Limits

In the following we consider an Eschenburg space E with parameters $k_1, k_2, k_3, l_1, l_2, l_3$. Write $|r| := |H^4(E)|$ and $s_2 := s_2(E)$.

1 Claim. *If $|r| \leq R$ and $|k_i|, |l_i| \leq P$, then in order for the computations of the invariants to be reliable, the data types used need to meet the following requirements:*

`INT_R`	*(signed) integer with capacity of at least ... bits*
`INT_P`	*(signed) integer with capacity of at least ... bits*
`INT_KS`	*(signed) integer with capacity of at least ... bits*
`FLOAT_KS`	*float with significand of at least ... bits*

These requirements are sufficient provided the implementation of the sin function `boost/math/special_functions/sin_pi.hpp` *is as exact as the data type* `FLOAT_KS` *permits.*

2 Example. *The default data types specified in* `config.h` *are:*

`INT_R`	*:= `long` (at east .. bit)*
`INT_P`	*:= `long` (at least .. bit)*
`INT_KS`	*:= `long long` (at least .. bit)*
`FLOAT_KS`	*:= `double` (53 bit significand)*

The implementation of `sin_pi` *for `double` using the GNU C++ compiler is exact*[1]*. Thus, by the above claim and the following note, computations are reliable in the following ranges:*

- *For analysing a single space with parameters $|k_i|, |l_i| \leq ....$*
- *For generating list of spaces in standard parametrization with $|r| \leq ....$*

3 Note. *For any Eschenburg space, $|k_i|, |l_i| \leq P$ implies $|r| \leq 6P^2$. For an Eschenburg space in standard presentation, $|k_i|, |l_i| \leq 2|r|^{1/2}$.*

4 Proposition. *Suppose E is a space in standard presentation. Then denominator and numerator of $|s_2|$ are bounded by $2^{17,1} \cdot |r|^{5/2}$. The absolute values of the integers d, q and $45\ell_i$ appearing in (1) are bounded by the same value.*

Suppose E is an Eschenburg space such that the absolute values of all paremeters are bounded by P. Then the denominator and the numerator of $|s_2|$ are bounded by $2^{16,7} \cdot |r|^5$.

0.1 Verification of the claim for the integer data types

The invariant s_2 is computed by a formula of the form

$$\begin{aligned} s_2 &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + [45\ell_1] + [45\ell_2] + [45\ell_3]}{45d} \end{aligned} \tag{1}$$

where ℓ_i are lens space invariants such that $45\ell_i$ is an integer.

5 Lemma. *Suppose the absolute values of $|k_i|$ and $|l_i|$ are bounded by P. Then the denominator and the numerator of $|s_2|$ are bounded as follows:*

$$|numerator| \leq 2 \cdot 3^3 \cdot 5 \cdot P^4$$
$$|denominator| \leq 2^7 \cdot 3^3 \cdot 5 \cdot |r| \cdot P^3$$

(provided P and $|r|$ are sufficienctly large). The absolute values of the integers d, q and $45\ell_i$ appearing in (1) are bounded by the same value.

[1] `http://www.boost.org/doc/libs/1_65_1/libs/math/doc/html/math_toolkit/powers/sin_pi.html`

Proof. The absolute value of q in (1) is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \leq 6(2P)^2 \leq 2^3 \cdot 3 \cdot P^2. \tag{2}$$

The absolute value of d is bounded by

$$\begin{aligned} |d| &\leq 3 \cdot 2^4 \cdot |r| \cdot (2P)^3 \\ &\leq 2^7 \cdot 3 \cdot |r| \cdot P^3 \end{aligned} \tag{3}$$

An upper bound for the lens invariants ℓ_i is estimated in Lemma 7 below. [...] Each lens invariant ℓ_i is a sum of p summands of the form

$$|cos(...) - 1| \cdot \left|1/sin\left(\frac{k\pi p_1}{p}\right)\right| \cdot \left|1/sin\left(\frac{k\pi p_2}{p}\right)\right| \cdot \left|1/sin\left(\frac{k\pi p_3}{p}\right)\right| \cdot \left|1/sin\left(\frac{k\pi p_4}{p}\right)\right|$$

When $|x|$ is small, $sin(x) \sim x$, so an upper bound for such a summand can be estimated as

$$|cos(...) - 1| \cdot \left|\frac{p}{k\pi p_1}\right| \cdot \left|\frac{p}{k\pi p_2}\right| \cdot \left|\frac{p}{k\pi p_3}\right| \cdot \left|\frac{p}{k\pi p_4}\right| \leq 2 \cdot \frac{|p|^4}{\pi^4 k^4} \leq 2^{-5}\frac{p^4}{k^4}$$

Summing over k, we obtain:

$$\begin{aligned} |\ell_i| &\leq 2^{-5}\,|p|^4 \sum_{k=1}^{|p|}\left(\frac{1}{k^4}\right) \\ &\leq 2^{-4}\,|p|^4\,. \end{aligned}$$

Finally, $|p| \leq 2P$ because the parameter p is a difference of two parameters of E, so

$$|\ell_i| \leq P^4. \tag{4}$$

Thus, altogether we obtain the following bounds for numerator and denominator of s_2:

$$\begin{aligned} |\text{numerator}| &\leq 45 \cdot (3 \cdot 2^3 \cdot P^2 + 3 \cdot P^4) \leq 2 \cdot 3^3 \cdot 5 \cdot P^4 \\ |\text{denominator}| &\leq 45 \cdot |d| \leq 2^7 \cdot 3^3 \cdot 5 \cdot RP^3 \end{aligned}$$

$\square$

Proof of Lemma 3. The first implication is clear from $r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3)$. For the second implication, note that all parameters execpt k_3 are bounded by $\sqrt{R}$ in the standard representation; the parameter k_3 is bounded only by $2\sqrt{R}$. $\square$

Proof of the proposition. The proposition is immediate from the two lemmas and the estimates of upper bounds for the values of q, d and $45\ell_i$ appearing in the proof of Lemma 5. In both cases, it is clear that for sufficiently large R and P the bound for the denominator of $|s_2|$ is the largest bound that occurs. For Eschenburg spaces in standard presentation, this bound is $2^{10} \cdot 3^3 \cdot R^{5/2} \leq 2^{17,1} \cdot R^{5/2}$. For general Eschenburg spaces, this bound is $2^8 \cdot 3^4 \cdot 5 \cdot P^5 \leq 2^{16,7} \cdot P^5$. $\square$

0.2 Verification of the claim for the float data type

Recall $\csc(x) := 1/\sin(x)$.

6 Lemma. *Let $\epsilon > 0$ be sufficiently small ($\leq 1/100$). Then $|\csc(\pi \cdot x)| \leq 2^{-1.6}\epsilon^{-1}$ for any real number x whose distance to the nearest integer is at least ϵ.*

Proof. It suffices to show that $\sin(\pi \cdot x) \geq 2^{1.6}\epsilon$ for any real $x \in [\epsilon, 1/2]$, where $\epsilon \in (0, 1/100)$ is some given lower bound. It is known that $\sin(\pi x) \geq \pi x \cdot \cos(\pi x)$ for all $x \in [0, 1/2]$, so for $x \in [\epsilon, 1/2]$ we find that

$$\sin(\pi x) \geq \sin(\pi\epsilon) \geq \pi\epsilon \cdot cos(\pi\epsilon).$$

If ϵ is sufficiently small, then $\pi \cdot \cos(\pi\epsilon)$ is close to π. The result is obtained by explicitly calculating this value for $\epsilon = 1/100$. $\square$

The lens invariants are computed using the function

$$L(x_0, x_1, x_2, x_3, x_4) := (\cos(\pi x_0) - 1) \cdot \textstyle\prod_{i=1}^{4} \csc(\pi x_i).$$

7 Lemma. *Let ϵ be as above. Suppose all coordinates of $\mathbf{x} = (x_0, x_1, x_2, x_3, x_4)$ have a distance of a least ϵ to the nearest integer. Then $|L(\mathbf{x})| \leq 2^{-5.4}\epsilon^{-4}$.*

Moreover, for any $\mathbf{y}$ satisfying the same assumptions and contained in a δ-cube around $\mathbf{x}$, $|L(\mathbf{x}) - L(\mathbf{y})| \leq 2^{-5.6}\epsilon^{-5}\delta$

Proof. The previous lemma implies that $|L(\mathbf{x})| \leq 2 \cdot (2^{-1.6}\epsilon^{-1})^4$, so the first claim is immediate. For the second claim, we use the multivariate mean value theorem. Assuming that the absolute values of the partial derivates $\partial_{x_i} L$ are bounded on the given δ-cube by some bound U', the theorem implies that

$$|L(\mathbf{x}) - L(\mathbf{y})| \leq 5 \cdot U' \cdot \delta.$$

for all $\mathbf{y}$ in the cube. The derivatives of L are easily computed using the fact that $\partial_x \csc(\pi x) = -\pi \cos(\pi x) \csc(\pi x)^2$. One easily sees that if $|\csc(\mathbf{y})| \leq U$ on the cube, then $|\partial_{x_i} L(\mathbf{y})| \leq \pi U^5$ on the cube. So we can take $U' := \pi U^5$ with U the upper bound from the previous lemma. This gives

$$|L(\mathbf{x}) - L(\mathbf{y})| \leq 5\pi \cdot (2^{-1.6}\epsilon^{-1})^5 \cdot \delta \leq 2^{-5.6}\epsilon^{-5}\delta,$$

as claimed. $\square$

The lens invariant ℓ_i is computed by

$$\ell_i := \sum_{v=1}^{|p|-1} L(\mathbf{x}^{(v)}) \tag{5}$$

where the coordinates of $\mathbf{x}^{(v)}$ have the form $\frac{vp_i}{p}$ and each of p, p_1, ..., p_4 is a sum difference of parameters $k_i - l_j$.

8 Proposition. *Suppose the parameters k_i, l_i are bounded by P. Then*

$$45\ell_i \leq 2^5 P^5.$$

Moreover, if $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, ... are the quintuples appearing in (5), and if $\mathbf{y}^{(1)}$, $\mathbf{y}^{(2)}$, ... are quintuples such that each $\mathbf{y}^{(v)}$ is contained in a δ-cube around $\mathbf{x}^{(v)}$ with $\delta \leq P^{-2}$, then $\left|45\ell_i(\mathbf{x}^{(v)}) - 45\ell_i(\mathbf{y}^{(v)})\right| \leq$ ...

Proof. If each of k_i and l_i is bounded by P, then each of the parameters p, p_1, ..., p_4 is bounded by $2P$. Thus, we can apply the previous lemma to each $L(\mathbf{x}^{(v)})$ with $\epsilon = 1/(2P)$ to obtain $L(\mathbf{x}^{(v)}) \leq 2^{-5.4}(2P)^4 = 2^{-1.4}P^4$. Taking a $(|p|-1)$-fold sum and multiplying by 45, we find p-fold sum –> factor $2P$

$$45\ell_i(\mathbf{x}) \leq 45 \cdot (2P) \cdot 2^{-1.4}P^4 \leq 2^5 P^5$$

$\square$

data type	bits	range	standard R	general P
int / long	32	$\pm 2^{31}$	47	7
long long	64	$\pm 2^{63}$	336 442	613
??	128	$\pm 2^{127}$	$1,7 \cdot 10^{13}$	4 372 418

Table 1: Different data types and the resulting bounds R on $|r|$ (for spaces in standard presentation) and P on the parameters (for any presentation), according to the above proposition. (The values in the first line of the table may not actually be "sufficiently large" for the proposition to apply.)