Limits

In the following we consider an Eschenburg space E with parameters k_1, k_2, k_3, l_1, l_2, l_3. Write $|r| := |H^4(E)|$ and $s_2 := s_2(E)$.

1 Claim. *If $|r| \leq R$ and $|k_i|, |l_i| \leq P$, then in order for the computations of the invariants to be reliable, the data types used need to meet the following requirements:*

`INT_R`	*(signed) integer with capacity of at least ... bits*
`INT_P`	*(signed) integer with capacity of at least ... bits*
`INT_KS`	*(signed) integer with capacity of at least ... bits*
`FLOAT_KS`	*float with significand of at least ... bits*

These requirements are sufficient provided the implementation of the sin function `boost/math/special_functions/sin_pi.hpp` *is as exact as the data type* `FLOAT_KS` *permits.*

2 Example. *The default data types specified in* `config.h` *are:*

`INT_R`	`:= long` *(at east .. bit)*
`INT_P`	`:= long` *(at least .. bit)*
`INT_KS`	`:= long long` *(at least .. bit)*
`FLOAT_KS`	`:= double` *(53 bit significand)*

The implementation of `sin_pi` *for* `double` *using the GNU C++ compiler is exact*[1]*. Thus, by the above claim, computations are reliable in the following ranges:*

- *For analysing a single space with parameters $|k_i|, |l_i| \leq ...$.*
- *For generating list of spaces in standard parametrization with $|r| \leq ...$.*

In the following we consider an Eschenburg space E with parameters k_1, k_2, k_3, l_1, l_2, l_3. Write $|r| := |H^4(E)|$ and $s_2 := s_2(E)$. The invariant s_2 is computed by a formula of the form

$$\begin{aligned} s_2 &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + [45\ell_1] + [45\ell_2] + [45\ell_3]}{45d} \end{aligned} \tag{1}$$

where ℓ_i are lens space invariants such that $45\ell_i$ is an integer.

In the following proposition, R and P need to be assumed sufficiently large.

3 Proposition. *Suppose E is a space in standard presentation with $|r| \leq R$. Then denominator and numerator of $|s_2|$ are bounded by $2^{17,1} \cdot R^{5/2}$. The absolute values of the integers d, q and $45\ell_i$ appearing in (1) are bounded by the same value.*

Suppose E is an Eschenburg space such that the absolute values of all paremeters are bounded by P. Then the denominator and the numerator of $|s_2|$ are bounded by $2^{16,7} \cdot P^5$.

What does “mantissa” mean? Should I make separate statements for the float values $45\ell_i$?

What does sufficiently large mean?

4 Lemma. *Suppose the absolute values of $|k_i|$ and $|l_i|$ are bounded by P, and $|r|$ is bounded by R. Then the denominator and the numerator of $|s_2|$ are bounded as follows:*

$$|numerator| \leq 2 \cdot 3^3 \cdot 5 \cdot P^4$$
$$|denominator| \leq 2^7 \cdot 3^3 \cdot 5 \cdot RP^3$$

[1] `http://www.boost.org/doc/libs/1_65_1/libs/math/doc/html/math_toolkit/powers/sin_pi.html`

(provided P and R are sufficiently large).

All intermediate integer variables used in the computation of s_2 in [...] are bounded by the same value.

5 Lemma. *For any Eschenburg space,*

$$|k_i|\,,|l_i| \leq P \quad \Rightarrow \quad |r| \leq 6P^2$$

For an Eschenburg space in standard presentation,

$$|k_i|\,,|l_i| \leq 2R^{1/2} \quad \Leftarrow \quad |r| \leq R$$

Proof of Lemma 4. The invariant s_2 is computed by a formula of the form

$$\begin{aligned} s_2 &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + [45\ell_1] + [45\ell_2] + [45\ell_3]}{45d} \end{aligned}$$

where ℓ_i are lens space invariants such that $45\ell_i$ is an integer. The absolute value of q in this formula is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \leq 6(2P)^2 \leq 2^3 \cdot 3 \cdot P^2. \tag{2}$$

The absolute value of d is bounded by

$$\begin{aligned} |d| &\leq 3 \cdot 2^4 \cdot R \cdot (2P)^3 \\ &\leq 2^7 \cdot 3 \cdot RP^3 \end{aligned} \tag{3}$$

Each lens invariant ℓ_i is a sum of p summands of the form

$$|cos(...) - 1| \cdot \left|1/sin\left(\frac{k\pi p_1}{p}\right)\right| \cdot \left|1/sin\left(\frac{k\pi p_2}{p}\right)\right| \cdot \left|1/sin\left(\frac{k\pi p_3}{p}\right)\right| \cdot \left|1/sin\left(\frac{k\pi p_4}{p}\right)\right|$$

When $|x|$ is small, $sin(x) \sim x$, so an upper bound for such a summand can be estimated as

$$|cos(...) - 1| \cdot \left|\frac{p}{k\pi p_1}\right| \cdot \left|\frac{p}{k\pi p_2}\right| \cdot \left|\frac{p}{k\pi p_3}\right| \cdot \left|\frac{p}{k\pi p_4}\right| \leq 2 \cdot \frac{|p|^4}{\pi^4 k^4} \leq 2^{-5}\frac{p^4}{k^4}$$

Summing over k, we obtain:

$$\begin{aligned} |\ell_i| &\leq 2^{-5}\,|p|^4 \sum_{k=1}^{|p|} \left(\frac{1}{k^4}\right) \\ &\leq 2^{-4}\,|p|^4\,. \end{aligned}$$

data type	bits	range	standard R	general P
int / long	32	$\pm 2^{31}$	47	7
long long	64	$\pm 2^{63}$	336 442	613
??	128	$\pm 2^{127}$	$1,7 \cdot 10^{13}$	4 372 418

Table 1: Different data types and the resulting bounds R on $|r|$ (for spaces in standard presentation) and P on the parameters (for any presentation), according to the above proposition. (The values in the first line of the table may not actually be "sufficiently large" for the proposition to apply.)

Finally, $|p| \leq 2P$ because the parameter p is a difference of two parameters of E, so

$$|\ell_i| \leq P^4. \tag{4}$$

Thus, altogether we obtain the following bounds for numerator and denominator of s_2:

$$\begin{aligned} |\text{numerator}| &\leq 45 \cdot (3 \cdot 2^3 \cdot P^2 + 3 \cdot P^4) \leq 2 \cdot 3^3 \cdot 5 \cdot P^4 \\ |\text{denominator}| &\leq 45 \cdot \ |d| \leq 2^7 \cdot 3^3 \cdot 5 \cdot RP^3 \end{aligned}$$

□

Proof of Lemma 5. The first implication is clear from $r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3)$. For the second implication, note that all parameters execpt k_3 are bounded by $\sqrt{R}$ in the standard representation; the parameter k_3 is bounded only by $2\sqrt{R}$. □

Proof of the proposition. The proposition is immediate from the two lemmas and the estimates of upper bounds for the values of q, d and $45\ell_i$ appearing in the proof of Lemma 4. In both cases, it is clear that for sufficiently large R and P the bound for the denominator of $|s_2|$ is the largest bound that occurs. For Eschenburg spaces in standard presentation, this bound is $2^{10} \cdot 3^3 \cdot R^{5/2} \leq 2^{17,1} \cdot R^{5/2}$. For general Eschenburg spaces, this bound is $2^8 \cdot 3^4 \cdot 5 \cdot P^5 \leq 2^{16,7} \cdot P^5$. □