Limits

In the following we consider an Eschenburg space E with parameters k_1, k_2, k_3, l_1, l_2, l_3. Write $|r| := |H^4(E)|$ and $s_2 := s_2(E)$.

1 Claim. *If $|r| \leq R$ and $|k_i|, |l_i| \leq P$, then in order for the computations of the invariants to be reliable, the data types used need to meet the following requirements:*

`INT_R`	*(signed) integer with capacity of at least ... bits*
`INT_P`	*(signed) integer with capacity of at least ... bits*
`INT_KS`	*(signed) integer with capacity of at least ... bits*
`FLOAT_KS`	*float with significand of at least $8\log_2(P) + 8$ bits*

These requirements are sufficient provided the implementation of the sin function `boost/math/special_functions/sin_pi.hpp` *is as exact as the data type* `FLOAT_KS` *permits.*

2 Claim. *For any Eschenburg space, $|k_i|, |l_i| \leq P$ implies $|r| \leq 6P^2$. For an Eschenburg space in standard presentation, $|k_i|, |l_i| \leq 2|r|^{1/2}$.*

3 Example. *The default data types specified in* `config.h` *are:*

`INT_R`	`:= long` *(at east .. bit)*
`INT_P`	`:= long` *(at least .. bit)*
`INT_KS`	`:= long long` *(at least .. bit)*
`FLOAT_KS`	`:= double` *(53 bit significand)*

The implementation of `sin_pi` *for* `double` *using the GNU C++ compiler is exact*[1]. *Thus, by the above claim and the following note, computations are reliable in the following ranges:*

- *For analysing a single space with parameters $|k_i|, |l_i| \leq$ ....*
- *For generating list of spaces in standard parametrization with $|r| \leq$ ....*

The invariant s_2 is computed by a formula of the form

$$\begin{aligned} s_2 &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + [45\ell_1] + [45\ell_2] + [45\ell_3]}{45d} \end{aligned} \tag{1}$$

where ℓ_i are lens space invariants such that $45\ell_i$ is an integer.

5 Lemma. *Suppose the absolute values of $|k_i|$ and $|l_i|$ are bounded by P. Then the denominator and the numerator of $|s_2|$ are bounded as follows:*

$$\begin{aligned} |numerator| &\leq 2 \cdot 3^3 \cdot 5 \cdot P^4 \\ |denominator| &\leq 2^7 \cdot 3^3 \cdot 5 \cdot \ |r| \ \cdot P^3 \end{aligned}$$

(provided P and $|r|$ are sufficienctly large). The absolute values of the integers d, q and $45\ell_i$ appearing in (are bounded by the same value.

Proof. The absolute value of q in (is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \leq 6(2P)^2 \leq 2^3 \cdot 3 \cdot P^2. \tag{2}$$

[1] `http://www.boost.org/doc/libs/1_65_1/libs/math/doc/html/math_toolkit/powers/sin_pi.html`

The absolute value of d is bounded by

$$\begin{aligned} |d| &\leq 3 \cdot 2^4 \cdot \ |r| \ \cdot (2P)^3 \\ &\leq 2^7 \cdot 3 \cdot \ |r| \ \cdot P^3 \end{aligned} \tag{3}$$

An upper bound for the lens invariants ℓ_i is estimated in Lemma

The lens invariants ℓ_1, ..., ℓ_n are computed as a sum

$$\ell_i := \sum_{v=1}^{|p|-1} L(\mathbf{x}^{(v)}), \tag{4}$$

where the coordinates of $\mathbf{x}^{(v)}$ have the form $\frac{vp_i}{p}$ and each of p, p_1, ..., p_4 is a difference of parameters $k_i - l_j$. The function L appearing in this sum is given by

$$eq : LL(x_0, x_1, x_2, x_3, x_4) := (\cos(\pi x_0) - 1) \cdot \ \textstyle\prod_{i=1}^{4} \csc(\pi x_i), \tag{5}$$

where $\csc(x) := 1/\sin(x)$. *Let $\epsilon > 0$ be sufficiently small ($\leq$ 1/100). Then $|\csc(\pi \cdot x)| \leq 2^{-1.6}\epsilon^{-1}$ for any real number x whose distance to the nearest integer is at least ϵ.*

Proof of the proposition. Proof. It suffices to show that $\sin(\pi \cdot x) \geq 2^{1.6}\epsilon$ for any real $x \in [\epsilon, 1/2]$, where $\epsilon \in (0, 1/100)$ is some given lower bound. It is known that $\sin(\pi x) \geq \pi x \cdot \ \cos(\pi x)$ for all $x \in [0, 1/2]$, so for $x \in [\epsilon, 1/2]$ we find that

$$\sin(\pi x) \geq \sin(\pi\epsilon) \geq \pi\epsilon \cdot cos(\pi\epsilon).$$

If ϵ is sufficiently small, then $\pi \cdot \ \cos(\pi\epsilon)$ is close to π. The result is obtained by explicitly calculating this value for $\epsilon = 1/100$. ☐

7 Lemma. *Let ϵ be as above. Suppose all coordinates of $\mathbf{x} = (x_0, x_1, x_2, x_3, x_4)$ have a distance of a least ϵ to the nearest integer. Then $|L(\mathbf{x})| \leq 2^{-5.4}\epsilon^{-4}$.*

Moreover, for any $\mathbf{y}$ satisfying the same assumptions and contained in a δ-cube around $\mathbf{x}$, $|L(\mathbf{x}) - L(\mathbf{y})| \leq 2^{-5.6}\epsilon^{-5}\delta$

Proof. The previous lemma implies that $|L(\mathbf{x})| \leq 2 \cdot (2^{-1.6}\epsilon^{-1})^4$, so the first claim is immediate. For the second claim, we use the multivariate mean value theorem. Assuming that the absolute values of the partial derivates $\partial_{x_i} L$ are bounded on the given δ-cube by some bound U', the theorem implies that

$$|L(\mathbf{x}) - L(\mathbf{y})| \leq 5 \cdot U' \cdot \delta.$$

for all $\mathbf{y}$ in the cube. The derivatives of L are easily computed using the fact that $\partial_x \csc(\pi x) = -\pi \cos(\pi x) \csc(\pi x)^2$. One easily sees that if $|\csc(\mathbf{y})| \leq U$ on the cube, then $|\partial_{x_i} L(\mathbf{y})| \leq \pi U^5$ on the cube. So we can take $U' := \pi U^5$ with U the upper bound from the previous lemma. This gives

$$|L(\mathbf{x}) - L(\mathbf{y})| \leq 5\pi \cdot (2^{-1.6}\epsilon^{-1})^5 \cdot \delta \leq 2^{-5.6}\epsilon^{-5}\delta,$$

as claimed. ☐

8 Proposition. *Suppose the parameters k_i, l_i of the space E are bounded by P. Let $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ... be the quintuples appearing in (, and let $\mathbf{y}^{(1)}$, $\mathbf{y}^{(2)}$, ... be quintuples such that each $\mathbf{y}^{(v)}$ is contained in a δ-cube around $\mathbf{x}^{(v)}$ for some positive $\delta <$ 1/2P. Then:*

$$\left| 45\ell_i(\mathbf{y}^{(v)}) \right| \leq 2^{5.1} P^5 \cdot (1 - 2P\delta)^{-4} \tag{6}$$

$$\left| 45\ell_i(\mathbf{x}^{(v)}) - 45\ell_i(\mathbf{y}^{(v)}) \right| \leq 2^{5.9} P^6 \cdot \delta \cdot (1 - 2P\delta)^{-5} \tag{7}$$

Proof. As each of k_i and l_i is bounded by P, each of the parameters p, p_1, …, p_4 used to define the quintuples $\mathbf{x}^{(v)}$ is bounded by $2P$. It follows that each coordinate of each $\mathbf{x}^{(v)}$ has a distance of at least $1/2P$ to the nearest integer, and hence each coordinate of each $\mathbf{y}^{(v)}$ has a distance of a least $1/2P - \delta$ to the nearest integer. Thus, we can apply the previous lemma to each $L(\mathbf{x}^{(v)})$ with $\epsilon = 1/(2P) - \delta$ to obtain:

$$\left|L(\mathbf{y}^{(v)})\right| \leq 2^{-5.4}(1/2P - \delta)^{-4} = 2^{-1.4}P^4 \cdot (1 - 2P\delta)^{-4}$$
$$\left|L(\mathbf{x}^{(v)}) - L(\mathbf{y}^{(v)})\right| \leq 2^{-5.6}(1/2P - \delta)^{-5}\delta = 2^{-0.6}P^5 \cdot \delta \cdot (1 - 2P\delta)^{-5}$$

Now take a $(|p| - 1)$-fold sum, multiply by 45 and note that $(|p| - 1) < 2P$. □

data type	bits	range	standard R	general P
int / long	32	$\pm 2^{31}$	47	7
long long	64	$\pm 2^{63}$	336 442	613
??	128	$\pm 2^{127}$	$1,7 \cdot 10^{13}$	4 372 418

Table 1: Different data types and the resulting bounds R on $|r|$ (for spaces in standard presentation) and P on the parameters (for any presentation), according to the above proposition. (The values in the first line of the table may not actually be “sufficiently large” for the proposition to apply.)