25.03.2021

SEMINAR 6

Aplicații liniare. Matricea asociată

OBS • $f: V_1 \to V_2$ liniară

$$R_1 = \{e_1, \dots, e_n\} \xrightarrow{A} R_2 = \{\overline{e}_1, \dots, \overline{e}_m\}$$

reper în V_1 reper în V_2.

$$f(e_i) = \sum_{j=1}^{m} a_{ji} \overline{e}_j, \ \forall i = \overline{1,n}, \quad A = (a_{ji})_{\substack{j=\overline{1,m} \\ i=\overline{1,n}}} \in \mathcal{M}_{m,n}(\mathbb{K})$$

• f aplicație liniară $\Leftrightarrow \exists A \in \mathcal{M}_{m,n}(\mathbb{K})$ a.î. $Y = AX$

$$f(x) = y, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad x = \sum_{i=1}^{n} x_i e_i$$

$$y = \sum_{j=1}^{m} y_j \overline{e}_j$$

T.dim $f: V_1 \to V_2$ liniară

$$\dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

OBS $f(V_1) \subseteq V_2$ subsp vect, $\dim f(V_1) \leq \dim V_1$

• f inj $\Leftrightarrow \dim V_1 = \operatorname{rg} A$

• f surj $\Leftrightarrow \dim V_2 = \operatorname{rg} A$

• f bij $\Leftrightarrow \dim V_1 = \dim V_2 = \operatorname{rg} A \Leftrightarrow A \in GL(n, \mathbb{K})$

$\Leftrightarrow [\forall R$ reper în $V_1 \Rightarrow f(R)$ reper în $V_2]$

✓ Ex1

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f(x_1,x_2,x_3) = (x_1+2x_2+x_3, -x_1-2x_2-x_3, x_1+x_2+x_3)$

a) $[f]_{R_0,R_0} = A$, $R_0 = \{e_1,e_2,e_3\}$ reperul canonic din $\mathbb{R}^3$.

b) $\dim \operatorname{Ker} f$, $\dim \operatorname{Im} f$

c) $f(V')$, $V' = \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \begin{cases} x_1-x_2+x_3=0 \\ x_1+2x_2-x_3=0 \end{cases}\right\}$

SOL

a) $f(e_1) = f(1,0,0) = (1,-1,1) = e_1 - e_2 + e_3$

$f(e_2) = f(0,1,0) = (2,-2,1) = 2e_1 - 2e_2 + e_3$

$f(e_3) = f(0,0,1) = (1,-1,1) = e_1 - e_2 + e_3$

$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$f(x) = y \Leftrightarrow Y = AX \Leftrightarrow \begin{pmatrix} x_1+2x_2+x_3 \\ -x_1-2x_2-x_3 \\ x_1+x_2+x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Leftrightarrow$ f liniară

b) $\operatorname{Ker} f = \{x\in\mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = \{x\in\mathbb{R}^3 \mid AX = 0\} = S(A)$

$\dim \operatorname{Ker} f = 3 - \operatorname{rg} A = 3 - 2 = 1$

$\det A = 0$

$\dim \mathbb{R}^3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$

c) $V' = \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid \begin{cases} x_1-x_2+x_3=0 \\ x_1+2x_2-x_3=0 \end{cases}\right\}$

$A_{V'} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$

$\begin{cases} x_1 - x_2 = -x_3 \\ x_1 + 2x_2 = x_3 \end{cases}$

$/\ 3x_2 = 2x_3$

$x_2 = \frac{2}{3}x_3$

$x_1 = \frac{2}{3}x_3 - x_3 = \frac{-x_3}{3}$

$V' = \left\{\left(-\frac{x_3}{3}, \frac{2}{3}x_3, x_3\right) = \frac{x_3}{3}(-1,2,3),\ x_3\in\mathbb{R}\right\} = \langle\{(-1,2,3)\}\rangle$

(dreaptă)

$R' = \{(-1,2,3)\}$ reper în V'

$f(-1,2,3) = (6,-6,4) \neq 0_{\mathbb{R}^3}$, $f(V') = \langle\{(6,-6,4)\}\rangle$

Ex 2

$f: \mathbb{R}^3 \to \mathbb{R}^3$,

$f(x_1, x_2, x_3) = (2x_1 + 2x_2, x_1 + x_3, x_1 + 3x_2 - 2x_3)$

a) f liniară, dar nu e izomorfism

b) $f/_{V'}: V' \to V''$ izomorfism,

$V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

$V'' = \{x \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$

c) $f(V' \cap V'')$

SOL

a) $f(x) = y \Leftrightarrow Y = AX$, $A = [f]_{R_0, R_0} \Leftrightarrow f$ liniară

$f(e_1) = f(1,0,0) = (2,1,1) = 2e_1 + e_2 + e_3$

$f(e_2) = f(0,1,0) = (2,0,3) = 2e_1 + 0e_2 + 3e_3$, $A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$

$f(e_3) = f(0,0,1) = (0,1,-2) = 0e_1 + e_2 - 2e_3$

SAU $y = f(x) \Leftrightarrow Y = AX$

$$\begin{pmatrix} 2x_1 + 2x_2 \\ x_1 + x_3 \\ x_1 + 3x_2 - 2x_3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow A \notin GL(3, \mathbb{R})$$

$\Rightarrow f$ nu e bijectivă

OBS • $\ker f = \{x \in \mathbb{R}^3 \mid AX = 0\}$

$\dim \ker f = 3 - rg A = 3 - 2 = 1 \Rightarrow f$ nu e inj

• $\dim \mathbb{R}^3 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$

$\Rightarrow f$ nu e surj

b) $f|_{V'} : V' \longrightarrow V''$ liniară

$V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$ $\quad A_{V'} = (1\ 1\ -1)$

$V'' = \{x \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$.

f bij $\Leftrightarrow [\forall \underset{\text{reper}}{R'}$ din $V' \Rightarrow f(R')$ reper în $V'']$.

$V' : x_3 = x_1 + x_2$

$V' = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\} = \langle \underbrace{\{(1,0,1), (0,1,1)\}}_{R'} \rangle$

$(x_1, x_2, x_1+x_2) = (x_1, 0, x_1) + (0, x_2, x_2) = x_1(1,0,1) + x_2(0,1,1)$

$\left.\begin{array}{l} \Rightarrow R' \text{ este SG pt } V' \\ \dim V' = 3 - \text{rg} A_{V'} = 3 - 1 = 2 = |R'| \end{array}\right\} \Rightarrow R'$ reper în V'

$f(R') = \{f(1,0,1), f(0,1,1)\}$.

$f(1,0,1) = (2,2,-1) \in V''$; $3 \cdot 2 - 4 \cdot 2 - 2(-1) = 8 - 8 = 0$

$f(0,1,1) = (2,1,1) \in V''$; $3 \cdot 2 - 4 \cdot 1 - 2 \cdot 1 = 6 - 6 = 0$

$V'' : 3x_1 - 4x_2 - 2x_3 = 0$

Dem că $f(R')$ este reper în V''

$\left.\begin{array}{l} \text{rg}\begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \max \Rightarrow f(R') \text{ este SLI} \\ \dim V'' = 3 - \text{rg} A_{V''} = 3 - 1 = 2 = |f(R')| \end{array}\right\} \Rightarrow f(R')$ reper în V''

$\left.\begin{array}{l} \text{Deci } f \text{ bij} \\ \text{dar } f \text{ lin} \end{array}\right\} \Rightarrow f|_{V'} : V' \longrightarrow V''$ izom. de sp. vect.

c) $f(V' \cap V'')$ $\quad V' \cap V'' = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases}\right\}$

$A_{V' \cap V''} = \left(\begin{array}{cc|c} \boxed{\begin{matrix} 1 & 1 \\ 3 & -4 \end{matrix}} & \begin{matrix} -1 \\ -2 \end{matrix} \end{array}\right)$

$\begin{cases} x_1 + x_2 = x_3 \quad \cdot 4 \\ 3x_1 - 4x_2 = 2x_3 \end{cases}$ $\oplus$

$7x_1 \quad = 6x_3$

$x_1 = \frac{6}{7} x_3$

$x_2 = x_3 - \frac{6}{7} x_3 = \frac{1}{7} x_3$

$V' \cap V'' = \left\{ \left(\frac{6}{7}x_3, \frac{1}{7}x_3, x_3 \right) = \frac{x_3}{7}(6,1,7),\ x_3 \in \mathbb{R} \right\} = \langle \{(6,1,7)\} \rangle$

$f(6,1,7) = (14,13,-5) \neq 0_{\mathbb{R}^3}$

$f(V' \cap V'') = \langle \{(14,13,-5)\} \rangle$ (o dreaptă).

✓ Ex

$f : \mathbb{R}_3[X] \longrightarrow \mathbb{R}_2[X]$, $f(P) = P'$

a) $[f]_{R_0, R_0'}$, $R_0 = \{1, X, X^2, X^3\}$ reper canonic în $\mathbb{R}_3[X]$

$R_0' = \{1, X, X^2\}$ — " — $\mathbb{R}_2[X]$

b) dim Ker f, dim Im f.

Sol

a) $f(1) = 0 = 0\cdot 1 + 0\cdot X + 0\cdot X^2$; $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$f(X) = 1 = 1\cdot 1 + 0\cdot X + 0\cdot X^2$;

$f(X^2) = 2X = 0\cdot 1 + 2\cdot X + 0\cdot X^2$

$f(X^3) = 3X^2 = 0\cdot 1 + 0\cdot X + 3\cdot X^2$

b) $\mathrm{Ker} f = \{ P \in \mathbb{R}_3[X] \mid P' = 0 \} = \mathbb{R}$

$\dim \mathrm{Ker} f = 4 - 3 = 1$

$\underbrace{\dim \mathbb{R}_3[X]}_{4} = \underbrace{\dim \mathrm{Ker} f}_{1} + \dim \mathrm{Im} f \Rightarrow \dim \mathrm{Im} f = 3$

f nu e inj; este surj.

OBS

$\mathrm{Ker} f = \langle \{1\} \rangle$

Extindem la $\{1, \underline{X, X^2, X^3}\}$ reper în $\mathbb{R}_3[X]$

$\{ \underbrace{f(X)}_{1}, \underbrace{f(X^2)}_{2X}, \underbrace{f(X^3)}_{3X^2} \}$ reper în Im f

(din dem Th dimensiunii)

T_3 (seminar)

① $f: \mathbb{R}^2 \to \mathbb{R}^2$ liniară

$f(1,1) = (3,5)$; $f(-1,2) = (0,1)$

a) $f(x) = ?$

b) Este izomorfism ?

② $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_3, 0, 2x_1 + x_2)$

a) $[f]_{R_0, R_0}$, $R_0 = \{e_1, e_2, e_3\}$ reper canonic în $\mathbb{R}^3$

b) $\operatorname{Ker} f$, $\operatorname{Im} f$, dim și câte un reper

③ $V' = \langle \{(1,1,1,0), (1,2,3,4)\} \rangle \subset \mathbb{R}^4$

a) Scrieți un sistem de ec. liniare care îl definesc pe V'

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$

c) $p: \mathbb{R}^4 \to \mathbb{R}^4$, p = proiecția pe V' de-a lg lui V''

$p(1,2,0,-1) = ?$

$s(1,2,0,-1) = ?$ $s = 2p - id_{\mathbb{R}^4}$.