

## Force Method, Stress Function, and Compatibility

### Introduction

The force method, as it's named, is the method that utilize stress  $\sigma$  as its independent variable to find the solution of governing equations under specified boundary conditions.

Once we find a set of stresses  $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$  that satisfy equilibrium, we can then find strains  $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$  through the constitutive law. The displacements  $(u, v, w)$  can finally be identified by combining integral and displacement boundary conditions.

### Governing Equations in Expression of Stress

Equilibrium  $\operatorname{div}(\sigma) + f = 0$

$$\begin{cases} \sigma_{x,x} + \tau_{yx,y} + \tau_{zx,z} + f_x = 0 \\ \tau_{xy,x} + \sigma_{y,y} + \tau_{zy,z} + f_y = 0 \\ \tau_{xz,x} + \tau_{yz,y} + \sigma_{z,z} + f_z = 0 \end{cases}$$

$$\begin{cases} \tau_{xy} = \tau_{yx} \\ \tau_{yz} = \tau_{zy} \\ \tau_{zx} = \tau_{xz} \end{cases}$$

Constitutive Law  $\epsilon = C : \sigma$

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu \\ -\mu & 1 & -\mu \\ -\mu & -\mu & 1 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}$$

$$\begin{bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1+\mu}{E} \begin{bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

Deformation Continuity  $\epsilon = \frac{1}{2}(\nabla u + u \nabla)$

$$\begin{cases} \epsilon_x = u_{,x} \\ \epsilon_y = v_{,y} \\ \epsilon_z = w_{,z} \end{cases}$$

$$\begin{cases} 2\gamma_{xy} = u_{,y} + v_{,x} \\ 2\gamma_{yz} = v_{,z} + w_{,y} \\ 2\gamma_{zx} = u_{,z} + w_{,x} \end{cases}$$

## Compatibility Equations

The compatibility equations are introduced to determine whether any stress function that satisfies equilibrium can be a potential solution to the problem.

Considering the governing equations, the relationship between stresses ( $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ ) and strains ( $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ ) is surjective, means term  $[\sigma]$  and  $[\epsilon]$  can be fully described by each other. However, the relationship between strains ( $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ ) and displacement ( $u, v, w$ ) is quite different. There must be some sort of dependency with six variables on one side, and three variables and derivatives on the other.

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ 2\gamma_{xy} \\ 2\gamma_{yz} \\ 2\gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

We can further reveal this dependency by defining the loose derivative symbol ( $\tilde{u}, \tilde{v}, \tilde{w}$ ) as the general derivative of ( $u, v, w$ ) under any path and using the same definition for ( $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ ).

$$\begin{bmatrix} \tilde{\epsilon}_x \\ \tilde{\epsilon}_y \\ \tilde{\epsilon}_z \\ 2\tilde{\gamma}_{xy} \\ 2\tilde{\gamma}_{yz} \\ 2\tilde{\gamma}_{zx} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix}$$

Then the expression under the special definition:

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\epsilon}_x \\ \tilde{\epsilon}_y \\ \tilde{\epsilon}_z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2\tilde{\gamma}_{xy} \\ 2\tilde{\gamma}_{yz} \\ 2\tilde{\gamma}_{zx} \end{bmatrix}$$

Dependency:

$$\begin{bmatrix} \widetilde{\gamma_{xy}} \\ \widetilde{\gamma_{yz}} \\ \widetilde{\gamma_{zx}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix}$$

$$\begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2\widetilde{\gamma_{xy}} \\ 2\widetilde{\gamma_{yz}} \\ 2\widetilde{\gamma_{zx}} \end{bmatrix}$$

It is expected that term  $(\widetilde{\gamma_{xy}}, \widetilde{\gamma_{yz}}, \widetilde{\gamma_{zx}})$  and  $(\widetilde{\epsilon_x}, \widetilde{\epsilon_y}, \widetilde{\epsilon_z})$  will have different expressions when describing each other, as their paths of derivative are not the same.

Now recover the strict expression from the loose definition.

$$\begin{aligned} \begin{bmatrix} \widetilde{\epsilon_x} \\ \widetilde{\epsilon_y} \\ \widetilde{\epsilon_z} \end{bmatrix} &= \begin{bmatrix} \sim \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \sim \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \sim \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ \begin{bmatrix} 2\widetilde{\gamma_{xy}} \\ 2\widetilde{\gamma_{yz}} \\ 2\widetilde{\gamma_{zx}} \end{bmatrix} &= \begin{bmatrix} \sim \frac{\partial}{\partial y} & \sim \frac{\partial}{\partial x} & 0 \\ 0 & \sim \frac{\partial}{\partial z} & \sim \frac{\partial}{\partial y} \\ \sim \frac{\partial}{\partial z} & 0 & \sim \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \end{aligned}$$

**Set 1**  $(\widetilde{\gamma_{xy}}, \widetilde{\gamma_{yz}}, \widetilde{\gamma_{zx}}) = f(\widetilde{\epsilon_x}, \widetilde{\epsilon_y}, \widetilde{\epsilon_z})$

$$2\widetilde{\gamma_{xy}} = \sim \frac{\partial}{\partial y} u + \sim \frac{\partial}{\partial x} v = \widetilde{\epsilon_x} + \widetilde{\epsilon_y} = \sim \frac{\partial}{\partial x} u + \sim \frac{\partial}{\partial y} v$$

$$C2\gamma_{xy} = C \frac{\partial}{\partial y} u + C \frac{\partial}{\partial x} v = A\epsilon_x + B\epsilon_y = A \frac{\partial}{\partial x} u + B \frac{\partial}{\partial y} v$$

*A, B, C are differential operator*

$$\begin{cases} u: C \frac{\partial}{\partial y} = A \frac{\partial}{\partial x} \\ v: C \frac{\partial}{\partial x} = B \frac{\partial}{\partial y} \end{cases}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix}_C = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix}_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix}_C = \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \end{bmatrix}_C = \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \end{bmatrix}$$

Define multiply of  $(A, B)$ 's coefficient  $C = \frac{\partial}{\partial x} \frac{\partial}{\partial y} C^{II}$ :

$$\begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \end{bmatrix} C^{II} = \begin{bmatrix} \frac{\partial}{\partial x} A \\ \frac{\partial}{\partial y} B \\ C \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \end{bmatrix} C^{II} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Plug back:

$$C2\gamma_{xy} = A\epsilon_x + B\epsilon_y \Rightarrow \frac{\partial}{\partial x} \frac{\partial}{\partial y} 2\gamma_{xy} = \frac{\partial}{\partial y} \frac{\partial}{\partial y} \epsilon_x + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \epsilon_y$$

Using the same method, we find:

$$\begin{cases} \frac{\partial}{\partial x} \frac{\partial}{\partial y} 2\gamma_{xy} = \frac{\partial}{\partial y} \frac{\partial}{\partial y} \epsilon_x + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \epsilon_y \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} 2\gamma_{yz} = \frac{\partial}{\partial z} \frac{\partial}{\partial z} \epsilon_y + \frac{\partial}{\partial y} \frac{\partial}{\partial y} \epsilon_z \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} 2\gamma_{xz} = \frac{\partial}{\partial z} \frac{\partial}{\partial z} \epsilon_x + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \epsilon_z \end{cases}$$

**Set 2** ( $\widetilde{\epsilon_x}, \widetilde{\epsilon_y}, \widetilde{\epsilon_z}$ ) =  $g(\widetilde{\gamma_{xy}}, \widetilde{\gamma_{yz}}, \widetilde{\gamma_{zx}})$

$$\widetilde{\epsilon_x} = \sim \frac{\partial}{\partial x} u = 2\widetilde{\gamma_{xy}} + 2\widetilde{\gamma_{yz}} + 2\widetilde{\gamma_{zx}} = \sim \left( \frac{\partial}{\partial y} u + \frac{\partial}{\partial x} v \right) + \sim \left( \frac{\partial}{\partial z} v + \frac{\partial}{\partial y} w \right) + \sim \left( \frac{\partial}{\partial z} u + \frac{\partial}{\partial x} w \right)$$

$$\begin{aligned} D\epsilon_x &= D \frac{\partial}{\partial x} u = A2\gamma_{xy} + B2\gamma_{yz} + C2\gamma_{xz} \\ &= A \left( \frac{\partial}{\partial y} u + \frac{\partial}{\partial x} v \right) + B \left( \frac{\partial}{\partial z} v + \frac{\partial}{\partial y} w \right) + C \left( \frac{\partial}{\partial z} u + \frac{\partial}{\partial x} w \right) \end{aligned}$$

$$\begin{cases} u: D \frac{\partial}{\partial x} = A \frac{\partial}{\partial y} + C \frac{\partial}{\partial z} \\ v: 0 = A \frac{\partial}{\partial x} + B \frac{\partial}{\partial z} \\ w: 0 = B \frac{\partial}{\partial y} + C \frac{\partial}{\partial x} \end{cases}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} C \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} C \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \\ 0 \\ 0 \end{bmatrix} D = \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} C \end{bmatrix}$$

$$\begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \end{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial x} D = \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} C \end{bmatrix}$$

Define multiply of  $(A, B, C)$ 's coefficient  $D = \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} D^{VI}$ :

$$\begin{bmatrix} 0.5 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \\ -0.5 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \\ 0.5 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \end{bmatrix} D^{VI} = \begin{bmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial y} A \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} B \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} 0.5 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \\ -0.5 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \\ 0.5 \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial}{\partial z} \end{bmatrix} D^{VI} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Plug back:

$$D\epsilon_x = A2\gamma_{xy} + B2\gamma_{yz} + C2\gamma_{xz} \Rightarrow \frac{\partial}{\partial y} \frac{\partial}{\partial z} \epsilon_x = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial z} \gamma_{xy} - \frac{\partial}{\partial x} \gamma_{yz} + \frac{\partial}{\partial y} \gamma_{xz} \right)$$

Using the same method, we find:

$$\begin{cases} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \epsilon_x = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial z} \gamma_{xy} - \frac{\partial}{\partial x} \gamma_{yz} + \frac{\partial}{\partial y} \gamma_{xz} \right) \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} \epsilon_y = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} - \frac{\partial}{\partial y} \gamma_{xz} \right) \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \epsilon_z = \frac{\partial}{\partial z} \left( -\frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} + \frac{\partial}{\partial y} \gamma_{xz} \right) \end{cases}$$

## Full Set

Combining these two sets, we obtained the compatibility equations:

$$\begin{cases} \frac{\partial}{\partial x} \frac{\partial}{\partial y} 2\gamma_{xy} = \frac{\partial}{\partial y} \frac{\partial}{\partial y} \epsilon_x + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \epsilon_y \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} 2\gamma_{yz} = \frac{\partial}{\partial z} \frac{\partial}{\partial z} \epsilon_y + \frac{\partial}{\partial y} \frac{\partial}{\partial y} \epsilon_z \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} 2\gamma_{xz} = \frac{\partial}{\partial z} \frac{\partial}{\partial z} \epsilon_x + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \epsilon_z \\ \\ \frac{\partial}{\partial y} \frac{\partial}{\partial z} \epsilon_x = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial z} \gamma_{xy} - \frac{\partial}{\partial x} \gamma_{yz} + \frac{\partial}{\partial y} \gamma_{xz} \right) \\ \frac{\partial}{\partial x} \frac{\partial}{\partial z} \epsilon_y = \frac{\partial}{\partial y} \left( \frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} - \frac{\partial}{\partial y} \gamma_{xz} \right) \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \epsilon_z = \frac{\partial}{\partial z} \left( -\frac{\partial}{\partial z} \gamma_{xy} + \frac{\partial}{\partial x} \gamma_{yz} + \frac{\partial}{\partial y} \gamma_{xz} \right) \end{cases}$$

Plug  $\epsilon = f(\sigma)$ , we shall have the compatibility equations of force method.