Machine Learning - Theoretical exercise 4

Téo Bouvard

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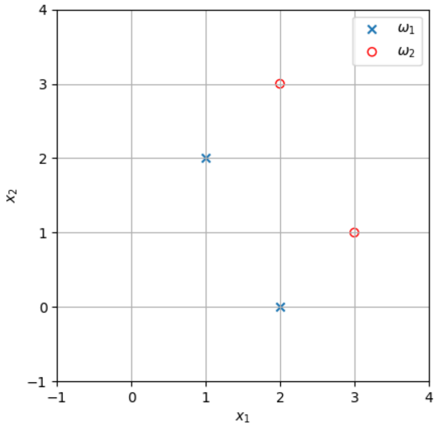
Problem 1

a) We have the same number of training samples for classes ω_1 and ω_2, thus the prior probabilities are equal for both classes i.e. $P(\omega_1) = 0.5$ and $P(\omega_2) = 0.5$



b) We compute θ according to the LS-method.

$$\theta = (X^T X)^{-1} X^T y \tag{1}$$

where

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}^T \qquad y = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

The steps to solve this equation are shown below.

$$X^T X = \begin{bmatrix} 18 & 11 & 8 \\ 11 & 14 & 6 \\ 8 & 6 & 4 \end{bmatrix}$$

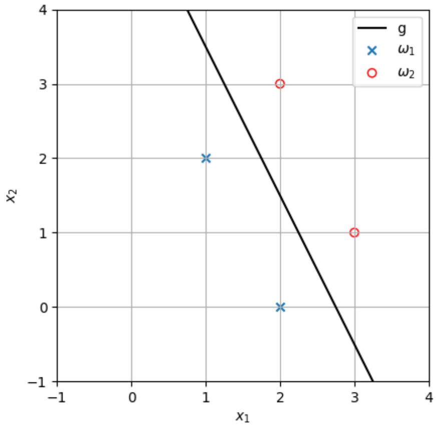
$$(X^T X)^{-1} = \begin{bmatrix} 18 & 11 & 8 \\ 11 & 14 & 6 \\ 8 & 6 & 4 \end{bmatrix}$$

$$(X^T X)^{-1} X^T = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{6} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\ \frac{5}{4} & -\frac{13}{12} & -\frac{3}{4} & -\frac{7}{12} \end{bmatrix}$$

$$\theta = \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{11}{3} \end{bmatrix}$$

To determine the decision boundary, we find the root of the discriminant function.

$$\begin{aligned} g(x) &= 0 \\ -\frac{4}{3}x_1 - \frac{2}{3}x_2 + \frac{11}{3} &= 0 \\ x_2 &= -2x_1 + \frac{11}{2} \end{aligned}$$

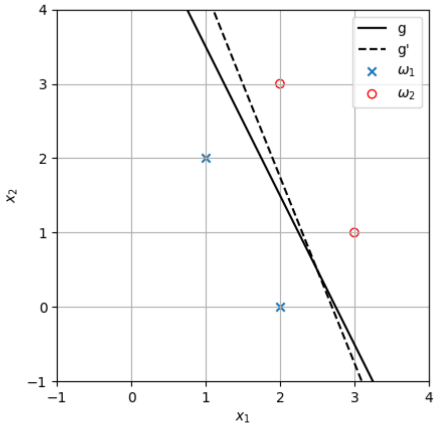


c) If we set $y_4 = -0.5$, we reduce the weight of the fourth training sample, which modifies θ.

$$\theta' = \begin{bmatrix} -\frac{5}{4} \\ -\frac{1}{2} \\ \frac{27}{8} \end{bmatrix}$$

Because $\theta' \neq \theta$, the decision boundary also changes.

$$\begin{aligned} g'(x) &= 0 \\ -\frac{5}{4}x_1 - \frac{1}{2}x_2 + \frac{27}{8} &= 0 \\ x_2 &= -\frac{5}{2}x_1 + \frac{27}{4} \end{aligned}$$



We can see that decreasing the weight of a training sample moves the decison boundary closer to it.

d) We now compute θ with the LMS-method. We use μ to denote the the learning rate. The descent vector at each iteration is denoted ∇.

Initialization

$$\mu^{(0)} = 0.5 \qquad \boldsymbol{\theta}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \theta = 1$$

Iteration 1

$$\begin{aligned}
\nabla &= \mu^{(1)}(y_1 - \boldsymbol{\theta}^{(0)T} y_1 x_1) y_1 x_1 \\
&= \frac{0.5}{1}\left(\begin{bmatrix}-1\end{bmatrix} - \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1\end{bmatrix}\begin{bmatrix}1 \\ 2 \\ 1\end{bmatrix}\right)\begin{bmatrix}-1\end{bmatrix}\begin{bmatrix}1 \\ 2 \\ 1\end{bmatrix} \\
&= -\frac{3}{2}\begin{bmatrix}1 \\ 2 \\ 1\end{bmatrix}
\end{aligned}$$

$$\left|\nabla\right| = \frac{3\sqrt{6}}{2} \approx 3.7 > \theta$$

$\left|\nabla\right|$ is greater than the threshold, so $\boldsymbol{\theta}$ is updated.

$$\begin{aligned}
\boldsymbol{\theta}^{(1)} &= \boldsymbol{\theta}^{(0)} + \nabla \\
&= -\frac{1}{2}\begin{bmatrix}1 \\ 4 \\ 1\end{bmatrix}
\end{aligned}$$

Iteration 2

$$\begin{aligned}
\nabla &= \mu^{(2)}(y_2 - \boldsymbol{\theta}^{(1)T} y_2 x_2) y_2 x_2 \\
&= \frac{0.5}{2}\left(\begin{bmatrix}-1\end{bmatrix} - \begin{bmatrix}-\frac{1}{2} & -2 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-1\end{bmatrix}\begin{bmatrix}2 \\ 0 \\ 1\end{bmatrix}\right)\begin{bmatrix}-1\end{bmatrix}\begin{bmatrix}2 \\ 0 \\ 1\end{bmatrix} \\
&= \begin{bmatrix}\frac{5}{4} \\ 0 \\ \frac{5}{8}\end{bmatrix}
\end{aligned}$$

$$\left|\nabla\right| = \frac{5\sqrt{5}}{8} \approx 1.4 > \theta$$

$\left|\nabla\right|$ is greater than the threshold, so $\boldsymbol{\theta}$ is updated.

$$\begin{aligned}
\boldsymbol{\theta}^{(2)} &= \boldsymbol{\theta}^{(1)} + \nabla \\
&= \begin{bmatrix}\frac{3}{4} \\ -2 \\ \frac{1}{8}\end{bmatrix}
\end{aligned}$$

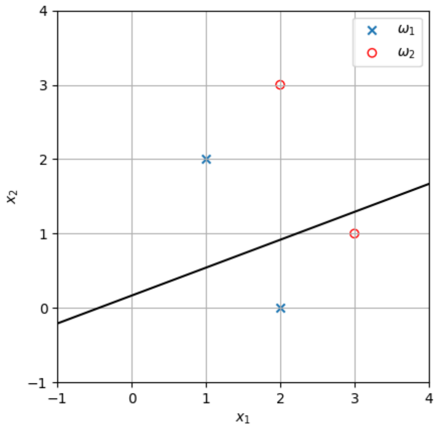
Iteration 3

$$
\begin{aligned}
\nabla &= \mu^{(3)}(y_3 - \boldsymbol{\theta}^{(2)T} y_3 x_3) y_3 x_3 \\
&= \frac{0.5}{3}\left(\begin{bmatrix}1\end{bmatrix} - \begin{bmatrix}\frac{3}{4} & -2 & \frac{1}{8}\end{bmatrix}\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}3\\1\\1\end{bmatrix}\right)\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}3\\1\\1\end{bmatrix} \\
&= \begin{bmatrix}\frac{5}{16}\\ \frac{5}{48}\\ \frac{5}{48}\end{bmatrix}
\end{aligned}
$$

$$
\left|\nabla\right| = \frac{5\sqrt{11}}{48} \approx 0.3 < \theta
$$

$\left|\nabla\right|$ is smaller than the threshold, so $\boldsymbol{\theta}$ is not updated and the algorithm terminates with $\boldsymbol{\theta}^{(2)} = \begin{bmatrix}\frac{3}{4}\\ -2\\ \frac{1}{8}\end{bmatrix}$.

If we plot the resulting decision boundary, we observe that the converged value of $\boldsymbol{\theta}$ does not discriminate between the classes.



Problem 2

a) An analytical solution to the equation $\boldsymbol{X\theta} = \boldsymbol{y}$ would be to find the inverse of $\boldsymbol{X}$ and compute $\boldsymbol{\theta} = \boldsymbol{X^{-1}y}$ directly. However, this solution assume that $\boldsymbol{X}$ is invertible. In practice $\boldsymbol{X}$ is often rectangular, with more rows (samples) than columns (features). Trying to find an exact solution would lead to have more equations than unknowns which is not solvable in general.

b) In order to find $\boldsymbol{\theta}$ minimizing the function $\|\boldsymbol{X\theta - y}\|^2$, we can differentiate this function with respect to $\boldsymbol{\theta}$ and find its root.

$$
\frac{\partial\|\boldsymbol{X\theta} - \boldsymbol{y}\|^2}{\partial\boldsymbol{\theta}} = 2\boldsymbol{X}^T(\boldsymbol{X\theta} - \boldsymbol{y})
$$

We now set the derivative to zero.

$$
\begin{aligned}
2\boldsymbol{X}^T(\boldsymbol{X\theta} - \boldsymbol{y}) &= 0 \\
\boldsymbol{X}^T\boldsymbol{X\theta} &= \boldsymbol{X^Ty} \\
\boldsymbol{\theta} &= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X^Ty}
\end{aligned}
$$

In this case, $\boldsymbol{X}^T\boldsymbol{X}$ is guaranteed to be invertible because it is a square symmetric matrix.

c) Let $\boldsymbol{\theta}_*$ be the value of $\boldsymbol{\theta}$ minimizing the squared error. We can rewrite the distance function as such.

$$\|\boldsymbol{X}\boldsymbol{\theta}_* - \boldsymbol{y}\|^2 = \|\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} - \boldsymbol{y}\|^2$$

d) For the problem to be lineraly separable, the distance function should be equal to zero.

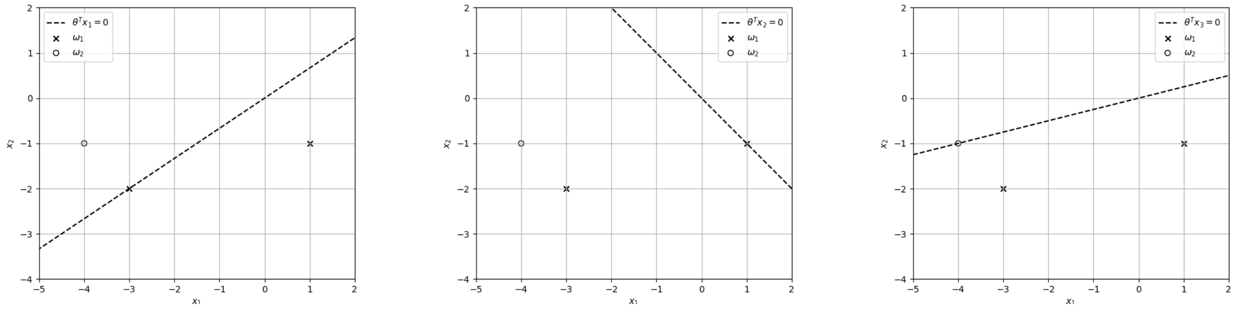
$$\|\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} - \boldsymbol{y}\|^2 = 0 \implies \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y} = \boldsymbol{y} \implies \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{I}$$

Problem 3

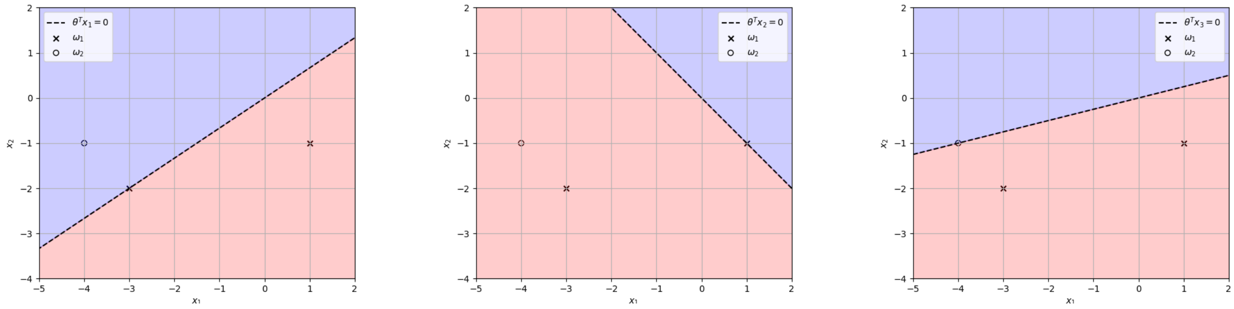
a) Samples are labeled as +1 if they belong to ω_1 and -1 if they belong to ω_2.

$$y_1 = 1 \qquad\qquad y_2 = 1 \qquad\qquad y_3 = -1$$

b) For each sample, we draw the decision boundary $\boldsymbol{\theta}_T x_i = 0$



c) If we colour the positive side of the discriminant function in red and the negative side in blue, we get the following regions.



The solution region lies between the decision boundaries shown in the first and the third plot.

d) We apply the Batch Perceptron algorithm