

“We have excessive computation power nowadays, so why are we as engineers still doing the simple one-way algebra instead of ruling out all the possibilities and to choose the optimal.”

## **Reinforced Concrete Beam Design**

**This guide covers the flexural strength design for reinforced concrete beam and one-way slab. The guide is organized with four chapters:**

- **Ch. 1 Singly Reinforced Section**
- **Ch. 2 Doubly Reinforced Section**
- **Ch. 3 T-Shaped Section**
- **Ch. 4 Computation Method for Arbitrarily Reinforced Section**

**The first three chapters use conventional methods that can be solved using hand calculation by assuming the moment is in one direction.**

**The fourth chapter covers the computation method that allows engineers to evaluate the flexural strength of arbitrarily reinforced sections.**

## Ch. 1 Singly Reinforced Section

### 1.1. Failure Model – Force Equilibrium in Expression of Reinforcement Area

Figure 2.1 Rectangular Stress Block<sup>3</sup>

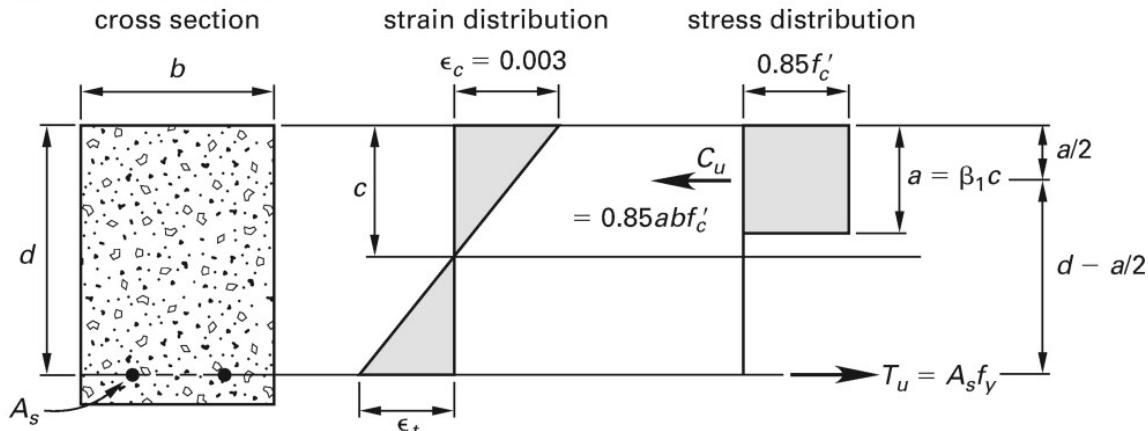


Figure 1.1.1 Singly Reinforced Cross Section Diagram [1]

Equilibrium (the design theory utilizes force equilibrium to find the strain distribution under plane assumption, then evaluate flexural strength).

$$0.85 * f'_c * b * a = f_y * A_s$$

$$a = \beta_1 * c = \frac{f_y * A_s}{0.85 * f'_c * b}$$

$$M_n = f_y * A_s * \left( d - \frac{a}{2} \right) = f_y * A_s * \left( d - \frac{f_y * A_s}{1.7 * f'_c * b} \right)$$

ACI 9.3.3.1 requires the steel strain in extreme fiber for  $P_u < 0.1 * f'_c * A_g$  to be:

$\epsilon_t \geq 0.004$  per 14<sup>th</sup>, maybe transition, see 14<sup>th</sup> Table 21.2.2.

$\epsilon_t \geq \epsilon_{ty} + 0.003$  per 19<sup>th</sup>, must be tension controlled, see 19<sup>th</sup> Table 21.2.1.

This requirement provides adequate ductility and has guaranteed the reinforcement reaches its yield strength  $f_y$  since  $\epsilon_{ty} \cong 0.002$ .

For back calc when knowing  $M_n$ :

$$\rho = \frac{0.85 * f'_c * \left( 1 - \sqrt{1 - \frac{4 * M_n}{1.7 * f'_c * b * d^2}} \right)}{f_y}$$

## 1.2. Failure Model – Strain Compatibility in Expression of Extreme Reinforcement Strain

We shall define the strain/stress/force is positive when it is in tension and is negative when it is in compression. The establishment of common frame is vital for programming and will significantly benefit the derivation.

$$c = \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d$$

Where  $\epsilon_{cu} = -0.003$

$$a = \beta_1 * c = \beta_1 * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d$$

$$A_s = \frac{0.85 * f'_c * b * a}{f_y} = \frac{0.85 * f'_c * b}{f_y} * \beta_1 * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d$$

$$M_n = f_y * A_s * \left(d - \frac{a}{2}\right) = 0.85 * f'_c * b * \beta_1 * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d * \left(d - \frac{\beta_1}{2} * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d\right)$$

Combine the equations:

$$\begin{cases} A_s = \frac{0.85 * f'_c * b}{f_y} * \beta_1 * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d \\ M_n = 0.85 * f'_c * b * \beta_1 * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d * \left(d - \frac{\beta_1}{2} * \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d\right) \\ \beta_1 = \begin{cases} 0.85 & \text{for } 2500 \leq f'_c < 4000 \\ 0.85 - 0.05 * \frac{f'_c - 4000}{1000} & \text{for } 4000 \leq f'_c < 8000 \\ 0.65 & \text{for } \epsilon_t > 0.005 \end{cases} \end{cases}$$

14<sup>th</sup> non-spiral

$$\phi = \begin{cases} 0.65 & \text{for } \epsilon_t < \epsilon_{ty} \\ 0.65 + 0.25 * \frac{\epsilon_t - \epsilon_{ty}}{0.005 - \epsilon_{ty}} & \text{for } \epsilon_{ty} < \epsilon_t < 0.005 \\ 0.9 & \text{for } \epsilon_t > 0.005 \end{cases}$$

19<sup>th</sup> non-spiral

$$\phi = \begin{cases} 0.65 & \text{for } \epsilon_t < \epsilon_{ty} \\ 0.65 + 0.25 * \frac{\epsilon_t - \epsilon_{ty}}{0.003} & \text{for } \epsilon_{ty} < \epsilon_t < \epsilon_{ty} + 0.003 \\ 0.9 & \text{for } \epsilon_t > \epsilon_{ty} + 0.003 \end{cases}$$

### 1.3. Reinforcement Limit

For upper limit, utilize the frame of strain compatibility.

ACI 9.3.3.1 requirement:

$$\epsilon_t \geq 0.004 \text{ per 14}^{\text{th}}$$

$$A_{s\text{upper}} = \frac{0.85 * f'_c * b}{f_y} * \beta_1 * \frac{0.003}{0.003 + 0.004} * d = 0.429 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} * b * d \right)$$

$$\rho_{\text{upper}} = 0.429 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} \right)$$

$$\epsilon_t \geq 0.005 + \Delta \text{ per 19}^{\text{th}}$$

$$A_{s\text{upper}} = \frac{0.85 * f'_c * b}{f_y} * \beta_1 * \frac{0.003}{0.003 + 0.005 + \Delta} * d \leq 0.375 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} * b * d \right)$$

$$\rho_{\text{upper}} \leq 0.375 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} \right)$$

Considering the quadratic function of the nominal strength:

$$f_y * A_s * \left( d - \frac{f_y * A_s}{1.7 * f'_c * b} \right) = f_y * d * A_s - \frac{f_y^2}{1.7 * f'_c * b} * A_s^2$$

$$A_s^* = \frac{f_y * d}{\frac{f_y^2}{1.7 * f'_c * b}} = \frac{0.85 * f'_c * b * d}{f_y}$$

$$\rho^* = \frac{A_s^*}{b * d} = \frac{0.85 * f'_c}{f_y}$$

Since

$$\rho^* = \frac{0.85 * f'_c}{f_y} \gg \begin{cases} \rho_{\text{upper}} = 0.429 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} \right) & \text{for 14th} \\ \rho_{\text{upper}} \leq 0.375 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} \right) & \text{for 19th} \end{cases}$$

Has guaranteed the strength increase with the area of reinforcement within the design limit.

For lower limit, ensure the flexural strength is greater than the uncracked concrete section by a margin.

$$f_r = 7.5 * \lambda * \sqrt{f'_c * psi}$$

$$M_{cr} = \frac{2 * f_r * I_g}{h} \cong \frac{1}{6} * f_r * b * d^2$$

$$M_n = f_y * \rho * b * d^2 * \left(1 - \frac{f_y * \rho}{1.7 * f'_c}\right)$$

$$\rho = \frac{0.85 * f'_c * \left(1 - \sqrt{1 - \frac{4 * M_n}{1.7 * f'_c * b * d^2}}\right)}{f_y} \cong \frac{0.85 * f'_c}{f_y} * \frac{1}{2} * \frac{4 * M_n}{1.7 * f'_c * b * d^2} = \frac{M_n}{f_y * b * d^2}$$

Assume equal design flexural strength:

$$\phi_{plain} = 0.6$$

$$\phi_{tension} = 0.9$$

$$\phi_{plain} * M_{cr} = \phi_{tension} * M_n$$

$$M_n = \frac{\phi_{plain}}{\phi_{tension}} * M_{cr} = \frac{5}{6} * \lambda * \sqrt{f'_c * psi} * b * d^2$$

Plug back:

$$\rho_{cr} = \frac{5 * \lambda * \sqrt{f'_c * psi}}{6 * f_y}$$

In order to provide flexural strength by a margin, manually define the  $\rho_{lower}$  to be:

$$\rho_{lower} = \max\left(\frac{3 * \sqrt{f'_c * psi}}{f_y}, \frac{200}{f_y}\right)$$

## 1.4. Summary

The nominal flexural strength is monotonically increasing with the reinforcement area when less than  $\rho^*$  and is considerably linear when far from  $\rho^*$ . However, the increasing rate of design flexural strength shows a drop with the truncation of strength reduction factor  $\phi$  around:

$$\rho_{upper} \cong 0.375 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} \right)$$

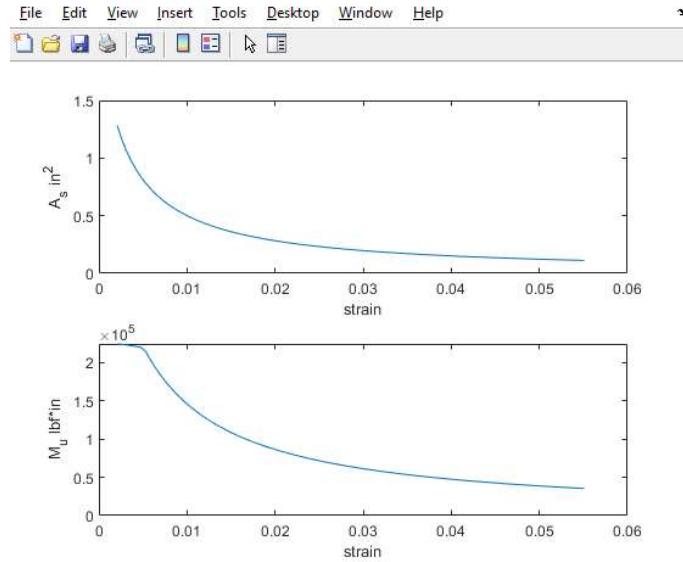


Figure 1.4.1 Flexural Strength Design ( $A_s, M_u$ ) vs.  $\epsilon_t$  per 19<sup>th</sup>

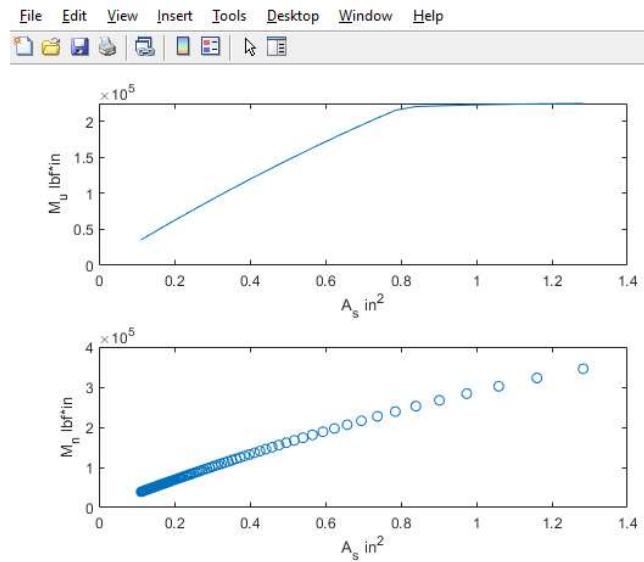


Figure 1.4.2 Flexural Strength Design ( $M_u, M_n$ ) vs.  $A_s$  per 19<sup>th</sup>

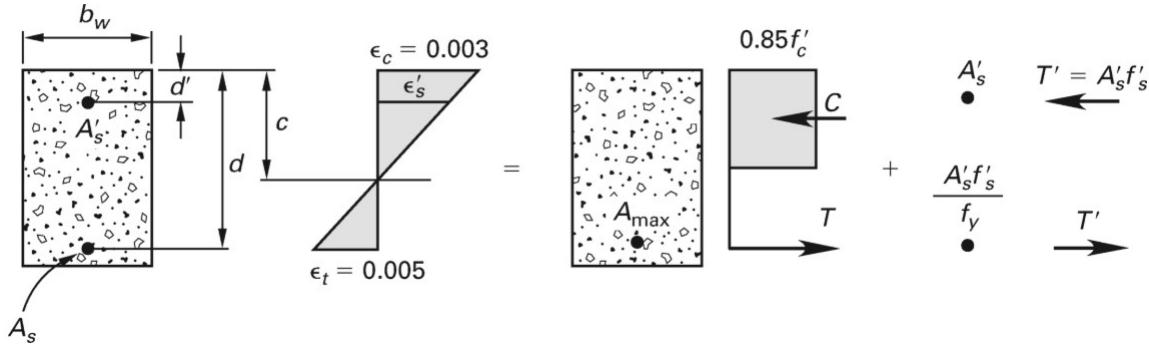
## Ch. 2 Doubly Reinforced Section

From Figure 1.4.2, it is obvious that the flexural strength potential for a singly reinforced section is limited and would reach its maximum when

$$\rho \rightarrow \rho_{upper} \cong 0.375 * \beta_1 * \left( 0.85 * \frac{f'_c}{f_y} \right)$$

That said, compression reinforcement must be added in order to increase the flexural strength of a section without changing the dimensions of it.

**Figure 2.2 Beam with Compression Reinforcement<sup>2</sup>**



**Figure 2.1 Doubly Reinforced Cross Section Diagram [1]**

Knowing the trend of flexural strength with the area of tension reinforcement, it is natural to claim that the optimal design would be reached when the section is right under tension-control, that is:

14<sup>th</sup>

$$\epsilon_t^* = 0.005$$

19<sup>th</sup>

$$\epsilon_t^* = \epsilon_{ty} + 0.003$$

The design procedure follows:

Step 1. Determining the “optimal” flexural strength  $M_{optimal}$  for singly reinforced section.

Please keep in mind that optimal flexural strength is defined with  $\epsilon_t^*$  as increasing reinforcement area can barely increase the flexural strength after this point.

Step 2. Evaluating the residual moment  $M_r = M_u - M_{optimal}$  that requires additional treatment.

Step 3. Calculating the required additional tension reinforcement  $A_{t,add}$  to balance this residual moment  $M_r$ .

$$A_{t,add} = \frac{M_r}{\phi * f_y * (d - d')}$$

Step 4. Evaluating the compressive stress in compression reinforcement  $f'_s$  then find the required compression reinforcement  $A_s'$ .

$$c = \frac{\epsilon_{cu}}{\epsilon_{cu} + \epsilon_t^*} * d$$

$$f'_s = \max\left(E_s * \left(1 - \frac{d'_c}{c}\right), f'_y\right)$$

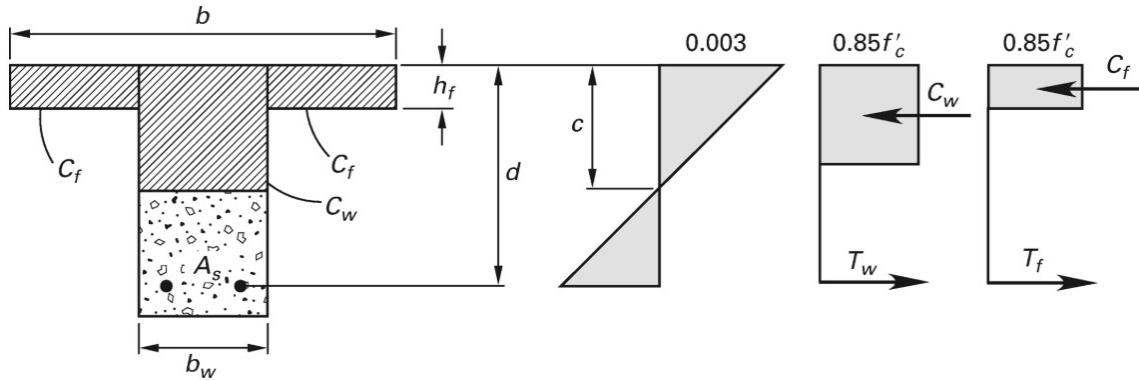
$$A'_s = A_{t,add} * \frac{f_y}{f'_s}$$

Note that in practice, the reinforcement area can rarely match the theoretical value. Instead, for a given  $[A_{t,add}] = [A_{t,doubly}] - A_{t,optimal}$ , the compression reinforcement  $[A'_s]$  can always be rounded up from  $A'_s = [A_{t,add}] * \frac{f_y}{f'_s}$  to yield a conservative design.

### Ch. 3 T-Shaped Section

Similar to the idea of doubly reinforced section, the T-shaped section can also be converted to rectangular section by assuming full flange compression.

**Figure 2.3 Flanged Section with Tension Reinforcement<sup>2</sup>**



**Figure 3.1 T-Shaped Singly Reinforced Cross Section Diagram [1]**

Step 1. Balancing the compression flange with  $A_{sf} = 0.85 * f'_c * h_f * (b - b_w)$ .

Step 2. Evaluating the residual moment  $M_r = M_u - M_{flange}$  that requires additional treatment.

$$M_{flange} = \phi * f_y * A_{sf} * \left(d - \frac{h_f}{2}\right)$$

Step 3. Calculating the required additional tension reinforcement using the equation from Ch. 1.1.

$$A_{sw} = \frac{0.85 * f'_c * b * d * \left(1 - \sqrt{1 - \frac{4 * M_r}{\phi * 1.7 * f'_c * b * d^2}}\right)}{f_y}$$

Lower bound: Making sure the compression block extends beyond the flange area  $c > h_f$ .

Step 4. Combining reinforcement  $A_s = A_{sw} + A_{sf}$ .

Upper bound: Making sure the section is under tension control  $c < \frac{\epsilon_{cu}}{\epsilon_{cu} + \epsilon_t^*} * d$  for  $[A_{sw}] = [A_s] - A_{sf}$ .

## Ch. 4 Computation Method for Arbitrarily Reinforced Section

Figure 2.1 Rectangular Stress Block<sup>3</sup>

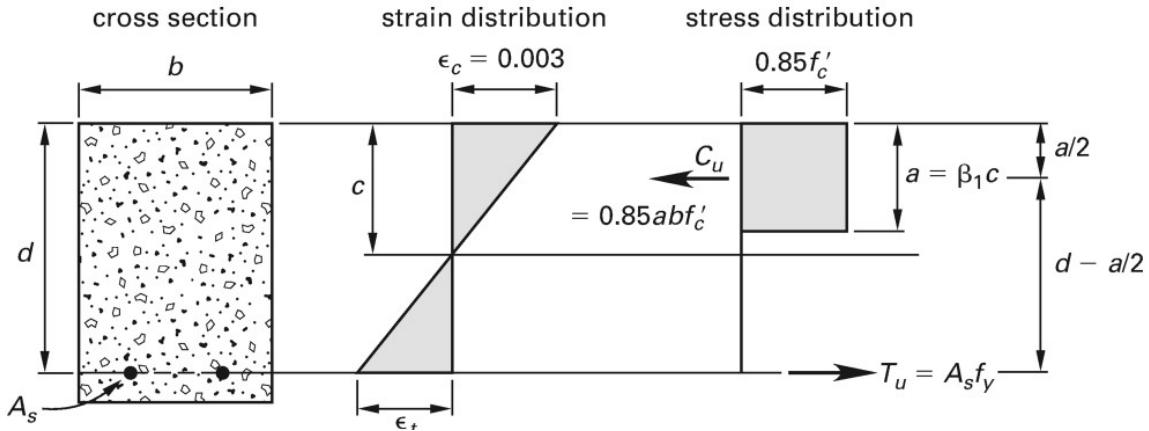


Figure 4.1 (Figure 1.1.1) Singly Reinforced Cross Section Diagram [1]

Anyone who is familiar with continuous mechanics would agree that it is always a privilege to know the strain field or/and its associated shape function before challenging the set of governing equations. Thus, it is natural to write this program using a controlling parameter that is directly related to the strain. There are three candidates for this purpose:

	$\epsilon_t$	$\theta'$	$c$
$\epsilon_t$	1	$\epsilon_t = \theta' * d + \epsilon_{cu}$	$\epsilon_t = \epsilon_{cu} * \left(1 - \frac{d}{c}\right)$
$\theta'$	$\theta' = \frac{\epsilon_t - \epsilon_{cu}}{d}$	1	$\theta' = -\frac{\epsilon_{cu}}{c}$
$c$	$c = \frac{\epsilon_{cu}}{\epsilon_{cu} - \epsilon_t} * d$	$c = -\frac{\epsilon_{cu}}{\theta'}$	1

$$*\theta' = \frac{d\theta}{ds} = \tan(\theta') \text{ when } \theta' \rightarrow 0$$

Table 4.1 Parametric Analysis

For numerical purposes we try to avoid placing our independent variables with their power less than zero (on the denominator side),  $\epsilon_t$  and  $\theta'$  satisfy this requirement. However, since  $\theta'$  is just an approximation of  $\tan(\theta')$ , we shall choose  $\epsilon_t$  as our control variable.

The derivation of the strain compatibility framework can be found in Ch 1.2. By adding stress-strain relationships for each reinforcement layer, we can compute the flexural strength for an arbitrarily reinforced section.

Figure 2.2 Beam with Compression Reinforcement<sup>2</sup>

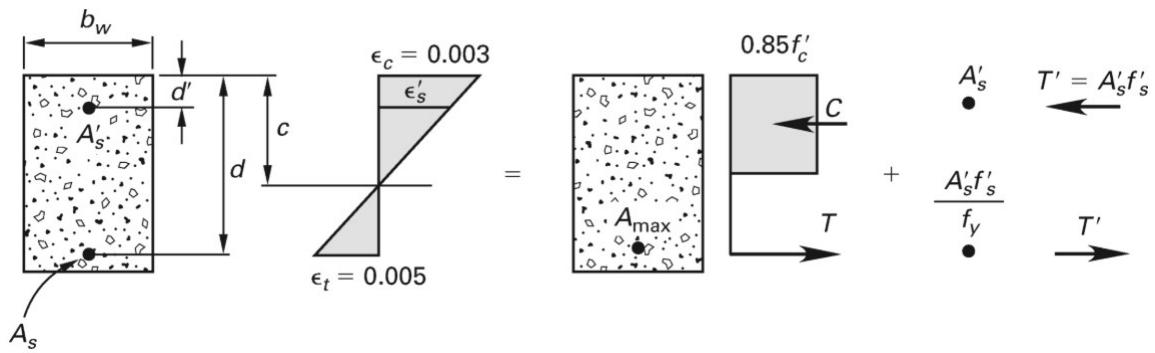


Figure 4.2 (Figure 2.1) Doubly Reinforced Cross Section Diagram [1]

Stress-Strain Relationship:

$$f_s = \begin{cases} -f_y & \text{for } \epsilon_s = -\epsilon_y \\ E_s * \epsilon_s & \text{for } -\epsilon_y < \epsilon_s < \epsilon_y \\ f_y & \text{for } \epsilon_s = \epsilon_y \end{cases}$$

Strain:

$$\epsilon_s = \epsilon_{cu} + \theta' * d_s = \epsilon_{cu} + \frac{\epsilon_t - \epsilon_{cu}}{d} * d_s$$

## Ch. 5 Prestressed Concrete

### Type of Prestressed Concrete

There are two types of prestressed concrete, named under their methodology:

#### Pre-tensioning

Steel tendons or strands are tensioned before the concrete is cast; these tendons are considered bonded with the adjacent concrete.

#### Post-tensioning

Steel tendons or strands are tensioned after the concrete is cast and cured; these tendons are connected through reserved holes and are considered unbonded with the adjacent concrete.

### Mechanical Model

Positive tension, negative compression, Moment is positive when distance times axial load is positive.

For post-tensioning, considering a beam-column with uniform sections is prestressed with tendons under initial tension  $T$  at distance  $e$  from the neutral axis of concrete (gross concrete subtract the tendons or reserved holes). The equilibrium would be reached when the concrete and tendon are deformed so that:

$$\begin{cases} F_c + T + \Delta T = 0 & \text{Equilibrium} \\ \Delta T = E_t * A_t * \epsilon_t & \text{Stress - Strain} \\ \epsilon_t = \frac{1}{L} \int_0^L \epsilon_{ti}(s) * ds & \text{Strain} \end{cases}$$

With proportional strain assumption, the concrete model can then be described as:

$$\epsilon_i = \frac{1}{E_c} * \left( \frac{F_c}{A_c} + \frac{F_c * e}{I_c} * y_i + \frac{M_{external}}{I_c} * y_i \right)$$

Plug back ( $y_i = e$ ):

$$\epsilon_{ti}(s) = - \left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) * \left( T + E_t * A_t * \frac{1}{L} \int_0^L \epsilon_{ti}(s) * ds \right) + \frac{e}{E_c * I_c} * M_{external}(s)$$

Solving this equation:

$$\begin{aligned}
\epsilon_t &= \frac{1}{L} \int_0^L \epsilon_{ti}(s) * ds \\
&= \frac{1}{L} \int_0^L \left[ -\left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) * (T + E_t * A_t * \epsilon_t) + \frac{e}{E_c * I_c} * M_{external}(s) \right] * ds \\
&= -\left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) * (T + E_t * A_t * \epsilon_t) + \frac{e}{E_c * I_c} * \int_0^L \frac{M_{external}(s)}{L} * ds
\end{aligned}$$

Parameterize:

$$\begin{cases} k = (E_t * A_t) \\ \phi = \left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) \\ f(s) = \frac{e}{E_c * I_c} * M_{external}(s) \\ F(s) = \int_0^L \frac{f(s)}{L} * ds = \frac{e}{E_c * I_c} * \int_0^L \frac{M_{external}(s)}{L} * ds \end{cases}$$

$$\epsilon_t = -\phi * (T + k * \epsilon_t) + F(s)$$

We get:

$$\begin{aligned}
\epsilon_t &= \frac{-\phi * T + F(s)}{1 + \phi * k} = \frac{-\left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) * T + \frac{e}{E_c * I_c} * \int_0^L \frac{M_{external}(s)}{L} * ds}{1 + \left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) * (E_t * A_t)} \\
\epsilon_{ti}(s) &= -\phi * (T + k * \epsilon_t) + f(s) = \frac{-\phi * T}{1 + \phi * k} + \frac{-\phi}{1 + \phi * k} * F(s) + f(s)
\end{aligned}$$

For  $\phi \ll 1, \phi * k \ll 1$ , we get the first order approximation which is adopted in the manual:

$$\epsilon_{ti}^{1st}(s) = -\phi * T + f(s)$$

For pre-tensioning with bonded tendons  $\Delta T = E_t * A_t * \epsilon_{ti}$ , the equation yields:

$$\begin{aligned}
\epsilon_{ti}(s) &= -\left( \frac{1}{E_c * A_c} + \frac{e^2}{E_c * I_c} \right) * (T + E_t * A_t * \epsilon_{ti}(s)) + \frac{e}{E_c * I_c} * M_{external}(s) \\
\epsilon_{ti}(s) &= \frac{-\phi * T}{1 + \phi * k} + \frac{f(s)}{1 + \phi * k}
\end{aligned}$$

For  $\phi \ll 1, \phi * k \ll 1$ , we get the same first order expression as the post-tensioning.

$$\epsilon_{ti}^{1st}(s) = -\phi * T + f(s)$$

## Stress Losses

$$\delta T$$

### Determination of Effective Prestress

Considering the complexity of concrete behavior when cross section reaching its failure, the material can no longer be idealized as linear elastics and thus leads to difficulty finding the accurate stress/strain of the tendons.

For unbonded tendons, general approach is to select the effective prestress as the stress after all losses when still in linear elastic. The tendons are then treated as the reinforcement with a fixed strength of  $T_{se} = T + \Delta T + \delta T$  to get the nominal flexural strength of the section.

It can be proof that this approach is conservative as the model:

$$M_n = T_{se} * \left( d - \frac{T_{se}}{1.7 * f'_c * b} \right)$$

Has its vertex at  $T_{se} = 0.85 * f'_c * b * d$  that would never be reached in a prestressed section (reaching the vertex means the concrete section is at compression failure even after all stress losses).

This limitation has guaranteed that  $M_{n,actual} > M_n$  as the  $T_{final} > T_{se}$  after the deformation.

Manual methods use approximate  $T_{final,approximate}$  for flexural strength.

For bonded tendons, we can also use the theoretical stress-strain relationship to derive more accurate strength. Please keep in mind that the initial strain of tendons should be set with all prestress losses except elastic shortening (the zero point of strain is defined when the concrete is free of deformation).

## **Reference**

- [1] Alan Williams (2023), PE Structural Reference Manual
- [2] American Concrete Institute (ACI 318-19), Building Code Requirements for Structural Concrete
- [3] Precast/Prestressed Concrete Institute (PCI), Bridge Design Manual 4<sup>th</sup> Edition