A HIGH-ORDER DISCONTINUOUS GALERKIN METHOD FOR THE BIDOMAIN PROBLEM OF CARDIAC ELECTROPHYSIOLOGY

Project N° 2

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Course of Numerical Analysis for Partial Differential Equations

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2. Dubiner basis implementation

The physical problem

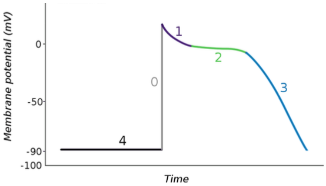
Mechanical contraction of the human heart

$\uparrow$

Electrical activation of the cardiac cells

$\downarrow$

Continuous electrical diffusion over the entire cardiac surface.



The mathematical model

Bidomain model + FitzHugh-Nagumo with Neumann B.C.

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_i \nabla \phi_i) + \chi_m I_{ion}(V_m, w) = I_i^{ext}, & \text{in } \Omega_{mus} \times (0,T], \\ -\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_e \nabla \phi_e) - \chi_m I_{ion}(V_m, w) = -I_e^{ext}, & \text{in } \Omega_{mus} \times (0,T], \\ I_{ion}(V_m, w) = kV_m(V_m - a)(V_m - 1) + w, & \text{in } \Omega_{mus} \times (0,T], \\ \frac{\partial w}{\partial t} = \epsilon(V_m - \gamma w), & \text{in } \Omega_{mus} \times (0,T], \\ \Sigma_i \nabla \phi_i \cdot n = b_i, & \text{on } \partial\Omega_{mus} \times (0,T], \\ \Sigma_e \nabla \phi_e \cdot n = b_e, & \text{on } \partial\Omega_{mus} \times (0,T], \\ \text{Initial conditions for } \phi_i, \phi_e, w, & \text{in } \Omega_{mus} \times \{t = 0\}. \end{cases}$$

Unknowns: $\phi_i,\ \phi_e,\ V_m = \phi_i - \phi_e,\ w$

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Our objectives

What had already been done:

- Implementation of a Discontinuous Galerkin with FEM basis for the Bidomain problem.
- Implementation of a Semi-Implicit temporal scheme.

What we did:

- Implementation of a Discontinuous Galerkin with **Dubiner** basis for the Bidomain problem.
- Implementation of further temporal schemes.
- Bugs corrections and optimizations.
- Pseudo-realistic simulations.

Analytical definition

Definition (Dubiner basis)

The Dubiner basis that generates the space $\mathbb{P}^p(\hat{K})$ of polynomials of degree p over the reference triangle is the set of functions:

$$\phi_{ij} : \hat{K} \to \mathbb{R},$$

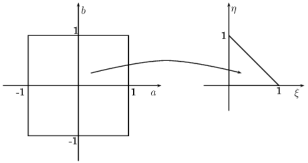
$$\phi_{ij}(\xi,\eta) = c_{ij}\, 2^j (1-\eta)^j J_i^{0,0}\left(\frac{2\xi}{1-\eta} - 1\right) J_j^{2i+1,0}(2\eta - 1),$$

for $i, j = 0, \ldots, p$ and $i + j \leq p$, where $c_{ij} := \sqrt{\frac{2(2i+1)(i+j+1)}{4^i}}$ and $J_i^{\alpha,\beta}(\cdot)$ is the i-th Jacobi polynomial.

Properties

- They consist in a pseudo tensor-product of Jacobi polynomials if the following transformation is then applied:

$$\xi = \frac{(1+a)(1-b)}{4}, \eta = \frac{(1+b)}{2}.$$



- They are $L^2(\hat{K})$ orthonormal ($\hat{K}$ is the reference triangle).

Main works

Remark

Dubiner basis coefficients of a discretized function have ***modal*** *meaning instead of a nodal meaning.*

Then, our main works regarded:

- Methods for the evaluation of the Dubiner functions and gradients in the reference points.
- Methods for the evaluation of the FEM coefficients of a discretized function starting from its Dubiner coefficients and viceversa.
  1. FEM $\rightarrow$ Dubiner is needed when we use the initial solution data into the Dubiner system.
  2. FEM $\leftarrow$ Dubiner is needed when we want to get the solution obtained from the Dubiner system in a comprehensible form.

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FEM-Dubiner conversion strategies

Consider:

- An element $\mathcal{K} \in \tau_h$
- $\{\psi_i\}_{i=1}^p$,$\{\varphi_j\}_{j=1}^q$ as the FEM and Dubiner functions with support in $\mathcal{K}$.
- $\{\hat{u}_i\}_{i=1}^p$,$\{\tilde{u}_j\}_{j=1}^q$ as the FEM and Dubiner coefficients of a function u_h.

FEM $\leftarrow$ Dubiner
Exploiting the nodal meaning of FEM, we compute its value in a point:

$$\hat{u}_i = \sum_{j=1}^{q} \tilde{u}_j \phi_j(x_i),$$

FEM $\rightarrow$ Dubiner
Exploiting the L^2-orthonormality of Dubiner, we compute its Fourier coeff.:

$$\tilde{u}_j = \int_{\mathcal{K}} u_h(x)\varphi_j(x)\,dx = \int_{\mathcal{K}} \sum_{i=1}^{p} \hat{u}_i\psi_i(x)\varphi_j(x)\,dx = \sum_{i=1}^{p} \left(\int_{\mathcal{K}} \psi_i(x)\varphi_j(x)\,dx \right) \hat{u}_i.$$