Limits

Consider an Eschenburg space E with parameters $(k_1, k_2, k_3, l_1, l_2, l_3)$. We say that the **parameters are bounded by** P if $|k_i| \leq P$ and $|l_i| \leq P$ for all i for some positive integer P. Similarly, we say that r **is bounded by** R if $|r(E)| \leq R$ for some positive integer R, where $|r(E)| = |H^4(E)|$. We say that E is given in **standard presentation** if the parameters satisfy the conditions of [CEZ07, Lemma 4.1]. (All spaces generated using `esch -r=XXXX` are in standard presentation.)

The default data types specified in `config.h` and their sizes are:[1]

`INT_P`	`:= boost::int_t<32>::exact`	32 bit
`INT_R`	`:= boost::int_t<64>::exact`	64 bit
`INT_KS`	`:= boost::multiprecision::int128_t`	$\geq$ 128 bit
`FLOAT_KS`	`:= long double`	64 bit significand

The implementation of the sin function used by `esch`[2] has a relative error of less than 1ϵ[3] for the data type `long double`.[4]

Claim 1. *With the above configurations, the output of the program* `esch` *is reliable in the following ranges:*

- *For computing the invariants of an arbitrary space with parameters bounded by $P = 1500$.*
- *For generating and analysing list of spaces in standard parametrization with r bounded by $R = 600\,000$.*

More generally, we claim the following:

Claim 2. *Suppose r is bounded by R and the parameters are bounded by P. Suppose further that the data types used meet the following minimum requirements:*

`INT_R`	*signed integer of size $\mathcal{E}_R$ bits*
`INT_P`	*signed integer of size $\mathcal{E}_P$ bits*
`INT_KS`	*signed integer of size $\mathcal{E}_{KS}$ bits*
`FLOAT_KS`	*base-2 float with significand of $\mathcal{S}_{KS}$ bits (including sign bit)*

Suppose further that, in the computation of the invariant $s_2(E)$, the sin*-values are computed with a relative error of at most $A\epsilon$. Then the computations of the invariants $r(E)$, $s(E)$, $p_1(E)$, $s_2(E)$ and $s_{22}(E)$ are exact provided each of the following inequalities is satisfied:*

$$P \leq 2^{\mathcal{E}_P - 1} \qquad (a) \qquad\qquad RP^8 \leq 2^{\mathcal{E}_{KS} - 15.3} \qquad (c)$$

$$R \leq 2^{\mathcal{E}_R - 1} \qquad (b) \qquad\qquad P^5 \leq 2^{\mathcal{S}_{KS} - 10.5} \cdot A^{-1} \qquad (d)$$

To obtain Claim 1 from Claim 2, we will use that the bounds P and R can be related as follows:

Note 1. *For any Eschenburg space,*

$$\text{parameters bounded by } P \quad \Rightarrow \quad r \text{ bounded by } R = 6P^2$$

For an Eschenburg space in standard presentation,

$$\text{parameters bounded by } P = 2R^{1/2} \quad \Leftarrow \quad r \text{ bounded by } R$$

[1] http://www.boost.org/doc/libs/1__65__1/libs/multiprecision/doc/html/boost__multiprecision/tut/ints/cpp__int.html
[2] We use `sin_pi` from `boost/math/special_functions/sin_pi.hpp`.
[3] Here, ϵ denotes the **machine epsilon**. See https://en.wikipedia.org/wiki/Machine__epsilon.
[4] http://www.boost.org/doc/libs/1__65__1/libs/math/doc/html/math__toolkit/powers/sin__pi.html

Verification of Claim 1 using Claim 2

We first verify Note 1. The first implication is clear from $r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3)$. For the second implication, note that while all parameters except k_3 are bounded by $\sqrt{R}$ in the standard presentation, the parameter k_3 is bounded only by $2\sqrt{R}$.

To find the bound for P when analysing a single space, we can replace R by $6P^2$ in all inequalities in Claim 2. With $\mathcal{E}_\mathrm{P} = 32$, $\mathcal{E}_\mathrm{R} = 64$ and $\mathcal{E}_\mathrm{KS} = 128$ and $A = 1$ the standard values specified above, these inequalities become:

$$P \leq 2^{31} \quad (a) \qquad 6P^{10} \leq 2^{112.7} \quad (c)$$
$$6P^2 \leq 2^{63} \quad (b) \qquad P^5 \leq 2^{53.5} \quad (d)$$

Here, the strongest inequality is inequality (d), which equates to $P \leq 1663$.

To find a bound for R when analysing spaces in standard presentation, repace P by $2\sqrt{R}$ in all inequalities in Claim 2. They become:

$$2\sqrt{R} \leq 2^{31} \quad (a) \qquad 2^8 \cdot R^5 \leq 2^{112.7} \quad (b)$$
$$R \leq 2^{63} \quad (a') \qquad 2^5 \sqrt{R}^5 \leq 2^{53.5} \quad (d)$$

Again, the strongest inequality is inequality (d). It equates to $R \leq 691\,802$.

Preliminary inequalities I (for integer types)

The invariants $s_2(E)$ and $s_{22}(E)$ are computed by a formulas of the following form [CEZ07, (2.1)]:

$$\begin{aligned} s_2(E) &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + ([45\ell_1] + [45\ell_2] + [45\ell_3])d}{45d} \end{aligned} \tag{1}$$
$$s_{22}(E) = 2\,|r(E)|\,s_2(E) \tag{2}$$

Here, ℓ_i are lens space invariants such that $45\ell_i$ is an integer [CEZ07, Prop. 3.13].

Proposition 1. *Suppose the parameters are bounded by P and r is bounded by R. Then the absolute values of the denominators and the numerators of $s_2(E)$ and $s_{22}(E)$, and the absolute values of the integers d, q and $45\ell_i$ appearing in* (1)*, are bounded by $2^{15.3}RP^8$.*

Proof. The integer d in (1) = [CEZ07, (2.1)] is a multiple of $2r(E)$. Thus, any bounds for numerator and generator of $s_2(E)$ will also be bounds for numerator and generator of $s_{22}(E)$.

The absolute value of q in (1) is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \leq 6(2P)^2 \qquad < 2^{4.6} \cdot P^2 \tag{3}$$

The absolute value of d is bounded by

$$|d| \leq 3 \cdot 2^4 \cdot R \cdot (2P)^3 < 2^{8.6} \cdot RP^3 \tag{4}$$

An upper bound for the values of $45\ell_i$ is estimated as $2^{5.1}P^5$ in Propsition 2 below. Thus, altogether we obtain the following bounds for numerator and denominator of s_2:

$$|\text{numerator}| \leq 45 \cdot 2^{4.6} \cdot P^2 + 3 \cdot 2^{5.1}P^5 \cdot 2^{8.6}RP^3 \approx 2^{15.3} \cdot RP^8$$
$$|\text{denominator}| \leq 45 \cdot |d| \leq 2^{14.1} \cdot RP^3$$

Clearly, the first bound is greater than the second. □

Preliminary inequalities II (for the float type)

The lens invariants ℓ_1, ℓ_2, ℓ_3 are computed as a sum

$$\ell_i := \sum_{v=1}^{|p|-1} L(\mathbf{x}^{(v)}), \tag{5}$$

where each $\mathbf{x}^{(v)} = (x_0^{(v)}, \ldots, x_4^{(v)})$ is a quintuple of real numbers and the function L is given by

$$L(\mathbf{x}) := (\cos(\pi x_0) - 1) \cdot \textstyle\prod_{i=1}^{4} \csc(\pi x_i), \tag{6}$$

where $\csc(x) := 1/\sin(x)$. The coordinates x_i on which this function is evaluated are given by $x_0^{(v)} = \frac{v}{p}$ and $x_i^{(v)} = \frac{vp_i}{p}$ for $i = 1, \ldots, 4$, with each of p, p_1, ..., p_4 a difference of parameters $k_i - l_j$.

Proposition 2. *If the parameters of the given Eschenburg space are bounded by P, then $|45\ell_i| \leq 2^{5.1}P^5$ for each $i \in \{1, 2, 3\}$.*

Proof. As each of k_i and l_i is bounded by P, each of the parameters p, p_1, ..., p_4 used to define the quintuples $\mathbf{x}^{(v)}$ is bounded by $2P$. It follows that each coordinate of each $\mathbf{x}^{(v)}$ has a distance of at least $1/2P$ to the nearest integer. Thus, we can apply Lemma 1 below to each csc-factor of $L(\mathbf{x}^{(v)})$ with $\epsilon = 1/2P$ to obtain

$$\left|L(\mathbf{x}^{(v)})\right| \leq 2 \cdot (2^{-1.6} \cdot 2P)^4 = 2^{-1.4}P^4$$

Now take a $(|p| - 1)$-fold sum and multiply by 45, and note that $(|p| - 1) < 2P$ and $45 < 2^{5.5}$. □

Lemma 1. *Let $\varepsilon > 0$ be sufficiently small ($\leq 1/100$). Then $|\csc(\pi \cdot x)| \leq 2^{-1.6}\varepsilon^{-1}$ for any real number x whose distance to the nearest integer is at least ε.*

Proof. It suffices to show that $\sin(\pi \cdot x) \geq 2^{1.6}\varepsilon$ for any real $x \in [\varepsilon, 1/2]$, where $\varepsilon \in (0, 1/100)$ is some given lower bound. As $\tan(\pi x) \geq \pi x$ for all $x \in [0, 1/2)$, we have $\sin(\pi x) \geq \pi x \cdot \cos(\pi x)$ for all $x \in [0, 1/2]$. So for $x \in [\varepsilon, 1/2]$ we find that

$$\sin(\pi x) \geq \sin(\pi\varepsilon) \geq \pi\varepsilon \cdot \cos(\pi\varepsilon).$$

If ε is sufficiently small, then $\pi \cdot \cos(\pi\varepsilon)$ is close to π. The result is obtained by explicitly calculating this value for $\varepsilon = 1/100$. □

Verification of claim 2

The inequalities (a) and (b) simply state that `INT_P` and `INT_K` need to be large enough to hold the values of the parameters and the value of $r(E)$, respectively. The data type `INT_KS` must be large enough to hold all integers used in computing $s_2(E)$, so inequality (c) follows directly from Proposition 1.

It remains to verify that the data type `FLOAT_KS` is sufficiently precise to compute the integer values $45\ell_i$ appearing in (1). By assumption, the sin-values in (5)/(6) are computed with a relative error of at most $\eta = A\epsilon$, where $\epsilon = 2^{1-\mathcal{S}_{\mathrm{KS}}}$. (Note that one bit of the significand is used to store the sign of the number, so we only have $\mathcal{S}_{\mathrm{KS}} - 1$ bits to store the value.) That is, the computed values of $\sin(\pi x_i)$ may differ from the actual values by a factor of at most $1 \pm \delta$. As the coordinates $x_i^{(v)}$ used as input to the sin-functions may also be exact only up to a factor of $1 \pm \delta$, we find that altogether the computed values of $\sin(\pi x_i)$ may differ from the actual values by a factor of $(1 \pm \delta)^2$, and likewise for the values of $\cos(\pi x_0)$. It follows that the computed values of L and ℓ_i differ from the actual values by a factor of $(1 \pm \delta)^2 (1 \mp \delta)^{-10} \approx 1 \pm 10\delta < 1 + A \cdot 2^{4.4-\mathcal{S}_{\mathrm{KS}}}$. By Proposition 2, this leads to an absolute error for $45\ell_i$ of at most

$$A \cdot 2^{4.4-\mathcal{S}_{\mathrm{KS}}} \cdot |45\ell_i| = A \cdot 2^{4.4-\mathcal{S}_{\mathrm{KS}}} \cdot 2^{5.1} P^5 = A \cdot 2^{9.5-\mathcal{S}_{\mathrm{KS}}} \cdot P^5.$$

To obtain the correct integer value of $45\ell_i$ after rounding, we need this absolute error to be less than $0.5 = 2^{-1}$. That is, we need $A \cdot 2^{9.5-\mathcal{S}_{\mathrm{KS}}} \cdot P^5 < 2^{-1}$, or, equivalently, $P^5 < A^{-1} \cdot 2^{\mathcal{S}_{\mathrm{KS}}-10.5}$. This is inequality (d).

Reference

[CEZ07] T. Chinburg, C. Escher, and W. Ziller, *Topological properties of Eschenburg spaces and 3-Sasakian manifolds*, Math. Ann. **339** (2007), no. 1, 3–20. [MR2317760]