pset 2 real

tfmollano

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1 0.4.7 let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. For each of the following in $X \times Y$, determine whether they are functions or not, and determine whether they are surjective, injective, or bijections.

(a): R_1 is a function. It is injective, verified by checking. (b): R_2 is not a function. Two elements map from $b \in X$ to two unique elements $\in$. (c): R_3 is a function. It is surjective, since for every $y \in Y$ there exists a $x \in X$, such that $y = f(x)$ (d): R_4 is a function. It is not injective since two $(a, 1), (c, 1) \in X \times Y$ and $a \neq c$ and not surjective since there does not exist $f(x) x \in X, f(x) = 3$

2 0.4.8: Let $f : X \to X$ be a function, and let $A \subseteq X$. Prove that $A \subseteq f^{-1}(A)$ if and only if $f(A) \subseteq A$

Proof. Let us first prove the first direction. Recall that the inverse image of A for $f : X \to X$ is $f^{-1}(A) = \{x \in X : f(x) \in A\}$. From our hypothesis, for all $a \in A$, $f(A) \subseteq A$, we note that by definition of the inverse image, the set A must be contained in $f^{-1}(A)$ hence $A \subseteq f^{-1}(A)$. Now let us consider the other direction. We note that $A \subseteq f^{-1}(A)$ implies that all elements of A are elements of the inverse image $f^{-1}(A)$. In other words for all $a \in A \subseteq X, f(a) \in A$. Therefore $f(A) \subseteq A$

□

3 0.4.10: Let $f : X \to Y$ be a function. a: Prove $f(f^{-1}(B)) \subseteq B$ for every $B \subseteq Y$. b: Prove that $f(f^{-1}(B)) = B$ when f is surjective.

Proof. Recall $f^{-1}(B) : \{x \in X : f(x) \in B\}$ for $B \subseteq Y$. By definition, then for all $q \in f^{-1}(B)$, $f(q) \in B$. Thus $f(f^{-1}(B)) \subseteq B$ as necessary. For any element of $b \in B$, f surjective implies there exists $f(x), x \in X$, such that $f(x) = b$. Therefore $f^{-1}(B)$ is the set of $x \in P \subseteq X$, such that $f(P) = B$. Thus, $f(f^{-1}(B)) = B$ precisely.

□

4 0.4.13: Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Prove that if f and g are surjective, then $g \circ f$ is surjective. Prove that if f and g are injective, then $g \circ f$ is injective

Proof. Let us assume the image of f is a subset of the domain of g. Otherwise $g \circ f$ is not defined. To prove $g \circ f$ surjective, we must show for all $z \in Z$, there exists $y \in f(X)$, such that $g(y) = z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective, there must exist $x \in X$, such that $f(x) = y$. Therefore, for all $z \in Z$, there exists $y \in f(X)$, such that $g(y) = z$

To prove that $g \circ f$ injective if f and g are, we must show for all $a, b \in Z$, $p, q \in X$, $g(f(p)) = a \neq b = g(f(q)$ when $p \neq q$. Since g is injective, for all $a, b \in Z, t, r \in Y$, $g(t) \neq g(r)$ when $t \neq r$. By assumption, consider t, r in the image of f(X). Then for some $p, q \in X, f(p) = t, f(r) = q$. Since f is injective, $t = f(p) \neq r = f(q)$ when $q \neq p$. Therefore, $a, b \in Z, t, r \in Y$, $g(t) \neq g(r)$ when $t \neq r$ is equivalent to $p, q \in X$, $g(f(p)) \neq g(f(q))$ when $p \neq q$ as necessary. Thus $f \circ g$ is injective.

□

5 0.5.3: Let $n, m \in \mathbb{N}$. Prove that if there exists a bijection $f : J_n \to J_m$, then $n = m$

Proof. We proceed by strong induction on n. Consider $n = 1, m \in \mathbb{N}$. Then $J_1 = 1$. Then there exists a bijection from $J_1 \to J_m$. If J_m is such that $m > 1$, then without loss of generality $1 \in J_1 \to 1 \in J_m$. However, there does not exist another element in J_1 so that $x \in J_1 \to p \in J_m$. Therefore, a bijection between J_1 and $J_{m>1}$ cannot exist, since this bijeciton would not be surjective. If $m = 1$, then the bijection $1 \in J_{n=1} \to 1 \in J_{m=1}$ does exist, and is bijective trivally. Therefore, the base case is proven We now may consider strong induction up to n.

Let us prove if there exists a bijection $f : J_{n+1} \to J_{m+1}$, then $n + 1 = m + 1$. We may assume if there exists a bijection $f : J_n \to J_m$, then $n = m$. Denote $a_1 = \{n + 1\} \in J_{n+1}$ and denote $b_1 = \{m + 1\} \in J_{m+1}$. From our inductive hypothesis, if there exists a bijection between $J_n \setminus a_1$ and $J_m \setminus b_1$, then then exists the following bijection taking $f : J_{n+1} \to J_{m+1}$, Consider the composition of mapping of $a_1 \to b_1$ which is bijective trivally with the bijection $J_n \setminus a_1$ and $J_m \setminus b_1$ from our inductive hypothesis. From problem 0.4.13, since both mappings are bijective and hence injective and surjective their composition also is both injective and surjective hence bijective. □

6 0.5.8: Let $\{F_n\}$ be the Fibonacci sequence. Prove that for each $n \in \mathbb{N}, \sum_{i=0}^{n} F_i = F_{n+2} - 1$

Proof. Let us proceed by induction on $n > 1$. Consider the $n = 1$ base case. Let us note then that the sum on the left hand side is $F_0 + F_1 = 2$ and the right hand side is $F_2 - 1 = 2$. Thus the base case holds. Assume the result holds for the nth case. Now let us prove the result also holds for $n + 1$th case. We assume $\sum_{i=0}^{n} F_i = F_{n+2} - 1$. Let us add F_{n+1} to both sides of this equation. Then we have $\sum_{i=0}^{n+1} F_i = F_{n+2} + F_{n+1} - 1$. Recall by definition of the Fibonacci sequence $F_{n+2} + F_{n+1} = F_{n+3}$. Thus $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$ as necessary, so the n+1 case holds. □

7 1.3.1: Write $\mathbb{N}$ as a union of infinitely many disjoint sets each of which is infinite

Let $A_2 = \{x : x \in 2\mathbb{N}\}$, $A_3 = \{x : x \in 3\mathbb{N}, x \notin A_2\}, A_4 = \{x : x \in 5\mathbb{N}, x \notin A_2, A_3\}, ...$ for A_p with $p > 1$ and p prime. Each set A_n is a subset of $\mathbb{N}$ so are all countable, and disjoint, by definition. Since there are infinitely many primes, there also must be infinitely many A_p. Note: $\mathbb{N} = A_2 \cup A_3 \cup ....$ Since for all $x \in \mathbb{N}$, x has a unique factorization into primes, then given some prime p in the factorisation, x must belong to A_p, since it it a multiple of that prime, or then belong to some other A_q if not. By definition of A_p, x divisible by p cannot belong to A_p only if x is already in another set A_q for $q < p$, by definition of the A_p's as being disjoint. Therefore, for all $x \in \mathbb{N}$, $x \in A_2 \cup A_3 \cup ...$

8 1.3.2: Let A, B be sets. Prove that if A is countable and if there exists a function $f : A \to B$ that is surjective, then B is countable

Proof. We suppose there exists $f : A \to B$ with f surjective. Let us note that if A is countable, then there exists a surjective function $g : \mathbb{N} \to A$ by proposition 1.3.4 (2). Now let us consider the following composition $g \circ f$. Then $g \circ f : \mathbb{N} \to B$. Since g and f are both surjective, $g \circ f$ also is by problem 0.4.13. According to proposition 1.3.4, if there exists a function $p : \mathbb{N} \to Q$ that is surjective, then Q must be countable. Therefore, B is countable. □

9 1.3.5: Let K be an infinite subset of $\mathbb{N}$. Prove that there does not exist $n \in \mathbb{N}$ such that $n > k$ for all $k \in K$

Proof. Suppose there exists $n \in \mathbb{N}$ such that $n > k$ for all $k \in K$. Then consider when $n = \max k$. Then there exists a bijection (identical to the one given in Lemma 1.3.1 pg 63) f between J_n and K given by $f(1) = \min K$, $f(2) = \min(K - \{f(1)\}), ...$ This function is injective since $f(i) \neq f(j)$ for $i, j \in \mathbb{N}$ by definition and surjective since if not, then for some $k \in K, f(p) \neq k$, so there exists $k < f(z)$ for z $\in \mathbb{N}$, $k, f(z) \in K$, which is impossible by definition of $f(z)$ as a minimum. Then we deduce $f(z) < k$ for all z, which is also impossible since we suppose there exists n such that $n > k$, thus f is bijective. Therefore, K, which is an infinite set, is bijective with a finite set, which contradicts the condition that there exists $n \in \mathbb{N}$ such that $n > k$ for all $k \in K$.

□