Principle of Superposition

Introduction

Principle of superposition is the very first property that a linear system has.

Proof

For a given solid mechanics problem with two independent sets of boundary conditions:

$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ (u, v, w) are unknown variables, others are known.

Equilibrium $div(\boldsymbol{\sigma}) + \boldsymbol{f} = \mathbf{0}, \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

Constitutive Law $\boldsymbol{\epsilon} = \boldsymbol{C}: \boldsymbol{\sigma}$

Deformation Continuity $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{u\nabla})$

Boundary Conditions:

Set 1

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{h}_1 \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q}_1 \quad on\ \boldsymbol{\Gamma_q}, \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma_q} = \boldsymbol{\partial\Omega} \\ \quad \boldsymbol{f} = \boldsymbol{f}_1 \quad in\ \boldsymbol{\Omega} \end{cases}$$

Solution 1

$$\begin{cases} \boldsymbol{u}_1 = \boldsymbol{u}_1 \\ \boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}_1 \\ \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \end{cases}$$

Set 2

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{h}_2 \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q}_2 \quad on\ \boldsymbol{\Gamma_q}, \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma_q} = \boldsymbol{\partial\Omega} \\ \quad \boldsymbol{f} = \boldsymbol{f}_2 \quad in\ \boldsymbol{\Omega} \end{cases}$$

Solution 2

$$\begin{cases} \boldsymbol{u}_2 = \boldsymbol{u}_2 \\ \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_2 \\ \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2 \end{cases}$$

Then for the boundary conditions:

Set 1+2

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{h_1} + \boldsymbol{h_2} \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q_1} + \boldsymbol{q_2} \quad on\ \boldsymbol{\Gamma_q}\ , \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma_q} = \boldsymbol{\partial\Omega} \\ \quad \boldsymbol{f} = \boldsymbol{f_1} + \boldsymbol{f_2} \quad in\ \boldsymbol{\Omega} \end{cases}$$

Solution 1+2

$$\begin{cases} \boldsymbol{u}_{1+2} = \boldsymbol{u}_1 + \boldsymbol{u}_2 \\ \boldsymbol{\epsilon}_{1+2} = \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 \\ \boldsymbol{\sigma}_{1+2} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 \end{cases}$$

Verify:

Equilibrium $div(\boldsymbol{\sigma}) + \boldsymbol{f} = \boldsymbol{0}$

$$div(\boldsymbol{\sigma}_{1+2}) + \boldsymbol{f}_{1+2} = [div(\boldsymbol{\sigma}_1) + \boldsymbol{f}_1] + [div(\boldsymbol{\sigma}_2) + \boldsymbol{f}_2] = \boldsymbol{0}$$

Constitutive Law $\boldsymbol{\epsilon} = \boldsymbol{C}: \boldsymbol{\sigma}$

$$\boldsymbol{\epsilon}_{1+2} = \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \boldsymbol{C}: \boldsymbol{\sigma}_1 + \boldsymbol{C}: \boldsymbol{\sigma}_2$$

Deformation Continuity $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{u\nabla})$

$$\boldsymbol{\epsilon}_{1+2} = \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 = \frac{1}{2}(\boldsymbol{\nabla u}_1 + \boldsymbol{u}_1\boldsymbol{\nabla}) + \frac{1}{2}(\boldsymbol{\nabla u}_2 + \boldsymbol{u}_2\boldsymbol{\nabla})$$

Boundary Conditions

$$\begin{cases} \quad \boldsymbol{u} = \boldsymbol{u}_{1+2} = \boldsymbol{h}_1 + \boldsymbol{h}_2 \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q}_{1+2} = \boldsymbol{q}_1 + \boldsymbol{q}_2 \quad on\ \boldsymbol{\Gamma_q}\ , \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma_q} = \boldsymbol{\partial\Omega} \\ \quad \boldsymbol{f} = \boldsymbol{f}_{1+2} = \boldsymbol{f}_1 + \boldsymbol{f}_2 \quad in\ \boldsymbol{\Omega} \end{cases}$$

For problems with two different boundaries $(\boldsymbol{\Gamma_h}, \boldsymbol{\Gamma_q})$, the boundary conditions of each set need to be treated inversely so that the stress/displacement of a set matches the others on boundaries.