Principle of Superposition

Introduction

Principle of superposition is the very first property that a linear system has.

Proof

For a given solid mechanics problem with two independent sets of boundary conditions:

$(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx})$ (u, v, w) are unknown variables, others are known.

Equilibrium $div(\boldsymbol{\sigma}) + \boldsymbol{f} = \boldsymbol{0}, \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$

Constitutive Law $\boldsymbol{\epsilon} = \boldsymbol{C} : \boldsymbol{\sigma}$

Deformation Continuity $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla)$

Boundary Conditions:

Set 1

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_1 & on\ \Gamma_u \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q}_1 & on\ \Gamma_q, \Gamma_u + \Gamma_q = \partial\Omega \\ \boldsymbol{f} = \boldsymbol{f}_1 & in\ \Omega \end{cases}$$

Solution 1

$$\begin{cases} \boldsymbol{u}_1 = \boldsymbol{u}_1 \\ \boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}_1 \\ \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \end{cases}$$

Set 2

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_1 & on\ \Gamma_u \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{q}_2 & on\ \Gamma_q, \Gamma_u + \Gamma_q = \partial\Omega \\ \boldsymbol{f} = \boldsymbol{f}_2 & in\ \Omega \end{cases}$$

Solution 2

$$\begin{cases} \boldsymbol{u}_2 = \boldsymbol{u}_2 \\ \boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_2 \\ \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2 \end{cases}$$

Then for the boundary conditions:

Set 1+2

$$\begin{cases} u(x) = u_1 + u_2 & on\ \Gamma_u \\ \sigma \cdot n = q_1 + q_2 & on\ \Gamma_q,\ \Gamma_u + \Gamma_q = \partial\Omega \\ f = f_1 + f_2 & in\ \Omega \end{cases}$$

Solution 1+2

$$\begin{cases} u_{1+2} = u_1 + u_2 \\ \epsilon_{1+2} = \epsilon_1 + \epsilon_2 \\ \sigma_{1+2} = \sigma_1 + \sigma_2 \end{cases}$$

Verify:

Equilibrium $div(\sigma) + f = 0$

$$div(\sigma_{1+2}) + f_{1+2} = [div(\sigma_1) + f_1] + [div(\sigma_2) + f_2] = 0$$

Constitutive Law $\epsilon = C:\sigma$

$$\epsilon_{1+2} = \epsilon_1 + \epsilon_2 = C:\sigma_1 + C:\sigma_2$$

Deformation Continuity $\epsilon = \frac{1}{2}(\nabla u + u\nabla)$

$$\epsilon_{1+2} = \epsilon_1 + \epsilon_2 = \frac{1}{2}(\nabla u_1 + u_1\nabla) + \frac{1}{2}(\nabla u_2 + u_2\nabla)$$

Boundary Conditions

$$\begin{cases} u = u_{1+2} = u_1 + u_2 & on\ \Gamma_u \\ \sigma \cdot n = q_{1+2} = q_1 + q_2 & on\ \Gamma_q,\ \Gamma_u + \Gamma_q = \partial\Omega \\ f = f_{1+2} = f_1 + f_2 & in\ \Omega \end{cases}$$

For problems with two different boundaries (Γ_u, Γ_q), the boundary conditions of each set need to be treated inversely so that the stress/displacement of a set matches the others on boundaries.