Lecture 4 - Solved Examples

Example 7:

Example 7: Find the Hermite Interpolation formula for the fct. $f(x) = xe^x$ for which we know: $f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$, $f(1) = 2.7183$.

Solution:

$x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$m = 2$,

$r_0 = 0$, $r_1 = 1$, $r_2 = 0$

$n = m + r_0 + r_1 + r_2 = 3$

$$(H_3 f)(x) = \sum_{k=0}^{2} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k)$$

$$= h_{00}(x) f(-1) + \cancel{h_{01}(x) f'(-1)} + h_{10}(x) f(0) + h_{11}(x) f'(0) + h_{20}(x) f(1) + \cancel{h_{21}(x) f'(1)}$$

We have h_{00}, h_{10}, h_{11} and h_{20} which fulfill the following relations:

$h_{kj}^{(p)}(x_\nu) = 0$, $\nu \neq k$, $p = \overline{0, r_\nu}$

$h_{kj}^{(p)}(x_k) = \delta_{jp}$, $p = \overline{0, r_k}$ for $j = \overline{0, r_k}$; $\nu, k = \overline{0, m}$

- $h_{00}(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1 \in \mathbb{P}_2$, $a_1, b_1, c_1, d_1 \in \mathbb{R}$

$$\begin{cases} h_{00}(x_0) = h_{00}(-1) = 1 \\ h_{00}(x_1) = h_{00}(0) = 0 \\ h'_{00}(x_1) = h'_{00}(0) = 0 \\ h_{00}(x_2) = h_{00}(1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -a_1 + b_1 - c_1 + d_1 = 1 \\ d_1 = 0 \\ c_1 = 0 \\ a_1 + b_1 + c_1 + d_1 = 0 \end{cases}$$

$\cancel{2b_1 = 1 \Rightarrow b = \frac{1}{2}}$

$$\begin{cases} b_1 - a_1 = 1 \\ a_1 + b_1 = 0 \end{cases}$$

$2b_1 = 1 \Rightarrow b = \frac{1}{2}$

$a = -\frac{1}{2}$

$$\boxed{h_{00}(x) = -\frac{x^3}{2} + \frac{x^2}{2}}$$

- $h_{10}(x) = a_2 x^3 + b_2 x^2 + c_2 x + d_2 \in \mathbb{P}_2$, $a_2, b_2, c_2, d_2 \in \mathbb{R}$

$$\begin{cases} h_{10}(x_0) = h_{10}(-1) = 0 \\ h_{10}(x_1) = h_{10}(0) = 1 \\ h'_{10}(x_1) = h'_{10}(0) = 0 \\ h_{10}(x_2) = h_{10}(1) = 0 \end{cases}$$

$$\begin{cases} -a_2 + b_2 - c_2 + d_2 = 0 \\ d_2 = 1 \\ c_2 = 0 \\ a_2 + b_2 + c_2 + d_2 = 0 \end{cases}$$

$-a_2 + b_2 = -1$

$a_2 + b_2 = -1$ $\oplus$

$2b_2 = -2$

$b_2 = -1$

$a_2 = 0$

$$\boxed{h_{10}(x) = -x^2 + 1}$$

$$\begin{cases} -a_4 + h_4 = 0 \\ a_4 + h_4 = 1 \end{cases}$$

$$2h_4 = 1$$

$$h_4 = \frac{1}{2}$$

$$a = \frac{1}{2}$$

$$\boxed{h_{20}(x) = \frac{1}{2}(x^3 + x^2)}$$

$$(H_3 f)(x) = -\frac{1}{2}(x^3 - x^2) \cdot f(-1) + \overbrace{(-x^2+1) \cdot f(0)}^{0} +$$

$$(-x^3 + x) \cdot f'(0) + \frac{1}{2}(x^3 + x^2) \cdot f(1)$$

$$(H_3 f)(x) = x^3 \left(\frac{0.3679}{2} - 1 + \frac{2.7183}{2}\right)$$

$$+ x^2 \left(\frac{-0.3679}{2} + \frac{2.7183}{2}\right) + x.$$

$$f\left(\tfrac{1}{2}\right) = (H_3 f)\left(\tfrac{1}{2}\right) + R_3 f\left(\tfrac{1}{2}\right)$$

$$\frac{\sqrt{e}}{2} = \frac{1}{8} \cdot 1{,}0862 + \frac{1}{4} \cdot 2.3504 + \frac{1}{2} + R_3 f\left(\tfrac{1}{2}\right)$$

Example 10:

Example 10:

Considering the following data:

x	0	2	3
$f(x)$	0	10	12
$f'(x)$	5	3	7

find the corresponding Hermite interpolation polynomial:

Solution:

We have the following double nodes:

$x_0 = 0$, $x_1 = 2$, $x_2 = 3$, where

$f(x_0) = 0$, $f(x_1) = 10$, $f(x_2) = 12$ and

$f'(x_0) = 5$, $f'(x_1) = 3$, $f'(x_2) = 7$.

$$\begin{cases} m = 2 \Rightarrow m = 2m+1 = 5 \\ r_0 = r_1 = r_2 = 1 \end{cases}$$

We construct the table:

		$D^1 f$	$D^2 f$	$D^3 f$	$D^4 f$	$D^5 f$
$z_0=0$	$f(0)=0$	$f'(0)=5$	$\frac{5-5}{z_2-z_0}=0$	$\frac{-1-0}{z_3-z_0}=-\frac{1}{2}$	$\frac{0-(-\frac{1}{2})}{z_4-z_0}=\frac{1}{6}$	$\frac{2-\frac{1}{6}}{z_5-z_0}=\frac{11}{18}$
$z_1=0$	$f(0)=0$	$\frac{f(2)-f(0)}{z_2-z_1}=5$	$\frac{f'(2)-5}{z_3-z_1}=-1$	$\frac{-1-(-1)}{z_4-z_1}=0$	$\frac{6-0}{z_5-z_1}=2$	
$z_2=2$	$f(2)=10$	$f'(2)=3$	$\frac{2-3}{z_4-z_2}=-1$	$\frac{5-(-1)}{z_5-z_2}=6$		
$z_3=2$	$f(2)=10$	$\frac{f(3)-f(2)}{z_4-z_3}=2$	$\frac{f'(3)-2}{z_5-z_3}=5$			
$z_4=3$	$f(3)=12$	$f'(3)=7$				
$z_5=3$	$f(3)=12$					

$$(H_5 f)(x) = f(z_0) + \sum_{i=1}^{5} (x-z_0)\cdots(x-z_{i-1})\,(D^i f)(z_0)$$

$$= f(z_0) + (x-z_0)(D^1 f)(z_0) + (x-z_0)(x-z_1)(D^2 f)(z_0) + (x-z_0)(x-z_1)(x-z_2)(D^3 f)(z_0)$$

$$(H_5 f)(x) = f(0) + (x-0)\cdot 5 + (x-0)(x-0)\cdot 0 + x^2(x-2)\cdot\left(-\frac{1}{2}\right) + x^2\cdot(x-2)^2\cdot\frac{1}{6} + x^2\cdot(x-2)^2\cdot(x-3)\cdot\frac{11}{18}$$

$$(H_5 f)(x) = 5x - \frac{x^3-2x^2}{2} + \frac{x^4-4x^3+4x^2}{6} + \frac{11}{18}\left(x^5-7x^4+16x^3-12x^2\right)$$