Limits

In the following we consider an Eschenburg space E with parameters $(\mathbf{k}, \mathbf{l}) = (k_1, k_2, k_3, l_1, l_2, l_3)$. We say that the **parameters are bounded by** P for some positive integer P if $|k_i| \leq P$ and $|l_i| \leq P$ for all i. Similarly, we say that r **is bounded by** R if $|r(E)| \leq R$ for some positive integer R, where $|r(E)| = |H^4(E)|$.

1 Lemma. *For any Eschenburg space,*

$$\textit{parameters bounded by } P \quad \Rightarrow \quad r \textit{ bounded by } R = 6P^2 \tag{1}$$

For an Eschenburg space in standard presentation,

$$\textit{parameters bounded by } P = 2R^{1/2} \quad \Leftarrow \quad r \textit{ bounded by } R \tag{2}$$

2 Claim. *Suppose r is bounded by R and the parameters are bounded by P. Suppose further that the data types used meet the following minimum requirements:*

`INT_R`	*signed integer with capacity of e_R bits*
`INT_P`	*signed integer with capacity of e_P bits*
`INT_KS`	*signed integer with capacity of e_{KS} bits*
`FLOAT_KS`	*base-2 float with significand of s_{KS} bits (including sign bit)*

Suppose further that, in the of the computation of the invariant $s_2(E)$, the sin*-values are computed with a relative error of at most $A\epsilon$.*[1] *Then the computations of the invariants $r(E)$, $s(E)$, $p_1(E)$ and $s_2(E)$ are exact provided each of the following inequalities is satisfied:*

$$P \leq 2^{e_P - 1} \qquad P^5 \leq 2^{e_{KS} - 7.7} \qquad P^5 \leq (2/A)^{s_{KS} - 9.5}$$
$$R \leq 2^{e_R - 1} \qquad RP^3 \leq 2^{e_{KS} - 15.1}$$

3 Example. *The default data types specified in* `config.h` *and their sizes on my system are:*

`INT_R`	`:= int`	32 *bit*
`INT_P`	`:= long`	64 *bit*
`INT_KS`	`:= long long`	64 *bit*
`FLOAT_KS`	`:= long double`	64 *bit significand*

The implementation of the sin function `boost/math/special_functions/sin_pi.hpp` *for the data type* `long double` *has a relative error of less than 1ϵ.*[2] *Thus, by the above claim and Lemma 1, computations are reliable in the following ranges:*

- *For analysing a single space with parameters bounded by $P = 146$.*
- *For generating and analysing list of spaces in standard parametrization with r bounded by $R = 336\,442$.*

Verification of Claim 2 for s_{22}. The sin-values in (5)/(6) are computed with a relative error of $\eta = A\epsilon$, where by assumption $\epsilon = 2^{1-s_{KS}}$. (Note that one bit of the significand is used to store the sign of the number, so we only have $s_{KS} - 1$ bits to store the value.) That is, the computed value of $\sin(\dots)$ differs from the actual value by a factor of at most $1 \pm \delta$. It follows that the computed values of L and ℓ_i differ from the acutal values by a factor of approximately $(1+\delta)(1-\delta)^{-4} \approx 1 + 5\delta < 1 + A \cdot 2^{3.4 - s_{KS}}$. By Proposition 7, this leads to

[1] Here, ϵ denotes the **machine epsilon**. See `https://en.wikipedia.org/wiki/Machine_epsilon`.

[2] `http://www.boost.org/doc/libs/1_65_1/libs/math/doc/html/math_toolkit/powers/sin_pi.html`

an absolute error of approximately $A \cdot 2^{3.4-s_{KS}} \cdot 2^{5.1}P^5 = A \cdot 2^{8.5-s_{KS}}$. We know that the exact value of $45\ell_i$ is an integer. To obtain the correct integer, we need this absolute error to be less than $0.5 = 2^{-1}$. That is, we need $A \cdot 2^{8.5-s_{KS}} \cdot P^5 < 2^{-1}$, or, equivalently, $P^5 < A^{-1} \cdot 2^{s_{KS}-9.5}$. This gives the above result. □

0.1 Verification of the claim for the integer data types

The invariant s_2 is computed by a formula of the form

$$\begin{aligned} s_2 &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + [45\ell_1] + [45\ell_2] + [45\ell_3]}{45d} \end{aligned} \tag{3}$$

where ℓ_i are lens space invariants such that $45\ell_i$ is an integer.

4 Lemma. *Suppose the parameters are bounded by P and r is bounded by R. Then the denominator and the numerator of $|s_2|$ are bounded as follows:*

$$\begin{aligned} |numerator| &\le 2^{6.7} \cdot P^5 \\ |denominator| &\le 2^{14.1} \cdot RP^3 \end{aligned}$$

The absolute values of the integers d, q and $45\ell_i$ appearing in (3) are bounded by the same value as the denominator of s_2.

Proof. The absolute value of q in (3) is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \le 6(2P)^2 \qquad < 2^{4.6} \cdot P^2 \tag{4}$$

The absolute value of d is bounded by

$$|d| \le 3 \cdot 2^4 \cdot R \cdot (2P)^3 < 2^{8.6} \cdot RP^3$$

An upper bound for the values of $45\ell_i$ is estimated as $2^{5.1}P^5$ in Propsition 7 below. Thus, altogether we obtain the following bounds for numerator and denominator of s_2:

$$\begin{aligned} |\text{numerator}| &\le 45 \cdot 2^{4.6} \cdot P^2 + 3 \cdot 2^{5.1}P^5 \approx 2^{6.7} \cdot P^5 \\ |\text{denominator}| &\le 45 \cdot |d| \le 2^{14.1} \cdot RP^3 \end{aligned}$$

□

Proof of Lemma 1. The first implication is clear from $r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3)$. For the second implication, note that while all parameters except k_3 are bounded by $\sqrt{R}$ in the standard representation, the parameter k_3 is bounded only by $2\sqrt{R}$. □

Proof of the proposition. The proposition is immediate from the two lemmas and the estimates of upper bounds for the values of q, d and $45\ell_i$ appearing in the proof of Lemma 4. In both cases, it is clear that for sufficiently large R and P the bound for the denominator of $|s_2|$ is the largest bound that occurs. For Eschenburg spaces in standard presentation, this bound is $2^{10} \cdot 3^3 \cdot R^{5/2} \le 2^{17,1} \cdot R^{5/2}$. For general Eschenburg spaces, this bound is $2^8 \cdot 3^4 \cdot 5 \cdot P^5 \le 2^{16,7} \cdot P^5$. □

0.2 Preliminary estimates for the float type

The lens invariants ℓ_1, ..., ℓ_n are computed as a sum

$$\ell_i := \sum_{v=1}^{|p|-1} L(\mathbf{x}^{(v)}), \tag{5}$$

where the coordinates of $\mathbf{x}^{(v)}$ have the form $\frac{vp_i}{p}$ and each of p, p_1, ..., p_4 is a difference of parameters $k_i - l_j$. The function L appearing in this sum is given by

$$L(x_0, x_1, x_2, x_3, x_4) := (\cos(\pi x_0) - 1) \cdot \prod_{i=1}^{4} \csc(\pi x_i), \tag{6}$$

where $\csc(x) := 1/\sin(x)$.

5 Lemma. *Let $\epsilon > 0$ be sufficiently small ($\leq 1/100$). Then $|\csc(\pi \cdot x)| \leq 2^{-1.6}\epsilon^{-1}$ for any real number x whose distance to the nearest integer is at least ϵ.*

Proof. It suffices to show that $\sin(\pi \cdot x) \geq 2^{1.6}\epsilon$ for any real $x \in [\epsilon, 1/2]$, where $\epsilon \in (0, 1/100)$ is some given lower bound. It is known that $\sin(\pi x) \geq \pi x \cdot \cos(\pi x)$ for all $x \in [0, 1/2]$, so for $x \in [\epsilon, 1/2]$ we find that

$$\sin(\pi x) \geq \sin(\pi\epsilon) \geq \pi\epsilon \cdot cos(\pi\epsilon).$$

If ϵ is sufficiently small, then $\pi \cdot \cos(\pi\epsilon)$ is close to π. The result is obtained by explicitly calculating this value for $\epsilon = 1/100$. □

6 Lemma. *Let ϵ be as above. Suppose all coordinates of $\mathbf{x} = (x_0, x_1, x_2, x_3, x_4)$ have a distance of a least ϵ to the nearest integer. Then $|L(\mathbf{x})| \leq 2^{-5.4}\epsilon^{-4}$.*

Moreover, for any $\mathbf{y}$ satisfying the same assumptions and contained in a δ-cube around $\mathbf{x}$, $|L(\mathbf{x}) - L(\mathbf{y})| \leq 2^{-5.6}\epsilon^{-5}\delta$

Proof. The previous lemma implies that $|L(\mathbf{x})| \leq 2 \cdot (2^{-1.6}\epsilon^{-1})^4$, so the first claim is immediate. For the second claim, we use the multivariate mean value theorem. Assuming that the absolute values of the partial derivates $\partial_{x_i} L$ are bounded on the given δ-cube by some bound U', the theorem implies that

$$|L(\mathbf{x}) - L(\mathbf{y})| \leq 5 \cdot U' \cdot \delta.$$

for all $\mathbf{y}$ in the cube. The derivatives of L are easily computed using the fact that $\partial_x \csc(\pi x) = -\pi \cos(\pi x) \csc(\pi x)^2$. One easily sees that if $|\csc(\mathbf{y})| \leq U$ on the cube, then $|\partial_{x_i} L(\mathbf{y})| \leq \pi U^5$ on the cube. So we can take $U' := \pi U^5$ with U the upper bound from the previous lemma. This gives

$$|L(\mathbf{x}) - L(\mathbf{y})| \leq 5\pi \cdot (2^{-1.6}\epsilon^{-1})^5 \cdot \delta \leq 2^{-5.6}\epsilon^{-5}\delta,$$

as claimed. □

7 Proposition. *Suppose the parameters are bounded by P. Then $|45\ell_i| \leq 2^{5.1}P^5$.*

Proof. As each of k_i and l_i is bounded by P, each of the parameters p, p_1, ..., p_4 used to define the quintuples $\mathbf{x}^{(v)}$ is bounded by $2P$. It follows that each coordinate of each $\mathbf{x}^{(v)}$ has a distance of at least $1/2P$ to the nearest integer, and hence each coordinate of each $\mathbf{y}^{(v)}$ has a distance of a least $1/2P - \delta$ to the nearest integer. Thus, we can apply the previous lemma to each $L(\mathbf{x}^{(v)})$ with $\epsilon = 1/(2P) - \delta$ to obtain:

$$\begin{aligned} \left|L(\mathbf{y}^{(v)})\right| &\leq 2^{-5.4}(1/2P - \delta)^{-4} = 2^{-1.4}P^4 \cdot (1 - 2P\delta)^{-4} \\ \left|L(\mathbf{x}^{(v)}) - L(\mathbf{y}^{(v)})\right| &\leq 2^{-5.6}(1/2P - \delta)^{-5}\delta = 2^{-0.6}P^5 \cdot \delta \cdot (1 - 2P\delta)^{-5} \end{aligned}$$

Now take a $(|p| - 1)$-fold sum, multiply by 45 and note that $(|p| - 1) < 2P$. □

data type	bits	range	standard R	general P
int / long	32	$\pm 2^{31}$	47	7
long long	64	$\pm 2^{63}$	336 442	613
??	128	$\pm 2^{127}$	$1,7 \cdot 10^{13}$	4 372 418

Table 1: Different data types and the resulting bounds R on $|r|$ (for spaces in standard presentation) and P on the parameters (for any presentation), according to the above proposition. (The values in the first line of the table may not actually be "sufficiently large" for the proposition to apply.)