Formulario per il progetto NAPDE

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1 Monodominio

1.1 Modelli analitici

Modello del monodominio

$$\begin{cases} \chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma \nabla V_m) + \chi_m I_{ion}(V_m, w) = I^{ext} & \text{in } \Omega_{mus} \times (0, T] \\ \frac{\partial w}{\partial t} = g(V_\mathrm{m}, w) & \text{in } \Omega_{mus} \times (0, T] \\ \Sigma \nabla V_\mathrm{m} \cdot n = b & \text{on } \partial\Omega_{mus} \times (0, T] \end{cases} \tag{1}$$

dove le incognite sono:

- $V_m = \Phi_i - \Phi_e$ (differenza tra potenziale interno e esterno)
- w ("*gating variable*")

e sono date le costanti : χ_m, C_m, Σ

Modello di FitzHugh-Nagumo

$$\begin{gathered} I_{ion}(V_m, w) = -kV_m(V_m - a)(V_m - 1) - w \\ g(V_m, w) = \epsilon(V_m - \gamma w) \end{gathered} \tag{2}$$

1.2 Modello numerico semi-discretizzato

$$\left.\begin{aligned} V_{ij} &= \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \\ I^T_{i,j} &= \sum_{F \in F_h^I} \int_F \{\{\nabla\varphi_j\}\} \cdot [[\varphi_i]] \\ I_{i,j} &= \sum_{F \in F_h^I} \int_F [[\varphi_j]] \cdot \{\{\nabla\varphi_i\}\} \\ S_{i,j} &= \sum_{F \in F_h^I} \int_F \gamma[[\varphi_j]] \cdot [[\varphi_i]]] \end{aligned}\right\} \quad A = \Sigma(V - I^T - \delta I + S) \tag{3}$$

$$M_{ij} = \sum_{K \in \tau_h} \int_K \varphi_j \varphi_i \tag{4}$$

$$C(u_h)_{ij} = -\sum_{K \in \tau_h} \int_K \chi_m k(u_h - 1)(u_h - a)\varphi_j \varphi_i \tag{5}$$

$$F_i = \int_\Omega f \varphi_i - \sum_{F \in F_h^B} \int_F b \varphi_i \tag{6}$$

Problema semi-discretizzato

$$\{\varphi_j\}_{j=1}^{N_h} \text{ base di } V_h^p = \{v_h \in L^2 : v_h|_K \in \mathbb{P}^{p_k}(K) \quad p_k \leq p \quad \forall K \in \tau_h\}$$
$$u_h(t) = \sum_{j=1}^{N_h} u_j(t)\varphi_j, \quad w_h(t) = \sum_{j=1}^{N_h} w_j(t)\varphi_j$$

$$\Rightarrow \quad \begin{array}{c} \boxed{\chi_m C_m M \dot{u} + Au + C(u_h)u - \chi_m M w = F} \\ \boxed{\dot{w} = \epsilon(u - \gamma w)} \end{array} \tag{7}$$

1.3 Modello numerico totalmente discretizzato ("$\boldsymbol{\theta - method}$")

Forma implicita ($\theta \in [0,1]$)

1.
$$\begin{aligned} \chi_m C_m M \frac{u^{k+1} - u^k}{\Delta t} + A\left(\theta u^{k+1} + (1-\theta)u^k\right) + C(u^k)\left(\theta u^{k+1} + (1-\theta)u^k\right) + \\ -\chi_m M w^{k+1} = \theta F^{k+1} + (1-\theta)F^k \end{aligned} \tag{8}$$

2.
$$\frac{w^{k+1} - w^k}{\Delta t} = \epsilon(u^k - \gamma w^{k+1}) \tag{9}$$

Forma esplicita ($\theta \in [0,1]$)

1.
$$\begin{aligned} \left[\chi_m C_m M + \theta \Delta t A + \theta \Delta t C(u^k)\right] \boldsymbol{u^{k+1}} = \theta \Delta t F^{k+1} + (1-\theta)\Delta t F^k + \\ \left[\chi_m C_m M - (1-\theta)\Delta t A - (1-\theta)\Delta t C(u^k)\right] u^k + \chi_m \Delta t M w^{k+1} \end{aligned} \tag{10}$$

2.
$$[1 + \epsilon\gamma\Delta t]\, \boldsymbol{w^{k+1}} = w^k + (\epsilon \Delta t)u^k \tag{11}$$

2 Bidominio

2.1 Modelli analitici

Modello del bidominio

$$
\begin{cases}
\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_i \nabla \phi_i) + \chi_m I_{ion}(V_m, w) = I_i^{ext} & \text{in } \Omega_{mus} \times (0,T] \\
-\chi_m C_m \frac{\partial V_m}{\partial t} - \nabla \cdot (\Sigma_e \nabla \phi_e) - \chi_m I_{ion}(V_m, w) = -I_e^{ext} & \text{in } \Omega_{mus} \times (0,T] \\
\frac{\partial w}{\partial t} = g(V_\mathrm{m}, w) & \text{in } \Omega_{mus} \times (0,T] \\
\Sigma_i \nabla \phi_i \cdot n = b_i & \text{on } \partial\Omega_{mus} \times (0,T] \\
\Sigma_e \nabla \phi_e \cdot n = b_e & \text{on } \partial\Omega_{mus} \times (0,T]
\end{cases}
\tag{12}
$$

dove le incognite sono:

- $\phi_i, \phi_e \quad (V_m = \phi_i - \phi_e)$
- w ("*gating variable*")

e sono date le costanti : $\chi_m, C_m, \Sigma_i, \Sigma_e$

Modello di FitzHugh-Nagumo

$$
\begin{gathered}
I_{ion}(V_m, w) = -kV_m(V_m - a)(V_m - 1) - w \\
g(V_m, w) = \epsilon(V_m - \gamma w)
\end{gathered}
\tag{13}
$$

2.2 Modello numerico semi-discretizzato

$$
\left.
\begin{aligned}
V_{ij} &= \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \\
I_{i,j}^T &= \sum_{F \in F_h^I} \int_F \{\{\nabla\varphi_j\}\} \cdot [[\varphi_i]] \\
I_{i,j} &= \sum_{F \in F_h^I} \int_F [[\varphi_j]] \cdot \{\{\nabla\varphi_i\}\} \\
S_{i,j} &= \sum_{F \in F_h^I} \int_F \gamma [[\varphi_j]] \cdot [[\varphi_i]]
\end{aligned}
\right\}
\quad
\begin{gathered}
A = (V - I^T - \theta I + S) \\
A_i = \Sigma_i A \\
A_e = \Sigma_e A
\end{gathered}
\tag{14}
$$

$$
M_{ij} = \sum_{K \in \tau_h} \int_K \varphi_j \varphi_i \tag{15}
$$

$$
C(u_h)_{ij} = -\sum_{K \in \tau_h} \int_K \chi_m k (u_h - 1)(u_h - a)\varphi_j \varphi_i \tag{16}
$$

$$
\begin{gathered}
F_{i,k} = \int_\Omega I_i^{ext} \varphi_k - \sum_{F \in F_h^B} \int_F b_i \varphi_k \\
F_{e,k} = -\int_\Omega I_e^{ext} \varphi_k - \sum_{F \in F_h^B} \int_F b_e \varphi_k
\end{gathered}
\tag{17}
$$

Problema semi-discretizzato

$$\{\varphi_j\}_{j=1}^{N_h} \text{ base di } V_h^k = \{v_h \in L^2 : v_h|_{\mathcal{K}} \in \mathbb{P}^k(\mathcal{K}) \quad \forall \mathcal{K} \in \tau_h\}$$
$$\Phi_h(t) = \begin{bmatrix} \Phi_i^h(t) \\ \Phi_e^h(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N_h} \Phi_{i,j}(t)\varphi_j \\ \sum_{j=1}^{N_h} \Phi_{e,j}(t)\varphi_j \end{bmatrix}, \quad w_h(t) = \sum_{j=1}^{N_h} w_j(t)\varphi_j$$

$$\Rightarrow \boxed{\begin{gathered} \chi_m C_m \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \boldsymbol{\dot{\Phi}_i^h(t)} \\ \boldsymbol{\dot{\Phi}_e^h(t)} \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi_i^h(t)} \\ \boldsymbol{\Phi_e^h(t)} \end{bmatrix} + \\ \begin{bmatrix} C(V_m^h) & -C(V_m^h) \\ -C(V_m^h) & C(V_m^h) \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi_i^h(t)} \\ \boldsymbol{\Phi_e^h(t)} \end{bmatrix} - \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w_h(t) \\ w_h(t) \end{bmatrix} = \begin{bmatrix} F_i^h \\ F_e^h \end{bmatrix} \end{gathered}} \tag{18}$$

$$\boxed{\dot{w}_h(t) = \epsilon(V_m^h(t) - \gamma w_h(t))} \tag{19}$$

2.3 Modelli numerici totalmente discretizzati

2.3.1 Metodo semi-implicito

Forma implicita

$$\begin{gathered} \chi_m C_m \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \frac{\Phi_i^{k+1} - \Phi_i^k}{\Delta t} \\ \frac{\Phi_e^{k+1} - \Phi_e^k}{\Delta t} \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} \Phi_i^{k+1} \\ \Phi_e^{k+1} \end{bmatrix} + \\ \begin{bmatrix} C(V_m^k) & -C(V_m^k) \\ -C(V_m^k) & C(V_m^k) \end{bmatrix} \begin{bmatrix} \Phi_i^{k+1} \\ \Phi_e^{k+1} \end{bmatrix} - \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} = \begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} \end{gathered} \tag{20}$$

$$\frac{w^{k+1} - w^k}{\Delta t} = \epsilon(V_m^k - \gamma w^{k+1}) \tag{21}$$

Forma esplicita

$$\begin{gathered} \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & A_e \end{bmatrix} + \begin{bmatrix} C(V_m^k) & -C(V_m^k) \\ -C(V_m^k) & C(V_m^k) \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\Phi_i^{k+1}} \\ \boldsymbol{\Phi_e^{k+1}} \end{bmatrix} = \\ \begin{bmatrix} F_i^{k+1} \\ F_e^{k+1} \end{bmatrix} + \chi_m \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} w^{k+1} \\ w^{k+1} \end{bmatrix} + \frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ -M & M \end{bmatrix} \begin{bmatrix} \Phi_i^k \\ \Phi_e^k \end{bmatrix} \end{gathered} \tag{22}$$

$$(1 + \epsilon\gamma\Delta t)w^{k+1} = w^k + \epsilon\Delta t V_m^k \tag{23}$$

2.3.2 Operator Splitting quasi-implicito

Forma implicita

I

$$\begin{aligned} \chi_m C_m M \frac{\tilde{V}_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^{n+1} - \chi_m M w^{n+1} = 0 \\ \frac{w^{n+1} - w^n}{\Delta t} = \epsilon (V_m^{n+1} - \gamma w^{n+1}) \end{aligned} \tag{24}$$

II

$$\begin{aligned} \chi_m C_m M \frac{V_m^{n+1} - \tilde{V}_m^{n+1}}{\Delta t} + A_i \Phi_i^{n+1} = F_i^{n+1} \\ -\chi_m C_m M \frac{V_m^{n+1} - \tilde{V}_m^{n+1}}{\Delta t} + A_e \Phi_e^{n+1} = F_e^{n+1} \end{aligned} \tag{25}$$

Forma esplicita

$$\begin{cases} \chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^{n+1} - \chi_m M w^{n+1} + A_i \Phi_i^{n+1} = F_i^{n+1} \\ \chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^{n+1} - \chi_m M w^{n+1} - A_e \Phi_e^{n+1} = -F_e^{n+1} \\ \frac{w^{n+1} - w^n}{\Delta t} = \epsilon (V_m^{n+1} - \gamma w^{n+1}) \end{cases} \tag{26}$$

$$\begin{aligned} \bullet \quad Q_n := \frac{\chi_m C_m}{\Delta t} M + C(V_m^n) - \frac{\epsilon \chi_m \Delta t}{1 + \epsilon \gamma \Delta t} M \\ \bullet \quad R_n := \frac{\chi_m C_m}{\Delta t} M V_m^n + \frac{\chi_m}{1 + \epsilon \gamma \Delta t} M w^n \end{aligned} \tag{27}$$

1.

$$\begin{aligned} \chi_m C_m M \frac{\Phi_i^{n+1} - \Phi_e^{n+1} - V_m^n}{\Delta t} + C(V_m^n)(\Phi_i^{n+1} - \Phi_e^{n+1}) + \\ -\chi_m M \left(\frac{w^n + \epsilon \Delta t (\Phi_i^{n+1} - \Phi_e^{n+1})}{1 + \epsilon \gamma \Delta t} \right) + A_i \Phi_i^{n+1} = F_i^{n+1} \end{aligned} \tag{28}$$

$$\Rightarrow \quad (Q_n + A_i) \Phi_i^{n+1} - Q_n \Phi_e^{n+1} = R_n + F_i^{n+1}$$

2.

$$\begin{aligned} \chi_m C_m M \frac{\Phi_i^{n+1} - \Phi_e^{n+1} - V_m^n}{\Delta t} + \cdot C(V_m^n)(\Phi_i^{n+1} - \Phi_e^{n+1}) + \\ -\chi_m M \left(\frac{w^n + \epsilon \Delta t (\Phi_i^{n+1} - \Phi_e^{n+1})}{1 + \epsilon \gamma \Delta t} \right) - A_e \Phi_e^{n+1} = -F_e^{n+1} \end{aligned} \tag{29}$$

$$\Rightarrow \quad Q_n \Phi_i^{n+1} - (Q_n + A_e) \Phi_e^{n+1} = R_n - F_e^{n+1}$$

3.

$$w^{n+1} = \frac{w^n + \epsilon \Delta t (\Phi_i^{n+1} - \Phi_e^{n+1})}{1 + \epsilon \gamma \Delta t} \tag{30}$$

$$
\Rightarrow \begin{cases}
\left(\begin{bmatrix} Q_n & -Q_n \\ Q_n & -Q_n \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{\Phi_i^{n+1}} \\ \boldsymbol{\Phi_e^{n+1}} \end{bmatrix} = \begin{bmatrix} R_n \\ R_n \end{bmatrix} + \begin{bmatrix} F_i^{n+1} \\ -F_e^{n+1} \end{bmatrix} \\
Q_n = \frac{\chi_m C_m}{\Delta t} M + \cdot C(V_m^n) - \frac{\epsilon \chi_m \Delta t}{1 + \epsilon\gamma\Delta t} M \\
R_n := \frac{\chi_m C_m}{\Delta t} M V_m^n + \frac{\chi_m}{1 + \epsilon\gamma\Delta t} M w^n \\
w^{n+1} = \frac{w^n + \epsilon\Delta t(\Phi_i^{n+1} - \Phi_e^{n+1})}{1 + \epsilon\gamma\Delta t}
\end{cases} \tag{31}
$$

2.3.3 Operator Splitting di Godunov

Forma implicita

I

$$
\begin{aligned}
\chi_m C_m M \frac{\hat{V}_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^n - \chi_m M w^n = 0 \\
\frac{w^{n+1} - w^n}{\Delta t} = \epsilon(V_m^n - \gamma w^n)
\end{aligned} \tag{32}
$$

II

$$
\begin{aligned}
\chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^{n+1}}{\Delta t} + A_i \Phi_i^{n+1} = F_i^{n+1} \\
-\chi_m C_m M \frac{V_m^{n+1} - \hat{V}_m^{n+1}}{\Delta t} + A_e \Phi_e^{n+1} = F_e^{n+1}
\end{aligned} \tag{33}
$$

Forma esplicita

$$
\begin{cases}
\chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^n - \chi_m M w^n + A_i \Phi_i^{n+1} = F_i^{n+1} \\
\chi_m C_m M \frac{V_m^{n+1} - V_m^n}{\Delta t} + C(V_m^n) V_m^n - \chi_m M w^n - A_e \Phi_e^{n+1} = -F_e^{n+1} \\
w^{n+1} = (1 - \epsilon\gamma\Delta t) w^n + \epsilon\Delta t V_m^n
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\left(\frac{\chi_m C_m}{\Delta t} M + A_i\right) \Phi_i^{n+1} - \frac{\chi_m C_m}{\Delta t} M \Phi_e^{n+1} = F_i^{n+1} + \chi_m M w^n + \left(\frac{\chi_m C_m}{\Delta t} M - C(V_m^n)\right) V_m^n \\
\frac{\chi_m C_m}{\Delta t} M \Phi_i^{n+1} - \left(\frac{\chi_m C_m}{\Delta t} M + A_e\right) \Phi_e^{n+1} = -F_e^{n+1} + \chi_m M w^n + \left(\frac{\chi_m C_m}{\Delta t} M - C(V_m^n)\right) V_m^n \\
w^{n+1} = (1 - \epsilon\gamma\Delta t) w^n + \epsilon\Delta t V_m^n
\end{cases} \tag{34}
$$

$$
\Rightarrow \begin{cases}
\left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & -M \\ M & -M \end{bmatrix} + \begin{bmatrix} A_i & 0 \\ 0 & -A_e \end{bmatrix}\right) \begin{bmatrix} \boldsymbol{\Phi_i^{n+1}} \\ \boldsymbol{\Phi_e^{n+1}} \end{bmatrix} = \begin{bmatrix} F_i^{n+1} \\ -F_e^{n+1} \end{bmatrix} + \\
\chi_m \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} w^n \\ w^n \end{bmatrix} + \left(\frac{\chi_m C_m}{\Delta t} \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} C(V_m^n) & 0 \\ 0 & C(V_m^n) \end{bmatrix}\right) \begin{bmatrix} V_m^n \\ V_m^n \end{bmatrix} \\
w^{n+1} = (1 - \epsilon\gamma\Delta t) w^n + \epsilon\Delta t V_m^n
\end{cases} \tag{35}
$$

3 Basis

Transformation On the reference triangle

$$\hat{K} = \{(\xi,\eta) : \xi,\eta \geq 0, \xi+\eta \leq 1\} \tag{36}$$

we consider the transformation between the reference square and the reference triangle given by

$$\xi := \frac{(1+a)(1-b)}{4}, \eta := \frac{(1+b)}{2} \tag{37}$$

and the inverse transformation is

$$a := \frac{2\xi - 1 + \eta}{1-\eta} = \frac{2\xi}{1-\eta} - 1, b := 2\eta - 1 \tag{38}$$

Dubiner Basis

$$\begin{aligned}\phi_{ij}(\xi,\eta) :&= c_{ij}(1-b)^j J_i^{0,0}(a) J_j^{2i+1,0}(b) = \\ &= c_{ij}2^j(1-\eta)^j J_i^{0,0}(\frac{2\xi}{1-\eta} - 1) J_j^{2i+1,0}(2\eta-1)\end{aligned} \tag{39}$$

for i,j=1,....,p and i+j $\leq$ p, where

$$c_{ij} := \sqrt{\frac{2(2i+1)(i+j+1)}{4^i}} \tag{40}$$

and $\mathrm{J}_i{}^{\alpha,\beta}(.)$ is the i-th Jacobi polynomial

Jacobian Basis $\mathrm{J}_i{}^{\alpha,\beta}(.)$ is orthogonal under the Jacobi weight $\mathrm{w(x)=(1\text{-}x)}^\alpha(1+\mathrm{x})^\beta$ i.e.

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta J_m^{\alpha,\beta} J_q^{\alpha,\beta}(x) dx = \frac{2}{2m+1}\delta_{mq} \tag{41}$$

Evaluate the basis for a vector z of dimension n:

$$J_0^{\alpha,\beta}(z) = ones(1,n) \tag{42}$$

$$J_1^{\alpha,\beta}(z) = \frac{1}{2}(\alpha - \beta + (\alpha+\beta+2) * z); \tag{43}$$

for n $\geq$ 2

$$\begin{aligned}J_n^{\alpha,\beta}(z) = &\sum_{k=2}^{n}(\frac{(2k+\alpha+\beta-1)(\alpha^2-\beta^2)}{2k(k+\alpha+\beta)(2k+\alpha+\beta-2)} + \frac{(2k+\alpha+\beta-2)(2k+\alpha+\beta-1)(2k+\alpha+\beta)}{2k(k+\alpha+\beta)(2k+\alpha+\beta-2)}) J_{k-1}^{\alpha,\beta}(z) \\ &- \frac{2(k+\alpha-1)(k+\beta-1)(2k+\alpha+\beta)}{2k(k+\alpha+\beta)(2k+\alpha+\beta-2)} J_{k-2}^{\alpha,\beta}(z)\end{aligned} \tag{44}$$

Gradient of Dubiner Basis for i=0 and j=0

$$\begin{aligned}\phi_{00}^{\xi}(\xi,\eta) = 0 \\ \phi_{00}^{\eta}(\xi,\eta) = 0\end{aligned} \tag{45}$$

for i=0 and j$\neq$0

$$\begin{aligned}\phi_{0j}^{\xi} &= 0 \\ \phi_{0j}^{\eta} &= c_{0j}(j+2) J_{j-1}^{2,1}(b)\end{aligned} \tag{46}$$

for i$\neq$0 and j=0

$$\begin{aligned}
\phi^{\xi}_{i0}(\xi,\eta) &= c_{i0}2^{i}(1-\eta)^{i-1}(i+1)J^{1,1}_{i-1}(a) \\
\phi^{\eta}_{i0}(\xi,\eta) &= c_{i0}2^{i}(-i(1-\eta)^{i-1}J^{0,0}_{i}(a)+\xi(1-\eta)^{i-2}(i+1)J^{1,1}_{i-1}(a))
\end{aligned} \tag{47}$$

for i$\neq$0 and j$\neq$0

$$\begin{aligned}
&\phi^{\xi}_{ij}(\xi,\eta) = c_{ij}2^{i}(1-\eta)^{i-1}(i+1)J^{1,1}_{i-1}(a)J^{2i+1,0}_{j}(b) \\
&\phi^{\eta}_{ij}(\xi,\eta) = c_{ij}2^{i}(-i(1-\eta)^{i-1}J^{0,0}_{i}(a)J^{2i+1,0}_{j}(b)+\xi(1-\eta)^{i-2}(i+1)J^{1,1}_{i-1}(a)J^{2i+1,0}_{j}(b) \\
&+(1-\eta)^{i}(2i+j+2)J^{0,0}_{i}(a)J^{2i+2,1}_{j-1}(b))
\end{aligned} \tag{48}$$