Stress Function

Introduction

The stress function, as it's named, is the function that summarizes the components of the stress $\boldsymbol{\sigma}$. It is introduced to convert the coupled stresses of equilibrium functions by a so-called stress function that contains all information of stresses.

Due to the limitations of the force method, we shall only introduce the Airy Stress Function for 2D problems here.

Airy Stress Function (2D)

Equilibrium $div(\boldsymbol{\sigma}) + \boldsymbol{f} = \boldsymbol{0}$

$$\begin{cases} \sigma_{x,x} + \tau_{yx,y} + f_x = 0 \\ \tau_{xy,x} + \sigma_{y,y} + f_y = 0 \end{cases}$$

$$\tau_{xy} = \tau_{yx}$$

The equilibrium equations contain three linearly independent equations with four unknown variables $(\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx})$, this implies that there is only one independent variable among the four variables.

Consider:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$-\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial \sigma_x}{\partial x} + f_x$$

It contains the $\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$ structure:

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y} = c$$

$$\Gamma = \iint c \, dxdy = \int A \, dy = \int B \, dx$$

$$\begin{cases} A = \dfrac{\partial \Gamma}{\partial y} \\ B = \dfrac{\partial \Gamma}{\partial x} \end{cases}$$

We have:

$$\begin{cases} \sigma_x + \int f_x \, dx = \dfrac{\partial \mathrm{A}}{\partial y} \\ \quad -\tau_{yx} = \dfrac{\partial \mathrm{A}}{\partial x} \end{cases}$$

Similarly:

$$\begin{cases} \sigma_y + \int f_y \, dy = \dfrac{\partial \mathrm{B}}{\partial x} \\ \quad -\tau_{xy} = \dfrac{\partial \mathrm{B}}{\partial y} \end{cases}$$

Recall:

$$\tau_{xy} = \tau_{yx}$$

$$\frac{\partial \mathrm{A}}{\partial x} = \frac{\partial \mathrm{B}}{\partial y}$$

$$\begin{cases} A = \dfrac{\partial \Phi}{\partial y} \\ B = \dfrac{\partial \Phi}{\partial x} \end{cases}$$

Finally:

$$\begin{cases} \sigma_x = \dfrac{\partial^2 \Phi}{\partial y^2} - \int f_x \, dx \\ \sigma_y = \dfrac{\partial^2 \Phi}{\partial x^2} - \int f_y \, dx \\ \tau_{xy} = \tau_{yx} = -\dfrac{\partial^2 \Phi}{\partial x \partial y} \end{cases}$$

This Φ is called Airy Stress Function.

If the body force stays constant, we have the simplified form:

$$\begin{cases} \sigma_x = \dfrac{\partial^2 \Phi}{\partial y^2} - f_x x \\ \sigma_y = \dfrac{\partial^2 \Phi}{\partial x^2} - f_y y \\ \tau_{xy} = \tau_{yx} = -\dfrac{\partial^2 \Phi}{\partial x \partial y} \end{cases} \quad or \quad \begin{cases} \sigma_x = \dfrac{\partial^2 \Phi}{\partial y^2} \\ \sigma_y = \dfrac{\partial^2 \Phi}{\partial x^2} \\ \tau_{xy} = \tau_{yx} = -\dfrac{\partial^2 \Phi}{\partial x \partial y} - f_x y - f_y x \end{cases}$$

Or any combination that satisfies the homogeneous solution and inhomogeneous (special) solution.

Compatibility Equations (2D)

Due to the reduced dimensions, the 2D compatibility equation only contains one equation instead of six in 3D.

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ 2\gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Compatibility:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} 2\gamma_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y}\epsilon_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\epsilon_y$$

For plane stress problem ($\sigma_z = 0$):

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\mu \\ -\mu & 1 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}$$

$$2\gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy}$$

Plug back:

$$2(1+\mu)\frac{\partial^2}{\partial x \partial y}\tau_{xy} = \frac{\partial^2}{\partial y^2}(\sigma_x - \mu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \mu\sigma_x)$$

Consider:

$$\begin{cases} -\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial \sigma_x}{\partial x} + f_x \\ -\frac{\partial \tau_{yx}}{\partial x} = \frac{\partial \sigma_y}{\partial y} + f_y \end{cases}$$

$$-(1+\mu)[\frac{\partial}{\partial x}\left(\frac{\partial \sigma_x}{\partial x} + f_x\right) + \frac{\partial}{\partial y}\left(\frac{\partial \sigma_x}{\partial y} + f_y\right)] = \frac{\partial^2}{\partial y^2}(\sigma_x - \mu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \mu\sigma_x)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\mu)(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y})$$

For plane strain problem ($\epsilon_z = 0$):

$$\epsilon_z = \frac{1}{E}\left(-\mu\sigma_x - \mu\sigma_y + \sigma_z\right) = 0$$

$$\sigma_z = \mu\left(\sigma_x + \sigma_y\right)$$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} = \frac{1+\mu}{E}\begin{bmatrix} 1-\mu & -\mu \\ -\mu & 1-\mu \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}$$

$$2\gamma_{xy} = \frac{2(1+\mu)}{E}\tau_{xy}$$

Plug back:

$$2\frac{\partial^2}{\partial x \partial y}\tau_{xy} = \frac{\partial^2}{\partial y^2}\left[(1-\mu)\sigma_x - \mu\sigma_y\right] + \frac{\partial^2}{\partial x^2}\left[(1-\mu)\sigma_y - \mu\sigma_x\right]$$

Consider:

$$\begin{cases} -\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial \sigma_x}{\partial x} + f_x \\ -\frac{\partial \tau_{yx}}{\partial x} = \frac{\partial \sigma_y}{\partial y} + f_y \end{cases}$$

$$-[\frac{\partial}{\partial x}\left(\frac{\partial \sigma_x}{\partial x} + f_x\right) + \frac{\partial}{\partial y}\left(\frac{\partial \sigma_x}{\partial y} + f_y\right)] = \frac{\partial^2}{\partial y^2}\left[(1-\mu)\sigma_x - \mu\sigma_y\right] + \frac{\partial^2}{\partial x^2}\left[(1-\mu)\sigma_y - \mu\sigma_x\right]$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1-\mu}(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y})$$

Compatibility equation for homogeneous solution:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0$$

Use Airy Stress Function:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi=0 \quad or \quad \Delta^2\Phi=0$$

Laplace operator:

$$\Delta=\nabla^2=\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}=\sum_{\boldsymbol{i}}\frac{\partial^2}{\partial x_i^2}$$