Machine Learning - Theoretical exercise 1

Téo Bouvard

January 9, 2020

1 Problem 1

a) From the sum rule, we have

$$\int_\Omega \rho(\omega)\, d\omega = 1$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \rho(x_1, x_2)\, dx_1 dx_2 = 1$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} c\, dx_1 dx_2 = 1 \qquad \text{p.d.f is zero outside of these bounds}$$

$$c(b_1 - a_1)(b_2 - a_2) = 1$$

$$\frac{1}{(b_1 - a_1)(b_2 - a_2)} = c$$

b) Using the formula to compute the expected value, we have

$$\begin{aligned}
\mathbb{E}(x) &= \int_\Omega \omega\rho(\omega)\, d\omega \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rho(x_1, x_2)\, dx_1 dx_2 \\
&= c\int_{a_2}^{b_2}\int_{a_1}^{b_1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dx_1 dx_2 \\
&= c\int_{a_2}^{b_2} \begin{bmatrix} \frac{b_1^2 - a_1^2}{2} \\ x_2(b_1 - a_1) \end{bmatrix} dx_2 \\
&= c\begin{bmatrix} \frac{b_1^2 - a_1^2}{2}(b_2 - a_2) \\ \frac{b_2^2 - a_2^2}{2}(b_1 - a_1) \end{bmatrix}
\end{aligned}$$

Using $a^2 - b^2 = (a - b)(a + b)$, we can factor factor this expression as

$$\mathbb{E}(x) = c\begin{bmatrix} \frac{(b_1 - a_1)(b_1 + a_1)(b_2 - a_2)}{2} \\ \frac{(b_1 - a_1)(b_2 + a_2)(b_2 - a_2)}{2} \end{bmatrix}$$

If we replace c with the value computed in the previous question, we can simplify it further as

$$\mathbb{E}(x) = \frac{1}{(b_1-a_1)(b_2-a_2)} \begin{bmatrix} \frac{(b_1-a_1)(b_1+a_1)(b_2-a_2)}{2} \\ \frac{(b_1-a_1)(b_2+a_2)(b_2-a_2)}{2} \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} b_1+a_1 \\ b_2+a_2 \end{bmatrix}$$

DO SKETCH

c) We compute the covariance matrix.

$$\mathrm{Cov}(x) = \mathbb{E}((x-\mu)(x-\mu)^T)$$
$$= \mathbb{E}(xx^T) - \mu\mu^T$$

We compute each term independently.

$$\mathbb{E}(xx^T) = \mathbb{E}(\begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix})$$

Not so sure about that ?

$$\mu\mu^T = \frac{1}{2} \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} a_1+b_1 & a_2+b_2 \end{bmatrix}$$
$$= \frac{1}{4} \begin{bmatrix} (a_1+b_1)^2 & (a_1+b_1)(a_2+b_2) \\ (a_1+b_1)(a_2+b_2) & (a_2+b_2)^2 \end{bmatrix}$$

2 Problem 2

a) We first compute the eigenvalues.