T1 curs

1. $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$

a) Să se afle A^{-1} utilizând Th. H-C

b) Dacă $B = A^6 + A^5 + A^4 + A + i_3$ atunci să se afle $a, b, c \in \mathbb{R}$ a.î. $B = aA^2 + bA + ci_3$

a) $A \in M_3(\mathbb{Z}) \overset{Th.\ H-C}{\Rightarrow} P_A(A) = O_3 \Rightarrow$

$\Rightarrow A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 i_3 = O_3$

$\sigma_1 = Tr\, A = 1 + 2 + 3 = 6$

$\sigma_2 = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} =$

$= \cancel{6} - (-1) + 3 - (-1) \cdot 0 + 2 - \cancel{6} = 1 + 3 + 2 = 6$

$\sigma_3 = \det A = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{vmatrix} \overset{l_2 - 3l_1}{=}$

$= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -4 & 2 \\ 0 & 1 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 1 \begin{vmatrix} -4 & 2 \\ 1 & 3 \end{vmatrix} =$

$= -12 - 2 = -14$

$\Rightarrow A^3 - 6A^2 + 6A + 14 i_3 = O_3$

$A^3 - 6A^2 + 6A = -14 i_3$ (la stânga)

$A(A^2 - 6A + 6i_3) = -14 i_3 \mid A^{-1} \cdot$

$$A^2 - 6A + 6 i_3 = -14 \cdot A^{-1} \quad \Big| \left(-\frac{1}{14}\right)$$

$$A^{-1} = -\frac{1}{14} A^2 + \frac{3}{7} A - \frac{3}{7} i_3$$

$$A^2 = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 5 & -6 \\ 9 & 9 & -8 \\ 3 & 5 & 8 \end{pmatrix}$$

$$A^{-1} = -\frac{1}{14} \begin{pmatrix} 7 & 5 & -6 \\ 9 & 9 & -8 \\ 3 & 5 & 8 \end{pmatrix} + \frac{3}{7} \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$- \begin{pmatrix} \frac{3}{7} & 0 & 0 \\ 0 & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{5}{14} & \frac{3}{7} \\ -\frac{9}{14} & -\frac{9}{14} & \frac{4}{7} \\ -\frac{3}{14} & -\frac{5}{14} & -\frac{4}{7} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{3}{7} & \frac{6}{7} & -\frac{3}{7} \\ \frac{9}{7} & \frac{6}{7} & -\frac{3}{7} \\ 0 & \frac{3}{7} & \frac{9}{7} \end{pmatrix} + \begin{pmatrix} -\frac{3}{7} & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & 0 & -\frac{3}{7} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{14} & \frac{1}{2} & 0 \\ \frac{9}{14} & \frac{3}{14} & \frac{1}{7} \\ -\frac{3}{14} & \frac{1}{14} & \frac{5}{7} \end{pmatrix} + \begin{pmatrix} -\frac{3}{7} & 0 & 0 \\ 0 & -\frac{3}{7} & 0 \\ 0 & 0 & -\frac{3}{7} \end{pmatrix} =$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{9}{14} & -\frac{3}{14} & \frac{1}{7} \\ -\frac{3}{14} & \frac{1}{14} & \frac{2}{7} \end{pmatrix}.$$

b) $B = A^6 + A^5 + A^4 + A + i_3$

$a, b, c \in \mathbb{R}$ a.î. $B = aA^2 + bA + ci_3$

Th. H-C: $A^3 = 6A^2 - 6A - 14 i_3$

$$\begin{array}{l|l} A^6 + A^5 + A^4 + A + i_3 & A^3 - 6A^2 + 6A + 14 i_3 \\ \underline{-A^6 + 6A^5 - 6A^4 - 14A^3} & \overline{A^3 + 7A^2 + 37A + 166} \\ = 7A^5 - 5A^4 - 14A^3 + A + i_3 & \\ \underline{-7A^5 + 42A^4 - 42A^3 - 98A^2} & \\ = 37A^4 - 56A^3 - 98A^2 + A + i_3 & \\ \underline{-37A^4 + 222A^3 - 222A^2 - 518A} & \\ = 166A^3 - 320A^2 - 51\ A + i_3 & \\ \underline{-166A^3 + 996A^2 - 996A - 2324 i_3} & \\ = 676A^2 - 1513A - 2323 i_3 & \end{array}$$

$A^6 + A^5 + A^4 + A + i_3 = (A^3 - 6A^2 + 6A + 14 i_3)(A^3 + 7A^2 + 37A + 166) + 676A^2 - 1513A - 2323 i_3 \Rightarrow$

$\Rightarrow B = O_3 \cdot (A^3 + 7A^2 + 37A + 166) + 676A^2 - 1513A - 2323 i_3$

$\Rightarrow B = 676A^2 - 1513A - 2323 i_3 \Rightarrow$

$\Rightarrow a = 676$

$b = -1513$

$c = -2323$

2. $A \in M_2(\mathbb{R})$

a) Dacă $Tr\, A = 0$ atunci $A^2 B = B A^2$, $\forall B \in M_2(\mathbb{R})$

Th. H-C: $A^2 - Tr\, A \cdot A + \det A \cdot I_2 = O_2$

$A^2 + \det A \cdot I_2 = O_2 \quad | \cdot B \Rightarrow$

înm. la dreapta $\Rightarrow$ $A^2 B = -\det A \cdot B$

înm. la stânga $\Rightarrow$ $B \cdot A^2 = B(-\det A) \cdot I_2 = -\det A \cdot B$

$\Rightarrow A^2 B = B \cdot A^2$ c.c.t.d.

b) Dacă $Tr\, A \neq 0$ și $A^2 B = B \cdot A^2$ atunci $AB = BA$

Th. H-C: $A^2 - Tr\, A \cdot A + \det A \cdot I_2 = O_2 \quad | \cdot B \Rightarrow$

înm. la dreapta $\Rightarrow$ $A^2 B - (Tr\, A) AB + \det A \cdot B = O_2$

înm. la stânga $\Rightarrow$ $B \cdot A^2 - (Tr\, A) B \cdot A + \det A \cdot B = O_2$

$\underbrace{A^2 B - B \cdot A^2}_{0} - (Tr\, A)(AB - BA) = O_2$

$-(Tr\, A)(AB - BA) = O_2 \Rightarrow$

$AB - BA = O_2 \Rightarrow AB = BA$ c.c.t.d.

3. Să se arate că sistemul are sol. unică

$$\begin{cases} \frac{1}{2}x = ax + by + cz \\ \frac{1}{2}y = cx + ay + bz \\ \frac{1}{2}z = bx + cy + az \end{cases}, \quad a, b, c \in \mathbb{Z}$$

$$\Rightarrow \begin{cases} (a-\frac{1}{2})x + by + cz = 0 \\ cx + (a-\frac{1}{2})y + bz = 0 \\ bx + cy + (a-\frac{1}{2})z = 0 \end{cases}$$

$$A = \begin{pmatrix} a-\frac{1}{2} & b & c \\ c & a-\frac{1}{2} & b \\ b & c & a-\frac{1}{2} \end{pmatrix}$$

$$\det A = \begin{vmatrix} a-\frac{1}{2} & b & c \\ c & a-\frac{1}{2} & b \\ b & c & a-\frac{1}{2} \end{vmatrix} =$$

$$= (a+b+c-\frac{1}{2}) \begin{vmatrix} 1 & 1 & 1 \\ c & a-\frac{1}{2} & b \\ b & c & a-\frac{1}{2} \end{vmatrix} =$$

$$= (a+b+c-\frac{1}{2})\left[(a-\frac{1}{2})^2 + c^2 + b^2 - (a-\frac{1}{2})b - bc - (a-\frac{1}{2})c\right] = (a+b+c-\frac{1}{2})(a^2 - a + \frac{1}{4} + c^2 + b^2 - ab + \frac{1}{2}b - bc - ac + \frac{1}{2}c) = \frac{1}{2}(a+b+c-\frac{1}{2})$$

$$(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac - 2a + b + c + \frac{1}{2})$$

$$= \frac{1}{2}(a+b+c-\frac{1}{2})((a-b)^2 + (a-c)^2 + (b-c)^2 - 2a + b + c + \frac{1}{2})$$

$a, b, c \in \mathbb{Z} \Rightarrow$

$$\Rightarrow \begin{cases} a+b+c \neq \frac{1}{2} \\ -2a+b+c \neq \frac{1}{2} \end{cases} \Rightarrow \det A \neq 0 \Rightarrow$$

$\Rightarrow$ SCD $\Rightarrow (x, y, z) = (0, 0, 0)$

sol. unică