# Variational Principle

## Introduction

**Please read "Derivative of Tensor" before starting this chapter.**

Variational principle is the foundation of analytical mechanics (Lagrangian Mechanics & Hamiltonian Mechanics). It originated from the famous Brachistochrone problem and is then generalized for mechanics and physics, especially in Quantum mechanics. It is no exaggeration that the introduction of variational principle has changed how human beings understand the physical world.

## Functional

A functional is a mapping from a space $\boldsymbol{X}$ into the field of real or complex numbers:

$$I = I[\boldsymbol{y}(\boldsymbol{x})]$$

Where functional $I \in \mathbb{C}$, function $\boldsymbol{y} \in \mathbb{C}^m$ and $\boldsymbol{x} \in \mathbb{C}^n$

## Variation of Function

Assume we have a function $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$, the variation of this function is $\boldsymbol{\delta y}(\boldsymbol{x}) = \epsilon \overline{\boldsymbol{y}}(\boldsymbol{x})$ so that:

$$\lim_{\epsilon \to 0}[\boldsymbol{y}(\boldsymbol{x}) + \epsilon \boldsymbol{\delta} \overline{\boldsymbol{y}}(\boldsymbol{x})] = \boldsymbol{y}(\boldsymbol{x})$$

The new function $\boldsymbol{y}(\boldsymbol{x}) + \boldsymbol{\delta}\overline{\boldsymbol{y}}(\boldsymbol{x})$ shall satisfies all boundary conditions of the original function $\boldsymbol{y}(\boldsymbol{x})$

This is equivalent to the directional differential as we discussed in the derivative of tensor chapter.

By definition we find that the order of variation and differential/derivative is interchangeable:

$$\boldsymbol{\delta}(\boldsymbol{y}') = \overline{\boldsymbol{y}}'(\boldsymbol{x}) - \boldsymbol{y}'(\boldsymbol{x}) = (\boldsymbol{\delta y})'$$

## Variation of Functional

The variation of a functional is defined upon the variation functions:

$$\delta I = \delta I[\boldsymbol{y}(\boldsymbol{x})] = I[\boldsymbol{\delta y}(\boldsymbol{x})]$$

Considering the infinite possibilities $\boldsymbol{\delta y}(\boldsymbol{x})$, this means we often cannot evaluate this variation using a stationary value unless the functional becomes stationary itself, meaning that all possible variations are equal to zero—that is, the functional is at an extremum.

### Brachistochrone problem

There is a very interesting story about how this problem got raised and solved, but let's focus on the problem.

$Problem$:

Find the function $y = y(x)$ so that:

$$I = \int_0^l \frac{1}{\sqrt{2g}} \sqrt{\frac{1 + [y'(x)]^2}{y(x)}}\, dx$$

$$\begin{cases} \delta I[y(x)] = 0 \\ \delta^2 I[y(x)] > 0 \end{cases}$$

Boundary Conditions:

$$\begin{cases} y(0) = 0 \\ y(l) = h \end{cases}$$

$Solution$:

$$\begin{cases} y = r(1 - \cos\theta) \\ x = r(\theta - sin\theta) \end{cases}$$

To solve this problem mathematically (not physically, check the story), we need to establish the general procedure of solving the extremum of a functional.

### Extremum of Functional

#### General Problem

$$\delta I = \int_0^a \delta f(x, y, y')\, dx = 0$$

Where:

$$\begin{cases} y = y(x) \\ y' = \dfrac{dy}{dx} = y'(x) \end{cases}$$

Boundary Conditions:

$$\begin{cases} y(0) = 0 \\ y(a) = b \end{cases}$$

**Euler-Lagrange Equation**

$$\delta I = \int_a^b \delta f(x, y, y')\, dx$$

Recall the directional derivative in the derivative of tensor chapter, the first order variation:

$$\begin{cases} \delta f(x, y, y') = \dfrac{\partial f}{\partial x}\delta x + \dfrac{\partial f}{\partial y}\delta y + \dfrac{\partial f}{\partial y'}\delta y' \\ \delta y' = (\delta y)' \end{cases}$$

The variation is applied on $y$ solely, $\delta x = 0$:

$$\begin{cases} \delta f(x, y, y') = \dfrac{\partial f}{\partial y}\delta y + \dfrac{\partial f}{\partial y'}\delta y' \\ \delta y' = (\delta y)' \end{cases}$$

$$\delta I = \int_a^a \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right) dx$$

Since $\delta y' = (\delta y)'$ is bounded, we especially want to express $\delta y'$ in terms of $\delta y$ so that the variation $\delta I$ can be written as $\delta I = \delta y(\,)$

Considering the irreplaceability of integral and differential operator, this can only be achieved by utilizing the integral outside of this expression:

$$\int_a^b \frac{\partial f}{\partial y'}\delta y'\, dx = \frac{\partial f}{\partial y'}\delta y\bigg|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\, dx$$

Finally:

$$\delta I = \int_a^b \left\{\delta y\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\right\} dx + \frac{\partial f}{\partial y'}\delta y\bigg|_a^b$$

To make the equation meaningful, we must have:

$$\frac{\partial f}{\partial y'}\delta y\bigg|_a^b = 0$$

Essential Boundary Conditions:

$$\begin{cases}\delta y(a) = 0\\ \delta y(b) = 0\end{cases}$$

Natural Boundary Conditions:

$$\begin{cases}\dfrac{\partial f}{\partial y'}(a) = 0\\ \dfrac{\partial f}{\partial y'}(b) = 0\end{cases}$$

Mixed Boundary Conditions:

$$\begin{cases}\delta y(a) = 0\\ \dfrac{\partial f}{\partial y'}(b) = 0\end{cases} \quad or \quad \begin{cases}\dfrac{\partial f}{\partial y'}(a) = 0\\ \delta y(b) = 0\end{cases}$$

$Euler - Lagrangian\ Equation$:

Recall the arbitrary selection of $\delta y$, we have the Euler-Lagrangian Equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Note that the Euler-Lagrangian Equation contains the first order term $\frac{\partial f}{\partial y}$and second order term $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y'}) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)y''$ , which would result in coupled terms $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)y'$ and $\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)y''$

To illustrate this, apply the Euler-Lagrangian Equation to Brachistochrone problem:

$$\frac{1}{2}(1 + y'^2) - y''y = 0$$

This equation is coupled with high order terms and is complicated to solve.

This result has driven us to find an alternative expression so that the term $f$ and $\frac{\partial f}{\partial y'}$ have the same order. This alternative is called the Beltrami Identity.

**Beltrami Identity**

Recall the integral and differential operators are irreplaceable by operators other than the integral and differential operators themselves.

$Target$:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$Materials$:

$$\left[f(x,y,y') \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial y'}\right]$$

$Irreplaceability$:

$$\frac{d}{dx}\left[f(x,y,y') \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial y'}\right] = \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, y' + \frac{\partial f}{\partial y'}\, y" \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]$$

It is obvious that the term $\frac{d}{dx}(\frac{\partial f}{\partial y})$ introduce the most unwanted terms, and the target $\frac{\partial f}{\partial y}$ is coupled with $y'$

$Reconstruction$:

$$\frac{d}{dx}\left[f(x,y,y') \quad \frac{\partial f}{\partial y'}y'\right] = [\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\, y' + \frac{\partial f}{\partial y'}\, y" \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'}\, y"]$$

This is the expression we are looking for.

$Beltrami\ Identity$:

$$\frac{d}{dx}\left(f(x,y,y') - \frac{\partial f}{\partial y'}y'\right) - \frac{\partial f}{\partial x} = y'\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] = 0 \Leftrightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Apply the Beltrami Identity to Brachistochrone problem:

$$y(1 + y'^2) = C^2$$