Derivative of Tensor

Introduction

The tensor derivative is the fundamental and foundation of tensor calculus when studying tensor functions.

Please note that the basis $(\boldsymbol{g}_i, \boldsymbol{g}^i)$ in this chapter is considered constant and does not change within the variable space (straight-line basis).

Finite Difference

Since the derivative of tensor has not yet been defined, try the derivative of ordinary functions:

$$(trail\ 1)\colon \boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v}) = \lim_{\boldsymbol{v}\to 0} \frac{\boldsymbol{F}(\boldsymbol{u}+\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u})}{\boldsymbol{v}}$$

This trail equation has two issues:

1. $\boldsymbol{v} \to 0$ is undefined
2. $\boldsymbol{v}$ as denominator is undefined

If we further define the derivative by replacing the vector $\boldsymbol{v}$ by its metric $\|\boldsymbol{v}\|_V$:

$$(trail\ 2)\colon \boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v}) = \lim_{\|\boldsymbol{v}\|_V\to 0} \frac{\boldsymbol{F}(\boldsymbol{u}+\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u})}{\|\boldsymbol{v}\|_V}$$

$$(trail\ 2)\colon \lim_{\|\boldsymbol{v}\|_V\to 0} \frac{\boldsymbol{F}(\boldsymbol{u}+\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u}) - \|\boldsymbol{v}\|_V \boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v})}{\|\boldsymbol{v}\|_V} = \boldsymbol{0}$$

$$(trail\ 2)\colon \lim_{\|\boldsymbol{v}\|_V\to 0} \frac{\|\boldsymbol{F}(\boldsymbol{u}+\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u}) - \|\boldsymbol{v}\|_V \boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v})\|_W}{\|\boldsymbol{v}\|_V} = 0$$

Note that $\boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v})$ itself only contains the direction of $\boldsymbol{v}$, and its magnitude is achieved by multiplying $\|\boldsymbol{v}\|_V$. This implies that there is a linear operator so that:

$$\boldsymbol{T}(\boldsymbol{v}) = \|\boldsymbol{v}\|_V \boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v})$$

This generalization has led us to the so-called Fréchet derivatives.

Fréchet Derivatives

Definition:

Let $f: V \to W$ are both metric spaces. Then f is Fréchet differentiable at $v \in V$ if there exists a linear operator $T \in L(V, W)$ such that:

$$\lim_{\|\boldsymbol{v}\|_V \to 0} \frac{\|\boldsymbol{F}(\boldsymbol{u}+\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{T}(\boldsymbol{v})\|_W}{\|\boldsymbol{v}\|_V} = 0$$

The operator T (which is unique) is called the differential of derivatives of f at u, denoted $T = df(u)$.

The definition of Fréchet derivatives is more like a theoretical criterion for judgement than a practical tool for evaluation. In such sense we would need another definition that can determine the derivatives.

Gateaux Derivatives

Tensor Differential

Using the idea of directional derivatives (Gateaux derivatives), the tensor differential (the limit of finite difference) is defined as the difference between the original tensor function and the function with an infinitesimal increment on the variable. Please note that this increment must have the same dimension as the variable itself.

$$\boldsymbol{dF}(\boldsymbol{u}; \boldsymbol{v}) = \lim_{h \to 0} \frac{\boldsymbol{F}(\boldsymbol{u} + h\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u})}{h}$$

Some textbooks might write this as:

$$D\boldsymbol{F}(\boldsymbol{u})[\boldsymbol{v}] = \lim_{h \to 0} \frac{\boldsymbol{F}(\boldsymbol{u} + h\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u})}{h}$$

Or:

$$D\boldsymbol{F}(\boldsymbol{u})_n^* \boldsymbol{v} = \lim_{h \to 0} \frac{\boldsymbol{F}(\boldsymbol{u} + h\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{u})}{h}$$

Properties

Linearity:

$$\boldsymbol{dF}(\boldsymbol{u}; a\boldsymbol{v}) = a\boldsymbol{dF}(\boldsymbol{u}; \boldsymbol{v})$$

$$\boldsymbol{dF}(\boldsymbol{u};\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{dF}(\boldsymbol{u};\boldsymbol{v}) + \boldsymbol{dF}(\boldsymbol{u};\boldsymbol{w})$$

Proof:

$$\boldsymbol{dF}(\boldsymbol{u};a\boldsymbol{v}) = \lim_{h\to 0}\frac{\boldsymbol{F}(\boldsymbol{u}+ah\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{u})}{h} = a\lim_{h\to 0}\frac{\boldsymbol{F}(\boldsymbol{u}+ah\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{u})}{ah} = a\boldsymbol{dF}(\boldsymbol{u};\boldsymbol{v})$$

$$\begin{aligned}\boldsymbol{dF}(\boldsymbol{u};\boldsymbol{v}+\boldsymbol{w}) &= \lim_{h\to 0}\frac{\boldsymbol{F}(\boldsymbol{u}+h\boldsymbol{v}+h\boldsymbol{w})-\boldsymbol{F}(\boldsymbol{u})}{h} \\ &= \lim_{h\to 0}\frac{\boldsymbol{F}(\boldsymbol{u}+h\boldsymbol{v}+h\boldsymbol{w})-\boldsymbol{F}(\boldsymbol{u}+h\boldsymbol{v})+\boldsymbol{F}(\boldsymbol{u}+h\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{u})}{h} \\ &= \boldsymbol{dF}(\boldsymbol{u};\boldsymbol{v}) + \boldsymbol{dF}(\boldsymbol{u};\boldsymbol{w})\end{aligned}$$

Tensor Derivative

Recall the definition of Fréchet derivatives. If the operation done by $\boldsymbol{T}(\boldsymbol{v})$ is known, with the linearity property, we can further define a derivative $\boldsymbol{F}'(\boldsymbol{u})$ from differential $\boldsymbol{dF}(\boldsymbol{u};\boldsymbol{v})$ that is independent of the direction $\boldsymbol{v}$

This derivative is also Fréchet differentiable.

Definition:

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{dF}'(\boldsymbol{u};\boldsymbol{v}) = \boldsymbol{F}'(\boldsymbol{u})_n^*\boldsymbol{v}$$

$$\boldsymbol{F}'(\boldsymbol{u}) = \frac{d\boldsymbol{F}(\boldsymbol{u})}{d\boldsymbol{u}} = \boldsymbol{F}'\big(\boldsymbol{u};\boldsymbol{g}_i\otimes\boldsymbol{g}_j \dots\big)\otimes(\boldsymbol{g}^i\otimes\boldsymbol{g}^j \dots)$$

$$\dim[\boldsymbol{F}'(\boldsymbol{u})] = m+n$$

Where:

$$m = \dim(\boldsymbol{F}) = \dim\,(\boldsymbol{dF})$$

$$n = \dim(\boldsymbol{u}) = \dim(\boldsymbol{v})$$

$$(\boldsymbol{g}_1\otimes\boldsymbol{g}_2 \dots\otimes\boldsymbol{g}_n)_n^*(\boldsymbol{h}_1\otimes\boldsymbol{h}_2 \dots\otimes\boldsymbol{h}_n) = \boldsymbol{g}_i\cdot\boldsymbol{h}_i$$

Proof:

Project the direction $\boldsymbol{v}$ onto a straight-line basis $(\boldsymbol{g}_i,\boldsymbol{g}^i)$

$$\boldsymbol{F}'(\boldsymbol{u};\boldsymbol{v}) = \boldsymbol{F}'\big(\boldsymbol{u};v^{ij\cdots}\boldsymbol{g}_i\otimes\boldsymbol{g}_j \dots\big) = v^{ij\cdots}\boldsymbol{F}'\big(\boldsymbol{u};\boldsymbol{g}_i\otimes\boldsymbol{g}_j \dots\big) = \boldsymbol{F}'\big(\boldsymbol{u};\boldsymbol{g}_i\otimes\boldsymbol{g}_j \dots\big)\otimes(\boldsymbol{g}^i\otimes\boldsymbol{g}^j \dots)_n^*\boldsymbol{v}$$

$$\boldsymbol{F}'(\boldsymbol{u}) = \boldsymbol{F}'\big(\boldsymbol{u};\boldsymbol{g}_i\otimes\boldsymbol{g}_j \dots\big)\otimes(\boldsymbol{g}^i\otimes\boldsymbol{g}^j \dots)$$