A Theoretical Investigation Into the Implications of a Non-Linear Approach to Alfvén Waves

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Abstract. By linearising the equations of ideal magnetohydrodynamics using small perturbations when considering an infinite, homogeneous, magnetised plasma, one finds solutions for small-amplitude oscillations in the form of magnetosonic modes (for a compressible plasma) and linear Alfvén waves (for an incompressible plasma)[1]. This report details the process by which these results are found, and compares them to a non-linear approach[2]. It is found that the non-linear approach also concludes linear wave solutions for Alfvén waves in an incompressible plasma but with more profound implications than the original linear solutions; the non-linear argument proposes waves non-limited in amplitude, but one implication suggests there is in fact an amplitude limitation; an Alfvén wave induces a pressure change which causes the temperature of the plasma in which the wave is propagating to drop. If the temperature drops low enough then the plasma completely recombines and the wave is killed instantly – there is thus a limiting amplitude for an Alfvén wave. Even in the case of only partial recombination, each half-wavelength of the wave recombines more plasma resulting in an increasingly small transmission coefficient for each half-wavelength, until the wave dissipates completely. This report details the mathematical theory from the linear argument for Alfvén waves to the expression for the degree of recombination and explicitly derives an expression for the amplitude limit of an Alfvén wave. Also detailed is a method for which an evolving model can be constructed. This model quantifies the effects of an Alfvén wave which only partially recombines the plasma, as well as the feedback effect on the rest of the Alfvén wave as it propagates through the modified plasma. A code was written in Matlab which could return various plasma and wave properties including the transmission coefficients for 3 consecutive pulses propagating through the plasma. However, using the general expressions given in this report, one could potentially do this for many consecutive pulses. A specific example demonstrates the reduction in transmission of each half-wavelength of Alfvén wave, as well as the increase in degree of recombination of plasma per pulse. The potential use of W.K.B.J. phase-integral methods is also discussed.

1. Introduction and Background

The analysis of the equations of ideal magnetohydrodynamics (MHD) reveal the presence of three normal modes – two compressional magnetosonic modes and a transverse oblique Alfvén wave. Alfvén waves are a means by which we can detect magnetic field properties and varying currents in the cosmos by measuring periodic changes in the direction of the magnetic field over long scale lengths (the Alfvén waves themselves do not reach the detector – only their effects can be seen). The fact they are virtually un-damped[4] at long wavelengths (frequencies much less than the ion cyclotron frequency) makes them long-lasting and thus quite prominent throughout the cosmos. This long-lasting property suggests they could be an effective means of energy transport.

The discovery of Alfvén waves was published by Nature in 1942[3]. A linear argument considering small pertubations of the variables within the equations of ideal MHD led Alfvén to find a wave-solution for the pertubations. The calculated velocity of such waves propagating outward from beneath the solar surface matched the velocity at which sunspots tend to move towards the equator as the sunspot cycle proceeds. This led Alfvén to hypothesise about the connection between sunspots and magnetism. He referred to the waves as “a kind of electromagnetic-hydrodynamic wave”. Alfvén reasoned that if particles in a conducting fluid in a constant magnetic field undergo a perturbation, an E.M.F. is generated which induces electric currents. These currents in turn produce forces which cause the fluid to undergo an oscillation.

Experiments to first detect Alfvén waves were underway well before the non-linear argument was treated – non-linear papers did not appear until the late 1950s. Lundquist (1951) and Lehnert (1954) respectively and independently used the electrically conducting liquids mercury and liquid sodium in an attempt to observe Alfvén waves. Unfortunately, the resemblance to a cold plasma, combined with the properties of mercury and sodium, damped the waves so much that results were unreliable[1]. In 1959 Wilcox and his colleagues conducted an experiment to detect Alfvén waves[5]. A slow hydromagnetic wave was excited at one end of a cyclinder containing a hydrogen plasma by discharging an external capacitor through an electrode mounted within one of the cylinder’s endplates. The wave was transmitted through the plasma parallel to an applied magnetic field and the transmitted effects were detected by a magnetic probe at the opposite side. The velocity of the wave was calculated and plotted against the strength of the magnetic field. *Figure 1* shows the results – Wilcox has clearly shown the dependency of the velocity of these waves with magnetic field strength, as Alfvén predicted.

Figure 1

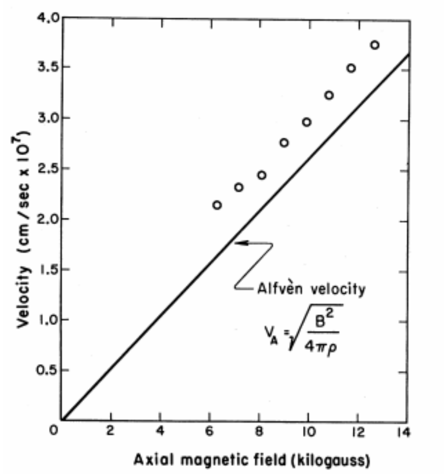

Figure 1. Wave velocity vs magnetic field. The circles are experimental points and the solid curve is the Alfvèn velocity. The hydrogen gas pressure is 100 microns.

2. Aims and Objectives

The primary aim of this project is to investigate the implications presented by the non-linear argument for Alfvén waves. The mathematics of the linear argument will be verified and compared to the non-linear argument. The implications of the pressure (and thus temperature) drop suggested by the non-linear argument will be explored and the effects of this on observations will be detailed.

Following the findings concerning the implications of the temperature drop, the limitations on the amplitude of an Alfvén wave which completely recombines the plasma will be investigated, as well as exploring the effects on an Alfvén wave which partially recombines a plasma.

3. List of Assumptions

There are a lot of assumptions to be made in this report – as such, it is useful to detail them as a list here for reference.

- The plasma is infinite and homogeneous.
- The plasma obeys the conditions for ideal MHD so the ion cyclotron and plasma frequencies are much higher than the Alfvén wave frequencies.
- The plasma, initially, is composed of completely ionised hydrogen.
- The displacement current $\frac{1}{c^2}\frac{\partial E}{\partial t}$ which normally appears in Maxwell’s equations is negligible because of the timescale we are working on and to be

consistent with the fluid approximation.
- Plasma cannot interact with neutral gas.
- The frequency of the Alfvén wave is much less than the ion cyclotron frequency.
- Recombination is a phase change with a latent energy associated with it – the re-ionisation of a recombined region thus requires more than just the same thermal input through diffusion from surrounding plasma since energy is lost as radiation during recombination. The re-ionisation timescale is thus much longer than the wave timescale.
- Recombination happens instantaneously and results in the responsible wave losing energy as radiation at the frequencies of Balmer recombination lines.
- The Alfvén wave source is unaffected by changes in the plasma and oscillates at a constant frequency and amplitude.
- The degree of ionisation from the Saha equation is 1 when the temperature is greater than the critical temperature T_c.
- Loss of plasma results in that region of plasma not being able to return to the initial temperature – the resulting temperature depends on the degree of recombination.
- The interference effects of reflected waves are ignored.

4. Verification of Linear Mathematics[1]

The existence of waves in a plasma can be extrapolated from the analysis of the equations of ideal MHD **(1-4)**, considering an infinite, homogeneous, perfectly conducting plasma, using Maxwell's equations **(5-7)** to assist:

(1) $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

(2) $$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla \mathrm{p} + \boldsymbol{J} \times \boldsymbol{B}$$

(3) $$\frac{D(p\rho^{-\gamma})}{Dt} = 0$$

(4) $$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0 \ (Ohm's\ Law)$$

(5) $$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$ **(6)** $$\nabla \cdot \boldsymbol{B} = 0$$

(7) $$\frac{\partial B}{\partial t} = -(\nabla \times \boldsymbol{E})$$ **(8)** $$P = p + p_B$$

(9) $$p_B = \frac{\boldsymbol{B}^2}{2\mu_0}$$

ρ is the mass density (approximately the ion mass density), $\boldsymbol{u}$ is the fluid velocity vector with equilibrium value $\boldsymbol{u}_0 = 0$ (we are in a frame in which the plasma, pre-perturbation, is stationary), P is total pressure (thermodynamic plus magnetic pressure) and is a scalar, $\boldsymbol{J}$ is the current density vector, $\boldsymbol{B}$ is the total magnetic field vector (with equilibrium value $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{x}}$), $\boldsymbol{E}$ is the electric field vector, t is time, $\gamma = 5/3$ is the heat capacity ratio and μ_0 is the permeability of free space. $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$ is the advective derivative. All these variables are in their normal SI units.

This is a closed set – it is possible to linearise these equations by suggesting that each variable consists of an equilibrium constant, uniform in space and time, which experiences a small additive perturbation of the first order (the second order is negligible) shown in the list of equations **(10)**.

(10) $$\boldsymbol{B} = \boldsymbol{B_0} + \boldsymbol{b} \qquad \boldsymbol{u} = 0 + \boldsymbol{u}'$$
$$p = p_0 + p' \qquad \rho = \rho_0 + \rho'$$
$$\boldsymbol{E} = 0 + \boldsymbol{E}' \qquad \boldsymbol{J} = \boldsymbol{J}_0 + \boldsymbol{J}'$$

Equating the orders and assuming the perturbations follow a wave-like behaviour (proportional to $\exp[i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t]$) where **k** is the wave-vector in the direction of propagation $\boldsymbol{k} = k\hat{\boldsymbol{r}}$, $\boldsymbol{r}$ is the position vector and ω is the angular frequency of the wave. From here, a set of linearised MHD equations can be found. By equating orders and neglecting second orders, some variables can be eliminated, leaving a single complete equation called "equation 11".

For an incompressible flow, there can be no velocity in the direction of propagation – velocity can only be perturbed parallel to the wavefront if it is to uphold the fundamental property of incompressibility **(12).**

(12) $$\nabla \cdot \boldsymbol{u} = 0 \rightarrow \hat{\boldsymbol{r}} \cdot \boldsymbol{u} = 0$$

This property can be applied to equation 11 to find a dispersion solution for shear (transverse) Alfvén waves **(13)** and an expression for the Alfvén velocity c_A **(14).**

(13) $\frac{\omega^2}{k^2} = c_A^2 cos^2\vartheta$ **(14)** $c_A^2 = \frac{B_0^2}{\mu_0 \rho}$

ϑ is the angle between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{x}}$. For the purposes of this report, only $\vartheta = 0$ will be considered so the wave runs parallel to the equilibrium magnetic field.
The following argument reveals more about the fundamental properties of a shear Alfvén wave: identity **(15)** with **(5)** can be used to show **(16)**.

(15) $$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}$$

(16) $$\mu_0(\boldsymbol{J} \times \boldsymbol{B}) = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{b} - \nabla(\boldsymbol{B}_0 \cdot \boldsymbol{b})$$

Again, small second order perturbation terms have been neglected. Then, using the momentum equation **(2)** for ideal MHD with **(16)**, the following argument can be presented:

$$\rho \frac{D\boldsymbol{u}'}{Dt} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}$$

$$\rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} = -\nabla\left(p' + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0}\right) + \frac{(\boldsymbol{B}_0 \cdot \nabla)}{\mu_0}\boldsymbol{b}$$

Using the property of incompressibility **(12)** and Maxwell's equations, it is found that

$$\nabla^2(p' + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0}) = 0$$

Considering a region of the plasma which is unperturbed, it is known that p' and $\boldsymbol{B}'$ are both zero. Potential theory tells us that if the perturbations are harmonic then $(p' + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0})$ is zero everywhere, thus:

(17) $$\nabla P = 0$$

$$\therefore \rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} = \frac{(\boldsymbol{B}_0 \cdot \nabla)}{\mu_0}\boldsymbol{b}$$

It is also possible to construct an argument for $\frac{\partial \boldsymbol{b}}{\partial t}$ by combining Ohm's Law **(4)** with Maxwell's equations **(1-4)** and the identity **(15)** as well as the condition for incompressibility **(12)**:

$$\frac{\partial \boldsymbol{b}}{\partial t} = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u}$$

This relation suggests that $\boldsymbol{b}$ is parallel to $\boldsymbol{u}$. Incompressibility implies that $\boldsymbol{B}_0 \cdot \boldsymbol{u} = 0$ so it is then known that $\boldsymbol{B}_0 \cdot \boldsymbol{b} = 0$. Recalling the definition $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{x}}$ it is seen that:

(18) $$\frac{\partial^2 \boldsymbol{b}}{\partial t^2} = c_A^2 \frac{\partial^2 \boldsymbol{b}}{\partial x^2} \qquad \frac{\partial^2 \boldsymbol{u}'}{\partial t^2} = c_A^2 \frac{\partial^2 \boldsymbol{u}'}{\partial x^2}$$

These two wave equations **(18)** confirm the suggestion that these are transverse waves (analogous to waves on a string): an obvious solution for $\boldsymbol{b}$ (and the one which shall be used for the rest of the report) is given by **(19)**:

(19) $$\boldsymbol{b}(x,t) = b_0 \sin(kx - \omega t)\hat{\boldsymbol{b}}$$

It has been assumed that the wave is propagating along x. Note that a simple argument where $\boldsymbol{b} = \lambda\boldsymbol{u}$ clearly satisfies **(18)** (if λ is a scalar). *Figure 2* and *Figure 3* below show the varying direction of the magnetic field due to the small perturbation $\boldsymbol{b}$.

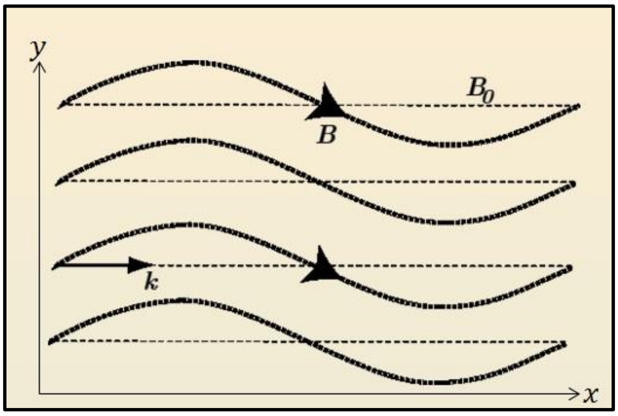

Figure 2

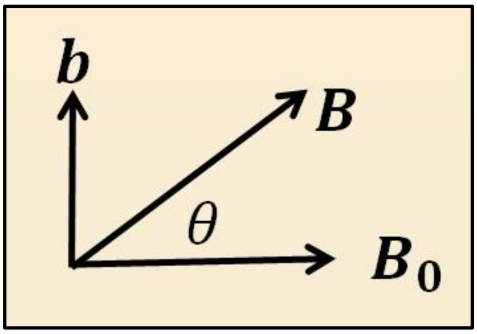

Figure 3

5. Non-Linear Argument and Implication[2]

5.1. Verification of the Non-Linear Mathematics

Here an approach similar to that of Sagdeev will be used. A list of assumptions is used for the non-linear argument:

- $\boldsymbol{b} = \lambda \boldsymbol{u}$.
- $\boldsymbol{B}_0 \cdot \boldsymbol{b} = 0$.

so $\boldsymbol{b}$ is transverse to the direction of the equilibrium magnetic field lines. $\boldsymbol{b}$ is also parallel and proportional to the perturbed velocity of the particles.

- Incompressible fluid $\nabla \cdot \boldsymbol{u} = 0$.
- Choose $\lambda^2 = \rho \mu_0$.
- $\nabla P = 0$ as it was in the linear case.
- Each equilibrium value experiences a perturbation but not necessarily small - thus second order perturbations are not negligible.

Simply using the assumptions with the identities **(15)** and **(16)**, along with equations **(1-9)**, it is simple to find the following:

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{\lambda^2}(\nabla \times \boldsymbol{b}) \times \boldsymbol{b}$$

$$(\nabla \times \boldsymbol{b}) \times \boldsymbol{B}_0 = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{b}$$

(20) $\quad \frac{\partial \boldsymbol{u}}{\partial t} = \frac{(\boldsymbol{B}_0 \cdot \nabla)}{\rho \mu_0}\boldsymbol{b} \qquad \frac{\partial \boldsymbol{b}}{\partial t} = (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{u}$

These results **(20)** give a wave solution for $\boldsymbol{b}$ as a function of position and time, which is exactly the same as was found in the linear case!

(19) $\quad \boldsymbol{b}(x,t) = b_0 \sin(kx - \omega t)\hat{\boldsymbol{b}}$

Interestingly, by considering a non-linear argument for non-limited perturbations, the result is a linearised solution.

5.2. Implied Temperature Variation

In the non-linear case, $\boldsymbol{b}^2$ is not assumed small and so is not neglected, unlike in the linear case. Recall $\boldsymbol{B}_0 \cdot \boldsymbol{b} = 0$ and consider the following, where **(17)** becomes important, and p_B is given by **(9)**.

$$P = p + p_B$$

$$P = p_0 + p' + \frac{B_0^2}{2\mu_0} + \frac{b^2}{2\mu_0} + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0}$$

$$\nabla P = 0 \rightarrow p' + \frac{b^2}{2\mu_0} = constant$$

If the constant is considered to be zero (since unperturbed regions will experience no change in total equilibrium pressure), then:

(21) $\quad p'(x,t) = \frac{-b^2(x,t)}{2\mu_0}$

Equation **(21)** differs from the linear case, where it was found that $p' = 0$. Clearly in the non-linear case, the thermodynamic pressure perturbation is now a function of position and time since the solution for $\boldsymbol{b}$ is. Also, the pressure perturbation is negative since b^2 is positive. If the plasma is considered to be an ideal gas then thermodynamics gives **(22)**

(22) $\quad p'(x,t) = n_0 k_B T'(x,t)$

where n_0 is the initial number density of ions and electrons per cubic metre, k_B is the Boltzmann constant and T' is the temperature displacement in Kelvin. Since n_0 is a constant, a change in pressure results in a change in temperature. The minus sign in **(21)** implies this temperature change will result in cooling the temperature of the plasma below its initial temperature T_0. Assuming the form for the temperature variation from equilibrium **(23)** and rearranging **(22)** gives **(24)**.

(23) $\quad T(x,t) = T_0 + T'(x,t)$

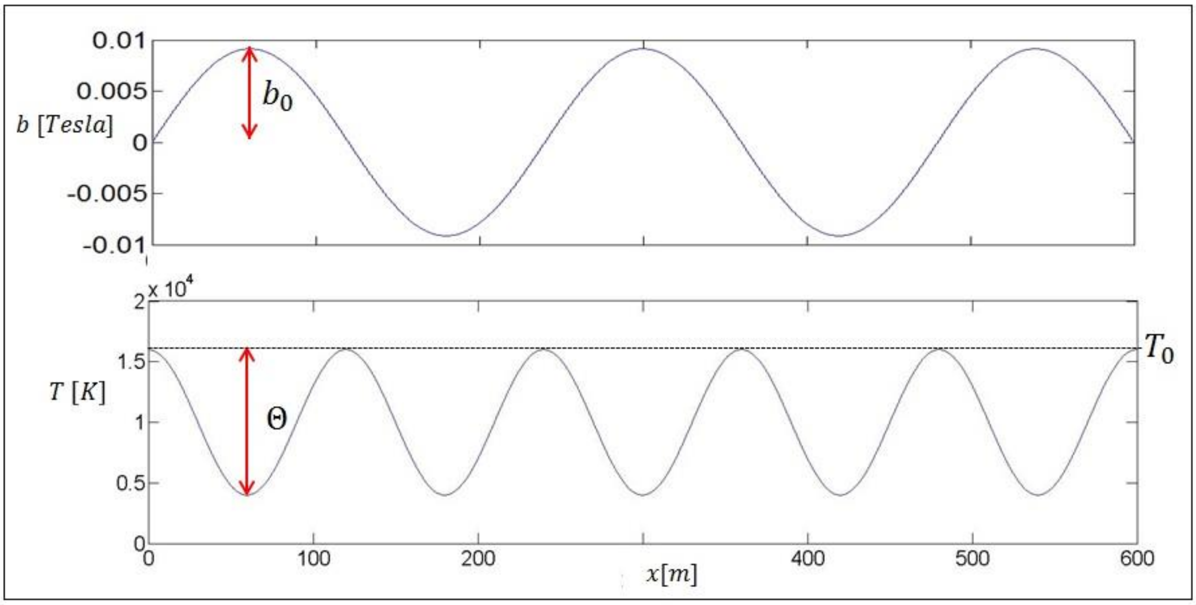

Figure 4

(24) $T(x,t) = T_0 - \frac{{b_0}^2}{2\mu_0 n_0 k_B}\sin^2(kx - \omega t)$

Here the sin wave solution for b **(19)** has been substituted. The amplitude of the Alfvén in terms of temperature can be collected into a constant Θ **(25)** which has units of Kelvin. The identity **(26)** for $\sin^2 x$ shows that the frequency of the temperature variation is twice the Alfvén wave frequency. *Figure 4* shows the temperature of the plasma **(27)** and the b component of the magnetic field **(19)** plotted against position x. Time dependency will be neglected from now on to simplify the case.

(25) $$\Theta \equiv \frac{b_0^2}{2\mu_0 n_0 k_B}$$

(26) $$sin^2 x = \frac{(1-\cos(2x))}{2}$$

(27) $$T(x) = T_0 - \frac{\Theta}{2}[1 - \cos(2kx)]$$

Intuitively, if Θ is large enough, the plasma can be cooled to such a degree that recombination occurs. This is discussed further in section 6.

5.3. Equipartition of energy

It was assumed in the non-linear case that $\boldsymbol{b} = \lambda\boldsymbol{u}$ and $\lambda^2 = \rho\mu_0$. It is simple then to see that energy is equipartitioned **(28)**:

(28) $$\frac{\rho u^2}{2} = \frac{b^2}{2\mu_0}$$

Clearly the left hand side of **(28)** is the kinetic energy density and the right hand side is the magnetic energy density – it is then clear that magnetic and kinetic energy are equipartitioned, so when an ion moves a magnetic energy is induced. This magnetic energy provides a restoring force to the ion in an attempt to bring it back to equilibrium.

6. Expressing the Degree of Recombination

In this section, the principles used to find an expression for the degree of recombination will be discussed. Section 7 investigates this expression further.

6.1. The Saha Ionisation Equation[6][7].

The Saha ionisation equation **(29)** relates the ratio of number density of particles in different states to the temperature when the medium considered is in thermal and chemical equilibrium.

$$\frac{n_{i+1}}{n_i} = \frac{2G_{i+1}}{n_e G_i}\left(\frac{2\pi m_e k_B}{h^2}\right)^{3/2} T^{3/2} e^{-\chi_0/k_B T}$$
(29)

Here n_i is the number density of atoms in state i. n_e is the electron number density (the same as the ionised hydrogen number density),

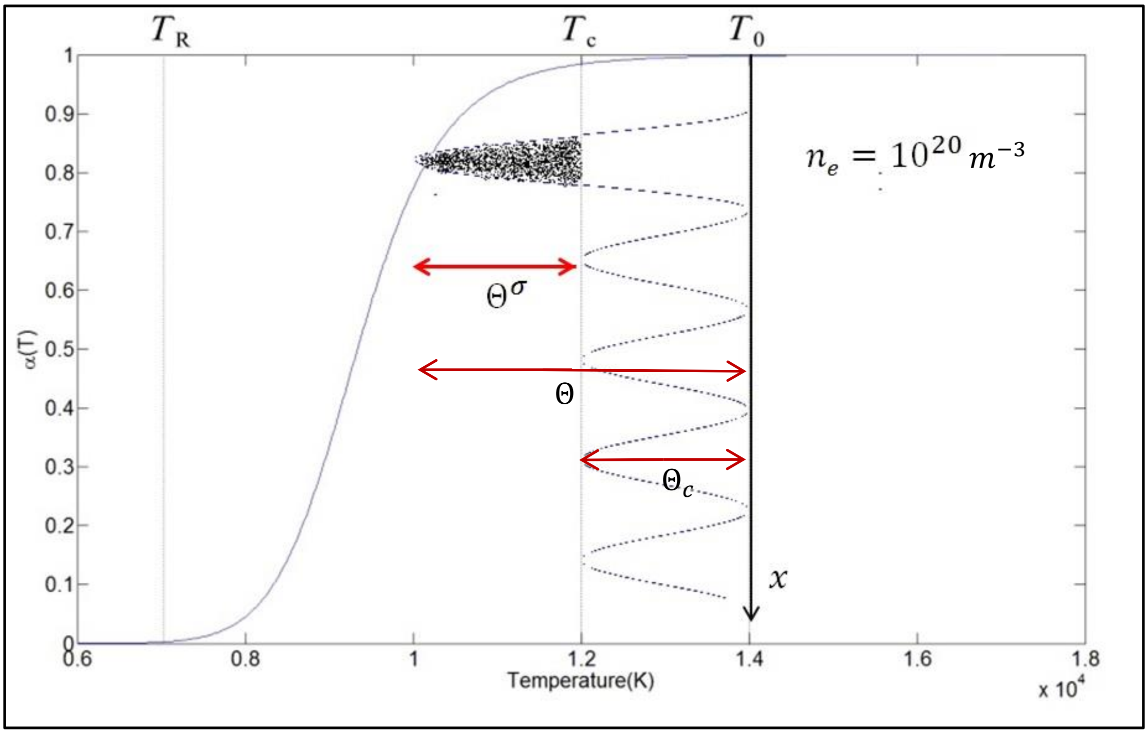

Figure 5

m_e is the mass of an electron, h is Planck's constant, G_i is the partition function for state i and χ_0 is the ionisation energy.

If we are assuming our plasma is composed of hydrogen then state 1 is neutral hydrogen and state 2 is ionised hydrogen. If we assume that all neutral hydrogen is in the ground state then the partition functions are $G_1 \sim 2$ and $G_2 = 1$. For ground state Hydrogen $\chi_0 = 13.6eV$.

The degree of ionisation α **(30)** is the ratio of ionised hydrogen to total hydrogen, which depends on the temperature.

(30) $$\alpha(T) = \frac{n_2}{n_2 + n_1}$$

Using **(29)** and **(30)** it is possible to plot the degree of ionisation α as a function of temperature for $n_e = 10^{20} m^{-3}$ as shown in *Figure 5.*

6.2. Applying the Saha Ionisation Equation.

The goal now is to use the Saha equation **(29)** to our advantage by applying it to the problem at hand. The aim is to find an expression for the degree of ionisation with position.

Note that alpha tends to 1 but never actually is 1. In order to simplify the case, alpha will be rounded to be exactly 1 above the critical temperature T_c. There is a degree of abstraction in T_c – without a rigorous definition and for the purposes of thinking about the problem, T_c is defined vaguely as the temperature below which interesting things (i.e. recombination) start to happen. Therefore T_c has been chosen at the point where the gradient of the graph (*Figure 5*) starts to vary more dramatically.

It was assumed that the plasma is initially completely ionised so the initial degree of ionisation $\alpha_0 = 1$. It follows then that the initial temperature $T_0 > T_c$. The case has also been simplified such that $\alpha = 0$ below the recombination temperature T_R. If the temperature is below T_R there is no plasma, only neutral gas.

There are three potential cases to consider. In order to discuss these cases, some definitions must be made **(31-34)**.

(31) $\Theta_c \equiv T_0 - T_c$ **(32)** $\Theta_R \equiv T_0 - T_R$

(33) $\Theta^{\sigma} \equiv \Theta - \Theta_c$ **(34)** $|t|_R \equiv \frac{\Theta_c}{\Theta}$

The first potential case is a temperature variation which begins at T_0 with an amplitude $\Theta < \Theta_c$, the critical amplitude **(31)**. In this case the temperature never falls below T_c so the plasma remains completely ionized and the wave propagates along the x-axis indefinitely as it experiences no damping.

The second potential case is a temperature variation which begins at T_0 with an amplitude $\Theta \geq \Theta_R$, the recombination amplitude **(32)**. In this case the temperature falls below T_R , resulting in complete loss of the plasma as well as instant death of the wave. This is discussed further in section 7.

The third potential case – and the one which we are interested in – is a temperature variation which begins at T_0 with an amplitude $\Theta \geq \Theta_c$ and $\Theta < \Theta_R$. In this case the wave brings the temperature into the region where recombination can occur – so some of the plasma recombines, indicated by the shaded region in *Figure 5*. The degree of recombination depends on how far below T_c the temperature goes, so we define Θ^{σ} **(33)**. This case is represented by the overlay on *Figure 5* of the temperature variation associated with an Alfvén wave. The wave propagates down the x axis with an amplitude of Θ.

Notice that once the wave propagates past the shaded region on the x-axis, the rest of the wave is of amplitude Θ_c rather than Θ. This is because energy is lost as radiation during recombination (assuming the plasma is optically thin). To simplify the case, the energy lost during recombination is quantified by the recombination transmission coefficient $|t|_R$ **(34)**, which clips the amplitude at Θ_c for positions of x after the wave has propagated half a phase cycle. This means no more plasma is recombined, and the wave loses no more energy.

6.3. Expressing the Degree of Ionisation as a Function of Position.

One could think that expressing α as a function of position would simply be a matter of substituting the derived expression for $T(x)$ **(27)** into the Saha ionisation equation **(29)** which would give $\alpha\big(T(x)\big) = \alpha(x)$ then $\rho(x) = \alpha(x)\rho_0$. However, Saha is an equilibrium equation – the problem being considered is evolving on timescales too fast for equilibrium to take place. Saha is still important though, because T_c is still the temperature above which no recombination will occur.

Consider the recombination as a phase change. Take melting ice as an analogy – as one heats up the ice, the temperature reaches the melting point. The temperature remains at the melting point until all the ice has melted into water and only then may the temperature increase further. Analogously, the plasma cools to the critical temperature at which it remains while the plasma recombines until all the plasma is lost.

To take this into account, the pressure function is now split into two functions as shown in *Figure 6*. The function $[p'(x)]^*$ labelled with $*$ is the original pressure variation function. $*$ is a condition which states: temperature is a function of x if $T > T_c$. σ is a region where if the original temperature variation **(27)** would have been $T \leq T_c$, the temperature for the function labelled by σ is instead $T = T_c$. In the region of the σ condition, the function $[p'(x)]^*$ is locked at its value for when $T = T_c$. The function $[p''(x)]$ labelled with σ is the phase change function at constant $T = T_c$. In the region of the $*$ condition the function $[p''(x)] = 0$ as there is no phase change occuring. *Figure 6* shows the application of the conditions $*$ and σ on the temperature variation and the region σ is labelled. The red line is at $T = T_c$. Recall also that after half a phase cycle the wave amplitude is clipped at Θ_c.

So in the region $*$ the total pressure variation $p'(x)$ would be given by **(35)**.

(35) $$[p'(x)] = [p'(x)]^*$$

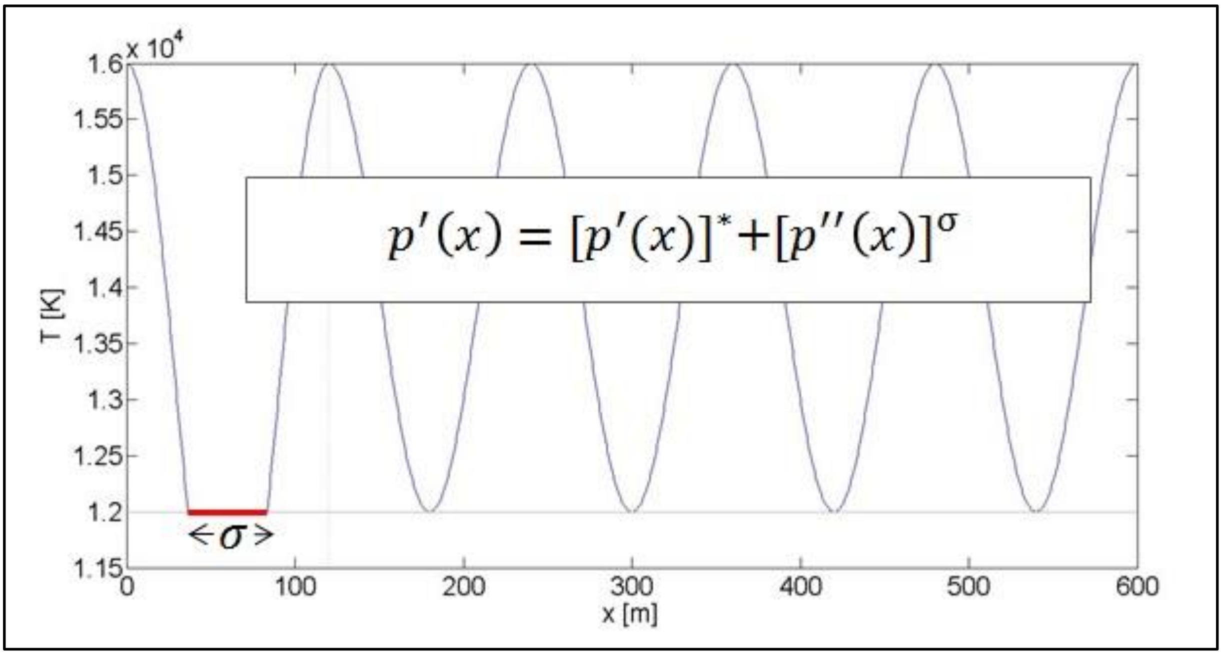

Figure 6

In the region σ the total pressure variation $p'(x)$ is given by **(36).**

(36) $$[p'(x)] = [p'(x)]^{*} + [p''(x)]^{\sigma}$$

Considering an ideal gas **(22)** and substituting the expressions for the functions in region σ and using $\rho = m_H n_0$ gives expression **(37)**.

(37) $$-[m_H n_0 k_B T'(x)]^{\sigma} = [m_H n_0 k_B \Theta_c]^{*} + [m_H n_0'(x) k_B T_c]^{\sigma}$$

The minus sign on the left hand side of **(37)** is important because $T'(x)$ is negative. Since $[p'(x)]^{*}$ is locked (constant) in region σ and temperature is constant for $p''(x)$ then clearly the only variable on the right hand side which can balance out the total pressure variation is the number density in $p''(x)$ and so it is found that the number density of the ions and electrons must change by $n_0'(x)$. The assumption of incompressibility could at this point seem to forbid this – excess plasma cannot spill into a different region. However, incompressibility is still conserved because the decrease in number density of ions (and electrons) is made up for in the increase in number density of neutral hydrogen atoms. There is now a mixture of plasma and neutral gas, though the plasma will still be treated as an MHD fluid as though the neutral gas were not there.

By rearranging **(37)** to find **(38)** then substituting the earlier expression for $T'(x)$ **(22)** in terms of the solution for $b(x)$ **(19)**, one obtains **(39)**, an expression for the change in number density of electrons and ions assuming condition σ:

(38) $$n_0'(x) = \frac{n_0}{T_c}[-T'(x) - \Theta_C]$$

(39) $$n_0'(x) = \frac{n_0}{T_c}\left[\frac{\Theta}{2}(1 - \cos 2kx) - \Theta_C\right]$$

Equation **(39)** appears to be normalised appropriately – the change in number density is proportional to how far below the critical temperature the plasma cools. Thus the expression for the new number density n_{new} is given by **(40)**.

(40) $$n_{new}(x) = n_0 - {n_0}'(x)$$

Obviously then the degree of ionisation is given by **(41).**

(41) $$\alpha(x) = \frac{n_{new}(x)}{n_0}$$

Therefore, by considering a phase change at the critical temperature T_c defined by using the Saha ionisation equation, an expression for the degree of ionisation with position has been found.

7. Complete Recombination – Expression of Maximum Amplitude

An Alfvén wave requires plasma to propagate. It is thus appropriate to define the condition for the maximum amplitude of an Alfvén wave to be that which completely recombines the plasma. Recall that the amplitude of the temperature variation Θ is proportional to the square of the amplitude of the Alfvén wave b_0 so if the maximum limit for Θ is found, the maximum for b_0 can be found.

The condition given by **(42)** for complete recombination is simple – if the minimum of $n_{new} = 0$ then no ions or free electrons are left and thus the plasma is completely lost.

(42) $$min[n_{new}(x)] = 0$$

The expression for the minimum of $n_{new}(x)$ is found by maximizing the $[1 - \cos(2kx)]$ term in **(38)** so the condition becomes **(43)**.

(43) $$n_0 - \frac{n_0}{T_c}(\Theta_{max} - \Theta_c) = 0$$

Some simple algebra and substitution of the definition for Θ_c **(31)** yields an expression for the maximum amplitude Θ_{max} **(44)**:

(44) $$\Theta_{max} = T_0$$

Substituting the definition of Θ **(25)** defined earlier, the expression for the characteristic maximum amplitude of an Alfvén wave, b_{0max} is given by **(45)**.

(45) $$b_{0max} = \sqrt{T_0(2\mu_0 n_0 k_B)}$$

In *Figure 3* there is an angle θ between $\boldsymbol{B_0}$ and $\boldsymbol{B}$. θ is the angle of deviation of the direction of the magnetic field from equilibrium. The limit on the amplitude of b_{0max} implies there is also a limit on the angle of deviation. The expression for the characteristic maximum angle of deviation of the magnetic field from equilibrium θ_{max} is given by **(46)**.

(46) $$\theta_{max} = tan^{-1}(\frac{b_{0max}}{B_0})$$

An observer should not see an Alfvén wave which induces an angular deviation of greater than θ_{max} because a wave of that amplitude would completely recombine the plasma, thus leaving no plasma for the wave to propagate in. Comparing the amplitudes/angles of Alfvén waves in research articles for specific plasma properties with the expressions derived for b_{0max} **(45)** and θ_{max} **(46)** could confirm whether the expressions are correct; if an amplitude/angle presented by a research article was greater than the derived maximum in this report then the derived expressions for the characteristic maximums would be put into question.

It was mentioned that a wave with a maximum amplitude recombines the plasma completely, leaving no plasma for the wave to propagate in. In this case the wave at the point of complete recombination dies instantly and the rest of the wave attempting to propagate through that point dies also. However, the leading phase of the wave which passed through the region in the time before the region recombined would still propagate. In principle there would be visible a one-time event where the magnetic field changes direction briefly as the leading phase of the Alfvén wave passes the line-of-sight of the observer. At this one-time event θ would increase up to θ_{max} and then decrease back to zero where it would remain.

8. Partial Recombination – General Method for Quantifying Recombination and Transmission

The case where complete recombination occurs is simple to consider. The case where the amplitude of the Alfvén wave is less than the maximum, so only partially recombines the plasma, is much more complicated; the leading half-wavelength of the Alfvén wave changes the plasma density so the next half-wavelength of the wave experiences a continuously varying plasma density. This means the velocity and wavelength of the next half-wavelength of Alfvén wave varies with position since it depends on density. Also, when a wave on a string experiences a change in impedance (a change in density and velocity) then transmission and reflection occurs. The reflections and transmissions result in interference and the varying velocity and wavelength due to varying density make the new plasma properties vary in such a way that they are no longer symmetrical – the complexity of finding the wavefunction for each wavelength and the effect it has on the plasma increases and the problem becomes very difficult, very fast. This section describes ways to simplify the problem and find the transmission coefficients for each part of the wave. The mathematics for this method are treated rigorously in section 9. Also mentioned are W.K.B.J. methods which could be useful if it were desired to restore some of the complexity of the problem.

The aim is to create an evolving model which can quantify the effect an Alfvén wave has on the plasma and the feedback effect the plasma has on the wave.

8.1. Iterative Pulses.

In order to quantify the effect an Alfvén wave would have on a plasma and the resulting effect it would have on the next part of the wave, it is necessary to think in iterative steps: dividing the Alfvén wave into half-wavelengths called 'pulses' means one pulse can be released at a time. The effects of that pulse on the plasma can be quantified. The effects of the new plasma properties on the wave-function of the next pulse, as well as the transmission effects on the amplitude of the pulse, are considered. The next pulse is then released into the modified plasma. This process can be repeated as many times as possible.

Since the temperature variation happens at twice the Alfvén wave frequency, it makes sense to define a pulse as half an Alfvén wavelength so a pulse has length ${}^{1}\lambda_0/2$ in an undisturbed plasma.

Intuitively, one could hypothesise that each pulse would ionise more plasma than the pulse before, resulting in each pulse having a smaller and smaller transmission coefficient until virtually no pulse is transmitted at all.

8.2. Notation

Before continuing, it is appropriate to detail the notation which was developed in order to undertake the problem. Understanding of the notation and the reasons for such notation will become clear in the following sections.

l is the pulse number. A plasma in state $l = 0$ means no pulse has been released and the plasma is undisturbed. $l = 1$ is the state of the plasma after one pulse has passed through it.

From now on, n denotes the region rather than the number density. $n = 0$ is the region of plasma which still possesses the initial undisturbed properties. Once the position of a region has been defined it does not change (apart from $n = 0$). The region $n = 1$ is defined as the length of the first pulse. Region n is at a new temperature ${}^{l}T_n$ after pulse l has passed. This means the critical amplitude in that region changes so for pulse l in region ${}^{l}\Theta_{cn} = {}^{l-1}T_n - T_c$.

New regions may be defined if the region $n = 0$ is disturbed. In this case the length of $n = 0$ is re-defined.

s denotes the section – sections are superseded by regions and denote positions in the plasma where the density of the plasma differs from other sections. Section s in region n is at a density ${}^{l}\rho_{ns}$ after pulse l has passed. The defined lengths and locations of sections can vary for different numbers of l, and new sections can appear.

Pulse l, propagating through region n, section s, has a wave-function ${}^{l}b_{ns}$. The properties of the wave are given by ${}^{l}c_{ns}$, ${}^{l}\lambda_{ns}$, ${}^{l}k_{ns}$ and ${}^{l}\Theta_{cn}$ where c has replaced c_A as the Alfvén velocity to keep things tidy. So for example, ${}^{2}b_{1c}$ is the wave-function and ${}^{2}\lambda_{1c}$ is the wavelength of the second pulse propagating through region 1, section c.

Pulse l propagates through a plasma with properties ${}^{l-1}\rho_{ns}$ and ${}^{l-1}T_n$ in region n, section s.

Sections meet each other at the critical points where the temperature function reaches T_c – this happens at two points along x for each pulse, the first and second denoted by ${}^{l}x_{c0}$ and ${}^{l}x_{c1}$ for pulse l.

Generally, section δ sits on the x-axis, adjacent and to the left of section ς.

When s is used neither as a subscript or superscript then s denotes the length of the section implied. The same goes for n.

8.3. Undisturbed Plasma ($l = 0$)

Figure 7 represents the ion mass density $\rho(x)$ against position x for the undisturbed plasma (so $l = 0$). The plasma is at a uniform density and temperature ${}^{0}\rho_0$ and ${}^{0}T_0$. The plasma properties are known, so the wave-function for the first pulse can be calculated. The first pulse is then released.

8.4. Disturbed Plasma ($l = 1$)

Figure 8a shows the state of the plasma after a pulse with an amplitude big enough to partially ionize the plasma has been released. The goal is to quantify what the pulse has done to the plasma so the wave-function for the next pulse can be calculated. The pulse has dug out a region of ions where the temperature reached T_c so the density in that section is now a continuous function of position whilst the rest of the plasma remains undisturbed. The region $n = 1$ has been defined as the length of the first pulse from the origin. In order to keep things simple (see the section on W.K.B.J. methods for more on this) the density function is averaged over that disturbed section and turned into a top hat function in *Figure 8b*.

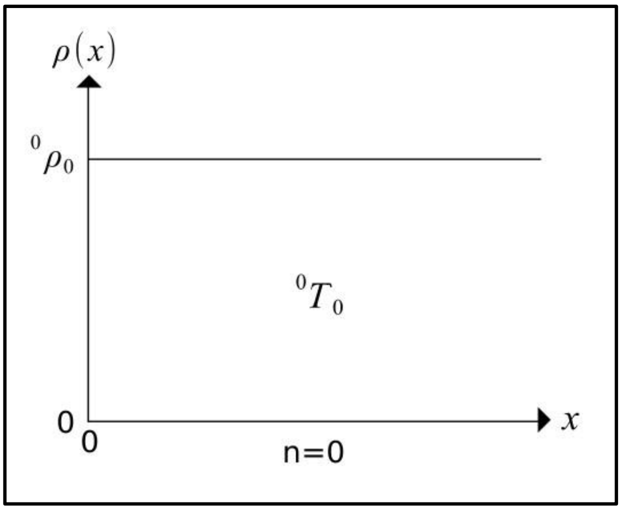

Figure 7

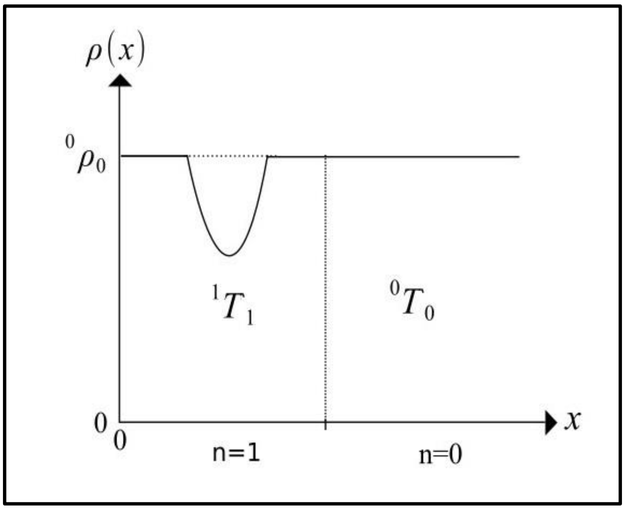

Figure 8a

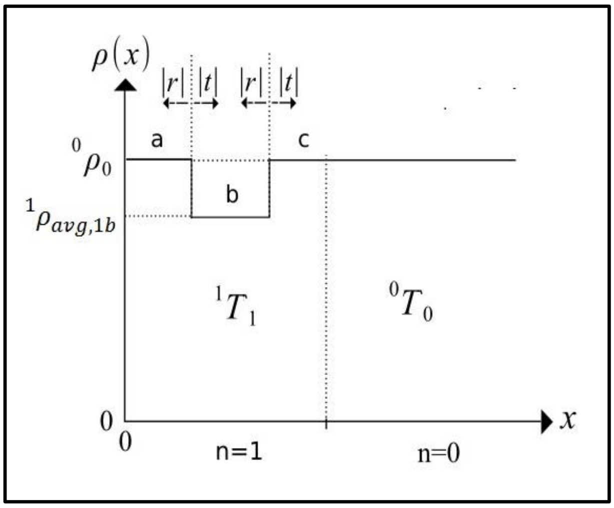

Figure 8b

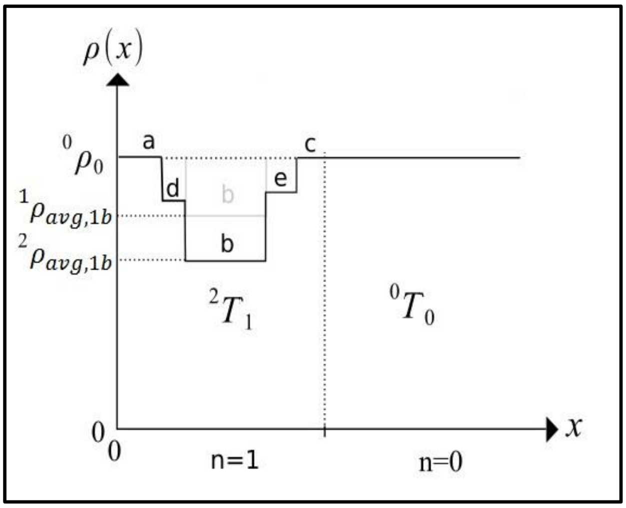

Figure 9

In *Figure 8b* region 1 has been divided into three sections a, b and c. Section b is the affected zone of the plasma – the zone over which the continuous density function has been averaged and is now a constant ${}^{1}\rho_{avg,1b}$ **(65)**. Section a and section c are undisturbed as the pulse did not bring the temperature below T_c. At the boundary between $n = 1$ and $n = 0$ the amplitude of the wave is clipped by $|t|_R$ **(34)** as described in section 6.2.

The loss of ions during the passing of the wave results in a loss of magnetic energy – normally this magnetic energy would be used to bounce the pressure and temperature back up to their equilibrium values, but if some of the ions are lost then the pressure and temperature cannot possibly reach their initial values again. The new average temperature ${}^{1}T_1$ across the whole of region 1 is calculated by weighting the degree of ionization **(78)**, as detailed in section 9.6.

Transmission and reflection coefficients $|t|$ and $|r|$ are labelled on *Figure 8b.* Since impedance **(69)** (discussed in section 9.4) depends on the linear density and wave velocity the wave encounters a change in impedance at the boundaries between sections. This results in partial reflection and transmission of the pulse – reflected components could interfere with other oncoming pulses which could either result in destructive or constructive interference. However, in order to simplify the case the interference effect of reflections will be neglected, but the transmission coefficients are still of interest – how much of each pulse gets through and how does it depend on the amplitude of the wave Θ and the initial temperature of the plasma ${}^{0}T_0$?

Once the plasma properties for $l = 1$ are known (see section 9 for details), it is possible to calculate the wave-function for the second pulse in each section .The second pulse is then released.

8.5. Disturbed Plasma ($l = 2$)

Figure 9 shows the state of the plasma after the second pulse has passed.

Sections a and c are still undisturbed, however their lengths have changed due to the addition of sections d and e, which appear because the initial temperature ${}^{1}T_1$ of the region is lower than ${}^{0}T_0$ so the wave hits the critical temperature at different points on the x-axis than it did before.

Section b remains the same length, however it is now at a lower average density than $l = 1$ since it has been attacked by 2 pulses.

Sections d and e are introduced, and are of different depths (in terms of density) and lengths. There are multiple reasons for this. The first reason is that the amplitude of the wave in section d is more than e due to the transmission coefficient. The second reason is that the wave now starts at ${}^{1}T_1$ so not only does the wave cut the critical temperature earlier on the x-axis than for ${}^{0}T_0$ but it also cuts deeper into the plasma. The third reason is that the wave has a greater wavelength in section b because the frequency of the wave is assumed constant but the density of plasma decreases. This means the wave is not symmetric and so the second critical point is shifted by a phase related to the degree of ionisation in section b. The wave-function of the second pulse thus needs to take this phase into account in order for the wave to be continuous in terms of amplitude, as shown in **(73,77)**.

In an extreme case, the wavelength of a pulse could increase enough that some of the plasma in region $n = 0$ is disturbed, in which case a new region $n = 2$ and new sections for that new region would be created. Properties in this new region would be calculated in the same fashion as in region $n = 1$.

All the new density functions in each section are averaged and a new temperature is assigned to the region (or regions) independently of the temperature in other regions. The new temperature depends on the initial temperature of the region and the amount of ionisation which occurred during the pulse passing **(78)**.

Now it is possible to calculate the wave-functions for pulse 3. Pulse 3 is then released.

<u>Note:</u> At this point the iterative method should be clear – if it were desired to take this

further it would be worth writing a computer code which could automatically create new sections and regions and manipulate them as required. It is possible to create and manipulate sections manually but the amount of work required each iteration increases because new sections are added each time.

8.6. Disturbed Plasma ($l = 3$)

For the release of the third pulse the focus will be on quantifying the transmission coefficients – the effect of the pulse on the plasma will not be quantified, though obviously it would be if it were desired to keep releasing pulses.

There are, in *Figure 9*, 4 changes in impedance for the third pulse. The impedance in each region is known (because the density and thus velocity are known) so it is quite simple to calculate the transmission coefficients for the wave.

8.7. W.K.B.J. Methods[9]

W.K.B.J. methods are mentioned as a more complex (and therefore accurate) alternative to the simplified version which is treated rigorously in the following sections. In *Figure 8a* the density function is a continuous function of position and so the Alfvén velocity c_A is a function of position. The wave equation **(20)** then becomes **(47)**.

(47) $$\frac{\partial \boldsymbol{b}}{\partial t} = c_A\,(x)\frac{\partial \boldsymbol{b}}{\partial x}$$

Normally c_A is constant, but the new position dependence of c_A (due to the position dependence of density and the relation **(14)**) makes the solution for $\boldsymbol{b}$ more difficult to solve. The continuous density function also means the reflection/transmission coefficients are not as simple as using **(69)** for a step change in impedance. One runs into difficulty when trying to find the transmission coefficients as two singularities are encountered – the W.K.B.J. methods provide a means by which these singularities can be avoided. The methods use the Stokes Phenomenon to solve the problem in the complex plane, taking a detour around the singularities. If one were to describe the problem in the manner of W.K.B.J. methods without averaging the density function, one could achieve more accurate results. However, the general means by which to calculate the effects of each pulse would become more and more complicated.

<u>Important Note:</u> Some variables have been used here which may have been used in other sections – the reason for this is partly to preserve the Heading notation and partly to do with the fact that there are not many letters or symbols left to use. From here until the end of section 8.7, any variables encountered should be considered explicitly self-consistent within this section and revert back to their original definitions out with section 8.7.

If one wished to proceed with the W.K.B.J. case they should consult the section in the book[9] entitled "Two Transition Points", case (i).

The general W.K.B.J. wave solutions to such a problem as **(47)** are given by **(48)**

(48) $$\boldsymbol{\varphi} = \frac{\mathbf{1}}{q^{1/4}}\exp(\pm ih\int q^{\frac{1}{2}}dx)$$

If one were to attempt to calculate the transmission/reflection coefficients then, as mentioned, two singularities would be encountered – all the hard work to avoid this has been done by W.K.B.J. so first of all one needs to convert **(47)** into the same form as Heading **(49)** using substitution.

(49) $$\frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} = -h^2 q(x)\boldsymbol{\varphi}$$

Obtaining this form is quite simple – differentiating **(47)** with respect to t gives **(50)**.

(50) $$\frac{\partial^2 \boldsymbol{b}}{\partial t^2} = c_A\,(x)\frac{\partial^2 \boldsymbol{b}}{\partial x \partial t}$$

$$\frac{\partial^2 \boldsymbol{b}}{\partial t^2} = c_A\,(x)\frac{\partial}{\partial x}(c_A\,(x)\frac{\partial \boldsymbol{b}}{\partial x})$$

Then, if variables differentiated by x are denoted by a dash e.g. $c_A\,'(x)$ and $\frac{\partial}{\partial t} \to -i\omega$ (here $\boldsymbol{b}$ is considered complex) then **(51)**.

(51) $$\boldsymbol{b}'' + \frac{c_A{}'(x)}{c_A(x)}\boldsymbol{b}' + \frac{\omega^2}{c_A^2(x)}\boldsymbol{b} = 0$$

(51) can be compared to the form of **(52)** – a substitution **(53)** is suggested for **(52)**. By extrapolating the coefficients from the simplified result **(54)**, one can obtain the substitution for $\boldsymbol{b}$ as $\boldsymbol{\varphi}$ **(55, 56)** in the same way as for u as y.

(52) $$u'' + \alpha u' + \beta u = 0$$

(53) $$y = uexp(0.5 \int \alpha \, dx)$$

(54) $$y'' = \left(\frac{\alpha^2}{4} - \beta\right) y$$

(55) $$\boldsymbol{\varphi}'' = \left(\frac{{c_A}'^2(x)}{4{c_A}^2(x)} - \frac{\omega^2}{c_A^2(x)}\right)\boldsymbol{\varphi}$$

(56) $$\boldsymbol{\varphi} = \boldsymbol{b}exp(0.5 \int \frac{c_A'(x)}{c_A(x)} dx)$$

The wavefunction for $\boldsymbol{b}$ is then found from **(56)** using the general solution for $\boldsymbol{\varphi}$ given by **(48)**. Comparing **(55)** to **(49)**, it is clear that $h = 1$ and:

$$q(x) = \left(\frac{{c_A}'^2(x)}{4{c_A}^2(x)} - \frac{\omega^2}{c_A^2(x)}\right)$$

The transmission coefficient $|t|$ is found by integrating over the limits at the two points of transition, a and $-a$, as defined by Heading **(57).**

(57) $$|t| = [-a, a] = \exp(ih \int_{-a}^{a} q^{1/2} dx)$$

The potential depth and accuracy which could be achieved through looking at the problem using W.K.B.J. methods is matched by the potential complexity the problem quickly develops in to. A code could potentially model the problem and find transmission coefficients for many pulses but it is beyond the scope of this project.

9. Partial Recombination – Rigorous General Expressions for Quantifying Recombination and Transmission

In this section the mathematics for quantifying properties in the model will be expressed rigorously. All one need do for a plasma with particular properties would be to plug the numbers in. A specific example for specific properties is shown in section 10, the results of which are found from a Matlab code in the Appendix which uses the expressions in this section to find the transmission coefficients and degree of ionization for the first three pulses. Refer to section 8.2 on notation for assistance.

9.1. Initial Expressions for $l = 0$ and First Pulse Wave-Function

The plasma is at uniform temperature ${}^{0}T_0$ and uniform density ${}^{0}\rho_0$. All pulses have an amplitude of Θ and frequency ν. The wave-function of the first pulse is **(58)** so the temperature function is given by **(59)** and the initial velocity is given by **(60)**

(58) $${}^{1}b_0(x) = b_0 \sin({}^{1}k_0 x)$$

(59) $${}^{1}T_1(x) = {}^{0}T_0 - \Theta sin^2 \, {}^{1}k_0 x$$

(60) $${}^{1}c_0^2 = \frac{B_0^2}{2\mu_0 \, {}^{0}\rho_0}.$$

9.2. General, Rigorous Expressions of Sections

The region $n = 1$ is defined from $x = 0$ to $x = {}^{1}\lambda_0/2$. A pulse is clipped once it has propagated a have a phase cycle for that particular pulse. It is necessary to calculate the sections a, b and c for $l = 1$, sections a, b, c, d and e for $l = 2$, sections a, b, c, d, e, f and g for $l = 3$ etc . To find these sections, the critical points ${}^{l}x_{c0}$ and ${}^{l}x_{c1}$ where ${}^{l}T_n(x) = T_c$ must be found. Some algebra reveals the general expression for the first critical point for pulse l **(61).**

(61) $${}^{l}x_{c0} = \frac{1}{{}^{1}k_0} sin^{-1}\left(\sqrt{\frac{{}^{l}\Theta_{cn}}{\Theta}}\right)$$

Due to symmetry, the second critical point can be found for $l = 1$ **(62)**.

(62) $${}^{1}x_{c1} = {}^{1}\lambda_0/2 - {}^{1}x_{c0}$$

The sections for $l = 1$ are defined as follows:

$a\colon x = [0,\ {}^{1}x_{c0}]$
$b\colon x = [\ {}^{1}x_{c0},\ {}^{1}x_{c1}]$
$c\colon x = [\ {}^{1}x_{c1},\ {}^{1}\lambda_0/2]$

The sections for $l = 2$ are defined as follows:

$a\colon\ x = [0,\ {}^{2}x_{c0}]$ since ${}^{2}x_{c0} < {}^{1}x_{c0}$
$b\colon\ x = [\ {}^{1}x_{c0},\ {}^{1}x_{c1}]$
$c\colon\ x = [\ {}^{2}x_{c1},\ {}^{1}\lambda_0/2]$ since ${}^{2}x_{c1} > {}^{1}x_{c1}$
$d\colon\ x = [\ {}^{2}x_{c0},\ {}^{1}x_{c0}]$
$e\colon\ x = [\ {}^{1}x_{c1},\ {}^{2}x_{c1}]$

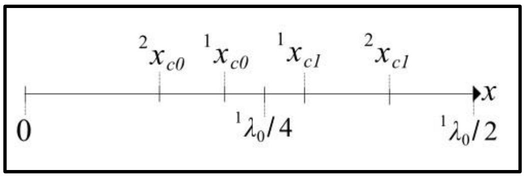

Figure 10

Refer to section 9.7 for the general expression of critical point ${}^{l}x_{c1}$ for $l > 1$ **(79)**, as it cannot be derived until the expressions for density, impedance and transmission have been rigorously defined. *Figure 10* shows generally what the order and scaling of critical points could look like.

9.3. General, Rigorous Expressions of Density

The section definitions are used in the calculation of the average density for a section to turn the continuous density function ${}^{l}\rho_{ns}(x)$ into the constant average density for the section. Using the expression derived for n_{new} **(39, 40)** with $\rho(x) = m_H n_{new}(x)$, the general, rigorous expression for density is given by **(63)** The temperature variation function is given by **(64)**.

$$ {}^{l}\rho_{ns}(x) = m_H {}^{l-1}n_{ns}\left(1 - \frac{[-\ {}^{l}T'_{ns}(x) -\ {}^{l}\Theta_{cn}]}{T_c}\right) \quad \textbf{(63)}$$

$$\textbf{(64)} \qquad {}^{l}T'_{ns}(x) = -\frac{{}^{l}b_{ns}^{2}(x)}{2\mu_0 n_0 k_0}$$

This gives the average ${}^{l}\rho_{avg,ns}$ for that section **(65)**:

$$\textbf{(65)} \qquad {}^{l}\rho_{avg,ns} = m_H {}^{l-1}n_{ns}\left\{1 - \frac{1}{T_c s}\int_{section} [-\ {}^{l}T'_{ns}(x) -\ {}^{l}\Theta_{cn}]dx\right\}$$

The degree of ionisation in a section due to the passing of pulse l is then given by **(66)**.

$$\textbf{(66)} \qquad {}^{l}\alpha_{ns} = \frac{{}^{l}\rho_{avg,ns}}{{}^{l-1}\rho_{avg,ns}}$$

where the cumulative degree of ionisation κ due to all the pulses is given by **(67)**.

$$\textbf{(67)} \qquad {}^{l}\kappa_{ns} = \prod_{l=0}^{l} {}^{l}\alpha_{ns}$$

9.4. General, Rigorous Expressions of Impedance and Transmission

The classic expression for the transmission at a change in impedance is given by[8] **(68)**:

$$\textbf{(68)} \qquad Z = \mu c$$

where μ is the linear density of the string and c is the wave velocity. In the case considered, the linear density is the cube root of the volumetric density ρ and the velocity is the Alfvén velocity c_A. It is also known that ${}^{l}\rho_{avg,ns} = {}^{l}\kappa_{ns}\ {}^{0}\rho_0$ from **(66)** and **(67)**. From **(14)**, it is seen that ${}^{l+1}c_{ns} \propto {}^{l}\rho_{avg,ns}^{-0.5}$ (since pulse l propagates through a plasma with density left by pulse $[l-1]$) it is seen that the impedance in a section after the passing of pulse l is given by **(69)** and is inversely proportional to the sixth route of the cumulative degree of ionisation **(70)**.

$$\textbf{(69)} \qquad {}^{l}Z_{ns} = {}^{l}\rho_{avg,ns}^{\frac{1}{3}}\ {}^{l+1}c_{ns}$$

$$\textbf{(70)} \qquad {}^{l}Z_{ns} \propto {}^{l}\kappa_{ns}^{-1/6}$$

A decrease in density causes an increase in the impedance. The definition for the transmission coefficient[8] $|t|$ is given by **(71)**.

$$\textbf{(71)} \qquad |t| = \frac{2Z_1}{Z_1 + Z_2}$$

The order is Z_1 and Z_2 is important – the transmission coefficient is defined for a wave travelling from Z_1 to Z_2. The wave travels from Z_1 to Z_2 so pulse l travelling from section δ to section ς - where section δ and ς are defined to be adjacent to each other - would have the transmission coefficient given by **(72)**.

(72) $$ {}^{l}|t|_{n\delta,n\varsigma} = \frac{2\,{}^{l-1}Z_{n\delta}}{{}^{l-1}Z_{n\delta} + {}^{l-1}Z_{n\varsigma}} $$

9.5. General, Rigorous Expressions of Wave-Functions and Phase Corrections

The wavelength of the wave is different in each section since each section has a different density, so the wave-functions must include a phase difference to account for this difference in wavelength. The amplitude change due to transmission coefficients must also be taken into account.

The general form of a wave-function in a section is given by **(73)**. The total transmission coefficient due to all the coefficients up to section ς is given by **(74)** where the product sum multiplies the transmission coefficients for all the impedance changes already encountered before reaching section s. The total transmission coefficient of a pulse which has completed half a phase cycle (so the clipping coefficient from **(34)** is included) is given by **(75)** – there is no subscript on the left hand side of the equation meaning the product sum on the right hand side is over every section boundary.

$$ {}^{l}b_{ns}(x) = b_0\ {}^{l}|T|_{n\delta,n\varsigma}\sin(\ {}^{l}k_{ns}x + \ {}^{l}\varphi_s) $$
(73)

(74) $$ {}^{l}|T|_{n\delta,n\varsigma} = \prod_{n\delta,=0,a}^{n,\varsigma=n,s} {}^{l}|t|_{n\delta,n\varsigma} $$

(75) $$ {}^{l}|T| = \ {}^{l}|t|_R \prod_{n\delta,=0,a}^{n,\varsigma=n,s} {}^{l}|t|_{n\delta,n\varsigma} $$

${}^{l}\varphi_s$ is the phase correction for the wave-function in section s.

As an example, the wave-function for $l = 2$ in region 1, section b is given by **(76)**.

(76) $$ {}^{2}b_{1b}(x) = b_0\ {}^{2}|t|_{1a,1d}\ {}^{2}|t|_{1d,1b}\sin(\ {}^{2}k_{1b}x + \ {}^{2}\varphi_{1b}) $$

The general expression for the phase correction is found by solving the boundary condition for transitions between sections whilst neglecting transmission coefficients so ${}^{l}|t|_{n\delta,n\varsigma} \equiv 1$. The boundary condition **(77)** is the continuity of amplitude across sections, whilst neglecting transmission coefficients.

$$ {}^{l}b_{n\delta}\left(x_{n\delta,n\varsigma},\ {}^{l}|t|_{n\delta,n\varsigma} \equiv 1\right) = {}^{l}b_{n\varsigma}\left(x_{n\delta,n\varsigma},\ {}^{l}|t|_{n\delta,n\varsigma} \equiv 1\right) $$
(77)

$x_{n\delta,n\varsigma}$ is the point where the two sections δ and ς meet and so will be a critical point (unless it is where two regions meet if there are more regions than $n = 1$ and $n = 0$). It is a simple process of systematic algebraic manipulation to find ${}^{l}\varphi_{n\varsigma}$. The wave-function for section a is always known. In order to find each wave-function for each section along the x-axis, one must first calculate the wave-function for the previous section.

9.6. General, Rigorous Expression for Assigning a New Temperature

Energy in the form of kinetic energy of ions is equipartitioned **(28)** with magnetic energy associated with the ions. If the ions receive the kinetic energy and then recombine then this restoring magnetic energy is no longer available. Therefore, if a pulse recombines some ions, the temperature cannot be pushed back up to the initial temperature. Thus the region in which this happens is assigned a new average temperature across the entire region. It would be ideal to assign the new temperature to sections but this would make the problem more complicated.

The method used to assign a new temperature T_n is to weight the degree of ionisation with the length of all the sections in that region, as shown in **(78)**.

(78) $$^{l}T_n = {}^{l}\Theta_{cn}\left(\frac{\sum_{all\,s} {}^{l}\alpha_{ns} s}{n}\right)$$

9.7. General, Rigorous Expressions of Critical Points for $l = 2$

Continuing where section 9.2 left off, it is now possible to discuss the general expressions for the second critical points, ${}^{l}x_{c1}$ for $l > 1$, as the wave-functions and phase corrections have now been defined.

The expression for ${}^{l}x_{c1}$ **(79)** is similar to ${}^{l}x_{c0}$ **(61)** but requires the calculation of the wave-function of pulse l in all the previous sections in order to compute the phase change ${}^{l}\varphi_{ns}$ in the section which ends at ${}^{l}x_{c1}$. The expression includes the transmission coefficients **(74)** up to the section ending ${}^{l}x_{c1}$.

$${}^{l}x_{c1} = \frac{{}^{1}\lambda_0}{2} - \frac{{}^{l}\varphi_{ns}}{{}^{1}k_0} - \frac{1}{{}^{1}k_0} sin^{-1}\left(\sqrt{\frac{{}^{l}\Theta_{c1}}{\Theta}} \frac{1}{{}^{l}|T|_{n\delta,n\varsigma}}\right)$$

(79)

10. Matlab Code and Specific Example

The Matlab code shown in the Appendix uses the rigorous expressions mentioned to calculate the transmission coefficients for 3 consecutive waves. The user can input the initial temperature of the plasma under consideration and the amplitude of the wave. The transmission coefficients for the first 3 waves are given by entering in the command window “t1”, “t2” or “t3” which respectively correspond to ${}^{1}|T|$, ${}^{2}|T|$ and ${}^{3}|T|$. Also available are the degrees of ionization for each section for each pulse, the lengths of sections and the wavelengths of pulses.

Specifically, a plasma with initial temperature ${}^{0}T_0 = 2 \times 10^4 K$, initial number density $n_0 = 2 \times 10^{20} m^{-3}$ (which from Saha gives a critical temperature of $T_c = 1.2 \times 10^4 K$) with a wave amplitude of $\Theta = 1.9 \times 10^4 K$ gives the following results.

$${}^{1}|T| = 0.4211 \qquad {}^{2}|T| = 0.2858$$

$${}^{3}|T| = 0.1801$$

Clearly the total transmission of each pulse decreases. Intuitively, if this were to continue, the transmission coefficient would eventually be very small. At this point the assumption that the plasma does not interact with neutral gas is no longer appropriate – if the plasma is nearly completely recombined then there will be a large amount of neutral gas which could dampen the movement of ions and thus dampen the waves. However, depending on the initial density, the increase in neutral gas collisions may not dominate the decrease in ranged collisional effects associated with the electrically conducting fluid. If the increase in neutral gas collisions did dominate the ranged collisional effects then very small degrees of ionisation can be considered to result in death of the wave.

11. Summary of Method and Results Concluded

By considering a non-linear argument for Alfvén waves, it is seen that an Alfvén wave propagating in a completely ionised plasma can cause the temperature of the plasma to vary **(24)**. If the temperature drops to the critical temperature T_c, defined using the Saha ionization equation **(29)**, then the plasma undergoes a phase change and begins to recombine **(39)**. If a plasma completely recombines then an Alfvén wave cannot propagate **(42)** – with this in mind, a limit was expressed for the characteristic maximum amplitude of an Alfvén wave **(45)** and thus the characteristic maximum angle of deviation **(46)** of the magnetic field from equilibrium.

If the plasma is only partially ionised then it is useful to split an Alfvén wave into pulses of half a wavelength and release them one at a time. A notation was set up in order to follow an iterative process of releasing pulses and then quantifying the effects they would have on the plasma. General expressions for each stage of the process were given – these expressions have the potential to describe the effect of many wavelengths of an Alfvén wave on a plasma as well as the feedback effect the plasma would have on the wave. It is possible to calculate the transmission coefficients for each pulse of the wave.

In the specific example in section 10, it was found that the transmission coefficient for each subsequent pulse is less than the previous pulse – intuitively this would continue until $|t|$ was very small and could be considered to be zero depending on whether or not the initial density was such that ion-neutral collisions would end up dominating the ranged collisional effects associated with the fluid when the degree of ionization were small.

W.K.B.J. methods were discussed in section 8.7. It would be advantageous to exploit the accuracy of W.K.B.J. methods but at the cost of a great increase in complexity. The complexity would be best dealt with by writing a code to process the transmission coefficients in a similar way to the code in the Appendix but using W.K.B.J. solutions with the un-simplified, continuous, spatial density functions.

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Appendix

```
%copy and paste this code into
matlab
%//////////////%SETUP%//////////
%Calculate transmission
coefficients for the first 3 waves.
%THE USER CAN CHOOSE VALUES FOR
THE FOLLOWING PARAMETERS:
%T0 - INITIAL TEMPERATURE OF
PLASMA
T0=3e4; %initial plasma
temperature
%theta - AMPLITUDE OF WAVE IN
KELVIN
theta=2.5e4; %wave amplitude

%PARAMETERS OF INTEREST - enter
any of these parameters in the
command
%window to see their value
%t1 - transmission coefficient of
wave 1
%t2 - transmission coefficient of
wave 2
%t3 - transmission coefficient of
wave 3
%tcritical2 - clipping coefficient
of wave 2
%tcritical3 - clipping coefficient
of wave 3
%alphab1 - degree of ionisation of
section b by first wave
%alphab2full - degree of
ionisation of section b by second
wave and first
%wave
%alphad2 - degree of ionisation of
section d by second wave
%alphae2 - degree of ionisation of
section e by second wave
%T1 - new temperature after first
wave
%T2 - new temperature after second
wave
%lambda1 - wavelength of first
wave
```

```
%lambda2 - wavelength of second
wave


Tc=1.2e4; %critical temperature

thetac1=T0-Tc; %critical amplitude
n0=2e20; %initial number density
lambda1=240; %initial wavelength
k1=(2*pi)/lambda1; %initial
wavenumber



t1=thetac1/theta; %transmission of
wave 1

%//////////////WAVE
1//////////////

%Create function for T - find two
points onexc0 and onexc1

x=(0:300/100:300);
  %plot points for wave 1
Tx1=T0-(theta/2)*(1-
cos(2*k1*x));          %wave
function for wave 1

%Find critical intersects

onexc0=(1/(2*k1))*acos(1-
(2*thetac1)/theta); %critical
intersects for wave 1
onexc1=(lambda1/2)-onexc0;

%Create sections

a1=onexc0;                    %sect
ions for wave 1
b1=onexc1-onexc0;
c1=(lambda1/2)-onexc1;

%Amplitude ifs

thetasigma1=theta-
thetac1;        %amplitude theta
sigma

if
thetasigma1<0 ;               %if
theta sigma is negative, nothing
happens
   thetasigma1=0;
end

%\\\\\\\\\\\\SECTION
B1\\\\\\\\\\\\\\\\
%number density change function
deltan0=-
(thetasigma1/(2*Tc))*n0*(1-
cos(2*k1*x));

%total number density
n1=n0+deltan0;

%calculate average number density
top hat
ik1=1/(2*k1); %the coefficient
from integration of cos
nb1avg=n0-
(thetasigma1/(2*Tc*b1))*n0*(b1-
ik1*(sin(2*k1*onexc1)-
sin(2*k1*onexc0)));

alphab1=nb1avg/n0;
if thetasigma1==Tc;
    alphab1=0;
end


%////////////WAVE
2////////////////////
%/////////////////////////////////
////////

%calculation of new properties


T1=thetac1*(a1+c1+b1*(alphab1))/(a
1+b1+c1)+Tc; %calculation of new
temperature due to wave 1
thetac2=T1-Tc;               %the
critical amplitude for wave 2

thetasigma2=theta-
thetac2;     %the normalised
amplitude for wave 2
if thetasigma2<0;             %if
theta sigma is negative, nothing
happens
   thetasigma2=0;
end

lambda2=1/(sqrt(alphab1))*lambda1;
  %the new wavelength for wave 2
k2=(2*pi)/lambda2; %the new
wavenumber for wave 2
```

```
ik2=(1/(2*k2)); %the integration
coefficient for wave 2 for k2

%insert transmissions for wave 2
here

za1=1     ;                %impedan
ce a1
zb1=1/(alphab1)^(1/6) ;        %imp
edance b1
zc1=1       ;              %impedan
ce c1

ta1=2*za1/(za1+zb1);      %transmis
sion coefficient for wave 2 at a1
tb1=2*zb1/(za1+zb1);         %trans
mission coefficient for wave 2 at
b1
tcritical2=thetac2/theta;
     %transmission coefficient due
to clipping for wave 2
t2=ta1*tb1*tcritical2;
%total transmission coefficient
for wave 2

%CALCULATION OF FIRST CRITICAL FOR
WAVE 2

twoxc0=ik1*acos(1-
(2*thetac2)/theta);

%Now know the following sections,
still need c2 and e2

a2=twoxc0;
b2=onexc1-onexc0;
d2=onexc0-twoxc0;

%Can now calculate first phase at
a1

phasea1=onexc0*(k1-k2); %phase
difference of wave 2 function for
k2 at end a1
phaseb1=onexc1*(k2-
k1)+phasea1; %phase of wave 2
function for k1 at end b1

%can now calculate the second
critical using wave 2 in section e
including
%transmission coefficients

firsttwoxc1=ik1*acos(1-
(2*thetac2)/(theta*(ta1)^2*(tb1)^2
))-phaseb1/k1;
twoxc1=lambda1/2-firsttwoxc1-
phaseb1/k1; %the second x position

%SO NOW HAVE SECTORS E2 AND C2

c2=(lambda1/2)-twoxc1;
e2=twoxc1-onexc1;

%NOW CAN CALCULATE DENSITY
FUNCTIONS DUE TO WAVE 2

%SECTIONS A,D,E,C BEGIN AT N0
%SECTION B BEGINS AT nb1avg

%//////////SECTION
D2///////////////

deltand2=-
((thetasigma2*n0)/(2*Tc))*(1-
cos(2*k1*x));

nd2avg=n0-
(thetasigma2/(2*Tc*d2))*n0*(d2-
ik1*(sin(2*k1*onexc0)-
sin(2*k1*twoxc0)));

alphad2=nd2avg/n0;

zd2=alphad2^(-1/6);

%////////////SECTION
B2///////////

deltanb2=-
((thetasigma2*nb1avg*ta1^2)/(2*Tc)
)*(1-cos(2*k2*x+phasea1));

nb2avg=nb1avg-
(thetasigma2*nb1avg*ta1^2/(2*Tc*b2
))*(b2-
ik2*(sin(2*k2*onexc1+phasea1)-
sin(2*k2*onexc0+phasea1)));

alphab2=nb2avg/nb1avg;
alphab2full=nb2avg/n0;

zb2=alphab2^(-1/6);

%///////////SECTION
E2/////////////////
```

```
deltane2=-
((thetasigma2*n0*ta1^2*tb1^2)/(2*T
c))*(1-cos(2*k1*x+phaseb1));

ne2avg=n0-
(thetasigma2*n0*ta1^2*tb1^2/(2*Tc*
e2))*(e2-
ik1*(sin(2*k1*twoxc1+phaseb1)-
sin(2*k1*onexc1+phaseb1)));

alphae2=ne2avg/n0;

ze2=alphae2^(-1/6);

%\\\\\\\\\\\\\\CALCULATION OF WAVE
3 TRANSMISSION AND
TEMPERATURE\\\\\\\\\\\\\\\\\\\\\\\
\


T2=((a2+alphad2*d2+alphab2*b2+alph
ae2*e2+c2)/(lambda1/2))*thetac2+Tc;

thetac3=T2-Tc;

ta2=2*za1/(za1+zd2);     %transmis
sion coefficient for wave 3 at a2
td2=2*zd2/(zd2+zb2);
tb2=2*zb2/(zb2+ze2);
te2=2*ze2/(ze2+zc1);
tcritical3=thetac3/theta;
     %transmission coefficient due
to clipping for wave 2
t3=ta2*tb2*td2*te2*tcritical3;

%A BUNCH OF IF COMMANDS WHICH
ABIDE BY THE RULES SET - IF THE
TEMPERATURE
%OF THE PLASMA DOES NOT DROP BELOW
TC THEN THE TRANSMISSION IS 1.
%IF THE MINIMUM TEMPERATURE DROPS
BELOW 0 THE TRANSMISSION IS 0.
%IF THE INITIAL TEMPERATURE OF THE
PLASMA IS BELOW TC THE
TRANSMISSION IS
%0, BUT SOME RECOMBINATION OCCURS.


if T0<Tc
    t1=0;
    t2=0;
    t3=0;
end
if T1<Tc
    t3=0;
    t2=0;
end
if T2<Tc
    t3=0;
end
if theta<T0-Tc
    t1=1;
    t2=1;
    t3=1;
end
if theta>T0
    t1=0;
    t2=0;
    t3=0;
end
if nb1avg<0
    nb1avg=0
end
if nd2avg<0
    nb1avg=0
end
if ne2avg<0
    nb1avg=0
end
if nb2avg<0
    nb1avg=0
end
```