Machine Learning - Theoretical exercise 1

Téo Bouvard

January 9, 2020

1 Problem 1

a) From the sum rule, we have

$$\begin{aligned}
\int_\Omega \rho(\omega)\, d\omega &= 1 \\
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(x_1, x_2)\, dx_1 dx_2 &= 1 \\
\int_{a_1}^{b_1} \int_{a_2}^{b_2} c\, dx_1 dx_2 &= 1 \\
c(b_1 - a_1)(b_2 - a_2) &= 1 \\
\frac{1}{(b_1 - a_1)(b_2 - a_2)} &= c
\end{aligned}$$

b) Using the formula to compute the expected value, we have

$$\begin{aligned}
\mathbb{E}(x) &= \int_\Omega \omega \rho(\omega)\, d\omega \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rho(x_1, x_2)\, dx_1 dx_2 \\
&= c \int_{a_2}^{b_2} \int_{a_1}^{b_1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dx_1 dx_2 \\
&= c \int_{a_2}^{b_2} \begin{bmatrix} \frac{b_1^2 - a_1^2}{2} \\ x_2(b_1 - a_1) \end{bmatrix} dx_2 \\
&= c \begin{bmatrix} \frac{b_1^2 - a_1^2}{2}(b_2 - a_2) \\ \frac{b_2^2 - a_2^2}{2}(b_1 - a_1) \end{bmatrix}
\end{aligned}$$

Using $a^2 - b^2 = (a - b)(a + b)$, we can factor factor this expression as

$$\mathbb{E}(x) = c \begin{bmatrix} \frac{(b_1 - a_1)(b_1 + a_1)(b_2 - a_2)}{2} \\ \frac{(b_1 - a_1)(b_2 + a_2)(b_2 - a_2)}{2} \end{bmatrix}$$

If we replace c with the value computed in the previous question, we can simplify it further as

$$
\begin{aligned}
\mathbb{E}(x) &= \frac{1}{(b_1 - a_1)(b_2 - a_2)} \begin{bmatrix} \frac{(b_1-a_1)(b_1+a_1)(b_2-a_2)}{2} \\ \frac{(b_1-a_1)(b_2+a_2)(b_2-a_2)}{2} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \end{bmatrix}
\end{aligned}
$$

DO SKETCH

c) We compute the covariance matrix.

$$
\begin{aligned}
\mathrm{Cov}(x) &= \mathbb{E}((x - \mu)(x - \mu)^T) \\
&= \mathbb{E}(xx^T) - \mu\mu^T
\end{aligned}
$$

We compute each term independently.

$$
\mathbb{E}(xx^T) = \mathbb{E}(\begin{bmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{bmatrix})
$$

Not so sure about that ?

$$
\begin{aligned}
\mu\mu^T &= \frac{1}{2} \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} (a_1 + b_1)^2 & (a_1 + b_1)(a_2 + b_2) \\ (a_1 + b_1)(a_2 + b_2) & (a_2 + b_2)^2 \end{bmatrix}
\end{aligned}
$$

2 Problem 2

a) We first compute the eigenvalues.