

## Strong & Weak Form

### Introduction

The introduction of work & energy has enabled us to review the mechanical systems analytically (analytical mechanics). To obtain this new expression of a system, we need to firstly convert the governing equations in terms of energy (force products displacement).

### Strong Form

The strong form in solid mechanics is the governing equations; it views the mechanical problem at point level and contains the strongest restriction which requires the equations to be satisfied at every point in the problem domain:

$$\text{Equilibrium } \operatorname{div}(\boldsymbol{\sigma}) + \mathbf{f} = \mathbf{0}, \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

$$\text{Constitutive Law } \boldsymbol{\epsilon} = \mathcal{C}: \boldsymbol{\sigma}$$

$$\text{Deformation Continuity } \boldsymbol{\epsilon} = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u} \nabla)$$

$$\text{Boundary Conditions} \begin{cases} \mathbf{u} = \mathbf{h} & \text{on } \Gamma_h \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{q} & \text{on } \Gamma_q, \Gamma_h + \Gamma_q = \partial\Omega \\ \mathbf{f} = \mathbf{f} & \text{in } \Omega \end{cases}$$

### Weak Form

We have used the weak form while proving the uniqueness of solution.

The weak form, although derived from and is identical with the strong form, does not contain point-to-point restriction explicitly; it instead views the mechanical problem as a whole (energy) system and only requires the equations to be satisfied systematically.

### Derivation

Dot product the equilibrium equation with displacement and integral over the entire domain:

$$\int \mathbf{u} \cdot \operatorname{div}(\boldsymbol{\sigma}) d\Omega + \int \mathbf{u} \cdot \mathbf{f} d\Omega = \mathbf{0}$$

Note that:

$$\mathbf{u} \cdot \operatorname{div}(\boldsymbol{\sigma}) = \operatorname{div}(\mathbf{u} \cdot \boldsymbol{\sigma}) - \operatorname{grad}(\mathbf{u}): \boldsymbol{\sigma}$$

$$\int \operatorname{div}(\mathbf{u} \cdot \boldsymbol{\sigma}) d\Omega + \int \mathbf{u} \cdot \mathbf{f} d\Omega = \int \operatorname{grad}(\mathbf{u}) : \boldsymbol{\sigma} d\Omega$$

Verify:

$$\operatorname{div}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \nabla = \frac{\partial \sigma_{ij}}{\partial x_j} e_i$$

$$u_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial u_i \sigma_{ij}}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \sigma_{ij}$$

Divergence theorem (right divergence):

$$\int \operatorname{div}(\mathbf{S}) d\Omega = \int \mathbf{S} \cdot \nabla d\Omega = \int \mathbf{S} \cdot \mathbf{n} d\partial\Omega$$

$$\int (\mathbf{u} \cdot \boldsymbol{\sigma}) \cdot \mathbf{n} d\partial\Omega + \int \mathbf{u} \cdot \mathbf{f} d\Omega = \int \operatorname{grad}(\mathbf{u}) : \boldsymbol{\sigma} d\Omega$$

$$\int \mathbf{u} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\partial\Omega + \int \mathbf{u} \cdot \mathbf{f} d\Omega = \int \operatorname{grad}(\mathbf{u}) : \boldsymbol{\sigma} d\Omega$$

Recall boundary conditions:

$$\begin{cases} \mathbf{u} = \mathbf{h} & \text{on } \Gamma_h \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{q} & \text{on } \Gamma_q, \Gamma_h + \Gamma_q = \partial\Omega \\ \mathbf{f} = \mathbf{f} & \text{in } \Omega \end{cases}$$

We have:

$$\begin{aligned} \int \mathbf{u} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_q &= \int \mathbf{u} \cdot \mathbf{q} d\Gamma_q \\ \int \mathbf{u} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_h &= \int \mathbf{h} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_h \\ \int \mathbf{u} \cdot \mathbf{f} d\Omega &= \int \mathbf{u} \cdot \mathbf{f} d\Omega \end{aligned}$$

Then:

$$\int \operatorname{grad}(\mathbf{u}) : \boldsymbol{\sigma} d\Omega = \int \mathbf{u} \cdot \mathbf{q} d\Gamma_q + \int \mathbf{h} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_h + \int \mathbf{u} \cdot \mathbf{f} d\Omega$$

Consider:

$$\int \operatorname{grad}(\mathbf{u}) : \boldsymbol{\sigma} d\Omega = \int \frac{\partial u_i}{\partial x_j} \sigma_{ij} d\Omega = \int \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \sigma_{ij} d\Omega = \int \boldsymbol{\epsilon} : \boldsymbol{\sigma} d\Omega$$

$$\int \boldsymbol{\epsilon} : \boldsymbol{\sigma} d\Omega = \int \mathbf{u} \cdot \mathbf{q} d\Gamma_q + \int \mathbf{h} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_h + \int \mathbf{u} \cdot \mathbf{f} d\Omega$$

Please note that the derivation is based solely on equilibrium, that is said, this weak form is the universal form for any solid mechanics problem that shares the same equilibrium equations.

If the material is linear elastic, the constitutive law:

$$\boldsymbol{\sigma} = \frac{E}{1+\mu} \left[ \frac{\mu}{1-2\mu} \operatorname{tr}(\boldsymbol{\epsilon}) \mathbf{I} + \boldsymbol{\epsilon} \right]$$

$$\boldsymbol{\epsilon} = -\frac{\mu}{E} \operatorname{tr}(\boldsymbol{\sigma}) \mathbf{I} + \frac{1+\mu}{E} \boldsymbol{\sigma}$$

Finally:

$$\frac{E}{1+\mu} \int \left[ \frac{\mu}{1-2\mu} \operatorname{tr}(\boldsymbol{\epsilon})^2 + \boldsymbol{\epsilon} : \boldsymbol{\epsilon} \right] d\Omega = \int \mathbf{u} \cdot \mathbf{q} d\Gamma_q + \int \mathbf{h} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_h + \int \mathbf{u} \cdot \mathbf{f} d\Omega$$

$$\frac{1}{E} \int \left[ -\frac{\mu}{E} \operatorname{tr}(\boldsymbol{\sigma})^2 + (1+\mu) \boldsymbol{\sigma} : \boldsymbol{\sigma} \right] d\Omega = \int \mathbf{u} \cdot \mathbf{q} d\Gamma_q + \int \mathbf{h} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) d\Gamma_h + \int \mathbf{u} \cdot \mathbf{f} d\Omega$$