

Limits

Consider an Eschenburg space E with parameters $(\mathbf{k}, \mathbf{l}) = (k_1, k_2, k_3, l_1, l_2, l_3)$. We say that the **parameters are bounded by P** for some positive integer P if $|k_i| \leq P$ and $|l_i| \leq P$ for all i . Similarly, we say that r **is bounded by R** if $|r(E)| \leq R$ for some positive integer R , where $|r(E)| = |H^4(E)|$. We say that E is given in **standard presentation** if the parameters satisfy the conditions of [CEZ07, Lemma 4.1]. (All spaces generated using `esch -r=XXXX` are in standard presentation.)

The default data types specified in `config.h` and their sizes are:¹

INT_P	:= boost::int_t<32>::exact	32 bit
INT_R	:= boost::int_t<64>::exact	64 bit
INT_KS	:= boost::multiprecision::int128_t	≥ 128 bit
FLOAT_KS	:= long double	64 bit significand

The implementation of the `sin` function used by `esch`² has a relative error of less than $1\epsilon^3$ for the data type `long double`.⁴

Claim 1. *With the above configurations, the output of the program `esch` is reliable in the following ranges:*

- For computing the invariants of an arbitrary space with parameters bounded by $P = 1500$.
- For generating and analysing list of spaces in standard parametrization with r bounded by $R = 600\,000$.

More generally, we claim the following:

Claim 2. *Suppose r is bounded by R and the parameters are bounded by P . Suppose further that the data types used meet the following minimum requirements:*

INT_R	signed integer of size \mathcal{E}_R bits
INT_P	signed integer of size \mathcal{E}_P bits
INT_KS	signed integer of size \mathcal{E}_{KS} bits
FLOAT_KS	base-2 float with significand of \mathcal{S}_{KS} bits (including sign bit)

Suppose further that, in the computation of the invariant $s_2(E)$, the `sin`-values are computed with a relative error of at most $A\epsilon$. Then the computations of the invariants $r(E)$, $s(E)$, $p_1(E)$, $s_2(E)$ and $s_{22}(E)$ are exact provided each of the following inequalities is satisfied:

$$\begin{array}{ll}
 P \leq 2^{\mathcal{E}_P-1} & (a) \qquad RP^8 \leq 2^{\mathcal{E}_{KS}-15.3} \qquad (c) \\
 R \leq 2^{\mathcal{E}_R-1} & (b) \qquad P^5 \leq 2^{\mathcal{S}_{KS}-10.5} \cdot A^{-1} \qquad (d)
 \end{array}$$

To obtain Claim 1 from Claim 2, we will use that the bounds P and R can be related as follows:

Note 1. *For any Eschenburg space,*

$$\text{parameters bounded by } P \quad \Rightarrow \quad r \text{ bounded by } R = 6P^2$$

For an Eschenburg space in standard presentation,

$$\text{parameters bounded by } P = 2R^{1/2} \quad \Leftarrow \quad r \text{ bounded by } R$$

¹http://www.boost.org/doc/libs/1_65_1/libs/multiprecision/doc/html/boost_multiprecision/tut/ints/cpp_int.html

²We use `sin_pi` from `boost/math/special_functions/sin_pi.hpp`.

³Here, ϵ denotes the **machine epsilon**. See https://en.wikipedia.org/wiki/Machine_epsilon.

⁴http://www.boost.org/doc/libs/1_65_1/libs/math/doc/html/math_toolkit/powers/sin_pi.html

Verification of Claim 1 using Claim 2

We first verify Note 1. The first implication is clear from $r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3)$. For the second implication, note that while all parameters except k_3 are bounded by \sqrt{R} in the standard presentation, the parameter k_3 is bounded only by $2\sqrt{R}$.

To find the bound for P when analysing a single space, we can replace R by $6P^2$ in all inequalities in Claim 2. With $\mathcal{E}_P = 32$, $\mathcal{E}_R = 64$ and $\mathcal{E}_{KS} = 128$ and $A = 1$ the standard values specified above, these inequalities become:

$$\begin{array}{lll} P \leq 2^{31} & (a) & 6P^{10} \leq 2^{112.7} \quad (c) \\ 6P^2 \leq 2^{63} & (b) & P^5 \leq 2^{53.5} \quad (d) \end{array}$$

Here, the strongest inequality is inequality (d), which equates to $P \leq 1663$.

To find a bound for R when analysing spaces in standard presentation, repack P by $2\sqrt{R}$ in all inequalities in Claim 2. They become:

$$\begin{array}{lll} 2\sqrt{R} \leq 2^{31} & (a) & 2^8 \cdot R^5 \leq 2^{112.7} \quad (b) \\ R \leq 2^{63} & (a') & 2^5 \sqrt{R}^5 \leq 2^{53.5} \quad (d) \end{array}$$

Again, the strongest inequality is inequality (d). It equates to $R \leq 691\,802$.

Preliminary inequalities I (for integer types)

The invariants $s_2(E)$ and $s_{22}(E)$ are computed by a formulas of the following form [CEZ07, (2.1)]:

$$\begin{aligned} s_2(E) &= (q - 2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q - 2) + ([45\ell_1] + [45\ell_2] + [45\ell_3])d}{45d} \end{aligned} \quad (1)$$

$$s_{22}(E) = 2|r(E)|s_2(E) \quad (2)$$

Here, ℓ_i are lens space invariants such that $45\ell_i$ is an integer [CEZ07, Prop. 3.13].

Proposition 1. *Suppose the parameters are bounded by P and r is bounded by R . Then the absolute values of the denominators and the numerators of $s_2(E)$ and $s_{22}(E)$, and the absolute values of the integers d , q and $45\ell_i$ appearing in (1), are bounded by $2^{15.3}RP^8$.*

Proof. The integer d in (1) = [CEZ07, (2.1)] is a multiple of $2r(E)$. Thus, any bounds for numerator and generator of $s_2(E)$ will also be bounds for numerator and generator of $s_{22}(E)$.

The absolute value of q in (1) is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \leq 6(2P)^2 < 2^{4.6} \cdot P^2 \quad (3)$$

The absolute value of d is bounded by

$$|d| \leq 3 \cdot 2^4 \cdot R \cdot (2P)^3 < 2^{8.6} \cdot RP^3 \quad (4)$$

An upper bound for the values of $45\ell_i$ is estimated as $2^{5.1}P^5$ in Propsition 2 below. Thus, altogether we obtain the following bounds for numerator and denominator of s_2 :

$$\begin{aligned} |\text{numerator}| &\leq 45 \cdot 2^{4.6} \cdot P^2 + 3 \cdot 2^{5.1}P^5 \cdot 2^{8.6}RP^3 \approx 2^{15.3} \cdot RP^8 \\ |\text{denominator}| &\leq 45 \cdot |d| \leq 2^{14.1} \cdot RP^3 \end{aligned}$$

Clearly, the first bound is greater than the second. \square

Preliminary inequalities II (for the float type)

The lens invariants ℓ_1, ℓ_2, ℓ_3 are computed as a sum

$$\ell_i := \sum_{v=1}^{|p|-1} L(\mathbf{x}^{(v)}), \quad (5)$$

where each $\mathbf{x}^{(v)} = (x_0^{(v)}, \dots, x_4^{(v)})$ is a quintuple of real numbers and the function L is given by

$$L(\mathbf{x}) := (\cos(\pi x_0) - 1) \cdot \prod_{i=1}^4 \csc(\pi x_i), \quad (6)$$

where $\csc(x) := 1/\sin(x)$. The coordinates x_i on which this function is evaluated are given by $x_0^{(v)} = \frac{v}{p}$ and $x_i^{(v)} = \frac{vp_i}{p}$ for $i = 1, \dots, 4$, with each of p, p_1, \dots, p_4 a difference of parameters $k_i - l_j$.

Proposition 2. *If the parameters of the given Eschenburg space are bounded by P , then $|45\ell_i| \leq 2^{5.1}P^5$ for each $i \in \{1, 2, 3\}$.*

Proof. As each of k_i and l_i is bounded by P , each of the parameters p, p_1, \dots, p_4 used to define the quintuples $\mathbf{x}^{(v)}$ is bounded by $2P$. It follows that each coordinate of each $\mathbf{x}^{(v)}$ has a distance of at least $1/2P$ to the nearest integer. Thus, we can apply Lemma 1 below to each \csc -factor of $L(\mathbf{x}^{(v)})$ with $\epsilon = 1/2P$ to obtain

$$|L(\mathbf{x}^{(v)})| \leq 2 \cdot (2^{-1.6} \cdot 2P)^4 = 2^{-1.4}P^4$$

Now take a $(|p|-1)$ -fold sum and multiply by 45, and note that $(|p|-1) < 2P$ and $45 < 2^{5.5}$. \square

Lemma 1. *Let $\epsilon > 0$ be sufficiently small ($\leq 1/100$). Then $|\csc(\pi \cdot x)| \leq 2^{-1.6}\epsilon^{-1}$ for any real number x whose distance to the nearest integer is at least ϵ .*

Proof. It suffices to show that $\sin(\pi \cdot x) \geq 2^{1.6}\epsilon$ for any real $x \in [\epsilon, 1/2]$, where $\epsilon \in (0, 1/100)$ is some given lower bound. As $\tan(\pi x) \geq \pi x$ for all $x \in [0, 1/2]$, we have $\sin(\pi x) \geq \pi x \cdot \cos(\pi x)$ for all $x \in [0, 1/2]$. So for $x \in [\epsilon, 1/2]$ we find that

$$\sin(\pi x) \geq \sin(\pi \epsilon) \geq \pi \epsilon \cdot \cos(\pi \epsilon).$$

If ϵ is sufficiently small, then $\pi \cdot \cos(\pi \epsilon)$ is close to π . The result is obtained by explicitly calculating this value for $\epsilon = 1/100$. \square

Verification of claim 2

The inequalities (a) and (b) simply state that `INT_P` and `INT_K` need to be large enough to hold the values of the parameters and the value of $r(E)$, respectively. The data type `INT_KS` must be large enough to hold all integers used in computing $s_2(E)$, so inequality (c) follows directly from Proposition 1.

It remains to verify that the data type `FLOAT_KS` is sufficiently precise to compute the integer values $45\ell_i$ appearing in (1). By assumption, the sin-values in (5)/(6) are computed with a relative error of at most $\eta = A\epsilon$, where $\epsilon = 2^{1-S_{KS}}$. (Note that one bit of the significand is used to store the sign of the number, so we only have $S_{KS} - 1$ bits to store the value.) That is, the computed values of $\sin(\pi x_i)$ may differ from the actual values by a factor of at most $1 \pm \delta$. As the coordinates $x_i^{(v)}$ used as input to the sin-functions may also be exact only up to a factor of $1 \pm \delta$, we find that altogether the computed values of $\sin(\pi x_i)$ may differ from the actual values by a factor of $(1 \pm \delta)^2$, and likewise for the values of $\cos(\pi x_0)$. It follows that the computed values of L and ℓ_i differ from the actual values by a factor of $(1 \pm \delta)^2(1 \mp \delta)^{-10} \approx 1 \pm 10\delta < 1 + A \cdot 2^{4.4-S_{KS}}$. By Proposition 2, this leads to an absolute error for $45\ell_i$ of at most

$$A \cdot 2^{4.4-S_{KS}} \cdot |45\ell_i| = A \cdot 2^{4.4-S_{KS}} \cdot 2^{5.1} P^5 = A \cdot 2^{9.5-S_{KS}} \cdot P^5.$$

To obtain the correct integer value of $45\ell_i$ after rounding, we need this absolute error to be less than $0.5 = 2^{-1}$. That is, we need $A \cdot 2^{9.5-S_{KS}} \cdot P^5 < 2^{-1}$, or, equivalently, $P^5 < A^{-1} \cdot 2^{S_{KS}-10.5}$. This is inequality (d).

Reference

- [CEZ07] T. Chinburg, C. Escher, and W. Ziller, *Topological properties of Eschenburg spaces and 3-Sasakian manifolds*, Math. Ann. **339** (2007), no. 1, 3–20. [\[MR2317760\]](#)