces an element in V for each element in U. In some cases, the map is invertible, so that one can find the inverse map (from V to U) such that $f^{-1}(f(u)) = u$. In this figure, the dimensions of the vector spaces are n and m, respectively. When an inverse exists, we necessarily have $n = m$.

The map f is *linear* if $f(a\vec{u}_1 + b\vec{u}_2) = af(\vec{u}_1) + bf(\vec{u}_2)$. The null space is defined to be the subspace of U consisting of all vectors in U that map to $\vec{0}$

in V. If the null space is just the identity element $\vec{0}$ in U, then the function or mapping is invertible and one can go backwards from V to U. (The fact that the map is linear requires the vector $\vec{0}$ in U to map to the vector $\vec{0}$ in V.)

Mappings are abstract functions. But with linear maps, we can always be concrete. If we use a set of basis vectors in U (and in V), then we can construct a matrix that faithfully represents the mapping f. The matrix is represented via a basis in U and one in V as follows: Suppose $\{\vec{e}_j^{\,u} : j = 1, ..., n\}$ = an orthonormal basis for U, and $\{\vec{e}_i^{\,v} : i = 1, ..., m\}$ = is an orthonormal basis for V. (Orthonormal means each basis vector has norm equal to 1 and they are mutually orthogonal with each other; this is the same as the conventional Cartesian basis we use in $\mathbb{R}^n$.) Then we define M_{ij} via

$$f(\vec{e}_j^{\,u}) = \sum_{j=1}^{m} M_{ij}\vec{e}_i^{\,v}, \tag{26.1}$$

with M_{ij} being numbers. Then, if $\vec{u} = u_1\vec{e}_1^{\,u} + u_2\vec{e}_2^{\,u} + ... + u_n\vec{e}_n^{\,u}$,

$$f(\vec{u}) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\vec{e}_i^{\,v} u_j = \sum_{i=1}^{m} v_i\vec{e}_i^{\,v}, \tag{26.2}$$

which we write as $M\vec{u} = \vec{v}$.

So, for the general case, we have that M is an $n \times m$ matrix. This is depicted schematically in Fig. 26.2. The inverse transformation, if it exists, is given by M^{-1} (which requires $m = n$).

26.2 Changing the bases of a map

What happens if we are working in a basis for a given map and we would like to use a new one? This might be because it simplifies the problem, or allows us to understand the relationship between two different ways of looking at things. Back in the late 1920's, it turns out that the relationship between matrix mechanics and wave mechanics was worked out when Schrödinger determined the transformation that connected the two. So this ends up being an important subject, especially for quantum mechanics.

Suppose we transform the bases of the vector spaces U and V as follows:

$$\{\vec{e}_j^{\,u}\} \xrightarrow{P} \{\vec{e}_j^{\,\prime u}\} \tag{26.3}$$

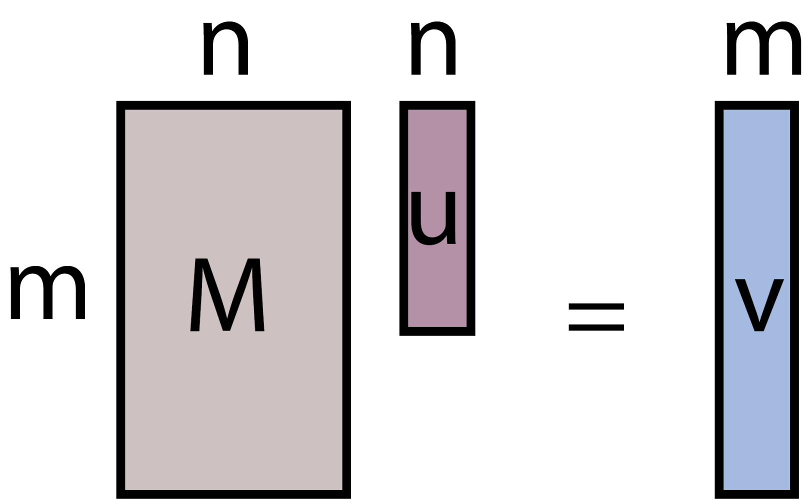

Figure 26.2: Schematic of the $n \times m$ matrix that maps vectors in U to V and represents the general linear map f. We can only have invertible maps when $n = m$ (but having $n = m$ does not guarantee that the map is invertible).

given by a matrix P_{ij}, which maps $U \to U$, and

$$\{\vec{e}_i^{\,v}\} \xrightarrow{Q} \{\vec{e}_i^{\,\prime v}\} \tag{26.4}$$

given by a matrix Q_{ij}, which maps $V \to V$. If a vector $\vec{u}$ is written as $(u_1, u_2, ..., u_n)$ in the basis $\{\vec{e}_j^{\,u}\}$ and as $(u'_1, u'_2, ..., u'_n)$ in the basis $\{\vec{e}_j^{\,\prime u}\}$, then

$$P\vec{u} = \vec{u}\,' \text{ or } \vec{u} = P^{-1}\vec{u}\,' \tag{26.5}$$

or, more pictorially, as

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & \ddots & \\ \vdots & & p_{nn} \end{bmatrix}^{-1} \begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_n \end{bmatrix}. \tag{26.6}$$

We have a similar result for $\vec{v}$ and $\vec{v}\,'$, with

$$Q\vec{v} = \vec{v}\,' \tag{26.7}$$

and Q an $m \times m$ matrix. But M denotes the transformation f in the original basis, so

$$\vec{v} = M\vec{u} \tag{26.8}$$

holds in general. Now we replace v and u in terms of v' and u' to find that

$$Q^{-1}\vec{v}\,' = MP^{-1}\vec{u}\,'. \tag{26.9}$$

We next multiply by Q on the left to obtain

$$\vec{v}\,' = QMP^{-1}\vec{u}\,'. \tag{26.10}$$

Hence, $M \to M' = QMP^{-1}$ is the transformed matrix when we change the bases of U and V. The case where we map $U \to U$, $(V = U)$ and $(m = n)$ is particularly interesting. In this case, we have

$$\vec{v} = \vec{u}, \quad \vec{v}\,' = \vec{u}\,', \quad \text{and} \quad Q = P, \text{ so} \tag{26.11}$$

$$M' = PMP^{-1}, \tag{26.12}$$

which is called a <u>*similarity*</u> <u>*transformation*</u>, and we say that M is similar to M'.

Some properties of similarity transformations:
1) If A is similar to B and B is similar to C then A is similar to C. The proof of this is obvious, if you just write it out directly.
2) If A is similar to B, then $\det(A) = \det(B)$.
Proof: Since $\det(AB) = \det(A)\det(B)$, we have

$$\det(PP^{-1}) = \det(I) = 1 = \det(P)\det(P^{-1}) \tag{26.13}$$

$$\Longrightarrow \det(P^{-1}) = \frac{1}{\det(P)}. \tag{26.14}$$

Hence, $\det(B) = \det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(A)$.
3) If A is similar to B, then A^k is similar to B^k. Proof:

$$B = PAP^{-1}. \tag{26.15}$$

Now look at the square of B,

$$B^2 = (PAP^{-1})(PAP^{-1}) = PA^2P^{-1} \tag{26.16}$$

since $P^{-1}P = 1$. This means A^2 is similar to B^2 with the same matrix in the similarity transformation! We continue this pattern and find immediately that

$$B^k = PA^kP^{-1}. \tag{26.17}$$

When P is an orthogonal matrix (which is common for these similarity transformations), then its rows (or columns) are orthonormal vectors, so that $P^TP = I \implies P^{-1} = P^T$. Then we have

$$\begin{aligned} P^TP = \begin{bmatrix} \leftarrow e_1 \rightarrow \\ \leftarrow e_2 \rightarrow \\ \vdots \\ \leftarrow e_n \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ e_1 & e_2 & ... & e_n \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix} &= \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 & ... & \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 & & \\ \vdots & & \ddots & \\ & & & \vec{e}_n \cdot \vec{e}_n \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}. \end{aligned} \tag{26.18}$$

If P is unitary, then its rows and columns are orthonormal basis vectors of a complex vector space and $PP^\dagger = I$, so $P^{-1} = P^\dagger$ where $\dagger$ denotes both the transpose and complex conjugation. These are two important types of similarity transformations that you will run into many times in your future physics courses.

Chapter 27

Eigenvalues and Eigenvectors

27.1 Main Idea

When a matrix multiplies a vector it can do two things: (i) it rotates the vector and (ii) it changes its length. Indeed, what else can it do?, as matrix multiplication maps a vector to another vector. What is *absolutely amazing* is that for a class of matrices called Hermitian matrices, one can find exactly n different orthogonal basis vectors for an $n \times n$ matrix such that when the matrix multiplies each of those basis vectors, there is no rotation, only scaling.

Hence, eigenvalues and eigenvectors are found by choosing the basis for the map such that when the matrix M multiplies one of the basis vectors $\vec{e}_i$ it results in a number λ_i times the basis vector $\vec{e}_i$. In equations, we have

$$M\vec{e}_i = \lambda_i \vec{e}_i. \tag{27.1}$$

In words we summarize as follows:

"A matrix times a vector equals a number times that vector."

Keep this mantra in mind to help you always understand what an eigenvalue and an eigenvector are. It is probably the most misunderstood concept by students of physics until they finally get it.

Again,

"A matrix times a vector equals a number times that vector."

How do we find these solutions? We rearrange the equations to have a shifted matrix annihilating a vector as follows:

$$\sum_{j=1}^{n} M_{ij}e_j = \lambda e_i \implies (M - \lambda I)\vec{e} = 0. \tag{27.2}$$

This has a nontrivial solution only if $\det(M - \lambda I) = 0$, which is an n-degree polynomial in λ. It will have n solutions, but if we work with a real vector space only the real solutions are valid, so it may have fewer than n *real* solutions.

27.2 Some Properties from the Spectral Theorem

If M is real and symmetric, it has exactly n real eigenvalues (this is called a symmetric matrix; the results actually hold a bit more broadly to include Hermitian matrices when matrix elements are complex valued). In addition, there might be multiple roots where $\lambda_i = \lambda_j$. When this occurs we say there is a *degeneracy*.

The eigenvectors are found by solving the equation

$$M\vec{e}_\lambda = \lambda\vec{e}_\lambda \tag{27.3}$$

for the specific λ value chosen (we already found the possible λ values when we found the roots to $\det(M - \lambda I) = 0$). Note, this becomes a row-reduction problem, which we have already solved.

If M is Hermitian, it also has exactly n real solutions. (A Hermitian matrix satisfies $M^{T*} = M$, which means we take the transpose and the complex conjugate of all elements; a symmetric matrix has all real values and satisfies $M^T = M$—obviously all symmetric matrices are also Hermitian). This result is called the spectral theorem. We will not prove the spectral theorem here.

One important property is that the the eigenvectors are orthogonal to each other if they have different eigenvalues. In the derivation below, we use a λ superscript to denote the eigenvalue associated with the eigenvector $\vec{e}^{\,\lambda}$. To see that the corresponding eigenvectors are orthogonal, note that the relation $\sum_j M_{ij}\vec{e}_j^{\,\lambda} = \lambda\vec{e}_i^{\,\lambda}$ can be rewritten as $\sum_j \vec{e}_j^{\,\lambda} M_{ji} = \lambda\vec{e}_i^{\,\lambda}$ because the matrix M is symmetric ($M_{ij} = M_{ji}$). So, let's compute $\sum_{ij} \vec{e}_i^{\,\lambda} M_{ij} \vec{e}_j^{\,\lambda'}$. By summing over j first, this becomes $\lambda' \vec{e}^{\,\lambda} \cdot \vec{e}^{\,\lambda'}$. But summing over i first, we have $\lambda \vec{e}^{\,\lambda} \cdot \vec{e}^{\,\lambda'}$. Of course, these two must be equal, so we learn that

$$(\lambda - \lambda')\vec{e}^{\,\lambda} \cdot \vec{e}^{\,\lambda'} = 0. \tag{27.4}$$

Hence, if $\lambda \neq \lambda'$, then we must have $\vec{e}^{\,\lambda} \cdot \vec{e}^{\,\lambda'} = 0$, implying the two vectors are orthogonal. It turns out that we can always find orthogonal eigenvectors when the eigenvalues are degenerate as well, so you can assume the eigenvectors are always orthogonal.

27.3 Examples

We begin with a simple example.

$$M = \begin{pmatrix} 10 & 6 \\ 6 & -10 \end{pmatrix} \tag{27.5}$$

M is real and symmetric, so it has 2 real roots (and hence, two eigenvectors). We find the eigenvalues by forming the shifted matrix

$$M - \lambda I = \begin{pmatrix} 10 - \lambda & 6 \\ 6 & -10 - \lambda \end{pmatrix} \tag{27.6}$$

and setting the determinant of this matrix to zero [$\det(M - \lambda I) = 0$]. Hence,

$$0 = (10 - \lambda)(-10 - \lambda) - 36 \tag{27.7}$$

$$0 = \lambda^2 - 100 - 36 = \lambda^2 - 136 \tag{27.8}$$

$$\lambda = \pm\sqrt{136} = \pm 2\sqrt{34}. \tag{27.9}$$

Now we find the eigenvectors using the procedure described above. First, we work out the eigenvectors for $\lambda = 2\sqrt{34}$:

$$\begin{pmatrix} 10 - 2\sqrt{34} & 6 \\ 6 & -10 - 2\sqrt{34} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{27.10}$$

So, working with the top row equation, we have

$$(10 - 2\sqrt{34})a + 6b = 0, \tag{27.11}$$

which gives us

$$b = -\frac{1}{6}(10 - 2\sqrt{34})a. \tag{27.12}$$

And, as you see here, we only need to solve *one* not *two* equations, because the second equation is also immediately solved (this follows because the determinant was zero); check this directly if you are unsure. To determine both coefficients we must *normalize* the eigenvector. So we want $a^2 + b^2 = 1$, and then

$$a^2 \left(1 + \frac{1}{36}(100 - 40\sqrt{34} + 136)\right) = 1. \tag{27.13}$$

Now, carrying out the rest of the algebra can get messy:

$$a^2 = \frac{36}{272 - 40\sqrt{34}} = \frac{18}{136 - 20\sqrt{34}} = \frac{9}{68 - 10\sqrt{34}} \tag{27.14}$$

$$a = \frac{3}{\sqrt{68 - 10\sqrt{34}}} \tag{27.15}$$

$$b = -\frac{1}{2}\frac{10 - 2\sqrt{34}}{\sqrt{68 - 10\sqrt{34}}}. \tag{27.16}$$

We change the sign of the square root term for the other eigenvector. You can then check that $\vec{e_1} \cdot \vec{e_2} = 0$. Summarizing, we have

$$\vec{e}_1 = \begin{pmatrix} 3 \\ -5 + \sqrt{34} \end{pmatrix} \frac{1}{\sqrt{68 - 10\sqrt{34}}} \tag{27.17}$$

and

$$\vec{e}_2 = \begin{bmatrix} 3 \\ -5 - \sqrt{34} \end{bmatrix} \frac{1}{\sqrt{68 - 10\sqrt{34}}}, \tag{27.18}$$

with $\lambda_1 = 2\sqrt{34}$ and $\lambda_2 = -2\sqrt{34}$. Note that one can essentially guess the second eigenvector by simply ensuring it is perpendicular to the first. Since we are working in a two-dimensional space, this is sufficient to determine the other eigenvector. One can also go through all the algebra again as well. If in doubt, you can do this, or simply check that $\vec{e}_1 \cdot \vec{e}_2 = 0$.

Here is another example. Find the eigenvalues and eigenvectors of the following matrix:

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 2 & -1 & -1 \end{pmatrix}. \tag{27.19}$$

We start by shifting and taking the determinant

$$\det(M - \lambda I) = 0, \tag{27.20}$$

which becomes

$$\det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & -\lambda & -1 \\ 2 & -1 & -1-\lambda \end{pmatrix} = 0. \tag{27.21}$$

There are many ways to evaluate the determinant. I like to do it the same way one calculates cross products, but many like to do it by minors. However you prefer to do it, you will end up with the result

$$\lambda - \lambda^3 + 4\lambda - 1 + \lambda = 0. \tag{27.22}$$

Simplifying, we have

$$\lambda^3 - 6\lambda + 1 = 0. \tag{27.23}$$

From here, we must go numerical to solve for the three different λ's and for the eigenvectors. This is often the case for these kinds of problems. You might have thought, wait this is a cubic, we can find the roots *analytically.* While true, often the formulas for the roots are so complicated that computing them numerically is better.

How do we make the transformation to "diagonalize" the matrix? It is a similarity transformation (which means multiply by one matrix on the left and by its inverse on the right). As before, we form P via

$$P = \begin{pmatrix} \cdots & — & e_1 & — & \cdots \\ \cdots & — & e_2 & — & \cdots \\ \vdots & & \vdots & & \vdots \\ \cdots & — & e_n & — & \cdots \end{pmatrix} \tag{27.24}$$

$$P^{-1} = \begin{pmatrix} | & | & \vdots & | \\ e_1 & e_2 & ... & e_n \\ | & | & \vdots & | \end{pmatrix} = P^T. \tag{27.25}$$

Then multiplying out and recalling that the eigenvectors are an orthonormal set yields

$$PMP^{-1} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ .0 & \lambda_2 & & \cdots \\ \vdots & & \vdots & \vdots \\ 0 & ... & 0 & \lambda_n \end{pmatrix}. \tag{27.26}$$

Knowing this allows us to form some interesting matrices.

For example,

$$e^M = P^{-1}\begin{pmatrix} e^{\lambda_1} & 0 & ... & 0 \\ .0 & e^{\lambda_2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e^{\lambda_{n-1}} & 0 \\ 0 & \cdots & 0 & e^{\lambda_n} \end{pmatrix} P \tag{27.27}$$

because

$$e^M = 1 + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + ... \tag{27.28}$$

$$= P^{-1}P\left(1 + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + ...\right)P^{-1}P \tag{27.29}$$

$$= P^{-1}\left(1 + PMP^{-1} + \frac{1}{2}(PMP^{-1})^2 + ...\right)P \tag{27.30}$$

$$= P^{-1}\begin{pmatrix} e^{\lambda_1} & 0 & ... & 0 \\ .0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & e^{\lambda_{n-1}} & 0 \\ 0 & \cdots & 0 & e^{\lambda_n} \end{pmatrix} P \tag{27.31}$$

. Similarly, we can compute the absolute value of a matrix (or, if you like, the square root of the square of a matrix). Here, we choose the unique root that has all nonnegative eigenvalues, which is also called the absolute value of a matrix. It is given by

$$|M| = P^{-1}\begin{pmatrix} |\lambda_1| & 0 & \cdots & 0 \\ .0 & |\lambda_2| & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & |\lambda_{n-1}| & 0 \\ 0 & \cdots & 0 & |\lambda_n| \end{pmatrix} P. \tag{27.32}$$

Note that, in general after multiplying by P^{-1} and P, we have that the ij element of $|M|_{ij}$ is often not equal to $|M_{ij}|$.

This approach can obviously be extended to many different functional forms, and so one can think about constructing matrix-valued functions in this fashion. Usually the exponential and the absolute value (and the square root) are the three most important ones to consider.

Chapter 28

Application of Eigenvalues and Eigenvectors

28.1 Landau-Zener Problem

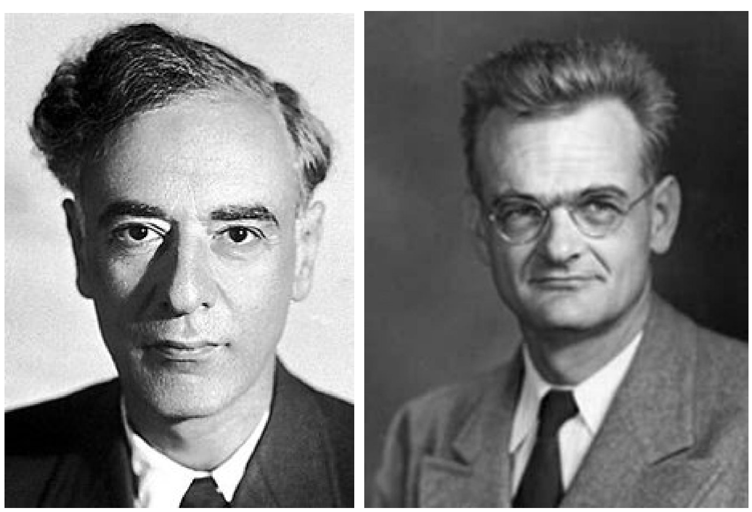

Figure 28.1: Lev Landau (left) and Clarence Zener (right). Landau solved this famous problem, but had an error in the coefficient of the exponent of his answer. Zener provided the correct derivation by mapping it to a complicated differential equation that had a special function solution (it is called the Weber equation and the solutions are parabolic cylinder functions).

We begin with a problem called the Landau-Zener problem. It is described by a 2×2 matrix

$$H = \begin{pmatrix} \delta t & v \\ v & -\delta t \end{pmatrix}. \tag{28.1}$$

Here, δ and v are real numbers and t is time. This matrix corresponds to the Hamiltonian of a two-state quantum-mechanical system whose uncoupled states are initially far away in energy, have their energy difference reduced linearly in time (with a rate given by δ) until it is zero at $t = 0$ and then it increases again to infinity as $t \to \infty$; but the energy level that is the lowest energy state switches. The v parameter describes how the two states are coupled together, which you should interpret as how easily can the system make a change from one state to another. If v is large, it does so easily. If v is small, it does so with difficulty. Don't worry if you don't understand why anything said above is correct or not, or what it means, if this seems too abstract to you. Just take what we do next as a purely mathematical exercise and know that it is meaningful for time-dependent quantum mechanics.

Th eigenvalues of this 2×2 matrix are found from solving

$$\det \begin{pmatrix} \delta t - E & v \\ v & -\delta t - E \end{pmatrix} = E^2 - (\delta t)^2 - v^2 = 0, \tag{28.2}$$

which is easily solved by

$$E = \pm\sqrt{(\delta t)^2 + v^2} \tag{28.3}$$

The plot, in Fig. 28.2, shows that the minimum energy gap is at $t = 0$ and the gap is equal to $2v$.

Note that as $t \to -\infty$, the lowest energy state has $E \to \delta t$ (recall, we have $t < 0$ here) and its eigenvector, called $\vec{e}_+$ is $\binom{1}{0}$. As $t \to +\infty$, the lowest-energy state has $E \to -\delta t$ and the eigenvector is $\vec{e}_-$ is $\binom{0}{1}$. So, as t runs from $-\infty$ to $+\infty$, *if the system stays in the lowest-energy state*, then the eigenstate will evolve from the initial one at $\vec{e}_+$ for $t = -\infty$ to the final one $\vec{e}_-$ for $t = +\infty$.

28.2 Quantum tunneling

In quantum mechanics, there can be tunneling, and the system can tunnel to the other state with some probability when $v \neq 0$. The Landau-Zener problem is to determine how to calculate this probability.

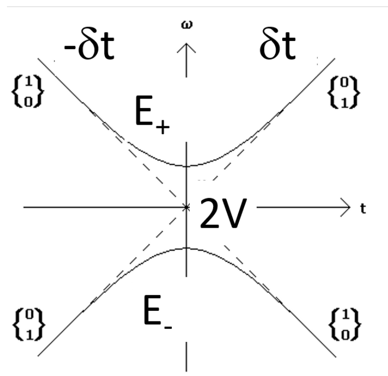

Figure 28.2: Schematic of the Landau-Zener transition. The solid lines, which are the two eigenvalues we calculated in Eq. (28.3), are the two instantaneous energy levels. They vary in time as shown above with what is called an avoided crossing at $t = 0$.

I won't be able to derive the full theory for how to solve this but will motivate the procedure for you. The parameter t is the time. Denote $\Psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ as the wave function at time t. We start with $\Psi(t \to -\infty) = \binom{1}{0}$. The time-dependent Schrodinger equation is

$$i\hbar \frac{d}{dt}\Psi(t) = H(t)\Psi(t). \tag{28.4}$$

To solve this approximately, we discretize the derivative to have

$$i\hbar \frac{\Psi(t+\Delta t) - \Psi(t)}{\Delta t} = H(t)\Psi(t). \tag{28.5}$$

Our goal is to reorganize this equation so we can determine $\Psi(t + \Delta t)$ from $\Psi(t)$. So, moving the $\Psi(t)$ terms to the right hand side, we have

$$\Psi(t + \Delta t) = \left(1 - \frac{i}{\hbar} H(t)\Delta t\right) \Psi(t) = e^{-\frac{i}{\hbar}H(t)\Delta t}\Psi(t), \tag{28.6}$$

where in the last line we used the Taylor series expansion for the exponential function and the fact that we choose the time step Δt to be small. We then repeat the process, replacing $\Psi(t)$ by $\Psi(t - \Delta t)$, continuing until we get to $\Psi(t_0)$ as the starting point. The result is

$$\Psi(t + \Delta t) = e^{-\frac{i}{\hbar}H(t)\Delta t} e^{-\frac{i}{\hbar}H(t-\Delta t)\Delta t} e^{-\frac{i}{\hbar}H(t-2\Delta t)\Delta t} \dots e^{-\frac{i}{\hbar}H(t_0)\Delta t}\Psi(t_0), \tag{28.7}$$

which is called the Trotter formula.

The probability to stay in the same eigenstate is $|\vec{e}_-(t=\infty)\cdot\Psi(t=\infty)|^2$. This is a postulate of quantum mechanics that you have to accept. To solve this problem, we start at an early time, pick a Δt, and multiply each of the "Trotter" factors to get to large time, which we approximate as infinity. Then we take the dot product and square it.

Unfortunately, we do not know how to compute this result analytically from the Trotter formula. It is true that the exponential of a 2×2 matrix can be computed exactly (which is a problem you work out in the lecture problems for this lecture). Unfortunately, we cannot determine the product of a chain of these exponentials. But, we can use this formula to determine the answer *numerically*. You need to experiment a bit with what value of t_0 you would want to start with, how small we should make Δt and how many steps we should include in the Trotter formula. But you will find, if you do this, that you can obtain a fairly accurate answer to the problem. Indeed, the correct result is that the probability to stay in the same eigenstate is $1-\exp(-\pi v^2/\delta)$. When Landau solved it, he did not correctly determine the factor π in the exponent. Give it a try and see if you get the right result!

28.3 Oscillations

We consider the problem of three ions confined in a one-dimensional trap. An image of ions in a trap is given in Fig. 28.3. I am always amazed by this. When I was in high school, textbooks would tell us that we could never image an atom. But, similar to the Dr. Seuss tale *Horton Hears a Who*, we can coax atoms to "tell us they are there." This is called a cycling transition. We shine light on atoms and they radiate the light out in all directions, similar to what you might call a nano reflector. So the figure shows images of *individual atoms* trapped in space!

There is a theorem in electromagnetism called Earnshaw's theorem that says one cannot trap a charged particle in a static electric field. So traps are constructed from time-varying fields that push in the right direction at the right time. They can be modeled by what are called pseudopotentials, which describe the trap using an effective static potential. It is simple, just a parabolic confining potential in one dimension with the repulsive Coulomb potential. The parabolic potential pulls the ions together, the Coulomb repulsion pushes them apart, the compromise is a lattice called a Wigner lattice.

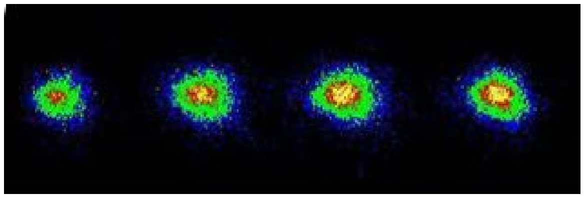

Figure 28.3: Image of light absorbed and then emitted from ions in a linear Paul trap undergoing a cycling transition. The balls of light come from individual ions in the trap.

We will work in dimensionless units where the potential is

$$V(x_1,x_2,x_3)=\frac{1}{2}k\left(x_1^2+x_2^2+x_3^2\right)+\frac{1}{|x_1-x_2|}+\frac{1}{|x_2-x_3|}+\frac{1}{|x_1-x_3|} \quad (28.8)$$

and the kinetic energy is

$$\frac{1}{2}\left(\frac{dx_1}{dt}\right)^2+\frac{1}{2}\left(\frac{dx_2}{dt}\right)^2+\frac{1}{2}\left(\frac{dx_3}{dt}\right)^2. \quad (28.9)$$

The dimensionless units allow us to set $e^2=1$ and $m=1$. You are not expected to see why without making the change to the dimensionless variables yourself.

For equilibrium, we need $\frac{\partial V}{\partial x_i}=0$ for each ion

$$\implies -kx_1-\frac{1}{(x_1-x_2)^2}-\frac{1}{(x_1-x_3)^2}=0 \quad (28.10)$$

$$\implies -kx_2+\frac{1}{(x_1-x_2)^2}-\frac{1}{(x_2-x_3)^2}=0 \quad (28.11)$$

$$\implies -kx_3+\frac{1}{(x_1-x_3)^2}+\frac{1}{(x_2-x_3)^2}=0. \quad (28.12)$$

The solution is $x_2^0=0$, $x_1^0=-x_3^0=-(\frac{5}{4k})^{\frac{1}{3}}$. Please work out this algebra yourself.

We examine oscillations about equilibrium $x_1=x_1^0+\delta x_1$, $x_2=x_2^0+\delta x_2$, $x_3=x_3^0+\delta x_3$. We substitute into the potential and expand through second order in δx_i^2 yielding

$$\begin{aligned} V(x_1,x_2,x_3) = \quad & \tfrac{1}{2}k(x_1^0)^2 + kx_1^0\delta x_1 + \tfrac{1}{2}k(\delta x_1)^2 \\ & +\tfrac{1}{2}k(x_2^0)^2 + kx_2^0\delta x_2 + \tfrac{1}{2}k(\delta x_2)^2 \\ & +\tfrac{1}{2}k(x_3^0)^2 + kx_3^0\delta x_3 + \tfrac{1}{2}k(\delta x_3)^2 \\ & +\frac{1}{|x_1^0-x_2^0|} + \frac{-\delta x_1+\delta x_2}{(x_1^0-x_2^0)^2} + \frac{1}{2}\frac{\delta x_1^2-2\delta x_1\delta x_2+\delta x_2^2}{|x_1^0-x_2^0|^3} \\ & +\frac{1}{|x_2^0-x_3^0|} + \frac{-\delta x_2+\delta x_3}{(x_2^0-x_3^0)^2} + \frac{1}{2}\frac{\delta x_2^2-2\delta x_2\delta x_3+\delta x_3^2}{|x_2^0-x_3^0|^3} \\ & +\frac{1}{|x_1^0-x_3^0|} + \frac{-\delta x_1+\delta x_3}{(x_1^0-x_3^0)^2} + \frac{1}{2}\frac{\delta x_1^2-2\delta x_1\delta x_3+\delta x_3^2}{|x_1^0-x_3^0|^3}, \end{aligned} \tag{28.13}$$

where the first terms in each line are constants and the total of the second terms in each line vanishes due to the condition of equilibrium. The potential can then be rewritten in the "compact" way

$$V(x_1,x_2,x_3) = V(x_1^0,x_2^0,x_3^0) + \\ \frac{1}{2}\begin{pmatrix} \delta x_1 & \delta x_2 & \delta x_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\left|x_1^0-x_2^0\right|^3}+\frac{1}{\left|x_1^0-x_3^0\right|^3}+k & -\frac{1}{\left|x_1^0-x_2^0\right|^3} & -\frac{1}{\left|x_1^0-x_3^0\right|^3} \\ -\frac{1}{\left|x_1^0-x_2^0\right|^3} & \frac{1}{\left|x_1^0-x_2^0\right|^3}+\frac{1}{\left|x_2^0-x_3^0\right|^3}+k & -\frac{1}{\left|x_2^0-x_3^0\right|^3} \\ -\frac{1}{\left|x_1^0-x_3^0\right|^3} & -\frac{1}{\left|x_2^0-x_3^0\right|^3} & \frac{1}{\left|x_1^0-x_3^0\right|^3}+\frac{1}{\left|x_2^0-x_3^0\right|^3}+k \end{pmatrix} \cdot \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix} \tag{28.14}$$

Using the notation $\delta\vec{x} = (\delta x_1, \delta x_2, \delta x_3)$, this is a vector "spring" problem with an equation of motion given by

$$m\frac{d^2\delta\vec{x}}{dt^2} = -\mathbb{K}\delta\vec{x}, \tag{28.15}$$

with the matrix $\mathbb{K}$ equal to the matrix in Eq. (28.14).

Our goal is to determine the normal modes of motion of the ions. These are the motions of the ions that are regular and repeating with the same period for each ion. Hence, we must have that the deviations from equilibrium satisfy $\delta x_j(t) = \delta x_j^0 e^{iwt}$, with δx^0 constants, for a normal mode. Substituting this form into the equation of motion yields

$$mw^2\delta\vec{x}^0 = \mathbb{K}\delta\vec{x}_0. \tag{28.16}$$

This is an eigenvalue problem (the eigenvalue is $m\omega^2$ since $\mathbb{K}$ is a matrix and $\delta\vec{x}_0$ is a vector) that we will solve on the homework.

Chapter 29

First Order Linear Differential Equations

29.1 Definition and Method of Solution

The most general linear first order differential equation is

$$\frac{d}{dt}y(t) + q(t)y(t) = r(t) \tag{29.1}$$

where we want to find the function $y(t)$ given $q(t)$ and $r(t)$. For most differential equations, but particularly for linear ones, the way we solve them is to first find a solution to the *homogeneous problem* $[r(t) = 0]$

$$\frac{d}{dt}y(t) + q(t)y(t) = 0 \tag{29.2}$$

and add to that solution a *particular solution* to

$$\frac{d}{dt}y(t) + q(t)y(t) = r(t) \tag{29.3}$$

called the *inhomogeneous problem.*

Here is how we solve the homogeneous problem:

$$\frac{d}{dy}y(t) + q(t)y(t) = 0 \implies \frac{d}{dt}y(t) = -q(t)y(t). \tag{29.4}$$

Divide both sides by $y(t)$

$$\frac{1}{y(t)}\frac{d}{dt}y(t) = -q(t), \tag{29.5}$$

and rewrite in terms of the logarithmic derivative

$$\frac{d}{dt} \ln y(t) = -q(t). \tag{29.6}$$

Now we integrate

$$\int_{t_0}^{t} d \ln y(t) = -\int_{t_0}^{t} q(\bar{t})\, d\bar{t} \tag{29.7}$$

to find

$$\ln \frac{y(t)}{y(t_0)} = -\int_{t_0}^{t} q(\bar{t}) d\bar{t} \tag{29.8}$$

or, after exponentiating

$$y(t) = y(t_0) \exp\left[-\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right] \tag{29.9}$$

where $y(t_0)$ is a constant determined by the value y has at $t = t_0$, since the exponential term when $t = t_0$ is 1.

How to solve the inhomogeneous problem:

$$\frac{dy}{dt} + q(t)y(t) = r(t) \tag{29.10}$$

We multiply by the integrating factor that we found above, $\exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right]$

$$\exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right] \frac{dy}{dt} + \exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right] q(t)y(t) = \exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right] r(t). \tag{29.11}$$

But, the left hand side is a perfect differential

$$\text{LHS} = \frac{d}{dt}\left(y(t) \exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right)\right]. \tag{29.12}$$

So

$$\int_{t_0}^{t} d\left(y(t) \exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right]\right) = \int_{t_0}^{t} \exp\left[\int_{t_0}^{t'} q(\bar{t}) d\bar{t}\right] r(t')\, dt' \tag{29.13}$$

$$y(t) \exp\left[\int_{t_0}^{t} q(\bar{t}) d\bar{t}\right] - y(t_0) = \int_{t_0}^{t} \exp\left[\int_{t_0}^{t'} q(\bar{t}) d\bar{t}\right] r(t')\, dt'. \tag{29.14}$$

Note that we integrate the q function to t' first and then integrate the entire integrand over t' on the right-hand side. Hence, we have

$$y(t) = y(t_0)e^{-\int_{t_0}^{t} q(\bar{t})\,d\bar{t}} + e^{-\int_{t_0}^{t} q(\bar{t})\,d\bar{t}} \int_{t_0}^{t} e^{\int_{t_0}^{t'} q(\bar{t})d\bar{t}} r(t')\,dt' \tag{29.15}$$

where the first term is the homogeneous solution (multiplied by a constant term) and the second is the particular solution. This is a general solution, with one parameter to fit, $y(t_0)$. This solution is also unique. The proof is given in the book. It requires $q(t)$ to be continuous and bounded on $[t_0, t]$. You may be surprised by the fact that we can find a closed-form solution for all first-order linear differential equations. But we can!

29.2 Examples

We never really understand how this works until we try ourselves. Let's start with some worked examples. Consider the following equation:

$$\frac{dy}{dt} + \frac{1}{1-t}y = 1 - t \tag{29.16}$$

for $0 \le t < 1$; we want the solution that also satisfies $y(0) = 1$.

Solving is essentially a "cookbook" given the form we have determined above. First find the integral of q:

$$\int_{t_0}^{t} q(\bar{t})\,d\bar{t} = \int_{0}^{t} \frac{1}{1-\bar{t}}\,d\bar{t} = -\ln(1-\bar{t})\Big|_0^t = -\ln(1-t). \tag{29.17}$$

Next, plug into the full solution

$$y(t) = y(t_0)\text{exp}\left[\ln(1-\bar{t})\right] + \text{exp}\left[\ln(1-t)\right] \int_0^t \text{exp}\left[-\ln(1-t')\right](1-t')\,dt' \tag{29.18}$$

$$= 1 \times (1-t) + (1-t)\int_0^t dt' \frac{1-t'}{1-t'} \tag{29.19}$$

$$= 1 - t + (1+t)t \tag{29.20}$$

$$= (1-t)(1+t) \tag{29.21}$$

$$= 1 - t^2. \tag{29.22}$$

OK, we worked through the solution, but it is relatively easy to make an error, so it is very wise to verify the solution actually solves the equation we set out to solve. So, we check the solution:

$$\frac{d}{dt}(1-t^2) = -2t \tag{29.23}$$

$$-2t + \frac{1-t^2}{1-t} = -2t + (1+t) = 1-t \tag{29.24}$$

which checks out.

Let's try another example.

$$\frac{d}{dt}y + 2y = e^{-t} \tag{29.25}$$

with $y(0) = 3$. First, we have

$$\int_0^t q(\bar{t})\, d\bar{t} = \int_0^t 2\, d\bar{t} = 2t. \tag{29.26}$$

Plugging into the solution yields

$$\begin{aligned} y(t) &= y(0)e^{-2t} + e^{-2t}\int_0^t e^{2t'}e^{-t'}\, dt' && (29.27)\\ &= 3e^{-2t} + e^{-2t}(e^t - 1) && (29.28)\\ &= 2e^{-2t} + e^{-t}. && (29.29) \end{aligned}$$

Check:

$$y(0) = 2 + 1 = 3, \tag{29.30}$$

which checks. In addition, we have

$$\frac{d}{dt}y(t) = -4e^{-2t} - e^{-t} \tag{29.31}$$

$$\frac{d}{dt}y(t) + 2y(t) = -4e^{-2t} - e^{-t} + 4e^{-2t} + 2e^{-t} = e^{-t}, \tag{29.32}$$

which also checks.

Here is our final example:

$$\frac{d}{dt}y - 2ty = t, \tag{29.33}$$

with $y(0) = 1$. We first integrate q

$$\int_0^t q(\bar{t})d\bar{t} = \int_0^t -2\bar{t}d\bar{t} = -t^2, \tag{29.34}$$

and then form the solution as before

$$y(t) = e^{t^2} + e^{t^2} \int_0^t e^{-t'^2} t'\, dt' \tag{29.35}$$

$$= e^{t^2} - \frac{1}{2} e^{t^2} \int_0^t (-2t) e^{-t^2}\, dt' \tag{29.36}$$

$$= e^{t^2} - \frac{1}{2} e^{t^2} \left(e^{-t^2} - 1 \right) \tag{29.37}$$

$$= \frac{3}{2} e^{t^2} - \frac{1}{2}, \tag{29.38}$$

since $y(0) = 1$. Now check

$$\frac{dy}{dt} - 2ty = 3te^{t^2} - 3te^{t^2} + t = t, \tag{29.39}$$

which also checks.

Essentially all linear first order differential equations can be solved by straightforward integrals. Nonlinear first order differential equations are another story. They will be covered next,

Chapter 30

Nonlinear first-order differential equations

30.1 Introduction to nonlinear differential equations

You might have thought that we are done with our simple world of first-order differential equations because we derived a formula to solve them all. But no!. This is one of the few cases where we actually can solve a number of nonlinear differential equations (in general, nonlinear differential equations are so hard people spend their entire scientific careers studying them). Nonlinear equations are very rich. All of the complexities of the weather, nonlinear effects like the so-called "butterfly effect" all derive from nonlinear differential equations. So, let's jump in and get started.

A nonlinear first order differential equation takes the form

$$\frac{dy}{dt} = f(t, y) = -\frac{M(t, y)}{N(t, y)} \tag{30.1}$$

so that

$$M(t, y)dt + N(t, y)dy = 0. \tag{30.2}$$

Note that the reduction into the M and N form is *not unique*. This is sometimes a cause for confusion, but it really is an opportunity that we will exploit later.

There are five known general methods to solve these nonlinear differential equations. If none of these work, you need to solve them numerically. But even that is not so simple ⋯

30.2 Type 1: Reducible to Linear

This is one of the easiest. Find a way to convert from a nonlinear to a linear and then solve it.

Example: The Bernouli equation

$$\frac{dy}{dt} + q(t)y = r(t)y^n. \tag{30.3}$$

Note how the nonlinearity arises from the power n on the last term. If it is zero, or one, we have a linear equation. Everything else is nonlinear.

Let $w = y^{1-n}$, Then $dw = (1-n)y^{-n}dy$, which implies that $\frac{dy}{dt} = \frac{1}{1-n}y^n\frac{dw}{dt}$ or, substituting into the original equation

$$\frac{1}{1-n}y^n\frac{dw}{dt} + q(t)y = r(t)y^n. \tag{30.4}$$

We then divide by y^n and multiply by $1 - n$ to obtain

$$\frac{dw}{dt} + (1-n)q(t)y^{1-n} = (1-n)r(t). \tag{30.5}$$

This still looks nonlinear, but express in terms of w to find

$$\frac{dw}{dt} + (1-n)q(t)w = (1-n)r(t) \tag{30.6}$$

which is a first-order *linear* differential equation!

We will have the chance to practice some problems on this type of differential equation in the homework.

30.3 Type 2: Separable

Separable equations occur when M *does not* depend explicitly on y and N *does not* depend explicitly on t. That is, $M(t, y) = M(t)$ and $N(t, y) = N(y)$. In this case, we have that the differential equation can be rewritten as

$$M(t)dt + N(y)dy = 0. \tag{30.7}$$

Move one term to the right and integrate both sides

$$\int_{t_0}^{t} M(\bar{t})d\bar{t} = -\int_{y_0}^{y} N(\bar{y})d\bar{y}, \quad \text{with} \quad y(t_0) = y_0. \tag{30.8}$$

If the integrals can be done, we now are left with an ordinary equation to solve.

Example: $\frac{dy}{dt} = ty^2$, which can be rewritten as $\frac{1}{y^2}dy - tdt = 0$, and we set $y(1) = a$, as the initial condition. Performing the integrals yields

$$\int_{a}^{y} \frac{1}{\bar{y}^2}d\bar{y} = \int_{1}^{t} \bar{t}d\bar{t} \implies -\frac{1}{\bar{y}}\bigg|_{a}^{y} = \frac{t^2-1}{2} \tag{30.9}$$

$$\frac{1}{a} - \frac{1}{y} = -\frac{1}{2}(1-t^2) \implies y(t) = \frac{1}{\frac{1}{a} + \frac{1}{2}(1-t^2)} = \frac{2a}{2+a-at^2}, \tag{30.10}$$

so we get

$$y(t) = \frac{2a}{2+a-at^2}. \tag{30.11}$$

Now comes the important point many students forget to do. We need to check to ensure we actually solved the original problem. $y(1) = a$ checks out, as well as

$$\frac{dy}{dt} = \frac{-2a \times 2at}{-(2+a-at^2)^2} = ty^2, \tag{30.12}$$

so everything works as it should.

30.4 Type 3: Reducible to separable

For this case, the restrictions on M and N are a bit looser. We require only that $M(t,y)$ and $N(t,y)$ are *homogeneous of degree k*. You have probably not encountered homogeneous functions yet. This condition requires

$$M(\lambda t, \lambda y) = \lambda^k M(t,y), \quad N(\lambda t, \lambda y) = \lambda^k N(t,y), \tag{30.13}$$

where λ is a number. To solve this type of problem, we let $y(t) = tu(t)$. The original equation

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0, \tag{30.14}$$

will be simplified by this transformation. First compute the derivative of y with respect to t using this substitution:

$$\frac{dy}{dt} = u + t\frac{du}{dt}. \tag{30.15}$$

Next, substitute into the differential equation, to give

$$M(t,y) + N(t,y)\frac{dy}{dt} = M(t,tu) + N(t,tu)(u + t\frac{du}{dt}) = 0. \tag{30.16}$$

Now, we are ready to use the homogeneity. Note we think of $t = \lambda$ to remove the factor of t from both arguments of M and N. This yields

$$t^k M(1,u) + t^k N(1,u)(u + t\frac{du}{dt}) = 0. \tag{30.17}$$

We divide out the common factor of t^k

$$M(1,u) + uN(1,u) + tN(1,u)\frac{du}{dt} = 0. \tag{30.18}$$

Now, rearrange the equation with all u-dependent terms on the right and all t-dependent terms on the left:

$$\frac{dt}{t} = -\frac{N(1,u)}{M(1,u) + uN(1,u)}du. \tag{30.19}$$

Here $\frac{dt}{t}$ only has t dependence and $-\frac{N(1,u)}{M(1,u)+uN(1,u)}$ only has u dependence. Then we just need to integrate both sides

$$\ln\left(\frac{t}{t_0}\right) = -\int_{u_0}^{u} d\bar{u}\frac{N(1,\bar{u})}{M(1,\bar{u}) + \bar{u}N(1,\bar{u})}, \quad \text{with} \quad u_0 = \frac{y(t_0)}{t_0} \tag{30.20}$$

This will solve the problem if we can do the integral and if we can solve the resulting algebraic problem.

Example: $(t^2 + y^2)\,dt + 2ty\,dy = 0$, with $y(t_0) = y_0$ when $t_0 > 0$. First note that $M = t^2 + y^2$ is homogeneous of degree 2 and $N = 2ty$ is also homogeneous of degree 2. Using our "cookbook" solution routine, we have

$$\ln\left(\frac{t}{t_0}\right) = -\int_{u_0}^{u} d\bar{u}\frac{2\bar{u}}{1 + \bar{u}^2 + 2\bar{u}^2} = -\int_{u_0}^{u} d\bar{u}\frac{2\bar{u}}{1 + 3\bar{u}^2}. \tag{30.21}$$

The integral can be done as follows:

$$= -\int_{u_0}^{u} \frac{d\bar{u}^2}{1+3\bar{u}^2} = -\frac{1}{3}\ln(1+3\bar{u}^2)\Big|_{u_0}^{u}. \tag{30.22}$$

Hence, we obtain

$$\left(\frac{t}{t_0}\right)^3 = \frac{1+3(u_0)^2}{1+3u^2}. \tag{30.23}$$

Now, we have to solve the algebraic equation for u (and eventually y):

$$t^3(1+3u^2) = (t_0)^3\left(1+3(u_0)^3\right). \tag{30.24}$$

Substituting in $y = tu$ gives

$$t^3 + 3ty^2 = (t_0)^3 + 3t_0(y_0)^2, \tag{30.25}$$

which is solved by

$$y = \sqrt{\frac{(t_0)^3 - t^3 + 3t_0(y_0)^2}{3t}}. \tag{30.26}$$

Now we check the answer: $y(t_0) = \sqrt{(y_0)^2} = y_0$, which checks out. Also,

$$\frac{dy}{dt} = \frac{1}{2}\frac{\left(-\frac{1}{3t^2}((t_0)^3 + 3t_0(y_0)^2) - \frac{2}{3}t\right)}{\sqrt{\frac{(t_0)^3 - t^3 + 3t_0(y_0)^2}{3t}}}\frac{t}{t} \tag{30.27}$$

$$\implies 2ty\frac{dy}{dt} = -\frac{1}{3t}((t_0)^3 - t^3 + 3t_0(y_0)^2) - t^2 = -(y^2 + t^2) \tag{30.28}$$

which also checks out!

30.5 Type 4: Exact differential

This case corresponds to the case when the functions M and N appear to come from derivatives of the same function F. We start with our nonlinear equation

$$M(t,y)dt + N(t,y)dy = 0. \tag{30.29}$$

If we have that M and N are so-called exact differentials, given by

$$M(t,y) = \frac{\partial F(t,y)}{\partial t} \quad \text{and} \quad N(t,y) = \frac{\partial F(t,y)}{\partial y}, \tag{30.30}$$

then, we have

$$M(t,y)dt + N(t,y)dy = \frac{\partial F(t,y)}{\partial t}dt + \frac{\partial F(t,y)}{\partial y}dy \tag{30.31}$$

$$= dF(t,y) = 0 \implies F(t,y) = \text{ constant } = F(t_0, y_0) \tag{30.32}$$

which solves the problem.

Example:

$$(y^2 - t^2)\, dt + 2ty\, dy = 0, \text{ with } y(t_0) = y_0. \tag{30.33}$$

By playing around a bit, we can see immediately that

$$F(t,y) = ty^2 - \frac{1}{3}t^3, \quad \frac{\partial F}{\partial t} = y^2 - t^2, \quad \frac{\partial F}{\partial y} = 2ty. \tag{30.34}$$

So,

$$ty^2 - \frac{1}{3}t^3 = t_0(y_0)^2 - \frac{1}{3}(t_0)^3, \tag{30.35}$$

because $F(t,y)$ is a constant. Rearranging to solve for $y(t)$ gives us

$$y^2 = \frac{t_0(y_0)^2 - \frac{1}{3}((t_0)^3 - t^3)}{t}, \tag{30.36}$$

and

$$y(t) = \sqrt{\frac{3t_0(y_0)^2 - (t_0)^3 + t^3}{3t}}. \tag{30.37}$$

Check: $y(t_0) = \sqrt{(y_0)^2} = y_0$, which checks out. Next, compute the derivative:

$$\frac{dy}{dt} = \frac{1}{2}\frac{-\frac{1}{3t^2}(3t_0(y_0)^2 - (t_0)^3 + t^3) + (\frac{1}{3t}) \times 3t^2}{\sqrt{\frac{3t_0(y_0)^2) - (t_0)^3 + t^3}{3t}}}, \tag{30.38}$$

Multiplying by $2ty$, we find

$$2ty\frac{dy}{dt} = -\frac{1}{3t}\left(3t_0y_0^2 - t_0^3 + t^3\right) + t^2 = -(y^2 - t^2) \tag{30.39}$$

which works!

30.6 Type 5: Reducible to exact

This is both the hardest type to recognize and the hardest to solve.

The technique is motivated by the so-called integrating factor that we used to integrate the linear one-dimensional problems and uses the fact that M and N are not uniquely determined, just their ratio is.

If we find an F and a Q such that

$$\frac{dF(t,y)}{dt} = Q(t,y)M(t,y), \quad \frac{dF(t,y)}{dy} = Q(t,y)N(t,y). \tag{30.40}$$

Then, we have

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial y}dy = 0 \implies \frac{dy}{dt} = -\frac{\frac{\partial F}{\partial t}}{\frac{\partial F}{\partial y}}, \tag{30.41}$$

or

$$\frac{dy}{dt} = -\frac{Q(t,y)M(t,y)}{Q(t,y)N(t,y)} = -\frac{M(t,y)}{N(t,y)}. \tag{30.42}$$

Then $F(t,y) = F(t_0,y_0)$ is a constant yields the solution again. To check if an equation is exact, check the mixed derivatives, which must be equal

$$\frac{\partial^2 F}{\partial t \partial y} = \frac{\partial^2 F}{\partial y \partial t} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{30.43}$$

Example:

$$(3t^4 - y)dt + tdy = 0. \tag{30.44}$$

We see that $\tilde{F} = \frac{3}{5}t^5 - yt$ does not work. But,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right) = \frac{1}{t}(-1-1) = -\frac{2}{t} = \text{ independent of } y. \tag{30.45}$$

So, let

$$Q(t) = e^{-\int^t \frac{2}{t'}dt'} = e^{-2\ln t} = \frac{1}{t^2} \tag{30.46}$$

be the integrating factor. Multiply by Q to get

$$\left(3t^2 - \frac{y}{t^2}\right)dt + \frac{1}{t}dy = 0. \tag{30.47}$$

Now,

$$\frac{\partial \tilde{M}}{\partial y} = -\frac{1}{t^2} = \frac{\partial \tilde{N}}{\partial t} \tag{30.48}$$

so

$$F = t^3 + \frac{y}{t}. \tag{30.49}$$

And, we have

$$t^3 + \frac{y}{t} = (t_0)^3 + \frac{y_0}{t_0} \implies y(t) = t\left((t_0)^3 - t^3 + \frac{y_0}{t_0}\right). \tag{30.50}$$

Check: $y(t_0) = y_0$ - this checks! Also,

$$\frac{dy}{dt} = \left((t_0)^3 - t^3 + \frac{y_0}{t_0}\right) + t(-3t^2) = -\left(-\frac{y}{t} + 3t^3\right) \implies \frac{1}{t}\frac{dy}{dt} + \left(3t^2 - \frac{y}{t^2}\right) = 0, \tag{30.51}$$

so it checks!

Chapter 31

Applications of First-Order Differential Equations

In this chapter, we will apply first-order differential equations to a number of different problems motivated by physics. This includes a description of air resistance in moving objects, Newton's law of cooling, and a proper treatment of rocket motion when we take into account the fact that the mass of the rocket changes as the fuel is burned and expelled.

31.1 Models of Air Resistance

Consider first the case of no air resistance—a mass m is dropped from a height in the presence of gravity (the acceleration due to gravity is g). If there is no air resistance then Newton says that

$$m\frac{dv}{dt} = mg, \tag{31.1}$$

where we take the positive direction to be *downward*. Rearranging, we find that

$$\frac{dv}{dt} = g \implies \int_0^v d\bar{v} = \int_0^t g\, d\bar{t} \implies v = gt, \tag{31.2}$$

under the assumption that the particle is dropped with no initial velocity.

Now assume the object feels air resistance proportional to $|v|$, $F = -k|v|$. Note that here we have $|v| \geq 0$ so $|v| = v$. This implies that

$$m\frac{dv}{dt} = mg - kv, \tag{31.3}$$

or, after rearranging

$$\frac{dv}{dt} = -\frac{k}{m}v + g. \tag{31.4}$$

This is a standard first-order linear differential equation. Let's use our methodology to solve it. First, we identify that

$$q(t) = \frac{k}{m} \tag{31.5}$$

and

$$r(t) = g. \tag{31.6}$$

Next, we find the integral of $q(\bar{t})$ subject to the initial and final conditions on the time variable:

$$\int_0^t d\bar{t}q(\bar{t}) = \frac{k}{m}t \tag{31.7}$$

$$e^{-\int_0^t d\bar{t}q(\bar{t})} = e^{-\frac{k}{m}t}. \tag{31.8}$$

Now we are ready to substitute into our general formula, which yields

$$v(t) = e^{-\frac{k}{m}t}\int_0^t dt' e^{\frac{k}{m}t'} g \tag{31.9}$$

$$= \frac{m}{k}ge^{-\frac{k}{m}t}(e^{\frac{k}{m}t} - 1) \tag{31.10}$$

$$= \frac{mg}{k}(1 - e^{-\frac{k}{m}t}). \tag{31.11}$$

Now we check our result.First the initial condition. Letting $t = 0$, we immediately fin that $v(0) = 0$. For the differential equation, we must take a derivative. We find that

$$\frac{dv}{dt} = -\frac{mg}{k}\left(-\frac{k}{m}\right)e^{-\frac{k}{m}t} = ge^{-\frac{k}{m}t}. \tag{31.12}$$

We use this to check the original equation and find that

$$g - \frac{k}{m}v = g - g(1 - e^{-\frac{k}{m}t}) = ge^{-\frac{k}{m}t} = \frac{dv}{dt}. \tag{31.13}$$

Hence, the solution is verified.

For air resistance proportional to v^2, we have a *nonlinear* first-order differential equation, given by

$$m\frac{dv}{dt} = mg - k'v^2. \tag{31.14}$$

Rearranging to put it into standard form, we have

$$\frac{dv}{dt} + \frac{k'}{m}v^2 = g, \tag{31.15}$$

or

$$dv + \left(\frac{k'}{m}v^2 - g\right) dt = 0. \tag{31.16}$$

This is an exact differential equation, so we can immediately solve it by isolating all v terms on the left and all t terms on the right, as follows:

$$\frac{dv}{\frac{k'}{m}v^2 - g} = -dt. \tag{31.17}$$

Now, we integrate both sides, being sure to incorporate the initial conditions for v and t

$$\int_0^v \frac{-d\bar{v}}{\frac{k'}{m}\bar{v}^2 - g} = \int_0^t dt = t. \tag{31.18}$$

Sinmplifying, we have

$$\frac{m}{k'}\int_0^v \frac{d\bar{v}}{\frac{gm}{k'} - \bar{v}^2} = t. \tag{31.19}$$

We have to perform the integral on the left hand side. Let $\frac{gm}{k'} = a^2$ and expand in partial fractions. This allows the integrals to be performed as logarithms:

$$\frac{m}{k'}\int_0^v \frac{d\bar{v}}{a^2 - \bar{v}^2} = \frac{m}{k'}\int_0^v \frac{d\bar{v}}{2a}\left(\frac{1}{a-\bar{v}} + \frac{1}{a+\bar{v}}\right) \tag{31.20}$$

$$= \frac{1}{2}\sqrt{\frac{m}{gk'}}\left(-\ln\frac{a-v}{a} + \ln\frac{a+v}{a}\right) \tag{31.21}$$

$$= \frac{1}{2}\sqrt{\frac{m}{gk'}}\ln\left(\frac{\sqrt{\frac{gm}{k'}} + v}{\sqrt{\frac{gm}{k'}} - v}\right). \tag{31.22}$$

Our next step is to exponentiate both sides

$$e^{2t\sqrt{\frac{gk'}{m}}} = \frac{\sqrt{\frac{gm}{k'}} + v}{\sqrt{\frac{gm}{k'}} - v} \tag{31.23}$$

and solve for $v(t)$

$$\left(1 + e^{2t\sqrt{\frac{gk'}{m}}}\right) v = \sqrt{\frac{gm}{k'}}\left(-1 + e^{2t\sqrt{\frac{gk'}{m}}}\right). \tag{31.24}$$

We obtain

$$v(t) = \sqrt{\frac{gm}{k'}}\left(\frac{1-e^{-2t\sqrt{\frac{gk'}{m}}}}{1+e^{-2t\sqrt{\frac{gk'}{m}}}}\right) \tag{31.25}$$

$$= \sqrt{\frac{gm}{k'}}\left(\frac{e^{t\sqrt{\frac{gk'}{m}}}-e^{-t\sqrt{\frac{gk'}{m}}}}{e^{t\sqrt{\frac{gk'}{m}}}+e^{-t\sqrt{\frac{gk'}{m}}}}\right) \tag{31.26}$$

$$= \sqrt{\frac{gm}{k'}}\tanh\sqrt{\frac{gk'}{m}}t, \tag{31.27}$$

where the last line follows by recalling that $\cosh(x) = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.

Now we need to check to verify the solution is correct. First check that $v(0) = 0$, which obviously holds since $\tanh(0) = 0$. Next, verify the differential equation. Th derivative of v with respect to t is

$$\frac{dv}{dt} = \sqrt{\frac{gm}{k'}}\sqrt{\frac{gk'}{m}}\operatorname{sech}^2\sqrt{\frac{gk'}{m}}t = g\operatorname{sech}^2\sqrt{\frac{gk'}{m}}t. \tag{31.28}$$

But we have a hyperbolic trig identity given by $1 - \tanh^2 x = \operatorname{sech}^2 x$, so we find that

$$\frac{dv}{dt} = g - \frac{k'}{m}v^2. \tag{31.29}$$

Hence, we have verified that the solution is indeed correct.

Note that this nonlinear equation is also a Bernoulli equation (and also a Ricatti equation). For the Bernoulli equation, we could use the standard method for those equations to solve this. As an exercise, you might want to verify that you obtain the same answer working with our general procedure for Bernoulli equations.

31.2 Newton's Law of Cooling

Let $T(t)$ be the temperature of an object at time t and T_0 be the ambient temperature of the surroundings. Then Newton's Law of Cooling is

$$\frac{dT}{dt} = -k(T - T_0) \tag{31.30}$$

where k is a constant. It says the rate of change of he temperature is proportional to the difference in temperature of the object and the surrounding "heat bath."

Suppose $T(0) = T_1$. Find the solution for all t. Starting from the differential equation $\frac{dT}{dt} + kT = kT_0$, we can identify the two functions $q(t) = +k$ and $r(t) = +kT_0$. Then we employ our standard procedure:

$$\int_0^t d\bar{t}\, q(\bar{t}) = kt \tag{31.31}$$

So we find that

$$T(t) = T_1 e^{-kt} + e^{-kt}\int_0^t dt' e^{kt'} kT_0. \tag{31.32}$$

Performing the integral yields

$$T(t) = T_1 e^{-kt} + e^{-kt} kT_0 \frac{1}{k}(e^{kt} - 1), \tag{31.33}$$

and after simplifying, we have

$$T(t) = T_1 e^{-kt} + T_0(1 - e^{-kt}). \tag{31.34}$$

As always, we need to check our results. First the initial condition is given by $T(0) = T_1$ and then the differential quation:

$$\begin{aligned} \frac{dT}{dt} &= -kT_1 e^{-kt} + kT_0 e^{-kt} && (31.35)\\ &= -k(T_1 e^{-kt} + T_0(1 - e^{-kt}) - T_0) && (31.36)\\ &= -k(T(t) - T_0). && (31.37) \end{aligned}$$

Hence, our solution checks.

Suppose the object cooling is a hot cup of coffee. $T_1 = 200°\text{F}$, $T_0 = 70°\text{F}$ and it cools to 190°F in one minute. How long before it reaches 150°?

Solution: Use our general solution after plugging in T_1 and T_0.

$$T(t) = 200e^{-kt} + 70(1 - e^{-kt}) \tag{31.38}$$

Use the criterion in he problem to determine k:

$$190 = 200e^{-k} + 70(1 - e^{-k}) \implies 120 = 130e^{-k}, \tag{31.39}$$

or, after solving for k

$$k = -\ln\frac{120}{130} = 0.08. \tag{31.40}$$

Now we have all the information needed to answer the posed question. We find

$$150 = 200e^{-0.08t} + 70(1 - e^{-0.08t}), \tag{31.41}$$

which simplifies to

$$80 = 130e^{-0.08t}. \tag{31.42}$$

Take the logarithm to find

$$0.08t = -\ln\frac{8}{13}. \tag{31.43}$$

Plugging in he numbers yields

$$t = 6.075 \text{ minutes.} \tag{31.44}$$

So drink your Starbucks quickly!

31.3 Variable Mass (Rocket) Problem

Newton says the time rate of change of the momentum is equal to he applied force, or

$$F = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}. \tag{31.45}$$

Note how the rate of change of momentum has two contributions. One from the acceleration and one from the changing mass of the rocket.

We consider a rocket with initial mass m_0 expelling fuel at a rate β. Then $m(t) = m_0 - \beta t$. Note that this is an *input* into our problem. We work with a fixed rate of fuel being expelled. Assume the gravitational constant is a constant g such that $F = -m(t)g$; this means the rocket is not going too far from the radius of the earth—of course more complicated treatments are possible. With all of these assumptions, Newton tells us that

$$(m_0 - \beta t)\frac{dv}{dt} - \beta v_{\text{exhaust}} = -(m_0 - \beta t)g \tag{31.46}$$

with an initial velocity $v(0) = v_0$. Rearranging the equation for the time derivative of the velocity gives us

$$\frac{dv}{dt} = -g + \frac{\beta v_{\mathrm{ex}}}{m_0 - \beta t}, \tag{31.47}$$

which is a separable differential equation. So we integrate both sides

$$\int_{v_0}^{v} d\bar{v} = \int_0^t \left(-g + \frac{\beta v_{\mathrm{ex}}}{m_0 - \beta \bar{t}}\right) d\bar{t}, \tag{31.48}$$

and find that

$$v - v_0 = -gt - \beta v_{\mathrm{ex}} \left(\frac{1}{\beta}\right) \ln\left(\frac{m_0 - \beta t}{m_0}\right) \tag{31.49}$$

or

$$v(t) = v_0 - gt + v_{\mathrm{ex}} \ln\left(\frac{m_0}{m_0 - \beta t}\right). \tag{31.50}$$

Increasing the exhaust velocity v_{exhaust} or increasing the rate that mass is expelled from the rocket β will increase the maximal speed at the time the fuel runs out.

One can extend this analysis to that of a *relativistic rocket*, but we will not do that here.

Chapter 32

Linear differential equations

32.1 Introduction

In this chapter, we discuss the general solution of linear differential equations. These equations can be of arbitrary order (involving arbitrary order derivatives), but the function and all derivatives appear *linearly*, so they can only be multiplied by a function of t not a function of y or any of its derivatives).

Hence, the inhomogeneous nth-order linear differential equation is given by

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y(t) = f(t). \tag{32.1}$$

There is an existence and uniqueness theorem about the solutions of these types of equations. If $a_n(t) \neq 0$ anywhere and all a's are continuous, then the solution, given a set of initial conditions, is *unique*. This is a very powerful theorem about these differential equation—it says that if we find a solution, it is *the* solution.

In general, we find that the solutions are given by linear combinations of the n linearly independent solutions of the homogeneous equation ($f = 0$) plus a particular solution of the inhomogeneous equation. This is just like what we did before for first-order linear differential equations.

32.2 Basis Set of Solutions

Define a set of n linearly independent solutions to the homogeneous equations with $\{y_i(t)|i = 1, \cdots, n\}$ such that for some t_0 we have

$$\left(y_1(t_0), y_1^{(1)}(t_0), y_1^{(2)}(t_0), \cdots, y_1^{(n-1)}(t_0)\right) = (1, 0, 0, \cdots, 0) \tag{32.2}$$

$$\left(y_2(t_0), y_2^{(1)}(t_0), y_2^{(2)}(t_0), \cdots, y_2^{(n-1)}(t_0)\right) = (0, 1, 0, \cdots, 0)$$

$$\vdots$$

$$\left(y_k(t_0), y_k^{(1)}(t_0), \cdots, y_k^{(k-1)}(t_0), \cdots, y_k^{(n-1)}(t_0)\right) = (0, 0, \cdots, \underset{k^{\text{th}}}{1}, \cdots, 0)$$

$$\vdots$$

$$\left(y_n(t_0), y_n^{(1)}(t_0), \cdots, y_n^{(n-1)}(t_0)\right) = (0, 0, 0, \cdots, 1) \tag{32.3}$$

Then, the Wronskian of these functions at t_0 is defined to be

$$W = \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = n, \tag{32.4}$$

so $W \neq 0$. This basis is useful for solving the homogeneous problem with $y(t_0) = c_1$, $y^{(1)}(t_0) = c_2$, ..., $y^{(n-1)}(t_0) = c_n$ because the solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + ... + c_n y_n(t), \tag{32.5}$$

as you can easily verify.

32.3 The Wronskian

Previously, when we discussed the Wronskian, we noted that if $W \neq 0$, the system of functions was linearly independent, but if $W = 0$, they might still be independent. We now show that if the functions in the set $\{y_1, \cdots, y_n\}$ solve a homogeneous differential equation, then the Wronskian cannot vanish. Here is why. Suppose $\{y_1, \cdots, y_n\}$ are linearly independent. Form $y(t) =$

$c_1y_1(t)+c_2y_2(t)+...+c_ny_n(t)$ for some set of constants $\{c_i|i=1,\cdots n\}$. We pick the constants such that

$$\begin{aligned} y(t_0)=c_1y_1(t_0)+c_2y_2(t_0)+\cdots+c_ny_n(t_0) &= 0 \\ y^{(1)}(t_0)=c_1y_1^{(1)}(t_0)+c_2y_2^{(1)}(t_0)+\cdots+c_ny_n^{(1)}(t_0) &= 0 \\ &\vdots \\ y^{(n-1)}(t_0)=c_1y_1^{(n-1)}(t_0)+c_2y_2^{(n-1)}(t_0)+\cdots+c_ny_n^{(n-1)}(t_0) &= 0, \end{aligned} \tag{32.6}$$

which can be done since we assumed that

$$\det\begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1^{(1)}(t_0) & y_2^{(1)}(t_0) & \cdots & y_n^{(1)}(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} = 0 \tag{32.7}$$

(recall that we are trying to show that we *cannot* have the Wronskian vanish if the functions are linearly independent). This means that $y(t)=0$ for all t, because $y(t)=0$ solves the differential equation *and* the initial conditions. Since the solution of the differential equations is unique, this implies that $y(t)=c_1y_1(t)+c_2y_2(t)+...+c_ny_n(t)=0$ for all t with coefficients c_i not all zero. But that implies $\{y_1,...,y_n\}$ are not linearly independent, which is a contradiction. So we cannot have $\{y_1,...,y_n\}$ linearly independent *and* the Wronskian vanishing at $t=t_0$ This implies that the Wronskian won't ever vanish for a set of linearly independent functions!

32.4 Method of variation of parameters

There is no simple way to say this, so I will be blunt. The method of variation of parameters is just a tortuous method to use to find a particular solution to a differential equation. It does always work, but one should use other techniques (which are essentially good guessing) if at all possible instead of this method. But, this method will *always* work. So if you have no other choice, it is a method of last resort.

Variation of parameters is used to find particular solutions to inhomogeneous differential equations. Suppose the differential equation that we want to solve is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y^{(1)} + a_0(t)y = f(t). \tag{32.8}$$

We seek to find a solution of the form

$$y_p(t) = A_1(t)y_1(t) + A_2(t)y_2(t) + \cdots + A_n(t)y_n(t). \tag{32.9}$$

Then, we have

$$\begin{aligned} \frac{dy_p}{dt} &= A_1(t)\dot{y}_1(t) + A_2(t)\dot{y}_2(t) + \cdots + A_n(t)\dot{y}_n(t) \\ &+ \dot{A}_1(t)y_1(t) + \dot{A}_2(t)y_2(t) + \cdots + \dot{A}_n(t)y_n(t). \end{aligned} \tag{32.10}$$

We choose to set $\dot{A}_1(t)y_1(t) + \dot{A}_2(t)y_2(t) + \cdots + \dot{A}_n(t)y_n(t) = 0$. This is a condition on the functions $A_1, \cdots, A_n$. Calculating the second derivative, we find

$$\begin{aligned} \frac{d^2y_p}{dt^2} &= A_1(t)\ddot{y}_1(t) + A_2(t)\ddot{y}_2(t) + \cdots + A_n(t)\ddot{y}_n(t) \\ &+ \dot{A}_1(t)\dot{y}_1(t) + \dot{A}_2(t)\dot{y}_2(t) + \cdots + \dot{A}_n(t)\dot{y}_n(t). \end{aligned} \tag{32.11}$$

Again, we choose to set $\dot{A}_1(t)\dot{y}_1(t) + \dot{A}_2(t)\dot{y}_2(t) + \cdots + \dot{A}_n(t)\dot{y}_n(t) = 0$.

We continue this way for the higher derivatives up to

$$\begin{aligned} \frac{d^ny_p}{dt^n} &= A_1y_1^{(n)} + A_2y_2^{(n)} + \cdots + A_ny_n^{(n)} \\ &+ \dot{A}_1y_1^{(n-1)} + \dot{A}_2y_2^{(n-1)} + \cdots + \dot{A}_ny_n^{(n-1)}. \end{aligned} \tag{32.12}$$

We also choose to set $\dot{A}_1y_1^{(n-1)} + \dot{A}_2y_2^{(n-1)} + ... + \dot{A}_ny_n^{(n-1)} = 0$.

Now, because all terms without derivatives with respect to t of A's vanish due to the fact that the y_i's solve the homogeneous equation, we find that the differential equation becomes the following set of equations (which includes the original equation and all of the "constraints" we chose earlier):

$$\dot{A}_1(t)y_1(t) + \dot{A}_2(t)y_2(t) + \cdots + \dot{A}_n(t)y_n(t) = 0 \tag{32.13}$$

$$\dot{A}_1(t)\dot{y}_1(t) + \dot{A}_2(t)\dot{y}_2(t) + \cdots + \dot{A}_n(t)\dot{y}_n(t) = 0 \tag{32.14}$$

$$\vdots$$

$$\dot{A}_1y_1^{(n-2)} + \dot{A}_2y_2^{(n-2)} + \cdots + \dot{A}_ny_n^{(n-2)} = 0 \tag{32.15}$$

$$\dot{A}_1y_1^{(n-1)} + \dot{A}_2y_2^{(n-1)} + \cdots + \dot{A}_ny_n^{(n-1)} = \frac{f(t)}{a_n(t)}. \tag{32.16}$$

This is a system of n equations in n unknowns with the determinant of the coefficient matrix being the matrix from which we compute the Wronskian of $\{y_1, \cdots, y_n\}$

$$W(t) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ \dot{y}_1 & \dot{y}_2 & \cdots & \dot{y}_n \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}. \tag{32.17}$$

But the Wronskian is never zero, so we can always solve for $\dot{A}_1, \dot{A}_2, \cdots, \dot{A}_n$ for each t value. This implies that these n these equations can each be solved. The solution y_p is the particular solution to the differential equation.

32.5 Example

As an example, consider solving the following third-order inhomogeneous linear differential equation:

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = -e^{-t} \tag{32.18}$$

The homogeneous equation is

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = 0. \tag{32.19}$$

We let $y = e^{\alpha t}$ and require $(\alpha^3 + 3\alpha^2 + 2\alpha)e^{\alpha t} = 0$ in order to solve the homogeneous equation (we try this ansatz because the equation has *constant* coefficients).

Our next step is to factor the polynomial

$$\alpha(\alpha^2 + 3\alpha + 2) = \alpha(\alpha + 2)(\alpha + 1) = 0. \tag{32.20}$$

The solution are $\alpha = 0, -1, -2$, so the homogeneous equation is solved by

$$y_1 = 1, \quad y_2 = e^{-t}, \quad \text{and} \quad y_3 = e^{-2t}. \tag{32.21}$$

The Wronskian becomes

$$W(t) = \det \begin{pmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{pmatrix} = -4e^{-3t} + 2e^{-3t} = -2e^{-3t} \neq 0. \tag{32.22}$$

This implies that the three solutions are independent.

To solve the variation of parameters method, we must find the derivatives of the A_i functions via

$$\begin{pmatrix} 1 & e^{-t} & e^{-2t} \\ 0 & -e^{-t} & -2e^{-2t} \\ 0 & e^{-t} & 4e^{-2t} \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -e^{-t} \end{pmatrix}. \tag{32.23}$$

Solving this equation for $\dot{A}_1, \dot{A}_2$, and $\dot{A}_3$, we obtain

$$\dot{A}_3 = -\frac{1}{2}e^{t} \implies A_3 = -\frac{1}{2}e^{t}, \tag{32.24}$$

$$-e^{-t}\dot{A}_2 = -e^{t} \implies \dot{A}_2 = 1 \implies A_2 = t \tag{32.25}$$

and

$$\dot{A}_1 + e^{-t} - \frac{1}{2}e^{-t} = 0 \implies \dot{A}_1 = -\frac{1}{2}e^{-t} \implies \frac{1}{2}e^{-t}. \tag{32.26}$$

So,

$$y_p(t) = \frac{1}{2}e^{-t} \times 1 + t \times e^{-t} - \frac{1}{2}e^{t} \times e^{-2t} = te^{-t}. \tag{32.27}$$

To check this, we need the higher derivatives as well

$$\dot{y}_p = e^{-t} - te^{-t} = (1-t)e^{-t}, \tag{32.28}$$

$$\ddot{y}_p = -e^{-t} - (1-t)e^{-t} = (-2+t)e^{-t}, \tag{32.29}$$

and

$$\dddot{y}_p = e^{-t} + (2-t)e^{-t} = (3-t)e^{-t}. \tag{32.30}$$

Then, we find

$$\dddot{y}_p + 3\ddot{y}_p + 2\dot{y}_p = (3 - t - 6 + 3t + 2 - 2t)e^{-t} = -e^{-t} \tag{32.31}$$

which checks!

Chapter 33

Linear Differential Equations with Constant Coefficients

33.1 Description of the basic method

Suppose we have an nth-order linear differential equation with constant coefficients

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1} y^{(1)} + p_n y = f(t) \tag{33.1}$$

and its associated homogeneous equation

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_{n-1} y^{(1)} + p_n y = 0. \tag{33.2}$$

Here $p_1, p_2, \cdots, p_n$ are constants. Recall we use the superscript (n) to denote an n-fold derivative. We will use D to denote $\frac{d}{dt}$ and then the differential equation becomes

$$(D^n + p_1 D^{n-1} + p_2 D^{n-2} + \cdots + p_{n-1} D + p_n) y = f, \tag{33.3}$$

which is a polynomial in D with real coefficients "acting" on the function y. The polynomial in D is called an *operator* and it *acts* on a function. Let

$$P(D) = D^n + p_1 D^{n-1} + p_2 D^{n-2} + ... + +p_{n-1} D + p_n. \tag{33.4}$$

We factorize it into products of monomials as follows:

$$P(D) = (D - r_1)^{k_1} (D - r_2)^{k_2} ... (D - r_m)^{k_m}, \tag{33.5}$$

where we have m distinct roots $\{r_i,\ i = 1, \cdots m\}$ and $k_1 + k_2 + ... + k_m = n$. The roots r_i are in general complex, but because the coefficients p_i are real, $P(D) = P^*(D)$, so the roots must come in *complex conjugate pairs* with $k_i = k_j$ if $r_i = r_j^*$. The corresponding homogeneous equation is

$$P(D)y = 0, \tag{33.6}$$

which can be solved by forming any linear combination of all n of the linearly independent solutions to the homogeneous equation (the inhomogeneous equation can be solved by the method of variation of parameters from Chapter 32). So if we find a y that satisfies

$$(D - r_i)^{k_i} y = 0, \tag{33.7}$$

then we must also have $P(D)y = 0$, because all derivative terms in the product of $P(D)$ commute with each other. This is due to the fact that the p_i (and r_i) are numbers and the derivative of a number is always zero: $Dp_i = Dr_i = 0$. (Be sure you understand this point, which is both subtle and important.)

Starting with the case where $k_i = 1$, we claim

$$(D - r_i)y = e^{r_i t} D(e^{-r_i t} y). \tag{33.8}$$

We check via the chain rule:

$$D(e^{-r_i t} y) = -r_i e^{-r_i t} y + e^{-r_i t} Dy \tag{33.9}$$

so

$$e^{r_i t} D(e^{-r_i t} y) = (D - r_i)y \tag{33.10}$$

which checks. Hence, we have

$$(D - r_i)^{k_i} y = (D - r_i)^{k_{i-1}} \left[e^{r_i t} D(e^{-r_i t} y)\right] \tag{33.11}$$

$$= (D - r_i)^{k_{i-2}} [e^{r_i t} D \left(e^{-r_i t} e^{r_i t} D(e^{-r_i t} y)\right] \tag{33.12}$$

$$= (D - r_i)^{k_{i-2}} e^{r_i t} D^2 (e^{-r_i t} y). \tag{33.13}$$

So we find the general identity

$$(D - r_i)^{k_i} y = e^{r_i t} D^{k_i} (e^{-r_i t} y). \tag{33.14}$$

This can be immediately integrated to find that

$$(D - r_i)^{k_i} y = 0 \implies e^{-r_i t} y = c_1 + c_2 t + c_3 t^2 + \cdots + c_{k_i} t^{k_i - 1} \tag{33.15}$$

or that

$$y(t) = c_1 e^{r_i t} + c_2 t e^{r_i t} + \cdots + c_{k_i} t^{k_i - 1} e^{r_i t} \tag{33.16}$$

is the general solution of $(D - r_i)^{k_i} y = 0$. Putting this together, the solution to $P(D)y = 0$ is a linear combination of the following functions:

$$\begin{aligned}
&r_1 : e^{r_1 t},\ t e^{r_1 t},\ t^2 e^{r_1 t}, \cdots, t^{k_1 - 1} e^{r_1 t} \\
&r_2 : e^{r_2 t},\ t e^{r_2 t},\ t^2 e^{r_2 t}, \cdots, t^{k_2 - 1} e^{r_2 t} \\
&\cdots \\
&r_m : e^{r_m t},\ t e^{r_m t},\ t^2 e^{r_m t}, \cdots, t^{k_m - 1} e^{r_m t}
\end{aligned}$$

where one can show, by calculating the Wronskian, that these functions are all independent. The full solution then becomes a linear combination of these functions plus any particular solution.

If all r_i are real, the above functions are also real. If any r_i are complex, they occur in complex conjugate pairs and when $r_i = r_j^*$ we also have $k_i = k_j$. In this case, changing the functions to

$$\frac{1}{2} t^\alpha (e^{r_i t} + e^{r_j t}) \tag{33.17}$$

and

$$\frac{1}{2i} t^\alpha (e^{r_i t} - e^{r_j t}) \tag{33.18}$$

and using $r_i = a_i + i b_i$ and $r_j = a_i - i b_i$ with a_i, b_i real, then gives the functions

$$t^\alpha e^{a_i t} \cos b_i t \tag{33.19}$$

and

$$t^\alpha e^{a_i t} \sin b_i t \tag{33.20}$$

as the *real function basis* to use for solving these problems. Note that here, we have $0 \leq \alpha \leq k_i - 1$.

33.2 Examples

One really needs to work through some examples before you can feel comfortable with being able to solve problems using this methodology. Here is our first example. Consider the quartic differential operator

$$D^4 - 4D^3 + 4D^2 = P(D). \tag{33.21}$$

Solve $P(D)y = 0$. We start by factorizing $P(D)$:

$$P(D) = D^2(D^2 - 4D + 4) = D^2(D-2)^2 \tag{33.22}$$

so functions that solve the homogeneous equation are $1, t, e^{2t}$,and te^{2t}. First check that these functions are indeed independent. The Wronskian becomes

$$\det W = \det \begin{pmatrix} 1 & t & e^{2t} & te^{2t} \\ 0 & 1 & 2e^{2t} & (1+2t)e^{2t} \\ 0 & 0 & 4e^{2t} & (4+4t)e^{2t} \\ 0 & 0 & 8e^{2t} & (12+8t)e^{2t} \end{pmatrix}. \tag{33.23}$$

We find the determinant by row reduction. Subtract twice the third row from the fourth and then take the product of the diagonal elements:

$$\det W = \det \begin{pmatrix} 1 & t & e^{2t} & te^{2t} \\ 0 & 1 & 2e^{2t} & (1+2t)e^{2t} \\ 0 & 0 & 4e^{2t} & (4+4t)e^{2t} \\ 0 & 0 & 0 & 4e^{2t} \end{pmatrix} = 16e^{4t} \neq 0. \tag{33.24}$$

So the functions are linearly independent.

Second example: Start from the differential operator

$$P(D) = D^5 + D^4 - D^3 - 3D^2 + 2. \tag{33.25}$$

We know we need to factorize this. Let us use a theorem from algebra—a polynomial with integer coefficients can only have rational roots of the form $r = \frac{p}{q}$, where p and q are integers. We must have p divides the constant term in the polynomial and q divides the coefficient of the highest power. So in this case $p = \pm 1, \ \pm 2$ (constant coefficient is 2), $q = \pm 1$ (highest-power coefficient is 1). Then our possible real roots of the polynomial are $\frac{p}{q} = 1, -1, 2$, and -2. The only way to see if these are roots is to substitute in for D and see if the polynomial vanishes.

Substituting into the polynomial shows that the only roots are $r = 1$ and $r = -1$ (which means there are multiple roots or complex roots). In any case, we know that $(D+1)(D-1) = D^2 - 1$ divides $P(D)$. Performing the division yields

$$P(D) = (D^2 - 1)(D^3 + D^2 - 2) \tag{33.26}$$

Repeating the p and q method on the cubic factor of $P(D)$, we find

$$P(D) = (D^2 - 1)^2(D + 1)(D^2 + 2D + 2). \tag{33.27}$$

Then using the quadratic formula, we find the remaining roots are $-1 \pm i$. So the solution y is a linear combination of $e^t, te^t, e^{-t}, e^{-t}\cos t$, and $e^{-t}\sin t$.

Our third example starts with the differential operator $P(D) = D^4 - 16$. We show how to solve the inhomogeneous equation $P(D)y = \cos t$.

First, we must solve the homogeneous equation, which can be factorized "by inspection":

$$D^4 - 16 = (D^2 - 4)(D^2 + 4) = (D - 2)(D + 2)(D - 2i)(D + 2i). \tag{33.28}$$

The general solution is then a linear combination of $e^{2t}, e^{-2t}, \cos 2t$, and $\sin 2t$ plus a particular solution.

To find the particular solution, we use variation of parameters. Recall this method is rather painful with all the algebraic manipulations we need to do. To begin, we write the particular solution as a linear combination of different functions multiplied by the homogeneous solutions, as follows:

$$y_p(t) = A_1(t)e^{2t} + A_2(t)e^{-2t} + A_3(t)\cos 2t + A_4(t)\sin 2t. \tag{33.29}$$

We find the four equations that we need to solve are (recall they involve higher order derivatives for each subsequent equation)

$$\begin{aligned}
\dot{A}_1 e^{2t} + \dot{A}_2 e^{-2t} + \dot{A}_3 \cos 2t + \dot{A}_4 \sin 2t &= 0 && (33.30)\\
\dot{A}_1 2e^{2t} + \dot{A}_2(-2)e^{-2t} + \dot{A}_3(-2)\sin 2t + \dot{A}_4(2)\cos 2t &= 0 && (33.31)\\
\dot{A}_1 4e^{2t} + \dot{A}_2 4e^{-2t} - \dot{A}_3 4\cos 2t - \dot{A}_4 4\sin 2t &= 0 && (33.32)\\
\dot{A}_1 8e^{2t} + \dot{A}_2(-8)e^{-2t} + \dot{A}_3 8\sin 2t - 8\dot{A}_4(2)\cos 2t &= \cos t. && (33.33)
\end{aligned}$$

We reorganize the equations into a matrix form given by

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 2e^{2t} & -2e^{-2t} & -2\sin 2t & 2\cos 2t \\ 4e^{2t} & 4e^{-2t} & -4\cos 2t & -4\sin 2t \\ 8e^{2t} & -8e^{-2t} & 8\sin 2t & -8\cos 2t \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix} \tag{33.34}$$

Now we use row reduction to solve it. This requires a lot of algebra. We zero out the first column

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 0 & -4e^{-2t} & -2(\cos 2t + \sin 2t) & 2(\cos 2t - \sin 2t) \\ 0 & 0 & -8\cos 2t & -8\sin 2t \\ 0 & -16e^{-2t} & 8(-\cos 2t + \sin 2t) & -8(\cos 2t + \sin 2t) \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}. \tag{33.35}$$

Then the second column

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 0 & -4e^{-2t} & -2(\cos 2t + \sin 2t) & 2(\cos 2t - \sin 2t) \\ 0 & 0 & -8\cos 2t & -8\sin 2t \\ 0 & 0 & 16\sin 2t & -16\cos 2t \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}. \tag{33.36}$$

And finally the third

$$\begin{pmatrix} e^{2t} & e^{-2t} & \cos 2t & \sin 2t \\ 0 & -4e^{-2t} & -2(\cos 2t + \sin 2t) & 2(\cos 2t - \sin 2t) \\ 0 & 0 & -8\cos 2t & -8\sin 2t \\ 0 & 0 & 0 & -16\sec 2t \end{pmatrix} \begin{pmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \dot{A}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{pmatrix}. \tag{33.37}$$

So, the solution is

$$\dot{A}_4 = -\frac{1}{16}\cos t \cos 2t \tag{33.38}$$

$$\dot{A}_3 = \frac{1}{16}\cos t \sin 2t \tag{33.39}$$

$$\begin{aligned} \dot{A}_2 &= \frac{1}{4}e^{2t}\left(-\frac{1}{8}\cos t \cos 2t \sin 2t\right. \\ &- \left.\frac{1}{8}\cos t \sin^2 2t - \frac{1}{8}\cos t \cos^2 2t + \frac{1}{8}\cos t \cos 2t \sin 2t\right), \end{aligned} \tag{33.40}$$

which becomes

$$\dot{A}_2 = -\frac{1}{32}e^{2t}\cos t. \tag{33.41}$$

Finally, we have

$$\dot{A}_1 = \frac{1}{32}e^{-2t}\cos t. \tag{33.42}$$

Each of these differential equations needs to be solved. We do this in turn next. For the first, we write the trig functions as sums of exponentials

$$\dot{A}_1 = \frac{1}{32}e^{-2t}\cos t = \frac{1}{64}\left(e^{(-2+i)t} + e^{(-2-i)t}\right), \tag{33.43}$$

which can be easily integrated to yield

$$A_1 = \frac{1}{64}\left(-\frac{1}{2-i}e^{(-2+i)t} - \frac{1}{2+i}e^{(-2-i)t}\right). \tag{33.44}$$

We do a similar thing to the second equation

$$\dot{A}_2 = -\frac{1}{32}e^{2t}\cos t = -\frac{1}{64}\left(e^{(2+i)t} + e^{(2-i)t}\right), \tag{33.45}$$

which yields

$$A_2 = -\frac{1}{64}\left(\frac{1}{2+i}e^{(2+i)t} + \frac{1}{2-i}e^{(2-i)t}\right). \tag{33.46}$$

Repeating for the third gives us

$$\dot{A}_3 = \frac{1}{16}\cos t \sin 2t = \frac{1}{64i}\left(e^{3it} + e^{it} - e^{-it} - e^{-3it}\right) \tag{33.47}$$

and

$$A_3 = -\frac{1}{64}\left(\frac{1}{3}e^{3it} + e^{it} + e^{-it} + \frac{1}{3}e^{-3it}\right). \tag{33.48}$$

And last, but not least, we have

$$\dot{A}_4 = -\frac{1}{16}\cos t \cos 2t = -\frac{1}{64}\left(e^{3it} + e^{it} + e^{-it} + e^{-3it}\right), \tag{33.49}$$

which is solved by

$$A_4 = -\frac{1}{64i}\left(\frac{1}{3}e^{3it} + e^{it} - e^{-it} - \frac{1}{3}e^{-3it}\right). \tag{33.50}$$

So we have our final results are

$$A_1 = -\frac{1}{32}\mathrm{Re}\left(\frac{e^{(-2+i)t}}{2-i}\right) \tag{33.51}$$

$$A_2 = -\frac{1}{32}\mathrm{Re}\left(\frac{e^{(2+i)t}}{2+i}\right) \tag{33.52}$$

$$A_3 = -\frac{1}{32}\left(\frac{1}{3}\cos 3t + \cos t\right) \tag{33.53}$$

$$A_4 = -\frac{1}{32}\left(\frac{1}{3}\sin 3t + \sin t\right) \tag{33.54}$$

and the particular solution becomes

$$A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4 = y_p(t). \tag{33.55}$$

We substitute in the results from the homogeneous solutions and simplify to

$$y_p(t) = -\frac{1}{40}\cos t - \frac{1}{96}\cos t - \frac{1}{32}\cos t = -\frac{1}{15}\cos t. \tag{33.56}$$

You need to use the cosine difference formula to find this.

Of course, now that we see what the final answer is (which is easy to check), we could have simply guessed it by trying a number times $\cos t$. But it is comforting to know that we could work through the full variation of parameters to reach the final solution. The procedure is never fun though.

Chapter 34

Method of Undetermined Coefficients

34.1 Introduction

As we saw last time, the method of variation of parameters is an exceedingly painful way to find the particular solution of a linear inhomogeneous differential equation, which we call y_p. While it will always work, there is an alternative, called the method of undetermined coefficients. It is essentially a procedure that allows you to guess and verify your guess using some standard procedures. You can think of it as providing a sufficiently general guess (with some adjustable parameters) and a procedure that shows you how to adjust the parameters to make it work.

The two realistic guesses that we can make are polynomials if $f(t)$ is a polynomial in t and exponentials if $f(t)$ has exponentials (or sines and cosines) in it. The only subtlety is if the guess is already one of the terms in the homogeneous solutions, then we need to multiply by powers of t that are higher than what was in the homogeneous solution set. Some examples will make these restrictions clear.

The method of undetermined coefficients is a method that is best taught by looking through a few examples to see the different types of issues that arise with the method. Since it involves guessing, it is only as good as the initial guess that is made. We illustrate a number of standard guesses that will work for certain inhomogeneous functions $f(t)$.

34.2 Example 1

Our first example is the same problem we tackled last time. The procedure was long and complicated before. Here we will find it will go a bit faster. We start from our old friend, the differential equation

$$(D^4 - 16)y = \cos(t) \tag{34.1}$$

We make the guess for $y_p(t)$ to be given by the form $y_p(t) = a\cos(t)$, with a a yet to be determined constant. Why does this seem reasonable? Because the fourth derivative of a cosine is proportional to a cosine. Hence, we have

$$D^4(a\cos(t)) = a\cos(t), \tag{34.2}$$

so, by substituting into the equation, we find that

$$a(1-16)\cos(t) = \cos(t) \implies a = -\frac{1}{15}, \quad y_p = -\frac{1}{15}\cos(t). \tag{34.3}$$

You can see that we simply force the given form to work and it yields the answer with very little work. This is much simpler than the solution we found before using variation of parameters.

34.3 Example 2

Here, we examine a differential equation, where the function f includes a term that is part of the solution of the homogeneous equation. This will require our guess to be a bit more sophisticated. The differential equation is

$$(D^3 - D)y = 1 + t. \tag{34.4}$$

We must first find the homogeneous solution. The polynomial in D is given by

$$P(D) = (D^3 - D) = D(D-1)(D+1), \tag{34.5}$$

and is easily factorized. The roots of the polynomial are $0, 1, -1$. Using the roots of the polynomial, we immediately find that the homogeneous solution is a linear combination of 1, e^t, and e^{-t}.

Now our first guess for y_p might have been $at + b$ but when we plug in b, we get $P(D)b = 0$. This occurs, because b solves the homogeneous equation.

So we need to multiply by an additional factor of t to try at^2+bt as our guess for y_p. We now plug into the differential equation to see if we can make it work. We operate the differential polynomial on y_p to find $D^3(at^2+bt)=0$ and $-D(at^2+bt)=-2at-b$. Hence to make $P(D)y_p=1+t$, we must pick $b=-1$ and $a=-\frac{1}{2}$. This gives us $y_p=-\frac{1}{2}t^2-t$, which we can immediately check. And it works!

34.4 Example 3

Here, we look at a more complicated example where the homogeneous solution represented in y_p is a nontrivial function of t. The differential equation we wish to solve is

$$(D^2-5D+6)y=e^{2t}+\cos(t). \tag{34.6}$$

We find the homogeneous solutions using our standard methods. The roots of $P(D)$ are

$$r_\pm=\frac{5}{2}\pm\frac{1}{3}\sqrt{25-24}=\frac{5}{2}\pm\frac{1}{2}=2,\ 3. \tag{34.7}$$

This means $y_{\text{homog}}(t)$ is a linear combination of e^{2t} and e^{3t}. Our initial guess for y_p would have been $ae^{2t}+be^{it}+ce^{-it}$, but e^{2t} is in the homogeneous solution, so we revise by multiplying the homogeneous solution by an extra power of t to give us our ansatz $y_p(t)=ate^{2t}+be^{it}+ce^{-it}$. You may wonder why we did not use just a number times $\cos(t)$. We are forced into this situation because there are both even order and odd order derivatives in the differential equation (recall, the derivative of a sin is a cos, and so on).

We operate the differential equation on our ansatz. This requires us to calculate a few different terms. We find that

$$\begin{aligned}
&(D^2-5D+6)(ate^{2t}+be^{it}+ce^{-it})=\\
&\quad=a(4+4t)e^{2t}-5a(1+2t)e^{2t}+6ate^{2t}-be^{it}-5ibe^{it}+6be^{it}\\
&\quad-ce^{-it}+5ice^{-it}+6ce^{-it}\\
&\quad=e^{2t}+\frac{1}{2}e^{it}+\frac{1}{2}e^{-it},
\end{aligned} \tag{34.8}$$

after using the definition of the cosine in terms of exponentials. Hence we have

$$a(4+4t)-a(5+10t)+a6t=1, \tag{34.9}$$

which implies that $a = -1$. We also have

$$-b - 5ib + 6b = \frac{1}{2} \tag{34.10}$$

or

$$b = \left(\frac{1}{10}\right) \frac{1}{1-i}. \tag{34.11}$$

Finally, we have

$$-c + 5ic + 6c = \frac{1}{2}, \tag{34.12}$$

which implies that

$$c = \left(\frac{1}{10}\right) \frac{1}{1+i}. \tag{34.13}$$

Putting this all together yields

$$y_p(t) = -te^{2t} + \frac{1}{5}\mathrm{Re}\left\{\frac{e^{it}}{1-i}\right\} = -te^{2t} + \left(\frac{1}{5}\right)\mathrm{Re}\left\{\frac{1+i}{2}e^{it}\right\}. \tag{34.14}$$

Calculating the real parts finally gives us our answer

$$y_p(t) = -te^{2t} + \frac{1}{10}[\cos(t) - \sin(t)]. \tag{34.15}$$

The full solution of the differential equation then becomes

$$y(t) = c_1 e^{2t} + c_2 e^{3t} - te^{2t} + \frac{1}{10}[\cos(t) - \sin(t)]. \tag{34.16}$$

I hope you can see that this method is far superior to variation of parameters when it works (that is, when you can make the right guess).

34.5 Example 4

Moving on, the next equation to solve is

$$(D^2 + 2D + 2)y = t\cos(2t) + \sin(2t) \tag{34.17}$$

The roots of $P(D)$ are

$$r_{\pm} = -1 \pm \frac{1}{2}\sqrt{4-8} = -1 \pm i \tag{34.18}$$

so $y_{\text{homog}}(t) =$ linear combination of $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$. Our guess for $y_p(t)$ is

$$y_p(t) = a\cos(2t) + bt\cos(2t) + c\sin(2t) + dt\sin(2t). \tag{34.19}$$

Again, it might seem odd to have both constants and linear terms multiplying the trig functions, but this is needed when we take the derivatives, as you will soon see. In other words, we add $a\cos(2t)$ because the derivatives will remove the power of t, and we add $dt\sin(2t)$, because we are likely to need it.

We now plug the ansatz into the equation to find

$$\begin{aligned}(D^2 + 2D + 2)y_p &= -4a\cos(2t) - 4a\sin(2t) + 2a\cos(2t) - 4tb\cos(2t)\\ &\quad - 4b\sin(2t) - 4tb\sin(2t) + 2b\cos(2t) + 2tb\cos(2t) - 4c\sin(2t)\\ &\quad + 4c\cos(2t) + 2c\sin(2t) - 4td\sin(2t) + 4d\cos(2t) + 4td\cos(2t)\\ &\quad + 2d\sin(2t) + 2td\sin(2t)\\ &\quad = t\cos(2t) + \sin(2t). \end{aligned}\tag{34.20}$$

We now have to combine terms and solve for the coefficients. The coefficients of the following functions must satisfy these coupled linear equations:

$$\begin{aligned}\cos(2t):&\qquad -2a + 2b + 4c + 4d = 0 &(34.21)\\ t\cos(2t):&\qquad 0a - 2b + 0c + 4d = 1 &(34.22)\\ \sin(2t):&\qquad -4a - 4b - 2c + 2d = 1 &(34.23)\\ t\sin(2t):&\qquad 0a - 4b + 0c - 2d = 0. &(34.24)\end{aligned}$$

We can rewrite this as the matrix equation

$$\begin{pmatrix} -2 & 2 & 4 & 4\\ 0 & -2 & 0 & 4\\ -4 & -4 & -2 & 2\\ 0 & -4 & 0 & -2 \end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix} 0\\ 1\\ 1\\ 0\end{pmatrix} \tag{34.25}$$

and use row reduction to solve it. This involves the following steps:

$$\left(\begin{array}{rrrr|c} -1 & 1 & 2 & 2 & 0\\ 0 & -1 & 0 & 2 & \frac{1}{2}\\ -2 & -2 & -1 & 1 & \frac{1}{2}\\ 0 & -2 & 0 & -1 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{rrrr|c} -1 & 1 & 2 & 2 & 0\\ 0 & -1 & 0 & 2 & \frac{1}{2}\\ 0 & -4 & -5 & -3 & \frac{1}{2}\\ 0 & -2 & 0 & -1 & 0 \end{array}\right) \tag{34.26}$$

$$\Longrightarrow \left(\begin{array}{cccc|c} -1 & 1 & 2 & 2 & 0 \\ 0 & -1 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & -5 & -11 & -\frac{3}{2} \\ 0 & 0 & 0 & -5 & -1 \end{array}\right) \tag{34.27}$$

We can now find the values of a, b, c, and d:

$$d = \frac{1}{5} \tag{34.28}$$

$$-5c - \frac{11}{5} = -\frac{3}{2} \Longrightarrow -5c = \frac{7}{10} \Longrightarrow c = -\frac{7}{50} \tag{34.29}$$

$$b = 2d - \frac{1}{2} = \frac{2}{5} - \frac{1}{5} \Longrightarrow b = -\frac{1}{10} \tag{34.30}$$

$$-a - \frac{1}{10} - \frac{7}{25} + \frac{2}{5} = 0 \Longrightarrow a = \frac{-5 - 14 + 20}{50} \Longrightarrow a = \frac{1}{50}. \tag{34.31}$$

Hence, we have the following for the general solution $y(t)$:

$$\begin{aligned} y(t) &= c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) \\ &+ \frac{1}{50}\cos(2t) - \frac{1}{10}t\cos(2t) - \frac{7}{50}\sin(2t) + \frac{1}{5}t\sin(2t). \end{aligned} \tag{34.32}$$

34.6 Example 5

Our final example is the equation

$$P(D)y = e^{\alpha t}, \tag{34.33}$$

where α is not equal to any of the roots $r_1, \cdots, r_n$ of $P(D)$. Then for the general solution $y(t) = y_{\text{homog}}(t) + y_p(t)$, we can make the guess $y_p(t) = ae^{\alpha t}$. The particular solution satisfies

$$P(D)ae^{\alpha t} = aP(\alpha)e^{\alpha t}, \tag{34.34}$$

implying the differential equation gets replaced by an *algebraic* equation. The solution follows immediately as

$$a = \frac{1}{P(\alpha)}, \tag{34.35}$$

since $P(\alpha) \neq 0$ due to the fact that α is not a root of $P(D)$. This shows us that when $f(t) =$ exponential whose exponent is not a root of $P(D)$, then finding $y_p(t)$ is **very easy** to do!

To summarize the method, if $f(t)$ is a polynomial, we guess a polynomial of the same degree unless there is a polynomial as a homogeneous solution, in which case extra factors of t will be needed. If $f(t)$ is an exponential, the above approach shows how to do it when none of the exponents are roots of $P(D)$. If any of them are, we need to once again include higher powers of t to be able to make it work. Finally, in many cases when $f(t)$ is a sine or cosine, we need to include both terms in the ansatz if the differential equation has any odd powers of D in it (and extra powers of t if they appear in the homogeneous solution $\cdots$).

Chapter 35

The Frenet-Serret Apparatus

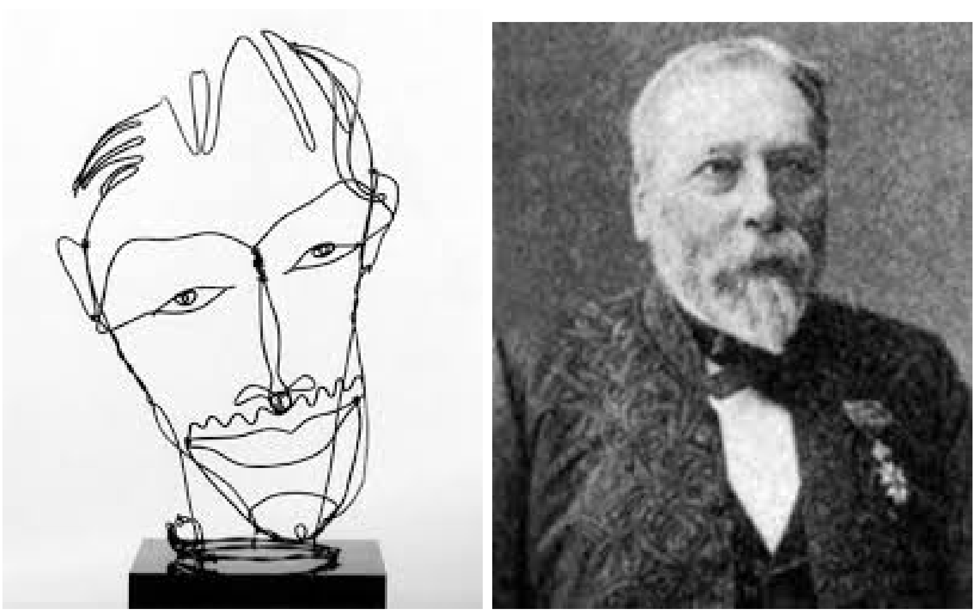

Figure 35.1: Unfortunately there is no easy to find picture of Jean Frédéric Frénet. So I substituted a wire sculpture by Alexander Calder that shows you the artwork in a three-dimensional curve. On the right is an image of Joseph Serret. These two mathematicians worked on the differential geometry of curves in the 19th century.

Differential geometry is a challenging subject. It incorporates the math behind Einstein's general relativity. You really need a full course to learn all of the details. And a lot of hard work! But there is one part of differential geometry that can be taught in just one lecture. It is the theory behind how we describe one-dimensional curves that lie in two or three dimensions. This includes the smooth curves you can draw on a piece of paper and those you can make by bending wire in three dimensions. Let's get started.

We discuss the differential geometry of 2D and 3D curves. Consider a curve in 3D (2D can be found by restricting to $\alpha_3(t) = 0$):

$$\vec{\alpha}(t) = \alpha_1(t)\hat{\imath} + \alpha_2(t)\hat{\jmath} + \alpha_3(t)\hat{k} = \Big(\alpha_1(t), \alpha_2(t), \alpha_3(t)\Big). \tag{35.1}$$

You should already know that $\frac{d\vec{\alpha}}{dt}$ is the velocity of the particle moving on the curve and $|\frac{d\vec{\alpha}}{dt}|$ is the speed. The velocity points in the direction of the unit tangent vector $\vec{T} = \frac{d\vec{\alpha}}{dt}/|\frac{d\vec{\alpha}}{dt}|$.

We start with a nontrivial curve to see how this approach works. Consider the helix $\alpha(t) = (r\cos t, r\sin t, ht)$. The velocity and tangent vector are

$$\frac{d\vec{\alpha}}{dt}(t_0) = (-r\sin t_0, r\cos t_0, h) \tag{35.2}$$

and

$$\vec{T} = (-r\sin t_0, r\cos t_0, h)\frac{1}{\sqrt{r^2+h^2}} \tag{35.3}$$

at time $t = t_0$.

Having to divide by the speed to determine the tangent vector can be cumbersome. Hence, the easiest curves to deal with are unit speed curves parametrized by the arc length s, which is given by the familiar formula from calculus

$$s(t) = \int_0^t dt\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \tag{35.4}$$

Note that we can think of this as integrating the *speed* of the curve as a function of time, which obviously leads to *how far* we moved along the curve.

Let's see an example for how to compute the arc length as a function of time. We have that the speed on our helix example is $\sqrt{r^2+h^2}$, so the arc length $s(t)$ satisfies $\int_0^t dt\sqrt{r^2+h^2} = t\sqrt{r^2+h^2}$. We next invert this relationship to find the *time* as a function of *arc length*. This is simple here and yields $t(s) = \frac{s}{\sqrt{r^2+h^2}}$. This means we can write

$$\vec{\alpha}(s) = \left(r\cos\frac{s}{\sqrt{r^2+h^2}}, r\sin\frac{s}{\sqrt{r^2+h^2}}, h\frac{s}{\sqrt{r^2+h^2}}\right), \tag{35.5}$$

by replacing t with s. When a curve is parametrized by its arc length, instead of some arbitrary time, it becomes a unit-speed curve and the velocity is automatically a unit vector, which is equal to the tangent vector and satisfies

$$\vec{T}(s) = \frac{d\vec{\alpha}}{ds} = \left(-r\sin\frac{s}{\sqrt{r^2+h^2}}, r\cos\frac{s}{\sqrt{r^2+h^2}}, h\right)\frac{1}{\sqrt{r^2+h^2}}. \tag{35.6}$$

You should check to confirm that this tangent vector is indeed a unit vector.

In general, the tangent vector tells us precisely where the particle is going in the next instant. To determine the curve for longer times, we need to know how the tangent vector changes with time. For a straight line $\vec{T}$ does not change but for something like a circle it does.

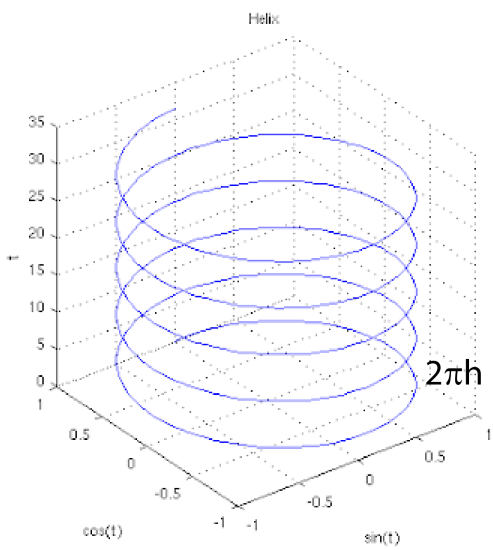

Figure 35.2: Schematic of a helix with radius r. As the curve winds around by an angle of 2π, the curve rises by a distance of $2\pi h$.

In general, when the path curves $\vec{T}$ changes with time. Since $\vec{T}$ is a unit vector, $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$ (be sure you know why). We define

$$\frac{d\vec{T}}{ds} = \kappa(s)\vec{N} \tag{35.7}$$

where $\kappa(s)$ is the curvature and $\vec{N}$ is the principal normal vector for the curve. Note that the derivative of a unit vector must be perpendicular to the unit vector, but it can have an arbitrary length. This is why we need to introduce $\kappa(s)$, which is given by the length of the derivative of the tangent vector.

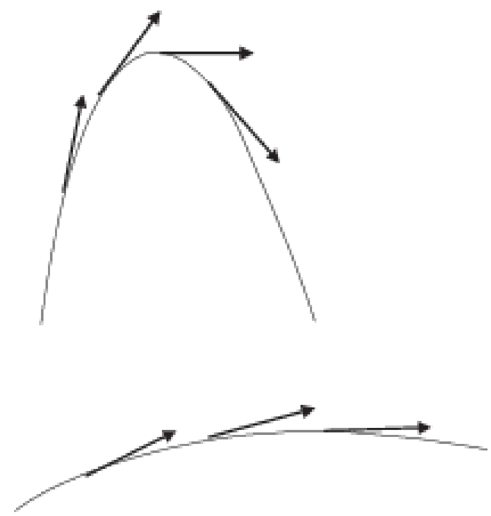

Figure 35.3: Illustration of different curvatures. In the top curve, the tangent vector rotates rapidly as we move along the curve. This is a curve with a large curvature. In the bottom curve, the tangent vector changes its direction more slowly. This has a small curvature and is closer to a straight line (which has no curvature).

Lets explore why we call κ a curvature. Consider the path of radius r that traces out a circle on the $x - y$ plane. It's unit speed parametrization ia given by

$$\vec{\alpha}(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0\right), \tag{35.8}$$

with

$$\vec{T}(s) = \frac{d\vec{\alpha}(s)}{ds} = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}, 0\right); \tag{35.9}$$

one can immediately verify that this is a unit speed curve because $|\vec{T}| = 1$. Now, we find the principal normal vector and the curvature via a second derivative:

$$\frac{d\vec{T}}{ds} = \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0\right). \tag{35.10}$$

By computing the length of the derivative, we find that $\kappa(s) = \frac{1}{r}$, which is the conventional curvature for a circle; we also find that $\vec{N}(s) = (-\cos\frac{s}{r}, \sin\frac{s}{r}, 0)$. This all makes sense because a circle with a small radius curves much more than a circle with a large radius.

For our helix, we find

$$\frac{d\vec{T}}{ds} = \left(-r\cos\frac{s}{\sqrt{r^2+h^2}}, -r\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right)\frac{1}{r^2+h^2} = \kappa(s)\vec{N}. \quad (35.11)$$

Hence, we learn that

$$\vec{N}(s) = \left(-\cos\frac{s}{\sqrt{r^2+h^2}}, -\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right) \quad (35.12)$$

and

$$\kappa(s) = \frac{r}{r^2+h^2}. \quad (35.13)$$

Note that the curvature of the helix is a constant, independent of s, but the helix is not a circle. In other words, not all curves with constant curvature are circles.

We have found two perpendicular vectors, $\vec{T}$ and $\vec{N}$, related to the motion of the particle on the curve. One more and we will have an orthonormal basis for the three-dimensional space. Obviously, we obtain the third vector via a cross product. So we define the binormal vector $\vec{B}(s)$ to satisfy $\vec{B} = \vec{T} \times \vec{N}$. Now for every s, we have $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ forming an orthonormal basis for the three-dimensional space. As s changes, so do $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$; but the three vectors always remain orthonormal and point in the directions of an *instantaneous* coordinate system determined by the curve. We already found that $\frac{d\vec{T}}{ds} = \kappa(s)\vec{N}$. When we take the derivative of the principal normal vector, all we know is that the derivative is perpendicular to $\vec{N}$. This means it can have components along $\vec{T}$ and $\vec{B}$. But the component along $\vec{T}$ is fixed already by the curvature. This is because $\vec{T}(s) \cdot \vec{N}(s) = 0$. If we differentiate that expression, we find

$$\frac{d\vec{T}}{ds} \cdot \vec{N} + \vec{T} \cdot \frac{d\vec{N}}{ds} = 0. \quad (35.14)$$

This means that

$$\vec{T} \cdot \frac{d\vec{N}}{ds} = -\frac{d\vec{T}}{ds} \cdot \vec{N} = -\kappa(s)\vec{N} \cdot \vec{N} = -\kappa(s), \quad (35.15)$$

because the principal normal vector is a unit vector. Hence, the component along $\vec{T}$ is $-\kappa(s)$. We define the component of the derivative of the principal normal along $\vec{B}$ to be the torsion. It satisfies

$$\frac{d\vec{N}}{ds} \cdot \vec{B} = \tau(s). \tag{35.16}$$

Then, noting that $\vec{N} \cdot \vec{B} = 0$, we find that

$$\frac{d\vec{B}}{ds} = -\tau(s)\vec{N}. \tag{35.17}$$

Be sure you understand why the derivative of the binormal vector has no component along $\vec{T}$, which comes from the definition of the principal normal.

For the helix, we have

$$\begin{aligned}
\vec{B}(s) &= \vec{T}(s) \times \vec{N}(s) \\
&= \left(-r\sin\frac{s}{\sqrt{r^2+h^2}}, r\cos\frac{s}{\sqrt{r^2+h^2}}, h\right)\frac{1}{\sqrt{r^2+h^2}} \\
&\qquad \times \left(-\cos\frac{s}{\sqrt{r^2+h^2}}, -\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right) \\
&= \det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\frac{1}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}} & r\frac{1}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}} & h\frac{1}{\sqrt{r^2+h^2}} \\ -\cos\frac{s}{\sqrt{r^2+h^2}} & -\sin\frac{s}{\sqrt{r^2+h^2}} & 0 \end{vmatrix} \\
&= \hat{i}\frac{h}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}} + \hat{j}\left(-\frac{h}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}}\right) \\
&\qquad + \hat{k}\left(\frac{r}{\sqrt{r^2+h^2}}\sin^2\frac{s}{\sqrt{r^2+h^2}} + \frac{r}{\sqrt{r^2+h^2}}\cos^2\frac{s}{\sqrt{r^2+h^2}}\right).
\end{aligned} \tag{35.18}$$

So, we find that

$$\vec{B}(s) = \frac{1}{\sqrt{r^2+h^2}}\left(h\sin\frac{s}{\sqrt{r^2+h^2}}, -h\cos\frac{s}{\sqrt{r^2+h^2}}, r\right). \tag{35.19}$$

To compute the torsion $\tau(s)$, we now compute the derivative of the binormal vector. Note that this is an easier way to obtain the torsion, because the derivative of the binormal lies along the principal normal vector and the length of the derivative immediately yields the torsion. Computing, we find that

$$\frac{d\vec{B}}{ds} = \frac{1}{\sqrt{r^2+h^2}}\left(\frac{h}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}}, \frac{h}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}}, 0\right). \tag{35.20}$$

We can either compute the length of this vector, or, in this case, we find it is easier to compute via the dot product with $\vec{N}$:

$$-\frac{d\vec{B}}{ds}\cdot\vec{N} = \tau(s) = -\frac{1}{r^2+h^2}\left(-h\cos^2\frac{s}{\sqrt{r^2+h^2}} - h\sin\frac{s}{\sqrt{r^2+h^2}}\right). \tag{35.21}$$

Hence, we find that

$$\tau(s) = \frac{h}{r^2+h^2}. \tag{35.22}$$

These five results, $\kappa(s)$, $\tau(s)$, $\vec{T}(s)$, $\vec{N}(s)$, and $\vec{B}(s)$ are called the Frenet-Serret apparatus. They can be painful to compute (lots of derivatives and lots of dot or cross products). But they provide the full description of a curve moving in a three dimensional space (in other words, no higher order derivatives are needed).

Furthermore, all the derivatives of these vectors are determined by κ and τ. We have already established this, but we show how these results follow directly with a compact notation given by

$$\frac{d}{ds}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{T}\\ \vec{N}\\ \vec{B}\end{pmatrix}. \tag{35.23}$$

Proof: We know by the definition of κ and $\vec{N}$, that $\frac{d\vec{T}}{ds} = \kappa\vec{N}$, so the first row is true. In addition, $\frac{d\vec{N}}{ds}$ is found via

$$0 = \vec{T}\cdot\vec{N} \implies \frac{d\vec{T}}{ds}\cdot\vec{N} + \vec{T}\cdot\frac{d\vec{N}}{ds} = 0. \tag{35.24}$$

Then we have that

$$\kappa\vec{N}\cdot\vec{N} + \vec{T}\cdot\frac{d\vec{N}}{ds} = 0, \tag{35.25}$$

from the definition of the principal normal vector and the curvature, Hence,

$$\vec{T}\cdot\frac{d\vec{N}}{ds} = -\kappa \tag{35.26}$$

since $\vec{N} \cdot \vec{N} = 1$. Similarly,

$$\vec{N} \cdot \vec{B} = 0 \implies \frac{d\vec{N}}{ds} \cdot \vec{B} + \vec{N} \cdot \frac{d\vec{B}}{ds} = 0 \implies \frac{d\vec{N}}{ds} \cdot \vec{B} = -\vec{N} \cdot \frac{d\vec{B}}{ds} = \tau \quad (35.27)$$

and $\vec{N} \cdot \vec{N} = 1 \implies \frac{d\vec{N}}{ds} \cdot \vec{N} = 0$, so the second row is true. For the third row, we know that $\frac{d\vec{B}}{ds} \cdot \vec{N} = -\tau$. Furthermore,

$$\vec{B} \cdot \vec{T} = 0 \implies \frac{d\vec{B}}{ds} \cdot \vec{T} + \vec{B} \cdot \frac{d\vec{T}}{ds} = 0 \implies \frac{d\vec{B}}{ds} \cdot \vec{T} = -\kappa \vec{B} \cdot \vec{N} = 0. \quad (35.28)$$

Finally, since a unit vector satisfies $\frac{d\vec{B}}{ds} \cdot \vec{B} = 0$, we have that the third row is true.

The equations above are the Frenet-Serret equations. We have some final definitions for jargon lovers: The *osculating plane* to α at s is the plane through $\alpha(s)$ that is perpendicular to $\vec{B}(s)$. The *normal plane* is the plane perpendicular to $\vec{T}$. The *rectifying plane* is the plane perpendicular to $\vec{N}$.

If $\alpha(s)$ is a planar curve, then $\vec{B} =$ constant and $\tau = 0$. The tangent vector shows the line the curve instantaneously moves on. The osculating plane is the plane the curve instantaneously moves on. The torsion describes how that plane rotates.

A helix can be defined to be a curve for which a unit vector $\vec{u}$ exists that satisfies $\vec{T} \cdot \vec{u} =$ constant . For our helix, $\vec{u} = (0, 0, 1)$. The book proves that if α is a helix, then $\tau = c\kappa$ where c is some constant. For our example above, we have that $c = h/r$. This result is called Lancret's Theorem.

I hope you enjoyed this short primer on differential geometry.

Chapter 36

Dirichlet's Problem and Poisson's Theorem


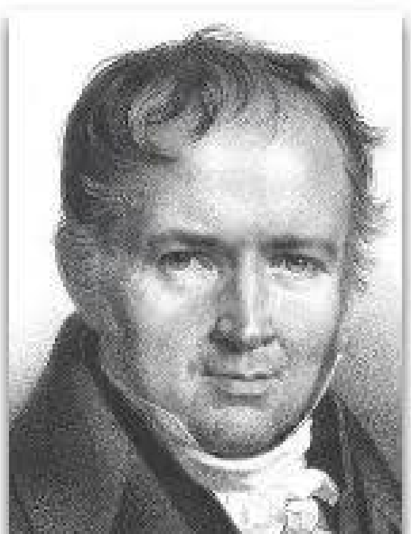

Figure 36.1: (Left) Peter Gustav Dirichlet and (right) Siméon-Denis Poisson. These are two famous mathematicians who worked on the Fourier series problem in the context of heat transport on a disk.

Consider a disc of radius $r = 1$ that has the temperature fixed with the boundary condition on the edge of the disc given by $U(r{=}1, \theta) = f(\theta)$. Here,

$U(r,\theta)$ is the temperature on the disc at the polar coordinates r and θ. Since we are in the steady state, the temperature distribution satisfies Laplace's equation:

$$\nabla^2 U(r,\theta) = 0. \tag{36.1}$$

In polar coordinates, we have that the Laplacian can be written in terms of derivatives with respect to r and θ as follows:

$$\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 U}{\partial \theta^2} = 0, \tag{36.2}$$

as we derived earlier. Note that $U(r,\theta)$ is continuous (and bounded) for $0 \le r \le 1$ and is periodic as θ increase by 2π, which implies that $U(r,\theta + 2\pi) = U(r,\theta)$.

We use the method of separation of variables to solve for $U(r,\theta)$. This technique is one to always consider when examining a partial differential equation that has a sum of derivatives with respect to independent variables. The approach makes an *ansatz* that the solution is a product of functions of each of the different independent variables. Here is how it works in this case. We let $U(r,\theta) = R(r)\Theta(\theta)$, then

$$\frac{\partial}{\partial r}\Big(rR'(r)\Theta(\theta)\Big) + \frac{1}{r}R(r)\Theta''(\theta) = 0. \tag{36.3}$$

Our next step is to divide both sides of the equation by $U/r = R\Theta/r$, which yields

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0. \tag{36.4}$$

Now we pause to look carefully at each term on the left hand side. The first term is a function of r only and the second term is a function of θ only. But the sum of both terms is equal to zero and this holds for *all* r and θ. This implies that we must have each function equal to opposite constants, so that

$$\frac{r^2R'' + rR'}{R} = n^2 = -\frac{\theta''}{\theta}. \tag{36.5}$$

Here, n^2 is a constant, but we allow it to be positive or negative. So

$$\Theta'' = -n^2\Theta. \tag{36.6}$$

This equation is easily solved; it is a differential equation you have seen many times. We have three cases:

$$\Theta(\theta) = \begin{cases} Ae^{in\theta} + Be^{-in\theta} & n^2 > 0 \\ A + B\theta & n^2 = 0 \\ Ae^{n\theta} + Be^{-n\theta} & n^2 < 0 \end{cases}. \tag{36.7}$$

The R equation becomes

$$r^2 R'' + rR' - n^2 R = 0. \tag{36.8}$$

This is a special type of equation called a Cauchy-Euler equation. You can see that each derivative term is multiplied by a corresponding power of r. Such an equation is solved by r raised to a power that is chosen carefully. The solutions are found as follows. We make the ansatz $R = r^\alpha$, which becomes an algebraic equation for α:

$$R(r) = r^\alpha \implies \alpha(\alpha-1)+\alpha-n^2 = 0 \implies \alpha^2-n^2 = 0 \implies \alpha = \pm n. \tag{36.9}$$

Hence, the solutions of this equation are

$$R(r) = \begin{cases} ar^n + br^{-n} & n^2 > 0 \\ a + b\ln(r) & n^2 = 0 \\ ar^{in} + br^{-in} & n^2 < 0 \end{cases}. \tag{36.10}$$

We now need to determine which of the different possible solutions are the correct ones to solve the problem. First, we need to require the periodicity requirement in θ, given by $\Theta(\theta + 2\pi) = \Theta(\theta)$. This implies that

$$n^2 > 0,\ \Theta(\theta) = Ae^{in\theta} + Be^{-in\theta} \text{ or } \Theta(\theta) = A,\ n^2 = 0. \tag{36.11}$$

The next requirement is that $R(r)$ remains bounded. We find that this implies that

$$R(r) = ar^n,\ n^2 > 0 \text{ or } a,\ n^2 = 0. \tag{36.12}$$

So the general result is

$$U(r,\theta) = \sum_{n=-\infty}^{+\infty} A_n r^{|n|} e^{in\theta}, \tag{36.13}$$

with A_n numbers that will be adjusted to solve the boundary condition at the edge of the disc. Note that the solutions for R with a given n^2 are correlated with the solutions for Θ with the same n^2. This is why the sum takes the given form above. Be sure you understand this clearly.

Then, for $r = 1$, we must require that

$$\sum_{n=-\infty}^{+\infty} A_n e^{in\theta} = f(\theta). \tag{36.14}$$

The question is, for arbitrary $f(\theta)$, can we find A_n's such that

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}. \tag{36.15}$$

This series is called the Fourier series. Obviously we can't make this work for arbitrary $f(\theta)$, but if $f(\theta)$ is continuous, then the answer turns out to be yes! It takes some work to show this.

Since we know the integral of the exponential function over 2π

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{in\theta} = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases} \tag{36.16}$$

We next interchange the summation and integration in the Fourier series to find

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) e^{-iN\theta} = \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta A_n e^{i(n-N)\theta} = A_N. \tag{36.17}$$

So the coefficients are found via

$$A_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) e^{-iN\theta}. \tag{36.18}$$

But can this interchange be justified? We show how to establish this rigorously using Poisson's kernel. Recall the geometric series, given by $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. So,

$$\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=-\infty}^{-1} r^{-n} e^{in\theta}. \tag{36.19}$$

When $r < 1$, we have

$$\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} = \frac{1}{1-re^{i\theta}} + \frac{1}{1-re^{-i\theta}} - 1 = \frac{2-2r\cos(\theta)}{1-2r\cos(\theta)+r^2} - 1 \tag{36.20}$$

(you need to recall that $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$). This sum is called Poisson's kernel $P(r,\theta)$ and is defined via

$$\frac{1-r^2}{1-2r\cos(\theta)+r^2} = 2\pi P(r,\theta). \tag{36.21}$$

Poisson's kernel satisfies the following properties:

1. $P(r,\theta+2\pi) = P(r,\theta)$
2. $P(r,-\theta) = P(r,\theta)$
3. The maximum of $P(r,\theta)$ is at $\theta = 0$ and equals $\frac{1}{2\pi}\frac{1+r}{1-r}$
4. $P(r,\theta)$ monotonically decreases from $\theta = 0$ to $\theta = \pi$. The minimum is $\frac{1}{2\pi}\frac{1-r}{1+r}$.
5. $\int_{-\pi}^{\pi} P(r,\theta)d\theta = 1$.

These results can be easily verified except for the last one, which requires significantly more work (it is probably easiest to carry out the integral using residues).

For our next step, observe that

$$P(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta} \tag{36.22}$$

so we have that

$$\begin{aligned}\int_{-\pi}^{\pi} d\theta\, P(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta} \\ &= \sum_{n=-\infty}^{+\infty} r^{|n|}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{in\theta} = \sum_{n=-\infty}^{+\infty} r^{|n|}\delta_{n,0} = 1.\end{aligned} \tag{36.23}$$

These results are all rigorously valid for $r < 1$.

So we consider the summation with A_n and substitute in our guess in terms of an integral of $f(\theta)$

$$\sum_{n=-\infty}^{\infty} r^{|n|}A_n e^{in\theta} = \sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta}\frac{1}{2\pi}\int_{-\pi}^{\pi} dt f(t)e^{-int} \tag{36.24}$$

For $r < 1$, we have

$$\sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta} = \int_{-\pi}^{\pi} dt \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-t)} f(t). \tag{36.25}$$

Now, we recall that $\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-t)} = P(r, \theta - t)$. We then get

$$\sum_{n=-\infty}^{\infty} r^{|n|} A_n e^{in\theta} = \int_{-\pi}^{\pi} dt P(r, \theta - t) f(t). \tag{36.26}$$

Now we want to consider what happens as $r \to 1$. Note that as $r \to 1$ one can see that $P(r, \theta - t)$ becomes very strongly peaked about $\theta = t$ (the maximum goes to ∞, while the integral over t remains 1).

We now want to show that $U(r, \theta)$ does approach $f(\theta)$ as $r \to 1$. To do this, we compute the absolute value of the difference

$$\begin{aligned} |U(r,\theta) - f(\theta)| &= \left| \int_{\theta-\pi}^{\theta+\pi} dt P(r, \theta - t)[f(t) - f(\theta)] \right| \\ &\leq \int_{\theta-\pi}^{\theta+\pi} dt P(r, \theta - t)|f(t) - f(\theta)|. \end{aligned} \tag{36.27}$$

The bound arises because the integral of P is one, P is nonnegative, and it is periodic in θ.

Now consider an interval of size δ about $t = \theta$: $|\theta - t| \leq \delta$. In this interval, the integral is dominated by the maximum of P. The remainder of the integral is dominated by $P(r, \delta)$, because P is monotonic. Note that if $|\theta - t| < \delta$ then $|f(t) - f(\theta)| < \frac{\epsilon}{2}$ for some ϵ and we can make ϵ as small as desired by reducing δ. So,

$$\int_{|\theta-t|\leq\delta} dt P(r, \theta - t)|f(t) - f(\theta)| \leq \int_{|\theta-t|\leq\delta} dt P(r, \theta - t)\frac{\epsilon}{2} \leq \frac{\epsilon}{2} \tag{36.28}$$

because $P(r, \theta - t) \to P \geq 0$ and when Poisson's kernel is integrated over all t it equals 1. The other piece of the integral satisfies

$$\int_{\delta\leq|\theta-t|\leq\pi} dt P(r, \theta - t)|f(t) - f(\theta)| \leq \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\delta) + r^2} \int_{-\pi}^{\pi} |f(t) - f(\theta)| dt. \tag{36.29}$$

Note that the first term on the RHS vanishes as $r \to 1$, so for $r > r_0$ the RHS is bounded by $\frac{\epsilon}{2}$ and the total integral is then less than ϵ. Hence, we have shown that

$$|U(r,\theta) - f(\theta)| < \epsilon \text{ for } 1 \geq r \geq r_0 \implies \lim_{r\to 1} |U(r,\theta) - f(\theta)| = 0. \quad (36.30)$$

This establishes that we have indeed found the solution to Dirichlet's problem.

One can prove that this solution is unique. (See section 1-7 of Seeley). One can also show that the Fourier series converges pointwise if $f(\theta)$ is differentiable. This proof is a bit more complicated and it is given in section 1-8 of Seeley.

Note that there are discontinuous (or piecewise differentiable) functions that have the Fourier series converge to the wrong result at the point of discontinuity. This is called the Gibbs phenomenon and it corresponds to an approximate 10 percent error at the point of discontinuity (elsewhere the Fourier series converges as we showed above).

Many of the notions of continuity and piecewise continuous or differentiable functions comes from the results of studying this problem to try to determine when and how the Fourier series converges. We do not have the time to go through these in detail though. If you fin this interesting, you should examine these ideas further in either a differential topology course or a real analysis course.

Chapter 37

Fourier Series and Separation of Variables

37.1 Formalism

Last time we showed "well behaved" functions of θ periodic on $0 \leq \theta \leq 2\pi$ can be written as a Fourier series

$$f(\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}, \tag{37.1}$$

with the Fourier coefficients determined by a simple integral

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta f(\theta) e^{-in\theta}. \tag{37.2}$$

One way to think of this is as an alternative expansion to the Taylor series expansion. Here we expand in a basis given by exponentials, not powers. The reason why is that the exponentials encompass *all* functions that share this periodicity (called completeness) and hence it is natural to believe such an expansion should exist. Since one can think of this as an expansion of a vector, given by the function $f(\theta)$, in terms of the basis, given by the exponentials, with the coordinates, or coefficients, given by the A_n values. Such expansions, generalized to arbitrary orthonormal and complete basis sets, are an important core concept in quantum mechanics, so it is well worth your time to learn these ideas now.

Now, we try to generalize these ideas a bit further, within the Fourier series concept. The first thing to note is that there is nothing sacred about the interval of length 2π. So if we have a function of x periodic on $-L \leq x \leq L$, then it can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{in\frac{\pi x}{L}} \tag{37.3}$$

with the Fourier coefficients given by

$$A_n = \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-in\frac{\pi x}{L}}. \tag{37.4}$$

We want to consider some special cases. Suppose $f(x)$ is an even function; that is, $f(-x) = f(x)$. Then we can rewrite the Fourier series expansion as

$$\begin{aligned} A_n &= \frac{1}{2L}\int_{-L}^{L} dx f(x) e^{-in\frac{\pi x}{L}} \\ &= \frac{1}{2L}\int_{0}^{L} dx \left[f(x)e^{-in\frac{\pi x}{L}} + f(-x)e^{in\frac{\pi x}{L}}\right] \\ &= \frac{1}{2L}\int_{0}^{L} dx f(x) 2\cos\frac{n\pi x}{L} \\ &= \frac{1}{L}\int_{0}^{L} dx f(x)\cos\frac{n\pi x}{L}. \end{aligned} \tag{37.5}$$

We immediately learn from the last form of the integral, that because cos is even, we have $A_n = A_{-n}$. Hence

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} A_n e^{in\frac{\pi x}{L}} \\ &= A_0 + \sum_{n=1}^{\infty}\left(A_n e^{in\frac{\pi x}{L}} + A_n e^{-in\frac{\pi x}{L}}\right) \\ &= A_0 + 2\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}. \end{aligned} \tag{37.6}$$

Define

$$a_n = \frac{2}{L}\int_0^L dx f(x)\cos\frac{n\pi x}{L} = 2A_n, \tag{37.7}$$

then we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \tag{37.8}$$

for even f, called the *Fourier cosine series.*

Similarly, if f is odd, we obtain

$$\begin{aligned} A_n &= \frac{1}{2L} \int_0^L dx \left[f(x) e^{-in\frac{\pi x}{L}} + f(-x) e^{in\frac{\pi x}{L}} \right] \\ &= \frac{1}{2L} \int_0^L dx f(x)(-2i) \sin \frac{n\pi x}{L}. \end{aligned} \tag{37.9}$$

Since sin is odd, we have $A_{-n} = -A_n$ and

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \left(A_n e^{in\frac{\pi x}{L}} + A_{-n} e^{-in\frac{\pi x}{L}} \right) \\ &= \sum_{n=1}^{\infty} A_n (2i) \sin \frac{n\pi x}{L}. \end{aligned} \tag{37.10}$$

Define

$$b_n = \frac{2}{L} \int_0^L dx f(x) \sin \frac{n\pi x}{L}, \tag{37.11}$$

so that $b_{-n} = b_n$. Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \tag{37.12}$$

which is called the *Fourier sine series.*

Since a general function can be written as the sum of an even function and an odd one, we have for the general case on $-L \leq x \leq L$, that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{37.13}$$

This is called the *Fourier series.* It is written entirely in terms of real objects here.

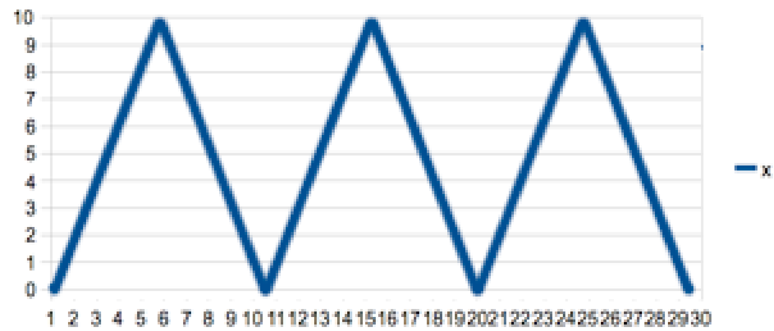

Figure 37.1: Schematic of the sawtooth wave (in this plot $L = 6$). This wave is given by an absolute value, so it always lies above zero. It is an even function. Over the half period from 0 to L, the function is linear and given simply by x.

37.2 Examples

We next consider two examples, to give you practice on the concepts.

The first example is what is called a sawtooth wave. The function is periodic, but has discontinuities at the end of each period. The sawtooth wave yields a Fourier cosine series. We can find the coefficients by integrating over a half period L. The integral is completed by integration by parts. For $n \neq 0$, we have

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L dx x \cos\frac{n\pi x}{L} \\
&= \frac{2}{L}\left[x\left(\sin\frac{n\pi x}{L}\frac{L}{n\pi}\right)\Bigg|_0^L - \int_0^L dx \frac{L}{n\pi}\sin\frac{n\pi x}{L}\right] \\
&= \frac{2}{L}\frac{L}{n\pi}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Bigg|_0^L \qquad (37.14)\\
&= \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right] \\
&= \begin{cases} 0 & n \text{ even} \\ -\frac{4L}{n^2\pi^2} & n \text{ odd,} \end{cases} \qquad (37.15)
\end{aligned}$$

while for $n = 0$, we find

$$\alpha_0 = \frac{2}{L}\int_0^L x dx = L. \tag{37.16}$$

So the Fourier series for the sawtooth wave becomes

$$f_{\mathrm{sawtooth}} = \frac{L}{2} - \sum_{n=0}^{\infty} \frac{4L}{(2n+1)^2\pi^2} \cos\frac{(2n+1)\pi x}{L}. \tag{37.17}$$

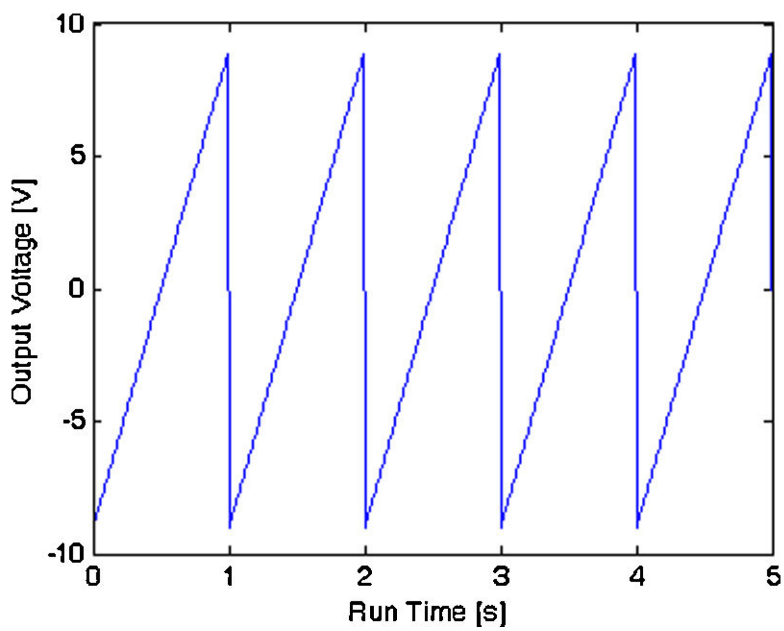

Figure 37.2: Schematic of the “modulo-line” wave. This wave is an odd function and here the period is 1.

The second example is the ”modulo line”. which is an odd function, so it is a sine series:

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L dx x \sin\frac{n\pi x}{L} \\
&= \frac{2}{L}\left[x\left(\cos\frac{n\pi x}{L}\frac{L}{n\pi}\right)\Big|_0^L - \int_0^L dx \frac{L}{n\pi}\cos\frac{n\pi x}{L}\right] \\
&= -\frac{2}{L}\frac{L}{n\pi}(-1)^n - \frac{2}{L}\frac{L}{n\pi}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L \\
&= \frac{2}{n\pi}(-1)^{n+1}.
\end{aligned}
\tag{37.18}
$$

So we have

$$
f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L} \tag{37.19}
$$

as the Fourier sine series of the "modulo-line" wave.

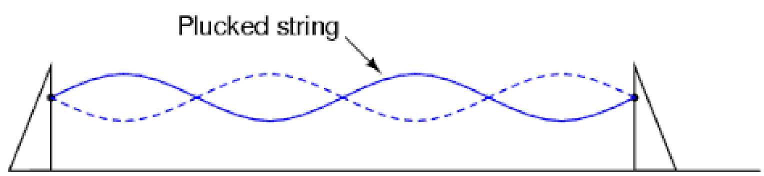

Figure 37.3: Schematic of a plucked string. The curved function is $U(x,t)$.

The third example, with more physics in it, is that of a plucked string that vibrates, as in a musical instrument. We have a demonstration of this in the videos as well, where we show a plucked guitar string.

The equation of motion for a taut string approximately satisfies the following differential equation:

$$
\tau\frac{\delta^2 U}{\delta x^2} = \rho\frac{\delta^2 U}{\delta t^2} \tag{37.20}
$$

where τ is the tension and ρ is the mass density of the string. Here, the left side of the equation can be thought of as acting like a potential energy, while the right side can be thought of as acting like a kinetic energy. So the equation can be interpreted as Total Energy = Constant. Note that a similar form holds in quantum mechanics for the Schrödinger equation.

The boundary conditions for the string correspond to it having "clamped" ends: $U(0,t) = 0$, and $U(L,t) = 0$ for all t. We also assume that the initial condition is $U(x,0) = f(x)$ and $\frac{\delta U(x,0)}{\delta t} = g(x)$. Our goal is to find $U(x,t)$ for all subsequent times.

The procedure we use is familiar—we employ separation of variables. To begin, we write the wavefunction as a product of two functions, which each depend on the independent variables x and t only:

$$U(x,t) = X(x)T(t). \tag{37.21}$$

Substituting into the differential equation and dividing by U, then yields

$$\frac{X''}{X} = \frac{\rho}{\tau}\frac{T''}{T} = -c \tag{37.22}$$

where c is a constant. The solution of each equation is given by exponentials if we choose $c > 0$, which is required to examine the oscillating behavior. This gives us

$$U(x,t) = X(x)T(t) = \left(Ae^{i\alpha x} + Be^{-i\alpha x}\right)\left(ae^{i\beta t} + be^{-i\beta t}\right) \tag{37.23}$$

with $\alpha = \sqrt{c}$ and $\beta = \sqrt{\frac{\tau c}{\rho}}$.

Now we need to solve the different boundary conditions. First, we find

$$X(0) = 0 \implies A + B = 0 \implies B = -A. \tag{37.24}$$

Then for the other edge, we have

$$X(L) = 0 \implies Ae^{i\alpha L} + Be^{-i\alpha L} = 0. \tag{37.25}$$

Combining these two together yields

$$A\left(e^{i\alpha L} - e^{-i\alpha L}\right) = 0 \implies \sin\alpha L = 0 \implies \alpha = \frac{n\pi}{L}. \tag{37.26}$$

So we have

$$U(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n e^{itn\omega} + b_n e^{-itn\omega}\right), \tag{37.27}$$

where $\omega = \frac{\pi}{L}\sqrt{\frac{\tau}{\rho}}$. Assuming we can differentiate under the summation, we obtain

$$\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(a_n + b_n) = f(x) \tag{37.28}$$

at $t = 0$ and

$$\sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} in\omega(a_n - b_n) = g(x) \tag{37.29}$$

for the derivative at $t = 0$. So to solve this, we compute the Fourier sine series for $f(x)$ with coefficients given by $B_n^{(f)}$ and for $g(x)$ with coefficients given by $B_n^{(g)}$. Plugging into the boundary conditions then shows that

$$a_n + b_n = B_n^{(f)} \implies b_n = B_n^{(f)} - a_n \tag{37.30}$$

and

$$in\omega(a_n - b_n) = B_n^{(g)}. \tag{37.31}$$

Combining these says that

$$2a_n - B_n^{(f)} = \frac{1}{in\omega} B_n^{(g)} \tag{37.32}$$

or

$$a_n = \frac{1}{2} B_n^{(f)} + \frac{1}{2in\omega} B_n^{(g)} \tag{37.33}$$

and

$$b_n = \frac{1}{2} B_n^{(f)} - \frac{1}{2in\omega} B_n^{(g)}. \tag{37.34}$$

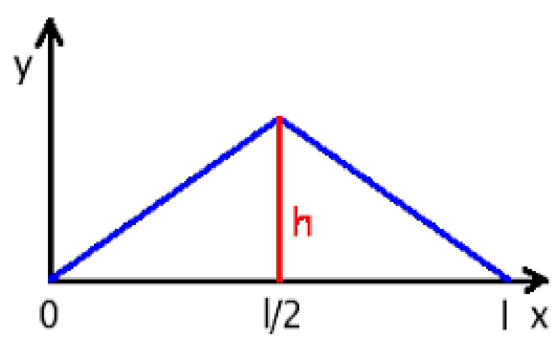

Figure 37.4: In this concrete example, the initial string is plucked at the midpoint, with a height given by $h = L/8$.

Lets work this up for a concrete example. Suppose the string is initially plucked as follows as indicated in Fig. 37.4. We assume it is not initially moving $\frac{\delta U}{\delta t}(x,0)=0$. Then $B_n^{(g)}=0$, and the solution preceeds as follows:

$$
\begin{aligned}
B_n^{(f)} &= \frac{2}{L}\int_0^{\frac{L}{2}} dx\frac{x}{4}\sin\frac{n\pi x}{L} + \frac{2}{L}\int_{\frac{L}{2}}^{L} dx\left(\frac{L}{4}-\frac{x}{4}\sin\frac{n\pi x}{L}\right)\\
&= \frac{1}{2L}\left[\int_0^{\frac{L}{2}} dx x\sin\frac{n\pi x}{L} + \int_{\frac{L}{2}}^{L} dx(L-x)\sin\frac{n\pi x}{L}\right]\\
&= \frac{1}{2L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}} dx\frac{L}{n\pi}\cos\frac{n\pi x}{L} - (L-x)\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{\frac{L}{2}}^{L}\right.\\
&\quad\left. - \int_{\frac{L}{2}}^{L} dx\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right]\\
&= \frac{1}{2L}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{\pi^2 n^2}\sin\frac{n\pi}{2} + \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{\pi^2 n^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)\right]\\
&= \frac{L}{n^2\pi^2}\sin\frac{n\pi}{2}\\
&= \begin{cases} \frac{L}{(2n+1)^2\pi^2}(-1)^n & n = \text{odd}\\ 0 & n = \text{even}.\end{cases}
\end{aligned}
\tag{37.35}
$$

So we finally obtain

$$
U(x,t) = \sum_{n=1}^{\infty} \sin\frac{(2n+1)\pi x}{L}\frac{L}{\pi^2(2n+1)^2}(-1)^n\sin(2n+1)\omega t. \tag{37.36}
$$

Note how the amplitudes decay rapidly (like $1/n^2$) indicating the majority of the signal is in the fundamental harmonic. The overtones, however, are often what leads to the richness in sound of an instrument.

The procedure we use here is a standard approach for how to solve partial differential equations. As you can see it can get quite lengthy and complicated. But, because we can work out analytically the functions in the different expansions, we can actually carry the solution all the way to the end. More complicated partial differential equations must be solved numerically and these solutions require significant computational power. Most of the high performance computing resources are used for solving these types of differential equations in fluid dynamics. These methods are used to design airplanes and cars, among other things. The field is quite active and we do not yet have the most efficient ways to solve these types of problems. It is an area you can consider going into if you are inspired by this type of work.

Chapter 38

Applications of Poisson's Theorem

Poisson's Theorem showed that a regularized Fourier series (solution of the heat equation at a radius $r \to 1$) could be brought as close as possible to any function $f(\theta)$ defined on the perimeter. If $f(\theta)$ was continuously differentiable, we could even prove *pointwise* convergence, which says at each θ, the Fourier series converges to $f(\theta)$.

Before going on, let me share a little story with you. When I was in graduate school, I took a graduate course in real analysis. The professor was a nice guy, but definitely did not feel like grading our work, so at the end of the term he simply posted a note, everyone in the class receives an "A." In this class, we spent much of the semester discussing different forms of continuity. I believe there were at least a dozen different forms of continuity—pointwise continuous, uniformly continuous, equicontinuous, piecewise continuous, and many, many more. It turns out that most of the ideas for different ways to define continuous and differentiable functions stem from extensions of Fourier's original work. So, we delve into this topic a bit further next.

We will discuss only two other forms of convergence (aside from pointwise convergence). The first is called *uniform* convergence, which says for some finite N, we can find a truncated Fourier series

$$S_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta} \tag{38.1}$$

with the standardly defined Fourier coefficients

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) e^{-in\theta}, \tag{38.2}$$

such that for large enough N,

$$|S_N(\theta) - f(\theta)| < \epsilon, \tag{38.3}$$

for *all* $0 \leq \theta \leq 2\pi$. Unlike pointwise continuity, where for each θ we can take the limit as N approaches infinity, here we want to keep N fixed and finite, but show the convergence nevertheless occurs for all θ. This is called the *Weierstrauss Theorem.*

The second kind of continuity is called *least squares* continuity. Here, we want to show that

$$\int_{-\pi}^{\pi} d\theta |S_N(\theta) - f(\theta)|^2 < \epsilon \tag{38.4}$$

or the average mean square deviation is small. Note that the results can err by a large amount at isolated points and not affect the integral's value. The least-squares continuity naturally leads to Parseval's equality, given by

$$2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 = \int_{-\pi}^{\pi} d\theta\, |f(\theta)|^2. \tag{38.5}$$

Parseval's identity can be very useful in applications. It is worthwhile to remember it. The detailed proof is given in the book. I encourage you to read it.

The main focus of this lecture is to tidy up some discussions we had earlier of thinking of the Fourier series as a change of basis in the infinite dimensional space of functions.

Here, we think of the orthonormal unit vectors as $u_n = e^{in\theta}$. This is a complex vector space so

$$u_n \cdot u_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \left(e^{-in\theta}\right)^* e^{-im\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{in\theta} e^{-im\theta} = \delta_{nm}. \tag{38.6}$$

The exponential functions are unit vectors that are orthogonal and there are an infinite number of them. So we can imagine expanding a function $f(\theta)$ in the basis of the u_n's. This means

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n u_n. \tag{38.7}$$

To find the expansion coefficients a_n, we take the inner product of $f(\theta)$ with u_m on the right:

$$f(\theta) \cdot u_m = \sum_{n=-\infty}^{\infty} a_n u_n \cdot u_m = \sum_{n=-\infty}^{\infty} a_n \delta_{nm} = a_m. \tag{38.8}$$

Hence, $a_m = f \cdot u_m = \int d\theta f(\theta) \exp(-im\theta)/2\pi$. This is the same function as we had before!

Parseval's equality becomes just an expansion for the norm of the vector f calculated two different ways:

$$\|f\|^2 = \frac{1}{2\pi} \int d\theta \, |f(\theta)|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2. \tag{38.9}$$

As we discussed earlier in the course, we can have dot products between two functions. Computing them is easiest done with the Fourier coefficients, as shown below:

$$\begin{aligned}
f \cdot g &= \frac{1}{2\pi} \int d\theta f(\theta) g^*(\theta) \\
&= \frac{1}{2\pi} \int d\theta \sum_n a_n e^{in\theta} \sum_m b_m^* e^{-im\theta} \\
&= \sum_{nm} a_n b_m^* \delta_{nm} \\
&= \sum_n a_n b_n^*.
\end{aligned} \tag{38.10}$$

Now, because $\|f\|^2 = \sum_n |a_n|^2$, $\|g\|^2 = \sum_n |b_n|^2$, we can define the angle between f and g via

$$\cos\alpha = \frac{f \cdot g}{\|f\| \, \|g\|} = \frac{\sum_n a_n b_n^*}{\sqrt{\left(\sum_n |a_n|^2\right)\left(\sum_m |b_m|^2\right)}} \tag{38.11}$$

and the cosine is well defined because the Schwartz inequality says

$$\left| \sum_n a_n b_n^* \right| \leq \sqrt{\left(\sum_n |a_n|^2\right)\left(\sum_m |b_m|^2\right)}. \tag{38.12}$$

The Fourier series also satisfies the triangle inequality, given by

$$\|f+g\| \leq \|f\| + \|g\| \tag{38.13}$$

or

$$\sqrt{\sum_n |a_n + b_m|^2} \leq \sqrt{\sum_n |a_n|^2} + \sqrt{\sum_n |b_n|^2}. \tag{38.14}$$

Proof:

$$\|f+g\|^2 = \|f\|^2 + 2f \cdot g + \|g\|^2 \leq \|f\|^2 + 2|f \cdot g| + \|g\|^2. \tag{38.15}$$

But the Schwartz inequality says $2|f \cdot g| \leq 2\|f\|\,\|g\|$, so that we have

$$\begin{aligned} \|f+g\|^2 &\leq \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 \\ &\leq (\|f\| + \|g\|)^2. \end{aligned} \tag{38.16}$$

This implies that $\|f+g\| \leq \|f\| + \|g\|$.

And that is it. We are done! I hope you enjoyed learning this material as much as I did in assembling it. Please go forward and pursue physics with your newfound knowledge. Be a practitioner, not a technician. Always think. Always ask why.