BABEŞ-BOLYAI UNIVERSITY
Faculty of Mathematics and Computer Science

Algorithms and Programming

Lecture 8 – Recursion. Computational complexity

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Course content

Big image in programming

- Introduction in the software development process
- Procedural programming
- Modular programming
- Abstract data types
- Software development principles
- Testing and debugging

Detailed image in programming

- **Recursion**
- **Complexity of algorithms**
- Search and sorting algorithms
- Backtracking and other problem solving methods
- Recap

Last time

- Testing
  - Black box testing
  - White box testing
  - Examples

- Design patterns
  - GRASP
  - Information Expert
  - Pure Fabrication (Repository)
  - Controller

Today

- Recursion
  - Basic concept
  - Mechanism
  - Examples

- Computational complexity
  - Why?
  - Examples
  - Analyzing the efficiency of a program

Recursion

- What is recursion?
  - A way to solve a problem by reducing it to simpler versions of itself
  - A programming technique where a function calls itself
- Basic concepts
  - Recursive element – an element that is defined by itself
  - Recursive algorithm – an algorithm that calls itself
  - Note: condition to stop recursion
- Recursion can be:
  - **Direct** – a function calls itself (`f` calls `f`)
  - **Indirect** – a function `f` calls a function `g`, function `g` calls `f`

Example: factorial

- n! = n*(n-1)*(n-2)*...*2*1
- n! = n*(n-1)! = n*(n-1)*(n-2)! =...

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n * (n-1)! & \text{otherwise} \end{cases}$$

```python
def factorial(n):
    '''
    Computes n! - the factorial for the given positive integer
    input: n - positive integer
    output: an integer 1 * 2 * 3 * ... * n
    '''
    # base case
    if n == 1:
        return 1
    else:
        # Recursive step: progresses toward the base case
        return n * factorial(n - 1)
```

```python
def testFactorial():
    assert factorial(5) == 120
    assert factorial(4) == 24
    assert factorial(1) == 1
    assert factorial(0) == 1

testFactorial()
```

Example: factorial

```python
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)

print(factorial(4))
```

- Function scope

Global scope

factorial | code for factorial

factorial scope (call with 4)

n | 4

factorial scope (call with 3)

n | 3

factorial scope (call with 2)

n | 2

factorial scope (call with 1)

n | 1

`print(factorial(4))`

`return 4*factorial(3)`

`return 3*factorial(2)`

`return 2*factorial(1)`

`return 1`

2*1=2

3*2=6

4*3=12

print(12)

Recursion: mechanism

- Main idea of developing a recursive algorithm for a problem of size n
  - **Base case**
    - How to stop recursion
    - Identify the base case solution (for n=1)
  - **Inductive step**
    - Break the problem into a simpler version of the same problem plus some other steps
    - e.g. A smaller problem (of size n-1) and some simple computations
    - e.g. Two smaller problems (of size n1, n2 such that n1+n2=n-1) and simple computations
- How recursion works
  - On each method invocation a new symbol table is created: it contains all the parameters and the local variables defined in the function
  - The symbol tables are stored in a stack: when a function is returning, the current symbol table is removed from the stack

Inductive reasoning

- How do you know that your recursive algorithm works?
- Mathematical induction
  - To prove a statement is true for all values of n:
    - Prove that the statement is true for smallest value of n (n=0 or n=1)
    - Assume that the statement is true for n, then prove that it is also true for n+1
- Same logic applies here:
  - ***Base case***: n=1, correctly returns 1
  - ***Recursive case***: assume that the answer is correct for problem size smaller than k, then for k+1 the answer will be (k+1)*k! = (k+1)!
  - By induction, the result returned is correct

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

Example: sum and product

Note: Inductive step

```
def sumList(lst):
    '''
    Compute the sum of the elements in the list
    input: lst - the list
    output: The sum of the elements
    '''
    # base case
    if len(lst) == 0:
        return 0
    else: # lst has at least one element
        return lst[0] + sumList(lst[1:])

def testSumList():
    assert sumList([]) == 0
    assert sumList([0]) == 0
    assert sumList([1, 2, 6]) == 9
    assert sumList([-1, 4, -100, 50]) == -47
    assert sumList([1, 2, 3, 4, 5, 6]) == 21

testSumList()
```

```
def product(lst):
    '''
    Computes the product of elements from a list
    input: a list of integers
    output: product of elements
    '''
    if (len(lst) == 0):
        return 1
    else:
        middle = len(lst) // 2
        return product(lst[0:middle]) * lst[middle] *
               product(lst[middle+1:])

def testProduct():
    assert product([1,2,3,4]) == 24
    assert product([]) == 1
    assert product([1,2,3,4,5]) == 120
    assert product([2]) == 2

testProduct()
```

Example: palindrome

Note: symbol table

```
>>> isPalindrom("noon")
49018080    {'s': 'noon'}
49171216    {'s': 'oo'}
48909600    {'s': ''}
True
```

```
>>> isPalindrom("redivider")
49020384    {'s': 'redivider'}
49018080    {'s': 'edivide'}
49171216    {'s': 'divid'}
48909600    {'s': 'ivi'}
48973552    {'s': 'v'}
True
```

```
def isPalindrom(s):
    '''
    Checks if a string is palindrom
    Input: a string
    Res: true, if str is palindrom and false, otherwise
    '''
    dico = locals()
    print(id(dico), " ", dico)

    if len(s) <= 1:
        return True
    else:
        return s[0]==s[-1] and isPalindrom(s[1:-1])

def testIsPalindrom():
    assert isPalindrom("abcba") == True
    assert isPalindrom("abccba") == True
    assert isPalindrom("abcdba") == False

testIsPalindrom()
```

"Able was I, ere I saw Elba" (Napoleon)
"Are we not drawn onward, we few, drawn onward to new era?" (Anne Michaels)

Example: belongs

Note: symbol table

```
>>> belongs(2, [4, 2, 3, 5])
49171168    {'l': [4, 2, 3, 5], 'el': 2}
48909648    {'l': [2, 3, 5], 'el': 2}
True
```

```
>>> belongs(7, [4, 2, 3, 5])
48909600    {'l': [4, 2, 3, 5], 'el': 7}
49171168    {'l': [2, 3, 5], 'el': 7}
48909648    {'l': [3, 5], 'el': 7}
49018080    {'l': [5], 'el': 7}
49171600    {'l': [], 'el': 7}
False
```

```python
def belongs(el, l):
    '''
    Checks if an element belongs to a list
    Input: an integer and a list of integers
    Output: true, if el is in list
            and false, otherwise
    '''
    dico = locals()
    print(id(dico), " ", dico)

    if (l == []):
        return False
    else:
        if (el == l[0]):
            return True
        else:
            return belongs(el, l[1:])

def testBelongs():
    assert belongs(5, []) == False
    assert belongs(5, [5,2,6,3]) == True
    assert belongs(5, [1,2,5,4,3]) == True
    assert belongs(5, [6,2,5]) == True
    assert belongs(5, [1,2,3]) == False

testBelongs()
```

Iteration vs. Recursion

```
def factorial_iter(n):
    res = 1
    for i in range(1,n+1):
        res *= i
    return res
```

```
def factorial(n):
    if n == 1:
        return 1
    else:
        return n * factorial(n - 1)
```

✓Recursive code is simple and intuitive

- However:
  - May be efficient from programming perspective, but
  - May not be efficient from the computer perspective
  - Why?
    - Large memory needed for in-depth recursion
    - Each function self call creates a new symbol table

Computational complexity

- What is it?
  - Study the efficiency of an algorithm from a mathematical perspective
- Why?
  - Problem: search for a number in a list
  - Solution:
    - Iterative search
    - Recursive search – one sub-problem
    - Recursive search – two sub-problems
  - **Which solution is better?**
    - **Algorithm efficiency**
    - Compare algorithms based on:
      - **Time needed for computations**
      - **Extra memory needed for temporary data**
    - Run time depends on:
      - Entry data (structure and size)
      - Changes from one run to another (due to hardware and software environment)
      - Hardware

Example: Fibonacci numbers

$F_n = F_{n-1} + F_{n-2}$ and $F_0 = F_1 = 1$

```python
def fibonacci_recursiv(n):
    '''
    Computes the Fibonacci number
    data: a positive integer
    res: fibonacci number of n
    '''
    if n == 0 or n == 1:
        return 1
    return fibonacci_recursiv(n-1) +
           fibonacci_recursiv(n-2)
```

```python
def fibonacci_iterativ(n):
    ''' ...
    '''
    s1 = 1
    s2 = 1
    fibo = 0
    for i in range(2, n + 1):
        fibo = s1 + s2
        s1 = s2
        s2 = fibo
    return fibo
```

```python
def timeFibonacci(n):
    import time
    start_time = time.time()
    print("computing Fibbonacci_iterativ(", n, ") = ",
                                          fibonacci_iterativ(n))
    end_time = time.time()
    print(" takes ", end_time - start_time, " seconds")
    start_time = time.time()
    print("computing Fibbonacci_recursiv(", n, ") = ",
                                          fibonacci_recursiv(n))
    end_time = time.time()
    print(" takes ", end_time - start_time, " seconds")
```

```
>>> timeFibonacci(23)
computing Fibbonacci_iterativ( 23 ) =  46368
 takes  0.07200431823730469  seconds
computing Fibbonacci_recursiv( 23 ) =  46368
 takes  0.13400769233703613  seconds
```

```
>>> timeFibonacci(33)
computing Fibbonacci_iterativ( 33 ) =  5702887
 takes  0.07800436019897461  seconds
computing Fibbonacci_recursiv( 33 ) =  5702887
 takes  11.261644124984741  seconds
```

Algorithm efficiency

- To analyse the efficiency of an algorithm (function):
  - The amount of resources used
  - **Time** and space efficiency of a program
  - Measure efficiency
    - **Asymptotic analysis**
      - Can provide the efficiency for all possible input data
      - Can not provide exact execution times
    - Empirical analysis
      - Can not predict the performance of the algorithm for all possible input data
      - Can determine the execution time for specific set of inputs
  - Run time is studied in connection with the size of the input data

Run time complexity

- Complexity in time
- Running time of an algorithm:
  - It is not a fixed number
  - It is a function *T(n)* that depends on the size *n* of the input data
  - Measures the basic steps the algorithm makes
- Example
  - Steps that take constant time:
    - Mathematical operations
    - Assignments
    - Comparisons
    - Accessing objects in memory
    - Return statement
  - Count the number of operations executed as a function of the input size

```python
def mySum(n):
    res = 0                  # 1
    for i in range(n):       # 2  } loop n times
        res = res + i        # 2  }
    return res               # 1
```

T(n)=2+4*n operations

An example

```
def searchElement(l, e):
    for i in l:
        if e == i:
            return True
    return False
```

- e is the first element in the list – best case
  - Minimum running time over all possible inputs
- e is not in the list – worst case
  - Maximum running time over all possible inputs
- e is found after about half of the list is searched – average case
  - Average running time over all possible inputs

Run time complexity

- **Best case (BC)**
  - For the entry data leading to minimum running time of the algorithm
  - *Algorithm complexity*: $BC(A) = \min_{I \in D_A} E_A(I)$
  - Gives a lower bound to the running time
- **Worst case (WC)**
  - For the entry data leading to maximum running time of the algorithm
  - *Algorithm complexity*: $WC(A) = \max_{I \in D_A} E_A(I)$
  - Gives an upper bound to the running time
- **Average case (AC)**
  - Average running time of the algorithm
  - *Algorithm complexity*: $AC(A) = \sum_{I \in D_A} P_A(I) E_A(I)$
  - Offers a prediction of the running time

where

A – algorithm

D_A – domain of the input data

I – an instance of input data

$E_A(I)$ – number of operations performed by algorithm A having input data I

$P_A(I)$ – probability that algorithm A receives input data I

Run time complexity: Examples

```python
def sumOfFirstNumbers(n):
    '''
    computes the sum of first n natural numbers
    data: a natural number
    res: the sum of first n numbers
    '''
    s = 0
    for i in range(1, n + 1):
        s = s + i
    return s
```

$$T(n) = \sum_{i=1}^{n} 1 = n$$

Case	T(n)
Best case	$\sum_{i=1}^{n} 1$
Worst case	$\sum_{i=1}^{n} 1$
Average case	$\sum_{i=1}^{n} 1$

Run time complexity: Examples

```python
def search(el, list):
    '''
    checks if an element belongs to a list
    data: an integer and a list of integers
    res: true, if elements belongs to list
    false, otherwise
    '''
    for i in range(0, len(list)):
        if (list[i] == el):
            return True
    return False
```

Case	len(list)=n	T(n)
Best case	el=list[0]	1
Worst case	el=list[n-1] el is not in list	$\sum_{i=0}^{n-1} 1 = \mathrm{n}$ $\sum_{i=0}^{n-1} 1 + 1 = \mathrm{n} + 1$
Average case	el=list[0] el=list[1] el=list[2] ... el=list[n-1] el is not in list	1 2 3 ... n n+1 ______ (1+2+3+...+n+(n+1))/(n+1)=(n+2)/2

Big Oh notation

- O(n) measure
  - How the running time grows depending on the input data size
  - Expression for the number of operations -> asymptotic behavior as the problem gets bigger
- Exact steps vs

Big Oh or

O() notation

```
def factorial_iter(n):
    i = 1
    res = 1
    while i <= n:
        res = res * i
        i = i -1
    return res
```

- Exact steps T(n) = 1 + 1 + n*5 + 1
- Worst case asymptotic complexity O(n)
  - Ignore additive constants
  - Ignore multiplicative constants

O()

- Drop constants and multiplicative factors
- Focus on dominant terms (consider the leading term)

O(n) $T(n) = 2n + 2$

O(n^2) $T(n) = n^2 + 2n + 2$

O(n^3) $T(n) = n^3 + 1000n + 3^{1000}$

O(n) $T(n) = n + \log n$

O(n log n) $T(n) = n \log n + 100n$

O(2^n) $T(n) = n^2 + 2^n$

?

$$T(n) = n^3 + 100n^2 + 10n \log_2 n + 2\sqrt{n} + 27^{100}$$

```python
for i in range(n):
    print(i)
for i in range(n):
    for j in range(n):
        print(i," ",j)
```

O(n)

O(n)*O(n)=O(n*n)=O(n^2)

O(n)+O(n^2)=O(n+n^2)=O(n^2)

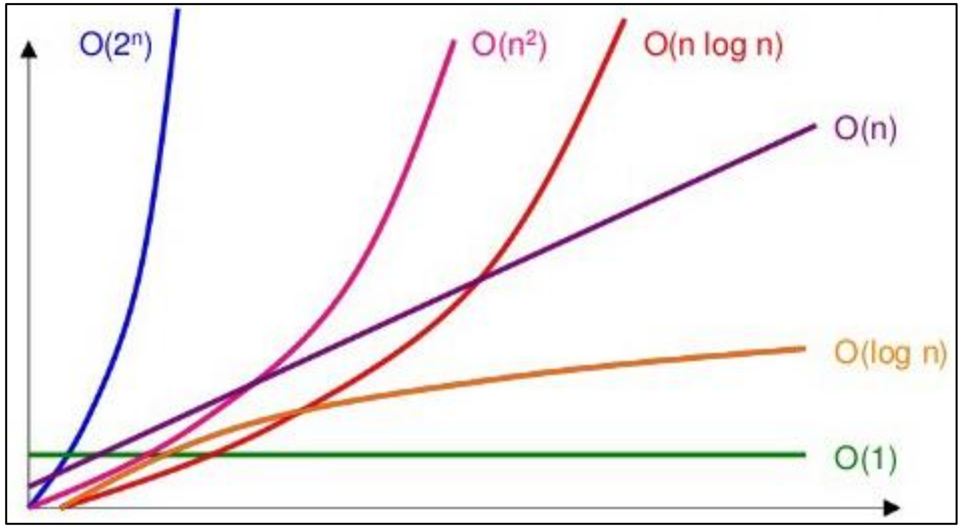
Run time complexity: theoretical aspects

- For a function $f: N \to R$ and $T: N \to N$, how to decide the complexity class of *T*?
- $T(n) \in O\big(f(n)\big)$ if there exist 2 positive and independent constants *c* and n_0 such that $0 \leq T(n) \leq c * f(n), \forall n \geq n_0$
- Examples
  - O(1) = 1, 5, 500
  - O(n) = n, 2n + 1, 5n – 200
  - $O(n^2) = n^2, n^2 + 5, 2n^2 + 3n - 1$
- If $T(n) \in O\big(f(n)\big)$ then $\lim_{n\to\infty} \frac{T(n)}{f(n)} < \infty$ (the limit is constant)

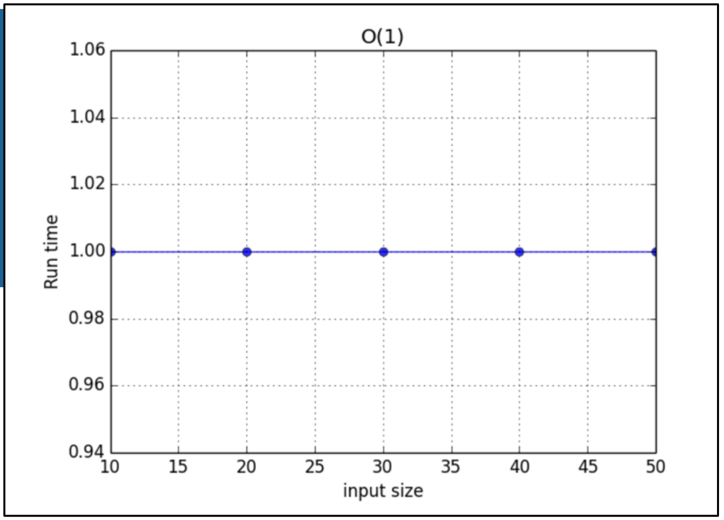
Complexity classes

O(1)	Constant running time	e.g. 1, 47, 100	Add an element to a list
O(log n)	Logarithmic running time	e.g. 10 + log n	Find an element in a sorted list
O(n)	Linear running time	e.g. n, 3n, 10n+100	Find an entry in an unsorted list
O(n log n)	Log-linear running time	e.g. n + n log n	Sort a list (MergeSort, QuickSort)
$O(n^c)$, c is constant	Polynomial running time	e.g. n^2+1, n^3+n^2+5n	Shortest path between two nodes
$O(c^n)$, c is constant	Exponential running time	e.g. 2^n+1, 3^n	Traveling Salesman Problem (TSP)

$O(n^2)$ - quadratic time
$O(n^3)$ - cubic time



Complexity classes: Constant



- $T(n) \in O(1)$
  - **Constant running time**
  - Very good complexity (the algorithm executes a constant number of steps regardless the size of input data)
  - Example: add an element to a list, access an element from a list, modify information in an object

```python
def isFirstElementNone(l):
    return l[0] == None
```

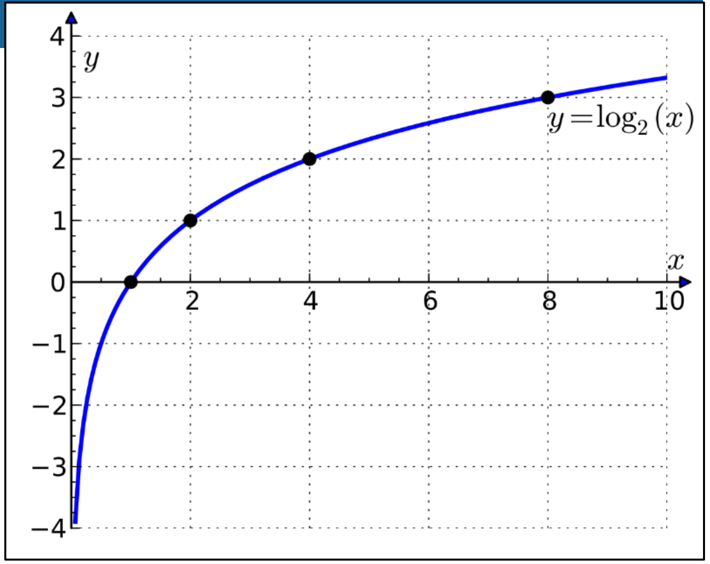
	n=1	n=10	n=100	n=1000
O(1)	1	1	1	1

If n doubles, O(1) remains unchanged

Complexity classes: Logarithmic

- $T(n) \in O(\log n)$
  - **Logarithmic running time**
  - Very good complexity

  - Example: binary search, the height of a binary tree

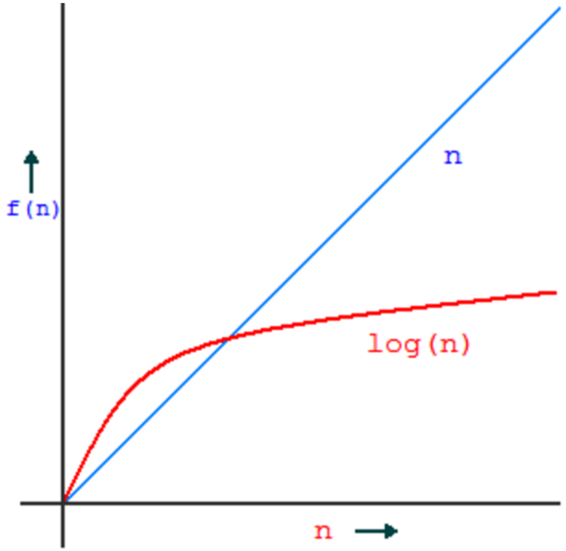


  - *Q: How many times to divide a problem of size n until arriving to a problem of size 1?*
  - *$n = 2^x$, x = ?*
  - *See the binary search algorithm in the next lecture*

	n=1	n=10	n=100	n=1000
O(log n)	0	1	2	3

If n doubles, O(log n) increases slightly

Complexity classes: Linear



- $T(n) \in O(n)$
  - Linear running time
  - Good complexity
  - Performance grows linearly and in direct proportion to the size of the input data set
  - Example: find the minimum / maximum in an unsorted list, find an element in a list

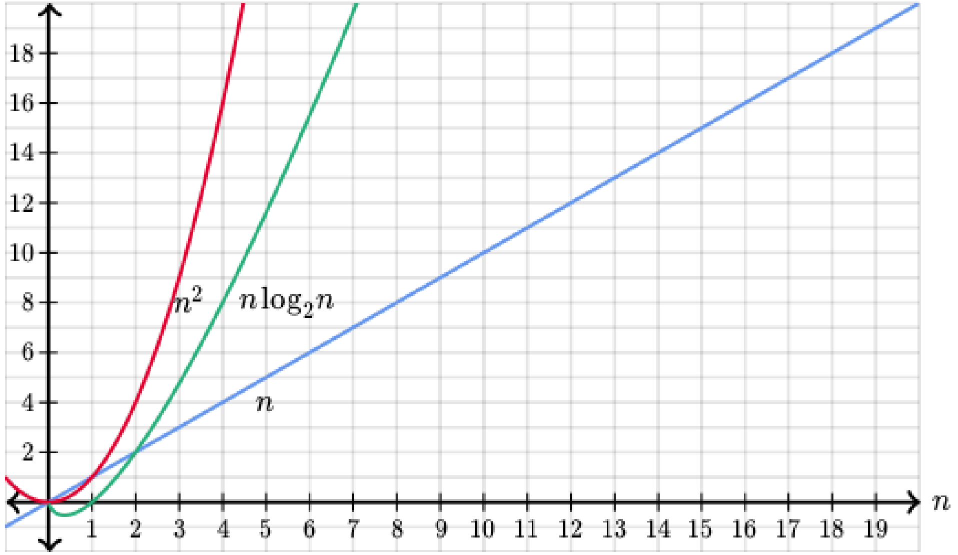
```
def containsValue(l, val):
    for i in range(0, len(l)):
        if (l[i] == val):
            return True
    return False
```

	n=1	n=10	n=100	n=1000
O(n)	1	10	100	1000

If n doubles,
O(n) doubles

Complexity classes: Log-linear

- $T(n) \in O(n \log n)$
  - Log-linear running time
  - Good complexity
  - Example: sort a list (MergeSort / QuickSort)
    - *See the algorithms in the next lecture*

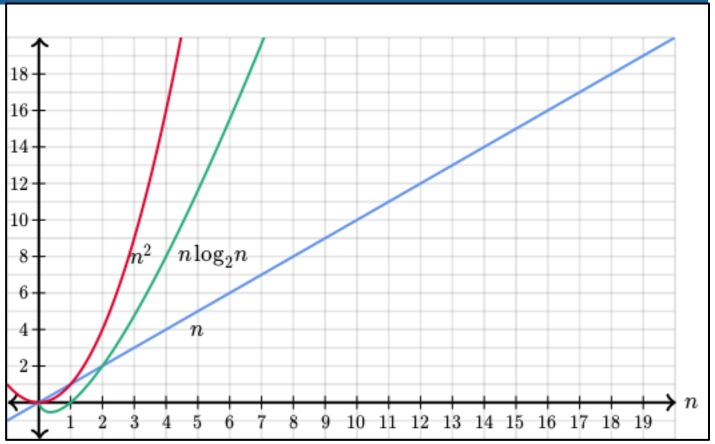


	n=1	n=10	n=100	n=1000
O(n log n)	0	10	200	3000

If n doubles,
O(n log n) slightly
more than doubles

Complexity classes: Quadratic

- $T(n) \in O(n^2)$
  - Quadratic running time
  - Good complexity if n is multiple of 1000
  - Bad complexity if n is multiple of 1 mil.
  - Example: sort a list (BubbleSort)
    - *See the algorithms in the next lecture*
  - Common when the algorithm involves nested iterations



Example

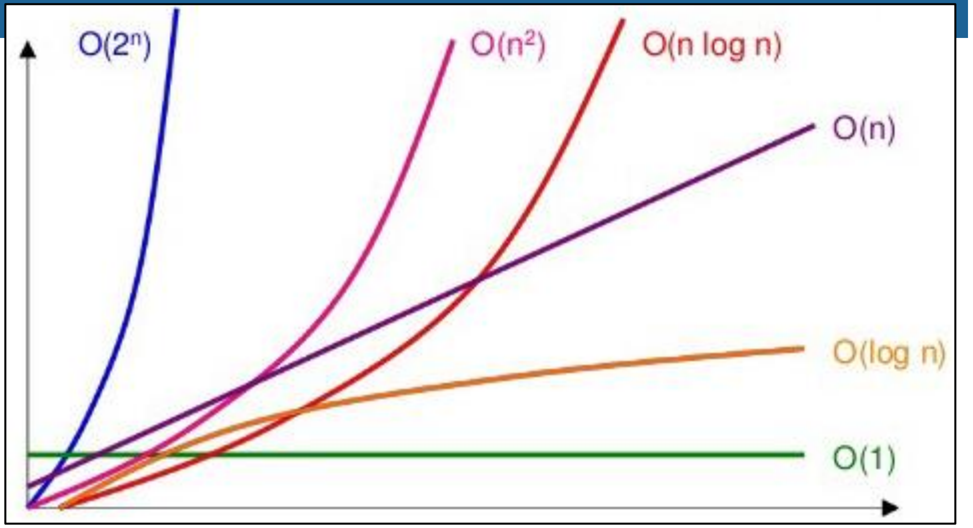
```
def containsDuplicates(l):
    for i in range(len(l)):
        for j in range(len(l)):
            if i != j and l[i]==l[j]:
                return True
    return False
```

	n=1	n=10	n=100	n=1000
$O(n^2)$	1	100	10000	1 mil.

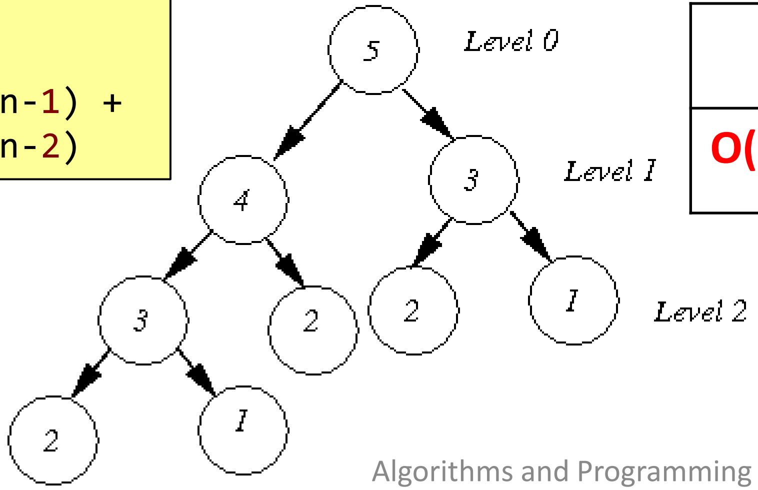
If n doubles,
$O(n^2)$ quadruples

Complexity classes: Exponential

- $T(n) \in O(2^n)$
  - **Exponential running time**
  - Bad complexity
  - Example: TSP, Fibonacci recursively




```python
def fibonacci_recursiv(n):
    if n == 0 or n == 1:
        return 1
    return fibonacci_recursiv(n-1) +
           fibonacci_recursiv(n-2)
```



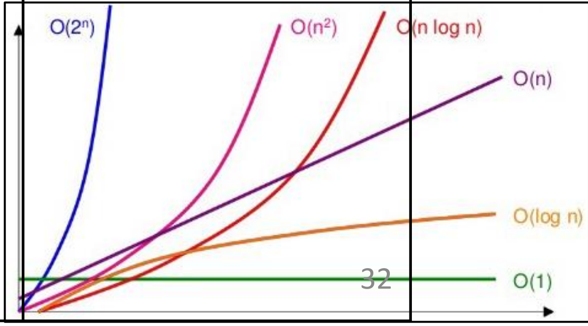
	n=1	n=10	n=100	n=1000
$O(2^n)$	2	2^{10}	2^{100}	2^{1000}

If n doubles,
$O(2^n)$ multiplies drastically

$2^{100} \approx 10^{30}$
HUGE!!

Complexity growth

Order of growth	n=10	n=100	n=1000	n=1000000
O(1)	1	1	1	1
O(log n)	1	2	3	6
O(n)	10	100	1000	1000000
O(n log n)	10	200	3000	6000000
$O(n^2)$	100	10000	1000000	1000000000000
$O(2^n)$	1024	1.267.650.600.228.229.401.496.703.205.376	10715086071862673209484250490600018105614048117055336074437503883703510511249361224931983788156958581275946729175531468251871452856923140435984577574698574803934567774824230985421074605062371141877954182153046474983581941267398767559165543946077062914571196477686542167660429831652624386837205668069376	??



Run time complexity analysis: example

```
def sumOfFirstNumbers(n):
    '''
    computes the sum of first n natural numbers
    data: a natural number
    res: the sum of first n numbers
    '''
    sum = 0
    for i in range(1, n + 1):
        sum = sum + i
    return sum

def test_sum():
    assert sumOfFirstNumbers(5) == 15
    assert sumOfFirstNumbers(1) == 1
```

Case	T(n)
Best case	$\sum_{i=1}^{n} 1 = \mathrm{n}$
Worst case	$\sum_{i=1}^{n} 1 = \mathrm{n}$
Average case	$\sum_{i=1}^{n} 1 = \mathrm{n}$

$\Rightarrow$ ***O(n)***

Run time complexity analysis: example

```python
def sumOfFirstNumbers(n):
    '''
    computes the sum of first n natural numbers
    data: a natural number
    res: the sum of first n numbers
    '''
    sum = 0
    i = 1
    while (i<=n):
        sum = sum + i
        i = i + 1
    return sum

def test_sum():
    assert sumOfFirstNumbers(5) == 15
    assert sumOfFirstNumbers(1) == 1
```

Case	**T(n)**
Best case	$\sum_{i=1}^{n} 1 = \mathrm{n}$
Worst case	$\sum_{i=1}^{n} 1 = \mathrm{n}$
Average case	$\sum_{i=1}^{n} 1 = \mathrm{n}$

O(n)

Run time complexity analysis: example

```python
def searchEven(list):
    '''
    checks if a list contains at least an even number
    data: a list of integers
    res: true, if list contains at least an even
    number false, otherwise
    '''
    i = 0
    while ((i < len(list)) and (list[i] % 2 != 0)):
        i = i + 1
    return (i < len(list))

def test_searchEven():
    assert searchEven([2,4,6]) == True
    assert searchEven([1,3,5]) == False
    assert searchEven([1,2,3]) == True
```

Case	**T(n)**
Best case	$1 + 1 = 2$
Worst case	$1+\sum_{i=1}^{n} 1 + 1 = n + 2$
Average case	$\sum_{i=1}^{n} i/n = (n+1)/2$

$\Rightarrow$ ***O(n)***

Run time complexity analysis: example

```
def sumOfElemFromMatrix(m):
    s = 0
    for i in range(0, len(m)):
        for j in range(0, len(m[i])):
            s = s+ m[i][j]
    return s

def test_sumOfElemFromMatrix():
    assert sumOfElemFromMatrix([[1,2],[4,5],[7,9]]) == 28
    assert sumOfElemFromMatrix([[1,2,3],[4,5,6],[7,8,9]]) == 45
```

Case	T(n)
Best case	$\sum_{i=1}^{n}\sum_{j=1}^{m} 1 = n * m$
Worst case	$\sum_{i=1}^{n}\sum_{j=1}^{m} 1 = n * m$
Average case	$\sum_{i=1}^{n}\sum_{j=1}^{m} 1 = n * m$

$\Rightarrow$ ***O(n²)***

Run time complexity analysis: example

```python
class Book:
    def __init__(self, t, na, a):
        self.title = t
        self.noAuthors = na
        self.authors = a

    def getAuthors(self):
        return self.authors

def searchBooksOfAnAuthor(books, author):
      res = []
      for b in books:
          authors = b.getAuthors()
          i = 0
          while (i < len(authors)):
              if (authors[i] == author):
                  res.append(b)
                  i = len(authors)
              else:
                  i = i + 1
      return res
```

```python
def test_searchBooksOfAnAuthor():
    b1 = Book("title1", 2, ["author1", "author2"])
    b2 = Book("title2", 3, ["author2", "author3", "author4"])
    b3 = Book("title3", 1, ["author4"])
    books = [b1, b2, b3]
    assert searchBooksOfAnAuthor(books, "a") == []
    assert searchBooksOfAnAuthor(books, "author5") == []
    assert searchBooksOfAnAuthor(books, "author1") == [b1]
    assert searchBooksOfAnAuthor(books, "author2") == [b1, b2]
    assert searchBooksOfAnAuthor(books, "author4") == [b2, b3]
```

Case	T(n)
Best case	$\sum_{i=1}^{n} 1 = n$
Worst case	$\sum_{i=1}^{n}\sum_{j=1}^{m} 1 = n * m$
Average case	$\sum_{i=1}^{n} m = m * n$

O(n²)

n - number of books
m - average number of authors for a book

Run time complexity analysis: example

```python
def sum(l):
    '''
    computes the sum of elements from a list
    data: a list of integers
    res: sum of elements
    '''
    if (len(l) == 0):
        return -1
    else: #len(l) > 0
        return l[0] + sum(l[1:])

def testSum():
    assert sum([1,2,3,4]) == 10
    assert sum([]) == 0
    assert sum([3]) == 3
```

$$T(n) = \begin{cases} 1, if\ n = 0 \\ T(n-1) + 1, otherwise \end{cases}$$

$$T(n) = T(n-1) + 1$$

$$T(n-1) = T(n-2) + 1$$

$$T(n-2) = T(n-3) + 1$$

$$\dots$$

$$T(1) = T(0) + 1$$

$$T(n) = n + 1$$

Complexity in **space**

- Estimates the **space** (memory) that an algorithm needs to store input data, output data and any temporary data
- Uses the same notations of run time complexity

```python
def minArray_v1():
    a = []
    n = int(input("n = "))
    for i in range(0, n):
        el = int(input("el = "))
        a.append(el)

    minim = a[0]
    for i in range(1, n):
        if (a[i] < minim):
            minim = a[i]
    print("minim is " + str(minim))
```

S(n)=1+n+1=n+2

O(n)

```python
def minArray_v2():
    n = int(input("n = "))
    minim = int(input("el = "))
    for i in range(1, n):
        el = int(input("el = "))
        if (el < minim):
            minim = el
    print("minim is " + str(minim))
```

S(n)=1+1+1=3

O(1)

Recap today

- Recursion
  - Basic concept
  - Mechanism
  - Examples
- Computational complexity
  - Examples
  - The efficiency of a program
  - Time and space complexity

Next time

- Algorithms
  - Sort
  - Search

Reading materials and useful links

1. The Python Programming Language - https://www.python.org/
2. The Python Standard Library - https://docs.python.org/3/library/index.html
3. The Python Tutorial - https://docs.python.org/3/tutorial/
4. M. Frentiu, H.F. Pop, Fundamentals of Programming, Cluj University Press, 2006.
5. MIT OpenCourseWare, Introduction to Computer Science and Programming in Python, https://ocw.mit.edu, 2016.
6. K. Beck, Test Driven Development: By Example. Addison-Wesley Longman, 2002. http://en.wikipedia.org/wiki/Test-driven_development
7. M. Fowler, Refactoring. Improving the Design of Existing Code, Addison-Wesley, 1999. http://refactoring.com/catalog/index.html