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1. $X \sim \begin{pmatrix} -5 & 3 & 5 \\ 0.44 & 0.52 & 0.04 \end{pmatrix}$

$Y \sim \begin{pmatrix} 3 & 4 & 7 \\ 0.78 & 0.15 & 0.07 \end{pmatrix}$

$X + Y \sim \begin{pmatrix} -2 & -1 & 2 & 6 \\ 0.3432 & 0.066 & 0.0308 & 0.4056 \\ 7 & 8 & 9 & 10 & 12 \\ 0.078 & 0.0312 & 0.006 & 0.0364 & 0.0028 \end{pmatrix}$

$X - Y \sim \begin{pmatrix} -12 & -9 & -8 & -4 \\ 0.0308 & 0.066 & 0.3432 & 0.0364 \\ -2 & -1 & 0 & 1 & 2 \\ 0.0028 & 0.078 & 0.4056 & 0.006 & 0.0312 \end{pmatrix}$

$X^2 \sim \begin{pmatrix} 25 & 9 \\ 0.48 & 0.52 \end{pmatrix}$ $\quad Y^2 \sim \begin{pmatrix} 9 & 16 & 49 \\ 0.78 & 0.15 & 0.07 \end{pmatrix}$

$4X^2 + 2Y^2 \sim \begin{pmatrix} 54 & 68 & 118 & 132 \\ 0.4056 & 0.078 & 0.3744 & 0.072 \\ 134 & 198 \\ 0.0364 & 0.0336 \end{pmatrix}$

$E[X] = -5 \cdot 0.44 + 3 \cdot 0.52 + 5 \cdot 0.04 = -0.44$

$E[Y] = 3 \cdot 0.78 + 4 \cdot 0.15 + 7 \cdot 0.07 = 3.43$

$Var(X) = E(X^2) - E(X)^2$

$E(X^2) = 25 \cdot 0.48 + 9 \cdot 0.52 = 16.68$

$E(Y^2) = 9 \cdot 0.78 + 16 \cdot 0.15 + 49 \cdot 0.07 = 12.85$

$Var(X) = 16.68 - 0.1936 = 16.4864$

$Var(Y) = 12.85 - 11.7649 = 1.0851$

$Var(6X - 3Y + 4) = Var(6X - 3Y)$

Notăm $Z = 6X - 3Y$

$$Z \sim \begin{pmatrix} -51 & -42 & -39 & -3 & 6 & 9 & 18 & 21 \\ 0.031 & 0.066 & 0.343 & 0.036 & 0.078 & 0.408 & 0.006 & 0.031 \end{pmatrix}$$

$$Z^2 \sim \begin{pmatrix} 9 & 36 & 81 & 324 & 441 & 1521 & 1764 & 2601 \\ 0.036 & 0.078 & 0.408 & 0.006 & 0.031 & 0.343 & 0.066 & 0.031 \end{pmatrix}$$

$Var(Z) = E(Z^2) - E(Z)^2 = 770.4612 - 167.1849$

$Var(Z) = Var(6X - 3Y + 4) = 603.2763$

$Var(5XY + 5) = Var(5XY) = 25\, Var(XY)$

$$XY \sim \begin{pmatrix} -35 & -20 & -15 & 9 & 12 & 15 & 20 & 21 & 35 \\ 0.031 & 0.066 & 0.343 & 0.406 & 0.078 & 0.031 & 0.006 & 0.036 & 0.003 \end{pmatrix}$$

$Var(XY) = E(XY^2) - E(XY)^2 = 214.338 - 2.27$

$$XY^2 \sim \begin{pmatrix} 81 & 144 & 225 & 400 & 441 & 1225 \\ 0.406 & 0.078 & 0.374 & 0.072 & 0.036 & 0.034 \end{pmatrix}$$

$\text{Var}(5XY + 5) = 212.06$

$\text{Cov}(X,Y) = E(XY) - E(X) \cdot E(Y) = -1.5092 - (-0.44) \cdot 3.43 =$

$= -1.5092 + 1.5092 = 0$

$\rho(X,Y) = \text{Cov}(X,Y) / (\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}) = 0$

2. X = numărul de succese înainte de al 5-lea eșec

$\Rightarrow$ repartiție negativ binomială

$P(X) = C_{X-1}^{m-1} \, p^m \, (1-p)^{X-m}$

p = prob. de succes $= 0.91 \Rightarrow q = 1 - p = 0.09$

= prob. de eșec

$P(X) = C_X^4 \cdot (0.09)^5 \cdot (0.91)^{X-5}$ repartiția lui X

$E(8X - 9) = 8 \cdot E(X) - 9 = 8 \cdot 50.5556 - 9 = 395.4448$

$E(X) = (p \cdot m) / (1-p) = (0.91 \cdot 5) / 0.09 = 4.55 / 0.09 = 50.(5)$

$\text{Var}(9X + 3) = 81 \, \text{Var}(X) = 81 \cdot 561.7284 = 45500.0004$

$\text{Var}(X) = (p \cdot m) / (1-p)^2 = (0.91 \cdot 5) / 0.0081 = 561.7284$

3. X = nr. de puncte de pe zar

Y = nr. de capete de pe monedă

$p = 0.75$ = prob. să pice cap

a) $P(Y = 4 \mid X = 5) = ?$

$P(Y = 4 \mid X = 5) = \frac{1}{6} \cdot C_5^4 (0.75)^4 \cdot (0.25)^1 =$

$= \frac{1}{6} \cdot \frac{5!}{4!} \cdot \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4} = \frac{5}{6} \cdot \frac{81}{256} \cdot \frac{1}{4} = \frac{135}{2048} =$

$= 0.0659$

b) $P(Y=1 \cup Y=5) = ?$

$P(Y=1) + P(Y=5) = P(Y=1 \cup Y=5)$

$P(Y=1) = \sum_{x=1}^{6} C_x^1 (0.75)^1 (0.25)^{x-1}$, x = nr. de puncte de pe zar

$P(Y=1) = C_1^1 \cdot (0.75) \cdot (0.25)^0 + C_2^1 \cdot (0.75) \cdot (0.25)^1 +$

$+ C_3^1 \cdot (0.75) \cdot (0.25)^2 + C_4^1 \cdot (0.75) \cdot (0.25)^3 + C_5^1 \cdot (0.75) \cdot$

$\cdot (0.25)^4 + C_6^1 \cdot (0.75) \cdot (0.25)^5$

$P(Y=1) = (0.75) \cdot (1 \cdot 1 + 2 \cdot 0.25 + 3 \cdot 0.0625 + 4 \cdot 0.015625$

$+ 5 \cdot 0.00390625 + 6 \cdot 0.00097656) = 0.75 (1 + 0.5 +$

$+0.1875 + 0.0625 + 0.01953125 + 0.00585936) =$

$= 0.75 \cdot 1.77539061 = 1.33154296$

$P(Y=5) = \sum_{x=5}^{6} C_x^5 \cdot (0.75)^5 \cdot (0.25)^{x-5}$, $x \geq 5$

deoarece e imposibil să pice 5 capete pt. $x<5$

$P(Y=5) = C_5^5 (0.75)^5 (0.25)^0 + C_6^5 \cdot (0.75)^5 \cdot (0.25)^1 =$

$= (0.75)^5 (C_5^5 + C_6^5 \cdot 0.25) = 0.23730469 \cdot$

$(1 + 6 \cdot 0.25) = 0.23730469 \cdot 2.5 = 0.59326172$

$P(Y=1 \cup Y=5) = 1.33154296 + 0.59326172 =$

$= 1.92480468$

c) $P(X=2 \mid Y=0) = P(Y=0 \mid X=2) \cdot P(X=2) / P(Y=0)$

$P(Y=0 \mid X=2) = C_2^0 (0.75)^0 (0.25)^2 = 1 \cdot 1 \cdot 0.0625$

$= 0.0625$

$P(X=2) = \frac{1}{6}$

$P(Y=0) = \sum_{x=1}^{6} C_x^0 (0.75)^0 \cdot (0.25)^{x-0} =$

$= \sum_{x=1}^{6} (0.25)^x = 0.368408200625$

5. X, Y v.a. cu densitatea comună

$$f(x,y) = \begin{cases} 6a e^{-y}, & \text{dacă } 0<x<y \\ 0, & \text{altfel} \end{cases}$$

a) $a \in \mathbb{R}$

$f(x,y)$ = dens. de repart. $\Rightarrow \int\int f(x,y) = 1 \Leftrightarrow$

$$\Leftrightarrow \int_x \int_y f(x,y)\,dy\,dx = 1 \Leftrightarrow \int_{-\infty}^{0} \underbrace{f(x,y)\,dy}_{0}\,dx +$$

$$+ \int_0^y \int_x^{+\infty} f(x,y)\,dy\,dx = 1 \Leftrightarrow \int_x \int_y 6a e^{-y}\,dy\,dx = 1$$

$$\Leftrightarrow 6a \int_0^y \int_x^{\infty} e^{-y}\,dy\,dx = 1 \Leftrightarrow 6a \int_0^y e^{-x}\,dx = 1$$

$$\Leftrightarrow 6a(-e^{-y}+1) = 1 \Leftrightarrow -6a \cdot e^{-y} + 6a = 1 \Leftrightarrow$$

$$\Leftrightarrow 6a = 1/(-e^{-y}+1) \Rightarrow$$

$$\Rightarrow a = \frac{1}{6-6e^{-y}} \in \mathbb{R}$$

4. X_m = nr. puncte zar negru, $p_m = \frac{1}{3}$

X_a = nr. puncte zar alb, $p_a = \frac{1}{3}$

a) repartiția comună (X_m, X_a)

X_m \ X_a	1	2	3	
1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
2	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$P(X_m = 1, X_a = 1) = p_m \cdot p_a = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

analog pentru restul

b) $V = \min(X_m, X_a)$

$W = \max(X_m, X_a)$

$$V \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{5}{9} & \frac{3}{9} & \frac{1}{9} \end{pmatrix} \qquad W \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{9} & \frac{3}{9} & \frac{5}{9} \end{pmatrix}$$

c) repartiție comună (V, W)

V \ W	1	2	3	
1	$\frac{5}{81}$	$\frac{15}{81}$	$\frac{25}{81}$	$\frac{5}{9}$
2	$\frac{3}{81}$	$\frac{9}{81}$	$\frac{15}{81}$	$\frac{3}{9}$
3	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{1}{9}$
	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$	

d) $E(9V) = 9 \cdot E(V) = 9^{1} \cdot \frac{14}{9_{1}} = 14$

$E(V) = \min(1,1) \cdot \frac{1}{9} + \min(1,2) \cdot \frac{1}{9} + \ldots + \min(3,3) \cdot \frac{1}{9} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{1}{9} + \frac{2}{9} + \frac{3}{9} = \frac{14}{9}$

$E(6W) = 6 \cdot E(W) = \overset{2}{6} \cdot \frac{22}{9_{3}} = \frac{44}{3} = 14.66667$

$E(W) = \max(1,1) \cdot \frac{1}{9} + \max(1,2) \cdot \frac{1}{9} + \ldots + \max(3,2) \cdot \frac{1}{9} + \max(3,3) \cdot \frac{1}{9} = \frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{2}{9} + \frac{2}{9} + \frac{3}{9} + \frac{3}{9} + \frac{3}{9} + \frac{3}{9} = \frac{22}{9}$

e) $R = |X_m - X_a|$, $S = X_m + X_a$

$$R \sim \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

$$S \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ \frac{1}{9} & \frac{2}{9} & \frac{3}{9} & \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

$$E(10R) = 10 \cdot E(R) = 10\left((-2)\cdot\frac{1}{9} + (-1)\cdot\frac{2}{9} + 0\cdot\frac{3}{9} + 1\cdot\frac{2}{9} + 2\cdot\frac{1}{9}\right) = 10 \cdot 0 = 0$$

$$E(8S) = 8 \cdot E(S) = 8\cdot\left(2\cdot\frac{1}{9} + 3\cdot\frac{2}{9} + 4\cdot\frac{3}{9} + 5\cdot\frac{2}{9} + 6\cdot\frac{1}{9}\right) = 8\cdot 4 = 32$$

$$\text{Var}(3R - 11S + 2) = \text{Var}(3R - 11S) = \text{Var}(3R) + \text{Var}(-11S) + 2\,\text{Cov}(3R, -11S)$$

$$= 9\,\text{Var}(R) + 121\,\text{Var}(S) + 2\,\text{Cov}(3R, -11S) =$$

$$= 9 \cdot 1.3333 + 121 \cdot 1.3333 + 2 \cdot 0 =$$

$$= 12 + 161.3333 = 173.3333$$