Spring 2018 MATH4080 Case Studies in Mathematical Modeling Mid-term Projects

Due Date: 13 May, 23:59

Instruction: Submit your answers and codes to SEPARATE links on iSpace. Late or incorrect submissions will not be graded. Each question carries the same weight.

1.(*Mixing tanks*) Two very large tanks *A* and *B* are each partially filled with 100 gallons of brine. Initially, 100 pounds of salt is dissolved in the solution in tank *A* and 50 pounds of salt is dissolved in the solution in tank *B*. The system is closed in that the well-stirred liquid is pumped only between the tanks, as shown in Figure 1.

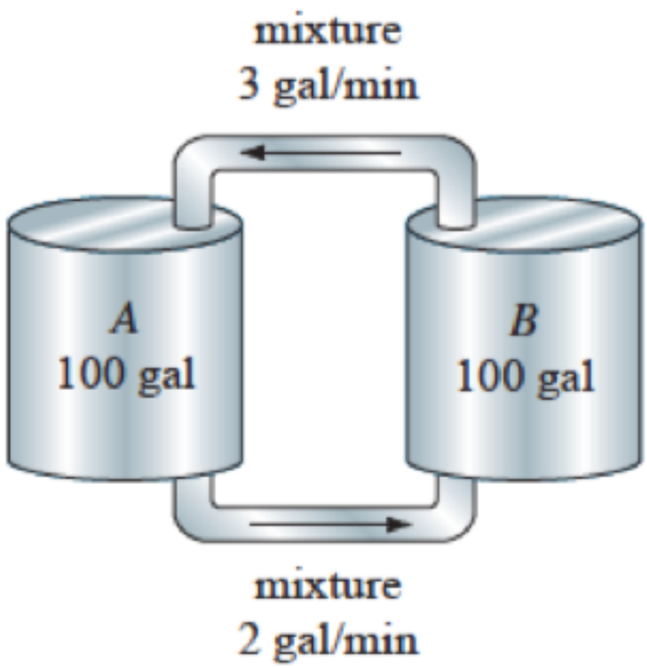

Figure 1: Container within a container.

(a)Use the information given in the figure to construct a mathematical model for the number of pounds of salt $x1(t)$ and $x2(t)$ at time t in tanks *A* and *B*, respectively.

(b)Find a relationship between the variables $x1(t)$ and $x2(t)$ that holds at time t. Explain why this relationship makes intuitive sense. Use this relationship to help find the amount of salt in tank B at t = 30 min.

Answer:

> (a) The differential equations corresponding to the given model are
>
> $$\frac{dx_1}{dt} = (3gal/min)(\frac{x_2}{100-t}lb/gal) - (2gal/min)(\frac{x_2}{100+t}lb/gal)$$
> $$= \frac{3x_2}{100-t} - \frac{2x_1}{100+t}$$
> $$\frac{dx_2}{dt} = (2gal/min)(\frac{x_1}{100+t}lb/gal) - (cgal/min)(\frac{x_2}{100-t}lb/gal)$$
> $$= \frac{2x_1}{100+t} - \frac{3x_2}{100-t}$$
>
> The the system is:
>
> $$\frac{dx_1}{dt} = \frac{3x_2}{100-t} - \frac{2x_1}{100+t}$$
> $$\frac{dx_2}{dt} = \frac{2x_1}{100+t} - \frac{3x_2}{100-t} \text{ a long with the intial conditons } x_1(0) = 100, x_2(0) = 50$$
>
> (b) According to the question 'The system is closed', that is to say the salt's quality will not be added or lost, we can get this equation: $x_1(t) + x_2(t) = 150$; at the point $t = 30$ we can get $x_2 = 47.4lb$

2.(*Newton's Law of cooling/warming*) As shown in Figure 2, a small metal bar is placed inside container *A*, and container *A* then is placed within a much larger container *B*. As the metal bar cools, the ambient temperature $T_A(t)$ of the medium within container A changes according to Newton's law of cooling. As container A cools, the temperature of the medium inside container B does not change significantly and can be considered to be a constant T_B . Construct a mathematical model for the temperatures $T(t)$ and $T_A(t)$, where $T(t)$ is the temperature of the metal bar inside container *A*. Find a solution of the system subject to the initial conditions $T(0) = T_0$, $T_A(0) = T_1$.

(Assume here that the temperature, $T(t)$ of the metal bar does not affect the temperature, $T_A(t)$ of the medium in container *A*)

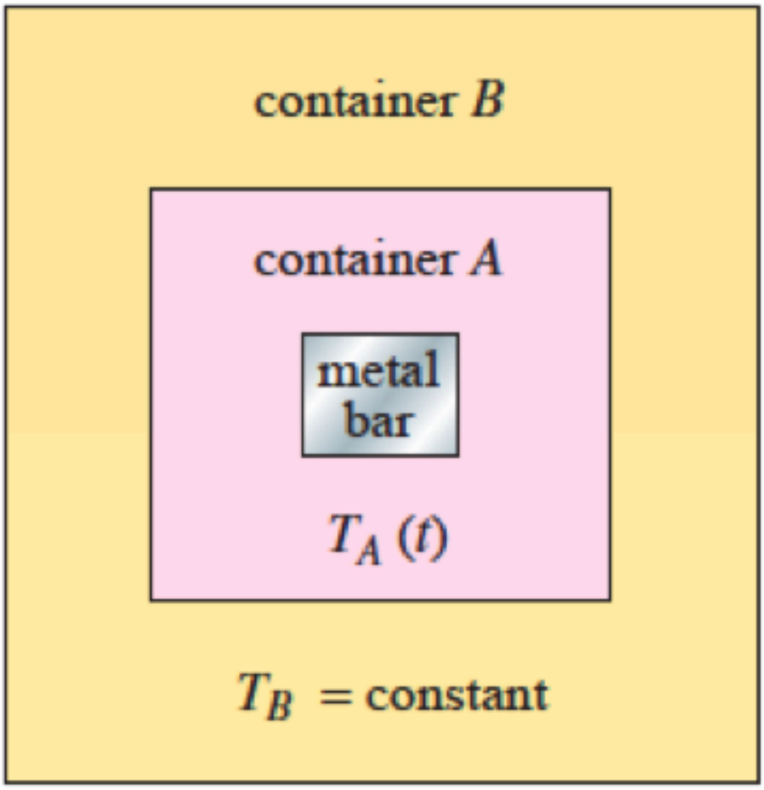

Figure 2: Container within a container.

Answer:

> Assume that the coefficient between container B and container A is k , According to the Newton Law of Cooling, we can get:
>
> The metal bar cools by conduction to container A, and loses heat at a rate proportional to the temperature difference so:
>
> $\frac{dT}{dt} = -k_1(T(t) - T_A(t))$
>
> Meanwhile container A has heat incoming from the metal bar, and is losing heat to container B, so
>
> $\frac{dT_A}{dt} = -k_2(T_A(t) - T_B) + k_1(T(t) - T_A(t)) = -k_2(T_A(t) - T_B) - \frac{dT}{dt}$
>
> So,

> $$\frac{dT_A(t)}{dt} = -k(T_A - T_B$$
> $$T_A - T_B = Ce^{-kt}$$
> $$T_A(0) - T_B = Ce^{-k0} = C$$
> $$T_1 = T_B = C$$
> $$\text{so, } T_A(t) = (T_1 - T_B)e^{-kt}$$
> $$\frac{dT}{dt} = -K(T - T_A)$$
> $$\frac{dT}{dt} + KT = KT_A$$
> $$e^{Kt}\frac{dT}{dt} + KT = KT_A$$
> $$\frac{d(e^{K1})}{dt} = KT_Ae^{K1}$$
> $$T = K(T_1 - T(B)\frac{e^{(K-k)t}}{(K-k)e^{Kt}} + C$$
> $$T(0) = T_0$$
> $$C = T_0 - \frac{K(T_1 - T_B)}{(K-k)}$$
> $$\text{We can get: } T(t) = K(T_1 - T_B)\frac{e^{(K-kt)}}{(K-k)e^{kt}} + T_o - \frac{K(T_1 - T_B)}{K-k}$$

3.(*Electoral system / Voting theory*) An electoral system is the system that determines how elections and referendums take place and how their results are arrived at. Different rules of voting may sometimes lead to different winner(s) even if the preferences of constituents remain the same.

Consider the following voting rules (assume only a single winner)

1. *Plurality method*: A single round of election is held. The candidate who polls the most among their counterparts is elected.
2. *Single runoff method*: A preliminary round of election is held and two candidates who receive the most number of votes advance to the final round. The final round is then based on the *plurality method*. The winner of the final round is chosen as the elect.
3. *Instant runoff method*: Multiple rounds of election are held. In each round, the candidate who receive the least number of votes is eliminated. When there are only two candidates remaining, use the *plurality method* to determine the elect.
4. *Coombs method:* Multiple rounds of election are held. In each round, the con- stituents are required to rank every candidate in terms of their preferences. The candidate who received the most number of votes for the last place is eliminated.

When there are only two candidates remaining, use the *plurality method* to determine the elect.

5. *Borda count:* A single round of election is held. The constituents are required to rank

every candidate in terms of their preferences. For each vote casted, the candidate in the last place gets 1 mark, the candidate in the second last place gets 2 marks, ..., the candidate in the first place gets *n* marks (*n* is the total number of candidates). The candidate who gets the highest mark is the elect.

(a)Consider the following voting results from 30 students on their favourite sports team. There are 4 teams and each student ranks them from the first (most like) and the fourth (least like). For example, 11 students vote B as the first, D as the second, C as the third and A as the fourth. Assume that the voters' preferences do not change if multiple rounds of voting are necessary. For each voting rule above, determine the winner.

Votes	**11**	**10**	**9**
First	B	C	A
Second	D	D	D
Third	C	A	C
Fourth	A	B	B

(b)Investigate the 2002 French presidential election. Which rule was used? Write a note to discuss the effect if 1% of all voters for Jean-Marie Le Pen instead voted for Jacques Chirac in the first round (other votes remain unchanged - see the tables below).

You will need to get additional information on your own to draw the conclusion.

Answer:

> (a)
>
> 1. Plurality method: B is the winner
> 2. Single runoff method: for the first round we will get B and C; I assume that the nine people who vote for A will vote C (because in the first round those 9 people think C is better then B) then the result is C get 19 point and B get 11; C is the winner
> 3.

Table 1: Real result (first round)

Candidate	**%**	**Rank**
Jacques Chirac	19.88%	1
Jean-Marie Le Pen	16.86%	2
Lionel Jospin	16.18%	3
.	.	

Table 2: Hypothesised result (first round)

Candidate	%	Rank
Jacques Chirac	20.88%	1
Lionel Jospin	16.18%	2
Jean-Marie Le Pen	15.86%	3
.	.	

4.(*Secretary problem / Marriage problem*) Use simulation to demonstrate the secretary/marriage problem (refer to case study 5) with $N = 10$ and $N = 50$ candidates, respectively. For each N , obtain the probability that the best candidate is chosen $p_N(k)$ for $k = 1, 2, \ldots N$

For each value of k, perform sufficient number of independent runs such that the 95% confidence interval with $U_r \leq 0.01$ is obtained for $p_N(k)$. Use your simulation results to plot a figure (see Figure 3 as an example, the vertical error bar at each point represents confidence interval) to illustrate the relationship between k an $p_N(k)$.

For more information, see: https://en.wikipedia.org/wiki/Secretary_problem or

http://www.math.uah.edu/stat/urn/Secretary.html

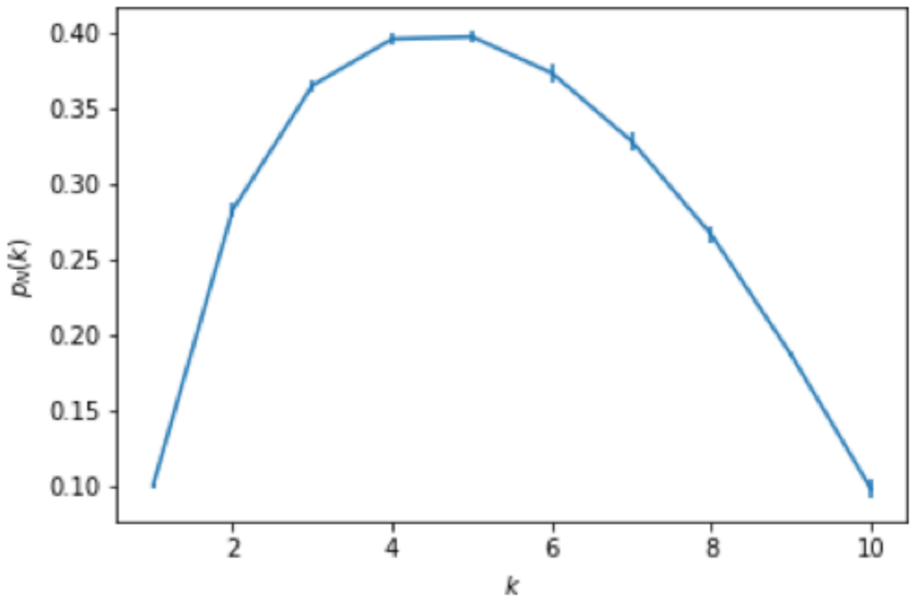

Figure 3: Expected output for Q4