Euler's Identity

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1 Introduction

Yesterday, a friend of mine asked me about i^i value. We were both surprised when we've learned that it is a real number. After that I started to read about exponential form of complex numbers, and then I bumped up into Euler's Identity.

2 Description

First of all, let's have a short talk on exponential form of complex numbers and justify it by writing z in the polar form and them expand sin and cos using Taylor three times. Let z a complex number:

$$z = r(cos(x) + isin(x)) = r(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots) + ir(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) = \quad (1)$$

$$r(1 + ix + \frac{(xi)^2}{2!} + \frac{(ix)^3}{3!} \dots) = re^{ix} \quad (2)$$

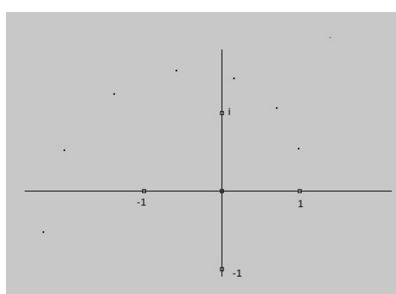

Figure 1: For N=6

Now, I will present a limit.

$$\lim_{N\to\infty}(1 + \frac{i\pi}{N})^N = e^{i\pi} = \cos\pi + i\sin\pi = -1; \quad (3)$$

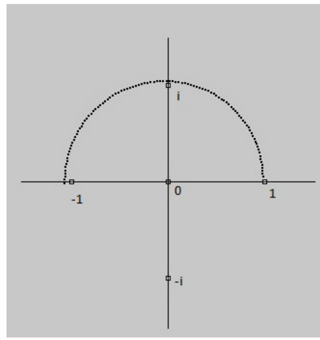

Figure 2: For N=100

That is indeed very interesting and for today I wanted to write some code that displays the image of this function for some N. Let:

$$f_N(n) = (1 + \frac{i\pi}{N})^n \tag{4}$$

where n is an integer between 1 and N. Notice that $f_N(n) = a_n + ib_n$ for some real a_n and b_n, so the coordinates of the points are going to be (a_n, b_n), not $(n, f(n))$, as we are probably used. It is obvious that the bigger N gets, the closer will be the last point to (-1,0).

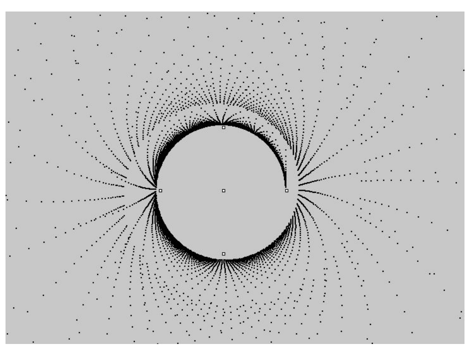

Figure 3: for $0 < n < 100$ and $0 < N < 100$

Unfortunately, I represented the function correctly (Figure 1 and Figure 2) after I realized I had one little mistake in my code which lead to the Figure 3. It was quite shocking to have this picture in front of my eyes, and instead of searching the bug, I changed again and again the values of N (because my code displayed the image of $f_N(n)$ for N, n between 1 and 100, independently defined).

I wanted to see what happens if I took more values for N, so I chose N=100, and displayed all the graphs of functions f_A (with A smaller than N=100). The mistake I've made came from the fact that for each A ($A < N$), $f_A(n)$ took values for $n < N = 100$.

For a better understanding check out Figure 4.

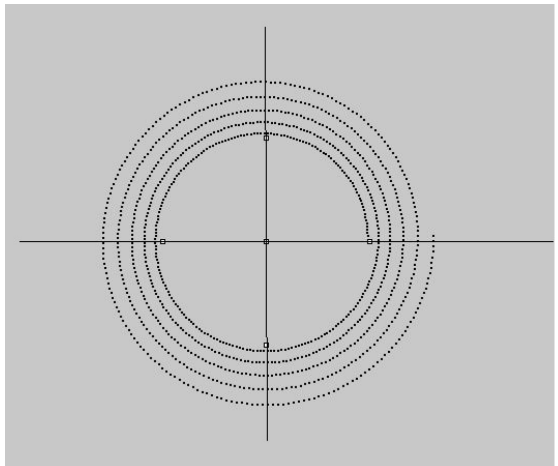

Figure 4: For N=100, $0 < n < 1000$

3 Code

In the next section I am going to present the fuction

```
void MyPaint(HDC hoDC)
{

    Rectangle(hoDC, 500,400,505,405);

    double e = 2.718281828, pi= 3.141592653;
    double a,b;
    double currentStep, zoom=100;
    Rectangle(hoDC, 500+1*zoom,400+0*zoom,505+1*zoom,405+0*zoom);
    Rectangle(hoDC, 500+0*zoom,400-1*zoom,505+0*zoom,405-1*zoom);
    Rectangle(hoDC, 500-1*zoom,400+0*zoom,505-1*zoom,405+0*zoom);
    Rectangle(hoDC, 500+0*zoom,400+1*zoom,505+0*zoom,405+1*zoom);
    double N=7000;
    for(int i=1;i<=5000;i++){
    a=1;b=pi/i;
    double a1,b1, a2=a, b2=b;
```

```
    for(int  j=1;j<=2*400*20;j++)
    {
            a1= a*a2-b*b2;
            b1= a*b2+b*a2;
            b2=b1;a2=a1;
    Rectangle(hoDC,500+a2*zoom,400-b2*zoom,502+a2*zoom,402-b2*zoom);
    }

    }
}
```

4 Church ceilings

In the next sections I'm going to present some really beautiful pictures of such functions. I'm not sure if they mean or not something but I'm totally happy with what I achieved for today.

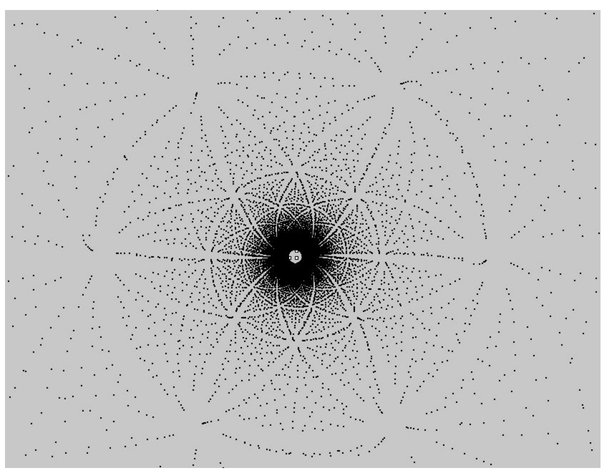

Figure 5: For $0 < N < 500$, $0 < n < 500$, and different scale

The little circle that is visible inside is the initial one, with the radius equal to 1. In the next few pictures it's not even going to be visible, because of the different scales. But the journey in this infinite world doesn't end here, as I've discovered that there exist some patterns, if different scales are used. Unexpectedly, the figure is kind of repetitive, and I strongly hope that the reader will agree with me after having a look on the next three pictures. All of them use the same intervals for N and n, the scaling is different.

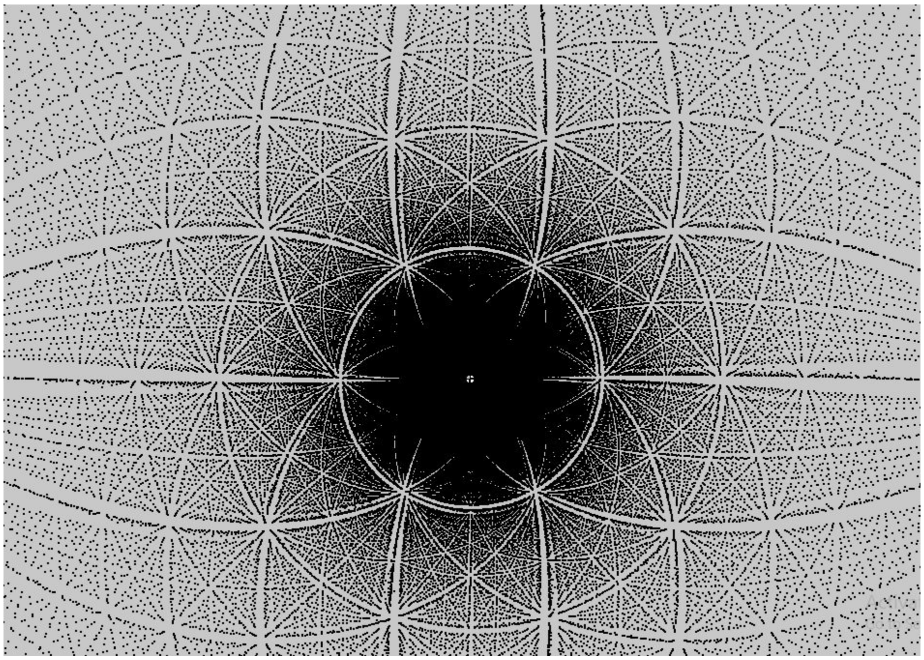

Figure 6: For $0 < N < 5000$, $0 < n < 5000$

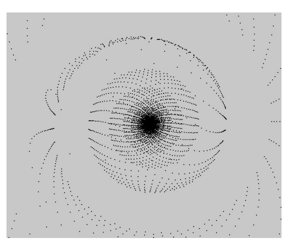

Figure 7: scale=0.1

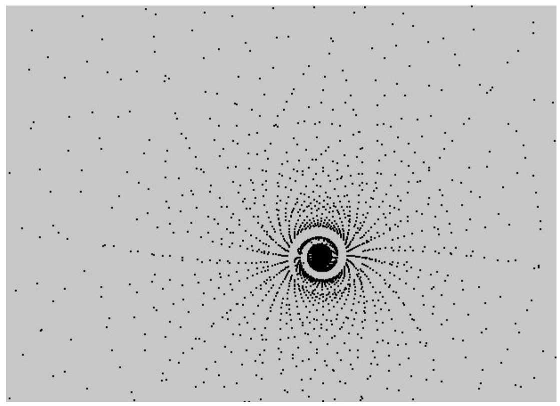

Figure 8: scale=0.0001

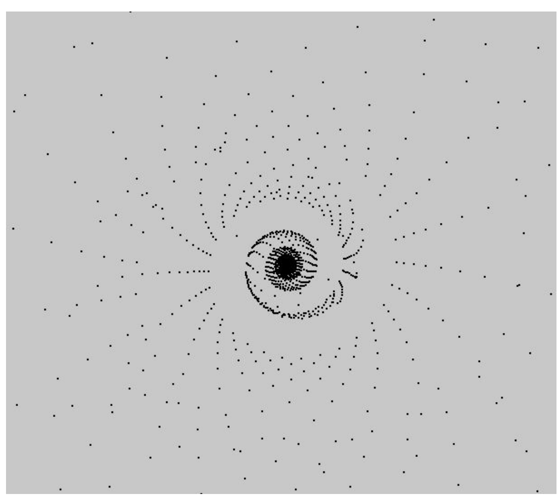

Figure 9: scale=0.00000001

5 a^b with a, b in $\mathbb{C}$

One of the many questions I had after computing i^i was related to the behavior that a^b has in general. It was expected that a dark but very interesting world was found. Let $a = r_1 e^{xi}$ and $b = r_2 e^{yi}$

$$a^b = (e^{log(a)})^b = e^{(log(r_1)+xi)b} \tag{5}$$

Something doesn't feel right about it. Are we really able to use logarithm in this situation? Is $blog(a)$ even existing? Let's try some numerical examples! From Euler's Identity we have:

$$log(-1) = \pi i \tag{6}$$

but if the function logarithm works as usual we also have that:

$$log(-1) = log(i^2) = 2log(i) = 2((-i)\frac{\pi}{2}) = -\pi i \tag{7}$$

If we replace i^2 with i^6, i^{10} so on, we will notice that $log(-1)$ takes an infinite number of values, thing which we are not used with. So the x in (5) may not be unique? Now we can either say this is impossible we shall stop here right now, or continue and assume it is possible for functions to be equal with more than one value, thing we have met before in complex space. But it is clear that we need to define it, in other way. For example clog(z) is the complex logarithm.

$$i = cos(\frac{\pi}{2} + 2\pi k) + isin(\frac{\pi}{2} + 2\pi k) = e^{i\frac{\pi}{2}+2\pi ik}$$

$$i^i = (e^{i\frac{\pi}{2}+2\pi ik})^i = e^{-\frac{\pi}{2}-2\pi k}$$

where k is an integer. Well, it seems like i^i takes infinitely many values as well, and that has to be the case for a^b. Now I guess it would be the best to end what I started at (5), but with $b = c + di$

$$a^b = (e^{log(a)})^b = e^{(log(r_1)+xi)b} =$$

$$= r^c e^{-dx} e^{i(log(r)d+xc)} = r^c e^{-dx} cis(log(r)d + xc)$$

Well, that looks pretty bad... and it's not correct either because the function should give multiple(infinite) values, but I'll end this section here, and maybe continue it later, as I want to turn back to my spirals and church ceilings.

6 Spirals and a mathematical approach

Feels good to look at these nice pictures again! I'll link one bellow, because I'm convinced you miss them as much as I do. In the figure 10 you can perceive a close look on what happens border. Less talk, and more about the good old $f_N(n)$ that generates the points. For n=1 we have:

$$\lim_{N\to\infty} f_N(1) = \lim_{N\to\infty} (1 + \frac{i\pi}{N})^1 = 1$$

I hope it is clear that all the points of $f_N(1)$ lie on the same line y=1, which is also tangent to the little grey circle inside, that is not defined yet. Let's take some N great enough, and P with coordinates $(1, \frac{\pi}{N})$ and O with (0,0) coordinates. $OP = \sqrt{1 + (\frac{\pi}{N})^2} > 1$, $tan(\alpha) = \frac{\pi}{N}$ so $cotan\frac{\pi}{N} = \alpha$ where α is the angle between OP and Ox. Let p be the complex coordinate of P.

$$f_N(n) = p^n = OP^n(cos(n\alpha) + isin(n\alpha))$$

and finally we now where the spiral comes from as

$$|f_N(n)| = OP^n > OP^{n-1} = |f_N(n-1)|$$

and the angle between OP^n and OP^{n-1} which is . Well let's define the circle

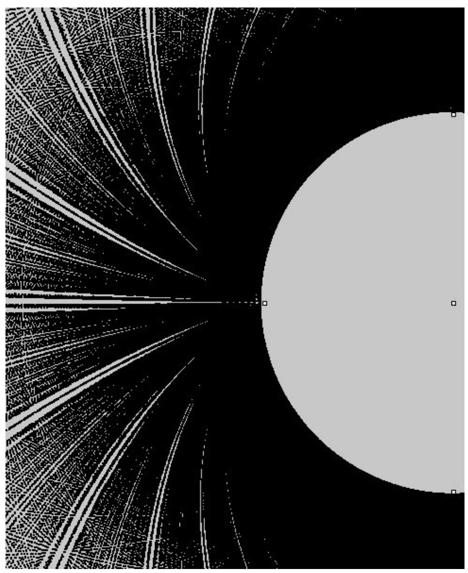

Figure 10: scale=200

from what we have written in the past. It is trivial that we can write $C(O, r)$ where r=1. I'm nor sure how can I prove mathematically that is the because the first point in the circle is going to be $(1, \frac{i\pi}{N})$ for N big enough.

$$\lim_{N\to\infty}(1 + \frac{i\pi}{N})^{\frac{N}{h}} = e^{\frac{i\pi}{h}} = \cos\frac{\pi}{h} + i\sin\frac{\pi}{h};$$

where h is a real number greater than 1. And that way we can obtain all real numbers between 0 and π, and I hope we are done.

But for sure the circle is not the star of my paper, but the spirals that form the beautiful church ceilings. And now we know that the graphic of the functions $f_N(n)$ with N great enough (let's say for $N > 500$) do not influence that much the final result, at least for the scale we are using.

Something I can't explain are the grey stripes. It would be the best to take a closer look.

7 Grey stripes?

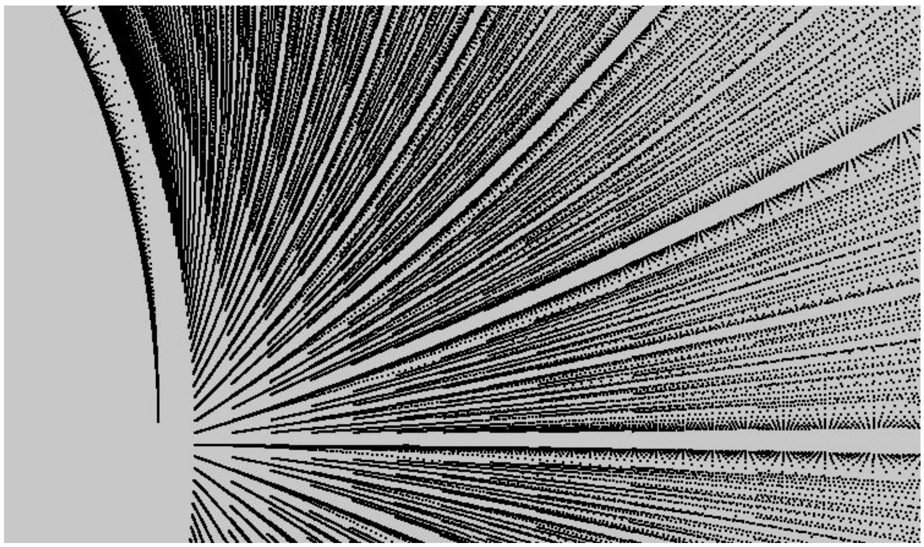

Figure 11: scale=200

There are places where the function doesn't draw points, and the little pattern that is visible on the circle's perimeter seems to repeat at different scales every where on each black line.