Practical Problems

1. A box contains 3 marbles: 1 red, 1 green, and 1 blue.Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

$$S = \{r, g, b\}^2$$
$$S = \{(a, b) \in \{r, g, b\}^2 \mid a \neq b\}$$

2. In an experiment, die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n?

What is $\left(\bigcup_1^\infty E_n\right)^{\complement}$?

$$S = \{1, 2, 3, 4, 5, 6\}^{\mathbb{N}}$$
$$E_n = \{s \in S : s_i < 6, s_n = 6, i < n\}$$

$\left(\bigcup_1^\infty E_n\right)^{\complement}$ es el evento $\{s \in S : \nexists i, s_i = 6\}$.

O sea tal que nunca se obtiene un 6.

3. Two dice are thrown. Let E be the event that the sum of the dice is odd, let F be the event that at least one of the dice lands on 1, and let G be the event that the sum is 5. Describe the events EF, $E \cup F$, FG, $EF^{\complement}$ and EFG.

- EF El evento tal que la suma de los dos dados es impar y al menos uno de los dados es 1.
- $E \cup F$ El evento tal que la suma de los dos dados es impar o uno de los dados es 1.
- FG El evento tal que al menos uno de los dados es 1 y la suma de ambos es 5.
- $EF^{\complement}$ El evento tal que la suma de los dados es impar y la suma de ambos no es 5.
- EFG El evento tal que la suma de los dos dados es 5 y al menos uno de los dados es 1.

4. A, B, and C take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \begin{cases} 1, 01, 001, 0001, \ldots, \\ 0000\cdots \end{cases}$$

i. Interpret the sample space.

Es el conjunto de secuencias formadas por 0 y 1, donde el i-ésimo componente representa el resultado de la tirada i-ésima, con 0 representando que el resultado es ceca y 1 representando que el resultado es cara.

ii. Define the following events in terms of S, assuming that A flips first, then B, then C, ...

a. A wins $= A$.

$$A = \{1, 0001, 0000001, \ldots\}$$

b. B wins $= B$.

$$B = \{01, 00001, 00000001, \ldots\}$$

c. $(A \cup B)^{\complement}$

$$(A \cup B)^{\complement} = \begin{cases} 001, 000001, \ldots \\ 0000\cdots \end{cases}$$

5. A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

i. How many outcomes are in the sample space of this experiment?

$$S = \{0, 1\}^5, \quad |S| = 2^5$$

ii. Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

$$11000, 11001, 11010, 11011, 11100,$$
$$11101, 11110, 11111, 00110, 00111,$$
$$01110, 01111, 10110, 10111, 10101$$

iii. Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

$$2^3$$

iv. Write out all the outcomes in the event AW.

$$11000, 11100$$

6. A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.

i. Give the sample space of this experiment.

$$S = \{(0, g), (0, f), (0, s), (1, g), (1, f), (1, s)\}$$

ii. Let A be the event that the patient is in serious condition. Specify the outcomes in A.

$$A = \{(0, s), (1, s)\}$$

iii. Let B be the event that the patient is uninsured. Specify the outcomes in B.

$$B = \{(0, g), (0, f), (0, s)\}$$

iv. Give all the outcomes in the event $B^{\mathsf{C}} \cup A$.

$$B^{\mathsf{C}} \cup A = \{(0, s), (1, g), (1, f), (1, s)\}$$

7. Consider an experiment that consists of determining the type of job—either blue collar or white collar—and the political affiliation—Republican, Democratic, or Independent—of the 15 members of an adult soccer team. How many outcomes are

i. in the sample space?

$$S = (\{b, w\} \times \{r, d, i\})^{15}$$
$$|S| = 6^{15}$$

ii. in the event that at least one of the team members is a blue-collar worker?

$$6^{15} - 3^{15}$$

iii. in the event that none of the team members considers himself or herself an Independent?

$$4^{15}$$

8. Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that

i. either A or B occurs?

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= .3 + .5 \\ &= .8 \end{aligned}$$

ii. A occurs but B does not?

$$\begin{aligned} P(AB^{\mathsf{C}}) &= P(A) - P(AB) \\ &= .3 - .0 \\ &= .3 \end{aligned}$$

iii. both A and B occur?

$$P(AB) = 0$$

9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

$$\begin{aligned} P(A \cup V) &= P(A) + P(V) - P(A \cap V) \\ &= .24 + .61 - .11 \\ &= .74 \end{aligned}$$

10. Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

i. a ring or a necklace?

$$\begin{aligned} P(R \cup N) &= 1 - P(R^{\mathsf{C}} \cap N^{\mathsf{C}}) \\ &= 1 - .6 \\ &= .4 \end{aligned}$$

ii. a ring and a necklace?

$$\begin{aligned} P(R \cap N) &= P(R) + P(N) - P(R \cup N) \\ &= .2 + .3 - .4 \\ &= .1 \end{aligned}$$

11. A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.

i. What percentage of males smokes neither cigars nor cigarettes?

$$\begin{aligned} P(A^{\mathsf{C}} \cap B^{\mathsf{C}}) &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - (.28 + .07 - .05) \\ &= .7 \end{aligned}$$

ii. What percentage smokes cigars but not cigarettes?

$$\begin{aligned} P(A^{\mathsf{C}} \cap B) &= P(B) - P(A \cap B) \\ &= .07 - .05 \\ &= .02 \end{aligned}$$

12. An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students who are in both Spanish and French, 4 who are in both Spanish and German, and 6 who are in both French and German. In addition, there are 2 students taking all 3 classes.

i. If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?

$$\begin{aligned} &P(S^{\mathsf{C}}F^{\mathsf{C}}G^{\mathsf{C}}) \\ &= 1 - P(S \cup F \cup G) \\ &= 1 - (.28 + .26 + .16 - .12 - .04 - .06 + .02) \\ &= 1 - .5 \\ &= .5 \end{aligned}$$

ii. If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?

$$\begin{aligned} &1 - P(S^{\mathsf{C}}F^{\mathsf{C}}G^{\mathsf{C}}) - \\ &P(SFG^{\mathsf{C}}) - P(SF^{\mathsf{C}}G) - P(S^{\mathsf{C}}FG) + \\ &P(SFG) \\ &= 1 - .5 - .12 - .04 - .06 + 2 \cdot .02 \\ &= 1 - .68 \\ &= .32 \end{aligned}$$

iii. If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?

$$1 - \frac{50}{100} \cdot \frac{49}{99}$$

13. A certain town with a population of 100000 has 3 newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows

	and I	and II	and III
I	10%	8%	2%
II	8%	30%	4%
III	2%	4%	5%

and 1% read I, II and III.

i. Find the number of people who read only one newspaper.

Sea E_i el evento tal que una persona lee exactamente i periódicos

$$\begin{aligned}&P(E_1)\\&= P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III}) -\\&\quad 2\cdot P(\mathrm{I\ II}) - 2\cdot P(\mathrm{I\ III}) - 2\cdot P(\mathrm{II\ III}) +\\&\quad 3\cdot P(\mathrm{I\ II\ III})\\&= .1 + .3 + .05 - 2\cdot(.08+.02+.04) + 3\cdot .01\\&= .2\end{aligned}$$

$$.2\cdot 100000 = 20000$$

ii. How many people read at least two newspapers?

$$\begin{aligned}&P(E_2\cup E_3)\\&= P(\mathrm{I\ II}) + P(\mathrm{I\ III}) + P(\mathrm{II\ III}) -\\&\quad 2\cdot P(\mathrm{I\ II\ III})\\&= .08 + .02 + .04 - 2\cdot .01\\&= .12\end{aligned}$$

$$.12\cdot 100000 = 12000$$

iii. If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?

$$\begin{aligned}&P((\mathrm{I\ II})\cup(\mathrm{III\ II}))\\&= P(\mathrm{I\ II}) + P(\mathrm{III\ II}) - P((\mathrm{I\ II})\cap(\mathrm{III\ II}))\\&= P(\mathrm{I\ II}) + P(\mathrm{III\ II}) - P(\mathrm{I\ II\ III})\\&= .08 + .04 - .01\\&= .11\end{aligned}$$

$$.11\cdot 100000 = 11000$$

iv. How many people do not read any newspapers?

$$\begin{aligned}&P(E_0)\\&= 1 - P(E_1) - P(E_2) - P(E_3)\\&= 1 - P(E_1) - P(E_2\cup E_3)\\&= 1 - .2 - .12\\&= .68\end{aligned}$$

$$.68\cdot 100000 = 68000$$

v. How many people read only one morning paper and one evening paper?

$$\begin{aligned}&P((\mathrm{I\ II\ III}^{\mathsf{C}})\cup(\mathrm{I}^{\mathsf{C}}\ \mathrm{II\ III}))\\&= P(\mathrm{I\ II\ III}^{\mathsf{C}}) + P(\mathrm{I}^{\mathsf{C}}\ \mathrm{II\ III})\\&= P(\mathrm{I\ II}) - P(\mathrm{I\ II\ III}) +\\&\quad P(\mathrm{II\ III}) - P(\mathrm{I\ II\ III})\\&= .08 - .01 + .04 - .01\\&= .1\end{aligned}$$

$$.1\cdot 100000 = 10000$$

14. The following data were given in a study of a group of 1000 subscribers to a certain magazine: In reference to job, marital status, and education, there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect.

Sea $P/M/G$ el evento tal que la persona es profesional/casado/graduado.

$$\begin{aligned}&P(P\cup M\cup C)\\&= P(P) + P(M) + P(C) -\\&\quad P(PM) - P(PC) - P(MC) + P(PMC)\\&= \tfrac{312}{1000} + \tfrac{470}{1000} + \tfrac{525}{1000} -\\&\quad \tfrac{86}{1000} - \tfrac{42}{1000} - \tfrac{147}{1000} + \tfrac{25}{1000}\\&= \tfrac{1057}{1000}\end{aligned}$$

Que contradice uno de los axiomas de la probabilidad $P(E)\leq 1$ para todo evento E.

15. In a game of poker, what is the probability of being dealt

i. a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)

$$\frac{\binom{4}{1} \cdot \binom{13}{5}}{\binom{52}{5}}$$

ii. one pair? (This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d are all distinct.)

$$\frac{\binom{13}{4} \cdot \binom{4}{1} \cdot \binom{4}{1}^3 \cdot \binom{4}{2}}{\binom{52}{5}}$$

iii. two pairs? (This occurs when the cards have denominations a, a, b, b, c, where a, b, and c are all distinct.)

$$\frac{\binom{13}{3} \cdot \binom{3}{2} \cdot \binom{4}{1} \cdot \binom{4}{2}^2}{\binom{52}{5}}$$

iv. three of a kind? (This occurs when the cards have denominations a, a, a, b, c, where a, b, and c are all distinct.)

$$\frac{\binom{13}{3} \cdot \binom{3}{1} \cdot \binom{4}{1}^2 \cdot \binom{4}{3}}{\binom{52}{5}}$$

v. four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

$$\frac{\binom{13}{2} \cdot \binom{2}{1} \cdot \binom{4}{1} \cdot \binom{4}{4}}{\binom{52}{5}}$$

16. Poker dice is played by simultaneously rolling 5 dice. Show that

i. $P\{\text{no two alike}\} = .0926$;

$$\frac{\binom{6}{5} \cdot 5!}{6^5}$$

ii. $P\{\text{one pair}\} = .4630$;

$$\frac{\binom{6}{4} \cdot \binom{4}{1} \cdot \binom{5}{2} \cdot 3!}{6^5}$$

iii. $P\{\text{two pair}\} = .2315$;

$$\frac{\binom{6}{3} \cdot \binom{3}{2} \cdot \binom{5}{2,2,1}}{6^5}$$

iv. $P\{\text{three alike}\} = .1543$;

$$\frac{\binom{6}{3} \cdot \binom{3}{1} \cdot \binom{5}{3,1,1}}{6^5}$$

v. $P\{\text{full house}\} = .0386$;

$$\frac{\binom{6}{2} \cdot \binom{2}{1} \cdot \binom{5}{3,2}}{6^5}$$

vi. $P\{\text{four alike}\} = .0193$;

$$\frac{\binom{6}{2} \cdot \binom{2}{1} \cdot \binom{5}{4,1}}{6^5}$$

vii. $P\{\text{five alike}\} = .0008$.

$$\frac{\binom{6}{1}}{6^5}$$

17. If 8 rooks (castles) are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. That is, compute the probability that no row or file contains more than one rook.

$$\frac{\prod_{i=1}^{8} i^2}{64 \cdot 63 \cdots 57}$$

18. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

$$\frac{\binom{4}{1} \cdot \binom{16}{1}}{\binom{52}{2}}$$

19. Two symmetric dice have had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color faceup?

$$\frac{2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 1}{6^2}$$

20. Suppose that you are playing blackjack against a dealer. In a freshly shuffled deck, what is the probability that neither you nor the dealer is dealt a blackjack?

Sea A el evento tal que el jugador obtiene blackjack y B el evento tal que el croupier obtiene blackjack

$$\begin{aligned}&P(A^{\mathsf{C}} \cap B^{\mathsf{C}})\\ &= 1 - P(A \cup B)\\ &= 1 - (P(A) + P(B) - P(A \cap B))\\ &= 1 - \left(\frac{4 \cdot 16}{\binom{52}{2}} + \frac{4 \cdot 16}{\binom{52}{2}} - \frac{4 \cdot 16 \cdot 3 \cdot 15}{\binom{52}{2} \cdot \binom{50}{2}} \right)\\ &\approx .9052\end{aligned}$$

21. A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children, and 1 has five children.

i. If one of these families is chosen at random, what is the probability it has i children, $i = 1, 2, 3, 4, 5$?

$$1.\ \frac{4}{20} \quad 2.\ \frac{8}{20} \quad 3.\ \frac{5}{20} \quad 4.\ \frac{2}{20} \quad 5.\ \frac{1}{20}$$

ii. If one of the children is randomly chosen, what is the probability that child comes from a family having i children, $i = 1, 2, 3, 4, 5$?

$$1.\ \frac{4}{48} \quad 2.\ \frac{16}{48} \quad 3.\ \frac{15}{48} \quad 4.\ \frac{8}{48} \quad 5.\ \frac{5}{48}$$

22. Consider the following technique for shuffling a deck of n cards: For any initial ordering of the cards, go through the deck one card at a time and at each card, flip a fair coin. If the coin comes up heads, then leave the card where it is; if the coin comes up tails, then move that card to the end of the deck. After the coin has been flipped n times, say that one round has been completed. For instance, if $n = 4$ and the initial ordering is 1, 2, 3, 4, then if the successive flips result in the outcome h, t, t, h, then the ordering at the end of the round is 1,4,2,3. Assuming that all possible outcomes of the sequence of n coin flips are equally likely, what is the probability that the ordering after one round is the same as the initial ordering?

Si $n = 1$ hay dos tiradas posibles.

Inductivamente, si para k cartas hay $k+1$ tiradas que dejan las cartas en su orden inicial.

Para $k + 1$ cartas, si la primer tirada es ceca, todas las subsecuentes tiradas deben ser ceca. Si la primer tirada es cara hay $k + 1$ tiradas que mantienen el orden de las siguientes k cartas. Por lo tanto hay $k + 2$ posibles tiradas que mantienen las cartas en su orden inicial.

Por lo tanto la probabilidad de dejar un mazo de n cartas en su orden inicial

$$\frac{n+1}{2^n}$$

23. A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?

$$\frac{5+4+3+2+1+0}{6^2} = \frac{15}{6^2}$$

24. If two dice are rolled, what is the probability that the sum of the upturned faces equals i? Find it for $i = 2, 3, \ldots, 11, 12$.

$$2.\ \frac{1}{6^2} \quad 3.\ \frac{2}{6^2} \quad 4.\ \frac{3}{6^2} \quad 5.\ \frac{4}{6^2} \quad 6.\ \frac{5}{6^2}$$
$$7.\ \frac{6}{6^2} \quad 8.\ \frac{5}{6^2} \quad 9.\ \frac{4}{6^2} \quad 10.\ \frac{3}{6^2} \quad 11.\ \frac{2}{6^2} \quad 12.\ \frac{1}{6^2}$$

25. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.

Hint: Let E_n denote the event that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $n-1$ rolls. Compute $P(E_n)$ and argue that $\sum_{n=1}^{\infty} P(E_n)$ is the desired probability.

Sea

$$\begin{aligned}
&P(E_n) \\
&= P(\{\text{no suma 5 o 7}\})^{n-1} \cdot P(\{\text{suma 5}\}) \\
&= \big(1 - P(\{\text{suma 5 o 7}\})\big)^{n-1} \cdot P(\{\text{suma 5}\}) \\
&= \left(\frac{26}{6^2}\right)^{n-1} \cdot \frac{4}{6^2}
\end{aligned}$$

entonces

$$\begin{aligned}
&\sum_{i=1}^{\infty} P(E_n) \\
&= \sum_{i=0}^{\infty} \left(\frac{26}{6^2}\right)^{i} \cdot \frac{4}{6^2} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{26}{6^2}\right)^{i} \cdot \frac{4}{6^2} \\
&= \frac{4}{6^2} \cdot \lim_{n\to\infty} \sum_{i=0}^{n} \frac{26^i}{36^i}
\end{aligned}$$

por formula cerrada de serie geométrica

$$\begin{aligned}
&= \frac{4}{6^2} \cdot \lim_{n\to\infty} \frac{1 - (\frac{26}{36})^{n+1}}{1 - \frac{26}{36}} \\
&= \frac{4}{6^2} \cdot \frac{1}{\frac{10}{36}} \\
&= \frac{2}{5}
\end{aligned}$$

26. The game of craps is played as follows: A player rolls two dice. If the sum of the dice is either a 2, 3, or 12, the player loses; if the sum is either a 7 or an 11, the player wins. If the outcome is anything else, the player continues to roll the dice until she rolls either the initial outcome or a 7. If the 7 comes first, the player loses, whereas if the initial outcome reoccurs before the 7 appears, the player wins. Compute the probability of a player winning at craps.

Hint: Let E_i denote the event that the initial outcome is i and the player wins. The desired probability is $\sum_{i=2}^{12} P(E_i)$. To compute $P(E_i)$, define the events $E_{i,n}$ to be the event that the initial sum is i and the player wins on the nth roll. Argue that $P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n})$.

$P(E_i) = 0$ para $i = 2, 3, 12$.

$P(E_7) = \frac{6}{36}$ y $P(E_{11}) = \frac{2}{36}$.

Para $i = 4, 5, 6, 8, 9, 10$

$$P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n}).$$

Sea $P(k^+)$ la probabilidad de sumar k para $k = 2, 3, \ldots, 11, 12$

$$\begin{aligned}
&P(E_{k,n}) \\
&= P(\{\text{no suma } k \text{ o } 7\})^{n-2} \cdot P(k^+) \\
&= \big(1 - P(\{\text{suma } k \text{ o } 7\})\big)^{n-2} \cdot P(k^+) \\
&= \big(1 - P(k^+) - P(7^+)\big)^{n-2} \cdot P(k^+) \\
&= \Big(1 - \frac{|k^+|}{36} - \frac{6}{36}\Big)^{n-2} \cdot \frac{|k^+|}{36} \\
&= \Big(\frac{30 - |k^+|}{36}\Big)^{n-2} \cdot \frac{|k^+|}{36}
\end{aligned}$$

entonces computando la probabilidad de obtener la tirada original de k

$$\begin{aligned}
&P(E_k) \\
&= \frac{|k^+|}{36} \cdot \sum_{i=0}^{\infty} P(E_{k,i}) \\
&= \frac{|k^+|}{36} \cdot \sum_{i=0}^{\infty} \frac{|k^+|}{36} \cdot \Big(\frac{30 - |k^+|}{36}\Big)^{i} \\
&= \Big(\frac{|k^+|}{36}\Big)^{2} \cdot \lim_{n\to\infty} \sum_{i=0}^{n} \cdot \Big(\frac{30 - |k^+|}{36}\Big)^{i}
\end{aligned}$$

por formula cerrada de serie geométrica

$$= \Big(\frac{|k^+|}{36}\Big)^{2} \cdot \lim_{n\to\infty} \frac{1 - \big(\frac{30-|k^+|}{36}\big)^{n+1}}{1 - \frac{30-|k^+|}{36}}$$

Como $0 < \frac{30-|k^+|}{36} < 1$

$$\begin{aligned}
&= \Big(\frac{|k^+|}{36}\Big)^{2} \cdot \frac{1}{\frac{36-(30-|k^+|)}{36}} \\
&= \frac{|k^+|}{36} \cdot \frac{|k^+|}{6 + |k^+|}
\end{aligned}$$

Por lo tanto

$$\begin{aligned} P(E_4) &= \frac{3}{36}\cdot\frac{3}{9} & P(E_5) &= \frac{4}{36}\cdot\frac{4}{10} \\ P(E_6) &= \frac{5}{36}\cdot\frac{5}{11} & P(E_8) &= \frac{5}{36}\cdot\frac{5}{11} \\ P(E_9) &= \frac{4}{36}\cdot\frac{4}{10} & P(E_{10}) &= \frac{3}{36}\cdot\frac{3}{9} \end{aligned}$$

Luego la probabilidad buscada es

$$\begin{aligned} &\sum_{i=2}^{12} P(E_i) \\ &= \frac{1}{36}\cdot\left(\frac{9}{9}+\frac{16}{10}+\frac{25}{11}+6+\frac{25}{11}+\frac{16}{10}+\frac{9}{9}+2\right) \\ &\approx .4\overline{92} \end{aligned}$$

27. An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

Sea $P(E_i)$, $i = 1, 2, \ldots, 10$ la probabilidad que gane A cuando se retira la i-ésima bolilla.

$$\begin{aligned} P(E_1) &= \frac{3}{10} \\ P(E_3) &= \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{3}{8} \\ P(E_5) &= \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6} \\ P(E_7) &= \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4} \\ P(E_9) &= 0 \end{aligned}$$

$$\begin{aligned} &P(E_1)+P(E_3)+P(E_5)+P(E_7) \\ &= \frac{3}{10}+\frac{7}{10}\cdot\frac{6}{9}\cdot\left(\frac{3}{8}+\frac{5}{8}\cdot\frac{4}{7}\cdot\left(\frac{3}{6}+\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}\right)\right) \\ &= \frac{7}{12} \end{aligned}$$

28. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be i. of the same color? ii. of different colors? Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as *sampling with replacement*.

i.

$$P(\{\text{iguales}\}) = \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}}$$

con reemplazo

$$P(\{\text{iguales}\}) = \frac{5^3+6^3+8^3}{19^3}$$

ii.

$$P(\{\text{distintas}\}) = \frac{\binom{5}{1}\cdot\binom{6}{1}\cdot\binom{8}{1}}{\binom{19}{3}}$$

con reemplazo

$$P(\{\text{distintas}\}) = 6\cdot\frac{5\cdot 6\cdot 8}{19^3}$$

29. An urn contains n white and m black balls, where n and m are positive numbers.

i. If two balls are randomly withdrawn, what is the probability that they are the same color?

$$\frac{\binom{n}{2}+\binom{m}{2}}{\binom{n+m}{2}}$$

ii. If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?

$$\frac{n^2+m^2}{(n+m)^2}$$

iii. Show that the probability in part ii. is always larger than the one in part i.

en i. la probabilidad de seleccionar otra del mismo color es de $\frac{n-1}{n+m}$ o $\frac{m-1}{n+m}$ mientras que en ii. la probabilidad es de $\frac{n}{n+m}$ o $\frac{m}{n+m}$.

30. The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between the two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that

i. Rebecca and Elise will be paired?

$$\frac{\binom{7}{3}\cdot\binom{8}{3}\cdot 3!}{\binom{8}{4}\cdot\binom{9}{4}\cdot 4!}$$

ii. Rebecca and Elise will be chosen to represent their schools but will not play each other?

$$\frac{\binom{7}{3}\cdot\binom{8}{3}\cdot(4!-3!)}{\binom{8}{4}\cdot\binom{9}{4}\cdot 4!}$$

iii. either Rebecca or Elise will be chosen to represent her school?

$$\frac{\binom{7}{3}\cdot\binom{8}{4}+\binom{7}{4}\cdot\binom{8}{3}}{\binom{8}{4}\cdot\binom{9}{4}}$$

31. A 3-person basketball team consists of a guard, a forward, and a center.

i. If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?

$$\frac{3}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}$$

ii. What is the probability that all 3 players selected play the same position?

$$\frac{3}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}$$

32. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the ith position, $1 \leq i \leq b+g$, is a girl?

Si fijamos el contenido de la posición i-ésima en chico entonces hay $\frac{(b+g-1)!}{(b-1)!\cdot g!}$ permutaciones tal que la i-ésima posición es un chico. Si fijamos el contenido de la posición i-ésima en chica entonces hay $\frac{(b+g-1)!}{b!\cdot(g-1)!}$ permutaciones tal que la i-ésima posición es una chica.

Por lo tanto la probabilidad de que la i-ésima posición sea una chica es de

$$\frac{\frac{(b+g-1)!}{b!\cdot(g-1)!}}{\frac{(b+g)!}{b!\cdot g!}} = \frac{(b+g-1)!\cdot b!\cdot g!}{b!\cdot(g-1)!\cdot(b+g)!} = \frac{g}{b+g}$$

33. A forest contains 20 elk, of which 5 are captured, tagged, and then released. A certain time later, 4 of the 20 elk are captured. What is the probability that 2 of these 4 have been tagged? What assumptions are you making?

$$\frac{\binom{5}{2}\cdot\binom{15}{2}}{\binom{20}{4}}$$

Asumimos que los hechos son independientes.

34. The second Earl of Yarborough is reported to have bet at odds of 1000 to 1 that a bridge hand of 13 cards would contain at least one card that is ten or higher. (By ten or higher we mean that a card is either a ten, a jack, a queen, a king, or an ace.) Nowadays, we call a hand that has no cards higher than 9 a Yarborough. What is the probability that a randomly selected bridge hand is a Yarborough?

$$\frac{\binom{32}{13}}{\binom{52}{13}}$$

35. Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability that

i. 3 red, 2 blue, and 2 green balls are withdrawn;

$$\frac{\binom{12}{3}\cdot\binom{16}{2}\cdot\binom{18}{2}}{\binom{46}{7}}$$

ii. at least 2 red balls are withdrawn;

$$1-\frac{\binom{34}{7}+\binom{12}{1}\cdot\binom{34}{6}}{\binom{46}{7}}$$

iii. all withdrawn balls are the same color;

$$\frac{\binom{12}{7}+\binom{16}{7}+\binom{18}{7}}{\binom{46}{7}}$$

iv. either exactly 3 red balls or exactly 3 blue balls are withdrawn.

$$\frac{\binom{12}{3}\cdot\left(\binom{18}{2}\cdot\binom{16}{2}+\binom{18}{3}\cdot\binom{16}{1}+\binom{18}{4}\right)}{\binom{46}{7}}+\frac{\binom{16}{3}\cdot\left(\binom{18}{2}\cdot\binom{12}{2}+\binom{18}{3}\cdot\binom{12}{1}+\binom{18}{4}\right)}{\binom{46}{7}}$$

36. Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they

i. are both aces?

$$\frac{\binom{4}{2}}{\binom{52}{2}}$$

ii. have the same value?

$$\frac{13\cdot\binom{4}{2}}{\binom{52}{2}}$$

37. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly

i. all 5 problems?

$$\frac{\binom{7}{5}}{\binom{10}{5}}$$

ii. at least 4 of the problems?

$$\frac{\binom{7}{4}\cdot\binom{3}{1}}{\binom{10}{5}}+\frac{\binom{7}{5}}{\binom{10}{5}}$$

38. There are n socks, 3 of which are red, in a drawer. What is the value of n if, when 2 of the socks are chosen randomly, the probability that they are both red is $\frac{1}{2}$.

$$\begin{aligned}\frac{1}{2}&=P(\{2\text{ red}\})\\&=\frac{\binom{3}{2}}{\binom{n}{2}}\\&=\frac{3\cdot 2}{(n-1)\cdot n}\end{aligned}$$

entonces

$$\begin{aligned}(n-1)\cdot n&=12\\n&=4\end{aligned}$$

39. There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different hotel? What assumptions are you making?

$$\frac{5\cdot 4\cdot 3}{5^3}$$

Asumimos que los hechos son independientes.

40. A town contains 4 people who repair televisions. If 4 sets break down, what is the probability that exactly i of the repairers are called? Solve the problem for $i=1,2,3,4$. What assumptions are you making?

1. $\frac{4}{4^4}$
2. $\frac{\binom{4}{2}\cdot\left(4+\binom{4}{2}+4\right)}{4^4}$
3. $\frac{\binom{4}{3}\cdot\binom{3}{1}\cdot\frac{4!}{2!}}{4^4}$
4. $\frac{4\cdot 3\cdot 2\cdot 1}{4^4}$

Asumimos que los hechos son independientes.

41. If a die is rolled 4 times, what is the probability that 6 comes up at least once?

$$\frac{\binom{4}{1}\cdot 5^3+\binom{4}{2}\cdot 5^2+\binom{4}{3}\cdot 5^1+\binom{4}{4}\cdot 5^0}{6^4}$$

42. Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large need n be to make this probability at least $\frac{1}{2}$?

$$1-P(\{\text{no doble 6 en } n \text{ tiros}\})$$
$$1-(\tfrac{35}{36})^n$$

La probabilidad es mayor a $\frac{1}{2}$ cuando $(\frac{35}{36})^n$ sea menor a $\frac{1}{2}$. Esto sucede cuando $n \geq 25$.

43. i. If N people, including A and B, are randomly arranged in a line, what is the probability that A and B are next to each other?

$$\frac{2 \cdot \binom{N-1}{1} \cdot (N-2)!}{N!} = \frac{2}{N}$$

ii. What would the probability be if the people were randomly arranged in a circle?

$$\frac{N}{\binom{N}{2}} = \frac{2}{N-1}$$

44. Five people, designated as A, B, C, D, E, are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that

i. there is exactly one person between A and B?

$$\frac{2 \cdot \binom{3}{1} \cdot 1! \cdot 3!}{5!}$$

ii. there are exactly two people between A and B?

$$\frac{2 \cdot \binom{3}{2} \cdot 2! \cdot 2!}{5!}$$

iii. there are three people between A and B?

$$\frac{2 \cdot \binom{3}{3} \cdot 3! \cdot 1!}{5!}$$

45. A woman has n keys, of which one will open her door.

i. If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?

$$\frac{1}{n}$$

ii. What if she does not discard previously tried keys?

$$\left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

46. How many people have to be in a room in order that the probability that at least two of them celebrate their birthday in the same month is at least $\frac{1}{2}$? Assume that all possible monthly outcomes are equally likely.

Para $1 \leq n \leq 12$

$$1 - \frac{\frac{12!}{(12-n)!}}{12^n}$$

Que es mayor a $\frac{1}{2}$ cuando $n \geq 5$.

47. If there are 12 strangers in a room, what is the probability that no two of them celebrate their birthday in the same month?

$$\frac{12!}{12^n}$$

48. Given 20 people, what is the probability that among the 12 months in the year, there are 4 months containing exactly 2 birthdays and 4 containing exactly 3 birthdays?

$$\frac{\binom{20}{2,2,2,2,3,3,3,3,0,0,0,0} \cdot \binom{12}{4,4,4}}{12^{20}}$$

49. A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

$$\frac{\binom{6}{3} \cdot \binom{6}{3}}{2^{12}}$$

50. In a hand of bridge, find the probability that you have 5 spades and your partner has the remaining 8.

$$\frac{\binom{13}{5,8} \cdot \binom{39}{13} \cdot \binom{13}{8,5}}{\binom{52}{26} \cdot \binom{26}{13}}$$

51. Suppose that n balls are randomly distributed into N compartments. Find the probability that m balls will fall into the first compartment. Assume that all N arrangements are equally likely.

$$\frac{\binom{n}{m} \cdot (N-1)^{n-m}}{N^n}$$

52. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be

i. no complete pair?

$$\frac{\binom{10}{8}\cdot 2^8}{\binom{20}{8}}$$

ii. exactly 1 complete pair?

$$\frac{\binom{10}{7}\cdot\binom{7}{6}\cdot 2^6}{\binom{20}{8}}$$

53. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

Sea E_i el evento tal que exactamente i parejas se sientan juntas, $i = 0, 1, 2, 3, 4$.

$$\begin{aligned} &P(E_0) \\ &= 1 - P(E_1 \cup E_2 \cup E_3 \cup E_4) \\ &= 1 - \binom{4}{1}\cdot\frac{2^1\cdot 7!}{8!} + \binom{4}{2}\cdot\frac{2^2\cdot 6!}{8!} \\ &\quad - \binom{4}{3}\cdot\frac{2^3\cdot 5!}{8!} + \binom{4}{4}\cdot\frac{2^4\cdot 4!}{8!} \end{aligned}$$

54. Compute the probability that a bridge hand is void in at least one suit. Note that the answer is not

$$\frac{\binom{4}{1}\cdot\binom{39}{13}}{\binom{52}{13}}$$

Why not?

Los casos no son necesariamente disjuntos, hay manos que no tienen ninguna carta de dos o tres palos.

Sea E_i el evento tal que la mano de bridge no tiene ninguna carta del palo i, $i = 1, 2, 3, 4$.

$$\begin{aligned} &P(E_1 \cup E_2 \cup E_3 \cup E_4) \\ &= \binom{4}{1}\cdot\frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2}\cdot\frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3}\cdot\frac{\binom{13}{13}}{\binom{52}{13}} \end{aligned}$$

55. Compute the probability that a hand of 13 cards contains

i. the ace and king of at least one suit.

Sea E_i el evento tal que la mano contiene el as y el rey del palo i, $i = 1, 2, 3, 4$.

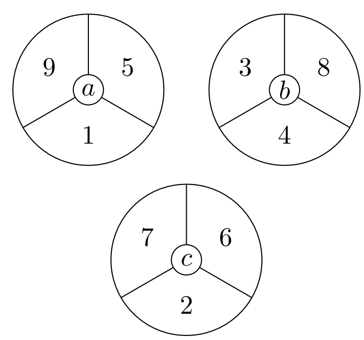
$$\begin{aligned} &P(E_1 \cup E_2 \cup E_3 \cup E_4) \\ &= \binom{4}{1}\cdot\frac{\binom{50}{11}}{\binom{52}{13}} - \binom{4}{2}\cdot\frac{\binom{48}{9}}{\binom{52}{13}} + \\ &\binom{4}{3}\cdot\frac{\binom{46}{7}}{\binom{52}{13}} - \binom{4}{4}\cdot\frac{\binom{44}{5}}{\binom{52}{13}} \end{aligned}$$

ii. all 4 of at least 1 of the 13 denominations.

Sea E_i el evento tal que la mano contiene las cuatro cartas numeradas i, $i = 1, 2, \ldots, 13$.

$$\begin{aligned} &P(\textstyle\bigcup_{i=1}^{13} E_i) \\ &= \binom{13}{1}\cdot\frac{\binom{48}{9}}{\binom{52}{13}} - \binom{13}{2}\cdot\frac{\binom{44}{5}}{\binom{52}{13}} + \\ &\binom{13}{3}\cdot\frac{\binom{40}{1}}{\binom{52}{13}} \end{aligned}$$

56. Two players play the following game: Player A chooses one of the three spinners pictured in Figure 6, and then player B chooses one of the remaining two spinners. Both players then spin their spinner, and the one that lands on the higher number is declared the winner. Assuming that each spinner is equally likely to land in any of its 3 regions, would you rather be player A or player B? Explain your answer!



Es mejor ser el jugador B, sea E_a, E_b, E_c el evento tal que A elige el disco correspondiente y B gana.

Si A elige el disco

a. y obtiene el siguiente valor B,

9. no puede ganar con ningún disco.
1. gana siempre con el disco c.
5. gana $\frac{2}{3}$ de las veces con el disco c.

$P(E_a) = \frac{1}{3} \cdot (0 + 1 + \frac{2}{3}) = \frac{5}{9} > \frac{1}{2}$

b. y obtiene el siguiente valor B,

3. gana $\frac{2}{3}$ de las veces con el disco a.
4. gana $\frac{2}{3}$ de las veces con el disco a.
8. gana $\frac{1}{3}$ de las veces con el disco a.

$P(E_a) = \frac{1}{3} \cdot (\frac{2}{3} + \frac{2}{3} + \frac{1}{3}) = \frac{5}{9} > \frac{1}{2}$

c. y obtiene el siguiente valor B,

7. gana $\frac{1}{3}$ de las veces con el disco b.
2. gana siempre con el disco b.
6. gana $\frac{1}{3}$ de las veces con el disco b.

$P(E_a) = \frac{1}{3} \cdot (\frac{1}{3} + 1 + \frac{1}{3}) = \frac{5}{9} > \frac{1}{2}$

Theoretical Problems

Prove the following relations

1. $EF \subset E \subset E \cup F$.

Demostración. de la relación anterior.

Si $x \in EF$ entonces $x \in E$ por definición.

Si $x \in E$ entonces $x \in E \cup F$ por definición. □

2. If $E \subset F$, then $F^{\mathsf{C}} \subset E^{\mathsf{C}}$.

Demostración. de la relación anterior.

Si $x \in F^{\mathsf{C}}$ entonces $x \notin F$. Supongamos que $x \in E$ entonces como $E \subset F$ ocurre que $x \in F$, absurdo, por lo tanto $x \notin E$ y $x \in F^{\mathsf{C}}$. □

3. $F = FE \cup FE^{\mathsf{C}}$ and $E \cup F = E \cup E^{\mathsf{C}}F$.

Demostración. de la relación anterior.

$$\begin{aligned} F &= FS \\ &= F(E \cup E^{\mathsf{C}}) \\ &= FE \cup FE^{\mathsf{C}} \end{aligned}$$

y

$$\begin{aligned} E \cup F &= (E \cup F)(E \cup E^{\mathsf{C}}) \\ &= ((E \cup F)E) \cup ((E \cup F)E^{\mathsf{C}}) \\ &= E \cup ((E^{\mathsf{C}}E) \cup (E^{\mathsf{C}}F)) \\ &= E \cup (\varnothing \cup (E^{\mathsf{C}}F)) \\ &= E \cup E^{\mathsf{C}}F \end{aligned}$$ □

4. $\left(\bigcup_{i=1}^{\infty} E_i\right) F = \bigcup_{i=1}^{\infty} (E_iF)$ y $\left(\bigcap_{i=1}^{\infty} E_i\right) \cup F = \bigcap_{i=1}^{\infty} (E_i \cup F)$.

Demostración.
$\left(\bigcup_{i=1}^{\infty} E_i\right) F = \bigcup_{i=1}^{\infty} (E_iF)$.

$\subset$ Si $x \in \left(\bigcup_{i=1}^{\infty} E_i\right) F$ entonces $x \in \bigcup_{i=1}^{\infty} E_i$ y $x \in F$. Como $x \in \bigcup_{i=1}^{\infty} E_i$ entonces $x \in E_k$ para algún k. Luego como $x \in E_k$ y $x \in F$ entonces $x \in E_kF$ por lo tanto $x \in \bigcup_{i=1}^{\infty} (E_iF)$.

$\supset$ Si $x \in \bigcup_{i=1}^{\infty} (E_iF)$ entonces $x \in E_kF$ para algún k. Luego como $x \in E_kF$ entonces $x \in E_k$ y $x \in F$. Como $x \in E_k$ entonces $x \in \bigcup_{i=1}^{\infty} E_i$. Como $x \in \bigcup_{i=1}^{\infty} E_i$ y $x \in F$ entonces $x \in \left(\bigcup_{i=1}^{\infty} E_i\right) F$. □

Demostración.
$\left(\bigcap_{i=1}^{\infty} E_i\right) \cup F = \bigcap_{i=1}^{\infty} (E_i \cup F)$.

$\subset$ Si $x \in \left(\bigcap_{i=1}^{\infty} E_i\right) \cup F$ entonces ó $x \in \bigcap_{i=1}^{\infty} E_i$ ó $x \in F$.
Si $x \in \bigcap_{i=1}^{\infty} E_i$ entonces $x \in E_k$ para todo $k \in \mathbb{N}$ y $x \in E_k \cup F$ para todo $k \in \mathbb{N}$ por lo tanto $x \in \bigcap_{i=1}^{\infty} (E_i \cup F)$.
Si $x \in F$ entonces $x \in E_k \cup F$ para todo $k \in \mathbb{N}$ por lo tanto $x \in \bigcap_{i=1}^{\infty} (E_i \cup F)$.

$\supset$ Si $x \in \bigcap_{i=1}^{\infty} (E_i \cup F)$ entonces $x \in E_k \cup F$ para todo k por lo tanto $x \in E_k$ para todo k ó $x \in F$.
Si $x \in F$ entonces $x \in \left(\bigcap_{i=1}^{\infty} E_i\right) \cup F$.
Si $x \in E_k$ para todo k entonces $x \in \bigcap_{i=1}^{\infty} E_i$ por lo tanto $x \in \left(\bigcap_{i=1}^{\infty} E_i\right) \cup F$. □

5. For any sequence of events $E_1, E_2, \ldots$, define a new sequence $F_1, F_2, \ldots$ of disjoint events (that is, events such that $F_iF_j = \varnothing$ whenever $i \neq j$) such that for all $n \geq 1$,

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$$

Sea

$$F_k = \begin{cases} E_1 & \text{si } k = 1 \\ E_k \left(\bigcap_{i=1}^{k-1} E_i^{\complement}\right) & \text{en otro caso} \end{cases}$$

Por inducción, F_2 es disjunto con F_1 ya que $F_2 = E_2E_1^{\complement}$ y

$$F_1 \cup F_2 = E_1 \cup (E_2E_1^{\complement})$$

por el ejercicio 3

$$= E_1 \cup E_2$$

Si $F_iF_j = \varnothing$ para $1 \leq i, j \leq k$, $i \neq j$ entonces, sea $j \leq k$

$$F_{k+1}F_j = \left(E_{k+1} \bigcap_{i=1}^{k} E_i^{\complement}\right) \left(E_j \bigcap_{i=1}^{j-1} E_i^{\complement}\right)$$

por asociatividad

$$= E_{k+1} \left(\bigcap_{i=1}^{k} E_i^{\complement}\right) E_j \left(\bigcap_{i=1}^{j-1} E_i^{\complement}\right)$$

como $j \leq k$ entonces $(\bigcap_{i=1}^{k} E_i^{\complement})E_j = \varnothing$

$$= \varnothing$$

además

$$\bigcup_{i=1}^{k+1} F_i$$

$$= \left(\bigcup_{i=1}^{k} F_i\right) \cup F_{k+1}$$

por hipótesis inductiva y definición de F_{k+1}

$$= \left(\bigcup_{i=1}^{k} E_i\right) \cup \left(E_{k+1} \bigcap_{i=1}^{k} E_i^{\complement}\right)$$

por distributividad

$$= \left(E_{k+1} \cup \bigcup_{i=1}^{k} E_i\right) \left(\left(\bigcap_{i=1}^{k} E_i^{\complement}\right) \cup \left(\bigcup_{i=1}^{k} E_i\right)\right)$$

por De Morgan

$$= \left(\bigcup_{i=1}^{k+1} E_i\right) \left(\left(\bigcup_{i=1}^{k} E_i\right)^{\complement} \cup \left(\bigcup_{i=1}^{k} E_i\right)\right)$$

$$= \left(\bigcup_{i=1}^{k+1} E_i\right) S$$

$$= \bigcup_{i=1}^{k+1} E_i$$

6. Let E, F, and G be three events. Find expressions so that, of E, F, and G,

i. only E occurs

$$E$$

ii. both E and G, but not F, occur

$$EF^{\complement}G$$

iii. at least one of the events occurs

$$E \cup F \cup G$$

iv. at least two of the events occur

$$EF \cup EG \cup FG$$

v. all three events occur

$$EFG$$

vi. none of the events occurs

$$E^{\complement}F^{\complement}G^{\complement}$$

vii. at most one of the events occurs

$$E^{\complement}F^{\complement}G^{\complement} \cup EF^{\complement}G^{\complement} \cup E^{\complement}FG^{\complement} \cup E^{\complement}F^{\complement}G$$

viii. at most two of the events occur

$$(EFG)^{\complement}$$

ix. exactly two of the events occur

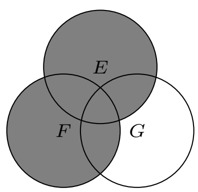
$$EFG^{\mathsf{C}} \cup EF^{\mathsf{C}}G \cup E^{\mathsf{C}}FG$$

x. at most three of the events occur

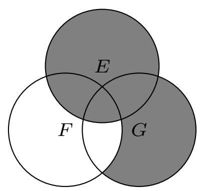
$$S$$

7. Use Venn diagrams

i. to simplify $(E \cup F)(E \cup F^{\mathsf{C}})$

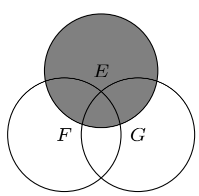


$E \cup F$



$E \cup F^{\mathsf{C}}$

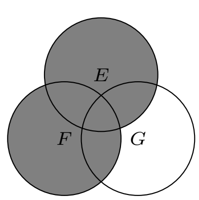
su intersección



$(E \cup F)(E \cup F^{\mathsf{C}})$

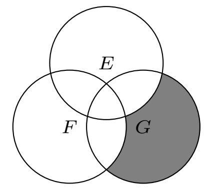
ii. to prove DeMorgan's laws for events E and F. (prove $(E \cup F)^{\mathsf{C}} = E^{\mathsf{C}}F^{\mathsf{C}}$, and $(EF)^{\mathsf{C}} = E^{\mathsf{C}} \cup F^{\mathsf{C}}$)

Demostración. $(E \cup F)^{\mathsf{C}} = E^{\mathsf{C}}F^{\mathsf{C}}$

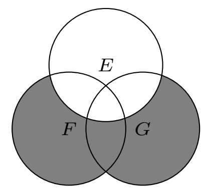


$E \cup F$

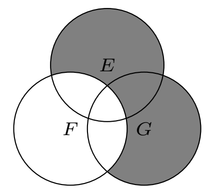
su complemento



$(E \cup F)^{\mathsf{C}}$

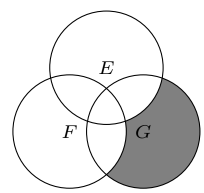


E^{C}



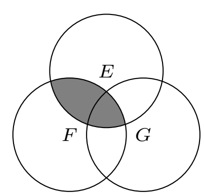
F^{C}

su intersección



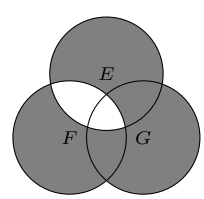
$E^{\mathsf{C}}F^{\mathsf{C}}$ □

Demostración. $(EF)^{\mathsf{C}} = E^{\mathsf{C}} \cup F^{\mathsf{C}}$

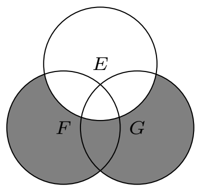


EF

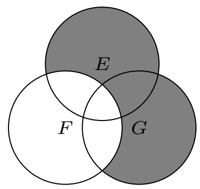
su complemento



$(EF)^{\mathsf{C}}$

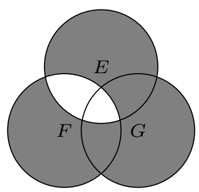


$E^{\complement}$



$F^{\complement}$

su unión



$E^{\complement} \cup F^{\complement}$ □

8. Let S be a given set. If, for some $k > 0$, $S_1, S_2, \ldots, S_k$ are mutually exclusive nonempty subsets of S such that $\bigcup_{i=1}^{k} S_i = S$, then we call the set $\{S_1, S_2, \ldots, S_k\}$ a partitation of S. Let T_n denote the number of different partitions of $\{1, 2, \ldots, n\}$. Thus, $T_1 = 1$ (the only partition being $S_1 = \{1\}$) and $T_2 = 2$ (the two partitions being $\{\{1,2\}\}, \{\{1\},\{2\}\}$).

i. Show, by computing all partitions, that $T_3 = 5$, $T_4 = 15$.

Particiones de $\{1, 2, 3\}$

$\{\{1,2,3\}\}$ $\{\{1,2\},\{3\}\}$
$\{\{1,3\},\{2\}\}$ $\{\{2,3\},\{1\}\}$
$\{\{1\},\{2\},\{3\}\}$

Particiones de $\{1, 2, 3, 4\}$

$\{\{1,2,3,4\}\}$ $\{\{1,2,3\},\{4\}\}$
$\{\{1,2,4\},\{3\}\}$ $\{\{1,3,4\},\{2\}\}$
$\{\{2,3,4\},\{1\}\}$ $\{\{1,2\},\{3,4\}\}$
$\{\{1,3\},\{2,4\}\}$ $\{\{1,4\},\{2,3\}\}$
$\{\{1\},\{2\},\{3,4\}\}$ $\{\{1\},\{3\},\{2,4\}\}$
$\{\{1\},\{4\},\{2,3\}\}$ $\{\{2\},\{3\},\{1,4\}\}$
$\{\{2\},\{4\},\{1,3\}\}$ $\{\{3\},\{4\},\{1,2\}\}$
$\{\{1\},\{2\},\{3\}\{4\}\}$

ii. Show that

$$T_{n+1} = 1 + \sum_{k=1}^{n} \binom{n}{k} T_k$$

and use this equation to compute T_{10}. *Hint:* One way of choosing a partition of $n+1$ items is to call one of the items special Then we obtain different partitions by first choosing k, $k = 0, 1, \ldots, n$, then a subset of size $n-k$ of the nonspecial items, and then any of the T_k partitions of the remaining k nonspecial items. By adding the special item to the subset of size $n-k$, we obtain a partition of all $n+1$ items.

Para cada $k = 1, \ldots, n$ tomamos todas las particiones de k elementos compuestas de k de los n elementos disponibles. A cada una le agregamos un último conjunto con los $n+1-k$ elementos restantes. Sumando la partición $\{1, \ldots, n+1\}$

Computando la serie $T_{10} = 115975$.

9. Suppose that an experiment is performed n times. For any event E of the sample space, let $n(E)$ denote the number of times that event E occurs and define $f(E) = \frac{n(E)}{n}$. Show that $f(\cdot)$ satisfies Axioms 1, 2, and 3.

Demostración. $f(\cdot)$ satisface los axiomas 1, 2, y 3.

1. $0 \leq f(E) \leq 1$

   Un experimento realizado n veces puede fallar como mucho todas las veces y como mínimo ninguna vez

$$0 \leq \frac{0}{n} \leq f(E) \leq \frac{n}{n} \leq 1$$

2. $f(S) = 1$

   Por definición $n(S) = n$ luego

$$f(S) = \frac{n(S)}{n} = \frac{n}{n} = 1$$

3. $f\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} f(E_i)$

supongamos que ocurre n_1 veces el evento E_1, n_2 veces el evento E_2, $\ldots$, entonces como los eventos son disjuntos

$$\begin{aligned}
f\left(\bigcup_{i=1}^{\infty} E_i\right) &= \frac{n\left(\bigcup_{i=1}^{\infty} E_i\right)}{n} \\
&= \frac{n_1 + n_2 + \cdots}{n} \\
&= \frac{n_1}{n} + \frac{n_2}{n} + \cdots \\
&= f(E_1) + f(E_2) + \cdots \\
&= \sum_{i=1}^{\infty} f(E_i)
\end{aligned}$$

□

10. Prove that

$$\begin{aligned}
&P(E \cup F \cup G) \\
&= P(E) + P(F) + P(G) - \\
&\quad P(E^{\mathsf{C}}FG) - P(EF^{\mathsf{C}}G) - P(EFG^{\mathsf{C}}) - \\
&\quad 2P(EFG)
\end{aligned}$$

Demostración. de la equivalencia anterior.

$$\begin{aligned}
&P(E \cup F \cup G) \\
&= P(E) + P(F \cup G) - P(E(F \cup G)) \\
&= P(E) + P(F) + P(G) - \\
&\quad P(FG) - P(E(F \cup G)) \\
&= P(E) + P(F) + P(G) - \\
&\quad P(FG) - P((EF) \cup (EG)) \\
&= P(E) + P(F) + P(G) - \\
&\quad P(FG) - P(EF) - P(EG) + P(EFG) \\
&= P(E) + P(F) + P(G) - \\
&\quad (P(EFG)P(E^{\mathsf{C}}FG)) - \\
&\quad (P(EFG) + P(EFG^{\mathsf{C}})) - \\
&\quad (P(EFG) + P(EF^{\mathsf{C}}G)) + \\
&\quad P(EFG) \\
&= P(E) + P(F) + P(G) - 3P(EFG) - \\
&\quad P(E^{\mathsf{C}}FG) - P(EFG^{\mathsf{C}}) - P(EF^{\mathsf{C}}G) + \\
&\quad P(EFG) \\
&= P(E) + P(F) + P(G) - \\
&\quad P(E^{\mathsf{C}}FG) - P(EFG^{\mathsf{C}}) - P(EF^{\mathsf{C}}G) - \\
&\quad 2P(EFG)
\end{aligned}$$

□

11. If $P(E) = .9$ and $P(F) = .8$, show that $P(EF) \geq .7$. In general, prove *Bonferroni's inequality*, namely,

$$P(EF) \geq P(E) + P(F) - 1$$

Demostración. de la *desigualdad de Bonferroni.*

$$P(EF) = P(E) + P(F) - P(E \cup F)$$

por axioma 1. $P(E \cup F) \leq 1$

$$\geq P(E) + P(F) - 1$$

□

12. Show that the probability that exactly one of the events E or F occurs equals $P(E) + P(F) -$

$2P(EF)$.

$$\begin{aligned}
&P(EF^{\complement} \cup E^{\complement}F)\\
&= P(EF^{\complement}) + P(E^{\complement}F) - P(EF^{\complement}E^{\complement}F)\\
&= P(E^{\complement} \cup F) + P(E \cup F^{\complement}) - 0\\
&= P(F) - P(EF) + P(E) - P(EF)\\
&= P(E) + P(F) - 2P(EF)
\end{aligned}$$

13. Prove that $P(EF^{\complement}) = P(E) - P(EF)$.

Demostración. de la equivalencia anterior.

$$\begin{aligned}
&P(E)\\
&= P(E(F \cup F^{\complement}))\\
&= P((EF) \cup (EF^{\complement}))\\
&= P(EF) + P(EF^{\complement}) - P(EFEF^{\complement})\\
&= P(EF) + P(EF^{\complement})
\end{aligned}$$

□

14. Prove Proposition 4.4 by mathematical induction.

Demostración. de la Proposición 4.4

Por inducción en n

Trivialmente para $n = 2$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1E_2)$$

Si para k ocurre que

$$\begin{aligned}
&P(E_1 \cup E_2 \cup \cdots \cup E_k)\\
&= \sum_{r=1}^{k}(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(\bigcap_{j=1}^{r} E_{i_j})
\end{aligned}$$

entonces

$$\begin{aligned}
&P(E_1 \cup E_2 \cup \cdots \cup E_{k+1})\\
&= P((E_1 \cup E_2 \cup \cdots \cup E_k) \cup E_{k+1})
\end{aligned}$$

por el caso base tenemos que

$$\begin{aligned}
&= P(E_1 \cup E_2 \cup \cdots \cup E_k) + P(E_{k+1}) -\\
&\quad P((E_1 \cup E_2 \cup \cdots \cup E_k)E_{k+1})\\
&= P(E_1 \cup E_2 \cup \cdots \cup E_k) + P(E_{k+1}) -\\
&\quad P(E_1E_{k+1} \cup E_2E_{k+1} \cup \cdots \cup E_kE_{k+1})
\end{aligned}$$

por hipótesis inductiva

$$\begin{aligned}
&= \sum_{r=1}^{k}(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(\bigcap_{j=1}^{r} E_{i_j}) +\\
&\quad P(E_{k+1}) -\\
&\quad \sum_{r=1}^{k}(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(\bigcap_{j=1}^{r} E_{i_j}E_{k+1})\\
&= \sum_{r=1}^{k}(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(\bigcap_{j=1}^{r} E_{i_j}) +\\
&\quad P(E_{k+1}) -\\
&\quad \sum_{r=1}^{k}(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(E_{k+1}\bigcap_{j=1}^{r} E_{i_j})\\
&= \sum_{r=1}^{k+1}(-1)^{r+1} \sum_{i_1<i_2<\cdots<i_r} P(\bigcap_{j=1}^{r} E_{i_j})
\end{aligned}$$

□

15. An urn contains M white and N black balls. If a random sample of size r is chosen, what is the probability that it contains exactly k white balls?

$$\frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}$$

16. Use induction to generalize *Bonferroni's inequality* to n events. Thats is show that

$$P(E_1E_2\cdots E_n) \geq P(E_1) + \cdots + P(E_n) - (n-1)$$

Demostración. de la generalización de la desigualdad de *Bonferroni.*

Por inducción, $P(E_1E_2) \geq P(E_1) + P(E_2) - 1$ por la desigualdad de *Bonferroni.*

Si

$$\begin{aligned}
&P(E_1E_2\cdots E_k)\\
&\geq P(E_1) + \cdots + P(E_k) - (k-1)
\end{aligned}$$

entonces

$$\begin{aligned}
&P(E_1E_2\cdots E_{k+1})\\
&= P(E_1(E_2\cdots E_{k+1}))
\end{aligned}$$

por la desigualdad de *Bonferroni*

$$= P(E_1) + P(E_2 \cdots E_{k+1}) - 1$$

por hipótesis inductiva

$$\begin{aligned} &= P(E_1) + \\ &\quad (P(E_2) + \cdots P(E_{k+1}) - (k-1)) - 1 \\ &= P(E_1) + \cdots + P(E_{k+1}) - k \end{aligned}$$

□

17. Consider the matching problem, Example 5m, and define A_n to be the number of ways in which the N men can select their hats so that no man selects his own. Argue that

$$A_N = (N-1)(A_{N-1} + A_{N-2})$$

This formula, along with the boundary conditions $A_1 = 0$, $A_2 = 1$, can then be solved for A_n, and the desired probability of no matches would be $A_n/N!$.

Hint: After the first man selects a hat that is not his own, there remain $N-1$ men to select among a set of $N-1$ hats that does not contain the hat of one of these men. Thus, there is one extra man and one extra hat. Argue that we can get no matches either with the extra man selecting the extra hat or with the extra man not selecting the extra hat.

Si el hombre i-ésimo elige el sombrero j-ésimo con $i \neq j$ entonces nos queda un hombre sin su sombrero y un sombrero sin su correspondiente dueño.

Para obtener un emparejamiento donde ningún hombre obtiene su sombrero podemos,

emparejar al j-ésimo hombre con el i-ésimo sombrero, en cuyo caso nos queda ordenar a $N-2$ hombres y sus sombreros, ó

no emparejar al j-ésimo hombre con el i-ésimo sombrero, en cuyo caso podemos pensar que el i-ésimo sombrero pertenecía al j-ésimo hombre y nos queda ordenar a $N-1$ hombres y sus sombreros.

Donde tenemos $N-1$ formas de seleccionar j tal que $i \neq j$.

18. Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear. Argue that

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2$$

where $f_0 = 1$, $f_1 = 2$.

Hint: How many outcomes are there that start with a head, and how many start with a tail? If P_n denotes the probability that successive heads never appear when a coin is tossed n times, find P_n (in terms of f_n) when all possible outcomes of the n tosses are assumed equally likely. Compute P_{10}.

Trivialmente $f_0 = 1$ y $f_1 = 2$, inductivamente si f_{k-2} y f_{k-1} son la cantidad de resultados donde la moneda no sale cara más de una vez seguida en $k-2$ y $k-1$ tiros respectivamente luego

si tiramos una moneda k veces puede ocurrir que el primer tiro

sea cara, en cuyo caso el siguiente tiro debe ser ceca y los $k-2$ restantes pueden formar cualquier secuencia tal que no salga cara más de una vez seguida (f_{k-2}) ó

sea ceca en cuyo caso los $k-1$ restantes pueden formar cualquier secuencia tal que no salga cara más de una vez seguida (f_{k-1}).

$$P_n = \frac{f_n}{2^n}$$

y entonces $P_{10} = \frac{f_{10}}{2^{10}} = \frac{144}{2^{10}}$

19. An urn contains n red and m blue balls. They are withdrawn one at a time until a total of r, $r \leq n$, red balls have been withdrawn. Find the probability that a total of k balls are withdrawn.

Hint: A total of k balls will be withdrawn if there are $r-1$ red balls in the first $k-1$ withdrawals and the kth withdrawal is a red ball.

$$\frac{\binom{n}{r} \cdot \binom{m}{k-r} \cdot r}{\binom{n+m}{k} \cdot k}$$

20. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?

Supongamos que todos los puntos del espacio muestral son equiprobables, y definamos los eventos E_i, el punto i-ésimo en un ordenamiento de S. Por el axioma 2

$$1 = P(S)$$

como los eventos son puntuales son disjuntos

$$= \sum_{i=1}^{\infty} P(E_i)$$

si suponemos que todos los puntos son equiprobables

$$= \lim_{n\to\infty} \sum_{i=1}^{n} c$$

$$= c \cdot \lim_{n\to\infty} n$$

si $c = 0$ entonces $0 \cdot \lim_{n\to\infty} n = 0 \neq 1$.

si $c > 0$ entonces $c \cdot \lim_{n\to\infty} n$ es divergente.

21. Consider Example 5o, which is concerned with the number of runs of wins obtained when n wins and m losses are randomly permuted. Now consider the total number of runs—that is, win runs plus loss runs—and show that

$$P(\{2k \text{ runs}\}) = 2 \cdot \frac{\binom{m-1}{k-1} \cdot \binom{n-1}{k-1}}{\binom{n+m}{n}}$$

$$P(\{2k{+}1 \text{ runs}\}) = \frac{\binom{n-1}{k} \cdot \binom{m-1}{k-1} + \binom{n-1}{k-1} \cdot \binom{m-1}{k}}{\binom{n+m}{n}}$$

Siguiendo el ejemplo 5o consideremos un vector de enteros positivos $x_1, \ldots, x_k$ con $x_1 + \cdots + x_k = n$ donde la seguidilla de victorias i-ésima es de longitud x_i, $i = 1, \ldots, k$.

De manera similar consideremos otro vector de enteros positivos $y_1, \ldots, y_k$ con $y_1 + \cdots + y_k = m$ donde la seguidilla de derrotas i-ésima es de longitud y_i, $i = 1, \ldots, k$.

Considerando la posibilidad de que la primer seguidilla sea de victorias o derrotas la cantidad de soluciones posibles es de

$$2 \cdot \binom{n-1}{k-1} \cdot \binom{m-1}{k-1}$$

y entonces

$$P(\{2k \text{ runs}\}) = 2 \cdot \frac{\binom{n-1}{k-1} \cdot \binom{m-1}{k-1}}{\binom{n+m}{n}}$$

Si ahora tenemos $2k + 1$ seguidillas entonces puede ocurrir que

la nueva seguidilla sea de victorias por lo tanto ahora tenemos $x_1, \ldots, x_{k+1}$ con $x_1 + \cdots + x_{k+1} = n$. Entonces las soluciones posibles son

$$\binom{n-1}{k} \cdot \binom{m-1}{k-1}$$

la nueva seguidilla sea de derrotas por lo tanto ahora tenemos $y_1, \ldots, y_{k+1}$ con $y_1 + \cdots + y_{k+1} = m$. Entonces las soluciones posibles son

$$\binom{n-1}{k-1} \cdot \binom{m-1}{k}$$

Combinando estos resultados tenemos que

$$P(\{2k{+}1 \text{ runs}\}) = \frac{\binom{n-1}{k} \cdot \binom{m-1}{k-1} + \binom{n-1}{k-1} \cdot \binom{m-1}{k}}{\binom{n+m}{n}}$$

Self-Test Problems

1. A cafeteria offers a three-course meal consisting of an entree, a starch, and a dessert. The possible choices aregiven in the following table

Course	Choices
Entree	Chicken Roast, Roast Beef
Starch	Pasta, rice, potatoes
Dessert	Ice cream, Jello, apple pie, a peach

A person is to choose one course from each category.

i. How many outcomes are in the sample space?

$$2 \cdot 3 \cdot 4$$

ii. Let A be the event that ice cream is chosen. How many outcomes are in A?

$$2 \cdot 3$$

iii. Let B be the event that chicken is chosen. How many outcomes are in B?

$$3 \cdot 4$$

iv. List all the outcomes in the event AB.

$$\begin{aligned} AB = \{&(\text{chicken, pasta, ice-cream}),\\ &(\text{chicken, rice, ice-cream}),\\ &(\text{chicken, potatoes, ice-cream})\} \end{aligned}$$

v. Let C be the event that rice is chosen. How many outcomes are in C?

$$2 \cdot 4$$

vi. List all the outcomes in the event ABC.

$$ABC = \{(\text{chicken, rice, ice-cream})\}$$

2. A customer visiting the suit department of a certain store will purchase a suit with probability .22, a shirt with probability .30, and a tie with probability .28. The customer will purchase both a suit and a shirt with probability .11, both a suit and a tie with probability .14, and both a shirt and a tie with probability .10. A customer will purchase all 3 items with probability .06. What is theprobability that a customer purchases

i. none of these items?

Sean A, B, C los eventos tales que el cliente compra el traje, la remera o una corbata respectivamente entonces buscamos $P(A^\complement B^\complement C^\complement)$.

$$\begin{aligned} &P(A^\complement B^\complement C^\complement)\\ &= 1 - P((A^\complement B^\complement C^\complement)^\complement)\\ &= 1 - P(A \cup B \cup C)\\ &= 1 - \Big(P(A) + P(B) + P(C) -\\ &\quad P(AB) - P(AC) - P(BC) + P(ABC)\Big)\\ &= 1 - (.22 + .3 + .28 - .11 - .14 - .10 + .06)\\ &= .49 \end{aligned}$$

ii. exactly 1 of these items?

$$\begin{aligned} &P(\{\text{exactamente 1 item}\})\\ &= 1 - \sum_{i \in \{0,2,3\}} P(\{\text{exactamente } i \text{ items}\})\\ &= 1 - \big(.49 + P(\{\text{exactamente 2 items}\}) + .06\big)\\ &= .45 - P(\{\text{exactamente 2 items}\})\\ &= .45 - \big(P(ABC^\complement \cup AB^\complement C \cup A^\complement BC)\big)\\ &= .45 - \big(P(ABC^\complement) + P(AB^\complement C) + P(A^\complement BC)\big)\\ &= .45 - P(AB) - P(AC) - P(BC) +\\ &\quad 3 \cdot P(ABC)\\ &= .45 - .11 - .14 - .10 + 3 \cdot .06\\ &= .28 \end{aligned}$$

3. A deck of cards is dealt out. What is the probability that the 14th card dealt is an ace? What is the probability that the first ace occurs on the 14th card?

$$\frac{4 \cdot \binom{51}{13}}{14 \cdot \binom{52}{14}}$$

la probabilidad de que el primer as ocurra en la carta catorceava

$$\frac{4 \cdot \binom{48}{13}}{14 \cdot \binom{52}{14}}$$

4. Let A denote the event that the midtown temperature in Los Angeles is 70°F, and let B denote the event that the midtown temperature in New York is 70°F. Also, let C denote the event that the maximum of the midtown temperatures in New York and in Los Angeles is 70°F. If $P(A) = .3$, $P(B) = .4$, and $P(C) = .2$, find the probability that the minimum of the two midtown temperatures is 70°F.

Sea D el evento tal que la temperatura mínima entre ambas ciudades sea 70°F.

$$\begin{aligned} &P(A \cup B) \\ &= P(A) + P(B) - P(AB) \\ &= .3 + .4 - P(AB) \end{aligned}$$

por otro lado

$$\begin{aligned} &P(C \cup D) \\ &= P(C) + P(D) - P(CD) \\ &= .2 + P(D) - P(CD) \end{aligned}$$

donde $C \cup D$ es el evento donde 70°F es el máximo o mínimo, que coincide con $A \cup B$ el evento donde alguna temperatura es 70°F. Similarmente CD el evento donde 70°F es la máxima y mínima temperatura coincide con AB el evento donde ambas temperaturas son 70°F. Si restamos $P(C \cup D)$ a $P(A \cup B)$

$$\begin{aligned} &0 \\ &= P(A \cup B) - P(C \cup D) \\ &= .7 - P(AB) - \big(.2 + P(D) - P(CD)\big) \\ &= .5 - P(D) \end{aligned}$$

Por lo tanto $P(D) = .5$

5. An ordinary deck of 52 cards is shuffled. What is the probability that the top four cards have

i. different denominations?

$$\frac{\binom{13}{4}\binom{4}{1}^4}{\binom{52}{4}}$$

ii. different suits?

$$\frac{\binom{13}{1}^4}{\binom{52}{4}}$$

6. Urn A contains 3 red and 3 black balls, whereas urn B contains 4 red and 6 black balls. If a ball is randomly selected from each urn, what is the probability that the balls will be the same color?

$$\begin{aligned} &P(\{\text{ambas bolillas son del mismo color}\}) \\ &= P(\{\text{ambas bolillas son rojas}\}) + \\ &\quad P(\{\text{ambas bolillas son negras}\}) \\ &= \tfrac{3}{6} \cdot \tfrac{4}{10} + \tfrac{3}{6} \cdot \tfrac{6}{10} \\ &= \tfrac{1}{2} \end{aligned}$$

7. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the $\binom{40}{8}$ combinations, what is the probability that a player has

i. all 8 of the numbers selected by the lottery commission?

$$\frac{\binom{8}{8}}{\binom{40}{8}}$$

ii. 7 of the numbers selected by the lottery commission?

$$\frac{\binom{8}{7}\binom{32}{1}}{\binom{40}{8}}$$

iii. at least 6 of the numbers selected by the lottery commission?

$$\frac{\binom{8}{6}\binom{32}{2}}{\binom{40}{8}} + \frac{\binom{8}{7}\binom{32}{1}}{\binom{40}{8}} + \frac{\binom{8}{8}}{\binom{40}{8}}$$

8. From a group of 3 first-year students, 4 sophomores, 4 juniors, and 3 seniors, a committee of size 4 is randomly selected. Find the probability that the committee will consist of

i. 1 from each class

$$\frac{\binom{3}{1}\binom{4}{1}\binom{4}{1}\binom{3}{1}}{\binom{14}{4}}$$

ii. 2 sophomores and 2 juniors

$$\frac{\binom{4}{2}\binom{4}{2}}{\binom{14}{4}}$$

iii. only sophomores or juniors

$$\frac{\binom{8}{4}}{\binom{14}{4}}$$

9. For a finite set A, let $N(A)$ denote the number of elements in A.

i. Show that

$$N(A \cup B) = N(A) + N(B) - N(AB)$$

ii. More generally, show that

$$N\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1<\cdots<i_r} N(\bigcap_{j=1}^{r} A_{i_j})$$

Sea $S = \bigcup_{i=1}^{n} A_i$ si realizamos el experimento de tomar al azar un elemento de S entonces

$$P(A) = \frac{N(A)}{N(S)}$$

luego

$$\begin{aligned}
& N(\bigcup_{i=1}^{n} A_i) \cdot 1 \\
&= N(S) \cdot P(S) \\
&= N(S) \cdot P(\bigcup_{i=1}^{n} A_i) \\
&= N(S) \cdot \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1<\cdots<i_r} P(\bigcap_{j=1}^{r} A_{i_j}) \\
&= N(S) \cdot \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1<\cdots<i_r} \frac{N(\bigcap_{j=1}^{r} A_{i_j})}{N(S)} \\
&= \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1<\cdots<i_r} N(\bigcap_{j=1}^{r} A_{i_j})
\end{aligned}$$

10. Consider an experiment that consists of 6 horses, numbered 1 through 6, running a race, and suppose that the sample space consists of the 6! possible orders in which the horses finish. Let A be the event that the number-1 horse is among the top three finishers, and let B be the event thatthe number-2 horse comes in second. How many outcomes are in the event $A \cup B$?

tenemos $N(A \cup B) = N(A) + N(B) - N(AB)$, $N(A) = 3 \cdot 5!$, $N(B) = 5!$, $N(AB) = 2 \cdot 4!$ por lo tanto

$$N(A \cup B) = 4 \cdot 5! - 2 \cdot 4!$$

11. A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

$$\frac{\binom{4}{1} \cdot \binom{13}{2} \cdot \binom{13}{1}^3}{\binom{52}{5}}$$

12. A basketball team consists of 6 frontcourt and 4 backcourt players. If players are divided into roommates at random, what is the probability that there will be exactly two roommate pairs made up of a backcourt and a frontcourt player?

$$\frac{\binom{6}{2}\binom{4}{2} \cdot 2 \cdot 3}{\frac{\binom{10}{2,2,2,2,2}}{5!}}$$

13. Suppose that a person chooses a letter at random from R E S E R V E and then chooses one at random from V E R T I C A L. What is the probability that the same letter is chosen?

$$\frac{2}{7} \cdot \frac{1}{8} + \frac{3}{7} \cdot \frac{1}{8} + \frac{1}{7} \cdot \frac{0}{8} + \frac{1}{7} \cdot \frac{1}{8} = \frac{3}{28}$$

14. Prove *Boole's inequality*

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} P(A_i)$$

Demostración. de la desigualdad de *Boole*

Utilizando el ejercicio teórico 5 existe una secuencia de eventos $B_1, B_2, \dots$ de eventos disjuntos tal que para todo $n \geq 1$

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big)$$
$$= P\Big(\bigcup_{i=1}^{\infty} B_i\Big)$$
$$= \sum_{i=1}^{\infty} P(B_i)$$

como $B_i \subseteq A_i$ para todo $i \in \mathbb{N}$

$$\leq \sum_{i=1}^{\infty} P(A_i) \qquad \square$$

15. Show that if $P(A_i) = 1$ for all $i \geq 1$, then

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) = 1.$$

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) = 1 - P\Big(\bigcup_{i=1}^{\infty} A_i^{\complement}\Big)$$

por el ejercicio anterior

$$\geq 1 - \sum_{i=1}^{\infty} P(A_i^{\complement})$$
$$\geq 1 - \lim_{n\to\infty} \sum_{i=1}^{n} 0$$
$$\geq 1$$

16. Let $T_k(n)$ denote the number of partitions of the set $\{1, \dots, n\}$ into k nonempty subsets, where $1 \leq k \leq n$. (See Theoretical Exercise 8 for the definition of a partition.) Argue that

$$T_k(n) = k \cdot T_k(n-1) + T_{k-1}(n-1)$$

Hint: In how many partitions is $\{1\}$ a subset, and in how many is 1 an element of a subset that contains other elements?

Si $\{n\}$ es un subconjunto de la partición entonces basta agregar el subconjunto a cada partición en $k-1$ subconjuntos de $n-1$.

Si $\{n\}$ no es un subconjunto de la partición entonces basta agregar n a alguno de los k subconjuntos de la partición en k subconjuntos de $n-1$.

17. Five balls are randomly chosen, without replacement, from an urn that contains 5 red, 6 white, and 7 blue balls. Find the probability that at least one ball of each color is chosen.

Sea A el evento donde elegimos aunque sea una bolilla roja, B el evento donde elegimos aunque sea una bolilla blanca, C el evento donde elegimos aunque sea una bolilla azul.

$$\begin{aligned}
&P(ABC)\\
&= 1 - P(A^{\complement} \cup B^{\complement} \cup C^{\complement})\\
&= 1 - \Big(P(A^{\complement}) + P(B^{\complement}) + P(C^{\complement}) -\\
&\quad P(A^{\complement}B^{\complement}) - P(A^{\complement}C^{\complement}) - P(B^{\complement}C^{\complement}) + P(A^{\complement}B^{\complement}C^{\complement})\Big)\\
&= 1 - \frac{\binom{13}{5} + \binom{12}{5} + \binom{11}{5} - \binom{7}{5} - \binom{6}{5} - \binom{5}{5} + 0}{\binom{18}{5}}
\end{aligned}$$

18. Four red, 8 blue, and 5 green balls are randomly arranged in a line.

i. What is the probability that the first 5 balls are blue?

$$\frac{\binom{8}{5} \cdot 5! \cdot 12!}{17!}$$

ii. What is the probability that none of the first 5 balls is blue?

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 12!}{17!}$$

iii. What is the probability that the final 3 balls are of different colors?

$$\frac{4 \cdot 8 \cdot 5 \cdot 3! \cdot 14!}{17!}$$

iv. What is the probability that all the red balls are together?

$$\frac{4! \cdot 14!}{17!}$$

19. Ten cards are randomly chosen from a deck of 52 cards that consists of 13 cards of each of 4 different suits. Each of the selected cards is put in one of 4 piles, depending onthe suit of the card.

i. What is the probability that the largest pile has 4 cards, the next largest has 3, the next largest has 2, and the smallest has 1 card?

$$\frac{4! \cdot \binom{13}{4}\binom{13}{3}\binom{13}{2}\binom{13}{1}}{\binom{52}{10}}$$

ii. What is the probability that two of the piles have 3 cards, one has 4 cards, and one has no cards?

$$\frac{\binom{4}{3}\binom{3}{2}\binom{13}{4}\binom{13}{3}\binom{13}{3}}{\binom{52}{10}}$$

20. Balls are randomly removed from an urn initially containing 20 red and 10 blue balls. What is the probability that all of the red balls are removed before all of the blue ones have been removed?

La última bolilla a sacar debe ser azul

$$\frac{10}{30}$$