Mean, Median and Mode in Binomial Distributions

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Summary While studying the median of the binomial distribution we discovered that the mean-median-mode inequality, recently discussed in Statistica Neerlandica by RUNNENBURG [4] and VAN ZWET [7] for continuous distributions, does not hold for the binomial distribution. If the mean is an integer, then mean = median = mode. In theorem 1 a sufficient condition is given for mode = median = rounded mean. If median and mode differ, the mean lies in between.

1 Introduction

Discrete distributions do not always have a unique median, contrary to a large class of continuous distributions considered by RUNNENBURG [4] and VAN ZWET [7]. A weak median of a random variable X will be a number m satisfying $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \leqslant \frac{1}{2}$. So, if for $m_1 < m_2$ both the left-hand tail $\mathrm{P}(X \leqslant m_1)$ and the right-hand tail $\mathrm{P}(X \geqslant m_2)$ equal $\frac{1}{2}$, then all m in the interval $[m_1, m_2]$ are weak medians. However, if such an interval does not exist, then there is a strong median, defined as the number m for which $\mathrm{P}(X \geqslant m-d) > \frac{1}{2}$ and $\mathrm{P}(X \leqslant m+d) > \frac{1}{2}$ for all positive d.

Before making some remarks on the median(s) of the binomial distribution in particular we give some notations.

$$\mathrm{b}(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for } k = 0, 1, 2, \ldots, n$$

$$\mathrm{B}(k; n, p) = \sum_{i=0}^{k} \mathrm{b}(i; n, p) \qquad \text{for } k = 0, 1, 2, \ldots, n$$

$$\lfloor x \rfloor = [x] = \text{largest integer not exceeding } x \text{ (entier)}$$

$$\lceil x \rceil = -[-x] = \text{smallest integer not less than } x$$

The binomial distribution $\mathrm{B}(n, p)$ has no strong median if and only if $\mathrm{B}(k-1; n, p) = \frac{1}{2}$ for some $k \in \{1, 2, \ldots, n\}$. Then $[k-1, k]$ is an interval of weak medians. The values $p_1, \ldots, p_n$ are defined by $\mathrm{B}(k-1; n, p_k) = \frac{1}{2}$. So $\mathrm{B}(n, p)$ has an interval of weak medians if $p = p_k$ for some k, otherwise it has a (unique) strong median.

Something similar holds for the mode. Consideration of the ratio $\mathrm{b}(k; n, p)/\mathrm{b}(k-1; n, p)$ shows that $\mathrm{b}(k; n, p)$ increases (as a function of k) for $k < (n+1)p$ and decreases for $k > (n+1)p - 1$. So, if $(n+1)p$ is not an integer, $\mathrm{b}(k; n, p)$ increases for $k \leqslant \lfloor (n+1)p \rfloor$ and decreases for $k \geqslant \lfloor (n+1)p \rfloor$ and $\lfloor (n+1)p \rfloor$ is the mode. If $(n+1)p$ is an integer $\mathrm{b}(k; n, p)$ increases for $k \leqslant (n+1)p - 1$ and decreases for $k \geqslant (n+1)p$; there are two neighbouring modes.

In this paper μ is used for the mean np, capital M will denote a (the) mode, and m stands for a (the) median. Whenever necessary the dependence on p of these quantities

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will be indicated by the notations $\mu(p)$, $M(p)$ and $m(p)$ respectively. Note that $M(p)$ and $m(p)$ will be considered here as more-valued functions.

2 A theorem on the median

Up to now not much was known of the median of the binomial distribution. NEUMANN [2, 3] proved, that, if the mean is an integer, it coincides with the median, not only for the binomial case but also for the Poisson and the hypergeometric case. The integer mean also equals the mode (one of the modes in the Poisson case), which the reader easily verifies, so then mean = median = mode, in perfect agreement with the familiar rule mean $\leqslant$ median $\leqslant$ mode.

For mean and mode usually simple arithmetic expressions exist, for the binomial median the following theorem gives a solution for about "half" of the values of p, as one sees by computing the total length of the p-intervals where the theorem is effective.

THEOREM 1. If X is $\mathrm{B}(n, p)$, the median can be found by rounding off np to, say, k, if the following condition holds:

$$|k-np| \leqslant \min\{p, 1-p\} \tag{2.1}$$

k is the strong median, except when $p = \frac{1}{2}$ and n is odd, then k is one of the end points of the interval of weak medians.

PROOF. If $p = \frac{1}{2}$ and n is odd, both $k = \frac{1}{2}(n-1)$ and $k = \frac{1}{2}(n+1)$ satisfy (2.1), the set of weak medians is the interval in between. If $p = \frac{1}{2}$ and n is even, $k = \frac{1}{2}n$ is the strong median. The case $p > \frac{1}{2}$ is reduced to $p < \frac{1}{2}$ by considering $n-X$, so $p < \frac{1}{2}$ is assumed from now on.

We shall show $\mathrm{P}(X \geqslant k) > \frac{1}{2}$ and $\mathrm{P}(X \leqslant k) > \frac{1}{2}$, which ensures k to be the strong median. Since $p < \frac{1}{2}$, (2.1) becomes

$$(n-1)p \leqslant k \leqslant (n+1)p \tag{2.2}$$

or

$$k/(n-1) \geqslant p \geqslant k/(n+1) \tag{2.3}$$

The proof depends on the following inequalities by UHLMANN [5, 6], to be found cited in [1]. (See also Remark 1 at the end of section 3.)

$$\mathrm{B}(k; n, k/(n-1)) > \tfrac{1}{2} > \mathrm{B}(k; n, (k+1)/(n+1)) \quad \textit{for } k < \tfrac{1}{2}(n-1) \tag{2.4}$$

For $k > \frac{1}{2}(n-1)$ the reverse inequalities hold, for $k = \frac{1}{2}(n-1)$ (n odd) all three terms are equal. Now by (2.2) $k-1 < \frac{1}{2}(n-1)$, so by the right-hand inequalities of (2.3) and (2.4)

$$P(X \geq k) = 1 - B(k-1; n, p) \geq 1 - B(k-1; n, k/(n+1)) > \tfrac{1}{2}$$

where the fact is used that $B(h; n, p)$ is decreasing in p (for $h = 0, 1, \ldots, n-1$), which follows from $(d/dp)\, B(h; n, p) = -nb(h; n-1, p)$. Next, now for $k < \frac{1}{2}(n-1)$, by the left-hand inequalities of (2.3) and (2.4)

$$P(X \leq k) = B(k; n, p) \geq B(k; n, k/(n-1)) > \tfrac{1}{2}$$

All that remains to be proved is $P(X \leq k) > \frac{1}{2}$ for $k = \frac{1}{2}(n-1)$ or $k = \frac{1}{2}n$. Since $p < \frac{1}{2}$, $B(k; n, p)$ exceeds $B(k; n, \frac{1}{2})$, which equals $\frac{1}{2}$ if $k = \frac{1}{2}(n-1)$ (n odd), or exceeds $\frac{1}{2}$ if $k = \frac{1}{2}n$.

3 The relative location of mean, median and mode

In section 1 it was pointed out that the median $m(p)$ is a step function of p with jumps in $p_1, \ldots, p_n$. The mode is a step function with jumps in $1/(n+1), 2/(n+1), \ldots, n/(n+1)$. With each jump median and mode pass the mean, the mean np passes median and mode in $1/n, 2/n, \ldots, (n-1)/n$. A consequence of theorem 1 is

$$p_i < i/(n+1) < i/(n-1) \leq p_{i+1} \qquad \text{for } 1 \leq i \leq \tfrac{1}{2}(n-1)$$

$$p_{i+1} \leq (i-2)/(n-1) < i/(n+1) < p_i \qquad \text{for } \tfrac{1}{2}(n+3) \leq i \leq n$$

This is illustrated in two figures for $n = 5$ and $n = 6$.

The figures clearly show, that median and mode either are equal or differ by 1, and that, if they differ, the median is closer to $\frac{1}{2}n$ than the mode, and the mean lies in between. These conclusions are formulated in the following two corollaries from theorem 1.

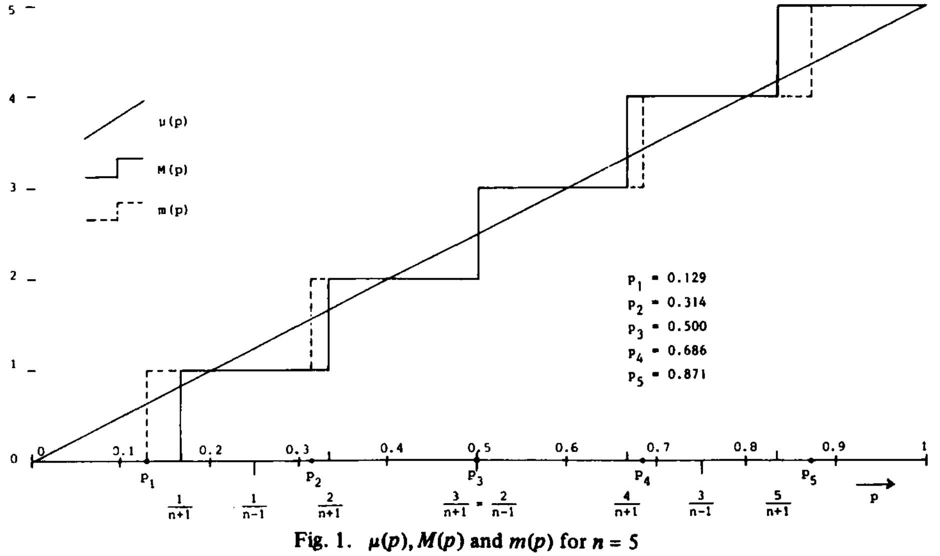

Fig. 1. $\mu(p)$, $M(p)$ and $m(p)$ for $n = 5$

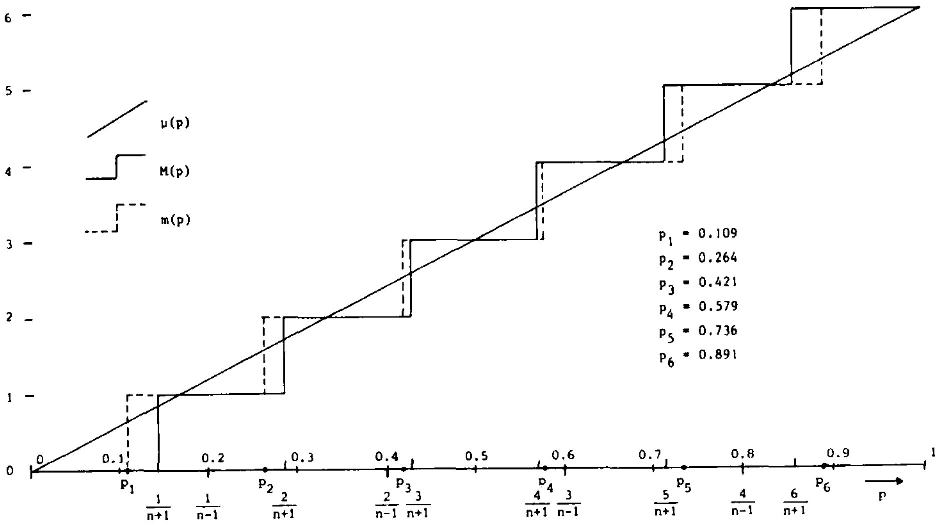

Fig. 2. $\mu(p), M(p)$ and $m(p)$ for $n = 6$

COROLLARY 1. For any median m, any mode M and the mean μ of the B(n, p) *distribution*

$$\lfloor \mu \rfloor \leqslant m, \mu, M \leqslant \lceil \mu \rceil \tag{3.1}$$

PROOF. If μ is an integer, then $\lfloor \mu \rfloor = \mu = m = M = \lceil \mu \rceil$, as was pointed out in the beginning of section 2. Therefore it is assumed that μ is not an integer, implying $\lfloor \mu \rfloor < \mu < \lceil \mu \rceil = \lfloor \mu \rfloor + 1$.

If $(n+1)p$ is an integer, then $(n+1)p$ and $(n+1)p-1$ are modes, both satisfying (3.1). If $(n+1)p$ is not an integer, then

$$\lfloor np \rfloor \leqslant \lfloor (n+1)p \rfloor = M \leqslant \lfloor np+1 \rfloor = \lceil \mu \rceil. \tag{3.2}$$

Now for the median, let $\lfloor np \rfloor = np_0$ and $\lceil np \rceil = np_1$. Being integers np_0 and np_1 are the strong medians of B(n, p_0) and B(n, p_1) respectively. Because B(h; n, p) is non-decreasing in p, it is obvious by the definition of the median that any median of a B(n,p) distribution is not less than that of a B(n, p_0) and not greater than that of a B(n, p_1) distribution, which completes the proof.

COROLLARY 2. Let a B(n, p) *distribution have a strong median m, a single mode M and mean μ. If median and mode differ, the mean lies in between. More specifically, if $m \neq M$ we have*

a) *if $p < \frac{1}{2}$ then $M < \mu < m$*
b) *if $p > \frac{1}{2}$ then $m < \mu < M$.*

PROOF. If $m \neq M$, then one of them equals $\lfloor np \rfloor$ and the other equals $\lceil np \rceil$ as a consequence of corollary 1. Then np cannot be an integer, and lies between m and M.

Suppose now $p < \frac{1}{2}$ and yet $m < M$. Then $M = \lceil np \rceil$, so $(n+1)p \geqslant \lfloor (n+1)p \rfloor = M = \lceil np \rceil \geqslant np$, which entails $|np - M| \leqslant p$ and then by theorem 1 $m = M$, which contradicts $m < M$. In the same way one may prove that $m \leqslant M$ if $p > \frac{1}{2}$.

One easily sees that the binomial mean and median do not differ by more than unity. A sharper bound is given below.

COROLLARY 3. For any median m of a $\mathrm{B}(n, p)$ *distribution*

$$|m - np| \leqslant \max\{p, 1-p\}. \tag{3.3}$$

PROOF. If np is an integer or if $p = \frac{1}{2}$, the situation is obvious. Therefore it is assumed that $p \neq \frac{1}{2}$ and $\lfloor np \rfloor < \lceil np \rceil$.

Now suppose that m does not satisfy (3.3). By (3.1) we have $\lfloor np \rfloor \leqslant m \leqslant \lceil np \rceil = \lfloor np \rfloor + 1$. So $|m - np| > \max\{p, 1-p\}$ entails that either $np - \lfloor np \rfloor < 1 - \max\{p, 1-p\} = \min\{p, 1-p\}$, or $\lceil np \rceil - np < \min\{p, 1-p\}$. But then, by theorem 1, either $\lfloor np \rfloor$ or $\lceil np \rceil$ is the strong median, which gives a contradiction.

Notice, that the inequality in (3.3) is strict, except for the case $p = \frac{1}{2}$ and n is odd.

REMARK 1. Actually UHLMANN's [5, 6] inequality (2.4) is a direct consequence of the mean-median-mode inequality for the Beta-distribution (see [4]). Let $Y_{r,s}$ have a Beta-distribution with density

$$y^{r-1}(1-y)^{s-1}\,\Gamma(r+s)\,\Gamma(r)^{-1}\,\Gamma(s)^{-1} \quad 0 < y < 1 \tag{3.4}$$

Mode and mean are

$$M_b = (r-1)/(r+s-2) \tag{3.5}$$

$$\mu_b = r/(r+s) \tag{3.6}$$

The connection between binomial and Beta-distributions is as follows.

$$\mathrm{B}(i; n, p) = \mathrm{P}(Y_{i+1,n-i} > p). \tag{3.7}$$

If $i + 1 < n - i$, i.e. $i < \frac{1}{2}(n-1)$, the inequality mode $<$ median $<$ mean holds, so

$$\mathrm{P}(Y_{i+1,n-i} > M_b) > \mathrm{P}(Y_{i+1,n-i} > m_b) > \mathrm{P}(Y_{i+1,n-i} > \mu_b)$$

or, by (3.5), (3.6) and the fact that m_b is the median,

$$\mathrm{P}(Y_{i+1,n-i} > i/(n-1)) > \tfrac{1}{2} > \mathrm{P}(Y_{i+1,n-i} > (i+1)/(n+1))$$

So, by (3.7),

$$B(i;n,i/(n-1)) > \tfrac{1}{2} > B(i;n,(i+1)/(n+1))$$

In a similar way it can be proved that the mean-median-mode inequality for gamma-distributions implies that an integer Poisson mean is also the median.

REMARK 2. VAN ZWET [7] proves, that, in a wide class of continuous distributions, a sufficient condition for $M \leqslant m \leqslant \mu$ is given by

$$P(X \leqslant m-d) \leqslant P(X \geqslant m+d) \quad \text{for all } d > 0 \tag{3.8}$$

It is easy to find a binomial distribution with strong median m not satisfying (3.8), e.g. the B(3, 0.24) distribution with $m = 1$, $P(X \leqslant m-1) > P(X \geqslant m+1)$, but $P(X \leqslant m-2) < P(X \geqslant m+2)$. Moreover, condition (3.8), though sufficient for $m \leqslant \mu$, is not sufficient for $M \leqslant m$. A counter example is B(6, 0.575), having $m = 3 < \mu = 3.45 < M = 4$ (see fig. 2, $4/7 < 0.575 < p_4$), but one easily verifies that (3.8) holds, because $p > \frac{1}{2}$ and $m = \frac{1}{2}n$.

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