# CHI-SQUARE TESTS

# Chi-Square Tests

- A Chi-Square Test for Independence
- Chi-Square Goodness-of-Fit Tests

# Contingency Table Example

Left-Handed vs. Gender

Dominant Hand: Left vs. Right

Gender: Male vs. Female

Sample results organized in a contingency table:

sample size = n = 300:

120 Females, 12 were left handed

180 Males, 24 were left handed

| Gender | Hand Preference | | |
|---|---|---|---|
| | Left | Right | |
| Female | 12 | 108 | 120 |
| Male | 24 | 156 | 180 |
| | 36 | 264 | 300 |

# Chi-square Test on Independence

Consider a contingency table with *r* rows and *c* columns, test

- $H_0$: The two categorical variables are independent, i.e., there is no relationship between them
- $H_1$: The two categorical variables are dependent.

The Chi-square test statistic is

$$\chi^2 = \sum_{\text{allcells}} \frac{(f_{ij} - \hat{E}_{ij})^2}{\hat{E}_{ij}}$$

- where:
- $f_{ij}$ = observed cell frequency for $i^{th}$ row and $j^{th}$ column
- $r_i$ = $i^{th}$ row total, $c_j$ = $j^{th}$ column total

$\hat{E}_{ij} = \frac{r_i c_j}{n} =$ expected cell frequency for $i^{th}$ row and $j^{th}$ column under independence

(Assumed: each cell in the contingency table has expected frequency of at least 5)

# Chi-square Test on Independence

Consider a contingency table with *r* rows and *c* columns,

The test statistic

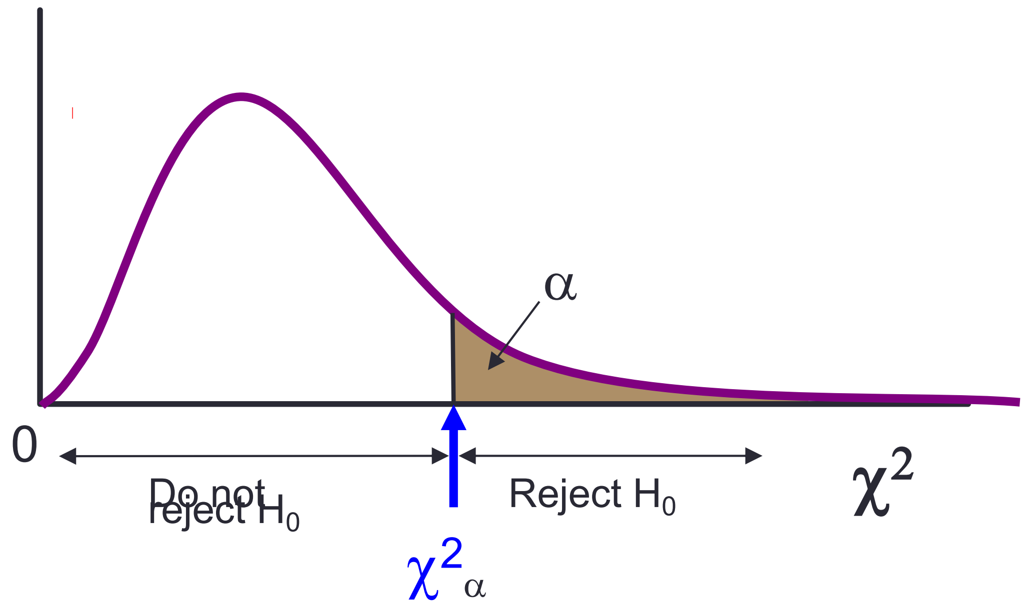
$$\chi^2 = \sum_{\text{all cells}} \frac{(f_{ij} - \hat{E}_{ij})^2}{\hat{E}_{ij}}$$

The null distribution is the chi-square distribution with *(r-1)(c-1)* degrees of freedom.

**Reject $H_0$ if**

$\chi^2 > \chi_\alpha^2$ or if p-value < $\alpha$

$\chi_\alpha^2$ and the p-value are based on (r-1)(c-1) degrees of freedom



# Observed vs. Expected Frequencies

| Gender | Hand Preference: Left | Hand Preference: Right | |
|---|---|---|---|
| Female | Observed = 12<br>Expected = 14.4 | Observed = 108<br>Expected = 105.6 | 120 |
| Male | Observed = 24<br>Expected = 21.6 | Observed = 156<br>Expected = 158.4 | 180 |
| | 36 | 264 | 300 |

# The Chi-Square Test Statistic

| Gender | Hand Preference | | |
|---|---|---|---|
| | Left | Right | |
| Female | Observed = 12<br>Expected = 14.4 | Observed = 108<br>Expected = 105.6 | 120 |
| Male | Observed = 24<br>Expected = 21.6 | Observed = 156<br>Expected = 158.4 | 180 |
| | 36 | 264 | 300 |

The test statistic is:

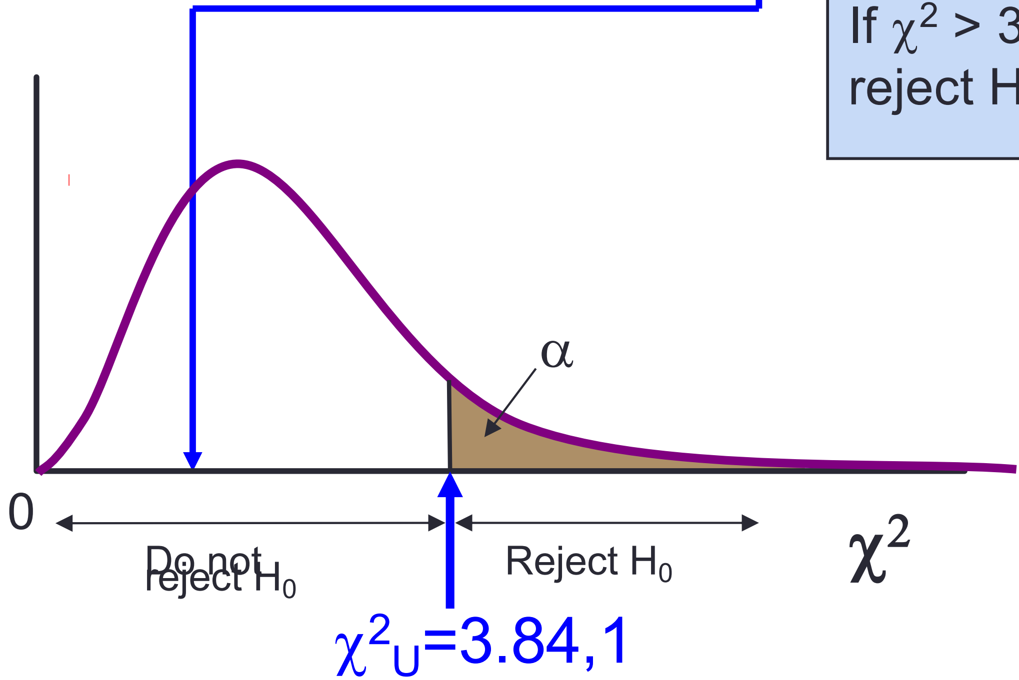
$$\chi^2 = \sum_{\text{all cells}} \frac{(f_{ij} - \hat{E}_{ij})^2}{\hat{E}_{ij}}$$

$$= \frac{(12-14.4)^2}{14.4} + \frac{(108-105.6)^2}{105.6} + \frac{(24-21.6)^2}{21.6} + \frac{(156-158.4)^2}{158.4} = 0.7576$$

# Decision Rule

The test statistic is $\chi^2 = \boxed{0.7576,}\quad \chi^2_\alpha \text{ with } 1 \text{ d.f.} = 3.841$

**Decision Rule:**
If $\chi^2 > 3.841$, reject $H_0$, otherwise, do not reject $H_0$




Here,
$\chi^2 = 0.7576 < \chi_\alpha^2 = 3.841$,
so we do not reject $H_0$ and conclude that there is not sufficient evidence that the two proportions are different at $\alpha = 0.05$

R code for $\chi_\alpha^2$ : qchisq(0.95,1)

# Example: The Client Satisfaction Case

| | Client Satisfaction | | | |
|---|---|---|---|---|
| Fund | High | Low | Med | All |
| Bond | 15 | 3 | 12 | 30 |
| Stock | 24 | 2 | 4 | 30 |
| TaxDef | 1 | 15 | 24 | 40 |
| All | 40 | 20 | 40 | 100 |

**$H_0$: client satisfaction is independent of fund type**

**$H_a$: client satisfaction depends upon fund type**

$$\chi^2 = \sum_{\text{allcells}} \frac{(f_{ij} - \hat{E}_{ij})^2}{\hat{E}_{ij}} \; ; \qquad E_{BH} = \frac{r_B c_H}{n} = \frac{(30)(40)}{100} = 12, \; \ldots \; , \; E_{TM} = \frac{r_T c_M}{n} = \frac{(40)(40)}{100} = 16$$

$$= \frac{(15-12)^2}{12} + \frac{(3-6)^2}{6} + \ldots + \frac{(24-16)^2}{16}$$

$$= 0.7500 + 1.5000 + \ldots + 4.0000 = \mathbf{46.4375}$$

$$\chi^2 = 46.4375 > 9.4877 = \chi^2_{.05}$$

Reject H0

$$p - value = P(\chi^2 > 46.4375) = 0.0000$$

## R function:  Chisq.test()

```
M=as.table(rbind(c(15,3,12),c(24,2,4),c(1,15,24)))
dimnames(M)=list(Fund= c("Bond", "Stock",'TaxDef'),
                 Satisfaction=c("High","Low", "Med"))

Xsq <- chisq.test(M))  # Prints test summary
Xsq$observed   # observed counts (same as M)
Xsq$expected   # expected counts under the null
```

```
> Xsq$observed   # observed counts (same as M)
        Satisfaction
Fund     High Low Med
  Bond     15   3  12
  Stock    24   2   4
  TaxDef    1  15  24

> Xsq$expected   # expected counts under the null
        Satisfaction
Fund     High Low Med
  Bond     12   6  12
  Stock    12   6  12
  TaxDef   16   8  16
```

```
> Xsq

        Pearson's Chi-squared test

data:  M
X-squared = 46.4375, df = 4, p-value = 1.997e-09
```

# Chi-Square Goodness-of-Fit Test

- Does sample data conform to a hypothesized distribution?
  - Examples:
    - Are technical support calls equal across all days of the week? (i.e., do calls follow a uniform distribution?)
    - Do measurements from a production process follow a normal distribution?

# Chi-Square Goodness-of-Fit Test

- Are technical support calls equal across all days of the week? (i.e., do calls follow a uniform distribution?)
  - Sample data for 10 days per day of week:

| | Sum of calls for this day: |
|---|---|
| Monday | 290 |
| Tuesday | 250 |
| Wednesday | 238 |
| Thursday | 257 |
| Friday | 265 |
| Saturday | 230 |
| Sunday | 192 |
| | $\Sigma = 1722$ |

# Logic of Goodness-of-Fit Test

- If calls **are** uniformly distributed, the 1722 calls would be expected to be equally divided across the 7 days:

$$\frac{1722}{7} = 246 \quad \text{expected calls per day if uniform}$$

# Observed vs. Expected Frequencies

| | Observed $f_o$ | Expected $f_e$ |
|---|---|---|
| Monday | 290 | 246 |
| Tuesday | 250 | 246 |
| Wednesday | 238 | 246 |
| Thursday | 257 | 246 |
| Friday | 265 | 246 |
| Saturday | 230 | 246 |
| Sunday | 192 | 246 |
| TOTAL | 1722 | 1722 |

# Chi-Square Test Statistic

$H_0$: The distribution of calls is uniform over days of the week

$H_1$: The distribution of calls is not uniform

- The test statistic is

$$\chi^2 = \sum_k \frac{(f_o - f_e)^2}{f_e} \qquad (\text{where } df = k - 1)$$

where:

k = number of categories
$f_o$ = observed frequency
$f_e$ = expected frequency

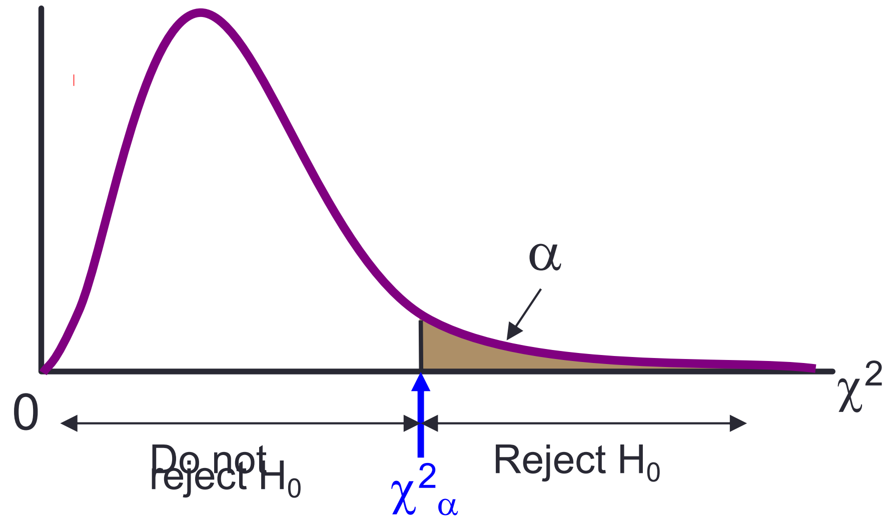
# The Rejection Region

$H_0$: The distribution of calls is uniform over days of the week

$H_1$: The distribution of calls is not uniform

$$\chi^2 = \sum_k \frac{(f_o - f_e)^2}{f_e}$$

- Reject $H_0$ if $\chi^2 > \chi^2_\alpha$



# Chi-Square Test Statistic

$H_0$: The distribution of calls is uniform over days of the week

$H_1$: The distribution of calls is not uniform

$$\chi^2 = \frac{(290-246)^2}{246} + \frac{(250-246)^2}{246} + \ldots + \frac{(192-246)^2}{246} = \boxed{23.05}$$

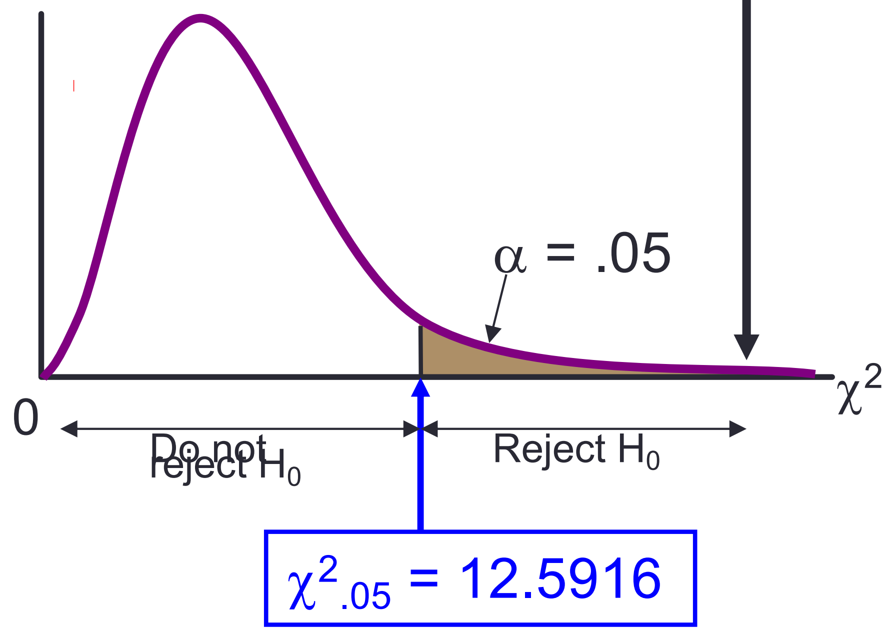
k – 1 = 6 (k = 7 days of the week) so use 6 degrees of freedom:

$\chi^2_{.05} = 12.5916$

**Conclusion:**

$\chi^2 = 23.05 > \chi^2_\alpha = 12.5916$

so **reject $H_0$** and conclude that the distribution is not uniform



# R function: Chisq.test()

- x<-c(290,250,238,257,265,230,192)
- prob=rep(1/length(x),length(x))
- chisq.test(x, p = prob)

```
> chisq.test(x, p = prob)

        Chi-squared test for given probabilities

data:  x
X-squared = 23.049, df = 6, p-value = 0.0007803
```

# Normal Distribution Example

- Do measurements from a production process follow a normal distribution with $\mu = 50$ and $\sigma = 15$?
- Process:
  - Get sample data
  - Group sample results into classes (cells) (Expected cell frequency must be at least 5 for each cell)
  - Compare actual cell frequencies with expected cell frequencies

# Normal Distribution Example

*(continued)*

- Sample data and values grouped into classes:

| 150 Sample Measurements |
|---|
| 80 |
| 65 |
| 36 |
| 66 |
| 50 |
| 38 |
| 57 |
| 77 |
| 59 |
| …etc… |

| Class | Frequency |
|---|---|
| less than 30 | 10 |
| 30 but < 40 | 21 |
| 40 but < 50 | 33 |
| 50 but < 60 | 41 |
| 60 but < 70 | 26 |
| 70 but < 80 | 10 |
| 80 but < 90 | 7 |
| 90 or over | 2 |
| TOTAL | 150 |

# Normal Distribution Example

*(continued)*

- What are the expected frequencies for these classes for a normal distribution with $\mu = 50$ and $\sigma = 15$?

| **Class** | **Frequency** | **Expected Frequency** |
|---|---|---|
| less than 30 | 10 | ? |
| 30 but < 40 | 21 | |
| 40 but < 50 | 33 | |
| 50 but < 60 | 41 | |
| 60 but < 70 | 26 | |
| 70 but < 80 | 10 | |
| 80 but < 90 | 7 | |
| 90 or over | 2 | |
| TOTAL | 150 | |

# Expected Frequencies

| Value | P(X < value) | Expected frequency |
|---|---|---|
| less than 30 | 0.09121 | 13.68 |
| 30 but < 40 | 0.16128 | 24.19 |
| 40 but < 50 | 0.24751 | 37.13 |
| 50 but < 60 | 0.24751 | 37.13 |
| 60 but < 70 | 0.16128 | 24.19 |
| 70 but < 80 | 0.06846 | 10.27 |
| 80 but < 90 | 0.01892 | 2.84 |
| 90 or over | 0.00383 | 0.57 |
| TOTAL | 1.00000 | 150.00 |

Combine class groups so no class has expected frequency <1

**Expected frequencies in a sample of size n=150, from a normal distribution with μ=50, σ=15**

**Example:**

$$P(x < 30) = P\left(z < \frac{30-50}{15}\right)$$

$$= P(z < -1.3333)$$

$$= .0912$$

$$(.0912)(150) = 13.68$$

# The Test Statistic

| Class | Frequency (observed, $f_o$) | Expected Frequency, $f_e$ |
|---|---|---|
| less than 30 | 10 | 13.68 |
| 30 but < 40 | 21 | 24.19 |
| 40 but < 50 | 33 | 37.13 |
| 50 but < 60 | 41 | 37.13 |
| 60 but < 70 | 26 | 24.19 |
| 70 but < 80 | 10 | 10.27 |
| 80 or over | 9 | 3.41 |
| TOTAL | 150 | 150.00 |

The test statistic is

$$\chi^2 = \sum_k \frac{(f_o - f_e)^2}{f_e}$$

- Reject $H_0$ if

$$\chi^2 > \chi^2_\alpha$$

(with k – 1 degrees of freedom)

# The Rejection Region

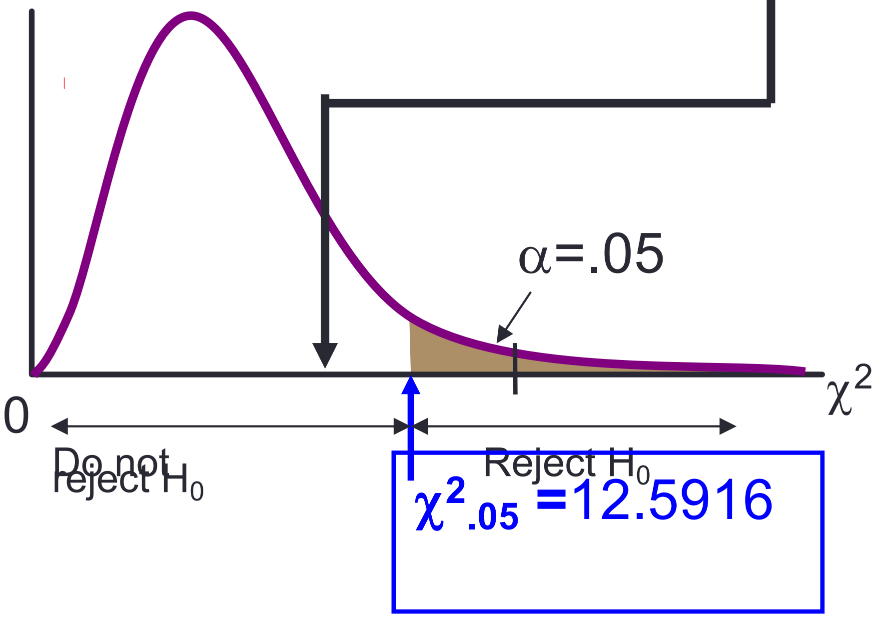
$H_0$: The distribution of values is normal with $\mu = 50$ and $\sigma = 15$

$H_1$: The distribution of calls does not have this distribution

$$\chi^2 = \sum_k \frac{(f_o - f_e)^2}{f_e} = \frac{(10 - 13.68)^2}{13.68} + \ldots + \frac{(9 - 3.41)^2}{3.41} = \boxed{11.580}$$

7 classes so use 7– 1 = 6 d.f.:

$\chi^2_{.05} = 12.5916$




**Conclusion:**

$\chi^2 = 11.580 < \chi^2_\alpha = 11.0705$ so not **reject $H_0$.** There is no sufficient evidence that the data are not normal with $\mu = 50$ and $\sigma = 15$

# Exercise

- Write a program to perform a test: Whether the following measurements from a production process follow a normal distribution with μ = 50 and σ = 15?

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 57 | 66 | 54 | 55 | 66 | 70 | 48 | 28 | 37 | 53 | 50 | 71 | 24 | 51 | 53 | 53 | 43 | 29 | 46 | 46 |
| 27 | 64 | 47 | 42 | 52 | 45 | 62 | 51 | 84 | 32 | 60 | 61 | 33 | 50 | 36 | 73 | 41 | 64 | 59 | 40 |
| 59 | 77 | 56 | 49 | 66 | 29 | 74 | 44 | 20 | 72 | 38 | 73 | 41 | 27 | 55 | 24 | 43 | 47 | 49 | 52 |
| 75 | 36 | 76 | 59 | 43 | 32 | 44 | 29 | 39 | 78 | 72 | 38 | 42 | 51 | 39 | 53 | 41 | 29 | 32 | 62 |
| 72 | 42 | 32 | 53 | 35 | 55 | 42 | 107 | 60 | 63 | 46 | 68 | 55 | 51 | 37 | 50 | 50 | 65 | 33 | 53 |
| 38 | 30 | 28 | 55 | 40 | 39 | 38 | 69 | 57 | 36 | 60 | 17 | 67 | 62 | 39 | 32 | 39 | 47 | 51 | 88 |
| 33 | 74 | 47 | 43 | 33 | 56 | 47 | 42 | 33 | 63 | 49 | 59 | 26 | 40 | 50 | 48 | 31 | 65 | 14 | 57 |
| 82 | 52 | 87 | 46 | 58 | 48 | 35 | 36 | 57 | 42 | | | | | | | | | | |

The data is assigned to 7classes:

```
x<-c(57,66,54,55,…)
```

| |
|---|
| less than 30 |
| 30 but < 40 |
| 40 but < 50 |
| 50 but < 60 |
| 60 but < 70 |
| 70 but < 80 |
| 80 or over |