Random Variables

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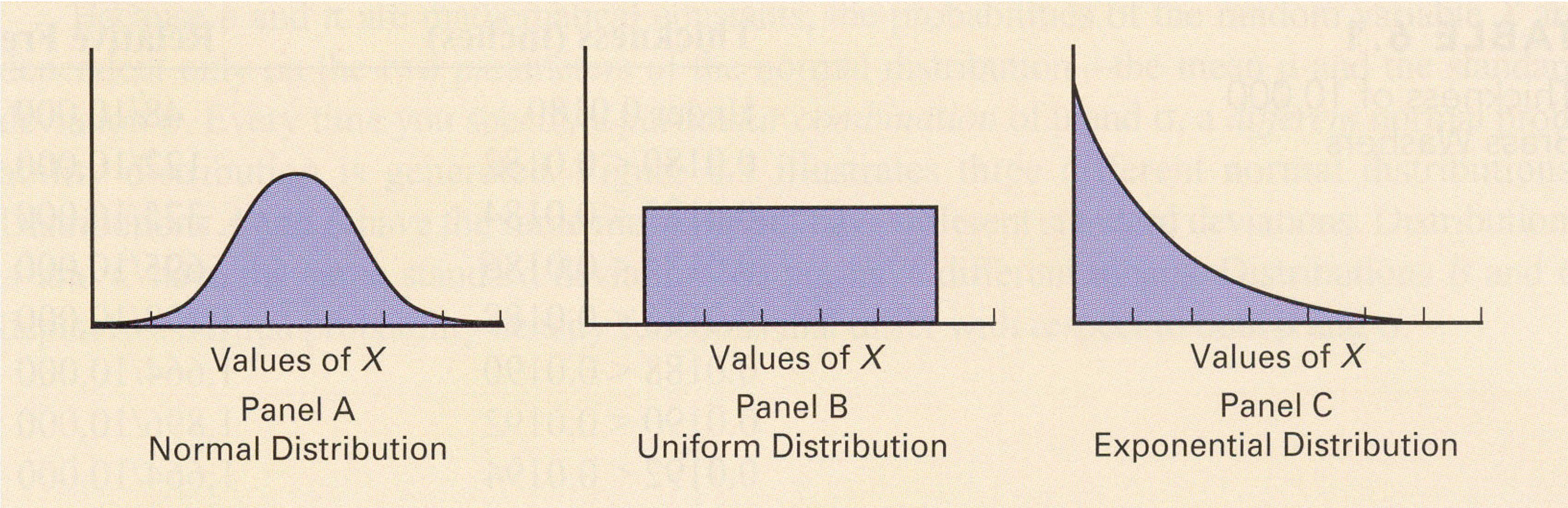
Random Variables

- Continuous random variables:
  - Uniform
  - Univariate normal
  - Gamma
  - Beta
  - Exponential

- Discrete random variables:
  - Poisson
  - Binomial

Continuous Random Variable

- A random variable is called **continuous** if the distribution function is continuous and is differentiable everywhere with the possible exception of a countable number of values.



Properties and Moments

- Properties of pdf:
  - $f(x) \geq 0$ for all x
  - $\int f(x)dx = 1$

- Expectation:

$$E(X) = \int xf(x)\,dx$$

- Variance:

$$Var(X) = \int x^2 f(x)\,dx - [E(X)]^2$$

Uniform Distribution

- The uniform distribution is used to represent a variable that is known to lie in an interval and equally likely to be found anywhere in the interval.

- $X \sim U(\alpha, \beta)$, boundaries α, β with $\beta > \alpha$
- $f(X) = \frac{1}{\beta - \alpha}$
- $E(X) = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} x \, dx = \frac{\alpha+\beta}{2}$
- $Var(X) = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} x^2 \, dx - \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{(\beta-\alpha)^2}{12}$

Univariate Normal Distribution

- The normal, or Gaussian, distribution is ubiquitous in statistics. Sample averages are approximately normally distributed by the central limit theorem.
- $X \sim N(\mu, \sigma^2)$, location μ, scale $\sigma > 0$
- $f(X) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$
- $E(X) = \int_\alpha^\beta x \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2)\, dx = \mu$
- $Var(X) = \int_\alpha^\beta x^2 \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2)\, dx - \mu^2 = \sigma^2$

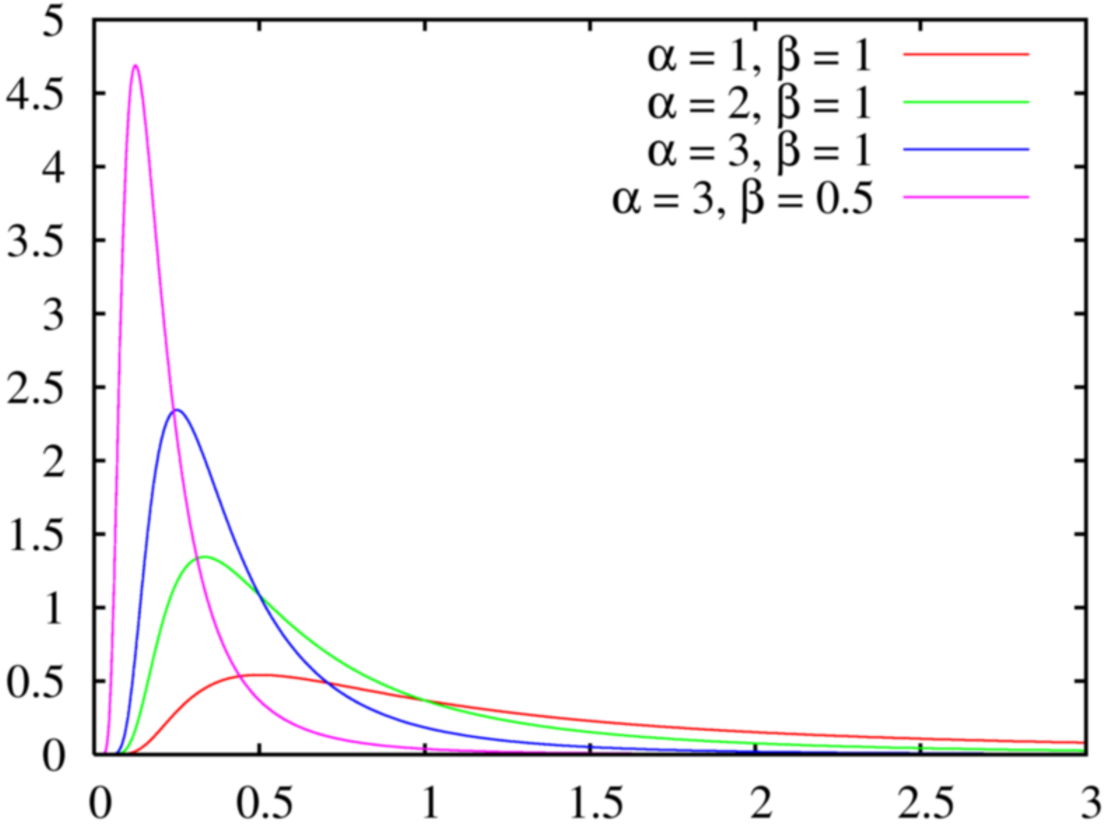
Univariate Normal Distribution

- Properties:
  - The sum of two independent normal random variables is normally distributed. If X_1 and X_2 are independent with $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ distributions, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

  - If $X_1 | X_2 \sim N(X_2, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$. Then $X_1 \sim N(\mu_2, \sigma_1^2 + \sigma_2^2)$

Gamma Distribution

- $X \sim Gamma(\alpha, \beta)$, shape $\alpha > 0$, inverse scale $\beta > 0$



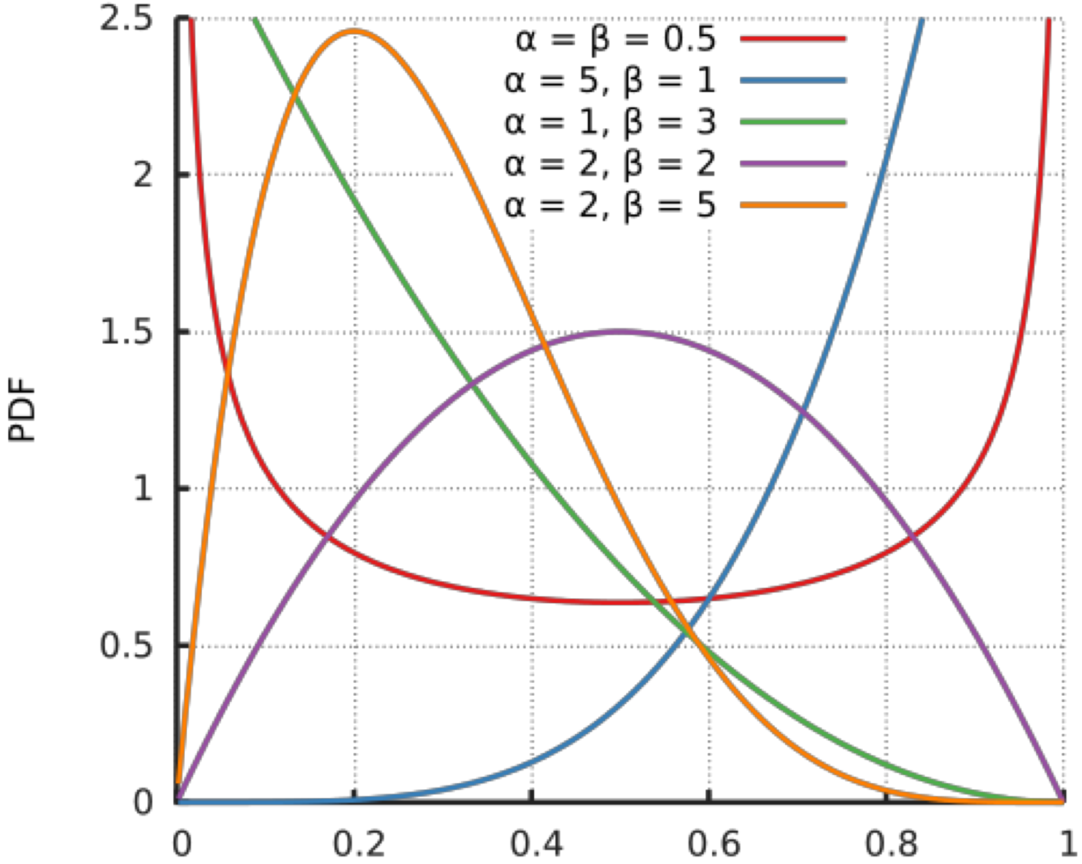
Gamma Distribution

- $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0$
- $E(X) = \int_0^{+\infty} x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx = \frac{\alpha}{\beta}$
- $Var(X) = \int_0^{+\infty} x^2 \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\, dx - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}$
- Property:
  – If X_1 and X_2 are independent with Gamma(α_1, β) and Gamma(α_2, β) distributions, then

    $X_1 + X_2 \sim$ Gamma(α_1+α_2, β).

Beta Distribution

- $X \sim Beta(\alpha, \beta)$, shape $\alpha > 0$, shape $\beta > 0$

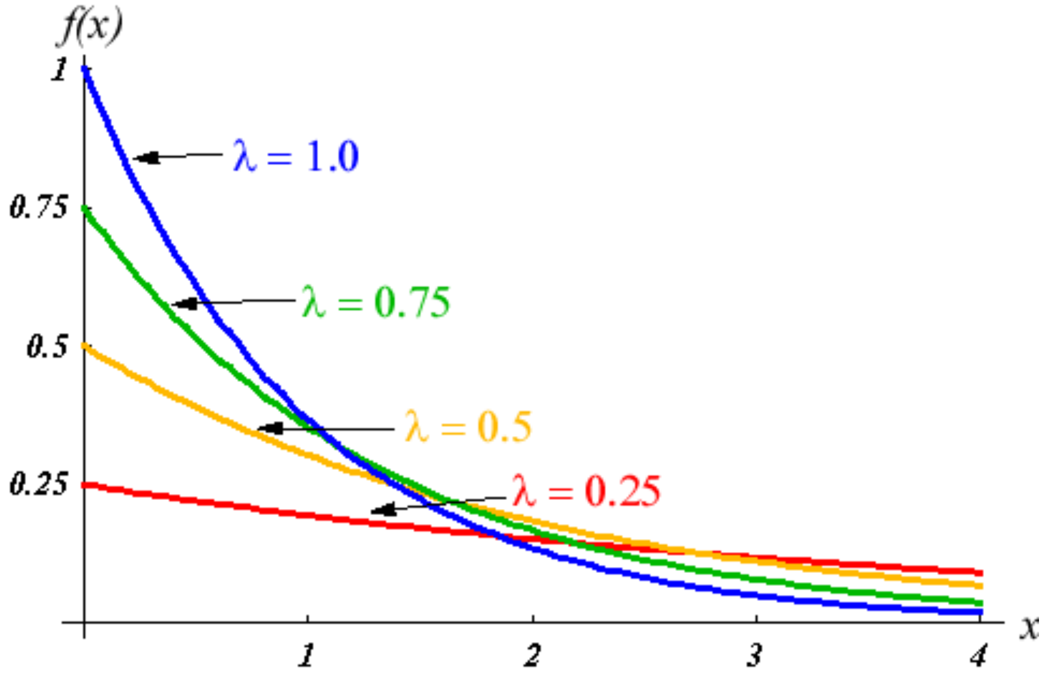


Beta Distribution

- $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}, x \in [0,1], \mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- $E(X) = \int x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} dx = \frac{\alpha}{\alpha+\beta}$
- $Var(X) = \int x^2 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)} dx - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Exponential Distribution

- The exponential distribution is the distribution of waiting times for the next event in a Poisson process and is a special case of the gamma distribution with $\alpha = 1$.
- $X \sim Expon(\lambda)$, inverse scale $\lambda > 0$



Exponential Distribution

- $f(X) = \lambda e^{-\lambda x}, x > 0$
- $E(X) = \int_0^{+\infty} x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}$
- $Var(X) = \int_0^{+\infty} x^2\lambda e^{-\lambda x}\,dx - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Discrete Random Variable

- The support of a random variable is the set of numbers that are possible values of the random variable.
- A random variable is called discrete if the support contains at most a countable number of values.

Discrete Random Variable

- Properties of probability function
  - For any value x of the random variable, $p(x) \geq 0$
  - The probabilities of all the events in the sample space must sum to 1, that is, $\sum_{all\ x} p(x) = 1$.

- Expectation $E(X) = \sum_{all\ x} xp(x)$
- Variance $\mathrm{Var}(X) = \sum_{all\ x} \big(x - E(X)\big)^2 p(x)$

Example

- Let Y be a discrete random variable with probability function given in the following table.

y_i	$f(y_i)$
0	0.20
1	0.15
2	0.25
3	0.35
4	0.05

- Find $E(Y)$
- Find $Var(Y)$

Example Solutions

- $E(Y) = 0\times 0.20 + 1\times 0.15 + 2\times 0.25 + 3\times 0.35 + 4\times 0.05 = 1.90$
- $Var(Y) = (0 - 1.90)^2\times 0.20 + (1 - 1.90)^2\times 0.15 + (2 - 1.90)^2\times 0.25 + (3 - 1.90)^2\times 0.35 + (4 - 1.90)^2\times 0.05 = 1.49$

Poisson Distribution

- The Poisson distribution is commonly used to represent count data, such as the number of arrivals in a fixed time period.
- $X \sim Poisson(\lambda)$
- $p(x) = \frac{1}{x!}\lambda^x \exp(-\lambda)\,, x = 0, 1, 2, \ldots$
- Expectation

$$E(X) = \sum_{k \geq 0} k \frac{1}{k!} \lambda^k e^{-\lambda}$$

$$E(X) = \lambda e^{-\lambda} \sum_{k \geq 1} \frac{1}{(k-1)!} \lambda^{k-1}$$ as the $k = 0$ term vanishes

$$= \boxed{\lambda e^{-\lambda} \sum_{j \geq 0} \frac{\lambda^j}{j!}}$$ putting $j = k - 1$

$$= \lambda e^{-\lambda} e^{\lambda}$$ Taylor Series Expansion for Exponential Function

$$= \lambda$$

Poisson Distribution

- Similarly, the variance is

$$Var(X) = E(X)^2 - (EX)^2 = \lambda$$

- Property:
  - if X_1 and X_2 are independent with Poisson(λ_1) and Poisson(λ_2) distributions, then

    $X_1 + X_2 \sim Poisson(\lambda_1 + \lambda_2)$.

Binomial Distribution

- The binomial distribution is commonly used to represent the number of 'successes' in a sequence of n independent and identically distributed Bernoulli trials, with probability of success p in each trial.
- $X \sim Bin(n, p)$
- $p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots n$
- Expectation

  $\mathrm{E(X)} = np$
- Variance

  $Var(X) = np(1-p)$

Thanks