Numerical Method for Coulomb's Theory

All the calculations below are based on active earth pressure, for passive conditions, choose $\begin{cases} \delta_p = -\delta_a \\ \varphi_p = -\varphi_a \end{cases}$

Equilibrium Equation:

$$p + w + b + q = 0$$

$$\begin{cases} p cos\delta - q cos(\theta + \varphi) = 0 \\ p sin\delta + q sin(\theta + \varphi) = W - B + S \end{cases}$$

- By divergence theorem, the buoyancy of a body is always towards the opposite direction of acceleration of fluid.

Where:

$$W = \frac{1}{2}\gamma h^2 tan\theta \ is\ the\ weight\ of\ the\ isolated\ soil\ body$$
$$B = \frac{1}{2}\gamma_w h_w^2 tan\theta \ is\ the\ buoyancy\ acting\ on\ it$$
$$S = shtan\theta \ is\ the\ uniform\ surcharge\ load$$

Written in Matrix Form:

$$\begin{bmatrix} cos\delta & -\cos(\theta + \varphi) \\ sin\delta & \sin(\theta + \varphi) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ W - B + S \end{bmatrix}$$

There are mainly two methods to search for the Global Maximum of p

Methods

1. Gradient Descent
   1.1. Matrix Solver

$$f(\theta): A(\theta)x(\theta) = b(\theta)$$

First order derivative:

$$A'x + Ax' = b'$$
$$x' = A^{-1}(b' - A'x)$$

Second order derivative:

$$A''x + 2A'x' + Ax'' = b''$$
$$x'' = A^{-1}(b'' - A''x - 2A'x')$$

Condition of Local Maximum:

$$\begin{cases} x' = A^{-1}(b' - A'x) = 0 \\ x'' = A^{-1}(b'' - A''x - 2A'x') < 0 \end{cases}$$

In practice, as we are doing numerical calculations, the condition of local maximum could be:

$$\begin{cases} {x'}_{1,i-1} > 0 \\ {x''}_{1,i} \le 0 \end{cases}$$

Where $(\,)_{i,j}$ means the ith component in jth step.

   1.2. Condensed Solver

As the matrix solver would only promise the local extreme and contains an unwanted component (we only care about p component), an alternative way is to condense the matrix before searching.

From Equilibrium Equation:

$$q = \frac{-psin\delta + W - B + S}{\sin(\theta + \varphi)}$$

$$pcos\delta - \left(\frac{-psin\delta + W - B + S}{\sin(\theta + \varphi)}\right)\cos(\theta + \varphi) = 0$$

$$pcos\delta tan(\theta + \varphi) + psin\delta = W - B + S$$

$$p = \frac{W - B + S}{cos\delta \tan(\theta + \varphi) + sin\delta} = \frac{1}{2}(\gamma h^2 - \gamma_w h_w^2 + sh)\frac{1}{cos\delta}\frac{tan\theta}{\tan(\theta + \varphi) + tan\delta}$$

$$p' = \frac{1}{2}(\gamma h^2 - \gamma_w h_w^2 + sh)\frac{1}{cos\delta}\frac{\sec^2\theta\tan(\theta + \varphi) - \sec^2(\theta + \varphi)\, tan\theta}{(\tan(\theta + \varphi) + tan\delta)^{\,2}}$$

If p' is monotonic in our domain, the first extreme we find would be the Global Extreme.

$$sign(p') = sign(\sec^2\theta\tan(\theta + \varphi) - \sec^2(\theta + \varphi)\, tan\theta)$$
$$= sign(\sin(2\theta + 2\varphi) - \sin(2\theta))$$

Thus, we could say that the Global Extreme exist. However, this conclusion is not obvious.

2. Search by Exhaustion

   As our domain is confined by $\tan(\theta + \varphi) + tan\delta > 0$, thus, our expression would be:

$$p = \frac{1}{2}(\gamma h^2 - \gamma_w h_w^2 + sh)\frac{1}{cos\delta}\frac{tan\theta}{\tan(\theta + \varphi) + tan\delta}\;\; \theta \in [0, \pi + \delta - \varphi)$$

Conclusion

1. In general, Search by Exhaustion is a better method for Coulomb's theory because of:
1. Easy to construct with various load conditions.
2. Do not need to consider the local extreme.
3. Computation cost is negligible with the power of modern computer.