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Course of Numerical Analysis for Partial Differential Equations

DISCONTINUOUS GALERKIN APPROXIMATION FOR CARDIAC ELECTROPHYSIOLOGY

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1 Introduction

1.1 Abstract

The aim of the project is to study and implement a suitable numerical scheme for the resolution of the *Bidomain Problem*, a famous system of equations that has been developed in the context of the electrophysiology of human heart.
This work is basically the continuation of a two-years-long study carried out by three past course projects ([3], [1], [5]). In particular, the very goal of this project is to improve the results obtained in [5] (Marta and Perego) for the Bidomain model. In fact, even if a *Discontinuous Galerkin* discretization has been successfully implemented, results are not satisfactory from the point of view of stability and convergence. We think this notice is noteworthy as this work is primarily based on these provided data and codes. Through this article, it will be illustrated how we managed to solve these problems extending, optimizing and correcting these past numerical strategies.

1.2 The physical problem

We intend to present the physical meaning of the Bidomain equations very briefly since it has already been widely shown in the previous project (Marta and Perego). For a more complete explanation, we instead refer to [6].
The mechanical contraction and expansion of human heart has its origin in the *electrical activation* of the cardiac cells. At every heart-beat, myocyties are activated and deactivated following a characteristic electrical cycle (fig 1).

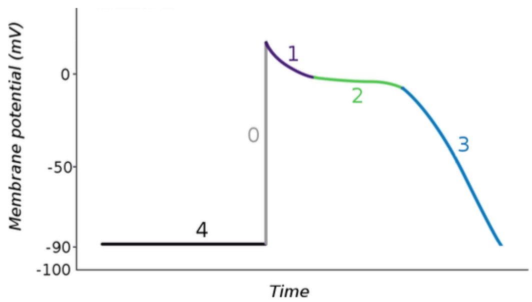

Figure 1: Membrane potential in function of time (one cardiac cycle)

The cell is initially at rest ($-90mV$, step 4). At a certain point, its potential increases rapidly ($\approx 2ms$) and reaches the value of $+20mV$: the cell is activated. Later, a plateau near $0mV$ is observed and then a slow repolarization to the initial potential.

From a microscopical point of view, we could study the dynamics acting in each single cell (as a consequence of the passage of chemical ions through specific channels, e.g. $Ca2+, Na+, K+$). From a macroscopical point of view, instead, one can think about it as a continuous electrical diffusion over the entire cardiac surface. Even if this consists in a very rapid phenomenon, the study of such propagation could be very interesting in order, for instance, to detect diseases in sick patients.

1.3 Mathematical models

Starting from the circuit in figure 2, applying some general electromagnetism laws and some calculations, the Bidomain model has been formulated (see [6] for more details and/or [4] for the complete passages).

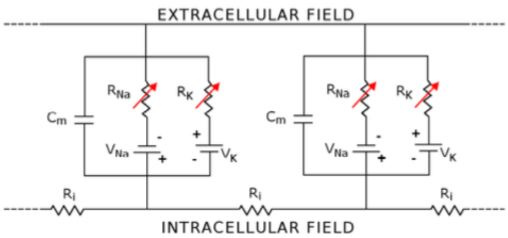

Figure 2: Simplified circuit to model the intracellular and extracellular potentials dynamics

The general formulation is then:

Definition 1 (Bidomain model)**.**

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_i \nabla \phi_i) + \chi_m I_{ion} = I_i^{ext} & in\ w_{mus} \times (0, T] \\ -\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_e \nabla \phi_e) - \chi_m I_{ion} = -I_e^{ext} & in\ w_{mus} \times (0, T] \end{cases}$$

where:

- $\boldsymbol{\phi_i, \phi_e}$ are the *Intracellular and Extracelllular Potentials* (unknowns)
- $V_m = \phi_i - \phi_e$ is the *Trans-membrane Potential*
- $\chi_m, C_m, \Sigma_i, \Sigma_e$ are known constants
- I_i^{ext}, I_e^{ext} are applied currents
- I_{ion} is the *Ionic Current*

– w_{mus} is the cardiac domain (myocardium + endocardium + epicardium)

Actually, this system is not complete since it misses boundary and initial conditions and a suitable model for I_{ion}. Initial conditions and Neumann boundary conditions for ϕ_i and ϕ_e are then imposed. For the definition of I_{ion}, instead, a *reduced ionic model* is chosen, in particular the *FitzHugh-Nagumo model*. Summing up:

Definition 2 (Bidomain + FitzHugh-Nagumo model with Neumann boundary conditions)**.**

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_i \nabla \phi_i) + \chi_m I_{ion}(V_m, w) = I_i^{ext} & in\ w_{mus} \times (0, T] \\ -\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_e \nabla \phi_e) - \chi_m I_{ion}(V_m, w) = -I_e^{ext} & in\ w_{mus} \times (0, T] \\ I_{ion}(V_m, w) = k V_m (V_m - a)(V_m - 1) - w & in\ w_{mus} \times (0, T] \\ \frac{\partial w}{\partial t} = \epsilon (V_m - \gamma w) & in\ w_{mus} \times (0, T] \\ \Sigma_i \nabla \phi_i \cdot n = b_i & on\ \partial w_{mus} \times (0, T] \\ \Sigma_e \nabla \phi_e \cdot n = b_e & on\ \partial w_{mus} \times (0, T] \\ Initial\ conditions\ for\ \phi_i, \phi_e, w & in\ w_{mus} \times \{t = 0\} \end{cases}$$

where:

– $\boldsymbol{w}$ is the *gating variable* (unknown)

– k, a, ϵ, γ are known constants

– b_i, b_e are the boundary conditions data

– n is the outward normal vector

From now on, the system of definition 2 will be the reference analytical problem for the development of numerical schemes.
To conclude, there exist other famous and useful models, such as the *Monodomain model*. But this is just a simplification of the Bidomain as in this case it is assumed that ϕ_i and ϕ_e are proportional. However, thanks to its simplicity, we often tested the code starting from the Monodomain implementation of the project [1] instead of analyzing directly the Bidomain.

1.4 Semi discretized numerical methods

1.4.1 DG discrete formulation

We have seen the bidomain model in a complete form in definition 2. We introduce now a triangulation τ_h over w, with $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$ set of the faces of the elements, which includes the internal and boundary faces respectively, and the DG space $V_h^k = \{v_h \in L^2 : v_h|_{\mathcal{K}} \in \mathbb{P}^k(\mathcal{K}) \quad \forall \mathcal{K} \in \tau_h\}$, where k is the degree of the piecewise continuos polynomial.

We obtain the semi discrete DG formulation:

For any time $t \in [0, T]$ find $\Phi_h(t) = [\phi_i^h(t), \phi_e^h(t)]^T \in [V_h^k]^2$ and $w_h(t) \in V_h^k$:

$$\sum_{K\in\tau_h} \int_K \chi_m C_m \frac{\partial V_m^h}{\partial t} V_h dw + a_i(\phi_i^h, v_h) + \sum_{K\in\tau_h} \int_K \chi_m k(V_m^h - 1)(V_m^h - a)V_m^h v_h dw +$$

$$+ \sum_{K\in\tau_h} \int_K \chi_m w_h v_h dw = (I_i^{ext}, v_h) \qquad \forall v_h \in V_h^p$$

$$- \sum_{K\in\tau_h} \int_K \chi_m C_m \frac{\partial V_m^h}{\partial t} V_h dw + a_e(\phi_e^h, v_h) - \sum_{K\in\tau_h} \int_K \chi_m k(V_m^h - 1)(V_m^h - a)V_m^h v_h dw +$$

$$- \sum_{K\in\tau_h} \int_K \chi_m w_h v_h dw = (-I_e^{ext}, v_h) \qquad \forall v_h \in V_h^p$$

$$\sum_{K\in\tau_h} \int_K \frac{\partial w_h}{\partial t} v_h dw = \sum_{k\in\tau_h} \int_K \epsilon(V_m^h - \gamma w_h) v_h dw \qquad \forall v_h \in V_h^p$$

where:

$$a_k(\phi_k^h, v_h) = \sum_{K\in\tau_h} \int_K (\Sigma_k \nabla_h \phi_k^h) \cdot \nabla_h v_h dw - \sum_{F\in\mathcal{F}_h^I} \int_F \{\{\Sigma_k \nabla_h \phi_k^h\}\} \cdot [[v_h]] d\sigma +$$

$$-\delta \sum_{F\in\mathcal{F}_h^I} \int_F \{\{\Sigma_k \nabla_h v_h\}\} \cdot [[\phi_k^h]] d\sigma + \sum_{F\in\mathcal{F}_h^I} \int_F \gamma [[\phi_k^h]] \cdot [[v_h]] d\sigma \qquad \text{for k=i,e}$$

$$(I_i^{ext}, v_h) = \sum_{K\in\tau_h} \int_K I_i^{ext} v_h dw + \int_{\partial w} b v_h d\sigma$$

$$(-I_e^{ext}, v_h) = - \sum_{K\in\tau_h} \int_K I_e^{ext} v_h dw + \int_{\partial w} b v_h d\sigma$$

Moreover, according to the choice of the coefficient δ we can define:

- $\delta = 1$: Symmetric Interior Penalty method (SIP)
- $\delta = 0$: Incomplete Interior Penalty method (IIP)

- $\delta = -1$: Non Symmetric Interior Penalty method (NIP)

To see in more detail the procedure see [5].

1.4.2 Algebraic formulation

Taking $\{\varphi_j\}_{j=1}^{N_h}$ base of V_h^k, so that we can write

$$\Phi_h(t) = \begin{bmatrix} \phi_i^h(t) \\ \phi_e^h(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N_h} \phi_{i,j}(t)\varphi_j \\ \sum_{j=1}^{N_h} \phi_{e,j}(t)\varphi_j \end{bmatrix}$$

$$w_h(t) = \sum_{j=1}^{N_h} w_j(t)\varphi_j$$

$$V_m^h(t) = \sum_{j=1}^{N_h} V_{m,j}(t)\phi_j = \sum_{j=1}^{N_h} (\phi_{i,j}(t) - \phi_{e,j}(t))\varphi_j$$

Also we can introduce the matrices:

$$\left.\begin{array}{ll} V_{ij} & = \int_w \nabla\varphi_j \cdot \nabla\varphi_i \\ I_{i,j}^T & = \sum_{F\in F_h^I} \int_F \{\{\nabla\varphi_j\}\} \cdot [[\varphi_i]] \\ I_{i,j} & = \sum_{F\in F_h^I} \int_F [[\varphi_j]] \cdot \{\{\nabla\varphi_i\}\} \\ S_{i,j} & = \sum_{F\in F_h^I} \int_F \gamma[[\varphi_j]] \cdot [[\varphi_i]] \end{array}\right\} \quad A = (V - I^T - \theta I + S) \tag{1}$$

$$A_i = \Sigma_i A \qquad \text{Intra-cellular stiffness matrix} \tag{2}$$

$$A_e = \Sigma_e A \qquad \text{Extra-cellular stiffness matrix} \tag{3}$$

$$M_{ij} = \sum_{K\in\tau_h} \int_K \qquad \text{Mass matrix} \varphi_j\varphi_i \tag{4}$$

$$C(u_h)_{ij} = \sum_{K\in\tau_h} \int_K \chi_m k(u_h - 1)(u_h - a)\varphi_j\varphi_i \qquad \text{Non linear matrix} \tag{5}$$

$$F_k = \begin{bmatrix} F_{i,k} \\ F_{e,k} \end{bmatrix} = \begin{bmatrix} \int_w I_i^{ext}\varphi_k - \sum_{F\in F_h^B} \int_F b_i\varphi_k \\ -\int_w I_e^{ext}\varphi_k - \sum_{F\in F_h^B} \int_F b_e\varphi_k \end{bmatrix} \tag{6}$$

Therefore our algebraic formulation is:
Find $\Phi_h(t) = [\phi_i^h(t), \phi_e^h(t)]^T \in [V_h^k]^2$ and $w_h(t) \in V_h^k$ for any $t \in (0;T]$ such that:

$$\begin{gathered} \chi_m CmM\dot{V}_m^h + A_i\phi_i^h + C(V_m^h)V_m^h + \chi_m Mw_h = F_i^h \\ -\chi_m CmM\dot{V}_m^h + A_e\phi_e^h - C(V_m^h)V_m^h - \chi_m Mw_h = F_e^h \\ M\dot{w_h}(t) = \epsilon M(V_m^h(t) - \gamma w_h(t)) \end{gathered} \tag{7}$$

Rewrite it with the block matrix and assuming that M is non singular: Find $\Phi_h(t) = [\phi_i^h(t), \phi_e^h(t)]^T \in [V_h^k]^2$ and $w_h(t) \in V_h^k$ for any $t \in (0; T]$ such that:

$$\chi_m C_m \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\phi_i^h(t)}} \\ \dot{\boldsymbol{\phi_e^h(t)}} \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi_i^h(t)} \\ \boldsymbol{\phi_e^h(t)} \end{bmatrix} +$$
$$\begin{bmatrix} C(V_m^h) & -C(V_m^h) \\ -C(V_m^h) & C(V_m^h) \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi_i^h(t)} \\ \boldsymbol{\phi_e^h(t)} \end{bmatrix} + \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w_h(t) \\ w_h(t) \end{bmatrix} = \begin{bmatrix} F_i^h \\ F_e^h \end{bmatrix} \tag{8}$$
$$\dot{w_h}(t) = \epsilon(V_m^h(t) - \gamma w_h(t))$$

1.5 Totally discretized numerical methods

The system is time dependent, so we divide the interval (0,T] into K subintervals $(t^k, t^k + 1]$ of length Δt such that $t^k = k\Delta t \quad \forall k = 0, \cdots, K-1$ and so we consider $V_m^k \approx V_m(t^k)$.

1.5.1 Semi implicit method

At first, we consider an implicit method for the time discretization, meanwhile for the non linear contribution of I^{ion} a semi implicit method, indeed we think about it as a cubic function, so we treat explicitely the quadratic term and the rest of the terms implicitly, that is:

$$I_{ion} = k(V_m^k - a)(V_m^k - 1)V_m^{k+1} + w^{k+1}$$

Moreover in the ODE of the model 7, we evaluate w implicitely and we consider V_m at the previous temporal step t^k:

$$M\frac{w^{k+1} - w^k}{\Delta t} = \epsilon M(V_m^k - \gamma w^{k+1})$$

Therefore we obtain:

$$\chi_m C_m \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \frac{\boldsymbol{\phi_i^{k+1}} - \boldsymbol{\phi_i^k}}{\boldsymbol{\Delta t}} \\ \frac{\boldsymbol{\phi_e^{k+1}} - \boldsymbol{\phi_e^k}}{\boldsymbol{\Delta t}} \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi_i^{k+1}} \\ \boldsymbol{\phi_e^{k+1}} \end{bmatrix} +$$
$$\begin{bmatrix} C(V_m^h) & -C(V_m^h) \\ -C(V_m^h) & C(V_m^h) \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi_i^{k+1}} \\ \boldsymbol{\phi_e^{k+1}} \end{bmatrix} + \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} = \begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} \tag{9}$$
$$M\frac{w^{k+1} - w^k}{\Delta t} = \epsilon M(V_m^k - \gamma w^{k+1})$$

where $V_m^k = \phi_i^k - \phi_e^k$.
We can rewrite it as:

$$
\begin{aligned}
&\left(\frac{\chi_m C_m}{\Delta t}\begin{bmatrix} M & -M \\ -M & M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} + \begin{bmatrix} C(V_m^k) & -C(V_m^k) \\ -C(V_m^k) & C(V_m^k) \end{bmatrix}\right)\begin{bmatrix} \boldsymbol{\phi_i^{k+1}} \\ \boldsymbol{\phi_e^{k+1}} \end{bmatrix} = \\
&\begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} - \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix}\begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} + \frac{\chi_m C_m}{\Delta t}\begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix}\begin{bmatrix} V_m^k \\ V_m^k \end{bmatrix}
\end{aligned} \tag{10}
$$

$$
(\frac{1}{\Delta t} + \epsilon\gamma) M w^{k+1} = \epsilon M V_m^k + \frac{M}{\Delta t} w^k \tag{11}
$$

Defining:

- $B = \frac{\chi_m C_m}{\Delta t}\begin{bmatrix} M & -M \\ -M & M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix}$
- $C_{nl}(V_m^k) = \begin{bmatrix} C(V_m^h) & -C(V_m^h) \\ -C(V_m^h) & C(V_m^h) \end{bmatrix}$
- $r^{k+1} = \begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} - \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix}\begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} + \frac{\chi_m C_m}{\Delta t}\begin{bmatrix} M & -M \\ -M & M \end{bmatrix}\begin{bmatrix} \phi_i^k \\ \phi_e^k \end{bmatrix}$

We get the final system:
Find $\Phi^{k+1} = [\phi_i^{k+1}\phi_e^{k+1}]^T$ and w^{k+1} $\forall k = 0, \cdots, T-1$ such that:

$$
\begin{cases} (\frac{1}{\Delta t} + \epsilon\gamma) M w^{k+1} = \epsilon M V_m^k + \frac{M}{\Delta t} w^k \\ (B + C_{nl}(\Phi^k))\Phi^{k+1} = r^{k+1} \end{cases} \tag{12}
$$

1.5.2 Quasi implicit operator splitting

The main characteristic of an operator splitting is to divide the problem into two different systems with two different operators, such that $L(u) = L_1(u) + L_2(u)$, and, starting from u^n, we find $\tilde(u^{n+1})$ through the first system, then the solution u^{n+1} through the second one. In the Quasi implicit operator splitting we treat in a explicit way any term, except the V_m^n in the non linear term.
Find $\tilde{V}_m^{k+1}$ and w^{k+1} such that:

$$
\begin{cases} \chi_m C_m M \frac{\tilde{V}_m^{k+1} - V_m^k}{\Delta t} + C(V_m^k) V_m^{k+1} + \chi_m M w^{k+1} = 0 \\ \frac{w^{k+1} - w^k}{\Delta t} = \epsilon(V_m^{k+1} - \gamma w^{k+1}) \end{cases} \tag{13}
$$

Find V_m^{k+1} such that:

$$
\begin{cases} \chi_m C_m M \frac{V_m^{k+1} - \tilde{V}_m^{k+1}}{\Delta t} + A_i \phi_i^{k+1} = F_i^{k+1} \\ -\chi_m C_m M \frac{V_m^{k+1} - \tilde{V}_m^{k+1}}{\Delta t} + A_e \phi_e^{k+1} = F_e^{k+1} \end{cases} \tag{14}
$$

Put into a unique system:

$$\begin{cases} \chi_m C_m M \frac{V_m^{k+1}-V_m^k}{\Delta t} + C(V_m^k)V_m^{k+1} + \chi_m M w^{k+1} + A_i \phi_i^{k+1} = F_i^{k+1} \\ \chi_m C_m M \frac{V_m^{k+1}-V_m^k}{\Delta t} + C(V_m^k)V_m^{k+1} + \chi_m M w^{k+1} - A_e \phi_e^{k+1} = -F_e^{k+1} \\ \frac{w^{k+1}-w^k}{\Delta t} = \epsilon(V_m^{k+1} - \gamma w^{k+1}) \end{cases} \tag{15}$$

Defining:

- $Q_k = \frac{\chi_m C_m}{\Delta t} M + C(V_m^k) + \frac{\epsilon \chi_m \Delta t}{1+\epsilon\gamma\Delta t} M$

- $R_k = \frac{\chi_m C_m}{\Delta t} M V_m^k - \frac{\chi_m}{1+\epsilon\gamma\Delta t} M w^k$

The equations in the system 15 can be written as:

1. $$\chi_m C_m M \frac{\phi_i^{k+1} - \phi_e^{k+1} - V_m^k}{\Delta t} + C(V_m^k)(\phi_i^{k+1} - \phi_e^{k+1}) + \chi_m M \left(\frac{w^k + \epsilon\Delta t(\phi_i^{k+1} - \phi_e^{k+1})}{1+\epsilon\gamma\Delta t} \right) +$$
$$+A_i \phi_i^{k+1} = F_i^{k+1}$$
$$\Rightarrow \quad (Q_k + A_i)\phi_i^{k+1} - Q_k \phi_e^{k+1} = R_k + F_i^{k+1}$$

2. $$\chi_m C_m M \frac{\phi_i^{k+1} - \phi_e^{k+1} - V_m^k}{\Delta t} + \cdot C(V_m^k)(\phi_i^{k+1} - \phi_e^{k+1}) + \chi_m M \left(\frac{w^k + \epsilon\Delta t(\phi_i^{k+1} - \phi_e^{k+1})}{1+\epsilon\gamma\Delta t} \right) +$$
$$-A_e \phi_e^{k+1} = -F_e^{k+1}$$
$$\Rightarrow \quad Q_k \phi_i^{k+1} - (Q_k + A_e)\phi_e^{k+1} = R_k - F_e^{k+1}$$

3. $$w^{k+1} = \frac{w^k + \epsilon\Delta t(\phi_i^{k+1} - \phi_e^{k+1})}{1+\epsilon\gamma\Delta t}$$

The final system becomes:
Find $\Phi^{k+1} = [\phi_i^{k+1} \phi_e^{k+1}]^T$ and w^{k+1} $\quad \forall k = 0, \cdots, K-1$ such that:

$$\begin{cases} \left(\begin{bmatrix} Q_k & -Q_k \\ Q_k & -Q_k \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\phi_i^{k+1}} \\ \boldsymbol{\phi_e^{k+1}} \end{bmatrix} = \begin{bmatrix} R_k \\ R_k \end{bmatrix} + \begin{bmatrix} F_i^{k+1} \\ -F_e^{k+1} \end{bmatrix} \\ \\ w^{k+1} = \frac{w^k + \epsilon\Delta t(\phi_i^{k+1} - \phi_e^{k+1})}{1+\epsilon\gamma\Delta t} \end{cases} \tag{16}$$

1.5.3 Godunov operator splitting

Another type of operator splitting is the Gudonov operator splitting where in this case the first system is written in an explicit way, meanwhile the second implicitly. Indeed:
Find $\tilde{V}_m^{k+1}$ and w^{k+1} such that:

$$\begin{cases} \chi_m C_m M \frac{\hat{V}_m^{k+1} - V_m^k}{\Delta t} + C(V_m^k)V_m^k + \chi_m M w^k = 0 \\ \frac{w^{k+1} - w^k}{\Delta t} = \epsilon(V_m^k - \gamma w^k) \end{cases}$$

Find V_m^{k+1} such that:

$$\begin{cases} \chi_m C_m M \frac{V_m^{k+1} - \hat{V}_m^{k+1}}{\Delta t} + A_i \phi_i^{k+1} = F_i^{k+1} \\ -\chi_m C_m M \frac{V_m^{k+1} - \hat{V}_m^{k+1}}{\Delta t} + A_e \phi_e^{k+1} = F_e^{k+1} \end{cases}$$

Put in a unique system:

$$\begin{cases} \chi_m C_m M \frac{V_m^{k+1} - V_m^k}{\Delta t} + C(V_m^k)V_m^k + \chi_m M w^k + A_i \phi_i^{k+1} = F_i^{k+1} \\ \chi_m C_m M \frac{V_m^{k+1} - V_m^k}{\Delta t} + C(V_m^k)V_m^k + \chi_m M w^k - A_e \Phi_e^{k+1} = -F_e^{k+1} \\ w^{k+1} = (1 - \epsilon\gamma\Delta t)w^k + \epsilon \Delta t V_m^k \end{cases} \tag{17}$$

The equations in the system 17 can be written as:

$$\begin{cases} \left(\frac{\chi_m C_m}{\Delta t} M + A_i\right) \phi_i^{k+1} - \frac{\chi_m C_m}{\Delta t} M \phi_e^{k+1} = F_i^{k+1} - \chi_m M w^k + \left(\frac{\chi_m C_m}{\Delta t} M - C(V_m^k)\right) V_m^k \\ \frac{\chi_m C_m}{\Delta t} M \phi_i^{n+1} - \left(\frac{\chi_m C_m}{\Delta t} M + A_e\right) \phi_e^{k+1} = -F_e^{k+1} - \chi_m M w^k + \left(\frac{\chi_m C_m}{\Delta t} M - C(V_m^k)\right) V_m^k \\ w^{k+1} = (1 - \epsilon\gamma\Delta t)w^k + \epsilon \Delta t V_m^k \end{cases}$$

In the end, find $\Phi^{k+1} = [\phi_i^{k+1} \phi_e^{k+1}]^T$ and w^{k+1} $\quad \forall k = 0, \cdots, K-1$ such that:

$$\begin{cases} \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ M & -M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{\phi_i^{k+1}} \\ \boldsymbol{\phi_e^{k+1}} \end{bmatrix} = \begin{bmatrix} F_i^{k+1} \\ -F_e^{k+1} \end{bmatrix} + \\ -\chi_m \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} w^k \\ w^k \end{bmatrix} + \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} C(V_m^k) & 0 \\ 0 & C(V_m^k) \end{bmatrix}\right) \begin{bmatrix} V_m^k \\ V_m^k \end{bmatrix} \\ \\ w^{k+1} = (1 - \epsilon\gamma\Delta t)w^k + \epsilon \Delta t V_m^k \end{cases} \tag{18}$$

2 Dubiner Basis

The most popular orthonormal basis on the reference triangle

$$\hat{K} = \{(\xi, \eta) : \xi, \eta \geq 0, \xi + \eta \leq 1\} \tag{19}$$

is the Dubiner polynomial basis [2]. We consider the transformation in figure 3 between the reference square

$$\hat{Q} = \{(a, b) : -1 \leq a \leq 1, -1 \leq b \leq 1\} \tag{20}$$

and the reference triangle given by

$$\xi := \frac{(1+a)(1-b)}{4}, \eta := \frac{(1+b)}{2} \tag{21}$$

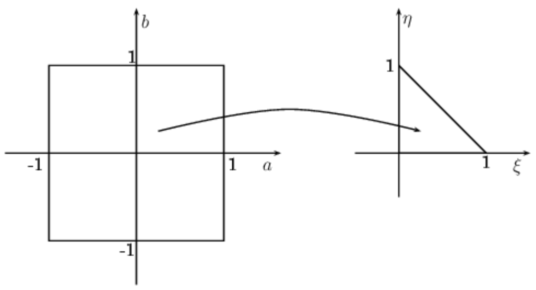

Figure 3: Transformation between the reference square to the reference triangle

The Dubiner basis is then constructed by a generalized tensor product of the Jacobi polynomials on the interval (-1, 1) to form a basis on the reference square, which is then transformed by the above "collapsing" mapping to a basis on the reference triangle.

Definition 3 (Jacobi polynomials)**.** *The Jacobi polynomials evaluated in $z \in \mathbb{R}^n$ are:*

– $n = 0$

$$J_0^{\alpha,\beta}(z) = \overbrace{\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}}^{n \text{ times}} \tag{22}$$

– $n = 1$

$$J_1^{\alpha,\beta}(z) = \frac{1}{2}(\alpha - \beta + (\alpha + \beta + 2) \cdot z); \tag{23}$$

– $n \geq 2$

$$J_n^{\alpha,\beta}(z) = \sum_{k=2}^{n} \Big[\frac{(2k+\alpha+\beta-1)(\alpha^2-\beta^2)}{2k(k+\alpha+\beta)(2k+\alpha+\beta-2)} + \frac{(2k+\alpha+\beta-2)(2k+\alpha+\beta-1)(2k+\alpha\beta)}{2k(k+\alpha+\beta)(2k+\alpha+\beta-2)} J_{k-1}^{\alpha,\beta}(z) + - \frac{2(k+\alpha-1)(k+\beta-1)(2k+\alpha+\beta)}{2k(k+\alpha+\beta)(2k+\alpha+\beta-2)} J_{k-2}^{\alpha,\beta}(z)\Big] \tag{24}$$

An important property of this polynomials is:

Proposition 1. *$J_i^{\alpha,\beta}(\cdot)$ is orthogonal w.r.t. the Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$:*

$$\int_{-1}^{1} (1-x)^\alpha(1+x)^\beta J_m^{\alpha,\beta} J_q^{\alpha,\beta}(x)dx = \frac{2}{2m+1}\delta_{mq} \tag{25}$$

Thanks to this definition, we can now look more accurately the formula of the Dubiner basis

Definition 4 (Dubiner Basis)**.**

$$\begin{aligned} \phi_{ij}(\xi,\eta) : &= c_{ij}(1-b)^j J_i^{0,0}(a) J_j^{2i+1,0}(b) = \\ &= c_{ij} 2^j (1-\eta)^j J_i^{0,0}(\frac{2\xi}{1-\eta} - 1) J_j^{2i+1,0}(2\eta - 1) \end{aligned} \tag{26}$$

for $i, j = 1, \ldots, p$ and $i + j \leq p$, where

$$c_{ij} := \sqrt{\frac{2(2i+1)(i+j+1)}{4^i}} \tag{27}$$

and $J_i^{\alpha,\beta}(\cdot)$ is the i-th Jacobi polynomial

We note that

Proposition 2. *$\phi_{ij}(\xi,\eta)$ is orthogonal w.r.t. Legendre internal product:*

$$\iint_{\hat{K}} \phi_{ij}(\xi,\eta)\phi_{mq}(\xi,\eta)d\xi d\eta = \delta_{im}\delta_{jq} \tag{28}$$

Indeed, we obtain a mass matrix diagonal as we can see in figure 4

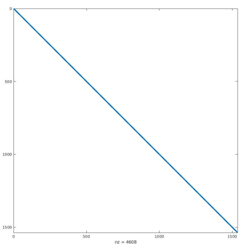

Figure 4: Elements non zero in the mass matrix w.r.t Dubiner basis

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