T1 seminar

1. $\begin{cases} x + y + az - t = 0 \\ 2x + y - z + t = 0 \\ 3x - y - z - t = 0 \\ ax - 2y - 2z - 2t = 0 \end{cases}$, $a \in \mathbb{R}$

$$\det A = \begin{vmatrix} 1 & 1 & a & -1 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & -1 \\ a & -2 & -2 & -2 \end{vmatrix} =$$

$$= \begin{vmatrix} 3 & 2 & a-1 & 0 \\ 2 & 1 & -1 & 1 \\ 5 & 0 & -2 & 0 \\ a+4 & 0 & -4 & 0 \end{vmatrix} = (-1)^{2+4} \cdot 1 \cdot$$

$$\begin{vmatrix} 3 & 2 & a-1 \\ 5 & 0 & -2 \\ a+4 & 0 & -4 \end{vmatrix} = (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 5 & -2 \\ a+4 & -4 \end{vmatrix} =$$

$$= -2(-20 + 2a + 8) = -2(-12 + 2a) = 24 - 4a$$

$$\Rightarrow \det A = 4(6 - a)$$

Dacă $\det A \neq 0 \iff 4(6-a) \neq 0 \iff a \in \mathbb{R} \setminus \{6\}$

SCD

Dacă $\det A = 0 \Rightarrow a = 6$

Pentru $a = 6 \Rightarrow A = \begin{pmatrix} 1 & 1 & 6 & -1 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & -1 \\ 6 & -2 & -2 & -2 \end{pmatrix}$

Cum $\begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{vmatrix} = -1-12-3-18-1+2 = -33 \neq 0 \Rightarrow$

$\Rightarrow$ rang $A = 3$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 6 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ 6 & -2 & -2 & -2 & 0 \end{pmatrix}$$

rang $\bar{A} \neq 4$ deoarece ultima coloană are doar zerouri $\Rightarrow$

$\Rightarrow$ rang $\bar{A} = 3$ (acelaşi minor)

Cum rang A = rang $\bar{A} = 3 \Rightarrow$ SCN

2. Să se arăte că $\det A = 4(a^2+b^2)(c^2+d^2)$ utilizând Laplace

$$A = \begin{pmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{pmatrix}$$

Pentru $p = 2$ luăm l_1, l_2 fixate

$$\Delta = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} (-1)^{1+2+1+2} \begin{vmatrix} c & -d \\ d & c \end{vmatrix} +$$

$$+ \begin{vmatrix} a & -a \\ b & -b \end{vmatrix} (-1)^{1+3+1+2} \begin{vmatrix} -d & -d \\ c & c \end{vmatrix} +$$

$$+ \begin{vmatrix} a & b \\ b & -a \end{vmatrix} (-1)^{1+4+1+2} \begin{vmatrix} -d & c \\ c & d \end{vmatrix} +$$

$$+ \begin{vmatrix} -b & -a \\ a & -b \end{vmatrix} (-1)^{2+3+1+2} \begin{vmatrix} c & -d \\ d & c \end{vmatrix} +$$

$$+ \begin{vmatrix} -b & b \\ a & -a \end{vmatrix} (-1)^{2+4+1+2} \begin{vmatrix} c & c \\ d & d \end{vmatrix} +$$

$$+ \begin{vmatrix} -a & b \\ -b & -a \end{vmatrix} (-1)^{3+4+1+2} \begin{vmatrix} c & -d \\ d & c \end{vmatrix} =$$

$$= (a^2+b^2)(c^2+d^2) + (-ab+ab)(-1)(-dc+dc) +$$
$$+ (-a^2-b^2)(-d^2-c^2) + \overset{0}{\shortparallel}\,(b^2+a^2)(c^2+d^2) +$$
$$+ (ab-ab)(-1)(cd-ca) + (a^2+b^2)(c^2+d^2) =$$
$$= 4(a^2+b^2)(c^2+d^2) \qquad \text{c.c.t.d.}$$

3. Fie $A = M_2(\mathbb{R})$ a.î. $A^2 = O_2$ şi P_A pol. caract.
Calculaţi $P_A(1) + \ldots + P_A(m)$, $m \in \mathbb{N}^*$

Fie $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$

$$A^2 = O_2 \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff$$

$$\iff \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & d^2+bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} a^2+bc=0 \\ ab+bd=0 \\ ac+cd=0 \\ bc+d^2=0 \end{cases} \Rightarrow \quad \begin{matrix} b(a+d)=0 \Rightarrow \\ c(a+d)=0 \Rightarrow \end{matrix}$$

$\Rightarrow$ $b=0$ sau $a+d=0$

$\Rightarrow$ $c=0$ sau $a+d=0$

Caz 1: $b = c = 0 \Rightarrow a = 0, d = 0 \Rightarrow A = O_2 \Rightarrow$

$\Rightarrow Tr A = \det A = 0$

Caz 2: $b \neq 0, c \neq 0 \Rightarrow a + d = 0 \Rightarrow a = -d$

$-a^2 = bc \Rightarrow$

$\Rightarrow c = -\dfrac{a^2}{b}$

$$\Rightarrow A = \begin{pmatrix} a & b \\ -\dfrac{a^2}{b} & -a \end{pmatrix} \Rightarrow \begin{matrix} Tr A = 0 \\ \det A = 0 \end{matrix} \Rightarrow A^2 = O_2$$

$P_A(x) = (-1)^2 (x^2 - Tr A \cdot x + \det A) = \det A - Tr A \cdot x + x^2$

$P_A(1) = \det A - Tr A + 1$

$P_A(2) = \det A - 2 Tr A + 4$

$P_A(3) = \det A - 3 Tr A + 9$

$\vdots$

$P_A(m) = \det A - m\, Tr A + m^2$

$n+4$

$P_A(1) + \ldots + P_A(m) = m \det A - (1 + 2 + \ldots + m) Tr A +$

$$\left.\begin{matrix} (1 + 4 + \ldots + m^2) \\ \det A = Tr A = 0 \end{matrix}\right\} \Rightarrow P_A(1) + \ldots + P_A(m) = 1 + 4 + \ldots + m^2$$

$$= \frac{m(m+1)(2m+1)}{6}$$