

Matemáticas para las Ciencias Aplicadas III
Tarea 2

27 de septiembre de 2018

Sección 14.2

15-18 Evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

18.

$\iint_R y \, dA$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

19-24. Evaluate the double integral in two ways using iterated integrals:

22. $\iint_R x \, dA$; R is the region enclosed by $y = \sin^{-1}x$, $x = \frac{1}{\sqrt{2}}$, and $y = 0$.

Solución:

Entonces $0 \leq y \leq \frac{\pi}{4}$ y x va desde $x = \sin y$ a $x = \frac{1}{\sqrt{2}}$ por lo que se tiene

$$\begin{aligned} \iint_R x \, dA; R &= \int_0^{\frac{\pi}{4}} \int_{\sin y}^{\frac{1}{\sqrt{2}}} x \, dx \, dy \\ &= \int_0^{\frac{\pi}{4}} \frac{x^2}{2} \Big|_{\sin y}^{\frac{1}{\sqrt{2}}} dy \\ &= \int_0^{\frac{\pi}{4}} \left(\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2} - \frac{(\sin y)^2}{2} \right) dy \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{4} - \frac{\sin^2 y}{2} dy \\ &= \frac{1}{4} \int_0^{\frac{\pi}{4}} 1 - 2\sin^2 y \, dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_0^{\frac{\pi}{4}} \cos 2y dy \\
&= \frac{1}{4} \left(\frac{1}{2} \sin 2y \Big|_0^{\frac{\pi}{4}} \right) \\
&= \frac{1}{8} \left(\sin 2 \left(\frac{\pi}{4} \right) - \sin 2(0) \right) \\
&= \frac{1}{8} (1 - 0) \\
&= \frac{1}{8}
\end{aligned}$$

28.

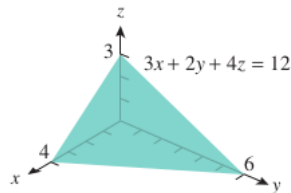
(a) By hand or with the help of a graphing utility, make a sketch of the region R enclosed between the curves $y = 4x^3 - x^4$ and $y = 3 - 4x + 4x^2$.

(b) Find the intersections of the curves in part (a).

(c) Find $\iint_R x \, dA$

37-38 Use double integration to find the volume of the solid.

37.



57. Try to evaluate the integral with a CAS using the stated order of integration, and then by reversing the order of integration.

Utilizaré Wolfram Mathematica para evaluar las integrales

$$(a) \int_0^4 \int_{\sqrt{x}}^2 \sin \pi y^3 \, dx dy$$

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In[6]:= (* 57 a) *)
(* Evaluando con el orden establecido*)
Integrate[Sin[Pi * y^3], {x, 0, 4}, {y, Sqrt[x], 2}]
Out[6]= 0

(*Evaluando con el orden invertido*)
Integrate[Sin[Pi * y^3], {y, 0, 2}, {x, 0, y^2}]
Out[4]= 0

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Obtuvimos que la integral siguiendo el orden establecido es 0 al igual que si invertimos el orden de la integral.

$$(b) \int_0^1 \int_{\sin^{-1}y}^{\frac{\pi}{2}} \sec \cos x^2 \, dx dy$$

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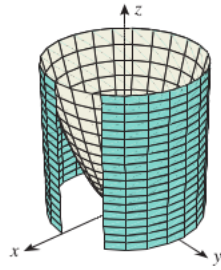
In[15]:= (* 57 b) *)
(* Evaluando con el orden establecido *)
Integrate[Sec[Cos[x]]^2, {y, 0, 1}, {x, ArcSin[y], Pi/2}]
Out[15]= Sec[Cos[x]]^2

(* Evaluando con el orden invertido *)
Integrate[Sec[Cos[x]]^2, {x, 0, Pi/2}, {y, 0, Sin[x]}]
Out[12]= Tan[1]

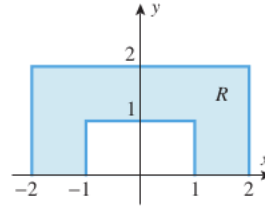
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La integral siguiendo el orden establecido no se puede calcular en Mathematica, y la integral invirtiendo el orden es $\tan(1)$

59. Evaluate $\iint_R xy^2 \, dA$ over the region R shown in the accompanying figure.



▲ Figure Ex-58



▲ Figure Ex-59

Solución:

Como podemos ver en la figura la podemos dividir en 3 partes

$$1. -2 \leq x \leq -1, 0 \leq y \leq 2$$

$$2. -1 \leq x \leq 1, 1 \leq y \leq 2$$

$$3. 1 \leq x \leq 2, 0 \leq y \leq 2$$

Entonces

$$\iint_R xy^2 \, dA = \int_{-2}^{-1} \int_0^2 xy^2 \, dy \, dx + \int_{-1}^1 \int_1^2 xy^2 \, dy \, dx + \int_1^2 \int_0^2 xy^2 \, dy \, dx \quad (1)$$

Ahora calcularemos cada una de las integrales por separado

- Calculemos la integral en $-2 \leq x \leq -1, 0 \leq y \leq 2$

$$\begin{aligned} \int_{-2}^{-1} \int_0^2 xy^2 \, dy \, dx &= \int_{-2}^{-1} \left(\frac{xy^3}{3} \right) \Big|_0^2 \, dx \\ &= \int_{-2}^{-1} \frac{8x}{3} \, dx \\ &= \frac{8}{3} \int_{-2}^{-1} x \, dx \\ &= \frac{8}{3} \left(\frac{x^2}{2} \right) \Big|_{-2}^{-1} \\ &= \frac{8}{3} \left(-\frac{3}{2} \right) \\ &= -4 \end{aligned}$$

- Calculemos la integral en $-1 \leq x \leq 1$, $1 \leq y \leq 2$

$$\begin{aligned}
 \int_{-1}^1 \int_1^2 xy^2 \, dy \, dx &= \int_{-1}^1 \left(\frac{xy^3}{3} \right) \Big|_1^2 \, dx \\
 &= \int_{-1}^1 \left(\frac{8x}{3} - \frac{x}{3} \right) \, dx \\
 &= \frac{7x}{3} \int_{-1}^1 x \, dx \\
 &= \frac{7x}{3} \left(\frac{x^2}{2} \right) \Big|_{-1}^1 \\
 &= \frac{7x}{3} \left(\frac{(1)^2}{2} - \frac{(-1)^2}{2} \right) \\
 &= \frac{7x}{3} (0) \\
 &= 0
 \end{aligned}$$

- Calculemos la integral en $1 \leq x \leq 2$, $0 \leq y \leq 2$

$$\begin{aligned}
 \int_1^2 \int_0^2 xy^2 \, dy \, dx &= \int_1^2 \left(\frac{xy^3}{3} \right) \Big|_0^2 \, dx \\
 &= \int_1^2 \frac{8x}{3} \, dx \\
 &= \frac{8}{3} \int_1^2 x \, dx \\
 &= \frac{8}{3} \left(\frac{x^2}{2} \right) \Big|_1^2 \\
 &= \frac{8}{3} \left(\frac{4}{2} - \frac{1}{2} \right) \\
 &= 4
 \end{aligned}$$

Entonces sustituyendo los valores en (1) se tiene

$$\iint_R xy^2 \, dA = 4 + 0 + 4 = 8$$

63. Suppose that the temperature in degrees Celsius at a point (x, y) on a flat metal plate is $T(x, y) = 5xy + x^2$, where x and y are in meters. Find the average temperature of the diamond-shaped portion of the plate for which $|2x + y| \leq 4$ and $|2x - y| \leq 4$.

Sección 14.5

12. $\iiint_G \cos \frac{z}{y} dV$, where G is the solid defined by the inequalities $\frac{\pi}{6} \leq y \leq \frac{\pi}{2}$, $y \leq x \leq \frac{\pi}{2}$, $0 \leq z \leq xy$.

37. Let G be the tetrahedron in the first octant bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, (a > 0, b > 0, c > 0)$$

(a) List six different iterated integrals that represent the volume of G .

Solución: Para este plano los trazos están en $x = a$, $y = b$ y $z = c$ y permutando el orden de integración, podemos escribir la integral de las siguientes 6 formas:

$$1. \text{ Volume of } G = \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} dx dy dz$$

$$2. \text{ Volume of } G = \int_0^b \int_0^{c(1-\frac{y}{b})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} dx dz dy$$

$$3. \text{ Volume of } G = \int_0^a \int_0^{c(1-\frac{x}{a})} \int_0^{b(1-\frac{x}{a}-\frac{z}{c})} dy dz dx$$

$$4. \text{ Volume of } G = \int_0^c \int_0^{a(1-\frac{z}{c})} \int_0^{b(1-\frac{x}{a}-\frac{z}{c})} dy dx dz$$

$$5. \text{ Volume of } G = \int_0^b \int_0^{a(1-\frac{y}{b})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz dx dy$$

$$6. \text{ Volume of } G = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz dy dx$$

(b) Evaluate any one of the six to show that the volume of G is $\frac{1}{6}abc$.

Solución: Calculemos el volumen de G con la siguiente integral

$$\begin{aligned}
 \text{Volume of } G &= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz \, dy \, dx \\
 &= \int_0^a \int_0^{b(1-\frac{x}{a})} c \left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \, dx \\
 &= c \left(\int_0^a \int_0^{b(1-\frac{x}{a})} 1 - \frac{x}{a} - \frac{y}{b} dy \, dx \right) \\
 &= c \left(\int_0^a \left(y - \frac{xy}{a} - \frac{y^2}{2b} \Big|_0^{b(1-\frac{x}{a})} \right) dx \right) \\
 &= c \left(\int_0^a \left(b \left(1 - \frac{x}{a}\right) - \frac{x \left(b \left(1 - \frac{x}{a}\right)\right)}{a} - \frac{\left(b \left(1 - \frac{x}{a}\right)\right)^2}{2b} \right) dx \right) \\
 &= c \left(\int_0^a \frac{b(a-x)}{a} - \frac{bx(a-x)}{a^2} - \frac{b(a-x)^2}{2a^2} dx \right) \\
 &= c \left(\int_0^a \frac{b(a-x)^2}{2a^2} dx \right) \\
 &= c \left(\int_0^a \frac{b}{2} - \frac{bx}{a} + \frac{bx^2}{2a^2} dx \right) \\
 &= c \left(\frac{bx}{2} - \frac{bx^2}{2a} + \frac{bx^3}{6a^2} \right) \Big|_0^a \\
 &= c \left(\frac{ab}{2} - \frac{ab}{2} + \frac{ab}{6} \right) \\
 &= \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{6} \\
 &= \frac{abc}{6}
 \end{aligned}$$

38. Use a triple integral to derive the formula for the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Hughes-Hallet

Sección 16.2

35.

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3} \, dx \, dy$$

37.

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x} \, dx \, dy$$

Solución: Cambiaremos el orden de integración para que sea más fácil integrar

$$\begin{aligned} \int_0^1 \int_{e^y}^e \frac{x}{\ln x} \, dx \, dy &= \int_1^e \int_0^{\ln x} \frac{x}{\ln x} \, dy \, dx = \int_1^e \frac{x}{\ln x} \cdot y \Big|_0^{\ln x} \, dy \, dx = \int_1^e \frac{x}{\ln x} \cdot \ln x \, dx \\ &= \int_1^e x \, dx = \frac{x^2}{2} \Big|_1^e = \frac{e^2}{2} - \frac{1^2}{2} = \frac{e^2 - 1}{2} \end{aligned}$$

60. Show that for a right triangle the average distance from any point in the triangle to one of the legs is one-third the length of the other leg. (The legs of a right triangle are the two sides that are not the hypotenuse.)

62. Find the area of the crescent-moon shape with circular arcs as edges and the dimensions shown in Figure 16.22.

Sección 16.3

In Problems 14–18, decide whether the integrals are positive, negative, or zero. Let S be the solid sphere $x^2 + y^2 + z^2 \leq 1$, and T be the top half of this

sphere (with $z \geq 0$), and B be the bottom half (with $z \leq 0$), and R be the right half of the sphere (with $x \geq 0$), and L be the left half (with $x \leq 0$).

14.

$$\int_T e^z \, dV$$

15.

$$\int_B e^z \, dV$$

16.

$$\int_S \sin z \, dV$$

17.

$$\int_T \sin z \, dV$$

18.

$$\int_R \sin z \, dV$$

31. A trough with triangular cross-section lies along the x -axis for $0 \leq x \leq 10$. The slanted sides are given by $z = y$ and $z = -y$ for $0 \leq z \leq 1$ and the ends by $x = 0$ and $x = 10$, where x, y, z are in meters. The trough contains a sludge whose density at the point (x, y, z) is $\delta = e^{-3x}$ kg per m^3 .

a) Express the total mass of sludge in the trough in terms of triple integrals.

Solución:

Encontremos la masa de lodo de un lado

$$\int_0^{10} \int_0^1 \int_z^1 e^{-3x} dy dz dx$$

Como la depresión es simétrica con el plano xz, podemos encontrar la masa de lado y duplicarla, por lo que la masa total esta dada por

$$\text{Total mass of sludge} = 2 \int_0^{10} \int_0^1 \int_z^1 e^{-3x} dy dz dx$$

b) Express the total mass of sludge in the trough in terms of triple integrals.

Solución:

Evalúemos la ecuación encontrada en el inciso anterior

$$\begin{aligned} \text{Total mass of sludge} &= 2 \int_0^{10} \int_0^1 \int_z^1 e^{-3x} dy dz dx = 2 \int_0^{10} \int_0^1 (ye^{-3x}) \Big|_z^1 dz dx \\ &= 2 \int_0^{10} \int_0^1 e^{-3x} - e^{-3x} z dz dx = 2 \int_0^{10} e^{-3x} z - \frac{e^{-3x} z^2}{2} \Big|_0^1 dz \\ &= 2 \int_0^{10} e^{-3x} - \frac{e^{-3x}}{2} dz = 2 \int_0^{10} \frac{e^{-3x}}{2} dz \end{aligned}$$

Sea $u = -3x$ se tiene que

$$= 2 \int_0^{30} -\frac{e^u}{6} du = 2 \left(-\frac{e^{-3x}}{6} \right) \Big|_0^{10} = 2 \left(-\frac{e^{-30}}{6} + 1 \right) = \frac{1 - e^{-30}}{3}$$

Problems 54–56 refer to Figure 16.27, which shows triangular portions of the planes $2x + 4y + z = 4$, $3x - 2y = 0$, $z = 2$, and the three coordinate planes $x = 0$, $y = 0$, and $z = 0$. For each solid region E, write down an iterated integral for the triple integral

$$\int_E f(x, y, z) dV$$

.

55. E is the region bounded by $x = 0$, $y = 0$, $z = 0$, $z = 2$, and $2x + 4y + z = 4$.

Solución:

Integraremos en dirección x . La parte trasera de E será descrita por el plano $x = 0$, ahora encontraremos la parte delantera de E

Despejamos a x en $2x + 4y + z = 4$

$$2x = 4 - 4y - z$$

$$x = 2 - 2y - \frac{z}{2}$$

Entonces la parte delantera de E será descrita por el plano $x = 2 - 2y - \frac{z}{2}$. Ahora integremos en dirección y . Para ello tenemos que encontrar los límites de integración, proyectando E en el plano yz se tiene que $y = 0$ y en $4y + z = 4$ despejamos y , entonces $y = 1 - \frac{z}{4}$ y como z va de 0 a 2 la integral iterada es

$$\int_0^2 \int_0^{1-\frac{z}{4}} \int_0^{2-2y-\frac{z}{2}} f(x, y, z) dx dy dz$$

57. Figure 16.28 shows part of a spherical ball of radius 5 cm. Write an iterated triple integral which represents the volume of this region.

Solución:

La ecuación de la esfera es $x^2 + y^2 + z^2 = 5^2$, queremos encontrar el volumen entre los planos $z = 2$ y $z = 3$, sustituimos el valor de z en $x^2 + y^2 + z^2 = 5^2$ para encontrar el círculo en el que $z = 3$ corta a la esfera.

$$x^2 + y^2 + 9 = 25$$

$$x^2 + y^2 = 25 - 9$$

$$x^2 + y^2 = 16$$

Ahora encontremos los límites de integración, $-4 \leq x \leq 4$, $-\sqrt{16-x^2} \leq y \leq \sqrt{16-x^2}$ y $3 \leq z \leq \sqrt{25-x^2-y^2}$ Por lo que la integral triple iterada es:

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_3^{\sqrt{25-x^2-y^2}}$$

66. Find the center of mass of the tetrahedron that is bounded by the xy , yz , xz planes and the plane $x + 2y + 3z = 1$. Assume the density is 1

gm/cm^3 and x, y, z are in centimeters.

Problems 67–69 concern a rotating solid body and its *moment of inertia* about an axis; this moment relates angular acceleration to torque (an analogue of force). For a body of constant density and mass m occupying a region W of volume V , the moments of inertia about the coordinate axes are

$$I_x = \frac{m}{V} \int_W (y^2 + z^2) dV$$

$$I_y = \frac{m}{V} \int_W (x^2 + z^2) dV$$

$$I_z = \frac{m}{V} \int_W (x^2 + y^2) dV$$

67. Find the moment of inertia about the z -axis of the rectangular solid of mass m given by $0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 3$.

Solución:

Tenemos que calcular

$$I_z = \frac{m}{V} \int_W (x^2 + y^2) dV \quad (1)$$

para ello primero encontraremos el valor de V

$$V = 1 \cdot 2 \cdot 3 = 6$$

Sustituyendo el valor de V en (1) se tiene

$$I_z = \frac{m}{6} \int_W (x^2 + y^2) dV$$

Sabemos por La integral triple como una integral iterada que

$$\int_W f dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x, y, z) dx \right) dy \right) dz$$

Entonces se tiene que

$$\frac{m}{6} \int_W (x^2 + y^2) dV = \frac{m}{6} \int_0^3 \left(\int_0^2 \left(\int_0^1 x^2 + y^2 dx \right) dy \right) dz$$

$$\begin{aligned}
&= \frac{m}{6} \int_0^3 \left(\int_0^2 \left(\frac{x^3}{3} + xy^2 \Big|_0^1 \right) dy \right) dz \\
&= \frac{m}{6} \int_0^3 \left(\int_0^2 \left(\frac{(1)^3}{3} + (1)y^2 \Big|_0^1 \right) dy \right) dz \\
&= \frac{m}{6} \int_0^3 \left(\int_0^2 \frac{1}{3} + y^2 dy \right) dz \\
&= \frac{m}{6} \int_0^3 \left(\frac{y}{3} + \frac{y^3}{3} \right) \Big|_0^2 dz \\
&= \frac{m}{6} \int_0^3 \left(\frac{2}{3} + \frac{2^3}{3} \right) dz \\
&= \frac{m}{6} \left(\frac{10}{3} z \right) \Big|_0^3 \\
&= \frac{m}{6} \left(\frac{10}{3} \beta \right) \\
&= \frac{5m}{3}
\end{aligned}$$

Are the statements in Problems 74–83 true or false? Give reasons for your answer.

75. The region of integration of the triple iterated integral $\int_0^1 \int_0^1 \int_0^x f dz dy dx$ lies above a square in the xy -plane and below a plane.

78. The iterated integrals $\int_{-1}^1 \int_0^1 \int_0^{1-x^2} f dz dy dx$ and $\int_0^1 \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} f dx dy dz$ are equal.

Solución:

Los limites de la primera integral $-1 \leq x \leq 1$, $0 \leq y \leq 1$ y $0 \leq z \leq 1 - x^2$

Los limites de la segunda integral $0 \leq z \leq 1$, $0 \leq y \leq 1$ y $-\sqrt{1-z} \leq x \leq \sqrt{1-z}$

Como podemos ver ambas integrales se encuentran debajo del cilindro parabólico debido a que $0 \leq z \leq 1 - x^2$ en la integral 1 y $-\sqrt{1-z} \leq x \leq \sqrt{1-z}$ en la integral 2, entonces $z = 1 - x^2$, además describen el rectángulo $-1 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$

Por lo tanto, es verdadera

80. If W is the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$ and $\int_W f \, dV = 0$, then $f = 0$ everywhere in the unit.