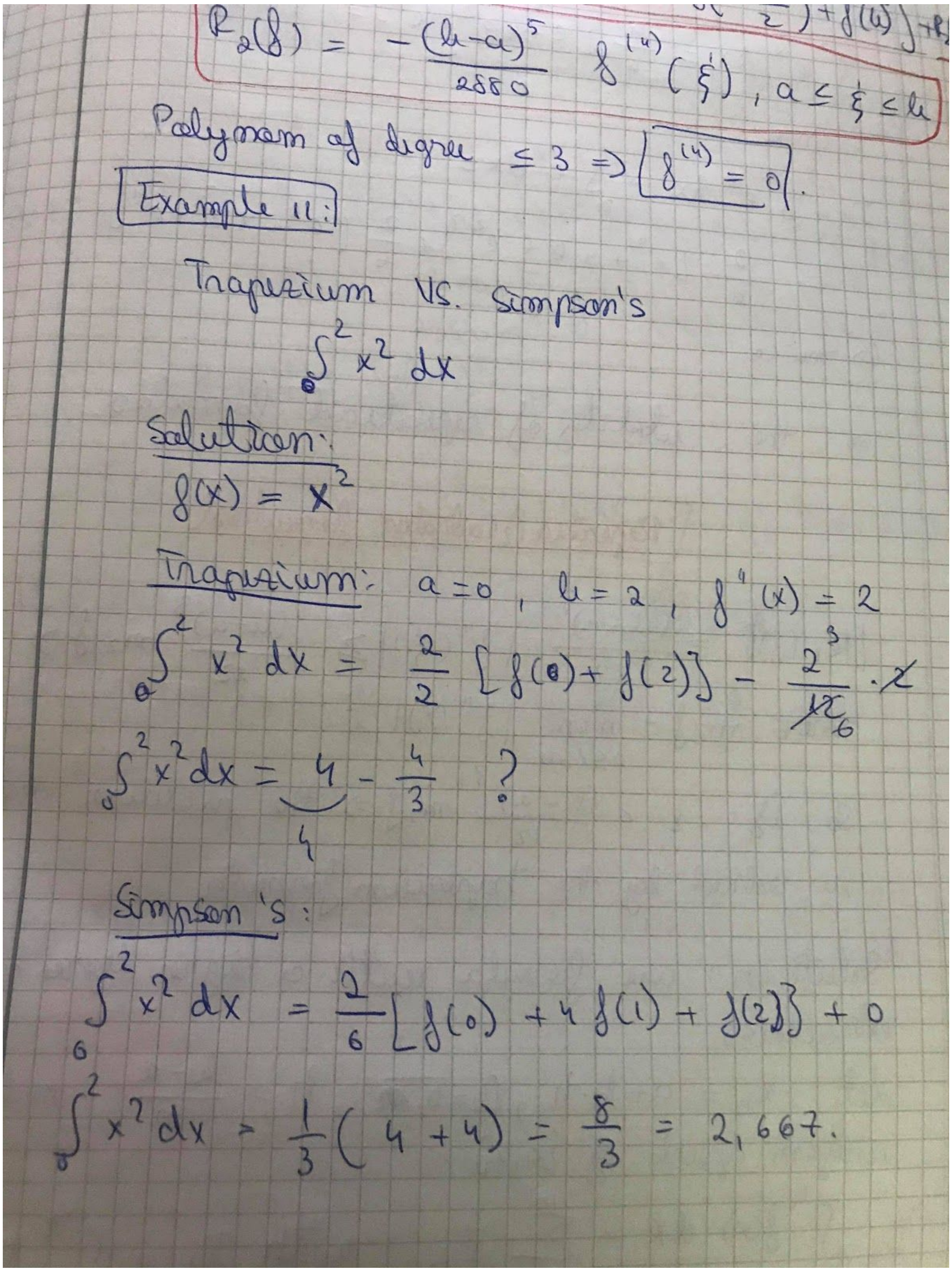
Lecture 7

La exemplul 11 din curs nu inteleg de ce nu se calculeaza si R la formula trapezului, pentru ca derivata de gradul 2 a functiei este 2, nu 0. Eu am facut in felul urmator si nu imi dau seama unde am gresit:



$$R_2(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad a \le \xi \le b$$

Polynom of degree $\le 3 \Rightarrow \boxed{f^{(4)} = 0}$.

Example 11:

Trapezium vs. Simpson's

$$\int_0^2 x^2\,dx$$

Solution:

$$f(x) = x^2$$

Trapezium: $a = 0$, $b = 2$, $f''(x) = 2$

$$\int_0^2 x^2\,dx = \frac{2}{2}\left[f(0) + f(2)\right] - \frac{2^3}{\cancel{12}_6} \cdot \cancel{2}$$

$$\int_0^2 x^2\,dx = \underbrace{4}_{4} - \frac{4}{3} \quad ?$$

Simpson's:

$$\int_0^2 x^2\,dx = \frac{2}{6}\left[f(0) + 4f(1) + f(2)\right] + 0$$

$$\int_0^2 x^2\,dx = \frac{1}{3}(4 + 4) = \frac{8}{3} = 2{,}667.$$

Example 13:

$$\frac{\quad}{288\, n^4} M_4 f .$$

Example 13: $\int_1^3 (2x+1)\,dx$, $n=2$, trapezium.

$$\int_1^3 (2x+1)\,dx = \frac{2}{4}\left[f(1)+f(3)+2\, f(2)\right]$$

$$\int_1^3 (2x+1)\,dx = \frac{1}{2}\left(3+7+2\cdot 5\right) = \frac{20}{2} = 10$$

$$\boxed{x_1 = 1 + 1\cdot 1 = 2}$$

$x_k = a + k\cdot h$, $a=1$, $k=1$, $h=\frac{3-1}{2}=1$

Example 15:

$$|R_n(f)| \le \frac{(1-0)^5}{2880\, n^4} M_4 f$$

$$M_4 f = \max_{x\in[0,1]} |f^{(4)}(x)|$$

$$f'(x) = \left(\frac{1}{1+x}\right)' = \left((1+x)^{-1}\right)' = \frac{-1}{(1+x)^2}$$

$$f''(x) = \left[-(1+x)^{-2}\right]' = \frac{2}{(1+x)^3} =$$

$$f^{(3)}(x) = -\frac{6}{(x+1)^4}$$

$$f^{(4)}(x) = \frac{24}{(x+1)^5}$$

$$M_4 f = 24$$

$h = \frac{1}{2}$

$x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$

$$|R_n(f)| \le \frac{1}{2880\, n^4}\cdot 24 = \frac{1}{120\cdot n^4} < 10^{-3}$$

$$\Rightarrow n^4 > \frac{10^{\cancel{3}\,2}}{12\cancel{0}} = 8{,}33 \Rightarrow n = 2$$

$$\int_0^1 f(x)\,dx = \frac{1}{12}\left(f(0)+f(1)+4\cdot f\left(\frac{0+\frac{1}{2}}{2}\right)+2\cdot f\left(\frac{1}{2}\right)\right)$$

$$\int_0^1 f(x)\,dx = \frac{1}{12}\left(1+\frac{1}{2}+4\cdot\frac{4}{5}+2\cdot\frac{2}{3}\right)\ \ldots$$

Lecture 8

Example 2:

Example 2 :

$$\int_0^1 f(x)\,dx = A\,f(a) + B\,f'(0) + C\,f(1) + R(f)$$

$$\int_a^b f(x)\,dx = \sum_{k=0}^{m} \sum_{j\in I_k} A_{kj}\, f^{(j)}(x_k) + R(f)$$

$a=0$, $b=1$

$A = A_{00} =$

? $B = A_{01} = \int_a^b (x-b)\,dx = -\frac{(a-b)^2}{2}$

$C = A_{10} = \int_a^b dx = b-a$

Lecture 9

Example 6:

$$Q_{D_0}(f),\ Q_{D_1}(f), \dots, Q_{D_k}(f), \dots \to I = \int_a^b f(x)\,dx$$

$$|Q_{D_m}(f) - Q_{D_{m-1}}(f)| \le \varepsilon .$$

<u>Example</u>: $\ln 2 = \int_1^2 \frac{1}{x}\,dx$, $\varepsilon = 10^{-2}$

– using the rectangle (midpoint) formula

<u>Solution</u>:

$$f(x) = \frac{1}{x}, \quad a = 1,\ b = 2$$

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) + R_n(f)$$

$$R_n(f) = \frac{(b-a)^3}{24n^2} f''(\xi), \quad \xi \in [a,b]$$

$$|R_n(f)| \le \frac{(b-a)^3}{24n^2} M_2 f$$

$$M_2 f = \max_{x \in [a,b]} |f''(x)|$$

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}$$

$$M_2 f = \max_{x \in [1,2]} \frac{2}{x^3} = 2, \quad x = 1$$

$$|R_n(f)| \leq \frac{(2-1)^3}{24n^2} \cdot 2 < 10^{-2}$$

$$\frac{1}{12 \cdot n^2} < 10^{-2} \Rightarrow n^2 > \frac{100}{12} = 8{,}33$$

$$\Downarrow$$

$$\boxed{n = 3}$$

$$x_1 = a + \frac{b-a}{2n} = 1 + \frac{1}{6} = \frac{7}{6}$$

$$x_2 = x_1 + (2-1)\frac{b-a}{n} = \frac{7}{6} + \frac{1}{3} = \frac{9}{6}$$

$$x_3 = x_1 + (3-1)\frac{1}{3} = \frac{7}{6} + \frac{2}{3} = \frac{11}{6}$$

$$\ln 2 = \int_1^2 \frac{1}{x}dx = \frac{1}{3}\left[f(x_1) + f(x_2) + f(x_3)\right]$$

$$\ln 2 = \frac{1}{3}\left(f\left(\frac{7}{6}\right) + f\left(\frac{9}{6}\right) + f\left(\frac{11}{6}\right)\right) + R_2(f)$$

$$\ln 2 = \frac{1}{3}\left(\frac{6}{7} + \frac{6}{9} + \frac{6}{11}\right) + R_2(f) = \frac{2}{7} + \frac{2}{9} + \frac{2}{11} + R_2(f)$$

$$\ln 2 = \frac{198 + 154 + 126}{693} + R_2(f) \approx 0{,}6897 + R_2(f)$$

$$R_2(f) \approx 0{,}009259$$

$$\ln 2 \approx 0{,}6989.$$

Lecture 10

Example 8:

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 - 2x_2 + 3x_3 = 5 \\ x_1 - x_2 + 4x_3 = 5 \end{cases}$$

Solution: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -2 & 3 & 5 \\ 1 & -1 & 4 & 5 \end{array}\right]$

The pivot is $a_{21} = 2$. We interchange L_1 and L_2 $\Rightarrow$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 4 \\ 1 & 1 & 1 & 5 \\ 1 & -1 & 4 & 5 \end{array}\right] \qquad \begin{array}{l} L_2 \leftarrow L_2 - \frac{1}{2} L_1 \\ L_3 \leftarrow L_3 - \frac{1}{2} L_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 5 \\ 0 & 2 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \end{array}\right]$$

$$\begin{cases} 2x_1 - 2x_2 + 3x_3 = 5 \\ 2x_2 - \frac{1}{2} x_3 = \frac{3}{2} \\ \frac{5}{2} x_3 = \frac{5}{2} \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = 1 \\ x_2 = 1 \\ x_1 = 2 \end{cases}$$

Lecture 11

Example 9:
Am facut mai multe iteratii, sperand ca voi ajunge mai aproape de rezultat.

Example 9 :

$$\begin{cases} 4x_1 + x_2 = 3 \\ 2x_1 + 5x_2 = 1 \end{cases}$$

$$x^{(0)} = (3, 11), \quad \omega = 1.25$$

Solution:

- ~~Jacobi~~
- Jacobi :

$$\begin{cases} x_1 = \frac{1}{4}(3 - x_2) \\ x_2 = \frac{1}{5}(1 - 2x_1) \end{cases}$$

$$\begin{cases} x_1^{(k+1)} = \frac{1}{4}(3 - x_2^{(k)}) \\ x_2^{(k+1)} = \frac{1}{5}(1 - 2x_1^{(k)}) \end{cases}$$

1st Iter.:
$$\begin{cases} x_1^{(1)} = \frac{1}{4}(3 - 11) = -2 \\ x_2^{(1)} = \frac{1}{5}(1 - 6) = -1 \end{cases}$$

2nd Iter:
$$\begin{cases} x_1^{(2)} = \frac{1}{4}(3 + 1) = 1 \\ x_2^{(2)} = \frac{1}{5}(1 - 2 \cdot (-2)) = 1 \end{cases}$$

3rd Iter.:
$$\begin{cases} x_1^{(3)} = \frac{1}{4}(3 - 1) = \frac{1}{2} \\ x_2^{(3)} = \frac{1}{5}(1 - 2 \cdot 1) = -\frac{1}{5} \end{cases}$$

4th Iter.:
$$\begin{cases} x_1^{(4)} = \frac{1}{4}(\overset{5}{3} + \frac{1}{5}) = \frac{16}{20} = \frac{4}{5} \\ x_2^{(4)} = \frac{1}{5}(1 - 2 \cdot \frac{1}{2}) = 0 \end{cases}$$

5th Iter:
$$\begin{cases} x_1^{(5)} = \frac{1}{4}(3 - 0) = \frac{3}{4} \\ x_2^{(5)} = \frac{1}{5}(\overset{5}{1} - 2 \cdot \frac{4}{5}) = \frac{-3}{25} \end{cases}$$

6th Iter:
$$\begin{cases} x_1^{(6)} = \frac{1}{4}(\overset{25}{3} + \frac{3}{25}) = \frac{78}{100} = \frac{39}{50} \\ x_2^{(6)} = \frac{1}{5}(\overset{2}{1} - \cancel{2} \cdot \frac{3}{\cancel{4}\,2}) = \frac{-1}{10} \end{cases}$$

$\vdots$

- Gauss-Seidel:

$$\begin{cases} x_1^{(k+1)} = \frac{1}{4}(3 - x_2^{(k)}) \\ x_2^{(k+1)} = \frac{1}{5}(1 - 2x_1^{(k+1)}) \end{cases}$$

1st Iter:
$$\begin{cases} x_1^{(1)} = \frac{1}{4}(3 - 11) = -2 \\ x_2^{(1)} = \frac{1}{5}(1 - 2 \cdot (-2)) = 1 \end{cases}$$

2nd Iter:
$$\begin{cases} x_1^{(2)} = \frac{1}{4}(3 - 1) = \frac{1}{2} \\ x_2^{(2)} = \frac{1}{5}(1 - 2\frac{1}{2}) = 0 \end{cases}$$

$x_1^{(3)} = \frac{1}{4}(3-0) = \frac{[illegible]}{4}$

3^{nd} Iter. :

$x_2^{(3)} = \frac{1}{5}\left(\overset{2}{1} - \cancel{2}\cdot\frac{3}{\cancel{2}}\right) = -\frac{1}{10}$

4^{th} Iter. :

$$\begin{cases} x_1^{(4)} = \frac{1}{4}\left(\overset{10}{3} + \frac{1}{10}\right) = \frac{31}{40} \\ x_2^{(4)} = \frac{1}{5}\left(\overset{20}{1} - \cancel{2}\cdot\frac{31}{\cancel{40}_{20}}\right) = \frac{-11}{5\cdot 20} = -\frac{11}{100} \end{cases}$$

5^{th} Iter :

$$\begin{cases} x_1^{(5)} = \frac{1}{4}\left(3 + \frac{11}{100}\right) = \frac{311}{400} \\ x_2^{(5)} = \frac{1}{5}\left(1 - \cancel{2}\cdot\frac{311}{\cancel{400}_{200}}\right) = \frac{-111}{10^3} \end{cases}$$

6^{th} Iter :

$$\begin{cases} x_1^{(6)} = \frac{1}{4}\left(3 + \frac{111}{10^3}\right) = \frac{3111}{4\cdot 10^3} \\ x_2^{(6)} = \frac{1}{5}\left(1 - \cancel{2}\cdot\frac{3111}{\cancel{4}_{2}\cdot 10^3}\right) = -\frac{1111}{10^4} \end{cases}$$

· <u>Relaxation</u> :

$$\begin{cases} x_1^{(k+1)} = \frac{1.25}{4}\left(3 - x_2^{(k)}\right) \\ x_2^{(k+1)} = \frac{1.25}{5}\left(1 - x_1^{(k+1)}\right) \end{cases}$$

1^{st} Iter :

$$\begin{cases} x_1^{(1)} = \frac{5}{16}(3 - 11) = -\frac{5}{2} \\ x_2^{(1)} = \frac{1}{4}\left(\overset{2}{1} + \frac{5}{2}\right) = \frac{7}{8} \end{cases}$$

2^{nd} Iter :

$$\begin{cases} x_1^{(2)} = \frac{5}{16}\left(\overset{8}{3} - \frac{7}{8}\right) = \frac{5}{16}\cdot\frac{17}{8} \\ x_2^{(2)} = \frac{1}{4}\left(1 - \frac{5\cdot 17}{128}\right) = \frac{43}{2^9} \end{cases}$$