Machine Learning - Theoretical exercise 2

Téo Bouvard

January 17, 2020

Problem 1

a) In the following, we use the notation $\lambda(\alpha_i \mid \omega_j) \Leftrightarrow \lambda_{ij}$

$\lambda_{11} = 0$	correct classification of toxic container
$\lambda_{12} = 10^5$	incorrect classification of toxic container
$\lambda_{22} = 0$	correct classification of non-toxic container
$\lambda_{21} = 250$	incorrect classification of non-toxic container

b)

$$R(\alpha_1 \mid x) = \lambda_{11} P(\omega_1 \mid x) + \lambda_{12} P(\omega_2 \mid x)$$
$$R(\alpha_2 \mid x) = \lambda_{21} P(\omega_1 \mid x) + \lambda_{22} P(\omega_2 \mid x)$$

As $\lambda_{11} = 0$ and $\lambda_{22} = 0$,

$$R(\alpha_1 \mid x) = \lambda_{12} P(\omega_2 \mid x)$$
$$R(\alpha_2 \mid x) = \lambda_{21} P(\omega_1 \mid x)$$

c) To determine the decision boundary that minimizes the average cost, we solve the equality of conditional loss functions.

$$\begin{aligned}
R(\alpha_1 \mid x) &= R(\alpha_2 \mid x) \\
\lambda_{12} P(\omega_2 \mid x) &= \lambda_{21} P(\omega_1 \mid x) \\
\frac{\lambda_{12}}{\lambda_{21}} \frac{P(\omega_2) P(x \mid \omega_2)}{P(x)} &= \frac{P(\omega_1) P(x \mid \omega_1)}{P(x)} && \text{(Bayes Rule)} \\
\frac{\lambda_{12} P(\omega_2)}{\lambda_{21} P(\omega_1)} P(x \mid \omega_2) &= P(x \mid \omega_1)
\end{aligned}$$

Let $K = \frac{\lambda_{12} P(\omega_2)}{\lambda_{21} P(\omega_1)}$. Furthermore, we know that $P(x \mid \omega_1) \sim \mathcal{N}(\mu_1, \sigma^2)$ and $P(x \mid \omega_2) \sim \mathcal{N}(\mu_2, \sigma^2)$.

$$\begin{aligned}
K\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu_1)^2}{2\sigma}} &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu_2)^2}{2\sigma}} \\
\ln\left(Ke^{-\frac{(x-\mu_1)^2}{2\sigma}}\right) &= \ln\left(e^{-\frac{(x-\mu_2)^2}{2\sigma}}\right) \\
\ln K - \frac{(x-\mu_1)^2}{2\sigma} &= -\frac{(x-\mu_2)^2}{2\sigma} \\
\ln K + \frac{1}{2\sigma}((x-\mu_2)^2 - (x-\mu_1)^2) &= 0 \\
\ln K + \frac{1}{2\sigma}(x^2 - 2\mu_2 x + \mu_2^2 - x^2 + 2\mu_1 x - \mu_1^2) &= 0 \\
\ln K + \frac{1}{2\sigma}(2x(\mu_1 - \mu_2) + \mu_2^2 - \mu_1^2) &= 0 \\
\ln K + \frac{\mu_1 - \mu_2}{\sigma}x + \frac{\mu_2^2 - \mu_1^2}{2\sigma} &= 0 \\
\frac{\mu_1 - \mu_2}{\sigma}x &= \frac{\mu_1^2 - \mu_2^2}{2\sigma} - \ln K \\
x &= \frac{\mu_1^2 - \mu_2^2}{2(\mu_1 - \mu_2)} - \sigma \ln K \\
x &= \frac{\mu_1 + \mu_2}{2} - \frac{\sigma}{\mu_1 - \mu_2}\ln K
\end{aligned}$$

Numerically solving this equation gives us the decsion boundary

$$\begin{aligned}
x &= \frac{0.4 + 0.2}{2} - \frac{10^{-4}}{0.4 - 0.2} \times \ln(\frac{25 \times 10^5}{250}) \\
x &= 0.2954
\end{aligned}$$

d) minimum average cost ?