Lecture 5

Example 5:

Note: Chiar daca intial am greis determinantul, calculele sunt corecte in continuare.

Example 5:

$f'(0) = 1, \; f(1) = 2$ and $f'(2) = 1$.

$f(\frac{1}{2}) = ?$

Solution:

$x_0 = 0, \; x_1 = 1, \; x_2 = 2$

$m = 2$

$I_0 = \{1\}, \; I_1 = \{0\}, \; I_2 = \{1\}$

$\Rightarrow n = 3 - 1 = 2.$

$P(x) = a_2 x^2 + a_1 x + a_0 \in \mathbb{P}_2$

$$\begin{cases} P'(0) = 2a_2 \cdot 0 + a_1 = a_1 = f'(0) \\ P(1) = a_2 \cdot 1 + a_1 + a_0 = f(1) \\ P'(2) = 2 \cdot a_2 \cdot 2 + a_1 = f'(2) \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = 1 \\ a_2 + a_1 + a_0 = 2 \Rightarrow a_0 = 1 \\ 4a_2 + a_1 = 1 \Rightarrow a_2 = 0 \end{cases}$$

$\Rightarrow P(x) = x + 1$

The det. of the system is: $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 4 \neq 0 \Downarrow$ we have a solution

?! $\cancel{\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0 + 0 + 0 - 0 - 0 - 0 = 0}$

$\Rightarrow$ ~~we have ~~more than one~~ no solution if det = 0~~

$(B_2 f)(x) = h_{01}(x) f'(0) + h_{10}(x) f(1) + h_{21}(x) f'(2)$

$h_{01}(x) = ax^2 + bx + c \in \mathbb{P}_2$

$$\begin{matrix} h_{01}'(0) = 1 \\ h_{01}(1) = 0 \\ h_{01}'(2) = 0 \end{matrix} \Rightarrow \begin{cases} 2 \cdot a \cdot 0 + b = 1 \Rightarrow b = 1 \\ a \cdot 1 + b \cdot 1 + c = 0 \\ 2 \cdot a \cdot 2 + b = 0 \Rightarrow a = -\frac{1}{4} \end{cases}$$

$-\frac{1}{4} + 1 + c = 0$

$c = \frac{1}{4} - 1 = -\frac{3}{4}$

$\cancel{h_{01}(x) = \frac{x^2}{4} \quad \frac{3x}{4} +}$

$h_{01}(x) = -\frac{x^2}{4} + x - \frac{3}{4}$

$\boxed{h_{01}(x) = -\frac{1}{4}(x^2 - 4x + 3)} \quad (1)$

$h_{10}(x) = a_1 x^2 + b_1 x + c_1 \in \mathbb{P}_2$

$$\begin{matrix} h_{10}'(0) = 0 \\ h_{10}(1) = 1 \\ h_{10}'(2) = 0 \end{matrix} \Rightarrow \begin{cases} 2a_1 \cdot 0 + b_1 = 0 \Rightarrow b_1 = 0 \\ a_1 \cdot 1 + b_1 \cdot 1 + c_1 = 1 \\ 2 \cdot a_1 \cdot 2 + b_1 = 0 \Rightarrow a_1 = 0 \end{cases}$$

$c_1 = 1$

$\boxed{h_{10}(x) = 1} \quad (2)$

$$l_{21}(x) = a_2x^2 + b_2x + c_2 \in \mathbb{P}_2$$

$$\begin{cases} l_{21}'(0) = 0 \\ l_{21}(1) = 0 \\ l_{21}'(2) = 1 \end{cases} \Rightarrow \begin{cases} b_2 = 0 \\ a_2 + b_2 + c_2 = 0 \\ 4a_2 + b_2 = 1 \end{cases}$$

$$\Rightarrow a_2 = \frac{1}{4}$$

$$c_2 = -\frac{1}{4}$$

$$\boxed{l_{21}(x) = \frac{1}{4}(x^2 - 1) = \frac{(x+1)(x-1)}{4}} \quad (3)$$

From (1), (2) and (3) $\Rightarrow$

$$\Rightarrow (B_2 f)(x) = -\frac{1}{4}(x^2 - 4x + 3) \cdot 1 + \frac{4}{2} + \frac{1}{4}(x+1)(x-1) \cdot 1$$

$$(B_2 f)(x) = \frac{1}{4}(-\cancel{x^2} + 4x - 3 + 8 + \cancel{x^2} - 1)$$

$$(B_2 f)(x) = \frac{4x + 4}{4} = x + 1 \quad ?!$$

$$\boxed{(B_2 f)\left(\frac{1}{2}\right) = \frac{3}{2}}$$

Example 14 & 15:

Note: Am gresit la calcule, dar ecuatiile sunt scrise corect.

14. Natural cubic Spline.

pts: $(1,2), (2,3), (3,5)$

Solution:

a) $S(x) = \begin{cases} S_0(x), & x \in [1,2] \\ S_1(x), & x \in [2,3] \end{cases}$

b) $\begin{cases} S_0(1) = f(1) = 2 \text{ and } S_0(2) = f(2) = 3 \\ S_1(2) = f(2) = 3 \text{ and } S_1(3) = f(3) = 5 \end{cases}$

c) $S_0(2) = S_1(2) = 3$

d) $S_0'(2) = S_1'(2)$

e) $S_0''(2) = S_1''(2)$

f) Natural spline $\Rightarrow$

$S''(1) = S''(3) = 0 \iff S_0''(1) = S_1''(3) = 0$

Using Remark 8, we can write:

$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$

$S_0(x) = a_0 + b_0 x - b_0 + c_0 x^2 - 2c_0 x + c_0 + d_0 x^3 - 3d_0 x^2 + 3d_0 x - d_0$

$S_0'(x) = b_0 + 2c_0 x - 2c_0 + 3d_0 x^2 - 6d_0 x + 3d_0$

$S_0'(x) = (b_0 - 2c_0 + 3d_0) + 2x(c_0 - 3d_0) + 3d_0 x^2$

$S_0''(x) = 2c_0 - 6d_0 + 6d_0 x$

$\begin{cases} S_0(1) = \boxed{a_0 = 2} \\ S_0(2) = a_0 + b_0 + c_0 + d_0 = 3 \\ S_0''(1) = 2c_0 = 0 \Rightarrow \boxed{c_0 = 0} \end{cases}$

$\Rightarrow \boxed{b_0 + d_0 = 3}$ (1)

$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3$

$S_1(x) = a_1 + b_1 x - 2b_1 + c_1 x^2 - 4c_1 x + 4c_1 + d_1 x^3 - 6d_1 x^2 + 12 d_1 x - 8d_1$

$S_1'(x) = b_1 + 2c_1 x - 4c_1 + 3d_1 x^2 - 12 d_1 x + 12 d_1$

$S_1'(x) = (b_1 - 4c_1 + 12 d_1) + 2x(c_1 - 6d_1) + 3d_1 x^2$

$S_1''(x) = 2c_1 - 12 d_1 + 6d_1 x$

$\begin{cases} S_1(2) = \boxed{a_1 = 3} \\ S_1(3) = a_1 + \cancel{d_1} + c_1 + d_1 = 5 \\ S_1''(3) = 2c_1 + 6d_1 = 0 \end{cases}$

$\boxed{c_1 + 3d_1 = 0}$ (3)

From d) $\Rightarrow S_0'(2) = S_1'(2)$

$b_0 - 2\overset{=0}{c_0} + 3d_0 + 4\overset{=0}{c_0} - \cancel{12d_0} + \cancel{12d_0} = b_1 - \cancel{4c_1} + \cancel{12d_1} + \cancel{4c_1} - \cancel{24d_1} + \cancel{12d_1}$

$\Rightarrow \boxed{b_0 + 3d_0 = b_1}$ (2)

From e) $\Rightarrow S_0''(2) = S_1''(2)$

$$2\overset{=0}{c_0} + 6d_0 = 2c_1$$

$$\boxed{3d_0 = c_1} \quad (4)$$

Until ~~more~~ now, we have the following relations:

$$\begin{cases} b_0 + d_0 = 3 \Rightarrow b_0 = 3 - d_0 \\ c_1 + 3d_1 = 0 \\ b_1 + c_1 + d_1 = 2 \\ b_0 + 3d_0 = b_1 \\ 3d_0 = c_1 \end{cases}$$

$$b_0 + 3d_0 = b_1$$

$$3 - d_0 + 3d_0 = b_1 \Rightarrow b_1 = 2d_0 + 3$$

$$c_1 = 3d_0$$

$$\cancel{3d}\, c_1 + 3d_1 = 0$$

$$3d_0 + 3d_1 = 0 \Rightarrow d_1 = -d_0$$

$$b_1 + c_1 + d_1 = 2$$

$$2d_0 + 3 + 3d_0 - d_0 = 2$$

$$4d_0 = -1$$

$$d_0 = -\frac{1}{4} \Rightarrow b_0 = 3 + \frac{1}{4} = \frac{13}{4}$$

$$\Rightarrow \boxed{S_0(x) = 2 + \frac{13}{4}(x-1) - \frac{1}{4}(x-1)^3}$$

$$\begin{cases} b_1 = 2d_0 + 3 = -\frac{2}{4} + 3 = \frac{5}{2} \\ c_1 = -\frac{3}{4} \\ d_1 = \frac{1}{4} \\ a_1 = 3 \end{cases}$$

$$\Rightarrow \boxed{S_1(x) = 3 + \frac{5}{2}(x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{4}(x-2)^3}$$

15. Clamped Spline

pts: $(1,2), (2,3), (3,5)$

$S'(1) = 2$ and $S'(3) = 1$.

Solution:

We will have the same conditions, except for f), which is:

$$S'(1) = f'(1) \text{ and } S'(3) = f'(3) \Leftrightarrow$$

$$\Leftrightarrow \begin{matrix} S_0'(1) = f'(1) = 2 \\ S_1'(3) = f'(3) = 1 \end{matrix} \quad \text{(from the problem statement)}$$

The systems for the clamped spline will look like:

$$\begin{cases} S_0(1) = a_0 = 2 \\ S_0(2) = a_0 + b_0 + c_0 + d_0 = 3 \\ S_0'(1) = b_0 - \cancel{2c_0} + 3d_0 \cancel{+2c_0} - 6\cancel{d_0} \cancel{+3d_0} = \\ \quad b_0 = 2 \end{cases}$$

We have : $\begin{cases} a_0 = 2 \\ b_0 = 2 \\ c_0 + d_0 = -1 \end{cases}$ $(*)$

$$\begin{cases} S_1(2) = a_1 = 3 \\ S_1(3) = a_1 + b_1 + c_1 + d_1 = 5 \\ S_1'(3) = b_1 - 4c_1 + 12d_1 + 6c_1 - 36d_1 + 27d_1 = \\ \quad b_1 + 2c_1 + 3d_1 = 1 \end{cases}$$

We have : $\begin{cases} a_1 = 3 \\ b_1 + c_1 + d_1 = 2 \\ b_1 + 2c_1 + 3d_1 = 1 \\ c_1 + 2d_1 = -1 \end{cases}$

From d) $\Rightarrow$ $b_0 + 3d_0 = b_1$

From e) $\Rightarrow$ $3d_0 = c_1$ $\Rightarrow 2 + c_1 = b_1$

$\cancel{c_1 = b_1}$

$d_1 = \dfrac{-1 - c_1}{2}$

$b_1 + c_1 + d_1 = 2$

$2 + c_1 + c_1 + \dfrac{-1 - c_1}{2} = 2 \quad |\cdot 2$

$\cancel{4} + 4c_1 - 1 - c_1 = \cancel{4}$

$\boxed{c_1 = \frac{1}{3}}$

$b_1 = 2 + c_1 = 2 + \dfrac{1}{3} = \dfrac{7}{3}$

$d_1 = \dfrac{-1 - \frac{1}{3}}{2} = -\dfrac{4}{6} = -\dfrac{2}{3}$

$\Rightarrow \boxed{S_1(x) = 3 + \frac{7}{3}(x-2) + \frac{(x-2)^2}{3} - \frac{2}{3}(x-2)^3}$

$d_0 = \dfrac{c_1}{3} = \dfrac{1}{9}$

$c_0 + d_0 = -1$

$c_0 = -1 - \dfrac{1}{9} = -\dfrac{10}{9}$

$\Rightarrow \boxed{S_0(x) = 2 + 2(x-1) - \frac{10}{9}(x-1)^2 + \frac{(x-1)^3}{9}}$