Limits

Consider an Eschenburg space E with parameters $(\mathbf{k}, \mathbf{l}) = (k_1, k_2, k_3, l_1, l_2, l_3)$. We say that the **parameters are bounded by** P for some positive integer P if $|k_i| \le P$ and $|l_i| \le P$ for all i. Similarly, we say that r **is bounded by** R if $|r(E)| \le R$ for some positive integer R, where $|r(E)| = |H^4(E)|$.

The default data types specified in `config.h` and their sizes on my system are:

`INT_R`	`:= int`	32 bit
`INT_P`	`:= long`	64 bit
`INT_KS`	`:= long long`	64 bit
`FLOAT_KS`	`:= long double`	64 bit significand

The implementation of the sin function `boost/math/special_functions/sin_pi.hpp` for the data type `long double` has a relative error of less than 1ϵ.[1]

Claim 1. *With the above configurations, the output of the program* `esch` *is reliable in the following ranges:*

- *For analysing a single space with parameters bounded by $P = 140$.*
- *For generating and analysing list of spaces in standard parametrization with r bounded by $R = 300\,000$.*

More generally, we claim the following:

Claim 2. *Suppose r is bounded by R and the parameters are bounded by P. Suppose further that the data types used meet the following minimum requirements:*

`INT_R`	*signed integer of size $\mathcal{E}_R$ bits*
`INT_P`	*signed integer of size $\mathcal{E}_P$ bits*
`INT_KS`	*signed integer of size $\mathcal{E}_{KS}$ bits*
`FLOAT_KS`	*base-2 float with significand of $\mathcal{S}_{KS}$ bits (including sign bit)*

Suppose further that, in the of the computation of the invariant $s_2(E)$, the sin*-values are computed with a relative error of at most $A\epsilon$.*[2] *Then the computations of the invariants $r(E)$, $s(E)$, $p_1(E)$ and $s_2(E)$ are exact provided each of the following inequalities is satisfied:*

$$P \le 2^{\mathcal{E}_P - 1} \quad (a) \qquad P^5 \le 2^{\mathcal{E}_{KS} - 7.7} \quad (b) \qquad P^5 \le (2/A)^{\mathcal{S}_{KS} - 9.5} \quad (c)$$

$$R \le 2^{\mathcal{E}_R - 1} \quad (a') \qquad RP^3 \le 2^{\mathcal{E}_{KS} - 15.1} \quad (b')$$

To obtain Claim 1 from Claim 2, we will use that the bounds P and R are related as follows:

Note 1. *For any Eschenburg space,*

$$\textit{parameters bounded by } P \quad \Rightarrow \quad r \textit{ bounded by } R = 6P^2$$

For an Eschenburg space in standard presentation,

$$\textit{parameters bounded by } P = 2R^{1/2} \quad \Leftarrow \quad r \textit{ bounded by } R$$

[1] http://www.boost.org/doc/libs/1_65_1/libs/math/doc/html/math_toolkit/powers/sin_pi.html

[2] Here, ϵ denotes the **machine epsilon**. See https://en.wikipedia.org/wiki/Machine_epsilon.

Verification of claim 1 using claim 2

We first verify Note 1. The first implication is clear from $r = \sigma_2(k_1, k_2, k_3) - \sigma_2(l_1, l_2, l_3)$. For the second implication, note that while all parameters except k_3 are bounded by $\sqrt{R}$ in the standard representation, the parameter k_3 is bounded only by $2\sqrt{R}$.

Now, to find the bound for P when analysing a single space, we can replace R by $6P^2$ in all inequalities in Claim 2. With $\mathcal{E}_\mathrm{P} = 32$, $\mathcal{E}_\mathrm{R} = 64$ and $\mathcal{E}_\mathrm{KS} = 64$ and $A = 1$ the standard values specified above, these inequalities become:

$$\begin{array}{rlrlrl} P \leq 2^{31} & (a) & P^5 \leq 2^{56.3} & (b) & P^5 \leq 2^{54.5} & (c) \\ 6P^2 \leq 2^{63} & (a') & 6P^5 \leq 2^{48.9} & (b') & & \end{array}$$

Here, the strongest inequality is inequality $(2')$, which equates to $P \leq 146$.

To find a bound for R when analysing spaces in standard presentation, repace P by $2\sqrt{R}$ in all inequalities in Claim 2. They become:

$$\begin{array}{rlrlrl} 2\sqrt{R} \leq 2^{31} & (a) & 2^5 \cdot \sqrt{R}^5 \leq 2^{56.3} & (b) & 2^5\sqrt{R}^5 \leq 2^{54.5} & (c) \\ R \leq 2^{63} & (a') & 2^3\sqrt{R}^5 \leq 2^{48.9} & (b') & & \end{array}$$

Again, the strongest inequality is inequality $(2')$. It equates to $R \leq 336\,442$.

Preliminary inequalities I (for integer types)

The invariant s_2 is computed by a formula of the form

$$\begin{aligned} s_2 &= (q-2)/d + \ell_1 + \ell_2 + \ell_3 \\ &= \frac{45(q-2) + [45\ell_1] + [45\ell_2] + [45\ell_3]}{45d} \end{aligned} \tag{1}$$

where ℓ_i are lens space invariants such that $45\ell_i$ is an integer.

Proposition 1. *Suppose the parameters are bounded by P and r is bounded by R. Then the denominator and the numerator of $|s_2|$ are bounded as follows:*

$$\begin{aligned} |\mathit{numerator}| &\leq 2^{6.7} \cdot P^5 \\ |\mathit{denominator}| &\leq 2^{14.1} \cdot RP^3 \end{aligned}$$

The absolute values of the integers d, q and $45\ell_i$ appearing in (1) are bounded by the same value as the denominator of s_2.

Proof. The absolute value of q in (1) is bounded by a sum of six squares of differences of parameters $(k_i - l_j)$, so

$$|q| \leq 6(2P)^2 \qquad < 2^{4.6} \cdot P^2 \tag{2}$$

The absolute value of d is bounded by

$$|d| \leq 3 \cdot 2^4 \cdot R \cdot (2P)^3 < 2^{8.6} \cdot RP^3$$

An upper bound for the values of $45\ell_i$ is estimated as $2^{5.1}P^5$ in Propsition 2 below. Thus, altogether we obtain the following bounds for numerator and denominator of s_2:

$$\begin{aligned} |\text{numerator}| &\leq 45 \cdot 2^{4.6} \cdot P^2 + 3 \cdot 2^{5.1} P^5 \approx 2^{6.7} \cdot P^5 \\ |\text{denominator}| &\leq 45 \cdot |d| \leq 2^{14.1} \cdot RP^3 \end{aligned} \qquad \square$$

Proof of the proposition. The proposition is immediate from the two lemmas and the estimates of upper bounds for the values of q, d and $45\ell_i$ appearing in the proof of Lemma 1. In both cases, it is clear that for sufficiently large R and P the bound for the denominator of $|s_2|$ is the largest bound that occurs. For Eschenburg spaces in standard presentation, this bound is $2^{10} \cdot 3^3 \cdot R^{5/2} \leq 2^{17,1} \cdot R^{5/2}$. For general Eschenburg spaces, this bound is $2^8 \cdot 3^4 \cdot 5 \cdot P^5 \leq 2^{16,7} \cdot P^5$. □

Preliminary inequalities II (for the float type)

The lens invariants ℓ_1, ℓ_2, ℓ_3 are computed as a sum

$$\ell_i := \sum_{v=1}^{|p_0|-1} L(\mathbf{x}^{(v)}), \tag{3}$$

where each $\mathbf{x}^{(v)} = (x_0^{(v)}, \dots, x_4^{(v)})$ is a quintuple of real numbers and the function L is given by

$$L(\mathbf{x}) := (\cos(\pi x_0) - 1) \cdot \textstyle\prod_{i=1}^4 \csc(\pi x_i), \tag{4}$$

where $\csc(x) := 1/\sin(x)$. The coordinates x_i on which this function is evaluated are given by $x_i^{(v)} = \frac{vp_i}{p}$ with each of p_0, p_1, …, p_4 a difference of parameters $k_i - l_j$.

Proposition 2. *If the parameters are bounded by P, then $|45\ell_i| \leq 2^{5.1}P^5$ for each $i \in \{1, 2, 3\}$.*

Proof. As each of k_i and l_i is bounded by P, each of the parameters p, p_1, …, p_4 used to define the quintuples $\mathbf{x}^{(v)}$ is bounded by $2P$. It follows that each coordinate of each $\mathbf{x}^{(v)}$ has a distance of at least $1/2P$ to the nearest integer. Thus, we can apply Lemma 1 below to each $L(\mathbf{x}^{(v)})$ with $\epsilon = 1/(2P)$ to obtain

$$\left|L(\mathbf{x}^{(v)})\right| \leq 2^{-5.4}(1/2P)^{-4} = 2^{-1.4}P^4$$

Now take a $(|p|-1)$-fold sum and multiply by 45, and note that $(|p|-1) < 2P$ and $45 < 2^{5.5}$. □

Lemma 1. *Let $\varepsilon > 0$ be sufficiently small ($\leq 1/100$). Then $|\csc(\pi \cdot x)| \leq 2^{-1.6}\varepsilon^{-1}$ for any real number x whose distance to the nearest integer is at least ε.*

Proof. It suffices to show that $\sin(\pi \cdot x) \geq 2^{1.6}\varepsilon$ for any real $x \in [\varepsilon, 1/2]$, where $\varepsilon \in (0, 1/100)$ is some given lower bound. It is known that $\sin(\pi x) \geq \pi x \cdot \cos(\pi x)$ for all $x \in [0, 1/2]$, so for $x \in [\varepsilon, 1/2]$ we find that

$$\sin(\pi x) \geq \sin(\pi\varepsilon) \geq \pi\varepsilon \cdot \cos(\pi\varepsilon).$$

If ε is sufficiently small, then $\pi \cdot \cos(\pi\varepsilon)$ is close to π. The result is obtained by explicitly calculating this value for $\varepsilon = 1/100$. □

Verification of claim 2

The inequalities (a), (a'), (b) and (b') follow directly from Proposition 1.

For inequality (c), we first note that by assumption the sin-values in (3)/(4) are computed with a relative error of at most $\eta = A\epsilon$, where $\epsilon = 2^{1-\mathcal{S}_{\mathrm{KS}}}$. (Note that one bit of the significand is used to store the sign of the number, so we only have $\mathcal{S}_{\mathrm{KS}} - 1$ bits to store the value.) That is, the computed value of $\sin(\dots)$ differs from the actual value by a factor of at most $1 \pm \delta$. It follows that the computed values of L and ℓ_i

differ from the acutal values by a factor of approximately $(1+\delta)(1-\delta)^{-4} \approx 1+5\delta < 1 + A \cdot 2^{3.4-\mathcal{S}_{\rm KS}}$. By Proposition 2, this leads to an absolute error of approximately $A \cdot 2^{3.4-\mathcal{S}_{\rm KS}} \cdot 2^{5.1} P^5 = A \cdot 2^{8.5-\mathcal{S}_{\rm KS}}$. We know that the exact value of $45\ell_i$ is an integer. To obtain the correct integer, we need this absolute error to be less than $0.5 = 2^{-1}$. That is, we need $A \cdot 2^{8.5-\mathcal{S}_{\rm KS}} \cdot P^5 < 2^{-1}$, or, equivalently, $P^5 < A^{-1} \cdot 2^{\mathcal{S}_{\rm KS}-9.5}$. This implies inequality (c).