Lecture 6

Example 4, a)

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Ex. 4.

x	-3	-1	2
$f(x)$	-4	-2	3

a) Least Squares:

$$E(a,b) = \sum_{i=0}^{2} [f(x_i) - P(x_i)]^2 =$$

$$= \sum_{i=0}^{2} [f(x_i) - (a x_i + b)]^2$$

We have to solve the following system:

$$\begin{cases} \dfrac{\partial E(a,b)}{\partial a} = 2 \sum_{i=0}^{2} [f(x_i) - (a x_i + b)] \cdot x_i = 0 \\ \dfrac{\partial E(a,b)}{\partial b} = 2 \sum_{i=0}^{2} [f(x_i) - (a x_i + b)] = 0 \end{cases}$$

$$2\Big([f(-3) - (-3a+b)]\cdot(-3) + [f(-1) - (-a+b)]\cdot(-1) + [f(2) - (2a+b)]\cdot 2 \Big) = 0$$

$$2\Big(-3[-4+3a-b] - [-2+a-b] + 2\cdot[3-2a-b]\Big) = 0$$

$$-12 - 9a + 3b + 2 - a + b + 6 - 4a - 2b = 0$$

$$-14a + 2b - 4 = 0 \quad |:2$$

$$\boxed{b - 7a - 2 = 0} \quad (*)$$

$$-4 + 3a - b - 2 + a - b + 3 - 2a - b = 0.$$

$$\boxed{2a - 3b - 3 = 0} \quad (**)$$

From $(*)$ and $(**)$ $\Rightarrow$

$$\begin{cases} b - 7a - 2 = 0 \quad |\cdot 3 \\ -3b + 2a - 3 = 0 \end{cases} \quad (+)$$

$$-19a - 9 = 0$$

$$a = -\frac{9}{19}$$

$$b = 2 + 7a = \frac{36 - 63}{19}$$

$$b = -\frac{27}{19}$$

$$E(a,b) = \Big[f(-3) + \frac{9}{19}\underbrace{(-3+3)}_{0}\Big]^2 + \Big[f(-1) + \frac{9}{19}(-1+3)\Big]^2 + \Big[f(2) + \frac{9}{19}(2+3)\Big]^2$$

$$= 16 + \left(\frac{-36+18}{19}\right)^2 + \left(\frac{57+45}{19}\right)^2 =$$

$$= 16 + \frac{18^2 + 102^2}{19^2}.$$

Example 4, b)

b) Least Squares polynomial

$$E(a_0,a_1,a_2) = \sum_{i=0}^{2} \left[f(x_i) - \sum_{k=0}^{2} a_k x_i^k \right]^2$$

$$\frac{\partial E(a_0,a_1,a_2)}{\partial a_j} = 0,\quad j=\overline{0,2}$$

$$\begin{cases} (1)\quad \dfrac{\partial E(a_0,a_1,a_2)}{\partial a_0} = 2\sum_{i=0}^{2} \left[f(x_i) - a_0 x_i^0 - a_1 x_i^1 - a_2 x_i^2 \right] \cdot (-x_i^0) \\ (2)\quad \dfrac{\partial E(a_0,a_1,a_2)}{\partial a_1} = 2\sum_{i=0}^{2} \left[f(x_i) - a_0 - a_1 x_i - a_2 x_i^2 \right] \cdot (-x_i) \\ (3)\quad \dfrac{\partial E(a_0,a_1,a_2)}{\partial a_2} = 2\sum_{i=0}^{2} \left[f(x_i) - a_0 - a_1 x_i - a_2 x_i^2 \right] \cdot (-x_i^2) \end{cases}$$

$$\Leftrightarrow \begin{cases} \overbrace{f(x_0) - a_0 - a_1 x_0 - a_2 x_0^2}^{A} + \overbrace{f(x_1) - a_0 - a_1 x_1 - a_2 x_1^2}^{B} + \\ + \underbrace{f(x_2) - a_0 - a_1 x_2 - a_2 x_2^2}_{C} = 0 \qquad (1) \\ A\cdot(-x_0) + B\cdot(-x_1) + C\cdot(-x_2) = 0 \qquad (2) \\ A\cdot(-x_0^2) + B\cdot(-x_1^2) + C\cdot(-x_2^2) = 0 \end{cases}$$

$$A = f(x_0) - a_0 - a_1 x_0 - a_2 x_0^2 = -4 - a_0 + 3a_1 - 9a_2$$

$$B = f(x_1) - a_0 - a_1 x_1 - a_2 x_1^2 = -2 - a_0 + a_1 - a_2$$

$$C = f(x_2) - a_0 - a_1 x_2 - a_2 x_2^2 = 3 - a_0 - 2a_1 - 4a_2$$

$$\begin{cases} (1)\quad A+B+C = -3 - 3a_0 + 2a_1 - 14a_2 = 0 \\ (2)\quad 3A + B - 2C = -12 - 3a_0 + 9a_1 - 27a_2 - 2 - a_0 \\ \quad + a_1 - a_2 - 6 + 2a_0 + 4a_1 + 8a_2 = 0 \\ \quad -20 - 2a_0 + 14a_1 - 20a_2 = 0 \quad |:2 \\ \quad -10 - a_0 + 7a_1 - 10a_2 = 0 \\ (3)\quad -9A - B - 4C = 36 + 9a_0 - 27a_1 + 81a_2 \\ \qquad 2 + a_0 - a_1 + a_2 \\ \qquad -12 + 4a_0 + 8a_1 + 16a_2 \\ \quad 26 + 14a_0 - 20a_1 + 98a_2 = 0 \end{cases}$$

$$\begin{cases} 3 + 3a_0 - 2a_1 + 14a_2 = 0 \\ 10 + a_0 - 7a_1 + 10a_2 = 0 \quad |\cdot(-3) \\ 26 + 14a_0 - 20a_1 + 98a_2 = 0 \end{cases} \text{ "-" }$$

$$\Rightarrow 3 - 30 - 2a_1 + 21a_1 + 14a_2 - 30a_2 = 0.$$

$$-27 + 19a_1 - 16a_2 = 0.$$

$$10\cdot 14 - 26 - (98+20)a_1 + (140-98)a_2 = 0.$$

$$114 - 118a_1 + 42a_2 = 0 \quad |:2$$

$$\begin{cases} 57 - 59a_1 + 21a_2 = 0 \\ 27 - 19a_1 + 16a_2 = 0 \end{cases} \quad \ominus$$

$$30 - 40a_1 + 5a_2 = 0 \quad |:5$$

$$6 - 8a_1 + a_2 = 0$$

$$a_2 = 8a_1 - 6$$

$$27 - 19a_1 + 128a_1 - 96 = 0$$

$$109 a_1 - 69 = 0$$

$$\boxed{a_1 = \frac{69}{109}}$$

$$a_2 = 8 a_1 - 6$$

$$\boxed{a_2 = -\frac{102}{109}}$$

$$a_0 = 7 a_1 - 10 a_2 - 10$$

$$a_0 = \frac{7 \cdot 69 + 1020 - 1090}{109} = \frac{413}{109}$$

$E(a_0, a_1, a_2) = (A + B + C)^2$, where

$$A = -4 - a_0 + 3 a_1 - 9 a_2 = -4 - \frac{413}{109} + \frac{3 \cdot 69}{109} - \frac{9 \cdot (-1\ldots}{109}$$

$$A = \frac{-436 - 413 + 207 + 918}{109} = \frac{276}{109}$$

$$B = -2 - a_0 + a_1 - a_2 = \frac{-218 - 413 + 69 + 102}{109}$$

$$\boxed{B = -\frac{460}{109}}$$

$$C = 3 - a_0 - 2 a_1 - 4 a_2 = \frac{327 - 413 - 138 + 408}{109}$$

$$C = \frac{184}{109}$$

$$\boxed{E(a_0, a_1, a_2) = \left(\frac{276 - 460 + 184}{109}\right)^2 = 0}$$