1.04.2021

SEMINAR 7

Endomorfisme. Vectori și valori proprii

Diagonalizare

✓ **Ex1** $f: \mathbb{R}^2 \to \mathbb{R}^2$ liniară și $A = [f]_{R_0,R_0} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

a) $f(x) = ?$ b) $A' = [f]_{R,R}$

unde $R = \{e_1' = 2e_1 - e_2,\ e_2' = e_1 - 2e_2\}$

$R_0 = \{e_1, e_2\}$ reperul canonic.

Sol

a) $f(x) = y \Leftrightarrow Y = AX \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \begin{pmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$ $\quad f(x_1, x_2) = (2x_1 - x_2, 3x_1 + 4x_2)$

OBS

$f(e_1) = 2e_1 + 3e_2$

$f(e_2) = -e_1 + 4e_2$

$f(x) = f(x_1 e_1 + x_2 e_2) = x_1 f(e_1) + x_2 f(e_2)$

$= x_1(2e_1 + 3e_2) + x_2(-e_1 + 4e_2) = e_1(2x_1 - x_2) + e_2(3x_1 + 4x_2)$

$= (2x_1 - x_2, 3x_1 + 4x_2)$

b) $f(e_1') = f(2e_1 - e_2) = f(2,-1) = (5,2) = a e_1' + b e_2'$

$(5,2) = a(2,-1) + b(1,-2) = (2a+b, -a-2b)$

$\begin{cases} 2a + b = 5 \\ -a - 2b = 2 \mid 2 \end{cases}$ $\oplus$

$\quad -3b = 9$

$b = -3$

$a = \frac{8}{2} = 4$

$A' = [f]_{R,R} = \begin{pmatrix} 4 & 1 \\ -3 & 2 \end{pmatrix}$

$f(e_2') = f(1,-2) = (4,-5) = c e_1' + d e_2' = (2c+d, -c-2d)$

$$\begin{cases} 2c+d=4 \\ -c-2d=-5 \quad |2 \end{cases} \oplus \qquad \begin{cases} d=2 \\ c=1 \end{cases}$$

$$/ \quad -3d=-6$$

OBS

$$\begin{array}{ccc} \mathcal{R}_0 & \xrightarrow{A} & \mathcal{R}_0 \\ C\downarrow & & \downarrow C \\ \mathcal{R} & \xrightarrow{A'} & \mathcal{R} \end{array} \qquad A' = C^{-1}AC$$

$$C: \begin{cases} e_1' = 2e_1 - e_2 \\ e_2' = e_1 - 2e_2 \end{cases} \qquad C = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$$

Ex2 $(\mathbb{R}_3[X], +, \cdot)_{/\mathbb{R}}$ $\quad V_1 = <\{1, 1-x\}$

$V_2 = <\{(1-x)^2, (1-x)^3\}>$

a) $\mathbb{R}_3[X] = V_1 \oplus V_2$

b) $p: V_1 \oplus V_2 \longrightarrow V_1$ proiecția pe V_1, de-a lungul lui V_2

$p(P) = p(\underset{\in V_1}{P_1} + \underset{\in V_2}{P_2}) = P_1$

$s: V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$ simetria pe V_1 — " —

$s = 2p - id_{\mathbb{R}_3[X]}$

$s(P) = 2p(P) - id_{\mathbb{R}_3[X]}(P) = 2P_1 - (P_1 + P_2) = P_1 - P_2$

$p(1 + 2X + X^2 + X^3) = ?$

$s(1 + 2X + X^2 + X^3) = ?$

Sol

a) $\mathcal{R} = \{1, 1-x, (1-x)^2, (1-x)^3\}$ reper in $\mathbb{R}_3[X]$ (dem)

$\mathcal{R}_0 = \{1, x, x^2, x^3\}$ reperul canonic

$$\left.\begin{array}{l} \text{Dem că } \mathcal{R} \text{ e SLI} \\ \dim \mathbb{R}_3[X] = 4 = |\mathcal{R}| \end{array}\right\} \Rightarrow \mathcal{R} \text{ reper.}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$(1-X)^2 = 1-2X+X^2$

$(1-X)^3 = 1-3X+3X^2-X^3$

$rg A = 4$ max $\Rightarrow \mathcal{R}$ e SLI

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{R}_1 = \{1, 1-X\}$, $\mathcal{R}_2 = \{(1-X)^2; (1-X)^3\}$.

$\mathbb{R}_3[X] = V_1 \oplus V_2$, $V_1 = \langle \mathcal{R}_1 \rangle$, $V_2 = \langle \mathcal{R}_2 \rangle$.

b) $1+2X+X^2+X^3 = \underbrace{a\cdot 1 + b(1-X)}_{P_1 \in V_1} + \underbrace{c(1-X)^2 + d(1-X)^3}_{P_2 \in V_2}$.

• $X=1 \Rightarrow 5 = a$

$2+2X+3X^2 = -b - 2c(1-X) - 3d(1-X)^2$

• $X=1 \Rightarrow -7 = b$.

$2+6X = 2c + 6d(1-X)$

• $X=1 \Rightarrow 8 = 2c \Rightarrow c = 4$

$6 = -6d \Rightarrow d = -1$.

$P = 1+2X+X^2+X^3$ are coord $(a,b,c,d) = (5,-7,4,-1)$ în raport cu $\mathcal{R}$.

$P = P_1 + P_2$, $P_1 = 5\cdot 1 - 7(1-X) = -2+7X \in V_1$

$P_2 = 4(1-X)^2 - (1-X)^3 = 3-5X+X^2+X^3 \in V_2$

$p(P) = P_1 = -2+7X$

$s(P) = P_1 - P_2 = -2+7X-3+5X-X^2-X^3 =$

$= -5+12X-X^2-X^3$

Ex3 $(\mathbb{R}^2, +, \cdot)/_{\mathbb{R}}$, $\mathcal{R} = \{e_1 = (1,0), e_2 = (0,1)\}$ reperul canonic

$\mathcal{R}' = \{e_1' = e_1 - e_2, e_2' = e_1 + 2e_2\}$ reper.

$((\mathbb{R}^2)^* = \{f : \mathbb{R}^2 \to \mathbb{R} \mid f \text{ liniară}\}, +, \cdot)/_{\mathbb{R}}$ spațiul dual

$\mathcal{R}^* = \{e_1^*, e_2^*\}$, $e_i^*(e_j) = \delta_{ij}$, $\forall i,j = \overline{1,2}$

$\mathcal{R}'^* = \{e_1'^*, e_2'^*\}$, $e_i'^*(e_j') = \delta_{ij}$ —//—

reperele duale în sp. dual

$\mathcal{R} \xrightarrow{C} \mathcal{R}'$, $\mathcal{R}^* \xrightarrow{D} \mathcal{R}'^*$

Precizați legătura dintre C și D.

SOL

$\mathcal{R} \xrightarrow{C} \mathcal{R}'$ $\qquad C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$$\begin{cases} e_1' = e_1 - e_2 \\ e_2' = e_1 + 2e_2 \end{cases}$$

$e_1^* : \mathbb{R}^2 \to \mathbb{R}$ liniară, $e_1^*(e_1) = 1$, $e_1^*(e_2) = 0$

$e_2^* : \mathbb{R}^2 \to \mathbb{R}$ liniară, $e_2^*(e_1) = 0$, $e_2^*(e_2) = 1$

$$e_1^*(x) = e_1^*(x_1 e_1 + x_2 e_2) = x_1 \underbrace{e_1^*(e_1)}_{1} + x_2 \underbrace{e_1^*(e_2)}_{0} = x_1$$

$$e_2^*(x) = e_2^*(x_1 e_1 + x_2 e_2) = x_1 \underbrace{e_2^*(e_1)}_{0} + x_2 \underbrace{e_2^*(e_2)}_{1} = x_2$$

$\mathcal{R}^* \xrightarrow{D} \mathcal{R}'^*$ $\qquad D = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\begin{cases} e_1'^* = a e_1^* + b e_2^* \\ e_2'^* = c e_1^* + d e_2^* \end{cases}$$

1) $e_1'^*(e_1') = 1 \Rightarrow (a e_1^* + b e_2^*)(e_1 - e_2) = 1$

$$a \underbrace{e_1^*(e_1)}_{1} - a \underbrace{e_1^*(e_2)}_{0} + b \underbrace{e_2^*(e_1)}_{0} - b \underbrace{e_2^*(e_2)}_{1} = 1 \Rightarrow \boxed{a - b = 1}$$

2) $e_1'^*(e_2') = 0 \Rightarrow (ae_1^* + be_2^*)(e_1 + 2e_2) = 0$

$a\underbrace{e_1^*(e_1)}_{1} + 2a\underbrace{e_1^*(e_2)}_{0} + b\underbrace{e_2^*(e_1)}_{0} + 2b\underbrace{e_2^*(e_2)}_{1} = 0 \Rightarrow \boxed{a+2b=0}$

$\begin{cases} a - b = 1 \\ a + 2b = 0 \end{cases}$ $\ominus$

$\diagup \quad 3b = -1$

$b = -\frac{1}{3}$

$a = 1 - \frac{1}{3} = \frac{2}{3}$

3) $e_2'^*(e_1') = 0 \Rightarrow (ce_1^* + de_2^*)(e_1 - e_2) = 0 \Rightarrow \boxed{c-d=0}$

4) $e_2'^*(e_2') = 1 \Rightarrow (ce_1^* + de_2^*)(e_1 + 2e_2) = 1 \Rightarrow \boxed{c+2d=1}$

$\begin{cases} c - d = 0 \\ c + 2d = 1 \end{cases}$ $\ominus$

$\diagup \quad 3d = 1$

$c = d = \frac{1}{3}$

$D = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$C^T = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \quad C^* = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}; \ \det C = 3$

$C^{-1} = \frac{1}{3} C^* = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

$D = (C^{-1})^T \qquad c_{ij}^* = (-1)^{i+j} \Delta_{ij}$

OBS $f \in End(V)$

$\exists$ un reper R în V ai $[f]_{R,R}$ diagonală

$\Leftrightarrow$ 1) răd dist $\lambda_1, \dots, \lambda_r$ ale polinomului caract $\in \mathbb{K}$

2) $\dim V_{\lambda_i} = m_i, \quad \forall i = \overline{1,r}$

$P(\lambda) = \det(A - \lambda I_n) = 0 \Rightarrow (\lambda - \lambda_1)^{m_1} \cdot \ldots \cdot (\lambda - \lambda_r)^{m_r} = 0$

$m_1 + \dots + m_r = n = \dim V$

x s.n. vector propriu al lui f dc. $\exists \lambda \in \mathbb{K}$ (valoare proprie)
a.î. $f(x) = \lambda x$

$V_{\lambda_i} = \{x \in V \mid f(x) = \lambda_i x\}$, $i = \overline{1,r}$ subsp. propriu

Ex4 $f: \mathbb{R}^4 \to \mathbb{R}^4$, $f(x) = (x_2 - x_3 + x_4,\ x_2 - x_3 + x_4,\ x_4,\ x_4)$

a) Să se afle valorile proprii

b) —"— subspațiile proprii

c) $\exists$ un reper $\mathcal{R}$ în $\mathbb{R}^4$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ diagonală?

SOL

a) $A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$f(x) = y \iff Y = AX$

$P(\lambda) = \det(A - \lambda I_4) = 0$

$$\begin{vmatrix} -\lambda & 1 & -1 & 1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^2(1-\lambda)^2 = 0 \Rightarrow \begin{cases} \lambda_1 = 0, & m_1 = 2 \\ \lambda_2 = 1, & m_2 = 2 \end{cases}$

b) $V_{\lambda_1} = \{x \in \mathbb{R}^4 \mid f(x) = 0 \cdot x = 0_{\mathbb{R}^4}\} = \operatorname{Ker} f = S(A)$

$\dim V_{\lambda_1} = 4 - \operatorname{rg} A = 4 - 2 = 2$

$\begin{cases} x_2 - x_3 + x_4 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = x_3 \\ x_4 = 0 \end{cases}$

$V_{\lambda_1} = \{(x_1, x_2, x_2, 0),\ x_1, x_2 \in \mathbb{R}\} = \langle \underbrace{\{(1,0,0,0),\ (0,1,1,0)\}}_{\mathcal{R}_1} \rangle$

$(x_1, x_2, x_2, 0) = x_1(1,0,0,0) + x_2(0,1,1,0)$

$\mathcal{R}_1$ e SG în V_{λ_1}; $\dim V_{\lambda_1} = 2 = |\mathcal{R}_1| \Rightarrow \mathcal{R}_1$ reper în V_{λ_1}

$V_{\lambda_2} = \{ x \in \mathbb{R}^4 \mid f(x) = 1 \cdot x \}$

$(x_2 - x_3 + x_4, x_2 - x_3 + x_4, x_4, x_4) = (x_1, x_2, x_3, x_4)$

$$\begin{cases} x_2 - x_3 + x_4 = x_1 \\ x_2 - x_3 + x_4 = x_2 \\ x_4 = x_3 \\ x_4 = x_4 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ x_4 = x_3 \end{cases}$$

$$V_{\lambda_2} = \{ (x_1, x_1, x_3, x_3) \mid x_1, x_3 \in \mathbb{R} \} = \langle \underbrace{\{(1,1,0,0), (0,0,1,1)\}}_{R_2}$$

$x_1(1,1,0,0) + x_3(0,0,1,1)$

$$\left.\begin{array}{l} R_2 \text{ este SG pt } V_{\lambda_2} \\ rg\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 2 \Rightarrow R_2 \text{ e SLI} \end{array}\right\} \Rightarrow R_2 \text{ reper în } V_{\lambda_2}$$

c)

$$\left.\begin{array}{l} 1)\ \lambda_1 = 0,\ \lambda_2 = 1 \in \mathbb{R} \\ 2)\ \dim V_{\lambda_1} = 2 = m_1,\ \dim V_{\lambda_2} = 2 = m_2 \end{array}\right\} \xrightarrow{\text{Th. diag}}$$

$\exists R = R_1 \cup R_2 = \{(1,0,0,0), (0,1,1,0), (1,1,0,0), (0,0,1,1)\}$

reper în $\mathbb{R}^4$ a.î. $[f]_{R,R} = A' = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_2 & \\ 0 & & & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\mathbb{R}^4 = V_{\lambda_1} \oplus V_{\lambda_2}$.

Ex 5 $f \in End(\mathbb{R}^3)$

Fie $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$ valorile proprii și $v_1 = (-3,2,1)$, $v_2 = (-2,1,0)$, $v_3 = (-6,3,1)$ vectori proprii corespunzători.

Să se determine $A = [f]_{R_0,R_0}$, R_0 reperul canonic.

SOL. v_1, v_2, v_3 vectori proprii corsp la valori proprii dist $\xrightarrow{PROP}$

$$\left.\begin{array}{l} R = \{v_1, v_2, v_3\} \text{ SLI} \\ \text{dar } \dim \mathbb{R}^3 = |R| = 3 \end{array}\right\} \Rightarrow R \text{ reper în } \mathbb{R}^3$$

$$\begin{cases} f(v_1) = \lambda_1 v_1 = 3v_1 \\ f(v_2) = \lambda_2 v_2 = -2v_2 \\ f(v_3) = \lambda_3 v_3 = v_3 \end{cases} \qquad A' = [f]_{R,R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_0 = \{e_1, e_2, e_3\} \xrightarrow{C} R = \{v_1, v_2, v_3\}.$$

$$\begin{cases} v_1 = (-3,2,1) = -3e_1 + 2e_2 + e_3 \\ v_2 = (-2,1,0) = -2e_1 + e_2 + 0e_3 \\ v_3 = (-6,3,1) = -6e_1 + 3e_2 + e_3 \end{cases} \qquad C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A' = C^{-1}AC \Rightarrow \boxed{A = CA'C^{-1}}$$

OBS
- $f(v_1) = \lambda_1 v_1 \Rightarrow$
  $f(-3,2,1) = f(-3e_1 + 2e_2 + e_3) = -3f(e_1) + 2f(e_2) + f(e_3) = 3(-3,2,1)$
- $f(-2e_1 + e_2) = -2f(e_1) + f(e_2) = -2(-2,1,0)$
- $f(-6e_1 + 3e_2 + e_3) = -6f(e_1) + 3f(e_2) + f(e_3) = (-6,3,1)$

ec1 - ec3: $3f(e_1) - f(e_2) = (-3,3,2)$

ec2: $-2f(e_1) + f(e_2) = (4,-2,0)$ (+)

$$f(e_1) \quad \cancel{} \quad = (1,1,2)$$

$$f(e_2) = (4,-2,0) + (2,2,4) = (6,0,4)$$

$$f(e_3) = (-9,6,3) + 3(1,1,2) - 2(6,0,4)$$

$$= (-9+3-12,\ 6+3,\ 3+6-8) = (-18,9,1)$$

$$\begin{cases} f(e_1) = e_1 + e_2 + 2e_3 \\ f(e_2) = 6e_1 + 0e_2 + 4e_3 \\ f(e_3) = -18e_1 + 9e_2 + e_3 \end{cases} \qquad A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & 6 & -18 \\ 1 & 0 & 9 \\ 2 & 4 & 1 \end{pmatrix}$$

Ex $f : \mathbb{R}^4 \to \mathbb{R}^4$ lin, $A = [f]_{R_0,R_0} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$

a) Să se afle valorile proprii și subsp. proprii.

b) $U = \langle \{ e_1 + 2e_2,\ e_2 + e_3 + 2e_4 \} \rangle$ este un subspațiu invariant al lui f

Sol

a) $P(\lambda) = \det(A - \lambda I_4) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 2 & -1 & -\lambda & 1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} = 0 \quad \xRightarrow{l_3' = l_3 - l_4}$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 & -1 \\ 0 & 1-\lambda & 4 & -2 \\ 0 & 0 & 1-\lambda & \lambda-1 \\ 2 & -1 & -1 & 2-\lambda \end{vmatrix} = 0 \qquad c_3' = c_3 + c_4$$

$$(\lambda-1) \begin{vmatrix} 1-\lambda & 0 & 1 & -1 \\ 0 & 1-\lambda & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\lambda-1) \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 2 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$-(\lambda-1)\left\{ (1-\lambda)\left[(1-\lambda)^2 + 2\right] - 2(1-\lambda) \right\} = 0$$

$$(1-\lambda)^2 \left[(1-\lambda)^2 + \cancel{2} - \cancel{2} \right] = 0 \Rightarrow (1-\lambda)^4 = 0$$

$\lambda_1 = 1,\ m_1 = 4$

$V_{\lambda_1} = \{ x \in \mathbb{R}^4 \mid f(x) = x \}$

$AX = X \Leftrightarrow (A - I_4) X = 0$

$$\begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & \boxed{0} & \boxed{4} & -2 \\ 2 & \boxed{-1} & \boxed{-1} & 1 \\ 2 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim V_{\lambda_1} = 4 - 2 = 2 \neq m_1$

Matricea nu se poate diagonaliza.

$$\begin{cases} 4x_3 - 2x_4 = 0 \\ 2x_1 - x_2 - x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = \frac{1}{2}x_4 \\ x_2 = 2x_1 - \frac{1}{2}x_4 + x_4 = 2x_1 + \frac{1}{2}x_4 \end{cases}$$

$$V_{\lambda_1} = \{(x_1, 2x_1 + \tfrac{1}{2}x_4, \tfrac{1}{2}x_4, x_4) \mid x_1, x_4 \in \mathbb{R}\}$$

$$= x_1\underbrace{(1,2,0,0)}_{e_1+2e_2} + \frac{x_4}{2}\underbrace{(0,1,1,2)}_{e_2+e_3+2e_4}$$

$V_{\lambda_1} = \langle\{e_1 + 2e_2,\ e_2 + e_3 + 2e_4\}\rangle = U$

V_{λ_1} = subspațiu invariant al lui f

$f(U) \subseteq U$.

- $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$

subsp. propriu coresp. valorii proprii λ