A HIGH-ORDER DISCONTINUOUS GALERKIN METHOD FOR THE BIDOMAIN PROBLEM OF CARDIAC ELECTROPHYSIOLOGY

Project N$^\circ$ 2

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Course of Numerical Analysis for Partial Differential Equations

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The physical problem

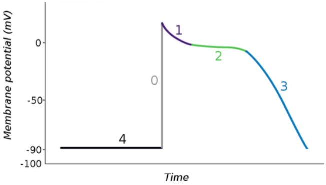
Mechanical contraction of the human heart

$\uparrow$

Electrical activation of the cardiac cells

$\downarrow$

Continuous electrical diffusion over the entire cardiac surface.



The mathematical model

Bidomain model + FitzHugh-Nagumo with Neumann B.C.

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_i \nabla \phi_i) + \chi_m I_{ion}(V_m, w) = I_i^{ext}, & \text{in } \Omega_{mus} \times (0, T], \\ -\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_e \nabla \phi_e) - \chi_m I_{ion}(V_m, w) = -I_e^{ext}, & \text{in } \Omega_{mus} \times (0, T], \\ I_{ion}(V_m, w) = kV_m(V_m - a)(V_m - 1) + w, & \text{in } \Omega_{mus} \times (0, T], \\ \frac{\partial w}{\partial t} = \epsilon(V_m - \gamma w), & \text{in } \Omega_{mus} \times (0, T], \\ \Sigma_i \nabla \phi_i \cdot n = b_i, & \text{on } \partial\Omega_{mus} \times (0, T], \\ \Sigma_e \nabla \phi_e \cdot n = b_e, & \text{on } \partial\Omega_{mus} \times (0, T], \\ \text{Initial conditions for } \phi_i, \phi_e, w, & \text{in } \Omega_{mus} \times \{t = 0\}. \end{cases}$$

Unknowns: $\phi_i,\ \phi_e,\ V_m = \phi_i - \phi_e,\ w$

Our objectives

What had already been done:

- Implementation of a Discontinuous Galerkin with FEM basis for the Bidomain problem.
- Implementation of a Semi-Implicit temporal scheme.

What we did:

- Implementation of a Discontinuous Galerkin with **Dubiner** basis for the Bidomain problem.
- Implementation of further temporal schemes.
- Bugs corrections and optimizations.
- Pseudo-realistic simulations.

Analytical definition

Definition (Dubiner basis)

The Dubiner basis that generates the space $\mathbb{P}^p(\hat{K})$ of polynomials of degree p over the reference triangle is the set of functions:

$$\phi_{ij} : \hat{K} \to \mathbb{R},$$

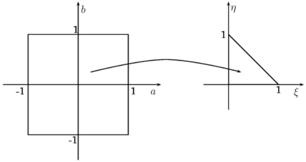
$$\phi_{ij}(\xi, \eta) = c_{ij}\, 2^j (1-\eta)^j J_i^{0,0}\left(\frac{2\xi}{1-\eta} - 1\right) J_j^{2i+1,0}(2\eta - 1),$$

for $i, j = 0, \ldots, p$ and $i + j \leq p$, where $c_{ij} := \sqrt{\frac{2(2i+1)(i+j+1)}{4^i}}$ and $J_i^{\alpha,\beta}(\cdot)$ is the i-th Jacobi polynomial.

Properties

- They consist in a pseudo tensor-product of Jacobi polynomials if the following transformation is then applied:

$$\xi = \frac{(1+a)(1-b)}{4}, \eta = \frac{(1+b)}{2}.$$



- They are $L^2(\hat{K})$ orthonormal ($\hat{K}$ is the reference triangle).

Main works

Remark

*Dubiner basis coefficients of a discretized function have **modal** meaning instead of a nodal meaning.*

Then, our main works regarded:

- Methods for the evaluation of the Dubiner functions and gradients in the reference points.
- Methods for the evaluation of the FEM coefficients of a discretized function starting from its Dubiner coefficients and viceversa.
  1. FEM $\rightarrow$ Dubiner is needed when we use the initial solution data into the Dubiner system.
  2. FEM $\leftarrow$ Dubiner is needed when we want to get the solution obtained from the Dubiner system in a comprehensible form.

FEM-Dubiner conversion strategies

Consider:

- An element $\mathcal{K} \in \tau_h$
- $\{\psi_i\}_{i=1}^{p}, \{\varphi_j\}_{j=1}^{q}$ as the FEM and Dubiner functions with support in $\mathcal{K}$.
- $\{\hat{u}_i\}_{i=1}^{p}, \{\tilde{u}_j\}_{j=1}^{q}$ as the FEM and Dubiner coefficients of a function u_h.

FEM $\leftarrow$ Dubiner
Exploiting the nodal meaning of FEM, we compute its value in a point:

$$\hat{u}_i = \sum_{j=1}^{q} \tilde{u}_j \phi_j(x_i),$$

FEM $\rightarrow$ Dubiner
Exploiting the L^2-orthonormality of Dubiner, we compute its Fourier coeff.:

$$\tilde{u}_j = \int_{\mathcal{K}} u_h(x)\varphi_j(x)\,dx = \int_{\mathcal{K}} \sum_{i=1}^{p} \hat{u}_i \psi_i(x)\varphi_j(x)\,dx = \sum_{i=1}^{p} \left(\int_{\mathcal{K}} \psi_i(x)\varphi_j(x)\,dx \right) \hat{u}_i.$$

Discretization

space-dependent: Discontinuous Galerkin method

$\nearrow$

Bidomain problem

$\searrow$

time-dependent: Semi-implicit, Godunov operator-splitting and quasi-implicit operator-splitting

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Semi-discrete Discontinuous Galerkin formulation

For any $t \in [0, T]$ find $\Phi_h(t) = [\phi_i^h(t), \phi_e^h(t)]^T \in [V_h^p]^2$ and $w_h(t) \in V_h^p$ such that:

$$
\begin{cases}
\sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \chi_m C_m \dfrac{\partial V_m^h}{\partial t} v_h d\omega + a_i(\phi_i^h, v_h) + \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \chi_m k(V_m^h - 1)(V_m^h - a) V_m^h v_h d\omega + \\
\qquad + \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \chi_m w_h v_h d\omega = (I_i^{ext}, v_h), \qquad \forall v_h \in V_h^p, \\
-\sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \chi_m C_m \dfrac{\partial V_m^h}{\partial t} v_h d\omega + a_e(\phi_e^h, v_h) - \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \chi_m k(V_m^h - 1)(V_m^h - a) V_m^h v_h d\omega + \\
\qquad - \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \chi_m w_h v_h d\omega = (-I_e^{ext}, v_h), \qquad \forall v_h \in V_h^p, \\
\sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \frac{\partial w_h}{\partial t} v_h d\omega = \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} \epsilon(V_m^h - \gamma w_h) v_h d\omega, \qquad \forall v_h \in V_h^p,
\end{cases}
$$

where:

- $$a_l(\phi_l^h, v_h) = \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} (\Sigma_l \nabla_h \phi_k^h) \cdot \nabla_h v_h d\omega - \sum_{F\in\mathcal{F}_h^I} \int_F \{\{\Sigma_l \nabla_h \phi_l^h\}\} \cdot [[v_h]] d\sigma + \\ - \delta \sum_{F\in\mathcal{F}_h^I} \int_F \{\{\Sigma_l \nabla_h v_h\}\} \cdot [[\phi_l^h]] d\sigma + \sum_{F\in\mathcal{F}_h^I} \int_F \Gamma [[\phi_l^h]] \cdot [[v_h]] d\sigma \qquad l = i, e,$$
- $$(I_i^{ext}, v_h) = \sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} I_i^{ext} v_h d\omega + \int_{\partial\Omega} b_i v_h d\sigma,$$
- $$(-I_e^{ext}, v_h) = -\sum_{\mathcal{K}\in\tau_h} \int_{\mathcal{K}} I_e^{ext} v_h d\omega + \int_{\partial\Omega} b_e v_h d\sigma.$$

Semi-implicit scheme

Idea:

- treat most of the terms of the PDE implicitly,
- treat the non-linear term semi-implictly,
- treat the ODE implictly with the exception of the term V_m.

Semi-implicit discretized system

Find $\Phi^{n+1} = [\phi_i^{n+1} \phi_e^{n+1}]^T$ and w^{n+1} $\forall n = 0, \cdots, N-1$ such that:

$$\begin{cases} (\frac{1}{\Delta t} + \epsilon\gamma) M w^{n+1} = \epsilon M V_m^n + \frac{M}{\Delta t} w^n, \\ (B + C_{nl}(V_m^n))\Phi^{n+1} = r^{n+1}. \end{cases}$$

Godunov operator-splitting scheme

The main feature is the sub-division of the problem into two different problems to be solved sequentially, such that $L(u) = L_1(u) + L_2(u)$. In our case:

1:

$$\begin{cases} \chi_m C_m M \frac{\hat{V}_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^n + \chi_m M w^n = 0, \\ \frac{w^{n+1} - w^n}{\Delta t} = \epsilon (V_m^n - \gamma w^n). \end{cases}$$

2:

$$\begin{cases} \chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^{n+1}}{\Delta t} + A_i \phi_i^{n+1} = F_i^{n+1}, \\ -\chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^{n+1}}{\Delta t} + A_e \phi_e^{n+1} = F_e^{n+1}. \end{cases}$$

Godunov operator-splitting discretized system

Find $\Phi^{n+1} = [\phi_i^{n+1} \phi_e^{n+1}]^T$ and w^{n+1} $\quad \forall n = 0, \cdots, N-1$ such that:

$$\begin{cases} \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ M & -M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix} \right) \begin{bmatrix} \phi_i^{n+1} \\ \phi_e^{n+1} \end{bmatrix} = \begin{bmatrix} F_i^{n+1} \\ -F_e^{n+1} \end{bmatrix} + \\ -\chi_m \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} w^n \\ w^n \end{bmatrix} + \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} C(V_m^n) & 0 \\ 0 & C(V_m^n) \end{bmatrix} \right) \begin{bmatrix} V_m^n \\ V_m^n \end{bmatrix}, \\ w^{n+1} = (1 - \epsilon \gamma \Delta t) w^n + \epsilon \Delta t V_m^n. \end{cases}$$

Quasi-implicit operator-splitting scheme

Idea:

- sub-division of the operator as Godunov operator-splitting
- treat implicitly all the terms except the cubic one

In this case:

1:

$$\begin{cases} \chi_m C_m M \frac{\tilde{V}_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^{n+1} + \chi_m M w^{n+1} = 0, \\ \frac{w^{n+1} - w^n}{\Delta t} = \epsilon (V_m^{n+1} - \gamma w^{n+1}). \end{cases}$$

2:

$$\begin{cases} \chi_m C_m M \frac{V_m^{n+1} - \tilde{V}_m^{n+1}}{\Delta t} + A_i \phi_i^{n+1} = F_i^{n+1}, \\ -\chi_m C_m M \frac{V_m^{n+1} - \tilde{V}_m^{n+1}}{\Delta t} + A_e \phi_e^{n+1} = F_e^{n+1}. \end{cases}$$

Quasi-implicit operator-splitting discretized system

Find $\Phi^{n+1} = [\phi_i^{n+1} \phi_e^{n+1}]^T$ and $w^{n+1} \quad \forall n = 0, \cdots, N-1$ such that:

$$\begin{cases} \left(\begin{bmatrix} Q_n & -Q_n \\ Q_n & -Q_n \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix} \right) \begin{bmatrix} \phi_i^{n+1} \\ \phi_e^{n+1} \end{bmatrix} = \begin{bmatrix} R_n \\ R_n \end{bmatrix} + \begin{bmatrix} F_i^{n+1} \\ -F_e^{n+1} \end{bmatrix}, \\ w^{n+1} = \frac{w^n + \epsilon \Delta t (\phi_i^{n+1} - \phi_e^{n+1})}{1 + \epsilon \gamma \Delta t}. \end{cases}$$

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Uniqueness

About uniqueness of the unkowns:

- V_m, w proved in *Existence and uniqueness of the solution for the bidomain model used in cardiac electrophysiology* by Y. Bourgault, Y. Coudière, and C. Pierre.
- ϕ_i, ϕ_e appear only through their difference V_m or their gradient. This means that there cannot be uniqueness.

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Uniqueness of potentials

Theorem

The classical solutions ϕ_i, ϕ_e are unique up to a constant depending only on time.

Namely:
Suppose now there exist two couples (ϕ_i^1, ϕ_e^1),(ϕ_i^2, ϕ_e^2) of potentials solutions of the Bidomain problem.

$$\exists \tilde{\varphi} : [0, T] \rightarrow \mathbb{R} \text{ such that } \phi_i^1(x, t) - \phi_i^2(x, t) = \phi_e^1(x, t) - \phi_e^2(x, t) = \tilde{\varphi}(t)$$

$$\forall x \in \Omega, \forall t \in [0, T].$$

SO HOW TO RESOLVE IT?

1. Imposition of the value of the function in a specific point.
2. Imposition of the function mean value.

Imposition of a value in a specific point

Focus on only one of the potentials ϕ_i.

$$\phi_i(\bar{x}, t) = \varphi(t) \quad \forall t \in [0, T] \qquad \xrightarrow[\text{with } \{u_j\} \text{ as vector of solution}]{\text{Numerical version}} \qquad u_l^n = \varphi(t^n) \quad \forall n \in \{1, N\}$$

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