Polar Decomposition & Singular Value Decomposition

Introduction

Consider displacement vector $\boldsymbol{u}$ and deformation tensor $\boldsymbol{F}$:

$$\boldsymbol{F} = \boldsymbol{\phi} \cdot \boldsymbol{\nabla}$$

Given a trail vector $\boldsymbol{l}$ in undeformed configuration $\boldsymbol{X}$, the vector after deformation $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X})$ becomes**:**

$$\boldsymbol{l}' = \boldsymbol{F} \cdot \boldsymbol{l}$$

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$$

Derivation of Polar Decomposition

$$\boldsymbol{C} = \sum_i \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i = \sum_i \mu_i^2 \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

$$\boldsymbol{F} = \sum_i \boldsymbol{N}_i \otimes \boldsymbol{n}_i = \sum_i \mu_i \overline{\boldsymbol{N}}_i \otimes \boldsymbol{n}_i$$

$$\boldsymbol{F} = \sum_i \mu_i \overline{\boldsymbol{N}}_i \otimes \boldsymbol{n}_i = \sum_i \mu_i \overline{\boldsymbol{N}}_i \otimes \overline{\boldsymbol{N}}_i \cdot \sum_j \overline{\boldsymbol{N}}_j \otimes \boldsymbol{n}_j = \boldsymbol{V} \cdot \boldsymbol{R}$$

$$\boldsymbol{V} = (\boldsymbol{F} \cdot \boldsymbol{F}^T)^{\frac{1}{2}}$$

$$\boldsymbol{F} = \sum_i \mu_i \overline{\boldsymbol{N}}_i \otimes \boldsymbol{n}_i = \sum_i \overline{\boldsymbol{N}}_i \otimes \boldsymbol{n}_i \cdot \sum_j \mu_j \boldsymbol{n}_j \otimes \boldsymbol{n}_j = \boldsymbol{R} \cdot \boldsymbol{U}$$

$$\boldsymbol{U} = (\boldsymbol{F}^T \cdot \boldsymbol{F})^{\frac{1}{2}}$$

Derivation of Singular Value Decomposition

$$\boldsymbol{F} = \sum_i \overline{\boldsymbol{N}}_i \otimes \boldsymbol{e}_i \cdot \sum_j \mu_j \boldsymbol{e}_j \otimes \boldsymbol{e}_j \cdot \sum_k \boldsymbol{e}_k \otimes \boldsymbol{n}_k = \widetilde{\boldsymbol{U}} \cdot \boldsymbol{\Sigma} \cdot \widetilde{\boldsymbol{V}}$$

SVD method does not require the “matrix to be square”. Thus, it can be used for more general purposes, especially in Principal Component Analysis (PCA).