

Numerical Method for Coulomb's Theory

All the calculations below are based on active earth pressure, for passive conditions, choose $\begin{cases} \delta_p = -\delta_a \\ \varphi_p = -\varphi_a \end{cases}$

Equilibrium Equation:

$$\begin{aligned} p + w + b + q &= 0 \\ \begin{cases} p \cos \delta - q \cos(\theta + \varphi) = 0 \\ p \sin \delta + q \sin(\theta + \varphi) = W - B + S \end{cases} \end{aligned}$$

- By divergence theorem, the buoyancy of a body is always towards the opposite direction of acceleration of fluid.

Where:

$$\begin{aligned} W &= \frac{1}{2} \gamma h^2 \tan \theta \text{ is the weight of the isolated soil body} \\ B &= \frac{1}{2} \gamma_w h_w^2 \tan \theta \text{ is the buoyancy acting on it} \\ S &= s h \tan \theta \text{ is the uniform surcharge load} \end{aligned}$$

Written in Matrix Form:

$$\begin{bmatrix} \cos \delta & -\cos(\theta + \varphi) \\ \sin \delta & \sin(\theta + \varphi) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ W - B + S \end{bmatrix}$$

There are mainly two methods to search for the Global Maximum of p

Methods

1. Gradient Descent

1.1. Matrix Solver

$$f(\theta): A(\theta)x(\theta) = b(\theta)$$

First order derivative:

$$\begin{aligned} A'x + Ax' &= b' \\ x' &= A^{-1}(b' - A'x) \end{aligned}$$

Second order derivative:

$$\begin{aligned} A''x + 2A'x' + Ax'' &= b'' \\ x'' &= A^{-1}(b'' - A''x - 2A'x') \end{aligned}$$

Condition of Local Maximum:

$$\begin{cases} x' = A^{-1}(b' - A'x) = 0 \\ x'' = A^{-1}(b'' - A''x - 2A'x') < 0 \end{cases}$$

In practice, as we are doing numerical calculations, the condition of local maximum could be:

$$\begin{cases} x'_{1,i-1} > 0 \\ x''_{1,i} \leq 0 \end{cases}$$

Where $()_{i,j}$ means the i th component in j th step.

1.2. Condensed Solver

As the matrix solver would only promise the local extreme and contains an unwanted component (we only care about p component), an alternative way is to condense the matrix before searching.

From Equilibrium Equation:

$$q = \frac{-psin\delta + W - B + S}{sin(\theta + \varphi)}$$

$$pcos\delta - \left(\frac{-psin\delta + W - B + S}{sin(\theta + \varphi)} \right) cos(\theta + \varphi) = 0$$

$$pcos\delta tan(\theta + \varphi) + psin\delta = W - B + S$$

$$p = \frac{W - B + S}{cos\delta tan(\theta + \varphi) + sin\delta} = \frac{1}{2}(\gamma h^2 - \gamma_w h_w^2 + 2sh) \frac{1}{cos\delta tan(\theta + \varphi) + tan\delta} \frac{tan\theta}{tan\theta}$$

$$p' = \frac{1}{2}(\gamma h^2 - \gamma_w h_w^2 + 2sh) \frac{1}{cos\delta} \frac{sec^2 \theta tan(\theta + \varphi) - sec^2(\theta + \varphi) tan\theta}{(tan(\theta + \varphi) + tan\delta)^2}$$

If p' is monotonic in our domain, the first extreme we find would be the Global Extreme.

$$sign(p') = sign(sec^2 \theta tan(\theta + \varphi) - sec^2(\theta + \varphi) tan\theta)$$

$$= sign(sin(2\theta + 2\varphi) - sin(2\theta))$$

Thus, we could say that the Global Extreme exist. However, this conclusion is not obvious.

2. Search by Exhaustion

As our domain is confined by $tan(\theta + \varphi) + tan\delta > 0$, thus, our expression would be:

$$p = \frac{1}{2}(\gamma h^2 - \gamma_w h_w^2 + 2sh) \frac{1}{cos\delta tan(\theta + \varphi) + tan\delta} \frac{tan\theta}{tan\theta} \quad \theta \in [0, \pi + \delta - \varphi)$$

Conclusion

1. In general, Search by Exhaustion is a better method for Coulomb's theory because of:
 1. Easy to construct with various load conditions.
 2. Do not need to consider the local extreme.
 3. Computation cost is negligible with the power of modern computer.