Quadratic Forms in Statistics

Kai-Tai Fang

BNU-HKBU United International College, UIC

Additional Chapter for Regression Analysis

February, 2014

Table of Contents

1. Introduction

1.1. Quadratic forms

Definition 1: A quadratic form on R^n is a polynomial function Q: $R^n \rightarrow R$ of the form

$$Q(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \tag{1}$$

in which each term has degree 2 and $\mathbf{x} = (x_1, \ldots, x_n)'$.

$\diamondsuit$ A homogeneous polynomial of degree 2 in R^n is called a quadratic form in n indeterminates.

1. Introduction

1.1. Quadratic forms - matrix presentation

Example 1.

$$\begin{aligned} Q &= x_1^2 - 5x_1x_2 \\ &= (x_1, x_2)\begin{pmatrix} 1 & -2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= (x_1, x_2)\begin{pmatrix} 1 & -\frac{5}{2} \\ -\frac{5}{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\equiv \mathbf{x}'\mathbf{A}\mathbf{x}. \end{aligned}$$

◊ For any quadratic form we always can express it as the above matrix form of $\mathbf{x}'\mathbf{A}\mathbf{x}$.

◊ Note $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{A}'\mathbf{x} = \mathbf{x}'(\frac{1}{2}(\mathbf{A} + \mathbf{A}'))\mathbf{x}$, we can choose $\mathbf{A}$ to be symmetric.

1. Introduction

1.1. Quadratic forms - matrix presentation

Definition 2: Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ symmetric matrix with real entries and $\mathbf{x} = (x_1, \ldots, x_n)'$ be a column vector. Then

- ♢ $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$ is said to be a quadratic form of $\mathbf{x}$,
- ♢ $\mathbf{A}$: the matrix of the quadratic form and
- ♢ rank($\mathbf{A}$): the rank of the quadratic form.

Question 1: Find the matrix form and rank of the following quadratic forms:

(1) $Q = x^2 + 2xy + 2xz + 3z^2 - yz + 7y^2$.

(2) $Q = x^2 + 8xy + 6y^2 - z^2$.

1.2. Quadratic forms in statistics

Example 2. Let $x_1, \ldots, x_n$ be a sample from the population $F(x)$. The sum of squares

$$Q = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \mathbf{x}'\mathbf{D}_n\mathbf{x}, \quad \mathbf{D}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n', \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)'$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\mathbf{I}_n$ the unit matrix of order n and $\mathbf{1}_n$ an n-vector of ones. Please prove that

(1) For any n-vector $\mathbf{x}$, $\mathbf{D}_n\mathbf{x} = (x_1 - \bar{x}, \ldots, \mathbf{x}_n - \bar{x})'$.

(2) For any n-vector $\mathbf{x}$ with $\bar{x} = 0$ we have $\mathbf{D}_n\mathbf{x} = \mathbf{x}$.

(3) $\mathbf{D}_n$ is a projection matrix with rank $n - 1$;

(4) $\mathbf{x}'(\frac{1}{n}\mathbf{1}_n\mathbf{1}_n')\mathbf{x}$ is a quadratic form and find its rank.

(5) The eigenvalues of a projection matrix are 0 or 1. The number of 1's equals to its rank.

1.2. Quadratic forms in statistics

Example 3. Let $x_1, \ldots, x_n$ be an *iid* sample from the population $N(\mu, \sigma^2)$. A unbiased estimators of μ and σ^2 are given by

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma^2} = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

What is the distribution of $\hat{\sigma^2}$? and How to prove the independence between $\bar{x}$ and s^2.

There are various quadratic forms in ANOVA, regression analysis, multivariate analysis and data mining. How to find their distributions and to show independence among them are very important in the statistics theory.

1. Introduction

1.3. Spectral decomposition

Let $\mathbf{A}$ be a symmetric matrix of order n. Its spectral decomposition is

$$\mathbf{A} = \mathbf{H}'\Lambda\mathbf{H} = \sum_{i=1}^{n} c_i \mathbf{h}_i \mathbf{h}_i', \tag{3}$$

where $\Lambda = \mathrm{diag}(c_1, \ldots, c_n)$ are the eigenvalues of $\mathbf{A}$, $\mathbf{H}' = [\mathbf{h}_1, \ldots, \mathbf{h}_n]$ are the corresponding normalized eigenvectors. It is easy to see that $\mathbf{H}'\mathbf{H} = \mathbf{H}\mathbf{H}' = \mathbf{I}_n$ Very often we can arrange $c_1 \geq c_2 \geq \ldots \geq c_n$.

♢ For a projection matrix $\mathbf{A}$ with rank r we have

$$\mathbf{A} = \sum_{i=1}^{r} \mathbf{h}_i \mathbf{h}_i' \equiv \mathbf{H}_1'\mathbf{H}_1, \tag{4}$$

where $\mathbf{H}_1 : r \times n$ and $\mathbf{H}_1\mathbf{H}_1' = \mathbf{I}_r$.

2. Distribution of A Quadratic Form

2.1 Non-central chi-square distribution

In this section we assume $\mathbf{x}$ follows a multivariate distribution.

$\diamondsuit$ $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, a general normal distribution with mean vector $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. See Chapter 2, Section 2.4, 32-36 of the Notes.

$\diamondsuit$ $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I}_n)$, the standard normal distribution

Definition 3: Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$. The distribution of $Q(\mathbf{x}) = \mathbf{x}'\mathbf{x} = x_1^2 + \ldots + x_n^2$ is called non-central chi-square distribution with n degrees of freedom and non-central parameter $\lambda = \boldsymbol{\mu}'\boldsymbol{\mu}$, and is denoted by $Q(\mathbf{x}) \sim \chi_n^2(\lambda)$. When $\lambda = 0$, the distribution of $Q(\mathbf{x})$ is called chi-square distribution with n degrees of freedom and is denoted as $Q(\mathbf{x}) \sim \chi_n^2$.

2. Distribution of A Quadratic Form

Properties of the non-central chi-square distribution

◇ Let $Y = Q(\mathbf{x}) \sim \chi^2_n(\lambda)$. Then

$$E(Y) = n + \lambda, \quad \mathrm{Var}(Y) = 2n + 4\lambda.$$

◇ Let $Y_i \sim \chi^2_{n_i}(\lambda_i), i = 1, \ldots, k$. Them

$$Y = \sum_{i=1}^{k} Y_i \sim \chi^2_n(\lambda),$$

where $n = n_1 + \ldots + n_k$ and $\lambda = \lambda_1 + \ldots + \lambda_k$.

◇ Let $Y \sim \chi^2_m(\lambda)$ and $Z \sim \chi^2_n$ be independent. The distribution of

$$F = \frac{n}{m}\frac{Y}{Z}$$

is called a non-central F-distribution with m, n degrees of freedom and non-central parameter λ and is denoted by $F_{m,n}(\lambda)$.

2. Distribution of A Quadratic Form

2.2 Expectation of a quadratic form

Lemma 1 : Let random vector $\mathbf{x}$ has $\boldsymbol{\mu} = E(\mathbf{x})$ and $\boldsymbol{\Sigma} = \mathrm{Cov}(\mathbf{x})$. Then

$$E(\mathbf{x}'\mathbf{A}\mathbf{x}) = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}, \qquad (5)$$

where $\mathrm{tr}(\mathbf{A})$ is the trace of $\mathbf{A}$. When $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I}_n)$, $E(\mathbf{x}'\mathbf{A}\mathbf{x}) = \mathrm{tr}(\mathbf{A})$. Further more, if $\mathbf{A}$ is a projection matrix in this case, the rank of $\mathbf{A}$ equals to $\mathrm{tr}(\mathbf{A})$.

Question 2: Assume that $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I}_n)$ and $\mathbf{A}$ is a symmetric matrix of order n. Under which condition we have $Q(\mathbf{x}) \sim \chi^2_r$ where $r \leq n$.

2. Distribution of A Quadratic Form

2.3 Cochran's theorem

Theorem 1 (Cochran): Assume that $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I}_n)$ and $\mathbf{A}$ is a symmetric matrix of order n. Then $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2_r$ if and only if $\mathbf{A}$ is a projection matrix with rank r.

Proof of Theorem 1

Sufficiency If $\mathbf{A}$ is a projection matrix with rank r, there exists a matrix $\mathbf{H}_1 : r \times n$ with $\mathbf{H}_1\mathbf{H}_1' = \mathbf{I}_r$ from (4). Then

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{H}_1'\mathbf{H}_1\mathbf{x} = \mathbf{y}'\mathbf{y},$$

where $\mathbf{y} = \mathbf{H}_1\mathbf{x} \sim N_r(\mathbf{0}, \mathbf{H}_1\mathbf{I}_n\mathbf{H}_1') = N_r(\mathbf{0}, \mathbf{I}_r)$ that implies $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{y}'\mathbf{y} \sim \chi^2_r$. $\sharp$

2. Distribution of A Quadratic Form

2.3 Cochran's theorem

For proving the necessity we need the following lemma.

Lemma 3: Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$ and $\mathbf{A}$ be a symmetric matrix of order n with the special decomposition $\mathbf{H}\Lambda\mathbf{H}' = \sum_{j=1}^n c_j \mathbf{h}_j \mathbf{h}_j'$ (see (3)). Then

$$E(\exp it\mathbf{x}'\mathbf{A}\mathbf{x}) = \prod_{j=1}^n (1 - 2itc_j)^{-1/2} \exp\left\{\frac{itc_j\lambda_j^2}{1 - 2itc_j}\right\}, \qquad (6)$$

where $\boldsymbol{\lambda} = (\lambda_1 \ldots, \lambda_n)' = \mathbf{H}\boldsymbol{\mu}$.

2. Distribution of A Quadratic Form

Proof of Lemma 3

Let $\mathbf{y} = \mathbf{H}\mathbf{x}$. Then $\mathbf{y} \sim N_n(\mathbf{H}\boldsymbol{\mu}, \mathbf{H}\mathbf{I}\mathbf{H}') = N_n(\mathbf{H}\boldsymbol{\mu}, \mathbf{I})$. We have

$$E(\exp it\mathbf{x}'\mathbf{A}\mathbf{x}) = E(\exp it\mathbf{y}'\Delta\mathbf{y}) = \prod_{j=1}^{n} E(\exp itc_j y_j^2).$$

The assertion follows from

$$E(\exp itc_j y_j^2) = (1-2itc_j)^{-1/2}\exp\left\{\frac{itc_j\lambda_j^2}{1-2itc_j}\right\}. \quad \sharp$$

Formula (6) can be expressed as

$$E(\exp it\mathbf{x}'\mathbf{A}\mathbf{x}) = |\mathbf{I} - 2it\mathbf{A}|^{-1/2}\exp\{it\boldsymbol{\mu}'(\mathbf{I} - 2it\mathbf{A})^{-1}\mathbf{A}\boldsymbol{\mu}\}. \qquad (7)$$

Cf. Zhang and Fang (1982).

2. Distribution of A Quadratic Form

Some special cases

◇ When $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I})$ we have

$$E(\exp it\mathbf{x}'\mathbf{A}\mathbf{x}) = |\mathbf{I} - 2it\mathbf{A}|^{-1/2}. \tag{8}$$

◇ When $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I})$ we have

$$E(\exp it\mathbf{x}'\mathbf{A}\mathbf{x}) = (1 - 2it)^{-r/2}. \tag{9}$$

if $\mathbf{A}$ is a projection matrix with rank r.

◇ When $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I})$ and $\mathbf{A}$ is a projection matrix with rank r. Then

$$E(\exp it\mathbf{x}'\mathbf{A}\mathbf{x}) = (1 - 2it)^{-r/2} \exp\left\{\frac{it\lambda}{1 - 2it}\right\}. \tag{10}$$

where $\lambda = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$.

By Lemma 1 we can prove more general Cochran's theorem.

2. Distribution of A Quadratic Form

Theorem 1A (Cochran):Assume that $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$ and $\mathbf{A}$ is a symmetric matrix of order n. Then $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi_r^2(\lambda)$ if and only if $\mathbf{A}$ is a projection matrix with trace r and $\lambda = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$.

Proof of Theorem 1A

The Sufficiency part is similar to what you given. For the necessity part assume $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi_r^2(\lambda)$. From (6) we have

$$(1-2it)^{-r/2} \exp\left\{\frac{it\lambda}{1-2it}\right\} = \prod_{j=1}^{n}(1-2itc_j)^{-1/2} \exp\left\{\frac{itc_j\lambda_j^2}{1-2itc_j}\right\}, \quad (11)$$

where $c_1, \ldots, c_n$ are the eigenvalues of $\mathbf{A}$. The sides of (11) are functions in $c_1, \ldots, c_n$. Comparing singular values between two sides we can find there are r's $c_j = 1$ and remaining to be zero. It implies $\mathbf{A}$ is a project matrix with rank r.

2. Distribution of A Quadratic Form

Corollary 1 (Cochran): Assume that $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$ and $\mathbf{A}$ is a symmetric matrix of order n. Then $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2_r$ if and only if $\mathbf{A}$ is a projection matrix with trace r and $\mathbf{A}\boldsymbol{\mu} = \mathbf{0}$.

Example 3 (cont): Let $x_1, \dots, x_n$ be an *iid* sample from the population $N(\mu, \sigma^2)$. Then $\mathbf{x} = (x_1 \dots, x_n)' \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$ with $\boldsymbol{\mu} = (\mu, \dots, \mu)' = \mu\mathbf{1}_n$ and $Q = \sum_{i=1}^n (x_i - \bar{x})^2 = \mathbf{x}'\mathbf{D}_n\mathbf{x}$, $\mathbf{D}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$. Clearly, $\mathbf{D}_n$ is a projection matrix with rank $n-1$ and

$$\mathbf{D}_n \cdot \boldsymbol{\mu} = \mu\mathbf{D}_n\mathbf{1} = \mathbf{0}.$$

From the Cochran's theorem we have

$$Q = \sum_{i=1}^n (x_i - \bar{x})^2 \sim \chi^2_{n-1}.$$

2. Distribution of A Quadratic Form

Example 3 (cont):
Let $\mathbf{B} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$. It is easy to see that $\mathbf{B}$ is a projection matrix with rank 1. Furthermore, we have $\mu\mathbf{1}_n'\mathbf{B}\mu\mathbf{1}_n = \mu^2\mathbf{1}_n'(\frac{1}{n}\mathbf{1}_n\mathbf{1}_n')\mathbf{1}_n = n\mu^2$. It results in $n\bar{x}\bar{x}' = \mathbf{x}'\mathbf{B}\mathbf{x} \sim \chi_1^2(n\mu^2)$.

Question 3: How to prove $\bar{x}$ and s^2 are independent?

3. Independence among Quadratic Forms

3.1 Craig's theorem

Theorem 2 (Craig): Assume that $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$ and $\mathbf{A}_j, j = 1, \ldots, m$ are $n \times n$ symmetric matrices. Then $\mathbf{x}'\mathbf{A}_j\mathbf{x}, j = 1, \ldots, m$ are mutually independent if and only if $\mathbf{A}_j\mathbf{A}_k = \mathbf{0}, j \neq k$.

It took a very long time to find two correct proofs, one of which was given by P.L. Hsu. The key point is the following Lemma.

Lemma 2 (P.L.Hsu): Let $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ symmetric matrices with non-zero eigenvalues $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots \beta_k\}$, respectively. If the non-zero eigenvalues of $\mathbf{A} + \mathbf{B}$ are $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots \beta_k\}$, then $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$.
The proof can refer to Zhang and Fang (1982).

3. Independence among Quadratic Forms

Corollary 2. Assume that $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I}_n)$ and $\mathbf{A}_j, j = 1, \ldots, m$ are $n \times n$ nonnegative definite matrices. Then $\mathbf{x}'\mathbf{A}_j\mathbf{x}, j = 1, \ldots, m$ are mutually independent if and only if $\mathrm{tr}(\mathbf{A}_j\mathbf{A}_k) = 0, j \neq k$.

Example 3 (cont): Take $\mathbf{A} = \mathbf{D}_n$ and $\mathbf{B} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$ in the consideration. It is easy to see

$$
\begin{aligned}
\mathbf{AB} &= (\mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n')\mathbf{B} = (\mathbf{I}_n - \mathbf{B})\mathbf{B} = \mathbf{B} - \mathbf{BB} = \mathbf{B} - \mathbf{B} = \mathbf{0}, \\
\mathbf{BA} &= \mathbf{B}(\mathbf{I}_n - \mathbf{B}) = \mathbf{B} - \mathbf{B} = \mathbf{0}.
\end{aligned}
$$

We can conclude $Q(\mathbf{x})$ and $n\bar{x}\bar{x}'$ are independent. Consequently, s^2 and $\bar{x}$ are independent.

3. Independence among Quadratic Forms

3.2. More Extensions

Lemma 3: Let $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be symmetric matrices of order n with rank $r_1, \ldots, r_k$, respectively. Let $\mathbf{A} = \mathbf{A}_1 + \ldots + \mathbf{A}_k$ and denote the rank of $\mathbf{A}$ by r. Consider the following conditions:

(a) $\mathbf{A}_j$'s are projection matrices;
(b) $\mathbf{A}_i\mathbf{A}_j = \mathbf{0}, i \neq j$;
(c) $\mathbf{A}$ is a projection matrix;
(d) $r = r_1 + \ldots, + r_k$.

Then we have
(I) Any two conditions of (a),(b),(c) and imply to the remaining one plus condition (d);
(II) (c)+(d)$\Rightarrow$ (a) and (b).

3. Independence among Quadratic Forms

Theorem 3: Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \mathbf{I})$ and let $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be symmetric matrices of order n with rank $r_1, \ldots, r_k$, respectively. Let $\mathbf{A}$ with rank r be the sum of $\mathbf{A}_1, \ldots, \mathbf{A}_k$. Consider the following conditions:

(a_1) $\mathbf{A}_j$'s are projection matrices;
(a_2) $\mathbf{A}_i\mathbf{A}_j = \mathbf{0}, i \neq j$;
(a_3) $\mathbf{A}$ is a projection matrix;
(b_1) $\mathbf{x}'\mathbf{A}_j\mathbf{x} \sim \chi^2_{r_j}(\boldsymbol{\mu}'\mathbf{A}_j\boldsymbol{\mu}), j = 1, \ldots, k$;
(b_2) $\{\mathbf{x}'\mathbf{A}_j\mathbf{x}, j = 1 \ldots, k\}$ are independent;
(b_3) $\mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2_r(\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu})$;
(c) $r = r_1 + \ldots + r_k$.

Then we have

(I) $(a_j) \Leftrightarrow (b_j), j = 1, 2, 3$;
(II) Any two of $(a_i), (b_j), i \neq j \Rightarrow$ the remaining conditions;
(III) (a_3)+(c) or (b_3)+(c) $\Rightarrow$ the remaining conditions.

3. Independence among Quadratic Forms

Corollary 3. Let $\mathbf{x} \sim N_n(\mathbf{0}, \mathbf{I})$ and $\mathbf{x}'\mathbf{x} = Q_1(\mathbf{x}) + Q_2(\mathbf{x})$, where Q_1 and Q_2 are quadratic forms of $\mathbf{x}$. The fact that $Q_1(\mathbf{x}) \sim \chi^2_r$ implies to $Q_2(\mathbf{x}) \sim \chi^2_{n-r}$.

Proof. Denote $Q_1(\mathbf{x}) = \mathbf{x}'\mathbf{A}_1\mathbf{x}$ and $Q_2(\mathbf{x}) = \mathbf{x}'\mathbf{A}_2\mathbf{x}$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are symmetric matrices. From Theorem 3, the fact that $Q_1(\mathbf{x}) \sim \chi^2_r$ implies that $\mathbf{A}_1$ is a projection matrix with rank r. Consequently, $\mathbf{A}_2 = \mathbf{I}_n - \mathbf{A}_1$ is a projection matrix with rank $n - r$, and $Q_2(\mathbf{x}) \sim \chi^2_{n-r}$.

3.2. More Extensions

Theorem 4: Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} > 0$ and $\mathbf{A}$ be a symmetric matrix. Then $\mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi_r^2(\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu})$ if and only if $\boldsymbol{\Sigma}^{1/2}\mathbf{A}\boldsymbol{\Sigma}^{1/2}$ is a projection matrix with rank r.

Corollary 4. Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} > 0$ and $\mathbf{A}$ be a symmetric matrix. Then $\mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi_r^2$ if and only if $\boldsymbol{\Sigma}^{1/2}\mathbf{A}\boldsymbol{\Sigma}^{1/2}$ is a projection matrix with rank r and $\mathbf{A}\boldsymbol{\mu} = \mathbf{0}$.

Theorem 5: Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} > 0$ and $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be $n \times n$ symmetric matrices. Then $\mathbf{x}'\mathbf{A}_1\mathbf{x}, \ldots, \mathbf{x}'\mathbf{A}_k\mathbf{x}$ are independent if and only if $\mathbf{A}_i\boldsymbol{\Sigma}\mathbf{A}_j = \mathbf{0}$, for any $i \neq j$.

3.2. More Extensions

Theorem 6: Let $\mathbf{x} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} > 0$ and $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be $n \times n$ symmetric matrices with rank $r_1, \ldots, r_k$, respectively. Let $\mathbf{A} = \sum_{j=1}^{n} \mathbf{A}_j$ and denote its rank by r. Consider the following conditions:

(a_1) $\boldsymbol{\Sigma}^{1/2}\mathbf{A}_j\boldsymbol{\Sigma}^{1/2}$'s are projection matrices;
(a_2) $\mathbf{A}_b S \mathbf{A}_j = \mathbf{0}, i \neq j$;
(a_3) $\boldsymbol{\Sigma}^{1/2}\mathbf{A}\boldsymbol{\Sigma}^{1/2}$ is a projection matrix;
(b_1) $\mathbf{x}'\mathbf{A}_j\mathbf{x} \sim \chi^2_{r_j}(\boldsymbol{\mu}'\mathbf{A}_j\boldsymbol{\mu}), j = 1, \ldots, k$;
(b_2) $\{\mathbf{x}'\mathbf{A}_j\mathbf{x}, j = 1 \ldots, k\}$ are independent;
(b_3) $\mathbf{x}'\mathbf{A}\mathbf{x} \sim \chi^2_r(\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu})$;
(c) $r = r_1 + \ldots + r_k$.
Then we have
(I) $(a_j) \Leftrightarrow (b_j), j = 1, 2, 3$;
(II) Any two of $(a_i), (b_j), i \neq j \Rightarrow$ the remaining conditions;
(III) (a_3)+(c) or (b_3)+(c) $\Rightarrow$ the remaining conditions.

4. Applications

Regression Analysis: Consider the linear model

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon$$
$$\varepsilon \sim N_n(\mathbf{0}, \sigma^2 \mathbf{I}_n) \tag{12}$$

where $\mathbf{X} : n \times p, \beta'' p \times 1$.

◇ The least squared estimator of β is given by $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}\mathbf{y}$,

◇ an unbiased estimator of σ^2 is given by $\hat{\sigma^2} = \frac{1}{n-p}\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}$,

where $p = \text{Rank}(\mathbf{X})$ and $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is the hat matrix.

◇ Unbiased estimator of $\mathbf{y}$ is given by $\hat{\mathbf{y}} = \mathbf{X}\hat{\beta} = \mathbf{H}\mathbf{y}$.

4. Applications

Regression Analysis:

◇ The decomposition of the sum squares of $SS_{total}, SS_{res}, SS_{reg}$ is given by

$$Q(\mathbf{y}) \equiv \sum_{j=1}^{n}(y_j - \bar{y})^2 = \sum_{j=1}^{n}(y_j - \hat{y}_j)^2 + \sum_{j=1}^{n}(\hat{y}_j - \bar{y})^2 \equiv Q_1(\mathbf{y}) + Q_2(\mathbf{y})$$

That can be expressed as

$$\begin{aligned} \mathbf{y}'\mathbf{D}_n\mathbf{y} &= \mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y} + \hat{y}'\mathbf{D}_n\hat{\mathbf{y}} \\ &= \mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y} + \mathbf{y}'\mathbf{H}\mathbf{D}_n\mathbf{H}\mathbf{y}. \end{aligned}$$

by the use of the fact $\bar{y} = \bar{\hat{y}}$, where $\mathbf{D}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_N\mathbf{1}_n'$.

4. Applications

Regression Analysis:

◊ As $\mathbf{D}_n$ is a projection matrix with rank $n-1$ and $\mathbf{y} \sim N_n(\mathbf{X}\beta, \sigma^2\mathbf{I}_n)$ we have $Q(\mathbf{y}) = \mathbf{y}'\mathbf{D}_n\mathbf{y} \sim \chi^2_{n-1}(\lambda)$, where $\lambda = \beta'\mathbf{X}'\mathbf{D}_n\mathbf{X}\beta$.

◊ As $(\mathbf{I}-\mathbf{H})$ is a projection matrix with rank $n-2$ and $(\mathbf{I}-\mathbf{H})\mathbf{X} = \mathbf{O}$, it implies that $SS_{res} = Q_1(\mathbf{y}) = \mathbf{y}'(\mathbf{I}-\mathbf{H})\mathbf{y} \sim \chi^2_{n-p}$.

◊ It shows that SS_{res} and SS_{reg} are independent by Theorem 2 (Craig) as $(\mathbf{I}-\mathbf{H})\mathbf{H}\mathbf{D}_n\mathbf{H} = \mathbf{O}$.

◊ From Theorem 3 we have that $\mathbf{H}\mathbf{D}_n\mathbf{H}$ is a projection matrix with rank $(n-1)-(n-p) = p-1$, and $SS_{reg} = \mathbf{y}'\mathbf{H}\mathbf{D}_n\mathbf{H}\mathbf{y} \sim \chi^2_{p-1}(\lambda_2), \lambda_2 = \beta'\mathbf{X}'\mathbf{H}\mathbf{D}_n\mathbf{H}\mathbf{X}\beta = \beta'\mathbf{X}'\mathbf{D}_n\mathbf{X}\beta$.

4. Applications

Regression Analysis:

Applying the theory of quadratic forms to model (12) prove the following facts. If some conditions are necessary, please show the related condition.

(1) The distributions of $Q(\mathbf{y})$, $Q_1(\mathbf{y})$ and $Q_2(\mathbf{y})$;

(2) Prove $Q_1(\mathbf{y})$ and $Q_2(\mathbf{y})$ are independent;

(3) Prove $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}\mathbf{y}$ and $\hat{\sigma^2}$ are independent.