

Formulario per il progetto NAPDE

Matteo Calafà e Federica Botta

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1 Monodominio

1.1 Modelli analitici

Modello del monodominio

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma \nabla V_m) + \chi_m I_{ion}(V_m, w) = I^{ext} & \text{in } \Omega_{mus} \times (0, T] \\ \frac{\partial w}{\partial t} = g(V_m, w) & \text{in } \Omega_{mus} \times (0, T] \\ \Sigma \nabla V_m \cdot n = b & \text{on } \partial\Omega_{mus} \times (0, T] \end{cases} \quad (1)$$

dove le incognite sono:

- $V_m = \Phi_i - \Phi_e$ (differenza tra potenziale interno e esterno)
- w ("gating variable")

e sono date le costanti : χ_m, C_m, Σ

Modello di FitzHugh-Nagumo

$$\begin{aligned} I_{ion}(V_m, w) &= -kV_m(V_m - a)(V_m - 1) - w \\ g(V_m, w) &= \epsilon(V_m - \gamma w) \end{aligned} \quad (2)$$

1.2 Modello numerico semi-discretizzato

$$\left. \begin{aligned} V_{ij} &= \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \\ I_{i,j}^T &= \sum_{F \in F_h^I} \int_F \{ \{ \nabla \varphi_j \} \} \cdot [[\varphi_i]] \\ I_{i,j} &= \sum_{F \in F_h^I} \int_F [[\varphi_j]] \cdot \{ \{ \nabla \varphi_i \} \} \\ S_{i,j} &= \sum_{F \in F_h^I} \int_F \gamma [[\varphi_j]] \cdot [[\varphi_i]] \end{aligned} \right\} \quad A = \Sigma(V - I^T - \delta I + S) \quad (3)$$

$$M_{ij} = \sum_{K \in \tau_h} \int_K \varphi_j \varphi_i \quad (4)$$

$$C(u_h)_{ij} = - \sum_{K \in \tau_h} \int_K \chi_m k(u_h - 1)(u_h - a) \varphi_j \varphi_i \quad (5)$$

$$F_i = \int_{\Omega} f \varphi_i - \sum_{F \in F_h^B} \int_F b \varphi_i \quad (6)$$

Problema semi-discretizzato

$$\{\varphi_j\}_{j=1}^{N_h} \text{ base di } V_h^p = \{v_h \in L^2 : v_h|_K \in \mathbb{P}^{p_k}(K) \quad p_k \leq p \quad \forall K \in \tau_h\}$$

$$u_h(t) = \sum_{j=1}^{N_h} u_j(t) \varphi_j, \quad w_h(t) = \sum_{j=1}^{N_h} w_j(t) \varphi_j$$

$$\Rightarrow \begin{array}{c} \boxed{\chi_m C_m M \dot{u} + Au + C(u_h)u - \chi_m M w = F} \\ \boxed{\dot{w} = \epsilon(u - \gamma w)} \end{array} \quad (7)$$

1.3 Modello numerico totalmente discretizzato (" θ - method")

Forma implicita ($\theta \in [0, 1]$)

1.

$$\begin{aligned} \chi_m C_m M \frac{u^{k+1} - u^k}{\Delta t} + A(\theta u^{k+1} + (1 - \theta)u^k) + C(u^k)(\theta u^{k+1} + (1 - \theta)u^k) + \\ - \chi_m M w^{k+1} = \theta F^{k+1} + (1 - \theta)F^k \end{aligned} \quad (8)$$

2.

$$\frac{w^{k+1} - w^k}{\Delta t} = \epsilon(u^k - \gamma w^{k+1}) \quad (9)$$

Forma esplicita ($\theta \in [0, 1]$)

1.

$$\begin{aligned} [\chi_m C_m M + \theta \Delta t A + \theta \Delta t C(u^k)] \mathbf{u}^{k+1} = \theta \Delta t F^{k+1} + (1 - \theta) \Delta t F^k + \\ [\chi_m C_m M - (1 - \theta) \Delta t A - (1 - \theta) \Delta t C(u^k)] u^k + \chi_m \Delta t M w^{k+1} \end{aligned} \quad (10)$$

2.

$$[1 + \epsilon \gamma \Delta t] \mathbf{w}^{k+1} = w^k + (\epsilon \Delta t) u^k \quad (11)$$

2 Bidominio

2.1 Modelli analitici

Modello del bidominio

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_i \nabla \phi_i) + \chi_m I_{ion}(V_m, w) = I_i^{ext} & \text{in } \Omega_{mus} \times (0, T] \\ -\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_e \nabla \phi_e) - \chi_m I_{ion}(V_m, w) = -I_e^{ext} & \text{in } \Omega_{mus} \times (0, T] \\ \frac{\partial w}{\partial t} = g(V_m, w) & \text{in } \Omega_{mus} \times (0, T] \\ \Sigma_i \nabla \phi_i \cdot n = b_i & \text{on } \partial\Omega_{mus} \times (0, T] \\ \Sigma_e \nabla \phi_e \cdot n = b_e & \text{on } \partial\Omega_{mus} \times (0, T] \end{cases} \quad (12)$$

dove le incognite sono:

- ϕ_i, ϕ_e ($V_m = \phi_i - \phi_e$)
- w ("gating variable")

e sono date le costanti : $\chi_m, C_m, \Sigma_i, \Sigma_e$

Modello di FitzHugh-Nagumo

$$\begin{aligned} I_{ion}(V_m, w) &= -kV_m(V_m - a)(V_m - 1) - w \\ g(V_m, w) &= \epsilon(V_m - \gamma w) \end{aligned} \quad (13)$$

2.2 Modello numerico semi-discretizzato

$$\left. \begin{aligned} V_{ij} &= \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \\ I_{i,j}^T &= \sum_{F \in F_h^I} \int_F \{ \{ \nabla \varphi_j \} \} \cdot [[\varphi_i]] \\ I_{i,j} &= \sum_{F \in F_h^I} \int_F [[\varphi_j]] \cdot \{ \{ \nabla \varphi_i \} \} \\ S_{i,j} &= \sum_{F \in F_h^I} \int_F \gamma [[\varphi_j]] \cdot [[\varphi_i]] \end{aligned} \right\} \quad \begin{aligned} A &= (V - I^T - \theta I + S) \\ A_i &= \Sigma_i A \\ A_e &= \Sigma_e A \end{aligned} \quad (14)$$

$$M_{ij} = \sum_{K \in \tau_h} \int_K \varphi_j \varphi_i \quad (15)$$

$$C(u_h)_{ij} = - \sum_{K \in \tau_h} \int_K \chi_m k (u_h - 1)(u_h - a) \varphi_j \varphi_i \quad (16)$$

$$\begin{aligned} F_{i,k} &= \int_{\Omega} I_i^{ext} \varphi_k - \sum_{F \in F_h^B} \int_F b_i \varphi_k \\ F_{e,k} &= - \int_{\Omega} I_e^{ext} \varphi_k - \sum_{F \in F_h^B} \int_F b_e \varphi_k \end{aligned} \quad (17)$$

Problema semi-discretizzato

$$\{\varphi_j\}_{j=1}^{N_h} \text{ base di } V_h^k = \{v_h \in L^2 : v_h|_{\mathcal{K}} \in \mathbb{P}^k(\mathcal{K}) \quad \forall \mathcal{K} \in \tau_h\}$$

$$\Phi_h(t) = \begin{bmatrix} \Phi_i^h(t) \\ \Phi_e^h(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N_h} \Phi_{i,j}(t) \varphi_j \\ \sum_{j=1}^{N_h} \Phi_{e,j}(t) \varphi_j \end{bmatrix}, \quad w_h(t) = \sum_{j=1}^{N_h} w_j(t) \varphi_j$$

$$\Rightarrow \begin{bmatrix} \chi_m C_m \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \dot{\Phi}_i^h(t) \\ \dot{\Phi}_e^h(t) \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} \Phi_i^h(t) \\ \Phi_e^h(t) \end{bmatrix} + \\ \begin{bmatrix} C(V_m^h) & -C(V_m^h) \\ -C(V_m^h) & C(V_m^h) \end{bmatrix} \begin{bmatrix} \Phi_i^h(t) \\ \Phi_e^h(t) \end{bmatrix} - \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w_h(t) \\ w_h(t) \end{bmatrix} = \begin{bmatrix} F_i^h \\ F_e^h \end{bmatrix} \end{bmatrix} \quad (18)$$

$$\boxed{\dot{w}_h(t) = \epsilon(V_m^h(t) - \gamma w_h(t))} \quad (19)$$

2.3 Modelli numerici totalmente discretizzati

2.3.1 Metodo semi-implicito

Forma implicita

$$\begin{aligned} & \chi_m C_m \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \frac{\Phi_i^{k+1} - \Phi_i^k}{\Delta t} \\ \frac{\Phi_e^{k+1} - \Phi_e^k}{\Delta t} \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} \Phi_i^{k+1} \\ \Phi_e^{k+1} \end{bmatrix} + \\ & \begin{bmatrix} C(V_m^k) & -C(V_m^k) \\ -C(V_m^k) & C(V_m^k) \end{bmatrix} \begin{bmatrix} \Phi_i^{k+1} \\ \Phi_e^{k+1} \end{bmatrix} - \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} = \begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} \end{aligned} \quad (20)$$

$$\frac{w^{k+1} - w^k}{\Delta t} = \epsilon(V_m^k - \gamma w^{k+1}) \quad (21)$$

Forma esplicita

$$\begin{aligned} & \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} + \begin{bmatrix} C(V_m^k) & -C(V_m^k) \\ -C(V_m^k) & C(V_m^k) \end{bmatrix} \right) \begin{bmatrix} \Phi_i^{k+1} \\ \Phi_e^{k+1} \end{bmatrix} = \\ & \begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} + \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} + \frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \Phi_i^k \\ \Phi_e^k \end{bmatrix} \end{aligned} \quad (22)$$

$$(1 + \epsilon \gamma \Delta t) w^{k+1} = w^k + \epsilon \Delta t V_m^k \quad (23)$$

2.3.2 Operator Splitting implicito

Forma implicita

I

$$\begin{aligned} \chi_m C_m M \frac{\tilde{V}_m^n - V_m^n}{\Delta t} + C(V_m^n) V_m^{n+1} - \chi_m M w^{n+1} &= 0 \\ \frac{\tilde{w}^n - w^n}{\Delta t} &= \epsilon(V_m^{n+1} - \gamma w^{n+1}) \end{aligned} \quad (24)$$

II

$$\begin{aligned} \chi_m C_m M \frac{\hat{V}_m^n - \tilde{V}_m^n}{\Delta t} + A_i \Phi_i^{n+1} &= F_i^{n+1} \\ -\chi_m C_m M \frac{\hat{V}_m^n - \tilde{V}_m^n}{\Delta t} + A_e \Phi_e^{n+1} &= -F_e^{n+1} \end{aligned} \quad (25)$$

III

$$\begin{aligned} \chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^n}{\Delta t} + C(V_m^n) V_m^{n+1} - \chi_m M w^{n+1} &= 0 \\ \frac{w^{n+1} - \tilde{w}^n}{\Delta t} &= \epsilon(V_m^{n+1} - \gamma w^{n+1}) \end{aligned} \quad (26)$$

Forma esplicita

$$\begin{cases} \chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + 2 \cdot C(V_m^n) V_m^{n+1} - 2\chi_m M w^{n+1} + A_i \Phi_i^{n+1} = F_i^{n+1} \\ \chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + 2 \cdot C(V_m^n) V_m^{n+1} - 2\chi_m M w^{n+1} - A_e \Phi_e^{n+1} = F_e^{n+1} \\ \frac{w^{n+1} - w^n}{\Delta t} = 2 \cdot \epsilon(V_m^{n+1} - \gamma w^{n+1}) \end{cases} \quad (27)$$

$$\begin{aligned} \bullet \quad Q_n &:= \frac{\chi_m C_m}{\Delta t} M + 2 \cdot C(V_m^n) - \frac{4\epsilon\chi_m\Delta t}{1+2\epsilon\gamma\Delta t} M \\ \bullet \quad T_n &:= \frac{\chi_m C_m}{\Delta t} M V_m^n + \frac{2\chi_m}{1+2\epsilon\gamma\Delta t} M w^n \end{aligned} \quad (28)$$

1.

$$\begin{aligned} \chi_m C_m M \frac{\Phi_i^{n+1} - \Phi_e^{n+1} - V_m^n}{\Delta t} + 2 \cdot C(V_m^n) (\Phi_i^{n+1} - \Phi_e^{n+1}) + \\ -2\chi_m M \left(\frac{w^n + 2\epsilon\Delta t(\Phi_i^{n+1} - \Phi_e^{n+1})}{1+2\epsilon\gamma\Delta t} \right) + A_i \Phi_i^{n+1} &= F_i^{n+1} \end{aligned} \quad (29)$$

$$\Rightarrow (Q_n + A_i) \Phi_i^{n+1} - Q_n \Phi_e^{n+1} = T_n + F_i^{n+1}$$

2.

$$\begin{aligned} \chi_m C_m M \frac{\Phi_i^{n+1} - \Phi_e^{n+1} - V_m^n}{\Delta t} + 2 \cdot C(V_m^n) (\Phi_i^{n+1} - \Phi_e^{n+1}) + \\ -2\chi_m M \left(\frac{w^n + 2\epsilon\Delta t(\Phi_i^{n+1} - \Phi_e^{n+1})}{1+2\epsilon\gamma\Delta t} \right) - A_e \Phi_e^{n+1} &= F_e^{n+1} \end{aligned} \quad (30)$$

$$\Rightarrow Q_n \Phi_i^{n+1} - (Q_n + A_e) \Phi_e^{n+1} = T_n + F_e^{n+1}$$

3.

$$w^{n+1} = \frac{w^n + 2\epsilon\Delta t(\Phi_i^{n+1} - \Phi_e^{n+1})}{1+2\epsilon\gamma\Delta t} \quad (31)$$

$$\Rightarrow \begin{cases} \left(\begin{bmatrix} Q_n & -Q_n \\ Q_n & -Q_n \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix} \right) \begin{bmatrix} \Phi_i^{n+1} \\ \Phi_e^{n+1} \end{bmatrix} = \begin{bmatrix} T_n \\ T_n \end{bmatrix} + \begin{bmatrix} F_i^{n+1} \\ F_e^{n+1} \end{bmatrix} \\ Q_n = \frac{\chi_m C_m}{\Delta t} M + 2 \cdot C(V_m^n) - \frac{4\epsilon\chi_m\Delta t}{1+2\epsilon\gamma\Delta t} M \\ T_n := \frac{\chi_m C_m}{\Delta t} M V_m^n + \frac{2\chi_m}{1+2\epsilon\gamma\Delta t} M w^n \\ w^{n+1} = \frac{w^n + 2\epsilon\Delta t(\Phi_i^{n+1} - \Phi_e^{n+1})}{1+2\epsilon\gamma\Delta t} \end{cases} \quad (32)$$

2.3.3 Operator Splitting di Godunov

Forma implicita

I

$$\begin{aligned} \chi_m C_m M \frac{\hat{V}_m^n - V_m^n}{\Delta t} + C(V_m^n) V_m^n - \chi_m M w^n &= 0 \\ \frac{w^{n+1} - w^n}{\Delta t} &= \epsilon(V_m^n - \gamma w^n) \end{aligned} \quad (33)$$

II

$$\begin{aligned} \chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^n}{\Delta t} + A_i \Phi_i^{n+1} &= F_i^{n+1} \\ -\chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^n}{\Delta t} + A_e \Phi_e^{n+1} &= -F_e^{n+1} \end{aligned} \quad (34)$$

Forma esplicita

$$\begin{cases} \chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^n - \chi_m M w^n + A_i \Phi_i^{n+1} = F_i^{n+1} \\ \chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^n - \chi_m M w^n - A_e \Phi_e^{n+1} = F_e^{n+1} \\ w^{n+1} = (1 - \epsilon\gamma\Delta t)w^n + \epsilon V_m^n \end{cases}$$

$$\Rightarrow \begin{cases} \left(\frac{\chi_m C_m}{\Delta t} M + A_i \right) \Phi_i^{n+1} - \frac{\chi_m C_m}{\Delta t} M \Phi_e^{n+1} = F_i^{n+1} + \chi_m M w^n + \left(\frac{\chi_m C_m}{\Delta t} M - C(V_m^n) \right) V_m^n \\ \frac{\chi_m C_m}{\Delta t} M \Phi_i^{n+1} - \left(\frac{\chi_m C_m}{\Delta t} M + A_e \right) \Phi_e^{n+1} = F_e^{n+1} + \chi_m M w^n - \left(\frac{\chi_m C_m}{\Delta t} M - C(V_m^n) \right) V_m^n \\ w^{n+1} = (1 - \epsilon\gamma\Delta t)w^n + \epsilon V_m^n \end{cases}$$

$$\Rightarrow \begin{aligned} &\left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ M & -M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix} \right) \begin{bmatrix} \Phi_i^{n+1} \\ \Phi_e^{n+1} \end{bmatrix} = \begin{bmatrix} F_i^{n+1} \\ F_e^{n+1} \end{bmatrix} + \\ &\chi_m \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} w^n \\ w^n \end{bmatrix} \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} C(V_m^n) & 0 \\ 0 & C(V_m^n) \end{bmatrix} \right) \begin{bmatrix} V_m^n \\ V_m^n \end{bmatrix} \end{aligned}$$

$$w^{n+1} = (1 - \epsilon\gamma\Delta t)w^n + \epsilon V_m^n \quad (35)$$