Exercițiul 4

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \dfrac{x^9 y^6}{x^{14} + y^{14}}, & x^2 + y^2 \neq 0 \\ 0, \text{ dacă } x = y = 0 \end{cases}$$

a) continuitatea lui $f(x,y)$

$f(x,y)$ este continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ deoarece este compunere de f. elementar[e]

Studiem în $(0,0)$:

$$y = x \Rightarrow \lim_{x \to 0} f(x,x) = \lim_{x \to 0} \frac{x^{15}}{2x^{14}} \; (x^{14} = \lim_{x \to 0} \frac{x}{2} = 0$$

$$y = -x \Rightarrow \lim_{x \to 0} f(x,-x) = \lim_{x \to 0} \frac{x^9 (-x)^6}{x^{14} + (-x)^{14}} =$$

$$= \lim_{x \to 0} \frac{x^{15}}{2x^{14}} \; (x^{14} = \lim_{x \to 0} \frac{x}{2} = 0$$

$$y = x^2 \Rightarrow \lim_{x \to 0} f(x,x^2) = \lim_{x \to 0} \frac{x^9 (x^2)^6}{x^{14} + (x^2)^{14}} =$$

$$= \lim_{x \to 0} \frac{x^{21}}{x^{14} + x^{28}} \; (x^{14} = \lim_{x \to 0} \frac{x^7}{1 + x^{14}} = \frac{0}{1+0} = 0$$

Cum toate limitele sunt egale vrem să demonstrăm că $\lim\limits_{(x,y) \to (0,0)} f(x) = 0$

Evaluăm $|f(x,y) - 0| = |f(x,y)| = \left| \dfrac{x^9 y^6}{x^{14} + y^{14}} \right| =$

$$= \frac{|x|^9 y^6}{x^{14} + y^{14}}$$

$$(x^6 - y^6)^2 \geq 0 \iff x^{12} + y^{12} - 2x^6 y^6 \geq 0 \Rightarrow$$

$$\Rightarrow x^{12} + y^{12} \geq 2x^6 y^6$$

$$|f(x,y)| = \frac{2\cdot|x|^9 y^6}{2\cdot x^{14}+y^{14}} \le \frac{(x^{14}+\cancel{y^{14}})^{1}\, x^2\cdot \frac{1}{y}}{2\,(x^{14}+\cancel{y^{14}})_{1}} \Rightarrow$$

$$x^{14}+y^{14} \geq 2x^7\cdot y^7$$

$$\Rightarrow 0 \le |f(x,y)| \le \frac{x^2}{2y}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{2y} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{(x,y)\to(0,0)} \frac{\cancel{2}^{1}x}{\cancel{2}_{1}} = 0$$

$$\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} |f(x,y)| \le \lim_{(x,y)\to(0,0)} \frac{x^2}{2y}$$

$$0$$

criteriul cleștelui

$$\left.\begin{array}{l} \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0 \\ f(0,0) = 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ f continuă în $(0,0)$

$\left.\begin{array}{l} f \text{ continuă în } (0,0) \\ f \text{ continuă pe } \mathbb{R}^2\setminus\{(0,0)\} \end{array}\right\} \Rightarrow$

$\Rightarrow$ f continuă pe $\mathbb{R}^2$

b) $\frac{\partial f}{\partial x} = \left(\frac{x^9 y^6}{x^{14}+y^{14}} \right)'_x = \frac{9x^8y^6(x^{14}+y^{14}) - x^9y^6 \cdot 14 \cdot x^{13}}{(x^{14}+y^{14})^2}$

$\frac{\partial f}{\partial x} \overset{not}{=} g(x,y),\ g: \mathbb{R}^2 \to \mathbb{R}$

$g(x,y)$ continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$g(x,y) = \frac{9x^8y^6(x^{14}+y^{14}) - 14x^{22}y^6}{(x^{14}+y^{14})^2}$$

Studiem continuitatea în $(0,0)$

$y = x \Rightarrow \lim\limits_{x\to 0} g(x,x) = \lim\limits_{x\to 0} \frac{9x^{14} \cdot 2 \cdot x^{14} - 14 \cdot x^{28}}{(2x^{14})^2} =$

$= \lim\limits_{x\to 0} \frac{18 \cdot x^{28} - 14x^{28}}{4x^{28}} = \lim\limits_{x\to 0} \frac{4x^{28}}{4x^{28}} = 1$

$y = -x \Rightarrow \lim\limits_{x\to 0} g(x,-x) = \lim\limits_{x\to 0} \frac{18 \cdot x^{28} - 14x^{28}}{4x^{28}} = 1$

$y = x^2 \Rightarrow \lim\limits_{x\to 0} g(x,x^2) = \lim\limits_{x\to 0} \frac{9 \cdot x^{20} \cancel{(x^{14}+x^{28})}}{(x^{14}+x^{28})^{\cancel{2}}} -$

$- \frac{14x^{22} \cdot x^{12}}{(x^{14}+x^{28})^2} = \lim\limits_{x\to 0} \frac{9 \cdot x^{20}}{x^{14}+x^{28}} - \frac{14 \cdot x^{34}}{(x^{14}+x^{28})^2} =$

$= \lim\limits_{x\to 0} \underbrace{\frac{9 \cdot \cancel{x^6}}{1+x^{14}}}_{0} - \frac{14 \cdot x^{34}}{(x^{14}+x^{28})^2} = \lim\limits_{x\to 0} - \frac{14 \cdot x^{34}}{x^{28}+2 \cdot x^{42}+x^{56}}$

$= \lim\limits_{x\to 0} - \frac{14 \cdot x^6}{1+2 \cdot x^{14}+x^{28}} = - \frac{0 \cdot 14}{1+0+0} = 0$

$\frac{\partial f}{\partial x} = g(x,y)$ cont. pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Cum $\lim\limits_{x\to 0} (g(x,x^2)) \neq \lim\limits_{x\to 0} g(x,x)$

$\Rightarrow g(x,y)$ nu e continuă în $(0,0)$

c) din punctul a => $f(x,y)$ cont. pe $\mathbb{R}^2$ =>

=> $f(x,y)$ continuă în $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_1) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t\cdot(1,0)) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(t,0)}{t} = \lim_{t\to 0} \frac{\frac{t^9\cdot 0^6}{t^{14}+0^{14}}}{t} \underset{l'H}{\overset{\frac{0}{0}}{=}}$$

$$= \lim_{t\to 0} \frac{\left(\frac{t^9\cdot 0}{t^{14}+0^{14}}\right)'}{t'} = \lim_{t\to 0} \frac{0'}{1} = \frac{0}{1} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f((0,0)+t\cdot(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)}{t} =$$

$$= \lim_{t\to 0} \frac{\frac{0^9\cdot t^6}{0^{14}+t^{14}}}{t} \underset{l'H}{\overset{\frac{0}{0}}{=}} \lim_{t\to 0} \frac{0}{1} = 0$$

$T: \mathbb{R}^2 \to \mathbb{R}$, $T(x,y) = a\cdot x + b\cdot y$

$a = \frac{\partial f}{\partial x}(0,0)$, $b = \frac{\partial f}{\partial y}(0,0)$

$$C = \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - T((x,y)-(0,0))|}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - 0 - 0|}{\sqrt{x^2+y^2}} =>$$

$$\Rightarrow L = \lim_{(x,y)\to(0,0)} \frac{x^9 y^6}{x^{14}+y^{14}} \cdot \frac{1}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} g(x)$$

studiem continuitatea lui $g(x)$

$g(x)$ continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Studiem continuitatea în $(0,0)$:

$$y = x \Rightarrow \lim_{x\to 0} (g(x,x)) = \lim_{x\to 0} \frac{\cancel{x^{15}}^{\,x}}{2\cancel{x^{14}}_{1}\sqrt{2x^2}} =$$

$$= \lim_{x\to 0} \frac{\cancel{x}^{1}}{\cancel{|x|}_{1}\, 2\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$y = -x \Rightarrow \lim_{x\to 0} (g(x,-x)) = \frac{1}{2\sqrt{2}}$$

$g(x)$ continuă în $(0,0) \Rightarrow$

$\Rightarrow f(x)$ derivabilă în $(0,0)$