Force Method & Compatibility

Introduction

The force method, as it's named, is the method that utilize stress $\boldsymbol{\sigma}$ as its independent variable to find the solution of governing equations under specified boundary conditions.

Once we find a set of stresses $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ that satisfy equilibrium, we can then find strains $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ through the constitutive law. The displacements (u, v, w) can finally be identified by combining integral and displacement boundary conditions.

Governing Equations in Expression of Stress

Equilibrium $div(\boldsymbol{\sigma}) + \boldsymbol{f} = \mathbf{0}$

$$\begin{cases} \sigma_{x,x} + \tau_{yx,y} + \tau_{zx,z} + f_x = 0 \\ \tau_{xy,x} + \sigma_{y,y} + \tau_{zy,z} + f_y = 0 \\ \tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} + f_z = 0 \end{cases}$$

$$\begin{cases} \tau_{xy} = \tau_{yx} \\ \tau_{yz} = \tau_{zy} \\ \tau_{zx} = \tau_{xz} \end{cases}$$

Constitutive Law $\boldsymbol{\epsilon} = \boldsymbol{C}\colon\boldsymbol{\sigma}$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu \\ -\mu & 1 & -\mu \\ -\mu & -\mu & 1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}$$

$$\begin{bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \frac{1+\mu}{E} \begin{bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}$$

Deformation Continuity $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{u\nabla})$

$$\begin{cases} \epsilon_x = u_{,x} \\ \epsilon_y = v_{,y} \\ \epsilon_z = w_{,z} \end{cases}$$

$$\begin{cases} 2\gamma_{xy} = u_{,y} + v_{,x} \\ 2\gamma_{yz} = v_{,z} + w_{,y} \\ 2\gamma_{xz} = u_{,z} + w_{,x} \end{cases}$$

Compatibility Equations

The compatibility equations are introduced to determine whether any stress function that satisfies equilibrium can be a potential solution to the problem.

Considering the governing equations, the relationship between stresses $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ and strains $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ is surjective, means term $[\boldsymbol{\sigma}]$ and $[\boldsymbol{\epsilon}]$ can be fully described by each other. However, the relationship between strains $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ and displacement (u, v, w) is quite different. There must be some sort of dependency with six variables on one side, and three variables and derivatives on the other.

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ 2\gamma_{xy} \\ 2\gamma_{yz} \\ 2\gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

We can further reveal this dependency by defining the loose derivative symbol $(\tilde{u}, \tilde{v}, \tilde{w})$ as the general derivative of (u, v, w) under any path and using the same definition for $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$.

$$\begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \\ 2\widetilde{\gamma_{xy}} \\ 2\widetilde{\gamma_{yz}} \\ 2\widetilde{\gamma_{zx}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}$$

Then the expression under the special definition:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2\widetilde{\gamma_{xy}} \\ 2\widetilde{\gamma_{yz}} \\ 2\widetilde{\gamma_{zx}} \end{bmatrix}$$

Dependency:

$$\begin{bmatrix} \widetilde{\gamma_{xy}} \\ \widetilde{\gamma_{yz}} \\ \widetilde{\gamma_{zx}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix}$$

$$\begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2\widetilde{\gamma_{xy}} \\ 2\widetilde{\gamma_{yz}} \\ 2\widetilde{\gamma_{zx}} \end{bmatrix}$$

It is expected that term $(\widetilde{\gamma_{xy}}, \widetilde{\gamma_{yz}}, \widetilde{\gamma_{zx}})$ and $(\widetilde{\epsilon_x}, \widetilde{\epsilon_y}, \widetilde{\epsilon_z})$ will have different expressions when describing each other, as their paths of derivative are not the same.

Now recover the strict expression from the loose definition.

$$\begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix} = \begin{bmatrix} \sim\dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \sim\dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \sim\dfrac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\begin{bmatrix} 2\widetilde{\gamma_{xy}} \\ 2\widetilde{\gamma_{yz}} \\ 2\widetilde{\gamma_{zx}} \end{bmatrix} = \begin{bmatrix} \sim\dfrac{\partial}{\partial y} & \sim\dfrac{\partial}{\partial x} & 0 \\ 0 & \sim\dfrac{\partial}{\partial z} & \sim\dfrac{\partial}{\partial y} \\ \sim\dfrac{\partial}{\partial z} & 0 & \sim\dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Set 1 $\left(\widetilde{\gamma_{xy}}, \widetilde{\gamma_{yz}}, \widetilde{\gamma_{zx}}\right) = f(\widetilde{\epsilon_x}, \widetilde{\epsilon_y}, \widetilde{\epsilon_z})$

$$2\widetilde{\gamma_{xy}} = \sim\frac{\partial}{\partial y}u + \sim\frac{\partial}{\partial x}v = \widetilde{\epsilon_x} + \widetilde{\epsilon_y} = \sim\frac{\partial}{\partial x}u + \sim\frac{\partial}{\partial y}v$$

$$C2\gamma_{xy} = C\frac{\partial}{\partial y}u + C\frac{\partial}{\partial x}v = A\epsilon_x + B\epsilon_y = A\frac{\partial}{\partial x}u + B\frac{\partial}{\partial y}v$$

$$A, B, C\ are\ differential\ operator$$

$$\begin{cases} u: C\dfrac{\partial}{\partial y} = A\dfrac{\partial}{\partial x} \\ v: C\dfrac{\partial}{\partial x} = B\dfrac{\partial}{\partial y} \end{cases}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix} C = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix} C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix} C = \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix} C = \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \end{bmatrix}$$

Define multiply of $(A, B)'s$ coefficient $C = \frac{\partial}{\partial x}\frac{\partial}{\partial y} C^{II}$:

$$\begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial y} \end{bmatrix} C^{II} = \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \\ C \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial y}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial y} \end{bmatrix} C^{II} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Plug back:

$$C 2\gamma_{xy} = A\epsilon_x + B\epsilon_y \Longrightarrow \frac{\partial}{\partial x}\frac{\partial}{\partial y} 2\gamma_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y}\epsilon_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\epsilon_y$$

Using the same method, we find:

$$\begin{cases} \frac{\partial}{\partial x}\frac{\partial}{\partial y}2\gamma_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y}\epsilon_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\epsilon_y \\ \frac{\partial}{\partial y}\frac{\partial}{\partial z}2\gamma_{yz} = \frac{\partial}{\partial z}\frac{\partial}{\partial z}\epsilon_y + \frac{\partial}{\partial y}\frac{\partial}{\partial y}\epsilon_z \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z}2\gamma_{xz} = \frac{\partial}{\partial z}\frac{\partial}{\partial z}\epsilon_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\epsilon_z \end{cases}$$

Set 2 $(\widetilde{\epsilon_x}, \widetilde{\epsilon_y}, \widetilde{\epsilon_z}) = g\left(\widetilde{\gamma_{xy}}, \widetilde{\gamma_{yz}}, \widetilde{\gamma_{zx}}\right)$

$$\widetilde{\epsilon_x} = \sim\frac{\partial}{\partial x}u = 2\widetilde{\gamma_{xy}} + 2\widetilde{\gamma_{yz}} + 2\widetilde{\gamma_{zx}} = \sim\left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v\right) + \sim\left(\frac{\partial}{\partial z}v + \frac{\partial}{\partial y}w\right) + \sim\left(\frac{\partial}{\partial z}u + \frac{\partial}{\partial x}w\right)$$

$$\begin{aligned} D\epsilon_x = D\frac{\partial}{\partial x}u &= A2\gamma_{xy} + B2\gamma_{yz} + C2\gamma_{xz} \\ &= A\left(\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v\right) + B\left(\frac{\partial}{\partial z}v + \frac{\partial}{\partial y}w\right) + C\left(\frac{\partial}{\partial z}u + \frac{\partial}{\partial x}w\right) \end{aligned}$$

$$\begin{cases} u: D\frac{\partial}{\partial x} = A\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} \\ v: 0 = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial z} \\ w: 0 = B\frac{\partial}{\partial y} + C\frac{\partial}{\partial x} \end{cases}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial y} & \frac{\partial}{\partial y}\frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y}\frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} C \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y}\frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} C \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y}\frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} C \end{bmatrix}$$

$$\begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \end{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} D = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y}\frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} C \end{bmatrix}$$

Define multiply of $(A, B, C)'s$ coefficient $D = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z} D^{VI}$:

$$\begin{bmatrix} 0.5 \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z} \\ -0.5 \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z} \\ 0.5 \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z} \end{bmatrix} D^{VI} = \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y}\frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x}\frac{\partial}{\partial z} C \\ D \end{bmatrix}$$

$$
\begin{bmatrix}
0.5\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z} \\
-0.5\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z} \\
0.5\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z} \\
\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z}
\end{bmatrix} D^{VI} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}
$$

Plug back:

$$
D\epsilon_x = A2\gamma_{xy} + B2\gamma_{yz} + C2\gamma_{xz} \Longrightarrow \frac{\partial}{\partial y}\frac{\partial}{\partial z}\epsilon_x = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\gamma_{xy} - \frac{\partial}{\partial x}\gamma_{yz} + \frac{\partial}{\partial y}\gamma_{xz}\right)
$$

Using the same method, we find:

$$
\begin{cases}
\frac{\partial}{\partial y}\frac{\partial}{\partial z}\epsilon_x = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\gamma_{xy} - \frac{\partial}{\partial x}\gamma_{yz} + \frac{\partial}{\partial y}\gamma_{xz}\right) \\
\frac{\partial}{\partial x}\frac{\partial}{\partial z}\epsilon_y = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\gamma_{xy} + \frac{\partial}{\partial x}\gamma_{yz} - \frac{\partial}{\partial y}\gamma_{xz}\right) \\
\frac{\partial}{\partial x}\frac{\partial}{\partial y}\epsilon_z = \frac{\partial}{\partial z}\left(-\frac{\partial}{\partial z}\gamma_{xy} + \frac{\partial}{\partial x}\gamma_{yz} + \frac{\partial}{\partial y}\gamma_{xz}\right)
\end{cases}
$$

Full Set

Combining these two sets, we obtained the compatibility equations:

$$
\begin{cases}
\frac{\partial}{\partial x}\frac{\partial}{\partial y}2\gamma_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial y}\epsilon_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\epsilon_y \\
\frac{\partial}{\partial y}\frac{\partial}{\partial z}2\gamma_{yz} = \frac{\partial}{\partial z}\frac{\partial}{\partial z}\epsilon_y + \frac{\partial}{\partial y}\frac{\partial}{\partial y}\epsilon_z \\
\frac{\partial}{\partial x}\frac{\partial}{\partial z}2\gamma_{xz} = \frac{\partial}{\partial z}\frac{\partial}{\partial z}\epsilon_x + \frac{\partial}{\partial x}\frac{\partial}{\partial x}\epsilon_z
\end{cases}
$$

$$
\begin{cases}
\frac{\partial}{\partial y}\frac{\partial}{\partial z}\epsilon_x = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}\gamma_{xy} - \frac{\partial}{\partial x}\gamma_{yz} + \frac{\partial}{\partial y}\gamma_{xz}\right) \\
\frac{\partial}{\partial x}\frac{\partial}{\partial z}\epsilon_y = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}\gamma_{xy} + \frac{\partial}{\partial x}\gamma_{yz} - \frac{\partial}{\partial y}\gamma_{xz}\right) \\
\frac{\partial}{\partial x}\frac{\partial}{\partial y}\epsilon_z = \frac{\partial}{\partial z}\left(-\frac{\partial}{\partial z}\gamma_{xy} + \frac{\partial}{\partial x}\gamma_{yz} + \frac{\partial}{\partial y}\gamma_{xz}\right)
\end{cases}
$$

Plug $\boldsymbol{\epsilon} = f(\boldsymbol{\sigma})$, we shall have the compatibility equations of force method.