Uniqueness of Solution

Introduction

The uniqueness of solution is one of another interesting properties that the linear system has. In most cases, no matter how we solve a solid mechanics problem theoretically or numerically, it requires some proper initial guesses to start with. By this meaning, it is vital to understand whether solution we find is the only solution that exists for the problem itself.

Proof

For a given complete solid mechanics problem:

$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ (u, v, w) are unknown variables, others are known.

Equilibrium $div(\boldsymbol{\sigma}) + \boldsymbol{f} = \boldsymbol{0}, \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

Constitutive Law $\boldsymbol{\epsilon} = \boldsymbol{C}\colon \boldsymbol{\sigma}$

Deformation Continuity $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{u\nabla})$

Boundary Conditions $\begin{cases} \boldsymbol{u} = \boldsymbol{h} \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q} \quad on\ \boldsymbol{\Gamma_q}, \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma_q} = \boldsymbol{\partial\Omega} \\ \boldsymbol{f} = \boldsymbol{f} \quad in\ \boldsymbol{\Omega} \end{cases}$

Assume $(\boldsymbol{u_1}, \boldsymbol{\epsilon_1}, \boldsymbol{\sigma_1})$ and $(\boldsymbol{u_2}, \boldsymbol{\epsilon_2}, \boldsymbol{\sigma_2})$ are two sets of solution for the problem, then the difference of these two sets:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{u_1} - \boldsymbol{u_2} \\ \boldsymbol{\epsilon} = \boldsymbol{\epsilon_1} - \boldsymbol{\epsilon_2} \\ \boldsymbol{\sigma} = \boldsymbol{\sigma_1} - \boldsymbol{\sigma_2} \end{cases}$$

Recall superposition, should be the solution for the elastica under new boundary conditions:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0} \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0} \quad on\ \boldsymbol{\Gamma_q}, \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma_q} = \boldsymbol{\partial\Omega} \\ \boldsymbol{f} = \boldsymbol{0} \quad in\ \boldsymbol{\Omega} \end{cases}$$

Dot product the equilibrium equation with displacement and integral over the entire domain:

$$\int \boldsymbol{u} \cdot div(\boldsymbol{\sigma})\, d\boldsymbol{\Omega} + \int \boldsymbol{u} \cdot \boldsymbol{f}\, d\boldsymbol{\Omega} = \boldsymbol{0}$$

Note that:

$$\boldsymbol{u} \cdot div(\boldsymbol{\sigma}) = div(\boldsymbol{u} \cdot \boldsymbol{\sigma}) - grad(\boldsymbol{u}) : \boldsymbol{\sigma}$$

$$\int div(\boldsymbol{u} \cdot \boldsymbol{\sigma})\, d\boldsymbol{\Omega} + \int \boldsymbol{u} \cdot \boldsymbol{f}\, d\boldsymbol{\Omega} = \int grad(\boldsymbol{u}) : \boldsymbol{\sigma}\, d\boldsymbol{\Omega}$$

Verify:

$$div(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} = \frac{\partial \sigma_{ij}}{\partial x_j} \boldsymbol{e}_i$$

$$u_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial u_i \sigma_{ij}}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \sigma_{ij}$$

Divergence theorem (right divergence):

$$\int div(\boldsymbol{S})\, d\boldsymbol{\Omega} = \int \boldsymbol{S} \cdot \boldsymbol{\nabla}\, d\boldsymbol{\Omega} = \int \boldsymbol{S} \cdot \boldsymbol{n}\, d\boldsymbol{\partial\Omega}$$

$$\int (\boldsymbol{u} \cdot \boldsymbol{\sigma}) \cdot \boldsymbol{n}\, d\boldsymbol{\partial\Omega} + \int \boldsymbol{u} \cdot \boldsymbol{f}\, d\boldsymbol{\Omega} = \int grad(\boldsymbol{u}) : \boldsymbol{\sigma}\, d\boldsymbol{\Omega}$$

$$\int \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, d\boldsymbol{\partial\Omega} + \int \boldsymbol{u} \cdot \boldsymbol{f}\, d\boldsymbol{\Omega} = \int grad(\boldsymbol{u}) : \boldsymbol{\sigma}\, d\boldsymbol{\Omega}$$

Recall boundary conditions:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{0} \quad on\ \boldsymbol{\Gamma_h} \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{0} \quad on\ \boldsymbol{\Gamma}_{\boldsymbol{q}}, \boldsymbol{\Gamma_h} + \boldsymbol{\Gamma}_{\boldsymbol{q}} = \boldsymbol{\partial\Omega} \\ \boldsymbol{f} = \boldsymbol{0} \quad in\ \boldsymbol{\Omega} \end{cases}$$

We have:

$$\int \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, d\boldsymbol{\Gamma}_{\boldsymbol{q}} = 0$$

$$\int \boldsymbol{u} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, d\boldsymbol{\Gamma_h} = 0$$

$$\int \boldsymbol{u} \cdot \boldsymbol{f}\, d\boldsymbol{\Omega} = 0$$

Then:

$$\int grad(\boldsymbol{u}) : \boldsymbol{\sigma}\, d\boldsymbol{\Omega} = 0$$

Consider:

$$\int grad(\boldsymbol{u}):\boldsymbol{\sigma}\, d\boldsymbol{\Omega} = \int \frac{\partial u_i}{\partial x_j}\sigma_{ij}\, d\boldsymbol{\Omega} = \int \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\sigma_{ij}\, d\boldsymbol{\Omega} = \int \boldsymbol{\epsilon}:\boldsymbol{\sigma}\, d\boldsymbol{\Omega}$$

$$\int \boldsymbol{\epsilon}:\boldsymbol{\sigma}\, d\boldsymbol{\Omega} = 0$$

The constitutive law:

$$\boldsymbol{\sigma} = \frac{E}{1+\mu}[\frac{\mu}{1-2\mu}tr(\boldsymbol{\epsilon})\boldsymbol{I} + \boldsymbol{\epsilon}]$$

Finally:

$$\frac{E}{1+\mu}\int [\frac{\mu}{1-2\mu}tr(\boldsymbol{\epsilon})^2 + \boldsymbol{\epsilon}:\boldsymbol{\epsilon}]\, d\boldsymbol{\Omega} = 0$$

In terms of $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$:

$$\frac{E}{1+\mu}\int [\frac{\mu}{1-2\mu}\left(\epsilon_x + \epsilon_y + \epsilon_z\right)^2 + (\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 + 2\gamma_{xy}^2 + 2\gamma_{yz}^2 + 2\gamma_{zx}^2)]\, d\boldsymbol{\Omega} = 0$$

The only possible solution is:

$$\boldsymbol{\epsilon} = \boldsymbol{0}$$

In terms of $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$:

$$\begin{cases} \epsilon_x = 0 \\ \epsilon_y = 0 \\ \epsilon_z = 0 \end{cases} \quad and \quad \begin{cases} \gamma_{xy} = 0 \\ \gamma_{yz} = 0 \\ \gamma_{zx} = 0 \end{cases}$$

Thus:

$$\begin{cases} \boldsymbol{u_1} = \boldsymbol{u_2} \\ \boldsymbol{\epsilon_1} = \boldsymbol{\epsilon_2} \\ \boldsymbol{\sigma_1} = \boldsymbol{\sigma_2} \end{cases}$$

The solution is unique.

Discussion

The uniqueness of solution has enabled us to use the inverse and semi-inverse methods, including numerical methods such as Finite Difference Method, Finite Element Method, and variation principle without worrying about the approach itself.