Practical Problems

1. i. How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

ii. Repeat part i. under the assumption that no letter or number can be repeated in a single license plate.

$$26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3,4,3,1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?

$$6 \cdot 6 \cdot 6 \cdot 6$$

3. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

$$20!$$

4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

Si todos pueden tocar los 4 instrumentos

$$4!$$

Si John y Jim pueden tocar los 4 instrumentos, pero Jay y Jack solo pueden tocar el piano y la batería

$$2! \cdot 2!$$

5. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?

$$8 \cdot 2 \cdot 9$$

Si el primer dígito es 4

$$2 \cdot 9$$

6. A well-known nursery rhyme starts as follows:

" As I was going to St. Ives
I met a man with 7 wives.
Each wife had 7 sacks.
Each sack had 7 cats.
Each cat had 7 kittens..."

How many kittens did the traveler meet?

$$7 \cdot 7 \cdot 7 \cdot 7$$

7. i. In how many ways can 3 boys and 3 girls sit in a row?

$$6!$$

ii. In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?

$$2! \cdot 3! \cdot 3!$$

iii. In how many ways if only the boys must sit together?

Hay 3! formas de ordenar a los chicos, luego tenemos 4! formas de ordenar al grupo y a las tres chicas

$$3! \cdot 4!$$

iv. In how many ways if no two people of the same sex are allowed to sit together?

Colocamos a un estudiante de los 6, luego uno de los tres posibles del género opuesto, luego uno de los dos restantes del mismo género que el primero, . . .

$$6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1$$

8. How many different letter arrangements can be made from the letters

i. Fluke

$$5!$$

ii. Propose

$$\frac{7!}{2!\cdot 2!}$$

iii. Mississippi

$$\frac{11!}{4!\cdot 4!\cdot 2!}$$

iv. Arrange

$$\frac{7!}{2!\cdot 2!}$$

9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

$$\frac{12!}{6!\cdot 4!}$$

10. In how many ways can 8 people be seated in a row if

i. there are no restrictions on the seating arrangement?

$$8!$$

ii. persons A and B must sit next to each other?

$$2\cdot 7!$$

iii. there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?

Siguiendo la lógica de 7. iv.

$$8\cdot 4\cdot 3\cdot 3\cdot 2\cdot 2$$

iv. there are 5 men and they must sit next to one another?

Hay 5! formas de sentar a los hombres entre sí, luego 4! formas de ordenar el grupo de hombres y las 3 mujeres.

$$5!\cdot 4!$$

v. there are 4 married couples and each couple must sit together?

Cada pareja tiene 2 formas de sentarse, luego hay 4! formas de ubicar a las parejas.

$$2^4\cdot 4!$$

11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if

i. the books can be arranged in any order?

$$6!$$

ii. the mathematics books must be together and the novels must be together?

$$3!\cdot 2!\cdot 3!$$

iii. the novels must be together, but the other books can be arranged in any order?

$$3!\cdot 4!$$

12. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if

i. a student can receive any number of awards?

$$30^5$$

ii. each student can receive at most 1 award?

$$30\cdot 29\cdot 28\cdot 27\cdot 26$$

13. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

$$\binom{20}{2}$$

14. How many 5-card poker hands are there?

$$\binom{52}{5}$$

15. A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

Formas de elegir 5 hombres y 5 mujeres

$$\binom{10}{5} \cdot \binom{12}{5}$$

formas de emparejarlos

$$5!$$

sigue que

$$\binom{10}{5} \cdot \binom{12}{5} \cdot 5!$$

16. A student has to sell 2 books from a collection of 6 maths, 7 science, and 4 economics books. How many choices are possible if

i. both books are to be on the same subject?

$$\binom{6}{2} + \binom{7}{2} + \binom{4}{2}$$

ii. the books are to be on different subjects?

$$6 \cdot 7 + 6 \cdot 4 + 7 \cdot 4$$

17. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

$$\binom{10}{7} \cdot 7!$$

18. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?

$$\binom{5}{2} \cdot \binom{6}{2} \cdot \binom{4}{3}$$

19. From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if

i. 2 of the men refuse to serve together?

$$\binom{8}{3} \cdot \left(\binom{6}{3} - \binom{4}{1} \right)$$

ii. 2 of the women refuse to serve together?

$$\binom{6}{3} \cdot \left(\binom{8}{3} - \binom{6}{1} \right)$$

iii. 1 man and 1 woman refuse to serve together?

$$\binom{6}{3} \cdot \binom{8}{3} - \binom{5}{2} \cdot \binom{7}{2}$$

20. A person has 8 friends, of whom 5 will be invited to a party.

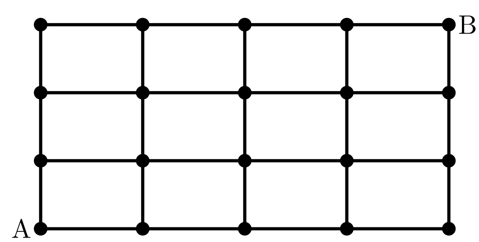
i. How many choices are there if 2 of the friends are feuding and will not attend together?

$$\binom{8}{5} - \binom{6}{3}$$

ii. How many choices if 2 of the friends will only attend together?

$$\binom{6}{5} + \binom{6}{3}$$

21. Consider the grid shown



Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?

Considerar el siguiente ordenamiento lineal

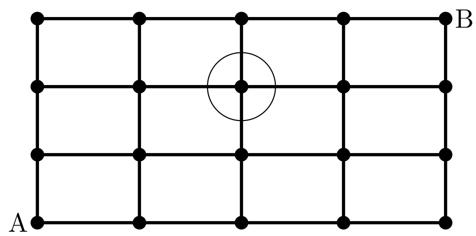
$$\rightarrow \rightarrow \rightarrow \rightarrow \uparrow \uparrow \uparrow$$

donde $\rightarrow$ indica un movimiento a la derecha y $\uparrow$ indica un movimiento hacia arriba.

Queremos todas las permutaciones únicas de esta secuencia

$$\frac{7!}{4! \cdot 3!}$$

22. In Problem 21, how many different paths are there from A to B that go through the point circled in the following lattice?



Considerar la secuencia

$$\rightarrow \rightarrow \uparrow \uparrow$$

las permutaciones únicas de esta secuencia son los posibles caminos de A al punto marcado.

Considerar la secuencia

$$\rightarrow \rightarrow \uparrow$$

las permutaciones únicas de esta secuencia son los posibles caminos del punto marcado a B.

Luego sigue que hay

$$\frac{4!}{2! \cdot 2!} \cdot \frac{3!}{2!}$$

posibles caminos de A a B pasando por el punto marcado.

23. A psychology laboratory conducting dream research contains 3 rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these 6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

$$3! \cdot 2^3$$

24. Expand $(3x^2 + y)^5$.

Usando el teorema del binomio

$$(3x^2 + y)^5 = \sum_{i=0}^{5} \binom{5}{i} \cdot (3x^2)^i \cdot y^{5-i}$$

25. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

$$\binom{52}{13, 13, 13, 13}$$

26. Expand $(x_1 + 2x_2 + 3x_3)^4$.

usando el teorema del multinomio $(x_1 + 2x_2 + 3x_3)^4$ equivale a

$$\sum_{\substack{(n_1,n_2,n_3): \\ n_1+n_2+n_3=4}} \binom{4}{n_1, n_2, n_3} \cdot x_1^{n_1} \cdot (2x_2)^{n_2} \cdot (3x_3)^{n_3}$$

27. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

$$\binom{12}{3, 4, 5}$$

28. If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

$$4^8$$

Si cada escuela debe recibir aunque sea 2 maestros

$$\binom{8}{2, 2, 2, 2}$$

29. Ten weight lifters are competing in a team weightlifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in top three and 2 in the bottom three?

$$\binom{10}{3,4,2,1}$$

Si requerimos que estados unidos tenga un competidor entre los tres primeros y 2 entre los tres últimos

$$\binom{3}{1}\cdot\binom{3}{2}\cdot\binom{7}{4,2,1}$$

30. Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other and the Russian and U.S. delegates are not to be next to each other?

Hay $2\cdot 9!$ formas de construir ordenamientos lineales donde Francia está al lado de Inglaterra. Luego hay $2\cdot 2\cdot 8!$ formas de construir ordenamientos lineales donde Francia está al lado de Inglaterra y Estados Unidos está al lado de Rusia, entonces

$$2\cdot 9! - 4\cdot 8!$$

31. If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?

Buscamos soluciones a

$$x_1+x_2+x_3+x_4=8,\ x_i\geq 0,\ 1\leq i\leq 4$$

entonces

$$\binom{8+4-1}{4-1}$$

si cada escuela debe recibir aunque sea 1 pizarrón

$$x_1+x_2+x_3+x_4=8,\ x_i\geq 1,\ 1\leq i\leq 4$$

entonces

$$\binom{8-1}{4-1}$$

32. An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

Buscamos soluciones a

$$x_1+x_2+x_3+x_4+x_5+x_6=8,\ x_i\geq 0,\ 1\leq i\leq 6$$

entonces

$$\binom{8+6-1}{6-1}$$

si las 8 personas consisten en 5 hombres y 3 mujeres y el operador puede diferenciar entre hombres y mujeres

$$\begin{cases} x_1+x_2+x_3+x_4+x_5+x_6=5,\ x_i\geq 0,\ 1\leq i\leq 6\\ y_1+y_2+y_3+y_4+y_5+y_6=3,\ y_i\geq 0,\ 1\leq i\leq 6\end{cases}$$

entonces

$$\binom{5+6-1}{6-1}\cdot\binom{3+6-1}{6-1}$$

33. We have \$20,000 that must be invested among 4 possible opportunities. Each investment must be integral in units of \$1000, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are \$2000, \$2000, \$3000, and \$4000. How many different investment strategies are available if

i. an investment must be made in each opportunity?

   Dado que debemos invertir en todas las oportunidades entonces debemos apartar \$11000 para cubrir las inversiones mínimas dejándonos \$9000 para invertir libremente. Buscamos soluciones a

$$x_1+x_2+x_3+x_4=9,\ x_i\geq 0,\ 1\leq i\leq 4$$

   entonces

$$\binom{9+4-1}{4-1}$$

ii. investments must be made in at least 3 of the 4 opportunities?

Considerar los casos

- Invirtiendo en las cuatro oportunidades
$$\binom{9+4-1}{4-1}$$
- No invirtiendo en alguna de las dos primeras oportunidades, nos quedan \$11000
$$\binom{11+3-1}{3-1}$$
- No invirtiendo en la tercer oportunidad, nos quedan \$12000
$$\binom{12+3-1}{3-1}$$
- No invirtiendo en la cuarta oportunidad, nos quedan \$13000
$$\binom{13+3-1}{3-1}$$

Combinando los casos disjuntos
$$\binom{12}{3}+2\cdot\binom{13}{2}+\binom{14}{2}+\binom{15}{2}$$

34. Suppose that 10 fish are caught at a lake that contains 5 distinct types of fish.

i. How many different outcomes are possible, where an outcome specifies the numbers of caught fish of each of the 5 types?

$$x_1+x_2+\cdots+x_5=10,\ x_i\geq 0,\ 1\leq i\leq 5$$

por lo tanto
$$\binom{10+5-1}{5-1}$$

ii. How many outcomes are possible when 3 of the 10 fish caught are trout?

$$x_1+x_2+\cdots+x_5=10,\ x_1=3,\ x_i\geq 0,\ 2\leq i\leq 5$$

por lo tanto
$$\binom{7+4-1}{4-1}$$

iii. How many when at least 2 of the 10 are trout?

$$x_1+x_2+\cdots+x_5=10,\ x_1\geq 2,\ x_i\geq 0,\ 2\leq i\leq 5$$

sea $y_1=x_1-2$, $y_i=x_i$, $2\leq i\leq 5$ entonces

$$y_1+\cdots+y_5=8,\ y_i\geq 0,\ 1\leq i\leq 5$$

por lo tanto
$$\binom{8+5-1}{5-1}$$

Theoretical Problems

1. Prove the generalized version of the basic counting principle.

Demostración. De la versión generalizada del principio de conteo básico.

Sea

$P(r)$:= La versión generalizada del principio de conteo básico es verdadera para r experimentos.

$P(2)$ es verdadero ya que es equivale al principio de conteo básico.

Sea $P(k)$ verdadera y sean $k+1$ experimentos a ser realizados tales que el primero tiene n_1 resultados posibles; y para cada uno de estos n_1 resultados posibles hay n_2 posibles resultados para el segundo experimento; y para cada uno de los posibles resultados de los dos primeros experimentos hay n_3 posibles resultados para el tercer experimento; … entonces es claro por $P(k)$ que los primeros k experimentos tienen $n_1\cdot n_2\cdots n_k$ posibles resultados.

Dado que el experimento $k+1$-ésimo tiene n_{k+1} resultados para todos los posibles resultados de los experimentos 1 a k entonces por el principio básico de conteo este experimento tiene $(n_1\cdot n_2\cdots n_k)\cdot n_{k+1}$ posibles resultados. □

2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

$$\sum_{i=1}^{m} n_i$$

3. In how many ways can r objects be selected from a set of n objects if the order of selection is considered relevant?

$$\frac{n!}{(n-r)!}$$

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n-r$ are white. Give a combinatorial explanation of this fact.

Considerar un arreglo lineal de los números 1 a n. Seleccionar r lugares y colocar en ellos las bolillas negras, en el resto ubicar las bolillas blancas.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

Si buscamos vectores tales que

$$\sum_{i=1}^{n} x_i = j$$

las formas de elegir j entre n para asignarles 1 son

$$\binom{n}{j}$$

Como queremos todos aquellos que sumen entre k y n entonces

$$\sum_{i=k}^{n} \binom{n}{i}$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

Seleccionar los $k+i$ primeros números, y entre ellos asignar a i el estatus de "ignorado".

$$\sum_{i=0}^{n-k} \binom{k+i}{i}$$

7. Give an analytic proof of

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Demostración. inductivamente en n.

$$\begin{aligned}
&\binom{n-1}{r-1} + \binom{n-1}{r} \\
&= \frac{(n-1)!}{(r-1)! \cdot (n-r)!} + \frac{(n-1)!}{r! \cdot (n-1-r)!} \\
&= \frac{n!}{r! \cdot (n-r)!} \cdot \left(\frac{r}{n} + \frac{n-r}{n}\right) \\
&= \binom{n}{r}
\end{aligned}$$

□

8. Prove that

$$\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}$$

Demostración. De la identidad anterior.

Considerar un ordenamiento lineal de los primeros $n+m$ números

$$1\ 2\ \ldots\ n+m$$

Todas las formas de tomar r números de entre $n+m$ eligen cierta cantidad entre los primeros n y otra entre los últimos m. Si quiero calcular el número de combinaciones donde elijo s números entre los primeros n y elijo $r-s$ números entre los últimos m.

$$\binom{n}{s}\binom{m}{r-s}$$

Si sumamos todas las posibles combinaciones entonces

$$\binom{n}{0}\binom{m}{r}+\binom{n}{1}\binom{m}{r-1}+\cdots+\binom{n}{r}\binom{m}{0} \quad \square$$

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n}=\sum_{i=0}^{n}\binom{n}{k}^2$$

Demostración. De la identidad anterior.

Usando el resultado del ejercicio 8

$$\begin{aligned}\binom{n+n}{n} &= \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{n!}{(n-k)!\cdot(n-n+k)!}\\ &= \sum_{k=0}^{n}\binom{n}{k}^2 \quad \square\end{aligned}$$

10. From a group of n people, suppose that we want to choose a committee of k, $k \leq n$, one of whom is to be designated as chairperson.

i. By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}\cdot k$ possible choices.

Hay $\binom{n}{k}$ formas de elegir un comité de k miembros y luego k formas de elegir un presidente entre los k miembros.

ii. By focusing first on the choice of the non chair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n-k+1)$ possible choices.

Hay $\binom{n}{k-1}$ formas de elegir un comité de $k-1$ miembros, lo que nos deja con $n-k+1$ personas entre las cuales elegir al presidente.

iii. By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\cdot\binom{n-1}{k-1}$ possible choices.

Hay n formas de elegir al presidente entre las n personas y luego $\binom{n}{k-1}$ formas de seleccionar un comité de $k-1$ miembros de entre las $n-1$ personas restantes.

iv. Conclude from parts i., ii., and iii. that

$$\binom{n}{k}\cdot k=\binom{n}{k-1}(n-k+1)=n\cdot\binom{n-1}{k-1}$$

Se concluye inmediatamente de i., ii. and iii.

v. Use the factorial definition of $\binom{m}{r}$ to verify the identity in part iv.

$$\begin{aligned}&\binom{n}{k}\cdot k\\ &=\frac{n!}{k!\cdot(n-k)!}\cdot k\\ &=\frac{n!}{(k-1)!\cdot(n-k+1)!}\cdot\frac{n-k+1}{k}\cdot k\end{aligned}$$

lo cual concluye la primera igualdad

$$\begin{aligned}&=\binom{n}{k-1}\cdot(n-k+1)\\ &=\frac{n!}{(k-1)!\cdot(n-k+1)!}\cdot(n-k+1)\\ &=n\cdot\frac{(n-1)!}{(k-1)!\cdot(n-k)!}\cdot\frac{n-k+1}{n-k+1}\end{aligned}$$

lo cual concluye la segunda igualdad

$$=n\cdot\binom{n-1}{k-1}$$

11. The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k}=\sum_{i=k}^{n}\binom{i-1}{k-1} \qquad n\geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.

Consideremos el conjunto de números naturales desde k hasta n, y elijamos uno arbitrariamente, i. De los $n-i$ números menores a i elijamos $k-1$ para completar una secuencia de k números tal que i sea el mayor de ellos.

Notar que esto abarca toda secuencia de k números.

12. i. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$$

Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.

Consideremos un comité de tamaño fijo, k, hay $\binom{n}{k}$ maneras de elegir tal comité y k posibles presidentes para este comité. Si consideramos todos los posibles tamaños de comité tenemos

$$\sum_{k=1}^{n} k \cdot \binom{n}{k}$$

Consideremos elegir un presidente entre las n personas y luego para el resto de personas decidir si son miembros del comité o nó

$$n \cdot 2^{n-1}$$

ii. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} \binom{n}{k} \cdot k^2 = 2^{n-2} \cdot n \cdot (n+1)$$

Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size a chairperson and a secretary for the committee (possible the same person).

Consideremos un comité de tamaño fijo, k, hay $\binom{n}{k}$ maneras de elegir tal comité, k posibles presidentes y k posibles secretarios para este comité. Si consideramos todos los posibles tamaños de comité tenemos

$$\sum_{k=1}^{n} \binom{n}{k} \cdot k^2$$

Consideremos los casos donde

El presidente y el secretario no son la misma persona
Hay n candidatos a presidente y $n-1$ candidatos a secretario, luego hay 2^{n-2} formas de determinar el resto de los miembros del comité de entre las $n-2$ personas que restan.

$$2^{n-2} \cdot n \cdot (n-1)$$

El presidente y el secretario son la misma persona
Hay n candidatos y 2^{n-1} formas de determinar el resto de los miembros del comité de entre las $n-1$ personas que restan.

$$2^{n-1} \cdot n$$

Combinando los casos disjuntos

$$\begin{aligned} &2^{n-2} \cdot n \cdot (n-1) + 2^{n-1} \cdot n \\ &= 2^{n-2} \cdot \big(n \cdot (n-1) + n \cdot 2\big) \\ &= 2^{n-2} \cdot n \cdot (n+1) \end{aligned}$$

iii. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} \binom{n}{k} \cdot k^3 = 2^{n-3} \cdot n^2 \cdot (n+3)$$

Present a combinatorial argument for this identity.

Consideremos un conjunto de n personas, ambos lados de la identidad representa el número de posible selecciones de un comité su presidente, vice-presidente y secretario (todos posiblemente la misma persona).
Consideremos un comité de tamaño fijo, k, hay $\binom{n}{k}$ maneras de elegir tal comité, k posibles presidentes, k posibles vice-presidentes y k posibles secretarios para este comité. Si consideramos todos los posibles tamaños de comité tenemos

$$\sum_{k=1}^{n}\binom{n}{k}k^3$$

Consideremos los casos donde

El presidente, vice-presidente y el secretario son la misma persona
Hay n maneras de elegir una persona que concentrará los tres cargos. Luego hay 2^{n-1} formas para decidir quienes de las restantes $n-1$ personas serán miembros del comité.

$$n\cdot 2^{n-1}$$

Dos personas se reparten los cargos de presidente, vice-presidente y secretario
Hay $\binom{n}{2}$ formas de seleccionar a las dos personas y 2^3-2 formas de repartir los cargos. Luego hay 2^{n-2} formas para decidir quienes de las restantes $n-2$ personas serán miembros del comité.

$$3\cdot n\cdot(n-1)\cdot 2^{n-2}$$

Tres personas se reparten los cargos de presidente, vice-presidente y secretario
Hay $\binom{n}{3}$ formas de seleccionar a las tres personas y 3! formas de repartir los cargos. Luego hay 2^{n-3} formas para decidir quienes de las restantes $n-2$ personas serán miembros del comité.

$$(n-2)\cdot(n-1)\cdot n\cdot 2^{n-3}$$

Combinando los casos disjuntos

$$\begin{aligned}&2^{n-3}\cdot(4\cdot n+6\cdot n\cdot(n-1)+(n-2)\cdot(n-1)\cdot n)\\ &=2^{n-3}\cdot((4n)+(6n^2-6n)+(n^3-3n^2+2n))\\ &=2^{n-3}\cdot(n^3+3n^2)\\ &=2^{n-3}\cdot n^2\cdot(n+3)\end{aligned}$$

13. Show that, for $n>0$,

$$\sum_{i=0}^{n}(-1)^i\cdot\binom{n}{i}=0$$

$$\sum_{i=0}^{n}(-1)^i\cdot\binom{n}{i}=\sum_{i=0}^{n}\binom{n}{i}\cdot(-1)^i\cdot 1^{n-i}$$

aplicando el teorema del binomio

$$\begin{aligned}&=(1-1)^n\\ &=0\end{aligned}$$

14. From a set of n people, a committee of size j is to be chosen, and from this committee, a subcommittee of size i, $i\leq j$ is also to be chosen.

i. Derive a combinatorial identity by computing, in two ways, the number of possible choices of the committee and subcommittee—first by supposing that the committee is chosen first and then the subcommittee is chosen, and second by supposing that the subcommittee is chosen first and then the remaining members of the committee are chosen.
Supongamos que primero seleccionamos el comité, hay $\binom{n}{j}$ posibles comités, y de estas j personas debemos seleccionar i para que formen parte del subcomité

$$\binom{n}{j}\cdot\binom{j}{i}$$

Supongamos que primero seleccionamos el subcomité, hay $\binom{n}{i}$ posibles subcomités, luego de las restantes $n-i$ personas debemos seleccionar $j-i$

$$\binom{n}{i}\cdot\binom{n-i}{j-i}$$

Finalmente concluimos que

$$\binom{n}{j}\cdot\binom{j}{i}=\binom{n}{i}\cdot\binom{n-i}{j-i}$$

ii. Use part i. to prove the following combinatorial identity:

$$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}=\binom{n}{i}\cdot 2^{n-i}\quad i\le n$$

por el inciso i.

$$\begin{aligned}\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}&=\sum_{j=i}^{n}\binom{n}{i}\binom{n-i}{j-i}\\&=\binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-i}\\&=\binom{n}{i}\sum_{k=0}^{n-i}\binom{n-i}{k}\end{aligned}$$

la suma equivale a todos los posibles subconjuntos de un conjunto con $n-i$ elementos.

$$=\binom{n}{i}\cdot 2^{n-i}$$

iii. Use part i. and Theoretical Exercise 13 to show that

$$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}(-1)^{(n-j)}=0\quad i<n$$

por el inciso i.

$$\sum_{j=i}^{n}\binom{n}{j}\binom{j}{i}(-1)^{(n-j)}$$

$$=\sum_{j=i}^{n}\binom{n}{i}\binom{n-i}{j-i}(-1)^{(n-j)}$$

la suma equivale a todos los posibles subconjuntos de un conjunto con $n-i$ elementos.

$$=\binom{n}{i}\cdot 2^{n-i}$$

15. Let $H_k(n)$ be the number of vectors $x_i,\ldots,x_k$ for which each x_i is a positive integer satisfying $1\le x_i\le n$ and $x_1\le x_2\le\cdots\le x_k$.

i. Without any computations, argue that

$$H_1(n)=n$$

$$H_k(n)=\sum_{j=1}^{n}H_{k-1}(j)\quad k>1$$

Es inmediato que $H_1(n)=n$.
Sea $H_k^\star(n)$ la cantidad de vectores $x_i,\ldots,x_{k+1}$ para los cuales x_i es un entero positivo, satisfaciendo $1\le x_i\le n$, $x_1\le x_2\le\cdots\le x_{k+1}$ y $x_{k+1}=n$.
Es claro que

$$H_k(n)=\sum_{j=1}^{n}H_k^\star(j)$$

Además $H_{k-1}(j)=H_k^\star(j)$ (basta agregar o quitar j al final de la secuencia).
Luego

$$H_k(n)=\sum_{j=1}^{n}H_k^\star(j)$$

como $H_{k-1}(j)=H_k^\star(j)$

$$=\sum_{j=1}^{n}H_{k-1}(j)$$

ii. Use the preceding recursion to compute $H_3(5)$.
Primero calculemos $H_2(n)$,

$$\begin{aligned}H_2(n)&=\sum_{j=1}^{n}H_1(j)\\&=\sum_{j=1}^{n}j\\&=\frac{n\cdot(n+1)}{2}\end{aligned}$$

luego usando el resultado anterior

$$\begin{aligned} H_3(5) &= \sum_{j=1}^{5} H_2(j) \\ &= \sum_{j=1}^{5} \frac{j \cdot (j+1)}{2} \\ &= \frac{1}{2} \cdot \big(2 + 6 + 12 + 20 + 30\big) \\ &= 35 \end{aligned}$$

16. Consider a tournament of n contestants in which the outcome is an ordering of these contestants, with ties allowed. That is, the outcome partitions the players into groups, with the first group consisting of the players who tied for first place, the next group being those who tied for the next-best position, and so on. Let $N(n)$ denote the number of different possible outcomes. For instance, $N(2) = 3$, since, in a tournament with 2 contestants, player 1 could be uniquely first, player 2 could be uniquely first, or they could tie for first.

i. List all the possible outcomes when $n = 3$.

1 1 1		
1 1 2	1 2 1	2 1 1
1 2 2	2 1 2	2 2 1
1 2 3	1 3 2	2 1 3
2 3 1	3 1 2	3 2 1

ii. With $N(0)$ defined to equal 1, argue, without any computations, that

$$N(n) = \sum_{i=1}^{n} \binom{n}{i} \cdot N(n-i)$$

Hint: How many outcomes are there in which i players tie for last place?

Considerar que en todo resultado de entre las n personas hay i personas, $i \leq n$ en la última posición. Construir todas las ocurrencias del caso donde hay i personas en la última posición tomando $N(n-i)$ y seleccionando i lugares entre los n para ubicar a los jugadores que terminan en la última posición.

iii. Show that the formula of part ii. is equivalent to the following

$$N(n) = \sum_{i=0}^{n-1} \binom{n}{i} N(i)$$

Inmediatamente por el mismo procedimiento construimos todas las ocurrencias del caso donde hay $n - i$ personas en la última posición tomando $N(i)$ y seleccionando i lugares para que no ocupen los jugadores que terminan en la última posición.

iv. Use the recursion to find $N(3)$ and $N(4)$.

Tenemos que $N(0) = 1$, $N(1) = 1$,

$$\begin{aligned} N(2) &= 3 \\ N(3) &= \sum_{i=1}^{3} \binom{3}{i} \cdot N(3-i) \\ &= 3 \cdot N(2) + 3 \cdot N(1) + 1 \cdot N(0) \\ &= 9 + 3 + 1 \\ &= 13 \\ N(4) &= \sum_{i=1}^{4} \binom{4}{i} \cdot N(4-i) \\ &= 4 \cdot N(3) + 6 \cdot N(2) + \\ &\quad 4 \cdot N(1) + 1 \cdot N(0) \\ &= 52 + 18 + 4 + 1 \\ &= 75 \end{aligned}$$

17. Present a combinatorial explanation of why

$$\binom{n}{r} = \binom{n}{r, n-r}$$

Al seleccionar r objetos entre n objetos queda únicamente determinado otro grupo de $n - r$ objetos, los que no fueron seleccionados.

18. Argue that

$$\binom{n}{n_1, n_2, \dots, n_r} = \binom{n-1}{n_1 - 1, n_2, \dots, n_r} + \binom{n-1}{n_1, n_2 - 1, \dots, n_r} + \cdots + \binom{n-1}{n_1, n_2, \dots, n_r - 1}$$

Numerar n_1 bolillas con 1, n_2 bolillas con 2, $\dots$, n_r bolillas con r. Para las bolillas numeradas i colocar una de ellas al principio del ordenamiento y obtener todos los posibles ordenamientos de las $n-1$ restantes.

De esta forma se obtienen todos los posibles ordenamientos de las n bolillas.

19. Prove the multinomial theorem.

Demostración. Del teorema multinomial.

preliminarmente

$$\binom{n}{n_1, \dots, n_r} = \frac{n!}{n_1! \cdots n_r!} = \frac{n!}{n_1! \cdots n_r!} \cdot \frac{(n-n_1)!}{(n-n_1)!} = \frac{n!}{n_1! \cdot (n-n_1)!} \cdot \frac{(n-n_1)!}{n_2! \cdots n_r!}$$

como $n_2 + \cdots + n_r = n - n_1$ entonces

$$= \binom{n}{n_1} \cdot \binom{n-n_1}{n_2, \dots, n_r}$$

Por inducción sobre r

si $r = 2$ entonces para todo n es un caso del teorema binomial.

si el teorema es válido para k entonces

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{n_1, \dots, n_k \in \mathbb{N}_0^k: \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \dots, n_k} \cdot x_1^{n_1} \cdots x_k^{n_k}$$

luego

$$\sum_{\substack{n_1, \dots, n_{k+1} \in \mathbb{N}_0^{k+1}: \\ n_1 + \cdots + n_{k+1} = n}} \binom{n}{n_1, \dots, n_{k+1}} \cdot x_1^{n_1} \cdots x_{k+1}^{n_{k+1}}$$

$$= \sum_{n_1=0}^{n} \sum_{\substack{n_2, \dots, n_{k+1} \in \mathbb{N}_0^{k}: \\ n_2 + \cdots + n_{k+1} = n - n_1}} \binom{n}{n_1, \dots, n_{k+1}} \cdot x_1^{n_1} \cdots x_{k+1}^{n_{k+1}}$$

$$= \sum_{n_1=0}^{n} \sum_{\substack{n_2, \dots, n_{k+1} \in \mathbb{N}_0^{k}: \\ n_2 + \cdots + n_{k+1} = n - n_1}} \binom{n}{n_1} \cdot \binom{n-n_1}{n_2, \dots, n_{k+1}} \cdot x_1^{n_1} \cdots x_{k+1}^{n_{k+1}}$$

$$= \sum_{n_1=0}^{n} \binom{n}{n_1} \cdot x_1^{n_1} \sum_{\substack{n_2, \dots, n_{k+1} \in \mathbb{N}_0^{k}: \\ n_2 + \cdots + n_{k+1} = n - n_1}} \binom{n-n_1}{n_2, \dots, n_{k+1}} \cdot x_2^{n_2} \cdots x_{k+1}^{n_{k+1}}$$

por hipótesis inductiva

$$= \sum_{n_1=0}^{n} \binom{n}{n_1} \cdot x_1^{n_1} \cdot \left(x_2 + \cdots + x_{k+1}\right)^{n-n_1}$$

por el teorema del binomio

$$= (x_1 + (x_2 + \cdots + x_{k+1}))^n$$
$$= (x_1 + \cdots + x_{k+1})^n$$

□

20. In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls for each $i = 1, \dots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

Primero colocamos las bolillas requeridas en las urnas, y nos restan $n - \sum_{i=1}^{r} m_i$ bolillas.

Luego debemos colocar las $n - \sum_{i=1}^{r} m_i$ bolillas restantes en las r urnas

$$\binom{n + r - 1 - \sum_{i=1}^{r} m_i}{r - 1}$$

21. Argue that there are exactly

$$\binom{r}{k} \cdot \binom{n-1}{n-r+k}$$

solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k of the x_i are equal to 0.

Primero elegir k de los r coeficientes para que sean 0

$$\binom{r}{k}$$

De los $r-k$ restantes, que son mayores o iguales a 1 obtener todas las combinaciones

$$\binom{n-1}{(r-k)-1} = \binom{n-1}{(n-1)-((r-k)-1)} = \binom{n-1}{n-r+k}$$

22. Consider a function $f(x_1, \ldots, x_n)$ of n variables. How many different partial derivatives of order r does f posses?

Si asumimos que $f \in C^r$ entonces buscamos las soluciones a

$$a_1 + a_2 + \cdots + a_n = r \quad x_i \geq 0, \quad i = 1, \ldots, n$$

Entonces

$$\binom{r+n-1}{n-1}$$

23. Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a nonnegative integer and

$$\sum_{i=1}^{n} x_i \leq k$$

La cantidad de soluciones para $\sum_{i=1}^{n} x_i = c$ es

$$\binom{c+n-1}{n-1}$$

Por lo tanto si $\sum_{i=1}^{n} x_i \leq k$

$$\sum_{i=0}^{k} \binom{i+n-1}{n-1}$$

Self-Test Problems

1. How many different linear arrangements are there of the letters A, B, C, D, E, F for which

i. A and B are next to each other?

$$2 \cdot 5!$$

ii. A is before B

$$\frac{6!}{2!}$$

iii. A is before B and B is before C?

$$\frac{6!}{3!}$$

iv. A is before B and C is before D?

$$\frac{6!}{2! \cdot 2!}$$

v. A and B are next to each other and C and D are also next to each other?

$$2^2 \cdot 3!$$

vi. E is not last in line?

$$5 \cdot 5!$$

2. If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

$$4! \cdot 3! \cdot 3! \cdot 3!$$

3. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if

i. there are no restrictions?

$$\binom{10}{3} \cdot 3!$$

ii. A and B will not serve together?

$$\binom{10}{3} \cdot 3! - 3 \cdot 2 \cdot 8$$

iii. C and D will serve together or not at all?

$$3 \cdot 2 \cdot 8 + \binom{8}{3} \cdot 3!$$

iv. E must be an officer?

$$\binom{9}{2} \cdot 3!$$

v. F will serve only if he is president?

$$\binom{9}{2} \cdot 2! + \binom{9}{3} \cdot 3!$$

4. A student is to answer 7 out of 10 questions in an examination. How many choice has she? How many is she must answer at least 3 of the first 5 questions?

$$\binom{10}{7}$$

Si debe responder al menos 3 de las 5 primeras preguntas

$$\binom{5}{3} \cdot \binom{5}{4} + \binom{5}{4} \cdot \binom{5}{3} + \binom{5}{5} \cdot \binom{5}{2}$$

5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

$$\binom{7}{3,2,2}$$

6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.

$$\binom{7}{3} \cdot 26^3 \cdot 10^4$$

7. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

Es un caso especial del ejercicio teórico 17.

8. Consider n-digit numbers where each digit is one of the 10 integers $0, 1, \ldots, 9$. How many such numbers are there for which

i. no two consecutive digits are equal?

$$10 \cdot 9^{n-1}$$

ii. 0 appears as a digit a total of i times, $i = 0, \ldots, n$?

$$\binom{n-1}{i} \cdot 9^{n-i}$$

9. Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.

i. How many choices are possible?

$$\binom{3n}{3}$$

ii. How many choices are there in which all 3 students are in the same class?

$$3 \cdot \binom{n}{3}$$

iii. How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?

$$3 \cdot \binom{n}{2} \cdot 2 \cdot \binom{n}{1}$$

iv. How many choices are there in which all 3 students are in different classes?

$$n \cdot n \cdot n$$

v. Using the results of parts i. through iv., write a combinatorial identity.

$$\binom{3n}{3} = 3 \cdot \binom{n}{3} + 3 \cdot \binom{n}{2} \cdot 2 \cdot \binom{n}{1} + n^3$$

10. How many 5-digit numbers can be formed from the integers $1, 2, \ldots, 9$ if no digit can appear more than twice? (For instance, 41434 is not allowed.)

$$\binom{9}{5} \cdot \frac{5!}{1!} + \binom{9}{4} \cdot \binom{4}{1} \cdot \frac{5!}{2!} + \binom{9}{3} \cdot \binom{3}{2} \cdot \frac{5!}{2!2!}$$

11. From 10 married couples, we want to select a group of 6 people that is not allowed to contain a married couple.

i. How many choices are there?

$$\binom{10}{6}\cdot 2^6$$

ii. How many choices are there if the group must also consist of 3 men and 3 women?

$$\binom{10}{3}\cdot\binom{7}{3}$$

12. A committee of 6 people is to be chosen from a group consisting of 7 men and 8 women. If the committee must consist of at least 3 women and at least 2 men, how many different committees are possible?

$$\binom{8}{3}\cdot\binom{7}{3}+\binom{8}{4}\cdot\binom{7}{2}$$

13. An art collection on auction consisted of 4 Dalis, 5 van Goghs, and 6 Picassos. At the auction were 5 art collectors. If a reporter noted only the number of Dalis, van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all of the works were sold?

$$\binom{4+5-1}{5-1}\cdot\binom{5+5-1}{5-1}\cdot\binom{6+5-1}{5-1}$$

14. Determine the number of vectors $(x_i,\dots,x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \le k$$

where $k\ge n$.

Si buscamos soluciones a

$$x_1+\cdots+x_n=k \quad x_i>0,\ i=1,\dots,n$$

entonces hay

$$\binom{k-1}{n-1}$$

soluciones, por lo tanto si buscamos soluciones menores o iguales a k

$$\sum_{i=n}^{k}\binom{i-1}{n-1}=\binom{k}{n}$$

15. A total of n students are enrolled in a review course for the actuarial examination in probability. The posted results of the examination will list the names of those who passed, in decreasing order of their scores. For instance, the posted result will be "Brown, Cho" if Brown and Cho are the only ones to pass, with Brown receiving the higher score. Assuming that all scores are distinct (no ties), how many posted results are possible?

$$\sum_{i=0}^{n}\binom{n}{i}\cdot i!$$

16. How many subsets of size 4 of the set $S=\{1,2.\dots,20\}$ contain at least one of the elements $1,2,3,4,5$?

$$\binom{20}{4}-\binom{15}{4}$$

17. Give an analytic verification of

$$\binom{n}{2}=\binom{k}{2}+k\cdot(n-k)+\binom{n-k}{2} \qquad 1\le k\le n$$

$$\begin{aligned}
&\binom{k}{2}+k\cdot(n-k)+\binom{n-k}{2}\\
&=\frac{k!}{2!\cdot(k-2)!}+k\cdot(n-k)+\frac{(n-k)!}{2!\cdot(n-k-2)!}\\
&=\frac{k^2-k+n^2-2nk+k^2-n+k+2kn-k^2}{2}\\
&=\frac{n^2-n}{2}\\
&=\binom{n}{2}
\end{aligned}$$

Now, give a combinatorial argument for this identity.

Suma de las formas de elegir 2 entre los k primeros, 1 entre los k primeros y 1 entre los $n-k$ últimos y 2 entre los $n-k$ últimos,

18. In a certain community, there are 3 families consisting of a single parent and 1 child, 3 families consisting of a single parent and 2 children, 5 families consisting of 2 parents and a single child, 7 families consisting of 2 parents and 2 children, and 6 families consisting of 2 parents and 3 children. If a parent and child from the same family are to be chosen, how many possible choices are there?

$$3 + 3 \cdot 2 + 5 \cdot 2 + 7 \cdot 2 \cdot 2 + 6 \cdot 2 \cdot 3$$

19. If there are no restrictions on where the digits and letters are placed, how many 8-place license plates consisting of 5 letters and 3 digits are possible if no repetitions of letters or digits are allowed? What if the 3 digits must be consecutive?

$$\binom{8}{3} \cdot \frac{26!}{21!} \cdot \frac{10!}{7!}$$

Si los 3 dígitos deben ser consecutivos entonces

$$6 \cdot \frac{26!}{21!} \cdot \frac{10!}{7!}$$

20. show that the following equality is valid

$$\sum_{\substack{x_1,\ldots,x_r \in \mathbb{N}_0^r: \\ x_1+\cdots+x_r=n}} \frac{n!}{x_1! \cdots x_r!} = r^n$$

Por definición

$$\sum_{\substack{x_1,\ldots,x_r \in \mathbb{N}_0^r: \\ x_1+\cdots+x_r=n}} \frac{n!}{x_1! \cdots x_r!} = \sum_{\substack{x_1,\ldots,x_r \in \mathbb{N}_0^r: \\ x_1+\cdots+x_r=n}} \binom{n}{x_1, \ldots, x_r}$$

Todas las maneras de seleccionar desde 1 hasta r subconjuntos de n objetos distintos.

De otra forma, asignar a los n uno de los r grupos, i.e. r^n.