Machine Learning - Theoretical exercise 2

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Problem 1

a) In the following, we use the notation $\lambda(\alpha_i \mid \omega_j) \Leftrightarrow \lambda_{ij}$

$\lambda_{11} = 0$	correct classification of toxic container
$\lambda_{21} = 10^5$	incorrect classification of toxic container
$\lambda_{22} = 0$	correct classification of non-toxic container
$\lambda_{12} = 250$	incorrect classification of non-toxic container

b)

$$R(\alpha_1 \mid x) = \lambda_{11}P(\omega_1 \mid x) + \lambda_{12}P(\omega_2 \mid x)$$
$$R(\alpha_2 \mid x) = \lambda_{21}P(\omega_1 \mid x) + \lambda_{22}P(\omega_2 \mid x)$$

As $\lambda_{11} = 0$ and $\lambda_{22} = 0$,

$$R(\alpha_1 \mid x) = \lambda_{12}P(\omega_2 \mid x)$$
$$R(\alpha_2 \mid x) = \lambda_{21}P(\omega_1 \mid x)$$

c) To determine the decision boundary that minimizes the average cost, we solve the equality of conditional loss functions.

$$R(\alpha_1 \mid x) = R(\alpha_2 \mid x)$$
$$\lambda_{12}P(\omega_2 \mid x) = \lambda_{21}P(\omega_1 \mid x)$$
$$\frac{\lambda_{12}}{\lambda_{21}}\frac{P(\omega_2)P(x \mid \omega_2)}{P(x)} = \frac{P(\omega_1)P(x \mid \omega_1)}{P(x)} \qquad \text{(Bayes Rule)}$$
$$\frac{\lambda_{12}P(\omega_2)}{\lambda_{21}P(\omega_1)}P(x \mid \omega_2) = P(x \mid \omega_1)$$

Let $K = \frac{\lambda_{12} P(\omega_2)}{\lambda_{21} P(\omega_1)}$. Furthermore, we know that $P(x \mid \omega_1) \sim \mathcal{N}(\mu_1, \sigma^2)$ and $P(x \mid \omega_2) \sim \mathcal{N}(\mu_2, \sigma^2)$.

$$\begin{aligned}
K \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma}} &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_2)^2}{2\sigma}} \\
\ln(Ke^{-\frac{(x-\mu_1)^2}{2\sigma}}) &= \ln(e^{-\frac{(x-\mu_2)^2}{2\sigma}}) \\
\ln K - \frac{(x-\mu_1)^2}{2\sigma} &= -\frac{(x-\mu_2)^2}{2\sigma} \\
\ln K + \frac{1}{2\sigma}((x-\mu_2)^2 - (x-\mu_1)^2) &= 0 \\
\ln K + \frac{1}{2\sigma}(x^2 - 2\mu_2 x + \mu_2^2 - x^2 + 2\mu_1 x - \mu_1^2) &= 0 \\
\ln K + \frac{1}{2\sigma}(2x(\mu_1 - \mu_2) + \mu_2^2 - \mu_1^2) &= 0 \\
\ln K + \frac{\mu_1 - \mu_2}{\sigma} x + \frac{\mu_2^2 - \mu_1^2}{2\sigma} &= 0 \\
\frac{\mu_1 - \mu_2}{\sigma} x &= \frac{\mu_1^2 - \mu_2^2}{2\sigma} - \ln K \\
x &= \frac{\mu_1^2 - \mu_2^2}{2(\mu_1 - \mu_2)} - \sigma \ln K \\
x &= \frac{\mu_1 + \mu_2}{2} - \frac{\sigma}{\mu_1 - \mu_2} \ln K
\end{aligned}$$

Numerically solving this equation gives us the decsion boundary

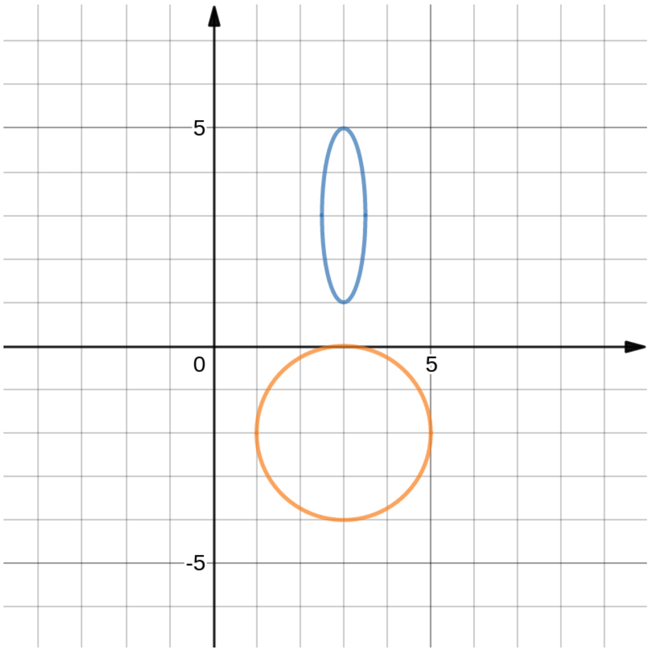
$$\begin{aligned}
x_{border} &= \frac{0.4 + 0.2}{2} - \frac{10^{-4}}{0.4 - 0.2} \times \ln(\frac{25 \times 250}{10^5}) \\
x_{border} &= 0.2954
\end{aligned}$$

d) To determine the minimum average cost R_{min} upon classification, we can use both equations derived in question b). Or P(error) with integral ?

$$\begin{aligned}
R_{min} &= \lambda_{12} P(\omega_2 \mid x_{border}) \\
&= \lambda_{12} \frac{P(\omega_2) P(x_{border} \mid \omega_2)}{P(x_{border})}
\end{aligned}$$

Problem 2

a)



b) The risk function g can be defined as $g_i(x) = \ln P(\omega_i) + \ln P(x \mid \omega_i)$ for $i = 1, 2$. In this exercise, priors are equal so they can be omitted when solving the equality of risk for both classes.

$$
\begin{aligned}
g_1(x) &= g_2(x) \\
\ln P(\omega_1) + \ln P(x \mid \omega_1) &= \ln P(\omega_2) + \ln P(x \mid \omega_2) \\
\ln P(x \mid \omega_1) &= \ln P(x \mid \omega_2)
\end{aligned}
$$

Furthermore, the feature vectors of the two classes are normally distributed, meaning that

$$
P(x \mid \omega_i) = \frac{1}{(2\pi)^{\frac{d}{2}} \left|\Sigma_i\right|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i)} \qquad\qquad i = 1, 2
$$

So the previous equality becomes

$$
\begin{aligned}
-\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln\left|\Sigma_1\right| - \frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) &= -\frac{d}{2}\ln 2\pi - \frac{1}{2}\ln\left|\Sigma_2\right| - \frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2) \\
-\frac{1}{2}\ln\left|\Sigma_1\right| - \frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) &= -\frac{1}{2}\ln\left|\Sigma_2\right| - \frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)
\end{aligned}
$$

The covariance matrices Σ_i have nice properties allowing us to simplify this equation.

$$
\left|\Sigma_1\right| = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{vmatrix} = 1 \implies \frac{1}{2}\ln\left|\Sigma_1\right| = 0 \tag{1}
$$

$$
\left|\Sigma_2\right| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \implies \frac{1}{2}\ln\left|\Sigma_2\right| = \ln 2 \tag{2}
$$

$$
\Sigma_1^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \text{because } \Sigma_1 \text{ is diagonal} \tag{3}
$$

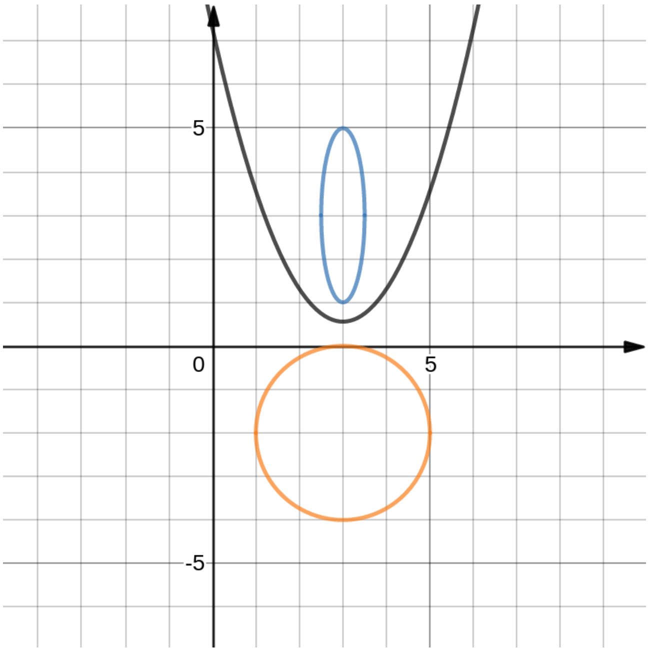
$$
\Sigma_2^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \text{because } \Sigma_2 \text{ is diagonal} \tag{4}
$$

This allows us to write the previous equality as

$$
\begin{aligned}
-\frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)+\frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)+\ln 2 &= 0\\
-\frac{1}{4}\begin{bmatrix}x_1-3\\x_2-3\end{bmatrix}^T\begin{bmatrix}4&0\\0&1\end{bmatrix}\begin{bmatrix}x_1-3\\x_2-3\end{bmatrix}+\frac{1}{4}\begin{bmatrix}x_1-3\\x_2+2\end{bmatrix}^T\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1-3\\x_2+2\end{bmatrix}+\ln 2 &= 0\\
-\frac{1}{4}[(4x_1-12)(x_1-3)+(x_2-3)^2]+\frac{1}{4}[(x_1-3)^2+(x_2+2)^2]+\ln 2 &= 0\\
-\frac{1}{4}(4x_1^2-24x_1+36+x_2^2-6x_2+9)+\frac{1}{4}(x_1^2-6x_1+9+x_2^2+4x_2+4)+\ln 2 &= 0\\
\frac{3}{4}x_1^2+\frac{9}{2}x_1-8+\ln 2 &= x_2
\end{aligned}
$$

Thus, the decision border is a parabola.

c)



Problem 3