Algorithm Theory - Assignment 1

Téo Bouvard

February 5, 2020

Problem 1

In this problem, the objective is to maximize the profit of selling products X and Y, while satisfying production constraints on these products. The profit can be computed as

$$\begin{aligned} profit &= revenue - cost \\ &= revenue - (time_{machine} \times cost_{machine} + time_{craftsman} \times cost_{craftsman}) \end{aligned}$$

We can compute profits for each of the products

$$\begin{aligned} profit(X) &= 200 - (\frac{15}{60} \times 100 + \frac{20}{60} \times 20) = \frac{505}{3} \\ profit(Y) &= 300 - (\frac{20}{60} \times 100 + \frac{30}{60} \times 20) = \frac{770}{3} \end{aligned}$$

And formulate the problem as a Linear Programming problem. Let n_X and n_Y be the number of products X and Y produced.

$$\begin{aligned} &\text{maximize } n_X \times profit(X) + n_Y \times profit(Y) \\ &\text{subject to } \begin{cases} n_X \times time_{machine}(X) + n_Y \times time_{machine}(Y) \leq 40 \times 60 \\ n_X \times time_{craftsman}(X) + n_Y \times time_{craftsman}(Y) \leq 35 \times 60 \\ n_X \geq 10 \\ n_X, n_Y \geq 0 \end{cases} \end{aligned}$$

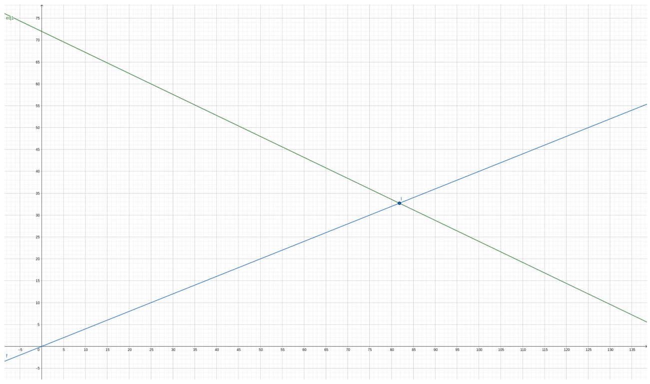
Which can be simplified as

$$\begin{aligned} &\text{maximize } n_X \times \frac{505}{3} + n_Y \times \frac{770}{3} \\ &\text{subject to } \begin{cases} 15 \times n_X + 20 \times n_Y \leq 2400 \\ 20 \times n_X + 30 \times n_Y \leq 2100 \\ n_X \geq 10 \\ n_Y \geq 0 \end{cases} \end{aligned}$$

Problem 2

a) Let n_A and n_B be the number of products A and B produced.

$$\begin{aligned} &\text{maximize } n_A \times 3 + n_B \times 5 \\ &\text{subject to } \begin{cases} 12 \times n_A + 25 \times n_B \leq 30 \times 60 \\ 2 \times n_B - 5 \times n_A \geq 0 \\ n_A, n_B \geq 0 \end{cases} \end{aligned}$$



By graphing the feasible region, we see three candidate points. The first one $I_0 = (0, 0)$ can be trivially discarded as it leads to a profit of 0\$. Let's evaluate the objective function at the two other points. At $I_1 = (72, 0)$ the profit is 216\$. To compute the coordinates of the last point, we solve the following system.

and at $I_2 = (81.8, 32.7)$ the profit is 408.9\$ which is the highest profit in this case. However, n_A and n_B are not integers. It is not specified in the exercise whether they should be or not, but as the problem is about a weekly production, we can assume that they do not need to be integer quantities as the weeks are continuous. If we wanted to optimal integer solution, we should pick $n_A = 81$ and $n_B = 33$, leading to a profit of 408\$.

b) By doubling the production capacity without modifying the other constraints, the resulting profit is doubled. The company should then pay less than 408\$ for renting an extra machine thath is profitable.

Problem 3

a) We want to minimize the total cost of transportation, while respecting constraints on the flights. This problem can be formulated as the following linear programming problem. Let n_A and n_B be the number of flights flown with aircrafts A and B respectively.

$$\text{minimize } n_A \times 10000 + n_B \times 12000$$
$$\text{subject to } \begin{cases} 30 \times n_A + 15 \times n_B \geq 300 \\ 500 \times n_A - 750 \times n_B \geq 9000 \\ n_A + n_B \leq 16 \\ n_A, n_B \geq 0 \end{cases}$$

With the graph method, we can identify the three points delimiting the feasible region. We first determine their coordinates, and then evaluate the objective function at these points.

$$I_1 : \begin{cases} 30x + 15y = 300 \\ x + \quad y = 16 \end{cases} \implies \begin{cases} x = 4 \\ y = 12 \end{cases} \implies \text{cost } = 184000\,\$$$
$$I_2 : \begin{cases} 30x + \ 15y = 300 \\ 500x + 750y = 9000 \end{cases} \implies \begin{cases} x = 6 \\ y = 8 \end{cases} \implies \text{cost } = 156000\,\$$$
$$I_3 : \begin{cases} x + \quad y = 16 \\ 500x + 750y = 9000 \end{cases} \implies \begin{cases} x = 12 \\ y = 4 \end{cases} \implies \text{cost } = 168000\,\$$$

The lowest cost is achieved by using 6 flights with A and 8 flights with B, for a total cost of 156000\$.

Problem 4

a) Graph method

$$I_2 : \begin{cases} x_1 \quad = 0 \\ \quad x_2 = 0 \end{cases} \implies \text{objective } = 0$$

$$I_3 : \begin{cases} x_1 \quad = 30 \\ \quad x_2 = 0 \end{cases} \implies \text{objective } = 90$$

$$I_3 : \begin{cases} x_1 \quad = 0 \\ \quad x_2 = 25 \end{cases} \implies \text{objective } = 125$$

$$I_1 : \begin{cases} x_1 + 2x_2 = 50 \\ 8x_1 + 3x_2 = 240 \end{cases} \implies \begin{cases} x_1 \quad = \frac{490}{13} \\ \quad x_2 = \frac{160}{13} \end{cases} \implies \text{objective } = \frac{2270}{13} \approx 174.6$$

b) Simplex algorithm

We first convert the linear problem its the slack form by introducing two variables s_1 and s_2.

$$\text{maximize } z = 3x_1 + 5x_2$$
$$\text{subject to } \begin{cases} x_1 + 2x_2 + s_1 = 50 \\ 8x_1 + 3x_2 + s_2 = 240 \\ x_1, x_2, s_1, s_2 \geq 0 \end{cases}$$

A basic feasible solution is $(x_1, x_2, s_1, s_2) = (0, 0, 50, 240)$. We choose x_2 as the entering variable as it has the highest coefficient in the objective function. To choose the leaving variable, we find the tightest constraint on x_2. In the first constraint, x_2 is limited to 25, and in the second constraint it is limited to 80. Thus the leaving variable is s_1 We rewrite the problem by switching the entering and the leaving variable. $x_2 = 25 - \frac{x_1}{2} - \frac{s_1}{2}$

$$\text{maximize } z = 125 + \frac{x_1}{2} - \frac{5}{2}s_1$$
$$\text{subject to } \begin{cases} 50 - x_1 - 2x_2 = s_1 \\ 8x_1 + 3x_2 + s_2 = 240 \\ x_1, x_2, s_1, s_2 \geq 0 \end{cases}$$

The only remaining entering variable able to increase the objective function is x_1.