Bayesian Analysis Assignment One

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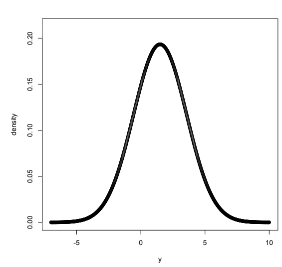
Question One Answer: For $\sigma = 2$, we can notice that:

$$
\begin{aligned}
p(y) &= Pr(\theta = 1)p(y|\theta = 1) + Pr(\theta = 2)p(y|\theta = 2) \\
&= 0.5N(y|1, 2^2) + 0.5N(y|2, 2^2)
\end{aligned}
$$

The graph can be drawn by R:

```r
y <- seq(-7,10,.02)
density <- 0.5*dnorm(y,1,2) + 0.5*dnorm(y,2,2)
plot (y, density, ylim=c(0,1.1*max(density)))
```

The plot is :



图：Marginal probability density of y

(b)

$$
\begin{aligned}
Pr(\theta = 1|y = 1) &= \frac{p(\theta = 1,\ y = 1)}{p(\theta = 1,\ y = 1) + p(\theta = 2,\ y = 1)} \\
&= \frac{Pr(\theta = 1)p(y = 1|\theta = 1)}{Pr(\theta = 1)p(y = 1|\theta = 1) + Pr(\theta = 2)p(y - 1|\theta = 2)} \\
&= \frac{0.5N(1|1, 2^2)}{0.5N(1|1, 2^2) + 0.5N(1|2, 2^2)} \approx 0.534
\end{aligned}
$$

(c) If $\sigma \rightarrow$, the posterior density for θ will approaches to the prior density and the probability will be $\frac{1}{2}$; if $\sigma \rightarrow 0$ the $0.5N(1|2,^2) \approx 0$ so the $Pr(\beta = 1|y = 1) \rightarrow 1$.

Question Two Answer: From the question, we can notice that: $Pr(fraternaltwins) = \frac{1}{125}$ and $Pr(identicaltwins) = \frac{1}{300}$. Since the fraternal twins are in different perm cell. the $Pr(bothboys) = \frac{1}{4}$

The conditional probability that Elvis was an identical twins is :

$$
\begin{aligned}
Pr(identicaltwins|twinsbrother) &= \frac{Pr(identicaltwins,\ twinsbrother)}{Pr(twinsbrother)} \\
&= \frac{\frac{1}{2} \times \frac{1}{300}}{\frac{1}{2} \times \frac{1}{300} + \frac{1}{4} \times \frac{1}{125}} \\
&= \frac{5}{11} \approx 45.5\%
\end{aligned}
$$

Question Three Answer: We can just classify the genes: Xx, xX in one group and let Xx represent the group. Pr(child is Xx | child has brown eyes

, the parents have brown eyes)

$$
\begin{aligned}
&= \frac{0\times(1-p)^4+\frac{1}{4}\times(1-p)^3+\frac{1}{2}\times 4p^2(1-p)^2}{1\times(1-p)^4+1\times 4p(1-p)^3+\frac{3}{4}\times 4p^2(1-p)^2}\\
&= \frac{2p(1-p)+2p^2}{(1-p)^2+4p(1-p)+3p^2}\\
&= \frac{2p}{1+2p}
\end{aligned}
$$

This is the poterior probability, but it is also the prior probability of the **probability that Judy is a heterozygote**, also, there added condition is that: **her n children are brown-eyed** $\Rightarrow Xx$ genes.

Pr(Judy is Xx | n children all have brown eyes , all previous information) = $\frac{\frac{2p}{1+2p}\times(\frac{3}{4})^n}{\frac{2p}{1+2p}\times(\frac{3}{4})^n+\frac{1}{1+2p}\times 1}$

Also, we know that the Judy's Children has no blue-eyed, if we want Judy's grand that means Judy needs to be a heterozygote or Judy's child is Xx and Judy is XX: Pr(Judy's child is Xx | all information) =

(Judy is Xx , Judy's child is Xx) + (Judy's child is XX , Judy's child is Xx)

$$
\frac{\frac{2p}{1+2p}\times(\frac{3}{4})^n}{\frac{2p}{1+2p}\times(\frac{3}{4})^n+\frac{1}{1+2p}}\times\frac{2}{3}+\frac{\frac{1}{1+2p}}{\frac{2p}{1+2p}\times\frac{3}{4}^n+\frac{1}{1+2p}}\times\frac{1}{2}
$$

Since her child is Xx, the probability of her grandchild having blue eyes is $0, \frac{1}{4}$ and $\frac{1}{2}$ as Judy's Child is XX,Xx or xx. so Pr(grandchild is xx|all the given information)

$$
\begin{aligned}
&= \frac{\frac{2}{3}\frac{2p}{1+2p}(\frac{3}{4})^n+\frac{1}{2}\frac{1}{1+2p}}{\frac{2p}{1+2p}(\frac{3}{4})^n+\frac{1}{1+2p}}(\frac{1}{4}2p(1-p)+\frac{1}{2}p^2)\\
&= \frac{\frac{2}{3}\frac{2p}{1+2p}(\frac{3}{4})^n+\frac{1}{2}\frac{1}{1+2p}}{\frac{2p}{1+2p}(\frac{3}{4})^n+\frac{1}{1+2p}}(\frac{1}{2}p)
\end{aligned}
$$

Quesion Four Answer: (a) The prior predictive distribution for y:

$$
\begin{aligned}
Pr(y=k) &= \int_0^1 Pr(y=k|\theta)d\theta \\
&= \int_0^1 \binom{n}{k}\theta^k(1-\theta)^{n-k}d\theta \\
&= \binom{n}{k}\frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{1}{n+1}
\end{aligned}
$$

(b) We can notice that the posterior mean is $\frac{\alpha+y}{\alpha+\beta+n}$. We assume that $\lambda \in (0,1)$ and $\frac{\alpha+y}{\alpha+\beta+n} = \lambda\frac{\alpha}{\alpha+\beta} + (1-\lambda)\frac{y}{n}$

$$
\begin{aligned}
\frac{\alpha+y}{\alpha+\beta+n} &= \lambda\frac{\alpha}{\alpha+\beta} + (1-\lambda)\frac{y}{n} \\
\frac{\alpha+y}{\alpha+\beta+n} - \frac{y}{n} &= \lambda(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}) \\
\lambda &= \frac{\alpha+\beta}{\alpha+\beta+n}
\end{aligned}
$$

(c) When we consider the uniform distribution, we can notice that $\alpha = \beta = 1$, so the prior variance is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{12}$. In this situation, we can calculate the posterior variance:

$$
\begin{aligned}
\text{Posterior Variance} &= \frac{(1+y)(1+n-y)}{(2+n)^2(3+n)} \\
&= (\frac{1+y}{2+n})(\frac{1+n-y}{2+n})(\frac{1}{3+n})
\end{aligned}
$$

The sum of $\frac{1+y}{2+n}$ and $\frac{1+n-y}{2+n}$ is not greater than $\frac{1}{4}$. And $n \geq 1$, so $\frac{1}{3+n}$ is less than $\frac{1}{4}$. So Posterior vairance $\neq \frac{1}{16}$

(d) If $n = 2$,$y = 1$,$\alpha = 1$ and $\beta = 1$ The prior vaiance $= \frac{1}{12}$ and the posterior variance is 0.025