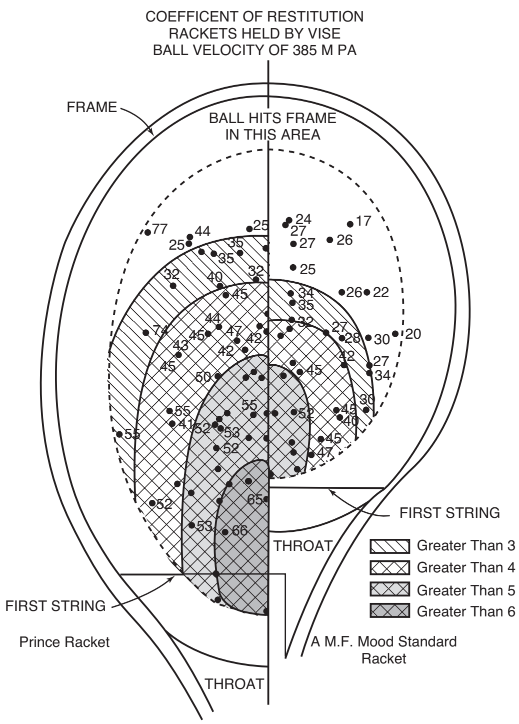
CHAPTER 13

Multiple Integrals



The design of modern sports equipment has become a sophisticated engineering enterprise. Many innovations can be traced back to a brilliant engineer but mediocre athlete named Howard Head. As an aircraft engineer in the 1940s, Head became frustrated learning to ski on the wooden skis of the day. Following years of experimentation, Head revolutionized the ski industry by introducing metal skis designed using principles borrowed from aircraft design.

By 1970, Head had retired from the Head Ski Company as a wealthy ski mogul. He quickly became frustrated by his slow progress learning to play tennis, a sport then played exclusively with wooden rackets. Head again focused on his equipment, reasoning that a larger racket would twist less and therefore be easier to control. However, years of experimentation showed that large wooden rackets either broke easily or were too heavy to swing.

Given that Head's metal skis were successful largely because they reduced the twisting of the skis in turns, it is not surprising that his experimentation turned to oversized metal tennis rackets. The rackets that Head eventually marketed as Prince rackets revolutionized tennis racket design. As the accompanying diagram shows, the sweet spot of the oversized racket is considerably larger than the sweet spot of the smaller wooden racket.

In this chapter, we introduce double and triple integrals, which are needed to compute the mass, moment of inertia and other important properties of three-dimensional solids. The moment of inertia is a measure of the resistance of an object to rotation. As shown in the exercises in section 13.2, compared to smaller rackets, the larger Head rackets have a larger moment of inertia and thus, twist less on off-center shots. Engineers use similar calculations as they test new materials for strength and weight for the next generation of sports equipment.

13.1 DOUBLE INTEGRALS

Before we introduce the idea of a double integral for a function of two variables, we first briefly remind you of the definition of definite integral for a function of a single variable and then generalize the definition slightly. Recall that we defined the definite integral while looking for the area A under the graph of a

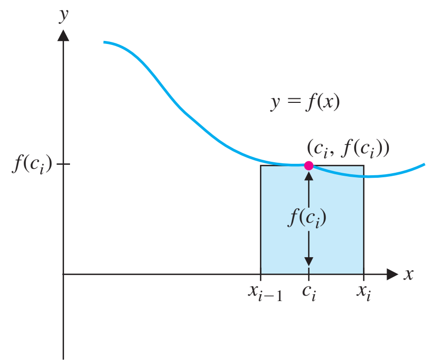


FIGURE 13.1a
Approximating the area on the subinterval $[x_{i-1}, x_i]$

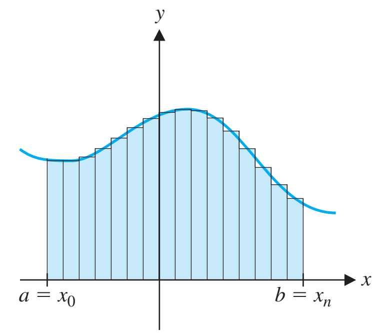


FIGURE 13.1b
Area under the curve

continuous function f defined on an interval $[a, b]$, where $f(x) \geq 0$ on $[a, b]$. We did this by *partitioning* the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$, of equal width $\Delta x = \dfrac{b-a}{n}$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

On each subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$, we approximated the area under the curve by the area of the rectangle of height $f(c_i)$, for some point $c_i \in [x_{i-1}, x_i]$, as indicated in Figure 13.1a. Adding together the areas of all of these rectangles, we obtain an approximation of the area, as indicated in Figure 13.1b:

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x.$$

Finally, taking the limit as $n \to \infty$ (which also means that $\Delta x \to 0$), we get the exact area (assuming that the limit exists and is the same for all choices of the evaluation points c_i):

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

We defined the definite integral as this limit:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x. \tag{1.1}$$

We generalize this by allowing partitions that are **irregular** (that is, where not all subintervals have the same width). We need this kind of generalization, among other reasons, for more sophisticated numerical methods for approximating definite integrals. This generalization is also needed for theoretical purposes; this is pursued in a more advanced course. We proceed essentially as above, except that we allow different subintervals to have different widths and define the width of the ith subinterval $[x_{i-1}, x_i]$ to be $\Delta x_i = x_i - x_{i-1}$ (see Figure 13.2 for the case where $n = 7$).

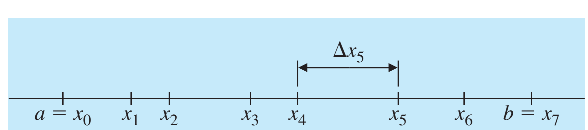


FIGURE 13.2
Irregular partition of $[a, b]$

An approximation of the area is then (essentially, as before)

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x_i,$$

for any choice of the evaluation points $c_i \in [x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. To get the exact area, we need to let $n \to \infty$, but since the partition is irregular, this alone will not guarantee that all of the Δx_i's will approach zero. We take a little extra care, by defining $\|P\|$ (the **norm of the partition**) to be the *largest* of all the Δx_i's. We then arrive at the following more general definition of definite integral.

DEFINITION 1.1

For any function f defined on the interval $[a, b]$, the **definite integral** of f on $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i,$$

provided the limit exists and is the same for all choices of the evaluation points $c_i \in [x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. In this case, we say that f is **integrable** on $[a, b]$.

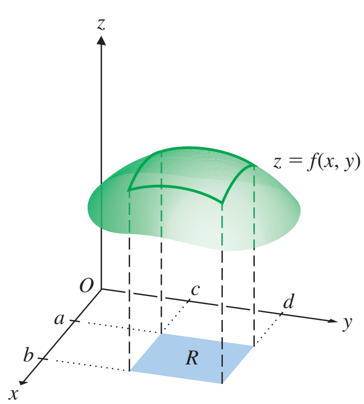


FIGURE 13.3
Volume under the surface $z = f(x, y)$

Here, by saying that the limit in Definition 1.1 equals some value L, we mean that we can make $\sum_{i=1}^{n} f(c_i)\,\Delta x_i$ as close as needed to L, just by making $\|P\|$ sufficiently small. How close must the sum get to L? We must be able to make the sum within any specified distance $\varepsilon > 0$ of L. More precisely, given any $\varepsilon > 0$, there must be a $\delta > 0$ (depending on the choice of ε), such that

$$\left| \sum_{i=1}^{n} f(c_i)\,\Delta x_i - L \right| < \varepsilon,$$

for *every* partition P with $\|P\| < \delta$. Notice that this is only a very slight generalization of our original notion of definite integral. All we have done is to allow the partitions to be irregular and then defined $\|P\|$ to ensure that $\Delta x_i \to 0$, for every i.

While you would likely never use Definition 1.1 to compute an area, your computer or calculator software probably does use irregular partitions to estimate integrals. Definition 1.1 will help us see how to generalize the notion of integral to functions of several variables.

Double Integrals over a Rectangle

We developed the definite integral as a natural by-product of our method for finding area under a curve in the xy-plane. Likewise, we are guided in our development of the double integral by a corresponding problem. For a function $f(x, y)$, where f is continuous and $f(x, y) \geq 0$ for all $a \leq x \leq b$ and $c \leq y \leq d$, we wish to find the *volume* of the solid lying below the surface $z = f(x, y)$ and above the rectangle $R = \{(x, y) | a \leq x \leq b$ and $c \leq y \leq d\}$ in the xy-plane (see Figure 13.3).

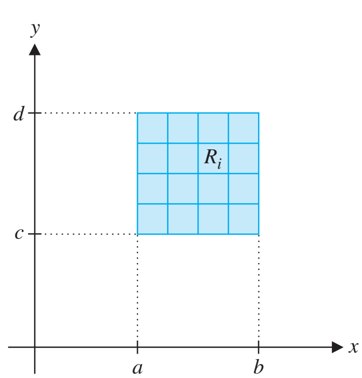


FIGURE 13.4a
Partition of R

We proceed essentially as we did to find the area under a curve. First, we partition the rectangle R by laying down a grid on top of R consisting of n smaller rectangles (see Figure 13.4a). (Note: The rectangles in the grid need not be all of the same size.) Call the smaller rectangles $R_1, R_2, \ldots, R_n$. (The order in which you number them is irrelevent.) For each rectangle R_i ($i = 1, 2, \ldots, n$) in the partition, we want to find an approximation to the volume V_i lying beneath the surface $z = f(x, y)$ and above the rectangle R_i. The sum of these approximate volumes is then an approximation to the total volume. Above each rectangle R_i in the partition, construct a rectangular box whose height is $f(u_i, v_i)$, for some point $(u_i, v_i) \in R_i$ (see Figure 13.4b). Notice that the volume V_i beneath the surface and above R_i is approximated by the volume of the box:

$$V_i \approx \text{Height} \times \text{Area of base} = f(u_i, v_i)\,\Delta A_i,$$

where ΔA_i denotes the area of the rectangle R_i.

The total volume is then approximately

$$V \approx \sum_{i=1}^{n} f(u_i, v_i)\,\Delta A_i. \tag{1.2}$$

As in our development of the definite integral in Chapter 4, we call the sum in (1.2) a **Riemann sum.** We illustrate the approximation of the volume under a surface by a Riemann sum in Figures 13.4c and 13.4d. Notice that the larger number of rectangles used in Figure 13.4d appears to give a better approximation of the volume.

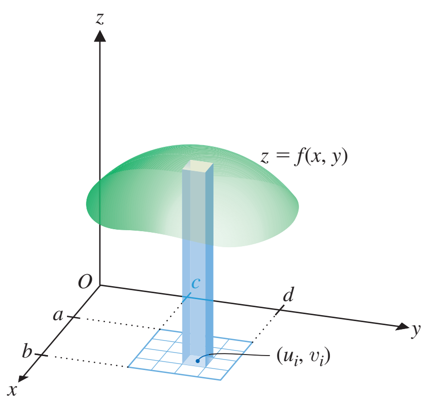


FIGURE 13.4b
Approximating the volume above R_i by a rectangular box

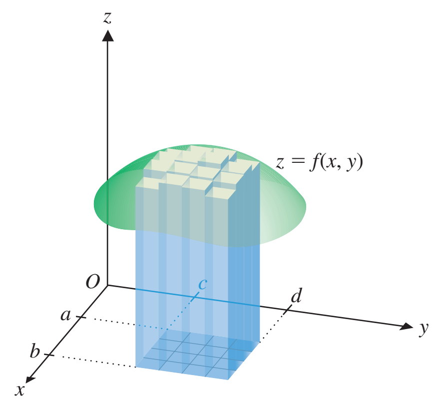


FIGURE 13.4c
Approximate volume

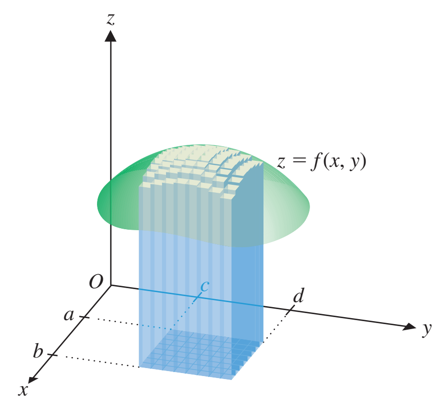


FIGURE 13.4d
Approximate volume

EXAMPLE 1.1 Approximating the Volume Lying Beneath a Surface

Approximate the volume lying beneath the surface $z = x^2 \sin \dfrac{\pi y}{6}$ and above the rectangle $R = \{(x, y) | 0 \le x \le 6, 0 \le y \le 6\}$.

Solution First, note that f is continuous and $f(x, y) = x^2 \sin \dfrac{\pi y}{6} \ge 0$ on R (see Figure 13.5a). Next, a simple partition of R is a partition into four squares of equal size, as indicated in Figure 13.5b. We choose the evaluation points (u_i, v_i) to be the centers of each of the four squares, that is, $\left(\frac{3}{2}, \frac{3}{2}\right)$, $\left(\frac{9}{2}, \frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{9}{2}\right)$ and $\left(\frac{9}{2}, \frac{9}{2}\right)$.

Since the four squares are the same size, we have $\Delta A_i = 9$, for each i. For $f(x, y) = x^2 \sin \dfrac{\pi y}{6}$, we have from (1.2) that

$$\begin{aligned} V &\approx \sum_{i=1}^{4} f(u_i, v_i)\,\Delta A_i \\ &= f\left(\frac{3}{2}, \frac{3}{2}\right)(9) + f\left(\frac{9}{2}, \frac{3}{2}\right)(9) + f\left(\frac{3}{2}, \frac{9}{2}\right)(9) + f\left(\frac{9}{2}, \frac{9}{2}\right)(9) \\ &= 9\left[\left(\frac{3}{2}\right)^2 \sin\left(\frac{\pi}{4}\right) + \left(\frac{9}{2}\right)^2 \sin\left(\frac{\pi}{4}\right) + \left(\frac{3}{2}\right)^2 \sin\left(\frac{3\pi}{4}\right) + \left(\frac{9}{2}\right)^2 \sin\left(\frac{3\pi}{4}\right)\right] \\ &= \frac{405}{2}\sqrt{2} \approx 286.38. \end{aligned}$$

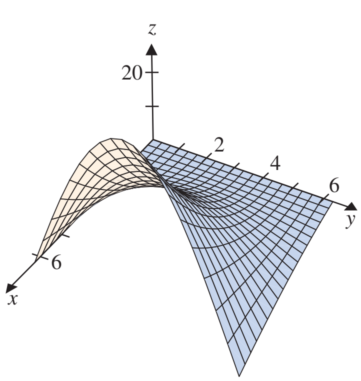


FIGURE 13.5a
$z = x^2 \sin \dfrac{\pi y}{6}$

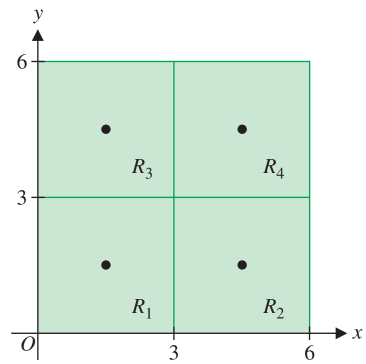


FIGURE 13.5b
Partition of R into four equal squares

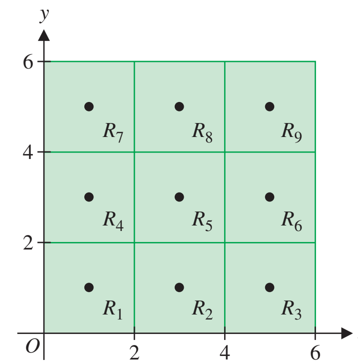


FIGURE 13.5c
Partition of R into nine equal squares

We can improve on this approximation by increasing the number of rectangles in the partition. For instance, if we partition R into nine squares of equal size (see Figure 13.5c) and again use the center of each square as the evaluation point, we have $\Delta A_i = 4$ for each i and

$$\begin{aligned} V &\approx \sum_{i=1}^{9} f(u_i, v_i)\,\Delta A_i \\ &= 4\,[f(1,1) + f(3,1) + f(5,1) + f(1,3) + f(3,3) + f(5,3) \\ &\quad + f(1,5) + f(3,5) + f(5,5)] \\ &= 4\left[1^2 \sin\left(\frac{\pi}{6}\right) + 3^2 \sin\left(\frac{\pi}{6}\right) + 5^2 \sin\left(\frac{\pi}{6}\right) + 1^2 \sin\left(\frac{3\pi}{6}\right) + 3^2 \sin\left(\frac{3\pi}{6}\right)\right. \\ &\quad \left. + 5^2 \sin\left(\frac{3\pi}{6}\right) + 1^2 \sin\left(\frac{5\pi}{6}\right) + 3^2 \sin\left(\frac{5\pi}{6}\right) + 5^2 \sin\left(\frac{5\pi}{6}\right)\right] \\ &= 280. \end{aligned}$$

No. of Squares in Partition	*Approximate Volume*
4	286.38
9	280.00
36	276.25
144	275.33
400	275.13
900	275.07

Continuing in this fashion to divide R into more and more squares of equal size and using the center of each square as the evaluation point, we construct continually better and better approximations of the volume (see the table in the margin). From the table, it appears that a reasonable approximation to the volume is slightly less than 275.07. In fact, the exact volume is $\frac{864}{\pi} \approx 275.02$. (We'll show you how to find this shortly.) ■

NOTES

The choice of the center of each square as the evaluation point, as used in example 1.1, corresponds to the Midpoint rule for approximating the value of a definite integral for a function of a single variable (discussed in section 4.7). This choice of evaluation points generally produces a reasonably good approximation.

Now, how can we turn (1.2) into an exact formula for volume? Note that it takes more than simply letting $n \to \infty$. We need to have *all* of the rectangles in the partition shrink to zero area. A convenient way of doing this is to define the **norm of the partition** $\|P\|$ to be the largest diagonal of any rectangle in the partition. Note that if $\|P\| \to 0$, then *all* of the rectangles must shrink to zero area. We can now make the volume approximation (1.2) exact:

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i)\,\Delta A_i,$$

assuming the limit exists and is the same for every choice of the evaluation points. Here, by saying that this limit equals V, we mean that we can make $\sum_{i=1}^{n} f(u_i, v_i)\,\Delta A_i$ as close as needed to V, just by making $\|P\|$ sufficiently small. More precisely, this says that given any

$\varepsilon > 0$, there is a $\delta > 0$ (depending on the choice of ε), such that

$$\left| \sum_{i=1}^{n} f(u_i, v_i)\, \Delta A_i - V \right| < \varepsilon,$$

for every partition P with $\|P\| < \delta$. More generally, we have the following definition, which applies even when the function takes on negative values.

DEFINITION 1.2

For any function $f(x, y)$ defined on the rectangle $R = \{(x, y) | a \le x \le b \text{ and } c \le y \le d\}$, we define the **double integral** of f over R by

$$\iint_R f(x, y)\, dA = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i)\, \Delta A_i,$$

provided the limit exists and is the same for every choice of the evaluation points (u_i, v_i) in R_i, for $i = 1, 2, \ldots, n$. When this happens, we say that f is **integrable** over R.

REMARK 1.1

It can be shown that if f is continuous on R, then it is also integrable over R. The proof can be found in more advanced texts.

There's one small problem with this new double integral. Just as when we first defined the definite integral of a function of one variable, we don't yet know how to compute it! For complicated regions R, this is a little bit tricky, but for a rectangle, it's a snap, as we see in the following.

We first consider the special case where $f(x, y) \ge 0$ on the rectangle $R = \{(x, y) | a \le x \le b \text{ and } c \le y \le d\}$. Notice that here, $\iint_R f(x, y)\, dA$ represents the volume lying beneath the surface $z = f(x, y)$ and above the region R. Recall that we already know how to compute this volume, from our work in section 5.2. We can do this by slicing the solid with planes parallel to the yz-plane, as indicated in Figure 13.6a. If we denote the area of the cross section of the solid for a given value of x by $A(x)$, then we have from equation (2.1) in section 5.2 that the volume is given by

$$V = \int_a^b A(x)\, dx.$$

Now, note that for each *fixed* value of x, the area of the cross section is simply the area under the curve $z = f(x, y)$ for $c \le y \le d$, which is given by the integral

$$A(x) = \int_c^d f(x, y)\, dy.$$

This integration is called a **partial integration** with respect to y, since x is held fixed and $f(x, y)$ is integrated with respect to y. This leaves us with

$$V = \int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx. \tag{1.3}$$

Likewise, if we instead slice the solid with planes parallel to the xz-plane, as indicated in Figure 13.6b, we get that the volume is given by

$$V = \int_c^d A(y)\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy. \tag{1.4}$$

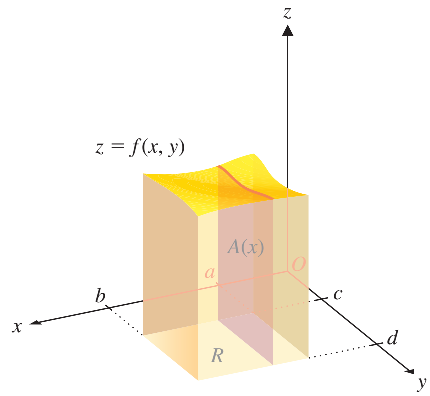


FIGURE 13.6a
Slicing the solid parallel to the *yz*-plane

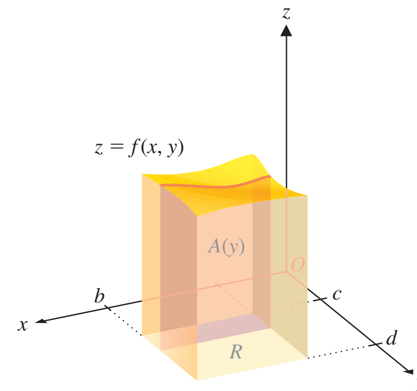


FIGURE 13.6b
Slicing the solid parallel to the *xz*-plane

The integrals in (1.3) and (1.4) are called **iterated integrals.** Note that each of these indicates a partial integration with respect to the inner variable (i.e., you first integrate with respect to the inner variable, treating the outer variable as a constant), to be followed by an integration with respect to the outer variable.

For simplicity, we ordinarily write the iterated integrals without the brackets:

$$\int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

and

$$\int_c^d \left[\int_a^b f(x, y)\, dx\right] dy = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

As indicated, these integrals are evaluated inside out, using the methods of integration we've already established for functions of a single variable. This now establishes the following result for the special case where $f(x, y) \geq 0$. The proof of the result for the general case is rather lengthy and we omit it.

HISTORICAL NOTES

Guido Fubini (1879–1943) Italian mathematician who made wide-ranging contributions to mathematics, physics and engineering. Fubini's early work was in differential geometry, but he quickly diversified his research to include analysis, the calculus of variations, group theory, non-Euclidean geometry and mathematical physics. Mathematics was the family business, as his father was a mathematics teacher and his sons became engineers. Fubini moved to the United States in 1939 to escape the persecution of Jews in Italy. He was working on an engineering textbook inspired by his sons' work when he died.

THEOREM 1.1 (Fubini's Theorem)

Suppose that f is integrable over the rectangle $R = \{(x, y) | a \leq x \leq b \text{ and } c \leq y \leq d\}$. Then we can write the double integral of f over R as either of the iterated integrals:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy. \tag{1.5}$$

Fubini's Theorem simply tells you that you can always rewrite a double integral over a rectangle as either one of a pair of iterated integrals. We illustrate this in example 1.2.

EXAMPLE 1.2 Double Integral over a Rectangle

If $R = \{(x, y) | 0 \leq x \leq 2 \text{ and } 1 \leq y \leq 4\}$, evaluate $\iint_R (6x^2 + 4xy^3)\, dA$.

Solution From (1.5), we have

$$\iint_R (6x^2 + 4xy^3)\,dA = \int_1^4 \int_0^2 (6x^2 + 4xy^3)\,dx\,dy$$

$$= \int_1^4 \left[\int_0^2 (6x^2 + 4xy^3)\,dx\right] dy$$

$$= \int_1^4 \left(6\frac{x^3}{3} + 4\frac{x^2}{2}y^3\right)\Bigg|_{x=0}^{x=2} dy$$

$$= \int_1^4 (16 + 8y^3)\,dy$$

$$= \left(16y + 8\frac{y^4}{4}\right)\Bigg|_1^4$$

$$= [16(4) + 2(4)^4] - [16(1) + 2(1)^4] = 558.$$

Note that we evaluated the first integral above by integrating with respect to x, while treating y as a constant. We leave it as an exercise to show that you get the same value by integrating first with respect to y, that is, that

$$\iint_R (6x^2 + 4xy^3)\,dA = \int_0^2 \int_1^4 (6x^2 + 4xy^3)\,dy\,dx = 558,$$

also. ■

Double Integrals over General Regions

So, what if we wanted to extend the notion of double integral to a bounded, nonrectangular region like the one shown in Figure 13.7a? (Recall that a region is bounded if it fits inside a circle of some finite radius.) We begin, as we did for the case of rectangular regions, by looking for the volume lying beneath the surface $z = f(x, y)$ and lying above the region R, where $f(x, y) \geq 0$ and f is continuous on R. First, notice that the grid we used initially to partition a rectangular region must be modified, since such a rectangular grid won't "fit" a nonrectangular region, as shown in Figure 13.7b.

We resolve this problem by considering only those rectangular subregions that lie *completely* inside the region R (see Figure 13.7c, where we have labeled these rectangles).

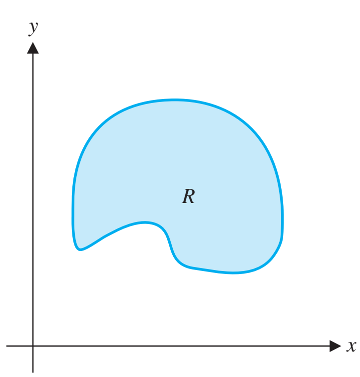


FIGURE 13.7a
Nonrectangular region

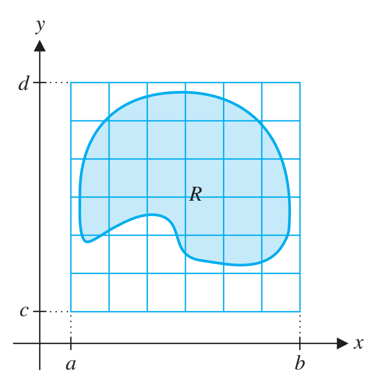


FIGURE 13.7b
Grid for a general region

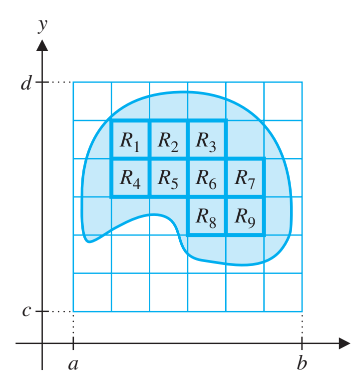


FIGURE 13.7c
Inner partition

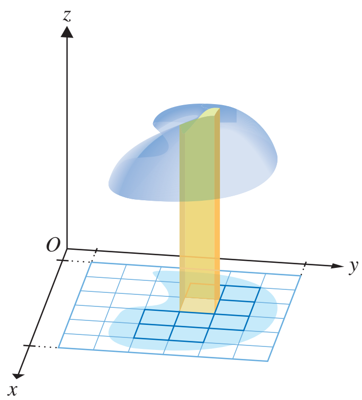


FIGURE 13.7d
Sample volume box

We call the collection of these rectangles an **inner partition** of R. For instance, in the inner partition indicated in Figure 13.7c, there are nine subregions.

From this point on, we proceed essentially as we did for the case of a rectangular region. That is, on each rectangular subregion R_i ($i = 1, 2, \ldots, n$) in an inner partition, we construct a rectangular box of height $f(u_i, v_i)$, for some point $(u_i, v_i) \in R_i$ (see Figure 13.7d for a sample box). The volume V_i beneath the surface and above R_i is then approximately

$$V_i \approx \text{Height} \times \text{Area of base} = f(u_i, v_i)\,\Delta A_i,$$

where we again denote the area of R_i by ΔA_i. The total volume V lying beneath the surface and above the region R is then approximately

$$V \approx \sum_{i=1}^{n} f(u_i, v_i)\,\Delta A_i. \tag{1.6}$$

We define the norm of the inner partition $\|P\|$ to be the length of the largest diagonal of any of the rectangles $R_1, R_2, \ldots, R_n$. Notice that as we make $\|P\|$ smaller and smaller, the inner partition fills in R nicely (see Figure 13.8a) and the approximate volume given by (1.6) should get closer and closer to the actual volume. (See Figure 13.8b.) We then have

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i)\,\Delta A_i,$$

assuming the limit exists and is the same for every choice of the evaluation points.

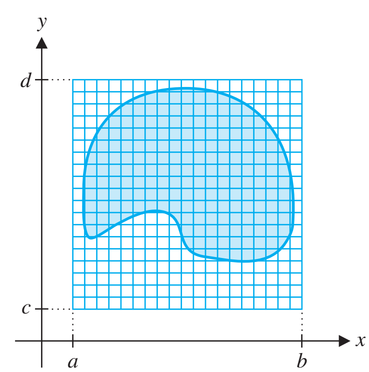


FIGURE 13.8a
Refined grid

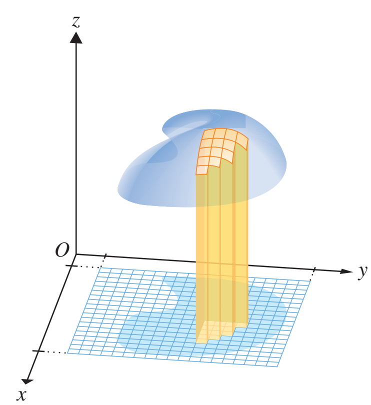


FIGURE 13.8b
Approximate volume

More generally, we have Definition 1.3.

DEFINITION 1.3

For any function $f(x, y)$ defined on a bounded region $R \subset \mathbb{R}^2$, we define the **double integral** of f over R by

$$\iint_R f(x, y)\,dA = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i)\,\Delta A_i, \tag{1.7}$$

provided the limit exists and is the same for every choice of the evaluation points (u_i, v_i) in R_i, for $i = 1, 2, \ldots, n$. In this case, we say that f is **integrable** over R.

REMARK 1.2

Once again, it can be shown that if f is continuous on R, then it is integrable over R, although the proof is beyond the level of this course.

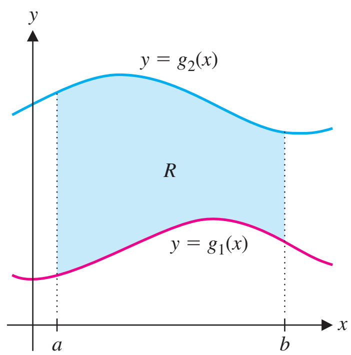


FIGURE 13.9a
The region R

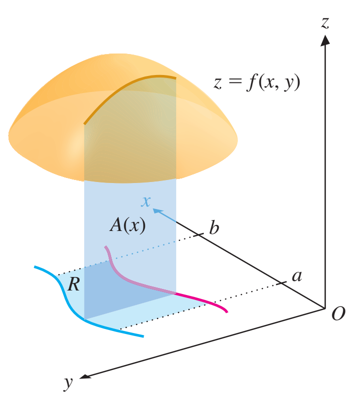


FIGURE 13.9b
Volume by slicing

CAUTION

Be sure to draw a reasonably good sketch of the region R before you try to write down the iterated integrals. Without doing this, you may be lucky enough (or clever enough) to get the first few exercises to work out, but you will be ultimately doomed to failure. It is *essential* that you have a clear picture of the region in order to set up the integrals correctly.

The question remains as to how we can calculate a double integral over a nonrectangular region. The answer is a bit more complicated than it was for the case of a rectangular region and depends on the exact form of R.

We first consider the case where the region R lies between the vertical lines $x = a$ and $x = b$, with $a < b$, has a top defined by the curve $y = g_2(x)$ and a bottom defined by $y = g_1(x)$, where $g_1(x) \le g_2(x)$ for all x in (a, b). That is, R has the form

$$R = \{(x, y) | a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}.$$

See Figure 13.9a for a typical region of this form lying in the first quadrant of the xy-plane. Think about this for the special case where $f(x, y) \ge 0$ on R. Here, the double integral of f over R gives the volume lying beneath the surface $z = f(x, y)$ and above the region R in the xy-plane. We can find this volume by the method of slicing, just as we did for the case of a double integral over a rectangular region.

From Figure 13.9b, observe that for each *fixed* $x \in [a, b]$, the area of the slice lying above the line segment indicated and below the surface $z = f(x, y)$ is given by

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

The volume of the solid is then given by equation (2.1) in section 5.2 to be

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Recognizing the volume as $V = \iint_R f(x, y)\, dA$ proves the following theorem, for the special case where $f(x, y) \ge 0$ on R.

THEOREM 1.2

Suppose that $f(x, y)$ is continuous on the region R defined by $R = \{(x, y) | a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$, for continuous functions g_1 and g_2, where $g_1(x) \le g_2(x)$, for all x in $[a, b]$. Then,

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Although the general proof of Theorem 1.2 is beyond the level of this text, the derivation given above for the special case where $f(x, y) \ge 0$ should help to make some sense of why it is true.

Notice that once again, we have managed to write a double integral as an iterated integral. This allows us to use all of our techniques of integration for functions of a single variable to help evaluate double integrals.

We illustrate the process of writing a double integral as an iterated integral in example 1.3.

EXAMPLE 1.3 Evaluating a Double Integral

Let R be the region bounded by the graphs of $y = x$, $y = 0$ and $x = 4$. Evaluate

$$\iint_R (4e^{x^2} - 5 \sin y)\, dA.$$

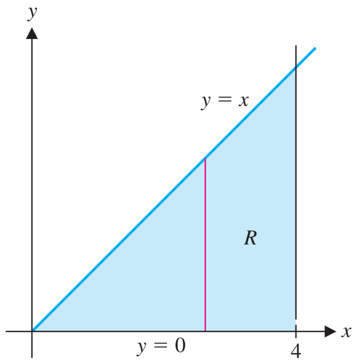


FIGURE 13.10
The region R

Solution First, we draw a graph of the region R in Figure 13.10. To help with determining the limits of integration, we have drawn a line segment illustrating that for each fixed value of x on the interval [0, 4], the y-values range from 0 up to x. From Theorem 1.2, we have

$$\iint_R (4e^{x^2} - 5\sin y)\,dA = \int_0^4 \int_0^x (4e^{x^2} - 5\sin y)\,dy\,dx \tag{1.8}$$

$$= \int_0^4 (4ye^{x^2} + 5\cos y)\Big|_{y=0}^{y=x} dx$$

$$= \int_0^4 [(4xe^{x^2} + 5\cos x) - (0 + 5\cos 0)]\,dx$$

$$= \int_0^4 (4xe^{x^2} + 5\cos x - 5)\,dx$$

$$= (2e^{x^2} + 5\sin x - 5x)\Big|_0^4$$

$$= 2e^{16} + 5\sin 4 - 22 \approx 1.78 \times 10^7.$$

Keep in mind that the inner integration above (with respect to y) is a partial integration with respect to y, so that we hold x fixed.

Be *very* careful here; there are plenty of traps to fall into. The most common error is to simply look for the minimum and maximum values of x and y and mistakenly write

$$\iint_R f(x, y)\,dA = \int_0^4 \int_0^4 f(x, y)\,dy\,dx. \quad \text{This is incorrect!}$$

Compare this last iterated integral to the correct expression in (1.8). Notice that instead of integrating over the region R shown in Figure 13.10, it corresponds to integration over the rectangle $0 \le x \le 4$, $0 \le y \le 4$. (This is close, but no cigar!) ■

As with any other integral, iterated integrals often cannot be evaluated symbolically (even with a very good computer algebra system). In such cases, we must rely on approximate methods. If you can, evaluate the inner integral symbolically and then use a numerical method (e.g., Simpson's Rule) to approximate the outer integral.

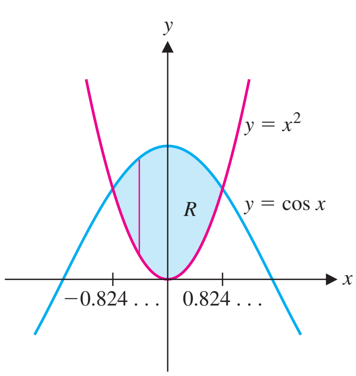


FIGURE 13.11
The region R

EXAMPLE 1.4 Approximate Limits of Integration

Evaluate $\iint_R (x^2 + 6y)\,dA$, where R is the region bounded by the graphs of $y = \cos x$ and $y = x^2$.

Solution We show a graph of the region R in Figure 13.11. Notice that the inner limits of integration are easy to see from the figure; for each fixed x, y ranges from x^2 up to $\cos x$. However, the outer limits of integration are not quite so clear. To find these, we must find the intersections of the two curves by solving the equation $\cos x = x^2$. We can't solve this exactly, but using a numerical procedure (e.g., Newton's method or one built into your calculator or computer algebra system), we get approximate intersections of $x \approx \pm 0.82413$. From Theorem 1.2, we now have

$$\iint_R (x^2 + 6y)\,dA \approx \int_{-0.82413}^{0.82413} \int_{x^2}^{\cos x} (x^2 + 6y)\,dy\,dx$$
$$= \int_{-0.82413}^{0.82413} \left(x^2 y + 6\frac{y^2}{2} \right) \Bigg|_{y=x^2}^{y=\cos x} dx$$
$$= \int_{-0.82413}^{0.82413} [(x^2 \cos x + 3\cos^2 x) - (x^4 + 3x^4)]\,dx$$
$$\approx 3.659765588,$$

where we have evaluated the last integral approximately, even though it could be done exactly, using integration by parts and a trigonometric identity. ■

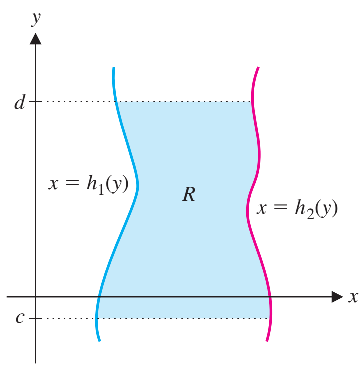


FIGURE 13.12
Typical region

Not all double integrals can be computed using the technique of examples 1.3 and 1.4. Often, it is necessary (or at least convenient) to think of the geometry of the region R in a different way.

Suppose that the region R has the form

$$R = \{(x, y) | c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}.$$

See Figure 13.12 for a typical region of this form. Then, much as in Theorem 1.2, we can write double integrals as iterated integrals, as in Theorem 1.3.

THEOREM 1.3

Suppose that $f(x, y)$ is continuous on the region R defined by $R = \{(x, y) | c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$, for continuous functions h_1 and h_2, where $h_1(y) \le h_2(y)$, for all y in $[c, d]$. Then,

$$\iint_R f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

The general proof of this theorem is beyond the level of this course, although the reasonableness of this result should be apparent from Theorem 1.2 and the analysis preceding that theorem, for the special case where $f(x, y) \ge 0$ on R.

TODAY IN MATHEMATICS

Mary Ellen Rudin (1924–)
An American mathematician who published more than 70 research papers while supervising Ph.D. students, raising four children and earning the love and respect of students and colleagues. As a child, she and her friends played games that were "very elaborate and purely in the imagination. I think actually that that is something that contributes to making a mathematician—having time to think and being in the habit of imagining all sorts of complicated things." She says, "I'm very geometric in my thinking. I'm not really interested in numbers." She describes her teaching style as, "I bubble and I get students enthusiastic."

EXAMPLE 1.5 Integrating First with Respect to *x*

Write $\iint_R f(x, y)\,dA$ as an iterated integral, where R is the region bounded by the graphs of $x = y^2$ and $x = 2 - y$.

Solution First, we sketch a graph of the region (see Figure 13.13a). Notice that integrating first with respect to y is not a very good choice, since the upper boundary of the region is $y = \sqrt{x}$ for $0 \le x \le 1$ and $y = 2 - x$ for $1 \le x \le 4$. A more reasonable choice is to use Theorem 1.3 and integrate first with respect to x. In Figure 13.13b, we have included a horizontal line segment indicating the inner limits of integration: for each fixed y, x runs from $x = y^2$ over to $x = 2 - y$. The value of y then runs between the values at the intersections of the two curves. To find these, we solve $y^2 = 2 - y$ or

$$0 = y^2 + y - 2 = (y + 2)(y - 1),$$

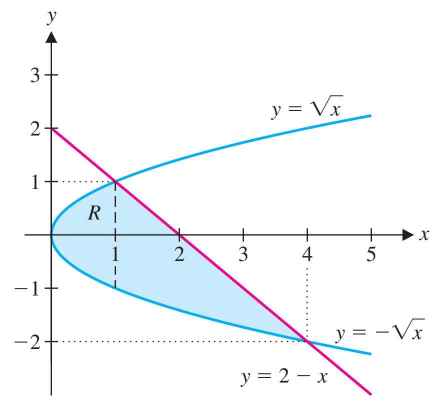


FIGURE 13.13a
The region R

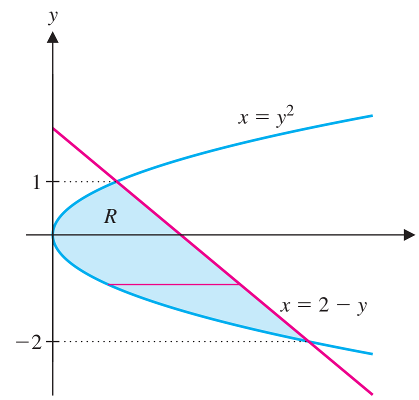


FIGURE 13.13b
The region R

so that the intersections are at $y = -2$ and $y = 1$. From Theorem 1.3, we now have

$$\iint_R f(x, y)\, dA = \int_{-2}^{1} \int_{y^2}^{2-y} f(x, y)\, dx\, dy.$$

You will often have to choose which variable to integrate with respect to first. Sometimes, you make your choice on the basis of the region. Often, a double integral can be set up either way but is much easier to calculate one way than the other. This is the case in example 1.6.

EXAMPLE 1.6 Evaluating a Double Integral

Let R be the region bounded by the graphs of $y = \sqrt{x}$, $x = 0$ and $y = 3$. Evaluate $\iint_R (2xy^2 + 2y\cos x)\, dA$.

Solution We show a graph of the region in Figure 13.14. From Theorem 1.3, we have

$$\begin{aligned}
\iint_R (2xy^2 + 2y\cos x)\, dA &= \int_0^3 \int_0^{y^2} (2xy^2 + 2y\cos x)\, dx\, dy \\
&= \int_0^3 (x^2y^2 + 2y\sin x)\Big|_{x=0}^{x=y^2} dy \\
&= \int_0^3 [(y^6 + 2y\sin y^2) - (0 + 2y\sin 0)]\, dy \\
&= \int_0^3 (y^6 + 2y\sin y^2)\, dy \\
&= \left(\frac{y^7}{7} - \cos y^2\right)\Big|_0^3 \\
&= \frac{3^7}{7} - \cos 9 + \cos 0 \approx 314.3.
\end{aligned}$$

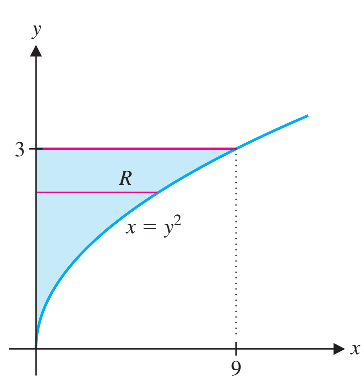


FIGURE 13.14
The region R

Alternatively, integrating with respect to y first, we get

$$\iint_R (2xy^2 + 2y\cos x)\,dA = \int_0^9 \int_{\sqrt{x}}^3 (2xy^2 + 2y\cos x)\,dy\,dx$$
$$= \int_0^9 \left(2x\frac{y^3}{3} + y^2\cos x\right)\Bigg|_{y=\sqrt{x}}^{y=3} dx$$
$$= \int_0^9 \left[\frac{2}{3}x(27 - x^{3/2}) + (3^2 - x)\cos x\right] dx,$$

which leaves you with an integration by parts to carry out. We leave the details as an exercise. Which way do you think is easier? ■

In example 1.6, we saw that changing the order of integration may make a given double integral easier to compute. As we see in example 1.7, sometimes you will *need* to change the order of integration in order to evaluate a double integral.

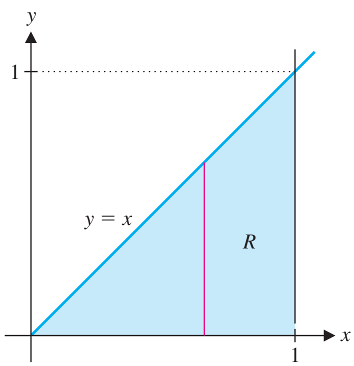


FIGURE 13.15
The region R

CAUTION

Carefully study the steps we used to change the order of integration in example 1.7. Notice that we did not simply swap the two integrals, nor did we just switch y's to x's on the inside limits. When you change the order of integration, it is extremely important that you sketch the region over which you are integrating, as in Figure 13.15. This allows you to see the orientation of the different parts of the boundary of the region. Failing to do this is the single most common error made by students at this point. This is a skill you need to practice, as you will use it throughout the rest of the course. (Sketching a picture takes only a few moments and will help you to avoid many fatal errors. So, do this routinely!)

EXAMPLE 1.7 A Case Where We Must Switch the Order of Integration

Evaluate the iterated integral $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$.

Solution First, note that we cannot evaluate the integral the way it is presently written, as we don't know an antiderivative for e^{x^2}. On the other hand, if we switch the order of integration, the integral becomes quite simple, as follows. First, recognize that for each fixed y on the interval $[0, 1]$, x ranges from y over to 1, giving us the triangular region of integration shown in Figure 13.15. If we switch the order of integration, notice that for each fixed x in the interval $[0, 1]$, y ranges from 0 up to x and we get the double iterated integral:

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx$$
$$= \int_0^1 e^{x^2} y\Big|_{y=0}^{y=x} dx$$
$$= \int_0^1 e^{x^2} x\,dx.$$

Notice that we can evaluate this last integral with the substitution $u = x^2$, since $du = 2x\,dx$ and the first integration has conveniently provided us with the needed factor of x. We have

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \frac{1}{2}\int_0^1 \underbrace{e^{x^2}}_{e^u}\underbrace{(2x)\,dx}_{du}$$
$$= \frac{1}{2}e^{x^2}\Big|_{x=0}^{x=1} = \frac{1}{2}(e^1 - 1).$$ ■

We complete the section by stating several simple properties of double integrals.

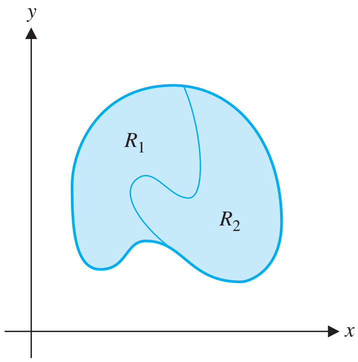


FIGURE 13.16
$R = R_1 \cup R_2$

THEOREM 1.4

Let $f(x, y)$ and $g(x, y)$ be integrable over the region $R \subset \mathbb{R}^2$ and let c be any constant. Then, the following hold:

(i) $\displaystyle\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA,$

(ii) $\displaystyle\iint_R [f(x, y) + g(x, y)]\,dA = \iint_R f(x, y)\,dA + \iint_R g(x, y)\,dA$ and

(iii) if $R = R_1 \cup R_2$, where R_1 and R_2 are nonoverlapping regions (see Figure 13.16), then

$$\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA.$$

Each of these follows directly from the definition of double integral in (1.7) and the proof are left as an exercise.

BEYOND FORMULAS

You should think of double integrals in terms of the Rule of Three: symbolic, graphical and numerical interpretations. Symbolically, you compute double integrals as iterated integrals, where the greatest challenge is correctly setting up the limits of integration. Graphically, the volume calculation that motivates Definition 1.2 is analogous to the area interpretation of single integrals. Numerically, double integrals can be approximated by Riemann sums. From your experience with single integrals and partial derivatives in Chapter 12, what percentage of double integrals do you expect to be able to evaluate symbolically?

EXERCISES 13.1

WRITING EXERCISES

1. If $f(x, y) \geq 0$ on a region R, then $\iint_R f(x, y)\,dA$ gives the volume of the solid above the region R in the xy-plane and below the surface $z = f(x, y)$. If $f(x, y) \geq 0$ on a region R_1 and $f(x, y) \leq 0$ on a region R_2, discuss the geometric meaning of $\iint_{R_2} f(x, y)\,dA$ and $\iint_R f(x, y)\,dA$, where $R = R_1 \cup R_2$.
2. The definition of $\iint_R f(x, y)\,dA$ requires that the norm of the partition $\|P\|$ approaches 0. Explain why it is not enough to simply require that the number of rectangles n in the partition approaches ∞.
3. When computing areas between curves in section 5.1, we discussed strategies for deciding whether to integrate with respect to x or y. Compare these strategies to those given in this section for deciding which variable to use as the inside variable of a double integral.
4. Suppose you (or your software) are using Riemann sums to approximate a particularly difficult double integral $\iint_R f(x, y)\,dA$. Further, suppose that $R = R_1 \cup R_2$ and the function $f(x, y)$ is nearly constant on R_1 but oscillates wildly on R_2, where R_1 and R_2 are nonoverlapping regions. Explain why you would need more rectangles in R_2 than R_1 to get equally accurate approximations. Thus, irregular partitions can be used to improve the efficiency of numerical integration routines.

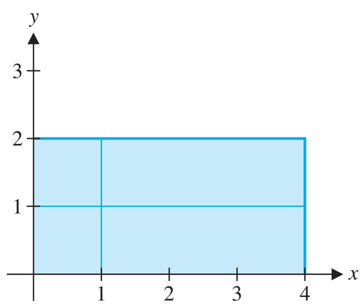
In exercises 1–4, compute the Riemann sum for the given function and region, a partition with n equal-sized rectangles and the given evaluation rule.

1. $f(x, y) = x + 2y^2$, $0 \leq x \leq 2$, $-1 \leq y \leq 1$, $n = 4$, evaluate at midpoint
2. $f(x, y) = 4x^2 + y$, $1 \leq x \leq 5$, $0 \leq y \leq 2$, $n = 4$, evaluate at midpoint
3. $f(x, y) = x + 2y^2$, $0 \leq x \leq 2$, $-1 \leq y \leq 1$, $n = 16$, evaluate at midpoint

4. $f(x, y) = 4x^2 + y$, $1 \le x \le 5$, $0 \le y \le 2$, $n = 16$, evaluate at midpoint

In exercises 5 and 6, compute the Riemann sum for the given function, the irregular partition shown and midpoint evaluation.

5. $f(x, y) = 3x - y$ **6.** $f(x, y) = 2x + y$



In exercises 7–10, evaluate the double integral.

7. $\iint_R (x^2 - 2y)\, dA$, where $R = \{0 \le x \le 2, -1 \le y \le 1\}$

8. $\iint_R 4xe^{2y} dA$, where $R = \{2 \le x \le 4, 0 \le y \le 1\}$

9. $\iint_R (1 - ye^{xy})\, dA$, where $R = \{0 \le x \le 2, 0 \le y \le 3\}$

10. $\iint_R (3x - 4x\sqrt{xy})\, dA$, where $R = \{0 \le x \le 4, 0 \le y \le 9\}$

In exercises 11–14, sketch the solid whose volume is given by the iterated integral.

11. $\int_{-1}^{1}\int_{0}^{1} (6 - 2x - 3y)\, dy\, dx$ **12.** $\int_{0}^{2}\int_{-1}^{1} (2 + x + 2y)\, dy\, dx$

13. $\int_{0}^{2}\int_{0}^{3} (x^2 + y^2)\, dy\, dx$ **14.** $\int_{-1}^{1}\int_{-1}^{1} (4 - x^2 - y^2)\, dy\, dx$

In exercises 15–22, evaluate the iterated integral.

15. $\int_{0}^{1}\int_{0}^{2x} (x + 2y)\, dy\, dx$ **16.** $\int_{0}^{2}\int_{0}^{x^2} (x + 3)\, dy\, dx$

17. $\int_{0}^{1}\int_{0}^{2y} (4x\sqrt{y} + y)\, dx\, dy$ **18.** $\int_{0}^{\pi}\int_{0}^{2} y \sin(xy)\, dx\, dy$

19. $\int_{0}^{2}\int_{0}^{2y} e^{y^2} dx\, dy$ **20.** $\int_{1}^{2}\int_{0}^{2/x} e^{xy} dy\, dx$

21. $\int_{1}^{4}\int_{0}^{1/x} \cos xy\, dy\, dx$ **22.** $\int_{0}^{1}\int_{0}^{y^2} \frac{3}{4 + y^3} dx\, dy$

23. Show that $\int_{0}^{1}\int_{0}^{2x} x^2\, dy\, dx \ne \int_{0}^{2}\int_{0}^{y/2} x^2 dx\, dy$.

24. Sketch the solids whose volumes are given in exercise 23 and explain why the volumes are not equal.

In exercises 25–32, find an integral equal to the volume of the solid bounded by the given surfaces and evaluate the integral.

25. $z = x^2 + y^2$, $z = 0$, $y = 1$, $y = 4$, $x = 0$, $x = 3$

26. $z = 3x^2 + 2y$, $z = 0$, $y = 0$, $y = 1$, $x = 1$, $x = 3$

27. $z = x^2 + y^2$, $z = 0$, $y = x^2$, $y = 1$

28. $z = 3x^2 + 2y$, $z = 0$, $y = 1 - x^2$, $y = 0$

29. $z = 6 - x - y$, $z = 0$, $x = 4 - y^2$, $x = 0$

30. $z = 4 - 2y$, $z = 0$, $x = y^4$, $x = 1$

31. $z = y^2$, $z = 0$, $y = 0$, $y = x$, $x = 2$

32. $z = x^2$, $z = 0$, $y = x$, $y = 4$, $x = 0$

In exercises 33–36, approximate the double integral.

33. $\iint_R (2x - y)\, dA$, where R is bounded by $y = \sin x$ and $y = 1 - x^2$

34. $\iint_R (2x - y)\, dA$, where R is bounded by $y = e^x$ and $y = 2 - x^2$

35. $\iint_R e^{x^2} dA$, where R is bounded by $y = x^2$ and $y = 1$

36. $\iint_R \sqrt{y^2 + 1}\, dA$, where R is bounded by $x = 4 - y^2$ and $x = 0$

In exercises 37–42, change the order of integration.

37. $\int_{0}^{1}\int_{0}^{2x} f(x, y)\, dy\, dx$ **38.** $\int_{0}^{1}\int_{2x}^{2} f(x, y)\, dy\, dx$

39. $\int_{0}^{2}\int_{2y}^{4} f(x, y)\, dx\, dy$ **40.** $\int_{0}^{1}\int_{0}^{2y} f(x, y)\, dx\, dy$

41. $\int_{0}^{\ln 4}\int_{e^x}^{4} f(x, y)\, dy\, dx$ **42.** $\int_{1}^{2}\int_{0}^{\ln y} f(x, y)\, dx\, dy$

In exercises 43–46, evaluate the iterated integral by first changing the order of integration.

43. $\int_{0}^{2}\int_{x}^{2} 2e^{y^2} dy\, dx$ **44.** $\int_{0}^{1}\int_{\sqrt{x}}^{1} \frac{3}{4 + y^3} dy\, dx$

45. $\int_{0}^{1}\int_{y}^{1} 3xe^{x^3} dx\, dy$ **46.** $\int_{0}^{1}\int_{\sqrt{y}}^{1} \cos x^3 dx\, dy$

47. Determine whether your CAS can evaluate the integrals $\int_{x}^{2} 2e^{y^2} dy$ and $\int_{0}^{2}\int_{x}^{2} 2e^{y^2} dy\, dx$.

48. Explain why a CAS would have trouble evaluating the first integral in exercise 47. Based on your result in exercise 47, can your CAS switch orders of integration to evaluate a double integral?

In exercises 49–52, sketch the solid whose volume is described by the given iterated integral.

49. $\int_{0}^{3}\int_{0}^{6-2x} (6 - 2x - y)\, dy\, dx$

50. $\int_0^4 \int_0^{4-x} (4-x-y)\,dy\,dx$

51. $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-y^2)\,dy\,dx$

52. $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

53. Explain why $\int_0^1 \int_0^{2x} f(x,y)\,dy\,dx$ is not generally equal to $\int_0^1 \int_0^{2y} f(x,y)\,dx\,dy$.

54. Give an example of a function for which the integrals in exercise 53 are equal. As generally as possible, describe what property such a function must have.

55. Compute the iterated integral by sketching a graph and using a basic geometric formula:

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\,dx.$$

56. Prove Theorem 1.4.

57. Prove that $\int_a^b \int_c^d f(x)g(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$ for continuous functions f and g.

58. Use the result of exercise 57 to quickly evaluate $\int_0^{2\pi} \int_{15}^{38} e^{-4y^2} \sin x\,dy\,dx$.

59. For the table of function values here, use upper-left corner evaluations to estimate $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$.

y \ x	0.0	0.25	0.5	0.75	1.0
0.0	2.2	2.0	1.7	1.4	1.0
0.25	2.3	2.1	1.8	1.6	1.1
0.5	2.5	2.3	2.0	1.8	1.4
0.75	2.8	2.6	2.3	2.2	1.8
1.0	3.2	3.0	2.8	2.7	2.5

60. Repeat exercise 59 with lower-right corner evaluations.

61. For the table of function values in exercise 59, use upper-left corner evaluations to estimate $\int_0^1 \int_0^{0.5} f(x,y)\,dy\,dx$.

62. Repeat exercise 61 with lower-right corner evaluations.

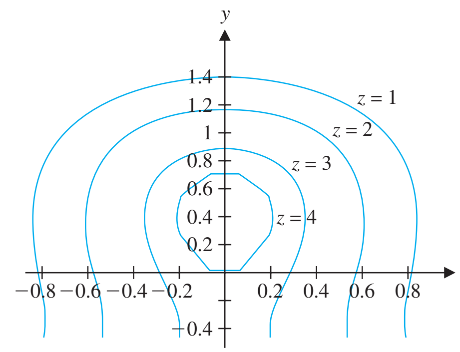
63. For the function in exercise 59, use an inner partition and lower-right corner evaluations to estimate $\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$.

64. Use the function in exercise 59, an inner partition and upper-right corner evaluations to estimate $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$.

65. Use the average of the function values at all four corners to approximate the integral in exercise 59.

66. Use the average of the function values at all four corners to approximate the integral in exercise 61.

67. Use the contour plot to determine which is the best estimate of $\int_{-1}^1 \int_0^1 f(x,y)\,dy\,dx$: (a) 1, (b) 2 or (c) 4.



68. Use the contour plot to determine which is the best estimate of $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$: (a) 1, (b) 2 or (c) 4.

69. From the Fundamental Theorem of Calculus, we have $\int_a^b f'(x)\,dx = f(b) - f(a)$. Find the corresponding rule for evaluating the double integral $\int_c^d \int_a^b f_{xy}(x,y)\,dx\,dy$. Use this rule to evaluate $\int_0^1 \int_0^1 24xy^2\,dx\,dy$, with $f(x,y) = 3x + 4x^2y^3 + y^2$.

70. Determine whether the rule from exercise 69 holds for double integrals over nonrectangular regions. Test it on $\int_0^1 \int_0^x 24xy^2\,dx\,dy$.

71. Evaluate $\int_0^2 [\tan^{-1}(4-x) - \tan^{-1} x]\,dx$ by rewriting it as a double integral and switching the order of integration.

72. Evaluate $\int_0^1 [\sin^{-1}(2-x) - \sin^{-1} x]\,dx$ by rewriting it as a double integral and switching the order of integration.

73. Evaluate $\int_0^2 \int_0^{2y} f(x,y)\,dx\,dy$ for $f(x,y) = \min\{2x, y\}$.

74. Evaluate $\int_0^2 \int_0^{2x} f(x,y)\,dy\,dx$ for $f(x,y) = \min\{y, x^2\}$.

EXPLORATORY EXERCISES

1. Set up a double integral for the volume of the solid bounded by the graphs of $z = 4 - x^2 - y^2$ and $z = x^2 + y^2$. Note that you actually have two tasks. First, the general rule for finding the volume between two surfaces is analogous to the general rule for finding the area between two curves. The greater challenge here is to find the limits of integration.

2. As mentioned in the text, numerical methods for approximating double integrals can be troublesome. The **Monte Carlo method** makes clever use of probability theory to approximate $\iint_R f(x, y)\,dA$ for a bounded region R. Suppose, for example, that R is contained within the rectangle $0 \le x \le 1, 0 \le y \le 1$. Generate two random numbers a and b from the uniform distribution on [0, 1]; this means that every number between 0 and 1 is in some sense equally likely. Determine whether or not the point (a, b) is in the region R and then repeat the process a large number of times. If, for example, 64 out of 100 points generated were within R, explain why a reasonable estimate of the area of R is 0.64 times the area of the rectangle $0 \le x \le 1, 0 \le y \le 1$. For each point (a, b) that is within R, compute $f(a, b)$. If the average of all of these function values is 13.6, explain why a reasonable estimate of $\iint_R f(x, y)\,dA$ is $(0.64)(13.6) = 8.704$. Use the Monte Carlo method to estimate $\int_1^2 \int_{\ln x}^{\sqrt{x}} \sin(xy)\,dy\,dx$. (Hint: Show that y is between $\ln 1 = 0$ and $\sqrt{2} < 2$.)

3. Improper double integrals can be treated much like improper single integrals. Evaluate $\int_0^\infty \int_0^\infty e^{-2x-3y}\,dx\,dy$ by first evaluating the inside integral as $\lim_{R\to\infty} \int_0^R e^{-2x-3y}\,dx$. To explore whether the integral is well defined, evaluate the integral as $\lim_{R\to\infty} \left(\int_0^R \int_0^R e^{-2x-3y}\,dx\,dy\right)$ and $\lim_{R\to\infty} \left(\int_0^{2R} \int_0^{R^2} e^{-2x-3y}\,dx\,dy\right)$. Then evaluate $\iint_R e^{-x^2-y}\,dA$, where R is the portion of the xy-plane with $0 \le x \le y$.

13.2 AREA, VOLUME AND CENTER OF MASS

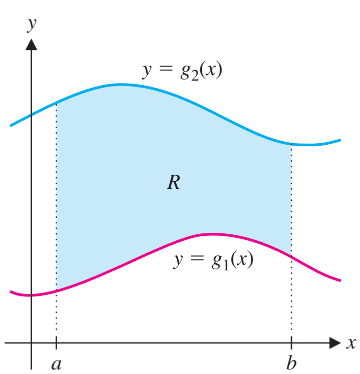


FIGURE 13.17
The region R

To use double integrals to solve problems, it's very important that you recognize what each component of the integral represents. For this reason, we pause briefly to set up a double iterated integral as a double sum. Consider the case of a continuous function $f(x, y) \ge 0$ on some region $R \subset \mathbb{R}^2$. If R has the form

$$R = \{(x, y) | a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\},$$

as indicated in Figure 13.17, then we have from our work in section 13.1 that the volume V lying beneath the surface $z = f(x, y)$ and above the region R is given by

$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx. \tag{2.1}$$

Here, for each fixed x, $A(x)$ is the area of the cross section of the solid corresponding to that particular value of x. Our aim is to write the volume integral in (2.1) in a slightly different way from our derivation in section 13.1. First, notice that by the definition of definite integral, we have that

$$\int_a^b A(x)\,dx = \lim_{\|P_1\|\to 0} \sum_{i=1}^n A(c_i)\,\Delta x_i, \tag{2.2}$$

where P_1 represents a partition of the interval $[a, b]$, c_i is some point in the ith subinterval $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$ (the width of the ith subinterval). For each fixed $x \in [a, b]$, since $A(x)$ is the area of the cross section, we have that

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy = \lim_{||P_2||\to 0} \sum_{j=1}^m f(x, v_j)\Delta y_j, \tag{2.3}$$

where P_2 represents a partition of the interval $[g_1(x), g_2(x)]$, v_j is some point in the jth subinterval $[y_{j-1}, y_j]$ of the partition P_2 and $\Delta y_j = y_j - y_{j-1}$ (the width of the jth

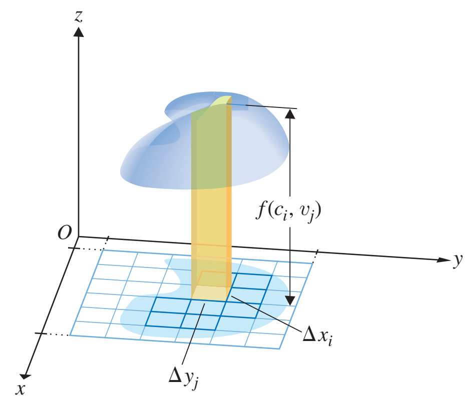


FIGURE 13.18
Volume of a typical box

subinterval). Putting (2.1), (2.2), and (2.3) together, we get

$$\begin{aligned} V &= \lim_{\|P_1\|\to 0} \sum_{i=1}^{n} A(c_i)\Delta x_i \\ &= \lim_{\|P_1\|\to 0} \sum_{i=1}^{n} \left[\lim_{\|P_2\|\to 0} \sum_{j=1}^{m} f(c_i, v_j)\Delta y_j \right] \Delta x_i \\ &= \lim_{\|P_1\|\to 0} \lim_{\|P_2\|\to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(c_i, v_j)\Delta y_j \Delta x_i . \end{aligned} \tag{2.4}$$

The double summation in (2.4) is called a **double Riemann sum.** Notice that each term corresponds to the volume of a box of length Δx_i, width Δy_j and height $f(c_i, v_j)$. (See Figure 13.18.) Observe that by superimposing the two partitions, we have produced an inner partition of the region R. If we represent this inner partition of R by P and the norm of the partition P by $\|P\|$, the length of the longest diagonal of any rectangle in the partition, we can write (2.4) with only one limit, as

$$V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(c_i, v_j)\Delta y_j \Delta x_i . \tag{2.5}$$

When you write down an iterated integral representing volume, you can use (2.5) to help identify each of the components as follows:

$$\begin{aligned} V &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{f(c_i, v_j)}_{\text{height}} \underbrace{\Delta y_j}_{\text{width}} \underbrace{\Delta x_i}_{\text{length}} \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \underbrace{f(x, y)}_{\text{height}} \underbrace{dy}_{\text{width}} \underbrace{dx}_{\text{length}} . \end{aligned} \tag{2.6}$$

You should make at least a mental picture of the components of the integral in (2.6), keeping in mind the corresponding components of the Riemann sum. We leave it as an exercise to show that for a region of the form

$$R = \{(x, y) | c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\},$$

we get a corresponding interpretation of the iterated integral:

$$V = \lim_{\|P\|\to 0} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{f(c_i, v_j)}_{\text{height}} \underbrace{\Delta x_i}_{\text{length}} \underbrace{\Delta y_j}_{\text{width}}$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \underbrace{f(x, y)}_{\text{height}} \underbrace{dx}_{\text{length}} \underbrace{dy}_{\text{width}}. \tag{2.7}$$

Observe that for any bounded region $R \subset \mathbb{R}^2$, $\iint_R 1\, dA$, which we sometimes write simply as $\iint_R dA$, gives the volume under the surface $z = 1$ and above the region R in the xy-plane. Since all of the cross sections parallel to the xy-plane are the same, the solid is a cylinder and so, its volume is the product of its height (1) and its cross-sectional area. That is,

$$\iint_R dA = (1)\,(\text{Area of } R) = \text{Area of } R. \tag{2.8}$$

So, we now have the option of using a double integral to find the area of a plane region.

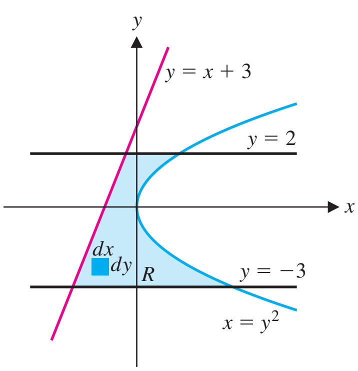


FIGURE 13.19
The region R

EXAMPLE 2.1 Using a Double Integral to Find Area

Find the area of the plane region bounded by the graphs of $x = y^2$, $y - x = 3$, $y = -3$ and $y = 2$ (see Figure 13.19).

Solution Note that we have indicated in the figure a small rectangle with sides dx and dy, respectively. This helps to indicate the limits for the iterated integral. From (2.8), we have

$$A = \iint_R dA = \int_{-3}^{2} \int_{y-3}^{y^2} dx\, dy = \int_{-3}^{2} x \Big|_{x=y-3}^{x=y^2} dy$$

$$= \int_{-3}^{2} [y^2 - (y-3)]\, dy = \left(\frac{y^3}{3} - \frac{y^2}{2} + 3y \right) \Bigg|_{-3}^{2}$$

$$= \frac{175}{6}.$$

Think about example 2.1 a little further. Recall that we had worked similar problems in section 5.1 using single integrals. In fact, you might have set up the desired area directly as

$$A = \int_{-3}^{2} [y^2 - (y-3)]\, dy,$$

exactly as you see in the second line of work above. While we will sometimes use double integrals to more easily solve familiar problems, double integrals will allow us to solve many new problems as well.

We have already developed formulas for calculating the volume of a solid lying below a surface of the form $z = f(x, y)$ and above a region R (of several different forms), lying in the xy-plane. So, what's the problem, then? As you will see in examples 2.2–2.4, the challenge in setting up the iterated integrals comes in seeing the region R that the solid lies above and then determining the limits of integration for the iterated integrals.

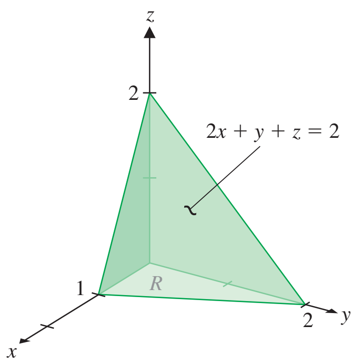


FIGURE 13.20a
Tetrahedron

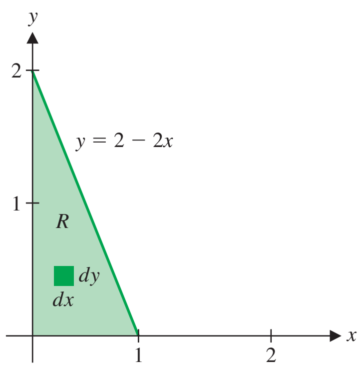


FIGURE 13.20b
The region R

EXAMPLE 2.2 Using a Double Integral to Find Volume

Find the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and the three coordinate planes.

Solution First, we need to draw a sketch of the solid. Since the plane $2x + y + z = 2$ intersects the coordinate axes at the points (1, 0, 0), (0, 2, 0) and (0, 0, 2), a sketch is easy to draw. Simply connect the three points of intersection with the coordinate axes and you'll get the graph of the tetrahedron (a four-sided object with all triangular sides) seen in Figure 13.20a. In order to use our volume formula, though, we'll first need to visualize the tetrahedron as a solid lying below a surface of the form $z = f(x, y)$ and lying above some region R in the xy-plane. Notice that the solid lies below the plane $z = 2 - 2x - y$ and above the triangular region R in the xy-plane, as indicated in Figure 13.20a. Although we're not simply handed R, you can see that R is the triangular region bounded by the x- and y-axes and the trace of the plane $2x + y + z = 2$ in the xy-plane. The trace is found by simply setting $z = 0$: $2x + y = 2$ (see Figure 13.20b). From (2.6), the volume is then

$$\begin{aligned} V &= \int_0^1 \int_0^{2-2x} \underbrace{(2 - 2x - y)}_{\text{height}} \underbrace{dy}_{\text{width}} \underbrace{dx}_{\text{length}} \\ &= \int_0^1 \left(2y - 2xy - \frac{y^2}{2}\right)\bigg|_{y=0}^{y=2-2x} dx \\ &= \int_0^1 \left[2(2-2x) - 2x(2-2x) - \frac{(2-2x)^2}{2}\right] dx \\ &= \frac{2}{3}, \end{aligned}$$

where we leave the routine details of the final calculation to you. ■

We cannot emphasize enough the need to draw reasonable sketches of the solid and particularly of the base of the solid in the xy-plane. You may be lucky enough to guess the limits of integration for a few of these problems, but don't be deceived: you need to draw good sketches and look carefully to determine the limits of integration correctly.

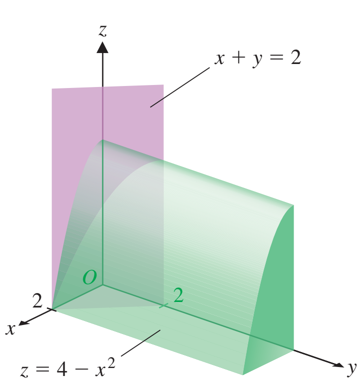


FIGURE 13.21a
Solid in the first octant

EXAMPLE 2.3 Finding the Volume of a Solid

Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$ and $z = 0$.

Solution First, draw a sketch of the solid. You should note that $z = 4 - x^2$ is a cylinder (since there's no y term), $x + y = 2$ is a plane and $x = 0$, $y = 0$ and $z = 0$ are the coordinate planes. (See Figure 13.21a.) Notice that the solid lies below the surface $z = 4 - x^2$ and above the triangular region R in the xy-plane formed by the x- and y-axes and the trace of the plane $x + y = 2$ in the xy-plane (i.e., the line

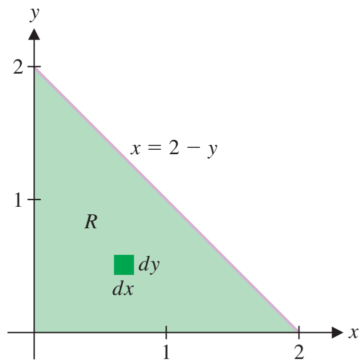


FIGURE 13.21b
The region R

$x + y = 2$). This is shown in Figure 13.21b. Although we could integrate with respect to either x or y first, we integrate with respect to x first. From (2.7), we have

$$\begin{aligned} V &= \int_0^2 \int_0^{2-y} \underbrace{(4-x^2)}_{\text{height}} \underbrace{dx}_{\text{length}} \underbrace{dy}_{\text{width}} \\ &= \int_0^2 \left(4x - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=2-y} dy \\ &= \int_0^2 \left[4(2-y) - \frac{(2-y)^3}{3}\right] dy \\ &= \frac{20}{3}. \end{aligned}$$

EXAMPLE 2.4 Finding the Volume of a Solid Bounded Above the *xy*-Plane

Find the volume of the solid bounded by the graphs of $z = 2$, $z = x^2 + 1$, $y = 0$ and $x + y = 2$.

Solution First, observe that the graph of $z = x^2 + 1$ is a parabolic cylinder with axis parallel to the y-axis. It intersects the plane $z = 2$ where $x^2 + 1 = 2$ or $x = \pm 1$. This forms a long trough, which is cut off by the planes $y = 0$ (the xz-plane) and $x + y = 2$. A sketch of the solid is shown in Figure 13.22a. The solid lies below $z = 2$ and above the cylinder $z = x^2 + 1$. You can view the integrand $f(x, y)$ in (2.6) as the height of the solid above the point (x, y). Drawing a vertical line from the xy-plane through the solid in Figure 13.22a shows that the height of the solid is the difference between 2 and $x^2 + 1$, so that $f(x, y) = 2 - (x^2 + 1) = 1 - x^2$. In Figure 13.22a, notice that the solid lies above the region R in the xy-plane bounded by $y = 0$, $x + y = 2$, $x = -1$ and $x = 1$. (See Figure 13.22b.)

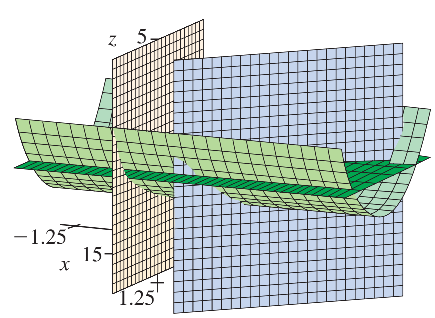


FIGURE 13.22a
The solid

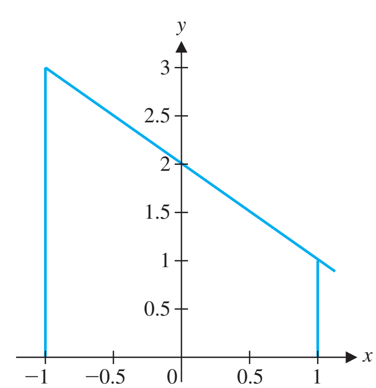


FIGURE 13.22b
The region R

It's easy to see from Figure 13.22b that we should integrate with respect to y first. For each fixed x in the interval $[-1, 1]$, y runs from 0 to $2 - x$. The volume is then

$$\begin{aligned} V &= \int_{-1}^{1}\int_{0}^{2-x} (1 - x^2)\, dy\, dx \\ &= \int_{-1}^{1} (1 - x^2)y\Big|_{y=0}^{y=2-x} dx \\ &= \int_{-1}^{1} (1 - x^2)(2 - x)\, dx \\ &= \frac{8}{3}. \end{aligned}$$

NOTES

Notice in example 2.4 that the limits of integration come from the two defining surfaces for y (that is, $y = 0$ and $y = 2 - x$) and the x-values for the intersection of the other two defining surfaces $z = 2$ and $z = x^2 + 1$. The defining surfaces and intersections are the sources of the limits of integration, but don't just guess which one to put where: use a graph of the surface to see how to arrange these elements.

Double integrals are used to calculate numerous quantities of interest in applications. We present one application in example 2.5, while others can be found in the exercises.

EXAMPLE 2.5 Estimating Population

Suppose that $f(x, y) = 20{,}000ye^{-x^2-y^2}$ models the population density (population per square mile) of a species of small animals, with x and y measured in miles. Estimate the population in the triangular-shaped habitat with vertices (1, 1), (2, 1) and (1, 0).

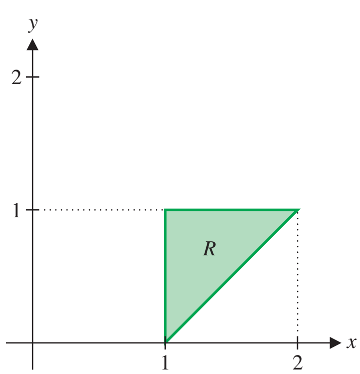


FIGURE 13.23
Habitat region

Solution The population in any region R is estimated by

$$\iint_R f(x, y)\, dA = \iint_R 20{,}000ye^{-x^2-y^2} dA.$$

[As a quick check on the reasonableness of this formula, note that $f(x, y)$ is measured in units of population per square mile and the area increment dA carries units of square miles, so that the product $f(x, y)\, dA$ carries the desired units of population.] Notice that the integrand is $20{,}000ye^{-x^2-y^2} = 20{,}000e^{-x^2}ye^{-y^2}$, which suggests that we should integrate with respect to y first. As always, we first sketch a graph of the region R (shown in Figure 13.23). Notice that the line through the points (1, 0) and (2, 1) has the equation $y = x - 1$, so that R extends from $y = x - 1$ up to $y = 1$, as x increases from 1 to 2. We now have

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_1^2 \int_{x-1}^{1} 20{,}000e^{-x^2} ye^{-y^2} dy\, dx \\ &= \int_1^2 10{,}000e^{-x^2}[e^{-(x-1)^2} - e^{-1}]\, dx \\ &\approx 698, \end{aligned}$$

where we approximated the last integral numerically.

Moments and Center of Mass

We close this section by briefly discussing a physical application of double integrals. Consider a thin, flat plate (a **lamina**) in the shape of the region $R \subset \mathbb{R}^2$ whose density (mass per unit area) varies throughout the plate (i.e., some areas of the plate are more dense than others). From an engineering standpoint, it's often important to determine where you could

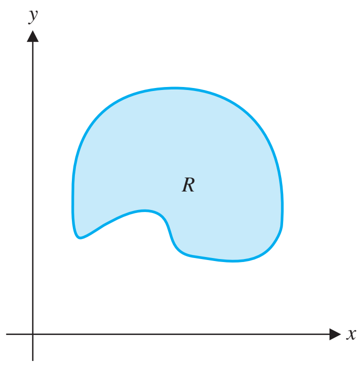


FIGURE 13.24a
Lamina

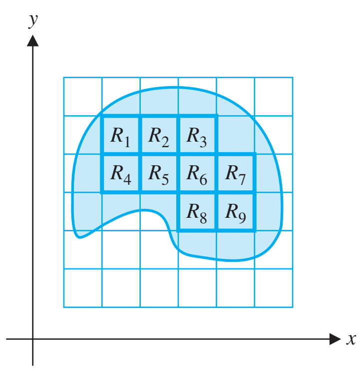


FIGURE 13.24b
Inner partition of R

place a support to balance the plate. We call this point the **center of mass** of the lamina. We'll first need to find the total mass of the plate. For a real plate, we'd simply place it on a scale, but for our theoretical plate, we'll need to be more clever. Suppose the lamina has the shape of the region R shown in Figure 13.24a and has mass density (mass per unit area) given by the function $\rho(x, y)$. Construct an inner partition of R, as in Figure 13.24b. Notice that if the norm of the partition $\|P\|$ is small, then the density will be nearly constant on each rectangle of the inner partition. So, for each $i = 1, 2, \ldots, n$, pick some point $(u_i, v_i) \in R_i$. Then, the mass m_i of the portion of the lamina corresponding to the rectangle R_i is given approximately by

$$m_i \approx \underbrace{\rho(u_i, v_i)}_{\text{mass/unit area}} \underbrace{\Delta A_i}_{\text{area}},$$

where ΔA_i denotes the area of R_i. The total mass m of the lamina is then given approximately by

$$m \approx \sum_{i=1}^{n} \rho(u_i, v_i)\,\Delta A_i.$$

Notice that if $\|P\|$ is small, then this should be a reasonable approximation of the total mass.

To get the mass exactly, we take the limit as $\|P\|$ tends to zero, which you should recognize as a double integral:

$$m = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \rho(u_i, v_i)\,\Delta A_i = \iint_R \rho(x, y)\,dA. \tag{2.9}$$

Notice that if you want to balance a lamina like the one shown in Figure 13.24a, you'll need to balance it both from left to right and from top to bottom. In the language of our previous discussion of center of mass in section 5.6, we'll need to find the first moments: both left to right (we call this the **moment with respect to the *y*-axis**) and top to bottom (the **moment with respect to the *x*-axis**). First, we approximate the moment M_y with respect to the y-axis. Assuming that the mass in the ith rectangle of the partition is concentrated at the point (u_i, v_i), we have

$$M_y \approx \sum_{i=1}^{n} u_i \rho(u_i, v_i)$$

(i.e., the sum of the products of the masses and their directed distances from the y-axis). Taking the limit as $\|P\|$ tends to zero, we get

$$M_y = \lim_{\|P\|\to 0} \sum_{i=1}^{n} u_i \rho(u_i, v_i) = \iint_R x\rho(x, y)\,dA. \tag{2.10}$$

Similarly, looking at the sum of the products of the masses and their directed distances from the x-axis, we get the moment M_x with respect to the x-axis,

$$M_x = \lim_{\|P\|\to 0} \sum_{i=1}^{n} v_i \rho(u_i, v_i) = \iint_R y\rho(x, y)\,dA. \tag{2.11}$$

The center of mass is the point $(\bar{x}, \bar{y})$ defined by

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}. \tag{2.12}$$

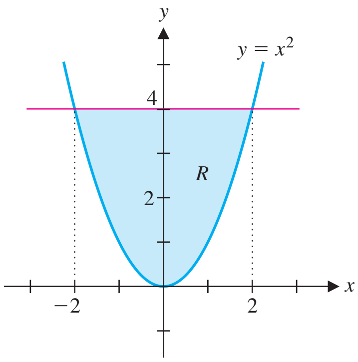


FIGURE 13.25
Lamina

EXAMPLE 2.6 Finding the Center of Mass of a Lamina

Find the center of mass of the lamina in the shape of the region bounded by the graphs of $y = x^2$ and $y = 4$, having mass density given by $\rho(x, y) = 1 + 2y + 6x^2$.

Solution We sketch the region in Figure 13.25. From (2.9), we have that the total mass of the lamina is given by

$$\begin{aligned} m &= \iint_R \rho(x, y)\, dA = \int_{-2}^{2} \int_{x^2}^{4} (1 + 2y + 6x^2)\, dy\, dx \\ &= \int_{-2}^{2} \left(y + 2\frac{y^2}{2} + 6x^2 y \right) \Bigg|_{y=x^2}^{y=4} dx \\ &= \int_{-2}^{2} [(4 + 16 + 24x^2) - (x^2 + x^4 + 6x^4)]\, dx \\ &= \frac{1696}{15} \approx 113.1. \end{aligned}$$

We compute the moment M_y from (2.10):

$$\begin{aligned} M_y &= \iint_R x\rho(x, y)\, dA = \int_{-2}^{2} \int_{x^2}^{4} x(1 + 2y + 6x^2)\, dy\, dx \\ &= \int_{-2}^{2} \int_{x^2}^{4} (x + 2xy + 6x^3)\, dy\, dx \\ &= \int_{-2}^{2} (xy + xy^2 + 6x^3 y) \Big|_{y=x^2}^{y=4} dx \\ &= \int_{-2}^{2} [(4x + 16x + 24x^3) - (x^3 + x^5 + 6x^5)]\, dx = 0. \end{aligned}$$

Note that from (2.12), this says that the x-coordinate of the center of mass is $\bar{x} = \frac{M_y}{m} = \frac{0}{113.1} = 0$. This should not surprise you since both the region *and* the mass density are symmetric with respect to the y-axis. [Notice that $\rho(-x, y) = \rho(x, y)$.] Next, from (2.11), we have

$$\begin{aligned} M_x &= \iint_R y\rho(x, y)\, dA = \int_{-2}^{2} \int_{x^2}^{4} y(1 + 2y + 6x^2)\, dy\, dx \\ &= \int_{-2}^{2} \int_{x^2}^{4} (y + 2y^2 + 6x^2 y)\, dy\, dx \\ &= \int_{-2}^{2} \left(\frac{y^2}{2} + 2\frac{y^3}{3} + 6x^2 \frac{y^2}{2} \right) \Bigg|_{y=x^2}^{y=4} dx \\ &= \int_{-2}^{2} \left[\left(8 + \frac{128}{3} + 48x^2 \right) - \left(\frac{x^4}{2} + \frac{2}{3}x^6 + 3x^6 \right) \right] dx \\ &= \frac{11{,}136}{35} \approx 318.2 \end{aligned}$$

and so, from (2.12) we have $\bar{y} = \frac{M_x}{m} \approx \frac{318.2}{113.1} \approx 2.8$. The center of mass is then located at approximately

$$(\bar{x}, \bar{y}) \approx (0, 2.8).$$

In example 2.6, we computed the first moments M_y and M_x to find the balance point (center of mass) of the lamina in Figure 13.25. Further physical properties of this lamina can be determined using the **second moments** I_y and I_x. Much as we defined the first moments in equations (2.10) and (2.11), the second moment about the y-axis (often called the **moment of inertia about the *y*-axis**) of a lamina in the shape of the region R, with density function $\rho(x, y)$ is defined by

$$I_y = \iint_R x^2 \rho(x, y)\, dA.$$

Similarly, the second moment about the x-axis (also called the **moment of inertia about the *x*-axis**) of a lamina in the shape of the region R, with density function $\rho(x, y)$ is defined by

$$I_x = \iint_R y^2 \rho(x, y)\, dA.$$

Physics tells us that the larger I_y is, the more difficult it is to rotate the lamina about the y-axis. Similarly, the larger I_x is, the more difficult it is to rotate the lamina about the x-axis. We explore this briefly in example 2.7.

EXAMPLE 2.7 Finding the Moments of Inertia of a Lamina

Find the moments of inertia I_y and I_x for the lamina in example 2.6.

Solution The region R is the same as in example 2.6 (see Figure 13.25), so that the limits of integration are the same. We have

$$\begin{aligned} I_y &= \int_{-2}^{2} \int_{x^2}^{4} x^2(1 + 2y + 6x^2)\, dy\, dx \\ &= \int_{-2}^{2} (20x^2 + 23x^4 - 7x^6)\, dx \\ &= \frac{2176}{15} \approx 145.07 \end{aligned}$$

and

$$\begin{aligned} I_x &= \int_{-2}^{2} \int_{x^2}^{4} y^2(1 + 2y + 6x^2)\, dy\, dx \\ &= \int_{-2}^{2} \left(\frac{448}{3} + 128x^2 - \frac{1}{3}x^6 - \frac{5}{2}x^8 \right) dx \\ &= \frac{61{,}952}{63} \approx 983.37. \end{aligned}$$

A comparison of the two moments of inertia shows that it is much more difficult to rotate the lamina of Figure 13.25 about the x-axis than about the y-axis. Examine the figure and the density function to be sure that this makes sense to you.

EXERCISES 13.2

WRITING EXERCISES

1. The double Riemann sum in (2.5) disguises the fact that the order of integration is important. Explain how the order of integration affects the details of the double Riemann sum.

2. Many double integrals can be set up in two steps: first identify the function $f(x, y)$, then identify the two-dimensional region R and set up the limits of integration. Explain how these two steps are separated in examples 2.2, 2.3 and 2.4.

3. The sketches in examples 2.2, 2.3 and 2.4 are essential, but somewhat difficult to draw. Explain each sketch, including which surface should be drawn first, second and so on. Also, when a previously drawn surface is cut in half by a plane, explain how to identify which half of the cut surface to keep.

4. The moment M_y is the moment about the y-axis, but is used to find the x-coordinate of the center of mass. Explain why it is M_y and not M_x that is used to compute the x-coordinate of the center of mass.

In exercises 1–6, use a double integral to compute the area of the region bounded by the curves.

1. $y = x^2, y = 8 - x^2$
2. $y = x^2, y = x + 2$
3. $y = 2x, y = 3 - x, y = 0$
4. $y = 3x, y = 5 - 2x, y = 0$
5. $y = x^2, x = y^2$
6. $y = x^3, y = x^2$

In exercises 7–18, compute the volume of the solid bounded by the given surfaces.

7. $2x + 3y + z = 6$ and the three coordinate planes
8. $x + 2y - 3z = 6$ and the three coordinate planes
9. $z = 4 - x^2 - y^2$ and $z = 0$, with $-1 \le x \le 1$ and $-1 \le y \le 1$
10. $z = x^2 + y^2, z = 0, x = 0, x = 1, y = 0, y = 1$
11. $z = 1 - y, z = 0, y = 0, x = 1, x = 2$
12. $z = 2 + x, z = 0, x = 0, y = 0, y = 1$
13. $z = 1 - y^2, x + y = 1$ and the three coordinate planes (first octant)
14. $z = 1 - x^2 - y^2, x + y = 1$ and the three coordinate planes
15. $z = x^2 + y^2 + 3, z = 1, y = x^2, y = 4$
16. $z = x^2 + y^2 + 1, z = -1, y = x^2, y = 2x + 3$
17. $z = x + 2, z = y - 2, x = y^2 - 2, x = y$
18. $z = 2x + y + 1, z = -2x, x = y^2, x = 1$

In exercises 19–22, set up a double integral for the volume bounded by the given surfaces and estimate it numerically.

19. $z = \sqrt{x^2 + y^2}, y = 4 - x^2$, first octant
20. $z = \sqrt{4 - x^2 - y^2}$, inside $x^2 + y^2 = 1$, first octant
21. $z = e^{xy}, x + 2y = 4$ and the three coordinate planes
22. $z = e^{x^2+y^2}, z = 0$ and $x^2 + y^2 = 4$

In exercises 23–28, find the mass and center of mass of the lamina with the given density.

23. Lamina bounded by $y = x^3$ and $y = x^2$, $\rho(x, y) = 4$
24. Lamina bounded by $y = x^4$ and $y = x^2$, $\rho(x, y) = 4$
25. Lamina bounded by $x = y^2$ and $x = 1$, $\rho(x, y) = y^2 + x + 1$
26. Lamina bounded by $x = y^2$ and $x = 4$, $\rho(x, y) = y + 3$
27. Lamina bounded by $y = x^2$ $(x > 0)$, $y = 4$ and $x = 0$, $\rho(x, y) =$ distance from y-axis
28. Lamina bounded by $y = x^2 - 4$ and $y = 5$, $\rho(x, y) =$ square of the distance from the y-axis
29. The laminae of exercises 25 and 26 are both symmetric about the x-axis. Explain why it is not true in both exercises that the center of mass is located on the x-axis.
30. Suppose that a lamina is symmetric about the x-axis. State a condition on the density function $\rho(x, y)$ that guarantees that the center of mass is located on the x-axis.
31. Suppose that a lamina is symmetric about the y-axis. State a condition on the density function $\rho(x, y)$ that guarantees that the center of mass is located on the y-axis.
32. Give an example of a lamina that *is* symmetric about the y-axis but that *does not* have its center of mass on the y-axis.
33. Suppose that $f(x, y) = 15{,}000xe^{-x^2-y^2}$ is the population density of a species of small animals. Estimate the population in the triangular region with vertices (1, 1), (2, 1) and (1, 0).
34. Suppose that $f(x, y) = 15{,}000xe^{-x^2-y^2}$ is the population density of a species of small animals. Estimate the population in the region bounded by $y = x^2$, $y = 0$ and $x = 1$.

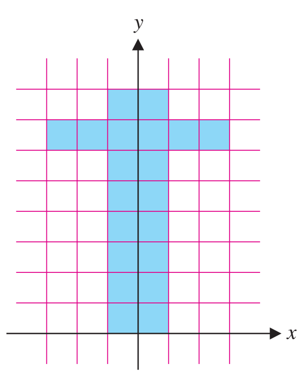
35. Suppose that $f(x, t) = 20e^{-t/6}$ is the yearly rate of change of the price per barrel of oil. If x is the number of billions of barrels and t is the number of years since 2000, compute and interpret $\int_0^{10}\int_0^4 f(x, t)\,dt\,dx$.

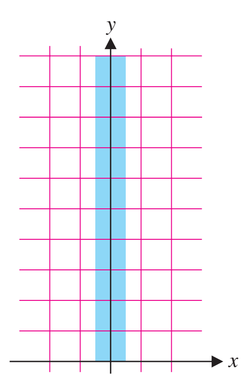
36. Repeat exercise 35 for $f(x, t) = \begin{cases} 20e^{-t/6}, & \text{if } 0 \le x \le 4 \\ 14e^{-t/6}, & \text{if } x > 4 \end{cases}$.

37. Find the mass and moments of inertia I_y and I_x for a lamina in the shape of the region bounded by $y = x^2$ and $y = 4$ with density $\rho(x, y) = 1$.

38. Find the mass and moments of inertia I_y and I_x for a lamina in the shape of the region bounded by $y = \frac{1}{4}x^2$ and $y = 1$ with density $\rho(x, y) = 4$. Comparing your answer with exercise 37, you should have found the same mass but different moments of inertia. Use the shapes of the regions to explain why this makes sense.

39. Figure skaters can control their rate of spin ω by varying their body positions, utilizing the principle of **conservation of angular momentum.** This states that in the absence of outside forces, the quantity $I_y\omega$ remains constant. Thus, reducing I_y by a factor of 2 will increase spin rate by a factor of 2. Compare the spin rates of the following two crude models of a figure skater, the first with arms extended (use $\rho = 1$) and the second with arms raised and legs crossed (use $\rho = 2$).





40. Lamina A is in the shape of the rectangle $-1 \le x \le 1$ and $-5 \le y \le 5$, with density $\rho(x, y) = 1$. It models a diver in the "layout" position. Lamina B is in the shape of the rectangle $-1 \le x \le 1$ and $-2 \le y \le 2$ with density $\rho(x, y) = 2.5$. It models a diver in the "tuck" position. Find the moment of inertia I_x for each lamina, and explain why divers use the tuck position to do multiple rotation dives.

41. Estimate the moment of inertia about the y-axis of the two ellipses R_1 bounded by $x^2 + 4y^2 = 16$ and R_2 bounded by $x^2 + 4y^2 = 36$. Assuming a constant density of $\rho = 1$, R_1 and R_2 can be thought of as models of two tennis racket heads. The rackets have the same shape, but the second racket is much bigger than the first (the difference in size is about the same as the difference between rackets of the 1960s and rackets of the 1990s).

42. For the tennis rackets in exercise 41, a rotation about the y-axis would correspond to the racket twisting in your hand, which is undesirable. Compare the tendency of each racket to twist. As related in Blandig and Monteleone's *What Makes a Boomerang Come Back,* the larger moment of inertia is what motivated a sore-elbowed Howard Head to construct large-headed tennis rackets in the 1970s.

In exercises 43–50, define the average value of $f(x, y)$ on a region R of area a by $\dfrac{1}{a}\displaystyle\iint_R f(x, y)\,dA$.

43. Compute the average value of $f(x, y) = y$ on the region bounded by $y = x^2$ and $y = 4$.

44. Compute the average value of $f(x, y) = y^2$ on the region bounded by $y = x^2$ and $y = 4$.

45. In exercise 43, compare the average value of $f(x, y)$ to the y-coordinate of the center of mass of a lamina with the same shape and constant density.

46. In exercise 44, R extends from $y = 0$ to $y = 4$. Explain why the average value of $f(x, y)$ corresponds to a y-value larger than 2.

47. Compute the average value of $f(x, y) = \sqrt{x^2 + y^2}$ on the region bounded by $y = x^2 - 4$ and $y = 3x$.

48. Interpret the geometric meaning of the average value in exercise 47. (Hint: What does $\sqrt{x^2 + y^2}$ represent geometrically?)

49. Suppose the temperature at the point (x, y) in a region R is given by $T(x, y) = 50 + \cos(2x + y)$, where R is bounded by $y = x^2$ and $y = 8 - x^2$. Estimate the average temperature in R.

50. Suppose the elevation at the point (x, y) in a region R is given by $h(x, y) = 2300 + 50 \sin x \cos y$, where R is bounded by $y = x^2$ and $y = 2x$. Estimate the average elevation in R.

51. Suppose that the function $f(x, y)$ gives the rainfall per unit area at the point (x, y) in a region R. State in words what (a) $\displaystyle\iint_R f(x, y)\,dA$ and (b) $\dfrac{\iint_R f(x, y)\,dA}{\iint_R 1\,dA}$ represent.

52. Suppose that the function $p(x, y)$ gives the population density at the point (x, y) in a region R. State in words what (a) $\displaystyle\iint_R p(x, y)\,dA$ and (b) $\dfrac{\iint_R p(x, y)\,dA}{\iint_R 1\,dA}$ represent.

53. A triangular lamina has vertices $(0, 0)$, $(0, 1)$ and $(c, 0)$ for some positive constant c. Assuming constant mass density, show that the y-coordinate of the center of mass of the lamina is independent of the constant c.

54. Find the x-coordinate of the center of mass of the lamina of exercise 53 as a function of c.

55. Let T be the tetrahedron with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Let B be the rectangular box with the same vertices plus $(a, b, 0)$, $(a, 0, c)$, $(0, b, c)$, and (a, b, c). Show that the volume of T is $\frac{1}{6}$ the volume of B.

56. Explain how to slice the box B of exercise 55 to get the tetrahedron T. Identify the percentage of volume that is sliced off each time.

In exercises 57–64, use the following definition of joint pdf (probability density function): a function $f(x, y)$ is a joint pdf on the region S if $f(x, y) \geq 0$ for all (x, y) in S and $\iint_S f(x, y)\,dA = 1$. Then for any region $R \subset S$, the probability that (x, y) is in R is given by $\iint_R f(x, y)\,dA$.

57. Show that $f(x, y) = e^{-x}e^{-y}$ is a joint pdf in the first quadrant $x \geq 0$, $y \geq 0$. (Hint: You will need to evaluate an improper double integral as iterated improper integrals.)

58. Show that $f(x, y) = 0.3x + 0.4y$ is a joint pdf on the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$.

59. Find a constant c such that $f(x, y) = c(x + 2y)$ is a joint pdf on the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 6)$.

60. Find a constant c such that $f(x, y) = c(x^2 + y)$ is a joint pdf on the region bounded by $y = x^2$ and $y = 4$.

61. Suppose that $f(x, y)$ is a joint pdf on the region bounded by $y = x^2$, $y = 0$ and $x = 2$. Set up a double integral for the probability that $y < x$.

62. Suppose that $f(x, y)$ is a joint pdf on the region bounded by $y = x^2$, $y = 0$ and $x = 2$. Set up a double integral for the probability that $y < 2$.

63. A point is selected at random from the region bounded by $y = 4 - x^2 (x > 0)$, $x = 0$ and $y = 0$. This means that the joint pdf for the point is constant, $f(x, y) = c$. Find the value of c. Then compute the probability that $y > x$ for the randomly chosen point.

64. A point is selected at random from the region bounded by $y = 4 - x^2 (x > 0)$, $x = 0$ and $y = 0$. Compute the probability that $y > 2$.

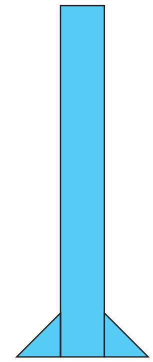
65. When solving projectile motion problems, we track the motion of an object's center of mass. For a high jumper, the athlete's entire body must clear the bar. Amazingly, a high jumper can accomplish this without raising his or her center of mass above the bar. To see how, suppose the athelete's body is bent into a shape modeled by the region between $y = \sqrt{9 - x^2}$ and $y = \sqrt{8 - x^2}$ with the bar at the point (0, 2). Assuming constant mass density, show that the center of mass is below the bar, but the body does not touch the bar.

66. Show that $V_1 = V_2$, where V_1 is the volume under $z = 4 - x^2 - y^2$ and above the xy-plane and V_2 is the volume between $z = x^2 + y^2$ and $z = 4$. Illustrate this with a graph.

EXPLORATORY EXERCISES

1. A function $f(x, y)$ is a **joint probability density function** on a region R if $f(x, y) \geq 0$ for all (x, y) in R and $\iint_R f(x, y)dA = 1$. Suppose that a person playing darts is aiming at the bull's-eye but is not very accurate. Suppose that the bull's-eye is centered at the origin and the dartboard is the region R bounded by $x^2 + y^2 = 64$ (units are inches), and the joint density function for the resulting position of the dart is $f(x, y) = ce^{-x^2-y^2}$, for some constant c. Estimate the value of the constant c such that $f(x, y)$ is a joint density function on R. For a region U contained within R, the probability that the dart lands in U is given by $\iint_U f(x, y)\,dA$. Estimate the probability that the dart hits inside the bull's-eye circle $x^2 + y^2 = \frac{1}{4}$. Estimate the probability that the dart accidentally lands in the "triple 20" band bounded by $x^2 + y^2 = 16$, $x^2 + y^2 = 14$, $y = 6.3x$ and $y = -6.3x$. Explain why all of the regions in this exercise would be easily described in polar coordinates. (Then start reading the next section!)

2. In this exercise, we explore an important issue in rocket design. We will work with the crude model shown, where the main tower of the rocket is 1 unit by 8 units and each triangular fin has height 1 and width w. First, find the y-coordinate y_1 of the center of mass, assuming a constant density $\rho(x, y) = 1$. Second, find the y-coordinate y_2 of the center of mass assuming the following density structure: the top half of the main tower has density $\rho = 1$, the bottom half of the main tower has density $\rho = 2$ and the fins have density $\rho = \frac{1}{4}$. Find the smallest value of w such that $y_1 < y_2$. In this case, if the rocket tilts slightly, air drag will push the rocket back in line. This stability criterion explains why model rockets have large, lightweight fins.



13.3 DOUBLE INTEGRALS IN POLAR COORDINATES

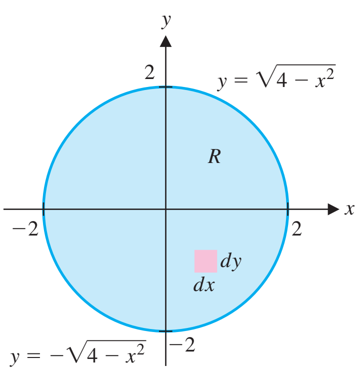


FIGURE 13.26
A circular region

Polar coordinates prove to be particularly useful for dealing with certain double integrals. This happens for several reasons. Most importantly, if the region over which you are integrating is in some way circular, polar coordinates may be exactly what you need for dealing with an otherwise intractable integration problem. For instance, you might need to evaluate

$$\iint_R (x^2 + y^2 + 3)\, dA.$$

This certainly looks simple enough, until we tell you that R is the circle of radius 2, centered at the origin, as shown in Figure 13.26. We write the top half of the circle as the graph of $y = \sqrt{4 - x^2}$ and the bottom half as $y = -\sqrt{4 - x^2}$. The double integral in question now becomes

$$\begin{aligned}\iint_R (x^2 + y^2 + 3)\, dA &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2 + 3)\, dy\, dx \\ &= \int_{-2}^{2} \left(x^2 y + \frac{y^3}{3} + 3y \right) \Bigg|_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx \\ &= 2 \int_{-2}^{2} \left[(x^2 + 3)\sqrt{4 - x^2} + \frac{1}{3}(4 - x^2)^{3/2} \right] dx. \qquad (3.1)\end{aligned}$$

We probably don't need to convince you that the integral in (3.1) is most unpleasant. On the other hand, as we'll see shortly, this double integral is simple when it's written in polar coordinates. We consider several types of polar regions.

Suppose the region R can be written in the form

$$R = \{(r, \theta) | \alpha \le \theta \le \beta \text{ and } g_1(\theta) \le r \le g_2(\theta)\},$$

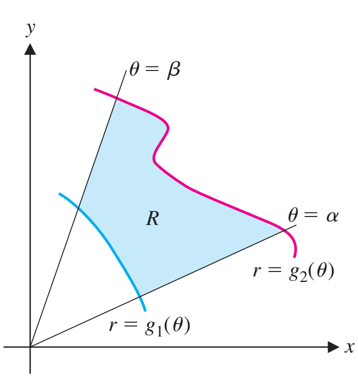


FIGURE 13.27a
Polar region R

where $0 \le g_1(\theta) \le g_2(\theta)$, for all θ in $[\alpha, \beta]$, as pictured in Figure 13.27a. As our first step, we partition R, but rather than use a rectangular grid, as we have done with rectangular

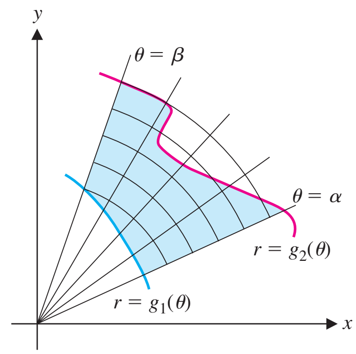


FIGURE 13.27b
Partition of R

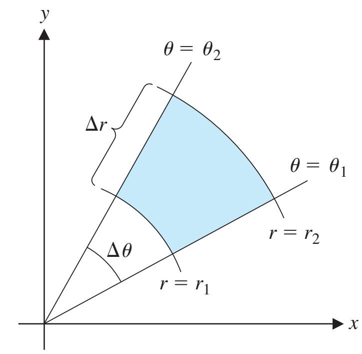


FIGURE 13.27c
Elementary polar region

coordinates, we use a partition consisting of a number of concentric circular arcs (of the form $r =$ constant) and rays (of the form $\theta =$ constant). We indicate such a partition of the region R in Figure 13.27b.

Notice that rather than consisting of rectangles, the "grid" in this case is made up of **elementary polar regions,** each bounded by two circular arcs and two rays (as shown in Figure 13.27c). In an **inner partition,** we include only those elementary polar regions that lie completely inside R.

We pause now briefly to calculate the area ΔA of the elementary polar region indicated in Figure 13.27c. Let $\bar{r} = \frac{1}{2}(r_1 + r_2)$ be the average radius of the two concentric circular arcs $r = r_1$ and $r = r_2$. Recall that the area of a circular sector is given by $A = \frac{1}{2}\theta r^2$, where $r =$ radius and θ is the central angle of the sector. Consequently, we have that

$$\begin{aligned}\Delta A &= \text{Area of outer sector} - \text{Area of inner sector}\\ &= \frac{1}{2}\Delta\theta r_2^2 - \frac{1}{2}\Delta\theta r_1^2\\ &= \frac{1}{2}\left(r_2^2 - r_1^2\right)\Delta\theta\\ &= \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta\\ &= \bar{r}\,\Delta r\,\Delta\theta. \qquad (3.2)\end{aligned}$$

As a familiar starting point, we first consider the problem of finding the volume lying beneath a surface $z = f(r, \theta)$, where f is continuous and $f(r, \theta) \geq 0$ on R. Using (3.2), we find that the volume V_i lying beneath the surface $z = f(r, \theta)$ and above the ith elementary polar region in the partition is then approximately the volume of the cylinder:

$$V_i \approx \underbrace{f(r_i, \theta_i)}_{\text{height}}\ \underbrace{\Delta A_i}_{\text{area of base}} = f(r_i, \theta_i)\, r_i\, \Delta r_i\, \Delta\theta_i,$$

where (r_i, θ_i) is a point in R_i and r_i is the average radius in R_i. We get an approximation to the total volume V by summing over all the regions in the inner partition:

$$V \approx \sum_{i=1}^{n} f(r_i, \theta_i)\, r_i\, \Delta r_i\, \Delta\theta_i.$$

As we have done a number of times now, we obtain the exact volume by taking the limit as the norm of the partition $\|P\|$ tends to zero and recognizing the iterated integral:

$$V = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(r_i, \theta_i)\, r_i\, \Delta r_i\, \Delta\theta_i = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

In this case, $\|P\|$ is the longest diagonal of any elementary polar region in the inner partition. More generally, we have the result in Theorem 3.1, which holds regardless of whether or not $f(r, \theta) \geq 0$ on R.

NOTES

Theorem 3.1 says that to write a double integral in polar coordinates, we write $x = r\cos\theta$, $y = r\sin\theta$, find the limits of integration for r and θ and replace dA by $r\,dr\,d\theta$. Be certain not to omit the factor of r in $dA = r\,dr\,d\theta$; this is a very common error.

THEOREM 3.1 (Fubini's Theorem)

Suppose that $f(r, \theta)$ is continuous on the region $R = \{(r, \theta) | \alpha \leq \theta \leq \beta$ and $g_1(\theta) \leq r \leq g_2(\theta)\}$, where $0 \leq g_1(\theta) \leq g_2(\theta)$ for all θ in $[\alpha, \beta]$. Then,

$$\iint_R f(r, \theta)\, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta. \tag{3.3}$$

The proof of this result is beyond the level of this text. However, the result should seem reasonable from our development for the case where $f(r, \theta) \geq 0$.

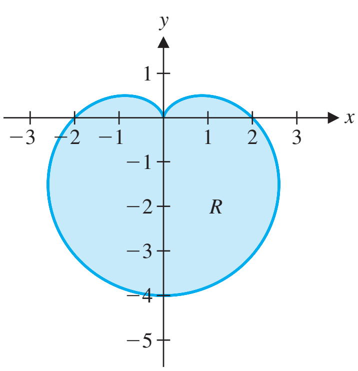


FIGURE 13.28
$r = 2 - 2\sin\theta$

EXAMPLE 3.1 Computing Area in Polar Coordinates

Find the area inside the curve defined by $r = 2 - 2\sin\theta$.

Solution First, we sketch a graph of the region in Figure 13.28. For each fixed θ, r ranges from 0 (corresponding to the origin) to $2 - 2\sin\theta$ (corresponding to the cardioid). To go all the way around the cardioid, exactly once, θ ranges from 0 to 2π. From (3.3), we then have

$$\begin{aligned} A &= \iint_R \underbrace{dA}_{r\,dr\,d\theta} = \int_0^{2\pi} \int_0^{2-2\sin\theta} r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{r^2}{2}\bigg|_{r=0}^{r=2-2\sin\theta} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} [(2 - 2\sin\theta)^2 - 0]\, d\theta = 6\pi, \end{aligned}$$

where we have left the details of the final calculation as an exercise. ■

We now return to our introductory example and show how the introduction of polar coordinates can dramatically simplify certain double integrals in rectangular coordinates.

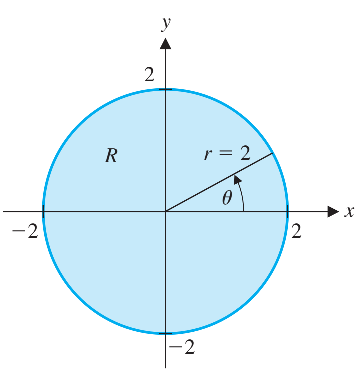


FIGURE 13.29
The region R

EXAMPLE 3.2 Evaluating a Double Integral in Polar Coordinates

Evaluate $\iint_R (x^2 + y^2 + 3)\, dA$, where R is the circle of radius 2 centered at the origin.

Solution First, recall from this section's introduction that in rectangular coordinates as in (3.1), this integral is extremely messy. From the region of integration shown in Figure 13.29, it's easy to see that for each fixed θ, r ranges from 0 (corresponding to the origin) to 2 (corresponding to a point on the circle). Then, in order to go around the circle exactly once, θ ranges from 0 to 2π. Finally, notice that the integrand contains the quantity $x^2 + y^2$, which you should recognize as r^2 in polar coordinates. From (3.3), we now have

$$\iint_R \underbrace{(x^2+y^2+3)}_{r^2+3}\,\underbrace{dA}_{r\,dr\,d\theta} = \int_0^{2\pi}\int_0^2 (r^2+3)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (r^3+3r)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left(\frac{r^4}{4} + 3\frac{r^2}{2}\right)\Bigg|_{r=0}^{r=2} d\theta$$
$$= \int_0^{2\pi} \left[\left(\frac{2^4}{4} + 3\frac{2^2}{2}\right) - 0\right] d\theta$$
$$= 10\int_0^{2\pi} d\theta = 20\pi.$$

Notice how simple this iterated integral was, as compared to the corresponding integral in rectangular coordinates in (3.1). ■

NOTES

For double integrals of the form $\int_a^b \int_c^d f(r)dr\,d\theta$, note that the inner integral does not depend on θ. As a result, we can rewrite the double integral as

$$\left(\int_a^b 1\,d\theta\right)\left(\int_c^d f(r)\,dr\right) = (b-a)\int_c^d f(r)\,dr.$$

When dealing with double integrals, you should always consider whether the region over which you're integrating is in some way circular. If it is a circle or some portion of a circle, consider using polar coordinates.

EXAMPLE 3.3 Finding Volume Using Polar Coordinates

Find the volume inside the paraboloid $z = 9 - x^2 - y^2$, outside the cylinder $x^2 + y^2 = 4$ and above the xy-plane.

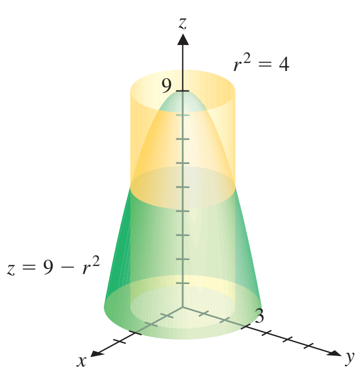


FIGURE 13.30a
Volume outside the cylinder and inside the paraboloid

Solution Notice that the paraboloid has its vertex at the point (0, 0, 9) and the axis of the cylinder is the z-axis. (See Figure 13.30a.) You should observe that the solid lies below the paraboloid and above the region in the xy-plane lying between the traces of the cylinder and the paraboloid in the xy-plane, that is, between the circles of radius 2 and 3, both centered at the origin. So, for each fixed $\theta \in [0, 2\pi]$, r ranges from 2 to 3. We call such a region a **circular annulus** (see Figure 13.30b). From (3.3), we have

$$V = \iint_R \underbrace{(9-x^2-y^2)}_{9-r^2}\,\underbrace{dA}_{r\,dr\,d\theta} = \int_0^{2\pi}\int_2^3 (9-r^2)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_2^3 (9r-r^3)\,dr\,d\theta = 2\pi\int_2^3 (9r-r^3)\,dr$$
$$= 2\pi\left(9\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=2}^{r=3} = \frac{25}{2}\pi.$$ ■

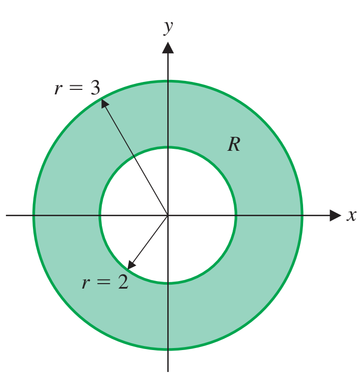


FIGURE 13.30b
Circular annulus

There are actually two things that you should look for when you are considering using polar coordinates for a double integral. The first is most obvious: Is the geometry of the region circular? The other is: Does the integral contain the expression $x^2 + y^2$ (particularly inside of other functions such as square roots, exponentials, etc.)? Since $r^2 = x^2 + y^2$, changing to polar coordinates will often simplify terms of this form.

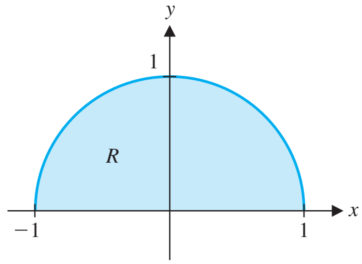


FIGURE 13.31
The region R

EXAMPLE 3.4 Changing a Double Integral to Polar Coordinates

Evaluate the iterated integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2(x^2+y^2)^2 dy\,dx$.

Solution First, you should recognize that evaluating this integral in rectangular coordinates is nearly hopeless. (Try it and see why!) On the other hand, it does have a term of the form x^2+y^2, which we discussed above. Even more significantly, the region over which you're integrating turns out to be a semicircle, as follows. Reading the inside limits of integration first, observe that for each fixed x between -1 and 1, y ranges from $y=0$ up to $y=\sqrt{1-x^2}$ (the top half of the circle of radius 1 centered at the origin). We sketch the region in Figure 13.31. From (3.3), we have

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2(x^2+y^2)^2 dy\,dx = \iint_R \underbrace{x^2}_{r^2\cos^2\theta}\underbrace{(x^2+y^2)^2}_{(r^2)^2}\underbrace{dA}_{r\,dr\,d\theta} \quad \text{Since } x = r\cos\theta.$$

$$= \int_0^{\pi}\int_0^1 r^7\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^8}{8}\bigg|_{r=0}^{r=1}\cos^2\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta \quad \text{Since } \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta).$$

$$= \frac{1}{16}\left(\theta+\frac{1}{2}\sin 2\theta\right)\bigg|_0^{\pi} = \frac{\pi}{16}.$$

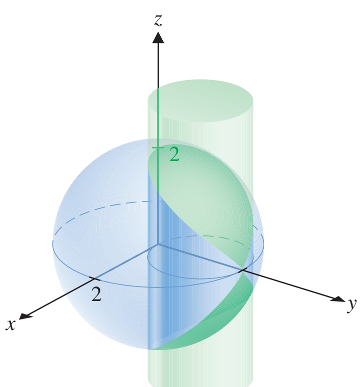


FIGURE 13.32a
Volume inside the sphere and inside the cylinder

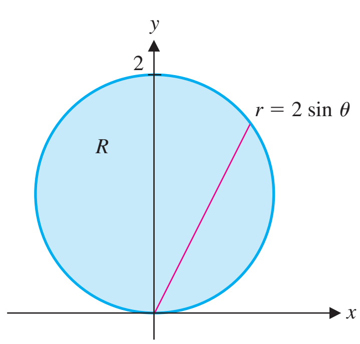


FIGURE 13.32b
The region R

EXAMPLE 3.5 Finding Volume Using Polar Coordinates

Find the volume cut out of the sphere $x^2+y^2+z^2=4$ by the cylinder $x^2+y^2=2y$.

Solution We show a sketch of the solid in Figure 13.32a. (If you complete the square on the equation of the cylinder, you'll see that it is a circular cylinder of radius 1, whose axis is the line: $x=0$, $y=1$, $z=t$.) Notice that equal portions of the volume lie above and below the circle of radius 1 centered at (0, 1), indicated in Figure 13.32b. So, we compute the volume lying below the top hemisphere $z=\sqrt{4-x^2-y^2}$ and above the region R indicated in Figure 13.32b and double it. We have

$$V = 2\iint_R \sqrt{4-x^2-y^2}\,dA.$$

Since R is a circle and the integrand includes a term of the form x^2+y^2, we introduce polar coordinates. Since $y=r\sin\theta$, the circle $x^2+y^2=2y$ becomes $r^2=2r\sin\theta$ or $r=2\sin\theta$. This gives us

$$V = 2\int_0^{\pi}\int_0^{2\sin\theta}\sqrt{4-r^2}\,r\,dr\,d\theta,$$

since the entire circle $r = 2\sin\theta$ is traced out for $0 \le \theta \le \pi$ and since for each fixed $\theta \in [0, \pi]$, r ranges from $r = 0$ to $r = 2\sin\theta$. Notice further that by symmetry, we get

$$\begin{aligned}
V &= 4\int_0^{\pi/2}\int_0^{2\sin\theta} \sqrt{4 - r^2}\, r\, dr\, d\theta \\
&= -2\int_0^{\pi/2} \left[\frac{2}{3}(4 - r^2)^{3/2}\right]_{r=0}^{r=2\sin\theta} d\theta \\
&= -\frac{4}{3}\int_0^{\pi/2} \left[(4 - 4\sin^2\theta)^{3/2} - 4^{3/2}\right] d\theta \\
&= -\frac{32}{3}\int_0^{\pi/2} [(\cos^2\theta)^{3/2} - 1]\, d\theta \\
&= -\frac{32}{3}\int_0^{\pi/2} (\cos^3\theta - 1)\, d\theta \\
&= -\frac{64}{9} + \frac{16}{3}\pi \approx 9.644.
\end{aligned}$$

There are several things to observe here. First, our use of symmetry was crucial. By restricting the integral to the interval $[0, \frac{\pi}{2}]$, we could write $(\cos^2\theta)^{3/2} = \cos^3\theta$, which is *not* true on the entire interval $[0, \pi]$. (Why not?) Second, if you think that this integral was messy, consider what it looks like in rectangular coordinates. (It's not pretty!) ■

EXAMPLE 3.6 Finding the Volume Between Two Paraboloids

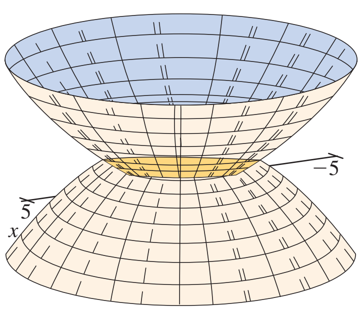


FIGURE 13.33a
Intersecting paraboloids

Find the volume of the solid bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

Solution Observe that the surface $z = 8 - x^2 - y^2$ is a paraboloid with vertex at $z = 8$ and opening downward, while $z = x^2 + y^2$ is a paraboloid with vertex at the origin and opening upward. The solid is shown in Figure 13.33a. At a given point (x, y), the height of the solid is given by

$$(8 - x^2 - y^2) - (x^2 + y^2) = 8 - 2x^2 - 2y^2.$$

We now have
$$V = \iint_R (8 - 2x^2 - 2y^2)\, dA,$$

where the region of integration R is the shadow of the solid in the xy-plane. The solid is widest at the intersection of the two paraboloids, which occurs where $8 - x^2 - y^2 = x^2 + y^2$ or $x^2 + y^2 = 4$. The region of integration R is then the disk shown in Figure 13.33b and is most easily described in polar coordinates. The integrand becomes $8 - 2x^2 - 2y^2 = 8 - 2r^2$ and we have

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^2 (8 - 2r^2) r\, dr\, d\theta \\
&= 16\pi.
\end{aligned}$$ ■

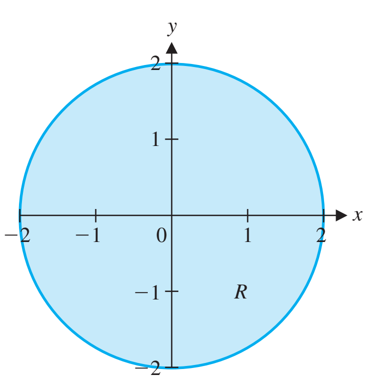


FIGURE 13.33b
The region R

Finally, we observe that we can also evaluate double integrals in polar coordinates by integrating first with respect to θ. Although such integrals are uncommon (given the way in which we change variables from rectangular to polar coordinates), we provide this for the sake of completeness.

Suppose the region R can be written in the form

$$R = \{(r, \theta) | 0 \le a \le r \le b \text{ and } h_1(r) \le \theta \le h_2(r)\},$$

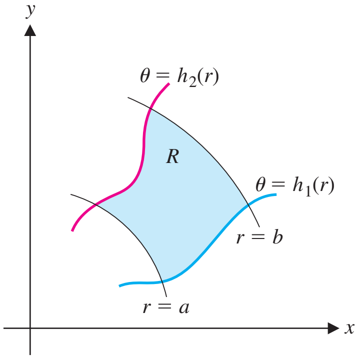


FIGURE 13.34
The region R

where $h_1(r) \le h_2(r)$, for all r in $[a, b]$, as pictured in Figure 13.34. Then, it can be shown that if $f(r, \theta)$ is continuous on R, we have

$$\iint_R f(r,\theta)\, dA = \int_a^b \int_{h_1(r)}^{h_2(r)} f(r,\theta)\, r\, d\theta\, dr. \tag{3.4}$$

BEYOND FORMULAS

This section may change the way you think of polar coordinates. While they allow us to describe a variety of unusual curves (roses, cardioids and so on) in a convenient form, polar coordinates are an essential computational tool for double integrals. In section 13.6, they serve the same role in triple integrals. In general, polar coordinates are useful in applications where some form of radial symmetry is present. Can you describe any situations in engineering, physics or chemistry where a structure or force has radial symmetry?

EXERCISES 13.3

WRITING EXERCISES

1. Thinking of $dy\,dx$ as representing the area dA of a small rectangle, explain in geometric terms why

$$\iint_R f(x,y)\, dA \ne \iint_R f(r\cos\theta, r\sin\theta)\, dr\, d\theta.$$

2. In all of the examples in this section, we integrated with respect to r first. It is perfectly legitimate to integrate with respect to θ first. Explain why it is unlikely that it will ever be necessary to do so. [Hint: If θ is on the inside, you need functions of the form $\theta(r)$ for the limits of integration.]

3. Given a double integral in rectangular coordinates as in example 3.2 or 3.4, identify at least two indications that the integral would be easier to evaluate in polar coordinates.

4. In section 9.5, we derived a formula $A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$ for the area bounded by the polar curve $r = f(\theta)$ and rays $\theta = a$ and $\theta = b$. Discuss how this formula relates to the formula used in example 3.1. Discuss which formula is easier to remember and which formula is more generally useful.

In exercises 1–6, find the area of the region bounded by the given curves.

1. $r = 3 + 2\sin\theta$
2. $r = 2 - 2\cos\theta$
3. one leaf of $r = \sin 3\theta$
4. $r = 3\cos\theta$
5. inside $r = 2\sin 3\theta$, outside $r = 1$, first quadrant
6. inside $r = 1$ and outside $r = 2 - 2\cos\theta$

In exercises 7–12, use polar coordinates to evaluate the double integral.

7. $\iint_R \sqrt{x^2+y^2}\, dA$, where R is the disk $x^2 + y^2 \le 9$
8. $\iint_R \sqrt{x^2+y^2+1}\, dA$, where R is the disk $x^2 + y^2 \le 16$
9. $\iint_R e^{-x^2-y^2} dA$, where R is the disk $x^2 + y^2 \le 4$
10. $\iint_R e^{-\sqrt{x^2+y^2}} dA$, where R is the disk $x^2 + y^2 \le 1$
11. $\iint_R y\, dA$, where R is bounded by $r = 2 - \cos\theta$
12. $\iint_R x\, dA$, where R is bounded by $r = 1 - \sin\theta$

In exercises 13–16, use the most appropriate coordinate system to evaluate the double integral.

13. $\iint_R (x^2+y^2)\, dA$, where R is bounded by $x^2 + y^2 = 9$
14. $\iint_R 2xy\, dA$, where R is bounded by $y = 4 - x^2$ and $y = 0$
15. $\iint_R (x^2+y^2)\, dA$, where R is bounded by $y = x$, $y = 0$ and $x = 2$
16. $\iint_R \cos\sqrt{x^2+y^2}\, dA$, where R is bounded by $x^2 + y^2 = 9$

In exercises 17–26, use an appropriate coordinate system to compute the volume of the indicated solid.

17. Below $z = x^2 + y^2$, above $z = 0$, inside $x^2 + y^2 = 9$
18. Below $z = x^2 + y^2 - 4$, above $z = 0$, inside $x^2 + y^2 = 9$

19. Below $z = \sqrt{x^2+y^2}$, above $z = 0$, inside $x^2+y^2=4$

20. Below $z = \sqrt{x^2+y^2}$, above $z = 0$, inside $x^2+(y-1)^2=1$

21. Below $z = \sqrt{4-x^2-y^2}$, above $z = 1$, inside $x^2+y^2=\frac{1}{4}$

22. Below $z = 8-x^2-y^2$, above $z = 3x^2+3y^2$

23. Below $z = 6-x-y$, in the first octant

24. Below $z = 4-x^2-y^2$, between $y = x$, $y = 0$ and $x = 1$

25. Below $z = 4-x^2-y^2$, above $z = x^2+y^2$, between $y = 0$ and $y = x$, in the first octant

26. Above $z = \sqrt{x^2+y^2}$, below $z = 4$, above the xy-plane, between $y = x$ and $y = 2x$, in the first octant

In exercises 27–32, evaluate the iterated integral by converting to polar coordinates.

27. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

28. $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sin(x^2+y^2)\,dy\,dx$

29. $\displaystyle\int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy\,dx$

30. $\displaystyle\int_{0}^{2}\int_{-\sqrt{4-x^2}}^{0} y\,dy\,dx$

31. $\displaystyle\int_{0}^{2}\int_{x}^{\sqrt{8-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

32. $\displaystyle\int_{0}^{2}\int_{y}^{\sqrt{2y-y^2}} x\,dx\,dy$

In exercises 33–36, compute the probability that a dart lands in the region R, assuming that the probability is given by $\iint_R \frac{1}{\pi}e^{-x^2-y^2}dA$.

33. A double bull's-eye, R is the region inside $r = \frac{1}{4}$ (inch)

34. A single bull's-eye, R bounded by $r = \frac{1}{4}$ and $r = \frac{1}{2}$

35. A triple-20, R bounded by $r = 3\frac{3}{4}$, $r = 4$, $\theta = \frac{9\pi}{20}$ and $\theta = \frac{11\pi}{20}$

36. A double-20, R bounded by $r = 6\frac{1}{4}$, $r = 6\frac{1}{2}$, $\theta = \frac{9\pi}{20}$ and $\theta = \frac{11\pi}{20}$

37. Find the area of the triple-20 region described in exercise 35.

38. Find the area of the double-20 region described in exercise 36.

39. Find the center of mass of a lamina in the shape of $x^2+(y-1)^2=1$, with density $\rho(x,y) = 1/\sqrt{x^2+y^2}$.

40. Find the center of mass of a lamina in the shape of $r = 2-2\cos\theta$, with density $\rho(x,y) = x^2+y^2$.

41. Suppose that $f(x,y) = 20{,}000\,e^{-x^2-y^2}$ is the population density of a species of small animals. Estimate the population in the region bounded by $x^2+y^2=1$.

42. Suppose that $f(x,y) = 15{,}000\,e^{-x^2-y^2}$ is the population density of a species of small animals. Estimate the population in the region bounded by $(x-1)^2+y^2=1$.

43. Find the moment of inertia I_y of the circular lamina bounded by $x^2+y^2=R^2$, with density $\rho(x,y)=1$. If the radius doubles, by what factor does the moment of inertia increase?

44. Repeat exercise 43 for the density function $\rho(x,y) = \sqrt{x^2+y^2}$.

45. Use a double integral to derive the formula for the volume of a sphere of radius a.

46. Use a double integral to derive the formula for the volume of a right circular cone of height h and base radius a. (Hint: Show that the desired volume equals the volume under $z = h$ and above $z = \frac{h}{a}\sqrt{x^2+y^2}$.)

47. Find the volume cut out of the sphere $x^2+y^2+z^2=9$ by the cylinder $x^2+y^2=2x$.

48. Find the volume of the wedge sliced out of the sphere $x^2+y^2+z^2=4$ by the planes $y = x$ and $y = 2x$. (Keep the portion with $x \ge 0$.)

49. Set up a double integral for the volume of the piece sliced off of the top of $x^2+y^2+z^2=4$ by the plane $y+z=2$.

50. Set up a double integral for the volume of the portion of $x+2y+3z=6$ cut out by the cylinder $x^2+4y^2=4$.

51. Show that the volume under the cone $z = k-r$ and above the xy-plane (where $k > 0$) grows as a cubic function of k. Show that the volume under the paraboloid $z = k-r^2$ and above the xy-plane (where $k > 0$) grows as a quadratic function of k. Explain why this volume increases less rapidly than that of the cone.

52. Show that the volume under the surface $z = k-r^n$ and above the xy-plane (where $k > 0$) approaches a linear function of k as $n \to \infty$. Explain why this makes sense.

53. Evaluate $\displaystyle\iint_R \frac{2}{1+x^2+y^2}dA$ where R is outside $r = 1$ and inside $r = 2\sin\theta$.

54. Evaluate $\displaystyle\iint_R \frac{\ln(x^2+y^2)}{x^2+y^2}dA$ where R is bounded by $r = 1$ and $r = 2$.

EXPLORATORY EXERCISES

1. Suppose that the following data give the density of a lamina at different locations. Estimate the mass of the lamina.

r \ θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\frac{1}{2}$	1.0	1.4	1.4	1.2	1.0
1	0.8	1.2	1.0	1.0	0.8
$\frac{3}{2}$	1.0	1.3	1.4	1.3	1.2
2	1.2	1.6	1.6	1.4	1.2

2. One of the most important integrals in probability theory is $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Since there is no antiderivative of e^{-x^2} among the elementary functions, we can't evaluate this integral directly. A clever use of polar coordinates is needed. Start by giving the integral a name,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = I.$$

Now, assuming that all the integrals converge, argue that $\int_{-\infty}^{\infty} e^{-y^2}\,dy = I$ and

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dy\,dx = I^2.$$

Convert the iterated integral to polar coordinates and evaluate it. The desired integral I is simply the square root of the iterated integral. Explain why the same trick can't be used to evaluate $\int_{-1}^{1} e^{-x^2}\,dx$.

13.4 SURFACE AREA

Recall that in section 5.4, we devised a method of finding the surface area for a surface of revolution. In this section, we consider how to find surface area in a more general setting. Suppose that $f(x, y) \geq 0$ and f has continuous first partial derivatives in some region R in the xy-plane. We would like to find a way to calculate the surface area of that portion of the surface $z = f(x, y)$ lying above R. As we have done innumerable times now, we begin by forming an inner partition of R, consisting of the rectangles $R_1, R_2, \ldots, R_n$. Our strategy is to approximate the surface area lying above each R_i, for $i = 1, 2, \ldots, n$ and then sum the individual approximations to obtain an approximation of the total surface area. We proceed as follows.

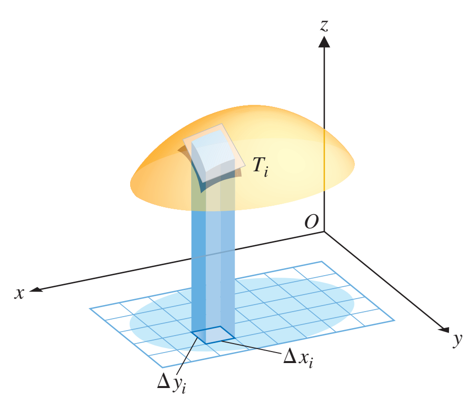


FIGURE 13.35a
Surface area

For each $i = 1, 2, \ldots, n$, let $(x_i, y_i, 0)$ represent the corner of R_i closest to the origin and construct the tangent plane to the surface $z = f(x, y)$ at the point $(x_i, y_i, f(x_i, y_i))$. Since the tangent plane stays close to the surface near the point of tangency, the area ΔT_i of that portion of the tangent plane that lies above R_i is an approximation to the surface area above R_i (see Figure 13.35a). Notice, too that the portion of the tangent plane lying above R_i is a parallelogram, T_i, whose area ΔT_i you should be able to easily compute. Adding together these approximations, we get that the total surface area S is approximately

$$S \approx \sum_{i=1}^{n} \Delta T_i.$$

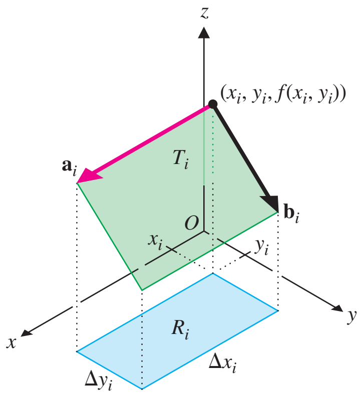


FIGURE 13.35b
Portion of the tangent plane above R_i

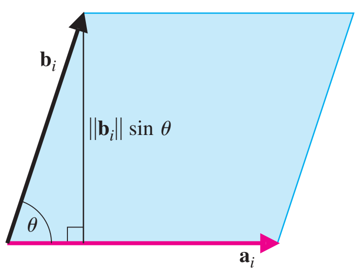


FIGURE 13.36
The parallelogram T_i

Also note that as the norm of the partition $\|P\|$ tends to zero, the approximations should approach the exact surface area and so we have

$$S = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta T_i, \tag{4.1}$$

assuming the limit exists. The only remaining question is how to find the values of ΔT_i, for $i = 1, 2, \ldots, n$. Let the dimensions of R_i be Δx_i and Δy_i, and let the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ form two adjacent sides of the parallelogram T_i, as indicated in Figure 13.35b. Recall from our discussion of tangent planes in section 12.4 that the tangent plane is given by

$$z - f(x_i, y_i) = f_x(x_i, y_i)(x - x_i) + f_y(x_i, y_i)(y - y_i). \tag{4.2}$$

Look carefully at Figure 13.35b; the vector $\mathbf{a}_i$ has its initial point at $(x_i, y_i, f(x_i, y_i))$. Its terminal point is the point on the tangent plane corresponding to $x = x_i + \Delta x_i$ and $y = y_i$. From (4.2), we get that the z-coordinate of the terminal point satisfies

$$\begin{aligned} z - f(x_i, y_i) &= f_x(x_i, y_i)(x_i + \Delta x_i - x_i) + f_y(x_i, y_i)(y_i - y_i) \\ &= f_x(x_i, y_i)\Delta x_i. \end{aligned}$$

This says that the vector $\mathbf{a}_i$ is given by

$$\mathbf{a}_i = \langle \Delta x_i, 0, f_x(x_i, y_i)\,\Delta x_i \rangle.$$

Likewise, $\mathbf{b}_i$ has its initial point at $(x_i, y_i, f(x_i, y_i))$, but has its terminal point at the point on the tangent plane corresponding to $x = x_i$ and $y = y_i + \Delta y_i$. Again, using (4.2), we get that the z-coordinate of this point is given by

$$\begin{aligned} z - f(x_i, y_i) &= f_x(x_i, y_i)\,(x_i - x_i) + f_y(x_i, y_i)\,(y_i + \Delta y_i - y_i) \\ &= f_y(x_i, y_i)\,\Delta y_i. \end{aligned}$$

This says that $\mathbf{b}_i$ is given by

$$\mathbf{b}_i = \langle 0, \Delta y_i, f_y(x_i, y_i)\Delta y_i \rangle.$$

Notice that ΔT_i is the area of the parallelogram shown in Figure 13.36, which you should recognize as

$$\Delta T_i = \|\mathbf{a}_i\|\|\mathbf{b}_i\| \sin\theta = \|\mathbf{a}_i \times \mathbf{b}_i\|,$$

where θ indicates the angle between $\mathbf{a}_i$ and $\mathbf{b}_i$. We have

$$\begin{aligned} \mathbf{a}_i \times \mathbf{b}_i &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_i & 0 & f_x(x_i, y_i)\,\Delta x_i \\ 0 & \Delta y_i & f_y(x_i, y_i)\,\Delta y_i \end{vmatrix} \\ &= -f_x(x_i, y_i)\,\Delta x_i\,\Delta y_i \mathbf{i} - f_y(x_i, y_i)\,\Delta x_i\,\Delta y_i \mathbf{j} + \Delta x_i\,\Delta y_i \mathbf{k}. \end{aligned}$$

This gives us

$$\Delta T_i = \|\mathbf{a}_i \times \mathbf{b}_i\| = \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1}\;\underbrace{\Delta x_i\,\Delta y_i}_{\Delta A_i},$$

where $\Delta A_i = \Delta x_i\,\Delta y_i$ is the area of the rectangle R_i. From (4.1), we now have that the total surface area is given by

$$\begin{aligned} S &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta T_i \\ &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{[f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2 + 1}\,\Delta A_i. \end{aligned}$$

You should recognize this limit as the double integral

Surface area

$$\boxed{S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA.} \tag{4.3}$$

There are several things to note here. First, you can easily show that the surface area formula (4.3) also holds for the case where $f(x, y) \le 0$ on R. Second, you should note the similarity to the arc length formula derived in section 5.4. Further, recall that $\mathbf{n} = \langle f_x(x, y), f_y(x, y), -1\rangle$ is a normal vector for the tangent plane to the surface $z = f(x, y)$ at (x, y). With this in mind, recognize that you can think of the integrand in (4.3) as $\|\mathbf{n}\|$, an idea we'll develop more fully in Chapter 14.

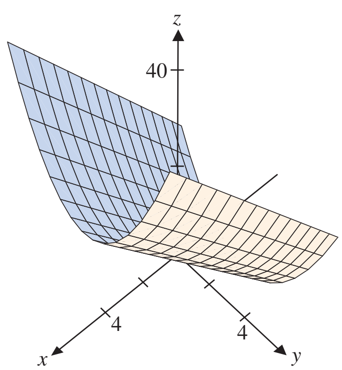


FIGURE 13.37a
The surface $z = y^2 + 4x$

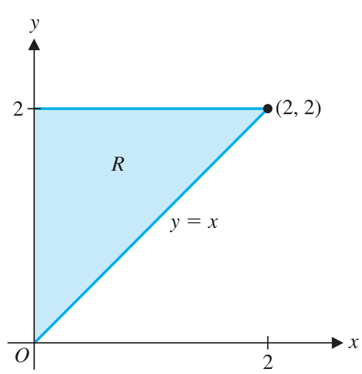


FIGURE 13.37b
The region R

EXAMPLE 4.1 Calculating Surface Area

Find the surface area of that portion of the surface $z = y^2 + 4x$ lying above the triangular region R in the xy-plane with vertices at (0, 0), (0, 2) and (2, 2).

Solution We show a computer-generated sketch of the surface in Figure 13.37a and the region R in Figure 13.37b. If we take $f(x, y) = y^2 + 4x$, then we have $f_x(x, y) = 4$ and $f_y(x, y) = 2y$. From (4.3), we now have

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA \\ &= \iint_R \sqrt{4^2 + 4y^2 + 1}\, dA. \end{aligned}$$

Looking carefully at Figure 13.37b, you can read off the limits of integration, to obtain

$$\begin{aligned} S &= \int_0^2 \int_0^y \sqrt{4y^2 + 17}\, dx\, dy = \int_0^2 \sqrt{4y^2 + 17}\, x \Big|_{x=0}^{x=y} dy \\ &= \int_0^2 y\sqrt{4y^2 + 17}\, dy = \frac{1}{8}(4y^2 + 17)^{3/2}\left(\frac{2}{3}\right)\Big|_0^2 \\ &= \frac{1}{12}\left[[4(2^2) + 17]^{3/2} - [4(0)^2 + 17]^{3/2}\right] \approx 9.956. \end{aligned}$$

Computing surface area requires more than simply substituting into formula (4.3). You will also need to carefully determine the region over which you're integrating and the best coordinate system to use, as in example 4.2.

EXAMPLE 4.2 Finding Surface Area Using Polar Coordinates

Find the surface area of that portion of the paraboloid $z = 1 + x^2 + y^2$ that lies below the plane $z = 5$.

Solution First, note that we have not given you the region of integration; you'll need to determine that from a careful analysis of the graph (see Figure 13.38a). Next, observe that the plane $z = 5$ intersects the paraboloid in a circle of radius 2, parallel to the xy-plane and centered at the point (0, 0, 5). (Simply plug $z = 5$ into the equation of the

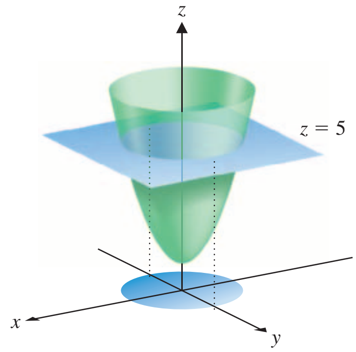


FIGURE 13.38a
Intersection of the paraboloid with the plane $z = 5$

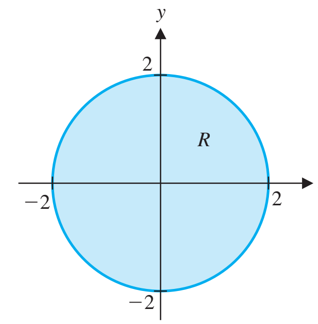


FIGURE 13.38b
The region R

paraboloid to see why.) So, the surface area *below* the plane $z = 5$ lies *above* the circle in the xy-plane of radius 2, centered at the origin. We show the region of integration R in Figure 13.38b. Taking $f(x, y) = 1 + x^2 + y^2$, we have $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$, so that from (4.3), we have

$$\begin{aligned} S &= \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA \\ &= \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA. \end{aligned}$$

Note that since the region of integration is circular and the integrand contains the term $x^2 + y^2$, polar coordinates are indicated. We have

$$\begin{aligned} S &= \iint_R \underbrace{\sqrt{4(x^2 + y^2) + 1}}_{\sqrt{4r^2+1}}\, \underbrace{dA}_{r\,dr\,d\theta} \\ &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \frac{1}{8} \int_0^{2\pi} \left(\frac{2}{3}\right) (4r^2 + 1)^{3/2} \Big|_{r=0}^{r=2} d\theta \\ &= \frac{1}{12} \int_0^{2\pi} (17^{3/2} - 1^{3/2})\, d\theta \\ &= \frac{2\pi}{12} (17^{3/2} - 1) \approx 36.18. \end{aligned}$$

We must point out that (just as with arc length) most surface area integrals cannot be computed exactly. Most of the time, you must rely on numerical approximations of the integrals. Although your computer algebra system no doubt can approximate even iterated integrals numerically, you should try to evaluate at least one of the iterated integrals and then approximate the second integral numerically (e.g., using Simpson's Rule). This is the situation in example 4.3.

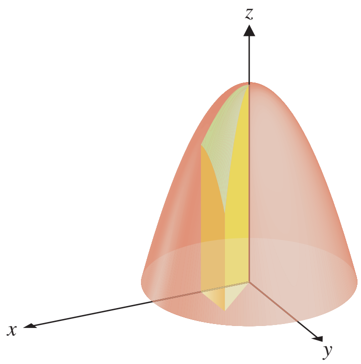


FIGURE 13.39a
$z = 4 - x^2 - y^2$

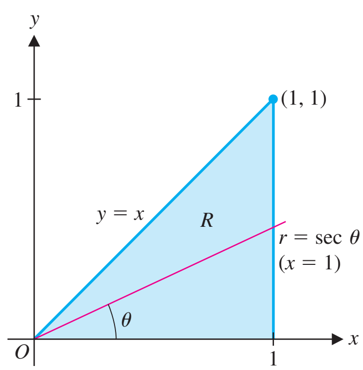


FIGURE 13.39b
The region R

EXAMPLE 4.3 Surface Area That Must Be Approximated Numerically

Find the surface area of that portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the triangular region R in the xy-plane with vertices at the points (0, 0), (1, 1) and (1, 0).

Solution We sketch the paraboloid and the region R in Figure 13.39a. Taking $f(x, y) = 4 - x^2 - y^2$, we get $f_x(x, y) = -2x$ and $f_y(x, y) = -2y$. From (4.3), we have

$$S = \iint_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$
$$= \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

Note that you have little hope of evaluating this double integral in rectangular coordinates. (Think about this!) Even though the region of integration is not circular, we'll try polar coordinates, since the integrand contains the term $x^2 + y^2$. We indicate the region R in Figure 13.39b. The difficulty here is in describing the region R in terms of polar coordinates. Look carefully at Figure 13.39b and notice that for each fixed angle θ, the radius r varies from 0 out to a point on the line $x = 1$. Since in polar coordinates $x = r\cos\theta$, the line $x = 1$ corresponds to $r\cos\theta = 1$ or $r = \sec\theta$, in polar coordinates. Further, θ varies from $\theta = 0$ (the x-axis) to $\theta = \frac{\pi}{4}$ (the line $y = x$). The surface area integral now becomes

$$S = \iint_R \underbrace{\sqrt{4x^2 + 4y^2 + 1}}_{\sqrt{4r^2+1}}\ \underbrace{dA}_{r\,dr\,d\theta}$$
$$= \int_0^{\pi/4} \int_0^{\sec\theta} \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$
$$= \frac{1}{8} \int_0^{\pi/4} \left(\frac{2}{3}\right) (4r^2 + 1)^{3/2} \Big|_{r=0}^{r=\sec\theta} d\theta$$
$$= \frac{1}{12} \int_0^{\pi/4} [(4\sec^2\theta + 1)^{3/2} - 1]\, d\theta$$
$$\approx 0.93078,$$

where we have approximated the value of the final integral, since no exact means of integration was available. You can arrive at this approximation using Simpson's Rule or using your computer algebra system. ■

BEYOND FORMULAS

The surface area calculations in this section are important in their own right. Builders often need to know the surface area of the structure they are designing. However, for our purposes the ideas in this section will assume more importance when we introduce surface integrals in section 14.6. For surface integrals, surface area is a basic component used in the setup of the integral. This is similar to how the arc length formula is incorporated in the formula for the surface area of a surface of revolution in section 5.4.

EXERCISES 13.4

WRITING EXERCISES

1. Starting at equation (4.1), there are several ways to estimate ΔT_i. Explain why it is important that we were able to find an approximation of the form $f(x_i, y_i)\Delta x_i \Delta y_i$.

2. In example 4.3, we evaluated the inner integral before estimating the remaining integral numerically. Discuss the number of calculations that would be necessary to use a rule such as Simpson's Rule to estimate an iterated integral. Explain why we thought it important to evaluate the inner integral first.

In exercises 1–12, find the surface area of the indicated surface.

1. The portion of $z = x^2 + 2y$ between $y = x$, $y = 0$ and $x = 4$.

2. The portion of $z = 4y + 3x^2$ between $y = 2x$, $y = 0$ and $x = 2$.

3. The portion of $z = 4 - x^2 - y^2$ above the xy-plane.

4. The portion of $z = x^2 + y^2$ below $z = 4$.

5. The portion of $z = \sqrt{x^2 + y^2}$ below $z = 2$.

6. The portion of $z = \sqrt{x^2 + y^2}$ between $y = x^2$ and $y = 4$.

7. The portion of $x + 3y + z = 6$ in the first octant.

8. The portion of $2x + y + z = 8$ in the first octant.

9. The portion of $x - y - 2z = 4$ with $x \geq 0$, $y \leq 0$ and $z \leq 0$.

10. The portion of $2x + y - 4z = 4$ with $x \geq 0$, $y \geq 0$ and $z \leq 0$.

11. The portion of $z = \sqrt{4 - x^2 - y^2}$ above $z = 0$.

12. The portion of $z = \sin x + \cos y$ with $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$.

In exercises 13–20, numerically estimate the surface area.

13. The portion of $z = e^{x^2+y^2}$ inside of $x^2 + y^2 = 4$.

14. The portion of $z = e^{-x^2-y^2}$ inside of $x^2 + y^2 = 1$.

15. The portion of $z = x^2 + y^2$ between $z = 5$ and $z = 7$.

16. The portion of $z = x^2 + y^2$ between $r = 2 - 2\cos\theta$.

17. The portion of $z = y^2$ below $z = 4$ and between $x = -2$ and $x = 2$.

18. The portion of $z = 4 - x^2$ above $z = 0$ and between $y = 0$ and $y = 4$.

19. The portion of $z = \sin x \cos y$ with $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$.

20. The portion of $z = \sqrt{x^2 + y^2 - 4}$ below $z = 1$.

21. In exercises 5 and 6, determine the surface area of the cone as a function of the area A of the base R of the solid and the height of the cone.

22. Use your solution to exercise 21 to quickly find the surface area of the portion of $z = \sqrt{x^2 + y^2}$ above the rectangle $0 \leq x \leq 2$, $1 \leq y \leq 4$.

23. In exercises 9 and 10, determine the surface area of the portion of the plane indicated as a function of the area A of the base R of the solid and the angle θ between the given plane and the xy-plane.

24. Use your solution to exercise 23 to quickly find the surface area of the portion of $z = 1 + y$ above the rectangle $-1 \leq x \leq 3$, $0 \leq y \leq 2$.

25. Generalizing exercises 17 and 18, determine the surface area of the portion of the cylinder indicated as a function of the arc length L of the base (two-dimensional) curve of the cylinder and the height h of the surface in the third dimension.

26. Use your solution to exercise 25 to quickly find the surface area of the portion of the cylinder with triangular cross sections parallel to the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and the origin and lying between the planes $z = 0$ and $z = 4$.

27. In example 4.2, find the value of k such that the plane $z = k$ slices off half of the surface area. Before working the problem, explain why $k = 3$ (halfway between $z = 1$ and $z = 5$) won't work.

28. Find the value of k such that the indicated surface area equals that of example 4.2: the surface area of that portion of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = k$.

Exercises 29–32 involve parametric surfaces.

29. Let S be a surface defined by parametric equations $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, for $a \leq u \leq b$ and $c \leq v \leq d$. Show that the surface area of S is given by $\int_c^d \int_a^b \|\mathbf{r}_u \times \mathbf{r}_v\| \, du\, dv$, where
$\mathbf{r}_u(u, v) = \left\langle \dfrac{\partial x}{\partial u}(u, v), \dfrac{\partial y}{\partial u}(u, v), \dfrac{\partial z}{\partial u}(u, v) \right\rangle$ and
$\mathbf{r}_v(u, v) = \left\langle \dfrac{\partial x}{\partial v}(u, v), \dfrac{\partial y}{\partial v}(u, v), \dfrac{\partial z}{\partial v}(u, v) \right\rangle$.

30. Use the formula from exercise 29 to find the surface area of the portion of the hyperboloid defined by parametric equations $x = 2\cos u \cosh v$, $y = 2\sin u \cosh v$, $z = 2\sinh v$ for $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$. (Hint: Set up the double integral and approximate it numerically.)

31. Use the formula from exercise 29 to find the surface area of the surface defined by $x = u$, $y = v\cos u$, $z = v\sin u$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

32. Use the formula from exercise 29 to find the surface area of the surface defined by $x = u$, $y = v + 2$, $z = 2uv$ for $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

EXPLORATORY EXERCISES

1. An old joke tells of the theoretical mathematician hired to improve dairy production who starts his report with the assumption, "Consider a spherical cow." In this exercise, we will approximate an animal's body with ellipsoids. Estimate the volume and surface area of the ellipsoids $16x^2 + y^2 + 4z^2 = 16$ and $16x^2 + y^2 + 4z^2 = 36$. Note that the second ellipsoid retains the proportions of the first ellipsoid, but the length of each dimension is multiplied by $\frac{3}{2}$. Show that the volume increases by a much greater proportion than does the surface area. In general, volume increases as the cube of length (in this case, $\left(\frac{3}{2}\right)^3 = 3.375$) and surface area increases as the square of length (in this case, $\left(\frac{3}{2}\right)^2 = 2.25$). This has implications for the sizes of animals, since volume tends to be proportional to weight and surface area tends to be proportional to strength. Explain why a cow increased in size proportionally by a factor of $\frac{3}{2}$ might collapse under its weight.

2. For a surface $z = f(x, y)$, recall that a normal vector to the tangent plane at $(a, b, f(a, b))$ is $\langle f_x(a, b), f_y(a, b), -1\rangle$. Show that the surface area formula can be rewritten as

$$\text{Surface area} = \iint_R \frac{\|\mathbf{n}\|}{|\mathbf{n} \cdot \mathbf{k}|}\, dA,$$

where $\mathbf{n}$ is the unit normal vector to the surface. Use this formula to set up a double integral for the surface area of the top half of the sphere $x^2 + y^2 + z^2 = 4$ and compare this to the work required to set up the same integral in exercise 17. (Hint: Use the gradient to compute the normal vector and substitute $z = \sqrt{4 - x^2 - y^2}$ to write the integral in terms of x and y.) For a surface such as $y = 4 - x^2 - z^2$, it is convenient to think of y as the dependent variable and double integrate with respect to x and z. Write out the surface area formula in terms of the normal vector for this orientation and use it to compute the surface area of the portion of $y = 4 - x^2 - z^2$ inside $x^2 + z^2 = 1$ and to the right of the xz-plane.

13.5 TRIPLE INTEGRALS

We developed the definite integral of a function of one variable $f(x)$ initially to compute the area under the curve $y = f(x)$. Similarly, we first devised the double integral of a function of two variables $f(x, y)$ to compute the volume lying beneath the surface $z = f(x, y)$. We have no comparable geometric motivation for defining the triple integral of a function of three variables $f(x, y, z)$, since the graph of $u = f(x, y, z)$ is a **hypersurface** in four dimensions. (We can't even visualize a graph in four dimensions.) Despite this lack of immediate geometric significance, integrals of functions of three variables have many very significant applications to studying the three-dimensional world in which we live. We'll consider two of these applications (finding the mass and center of mass of a solid) at the end of this section.

We pattern our development of the triple integral of a function of three variables after our development of the double integral of a function of two variables. We first consider the relatively simple case of a function $f(x, y, z)$ defined on a rectangular box Q in three-dimensional space defined by

$$Q = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d \text{ and } r \leq z \leq s\}.$$

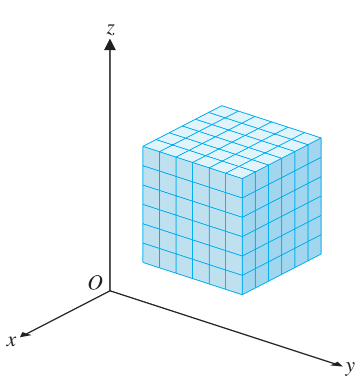


FIGURE 13.40a
Partition of the box Q

We begin by partitioning the region Q by slicing it by planes parallel to the xy-plane, planes parallel to the xz-plane and planes parallel to the yz-plane. Notice that this divides Q into a number of smaller boxes (see Figure 13.40a). Number the smaller boxes in any order: $Q_1, Q_2, \ldots, Q_n$. For each box Q_i $(i = 1, 2, \ldots, n)$, call the x, y and z dimensions of the box Δx_i, Δy_i and Δz_i, respectively (see Figure 13.40b). The volume of the box Q_i is then $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$. As we did in both one and two dimensions, we pick any point

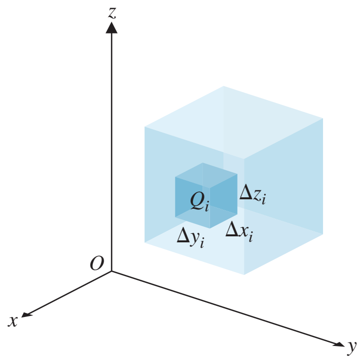


FIGURE 13.40b
Typical box Q_i

(u_i, v_i, w_i) in the box Q_i and form the Riemann sum

$$\sum_{i=1}^{n} f(u_i, v_i, w_i)\Delta V_i.$$

In this three-dimensional case, we define the norm of the partition $\|P\|$ to be the longest diagonal of any of the boxes Q_i, $i = 1, 2, \ldots, n$. We can now define the triple integral of $f(x, y, z)$ over Q.

DEFINITION 5.1

For any function $f(x, y, z)$ defined on the rectangular box Q, we define the **triple integral** of f over Q by

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(u_i, v_i, w_i)\, \Delta V_i, \tag{5.1}$$

provided the limit exists and is the same for every choice of evaluation points (u_i, v_i, w_i) in Q_i, for $i = 1, 2, \ldots, n$. When this happens, we say that f is **integrable** over Q.

REMARK 5.1

It can be shown that as long as f is continuous over Q, f will be integrable over Q.

Now that we have defined a triple integral, how can we calculate the value of one? The answer should prove to be no surprise. Just as a double integral can be written as two iterated integrals, a triple integral turns out to be equivalent to *three* iterated integrals.

THEOREM 5.1 (Fubini's Theorem)

Suppose that $f(x, y, z)$ is continuous on the box Q defined by $Q = \{(x, y, z) | a \le x \le b, c \le y \le d \text{ and } r \le z \le s\}$. Then, we can write the triple integral over Q as a triple iterated integral:

$$\iiint_Q f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz. \tag{5.2}$$

As was the case for double integrals, the three iterated integrals in (5.2) are evaluated from the inside out, using partial integrations. That is, in the innermost integral, we hold y and z fixed and integrate with respect to x and in the second integration, we hold z fixed and integrate with respect to y. Notice also that in this simple case (where Q is a rectangular box) the order of the integrations in (5.2) is irrelevant, so that we might just as easily write the triple integral as

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx,$$

or in any of the four remaining orders.

EXAMPLE 5.1 Triple Integral Over a Rectangular Box

Evaluate the triple integral $\iiint_Q 2xe^y \sin z\, dV$, where Q is the rectangle defined by

$$Q = \{(x, y, z) \,|\, 1 \le x \le 2, 0 \le y \le 1 \text{ and } 0 \le z \le \pi\}.$$

Solution From (5.2), we have

$$\iiint_Q 2xe^y \sin z \, dV = \int_0^\pi \int_0^1 \int_1^2 2xe^y \sin z \, dx \, dy \, dz$$
$$= \int_0^\pi \int_0^1 e^y \sin z \frac{2x^2}{2}\bigg|_{x=1}^{x=2} dy \, dz$$
$$= 3\int_0^\pi \sin z \, e^y \bigg|_{y=0}^{y=1} dz$$
$$= 3(e^1 - 1)(-\cos z)\bigg|_{z=0}^{z=\pi}$$
$$= 3(e-1)(-\cos\pi + \cos 0)$$
$$= 6(e-1).$$

You should pick one of the other five possible orders of integration and show that you get the same result. ■

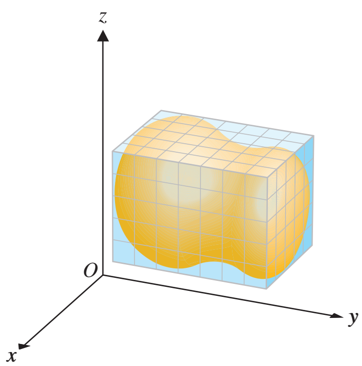


FIGURE 13.41a
Partition of a solid

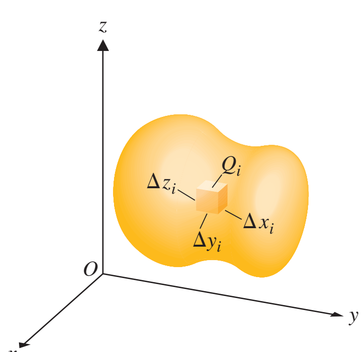


FIGURE 13.41b
Typical rectangle in inner partition of solid

As we did for double integrals, we can define triple integrals for more general regions in three dimensions by using an inner partition of the region. For any bounded solid Q in three dimensions, we partition Q by slicing it with planes parallel to the three coordinate planes. As in the case where Q was a box, these planes form a number of boxes (see Figures 13.41a and 13.41b). In this case, we consider only those boxes $Q_1, Q_2, \ldots, Q_n$ that lie *entirely* in Q and call this an **inner partition** of the solid Q. For each $i = 1, 2, \ldots, n$, we pick any point $(u_i, v_i, w_i) \in Q_i$ and form the Riemann sum

$$\sum_{i=1}^{n} f(u_i, v_i, w_i)\,\Delta V_i,$$

where $\Delta V_i = \Delta x_i \, \Delta y_i \, \Delta z_i$ represents the volume of Q_i. We can then define a triple integral over a general region Q as the limit of Riemann sums, as follows.

DEFINITION 5.2

For a function $f(x, y, z)$ defined on the (bounded) solid Q, we define the triple integral of $f(x, y, z)$ over Q by

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(u_i, v_i, w_i)\,\Delta V_i, \qquad (5.3)$$

provided the limit exists and is the same for every choice of the evaluation points (u_i, v_i, w_i) in Q_i, for $i = 1, 2, \ldots, n$. When this happens, we say that f is **integrable** over Q.

Observe that (5.3) is identical to (5.1), except that in (5.3), we are summing over an inner partition of Q.

The (very) big remaining question is how to evaluate triple integrals over more general regions. The fact that there are six different orders of integration possible in a triple iterated integral makes it difficult to write down a single result that will allow us to evaluate all triple integrals. So, rather than write down an exhaustive list, we'll indicate the general idea by

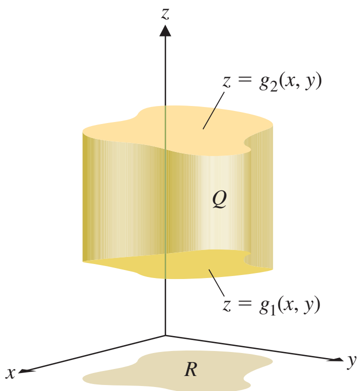


FIGURE 13.42
Solid with defined top and bottom surfaces

looking at several specific cases. For instance, if the region Q can be written in the form

$$Q = \{(x, y, z) \mid (x, y) \in R \text{ and } g_1(x, y) \le z \le g_2(x, y)\},$$

where R is some region in the xy-plane and where $g_1(x, y) \le g_2(x, y)$ for all (x, y) in R (see Figure 13.42), then it can be shown that

$$\boxed{\iiint_Q f(x, y, z)\, dV = \iint_R \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\, dA.} \tag{5.4}$$

As we have seen before, the innermost integration in (5.4) is a partial integration, where we hold x and y fixed and integrate with respect to z, and the outer double integral is evaluated using the methods we have already developed in sections 13.1 and 13.3.

EXAMPLE 5.2 Triple Integral Over a Tetrahedron

Evaluate $\iiint_Q 6xy\, dV$, where Q is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 4$ (see Figure 13.43a).

Solution Notice that each point in the solid lies above the triangular region R in the xy-plane indicated in Figures 13.43a and 13.43b. You can think of R as forming the *base* of the solid. Notice that for each fixed point $(x, y) \in R$, z ranges from $z = 0$ up to $z = 4 - 2x - y$. It helps to draw a vertical line from the base and through the top surface of the solid, as we have indicated in Figure 13.43a. The line first enters the solid on the xy-plane ($z = 0$) and exits the solid on the plane $z = 4 - 2x - y$. This tells you that the innermost limits of integration (given that the first integration is with respect to z) are $z = 0$ and $z = 4 - 2x - y$. From (5.4), we now have

$$\iiint_Q 6xy\, dV = \iint_R \int_0^{4-2x-y} 6xy\, dz\, dA.$$

This leaves us with setting up the double integral over the triangular region shown in Figure 13.43b. Notice that for each fixed $x \in [0, 2]$, y ranges from 0 up to $y = 4 - 2x$.

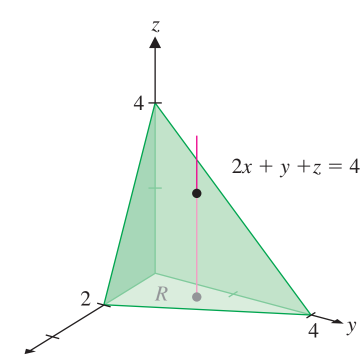


FIGURE 13.43a
Tetrahedron

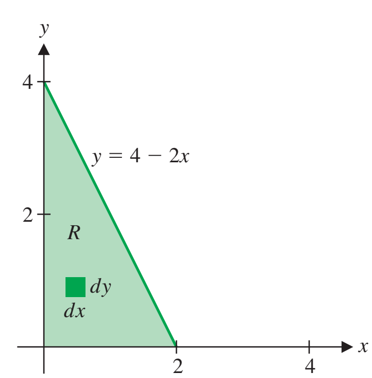


FIGURE 13.43b
The base of the solid in the xy-plane

NOTES

Observe that in example 5.2, the boundary of R consists of $x = 0$ and $y = 0$ (corresponding to the defining surfaces of Q that *do not* involve z) and $y = 4 - 2x$ (corresponding to the intersection of the two defining surfaces of Q that *do* involve z). The limits of integration for the outer two integrals can typically be found in this fashion.

We now have

$$\begin{aligned}\iiint_Q 6xy\,dV &= \iint_R \int_0^{4-2x-y} 6xy\,dz\,dA \\ &= \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} 6xy\,dz\,dy\,dx \\ &= \int_0^2 \int_0^{4-2x} (6xyz)\Big|_{z=0}^{z=4-2x-y} dy\,dx \\ &= \int_0^2 \int_0^{4-2x} 6xy\,(4-2x-y)\,dy\,dx \\ &= \int_0^2 6\left(4x\frac{y^2}{2} - 2x^2\frac{y^2}{2} - x\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=4-2x} dx \\ &= \int_0^2 \left[12x(4-2x)^2 - 6x^2(4-2x)^2 - 2x(4-2x)^3\right] dx \\ &= \frac{64}{5},\end{aligned}$$

where we leave the details of the last integration to you. ■

The greatest challenge in setting up a triple integral is to get the limits of integration correct. You can improve your chances of doing this by taking the time to draw a good sketch of the solid and identifying either the base of the solid in one of the coordinate planes (as we did in example 5.2) or top and bottom boundaries of the solid when both lie above or below the same region R in one of the coordinate planes. In particular, if the solid extends from $z = f(x, y)$ to $z = g(x, y)$ for each (x, y) in some two-dimensional region R, then z is a good choice for the innermost variable of integration. This may seem like a lot to keep in mind, but we'll illustrate these ideas generously in the examples that follow and in the exercises. Be sure that you don't rely on making guesses. Guessing may get you through the first several exercises, but will not work in general.

Once you have identified a base or a top and bottom surface of a solid, draw a line from a representative point in the base (or bottom surface) through the top surface of the solid, as we did in Figure 13.43a, indicating the limits for the innermost integral. To illustrate this, we take several different views of example 5.2.

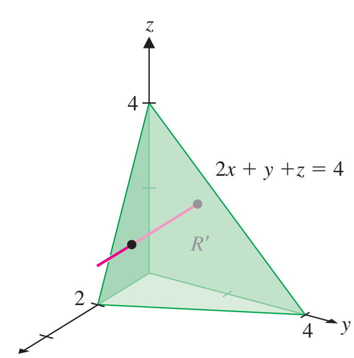


FIGURE 13.44a
Tetrahedron viewed with base in the yz-plane

EXAMPLE 5.3 A Triple Integral Where the First Integration Is with Respect to *x*

Evaluate $\iiint_Q 6xy\,dV$, where Q is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 4$, as in example 5.2, but this time, integrate first with respect to x.

Solution You might object that our only evaluation result for triple integrals (5.4) is for integration with respect to z first. While this is true, you need to realize that x, y and z are simply variables that we represent by letters of the alphabet. Who cares which letter is which? Notice that we can think of the tetrahedron as a solid with its base in the triangular region R' of the yz-plane, as indicated in Figure 13.44a. In this case, we draw a line orthogonal to the yz-plane, which enters the solid in the yz-plane $(x = 0)$ and exits in the plane $x = \frac{1}{2}(4 - y - z)$. Adapting (5.4) to this situation

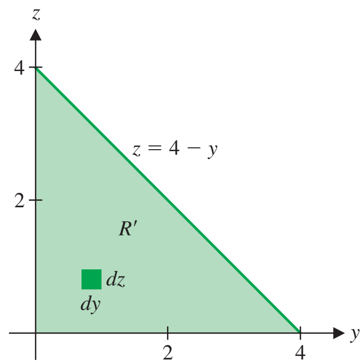


FIGURE 13.44b
The region R'

(i.e., interchanging the roles of x and z), we have

$$\iiint_Q 6xy\,dV = \iint_{R'} \int_0^{\frac{1}{2}(4-y-z)} 6xy\,dx\,dA$$

$$= \iint_{R'} \left(6\frac{x^2}{2}y\right)\Bigg|_{x=0}^{x=\frac{1}{2}(4-y-z)} dA$$

$$= \iint_{R'} 3\,\frac{(4-y-z)^2}{4}\,y\,dA.$$

To evaluate the remaining double integral, we look at the region R' in the yz-plane, as shown in Figure 13.44b. We now have

$$\iiint_Q 6xy\,dV = \frac{3}{4}\int_0^4 \int_0^{4-y} (4-y-z)^2 y\,dz\,dy = \frac{64}{5},$$

where we have left the routine details for you to verify. Finally, we leave it to you to show that we can also write this triple integral as a triple iterated integral where we integrate with respect to y first, as in

$$\iiint_Q 6xy\,dV = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-z} 6xy\,dy\,dz\,dx.$$

NOTES

Notice in example 5.3 that the boundary of R' consists of $y = 0$ and $z = 0$ (the defining surfaces of Q that *do not* involve x) and $z = 4 - y$ (the intersection of the two defining surfaces that *do* involve x). These are the typical sources of surfaces for the limits of integration in the outer two integrals.

We want to emphasize again that the challenge here is to get the correct limits of integration. While you can always use a computer algebra system to evaluate the integrals (at least approximately), no computer algebra system will set up the limits of integration for you! Keep in mind that the innermost limits of integration correspond to two three-dimensional surfaces. (You can think of these as the top and the bottom of the solid, if you orient yourself properly.) These limits can involve either or both (or neither) of the two outer variables of integration. The limits of integration for the middle integral represent two curves in one of the coordinate planes and can involve only the outermost variable of integration. Realize, too, that once you integrate with respect to a given variable, that variable is eliminated from subsequent integrations (since you've evaluated the result of the integration between two specific values of that variable). Keep these ideas in mind as you work through the examples and exercises and make sure you work lots of problems. Triple integrals can look intimidating at first and *the only way to become proficient with these is to work plenty of problems*! Multiple integrals form the basis of much of the remainder of the book, so don't skimp on your effort now.

EXAMPLE 5.4 Evaluating a Triple Integral by Changing the Order of Integration

Evaluate $\displaystyle\int_0^4 \int_x^4 \int_0^y \frac{6}{1+48z-z^3}\,dz\,dy\,dx.$

Solution First, notice that evaluating the innermost integral requires a partial fractions decomposition, which produces three natural logarithm terms. The second integration is not pretty. We can significantly simplify the integral by changing the order of integration, but we must first identify the surfaces that bound the solid over which we

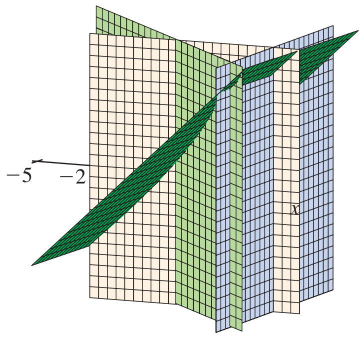


FIGURE 13.45a

are integrating. Starting with the inside limits, observe that the slanted plane $z = y$ forms the top of the solid and $z = 0$ forms the bottom.

The middle limits of integration indicate that the solid is also bounded by the planes $y = x$ and $y = 4$. The outer limits of $x = 0$ and $x = 4$ indicate that the solid is also bounded by the plane $x = 0$. (Here, $x = 4$ corresponds to the intersection of $y = x$ and $y = 4$.) A sketch of the solid is shown in Figure 13.45a. Notice that since y is involved in three different boundary planes, it is a poor choice for the inner variable of integration. To integrate with respect to x first, notice that a ray in the direction of the positive x-axis enters the solid through the plane $x = 0$ and exits through the plane $x = y$. We now have

$$\int_0^4 \int_x^4 \int_0^y \frac{6}{1+48z-z^3}\,dz\,dy\,dx = \iint_R \int_0^y \frac{6}{1+48z-z^3}\,dx\,dA,$$

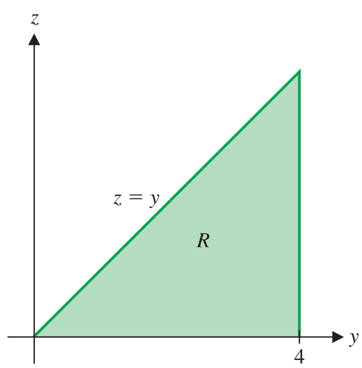


FIGURE 13.45b
The region R

where R is the triangle bounded by $z = y$, $z = 0$ and $y = 4$. (See Figure 13.45b.) In R, y extends from $y = z$ to $y = 4$, as z ranges from $z = 0$ to $z = 4$. The integral then becomes

$$\begin{aligned}\int_0^4 \int_x^4 \int_0^y \frac{6}{1+48z-z^3}\,dz\,dy\,dx &= \int_0^4 \int_z^4 \int_0^y \frac{6}{1+48z-z^3}\,dx\,dy\,dz\\ &= \int_0^4 \int_z^4 \frac{6}{1+48z-z^3}\,y\,dy\,dz\\ &= \int_0^4 \frac{6}{1+48z-z^3}\,\frac{y^2}{2}\bigg|_{y=z}^{y=4} dz\\ &= \int_0^4 \frac{48-3z^2}{1+48z-z^3}\,dz\\ &= \ln\left|1+48z-z^3\right|\Big|_{z=0}^{z=4}\\ &= \ln 129.\end{aligned}$$

NOTES

Try the original triple integral in example 5.4 on your CAS. Many integration packages are unable to evaluate this triple integral exactly. However, most packages will correctly return ln 129 if you ask them to evaluate the integral with the order of integration reversed. Technology does not replace an understanding of calculus techniques.

As you can see from example 5.4, there are clear advantages to considering alternative approaches for calculating triple integrals. So, take an extra moment to look at a sketch of a solid and consider your alternatives before jumping into the problem (i.e., look before you leap).

Recall that for double integrals, we had found that $\iint_R dA$ gives the area of the region R. Similarly, observe that if $f(x, y, z) = 1$ for all $(x, y, z) \in Q$, then from (5.3), we have

$$\iiint_Q 1\,dV = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \Delta V_i = V, \tag{5.5}$$

where V is the volume of the solid Q.

EXAMPLE 5.5 Using a Triple Integral to Find Volume

Find the volume of the solid bounded by the graphs of $z = 4 - y^2$, $x + z = 4$, $x = 0$ and $z = 0$.

Solution We show a sketch of the solid in Figure 13.46a. First, observe that we can consider the base of the solid to be the region R formed by the projection of the solid onto the yz-plane ($x = 0$). Notice that this is the region bounded by the parabola $z = 4 - y^2$ and the y-axis (see Figure 13.46b). Then, for each fixed y and z, the

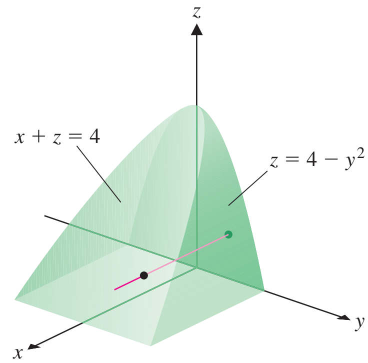


FIGURE 13.46a
The solid Q

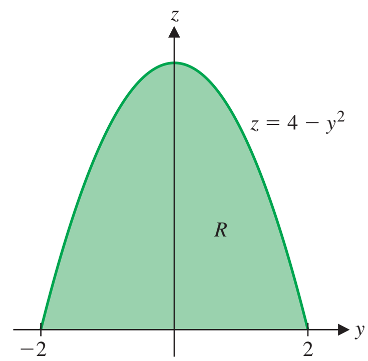


FIGURE 13.46b
Base R of the solid

corresponding values of x range from 0 to $4 - z$. The volume of the solid is then given by

$$
\begin{aligned}
V &= \iiint_Q dV = \iint_R \int_0^{4-z} dx\, dA \\
&= \int_{-2}^{2} \int_0^{4-y^2} \int_0^{4-z} dx\, dz\, dy \\
&= \int_{-2}^{2} \int_0^{4-y^2} (4-z)\, dz\, dy \\
&= \int_{-2}^{2} \left(4z - \frac{z^2}{2}\right)\Bigg|_{z=0}^{z=4-y^2} dy \\
&= \int_{-2}^{2} \left[4(4-y^2) - \frac{1}{2}(4-y^2)^2\right] dy \\
&= \frac{128}{5},
\end{aligned}
$$

where we have left the details of the last integration to you. ■

NOTES

To integrate with respect to z first, you must identify surfaces forming the top and bottom of the solid. To integrate with respect to y first, you must identify surfaces forming (from the standard viewpoint) the right and left sides of the solid. To integrate with respect to x first, you must identify surfaces forming the front and back of the solid. Often, the easiest pair of surfaces to identify will indicate the best choice of variable for the innermost integration.

Mass and Center of Mass

In section 13.2, we discussed finding the mass and center of mass of a lamina (a thin, flat plate). We now pause briefly to extend these results to three dimensions. Suppose that a solid Q has mass density given by $\rho(x, y, z)$ (in units of mass per unit volume). To find the total mass of a solid, we proceed (as we did for laminas) by constructing an inner partition of the solid: $Q_1, Q_2, \ldots, Q_n$. Realize that if each box Q_i is small (see Figure 13.47 on the following page), then the density should be nearly constant on Q_i and so, it is reasonable to approximate the mass m_i of Q_i by

$$m_i \approx \underbrace{\rho(u_i, v_i, w_i)}_{\text{mass/unit volume}} \underbrace{\Delta V_i}_{\text{volume}},$$

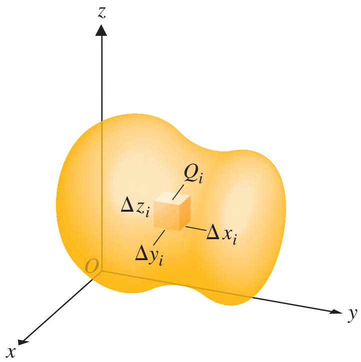


FIGURE 13.47
One box Q_i of the inner partition of Q

for any point $(u_i, v_i, w_i) \in Q_i$, where ΔV_i is the volume of Q_i. The total mass m of Q is then given approximately by

$$m \approx \sum_{i=1}^{n} \rho(u_i, v_i, w_i)\,\Delta V_i.$$

Letting the norm of the partition $\|P\|$ approach zero, we get the exact mass, which we recognize as a triple integral:

$$m = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \rho\,(u_i, v_i, w_i)\,\Delta V_i = \iiint_Q \rho\,(x, y, z)\,dV. \tag{5.6}$$

Now, recall that the center of mass of a lamina was the point at which the lamina will balance. For an object in three dimensions, you can think of this as balancing it left to right (i.e., along the y-axis), front to back (i.e., along the x-axis) and top to bottom (i.e., along the z-axis). To do this, we need to find first moments with respect to each of the three coordinate planes. We define these moments as

$$M_{yz} = \iiint_Q x\rho(x, y, z)\,dV, \quad M_{xz} = \iiint_Q y\rho(x, y, z)\,dV \tag{5.7}$$

and

$$M_{xy} = \iiint_Q z\rho(x, y, z)\,dV, \tag{5.8}$$

the **first moments** with respect to the yz-plane, the xz-plane and the xy-plane, respectively. The **center of mass** is then given by the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}. \tag{5.9}$$

Notice that these are straightforward generalizations of the corresponding formulas for the center of mass of a lamina.

EXAMPLE 5.6 Center of Mass of a Solid

Find the center of mass of the solid of constant mass density ρ bounded by the graphs of the right circular cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 4$ (see Figure 13.48a).

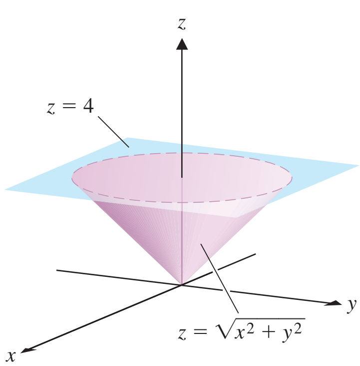


FIGURE 13.48a
The solid Q

Solution Notice that the projection R of the solid onto the xy-plane is the disk of radius 4 centered at the origin (see Figure 13.48b). Further, for each $(x, y) \in R$, z ranges from the cone $(z = \sqrt{x^2 + y^2})$ up to the plane $z = 4$. From (5.6), the total mass of the solid is given by

$$\begin{aligned} m &= \iiint_Q \rho\,(x, y, z)\,dV = \rho \iint_R \int_{\sqrt{x^2+y^2}}^{4} dz\,dA \\ &= \rho \iint_R \left(4 - \sqrt{x^2 + y^2}\right) dA, \end{aligned}$$

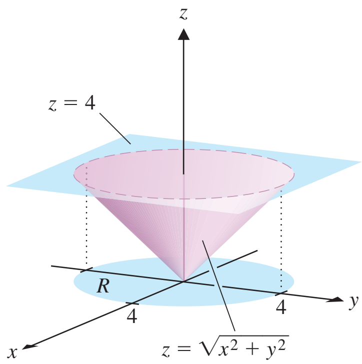


FIGURE 13.48b
Projection of the solid onto the xy-plane

where R is the disk of radius 4 in the xy-plane, centered at the origin, as indicated in Figure 13.48b. Since the region R is circular and since the integrand contains a term of the form $\sqrt{x^2+y^2}$, we use polar coordinates for the remaining double integral. We have

$$\begin{aligned} m &= \rho \iint_R \Big(4 - \underbrace{\sqrt{x^2+y^2}}_{r}\Big) \underbrace{dA}_{r\,dr\,d\theta} \\ &= \rho \int_0^{2\pi} \int_0^4 (4-r)\,r\,dr\,d\theta \\ &= \rho \int_0^{2\pi} \left(4\frac{r^2}{2} - \frac{r^3}{3}\right)\Bigg|_{r=0}^{r=4} d\theta \\ &= \rho\left(32 - \frac{4^3}{3}\right)(2\pi) = \frac{64}{3}\pi\rho. \end{aligned}$$

From (5.8), we get that the moment with respect to the xy-plane is

$$\begin{aligned} M_{xy} &= \iiint_Q z\,\rho(x,y,z)\,dV = \rho \iint_R \int_{\sqrt{x^2+y^2}}^4 z\,dz\,dA \\ &= \rho \iint_R \frac{z^2}{2}\Bigg|_{\sqrt{x^2+y^2}}^4 dA \\ &= \frac{\rho}{2}\iint_R [16 - (x^2+y^2)]\,dA. \end{aligned}$$

For the same reasons as when we computed the mass, we change to polar coordinates in the double integral to get

$$\begin{aligned} M_{xy} &= \frac{\rho}{2}\iint_R \Big[16 - \underbrace{(x^2+y^2)}_{r^2}\Big] \underbrace{dA}_{r\,dr\,d\theta} \\ &= \frac{\rho}{2}\int_0^{2\pi}\int_0^4 (16-r^2)\,r\,dr\,d\theta \\ &= \frac{\rho}{2}\int_0^{2\pi}\left(16\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=4} d\theta \\ &= 32\rho(2\pi) = 64\pi\rho. \end{aligned}$$

Notice that the solid is symmetric with respect to both the xz-plane and the yz-plane and so, the moments with respect to both of those planes are zero, since the density is constant. (Why does constant density matter?) That is, $M_{xz} = M_{yz} = 0$. From (5.9), the center of mass is then given by

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(0, 0, \frac{64\pi\rho}{64\pi\rho/3}\right) = (0, 0, 3).$$

EXERCISES 13.5

WRITING EXERCISES

1. Discuss the importance of having a reasonably accurate sketch to help determine the limits (and order) of integration. Identify which features of a sketch are essential and which are not. Discuss whether it's important for your sketch to distinguish between two surfaces like $z = 4 - x^2 - y^2$ and $z = \sqrt{4 - x^2 - y^2}$.
2. In example 5.2, explain why all six orders of integration are equally simple. Given this choice, most people prefer to integrate in the order of example 5.2 ($dz\,dy\,dx$). Discuss the visual advantages of this order.
3. In example 5.4, identify any clues in the problem statement that might indicate that y should be the innermost variable of integration. In example 5.5, identify any clues that might indicate that z should *not* be the innermost variable of integration. (Hint: With how many surfaces is each variable associated?)
4. In example 5.6, we used polar coordinates in x and y. Explain why this is permissible and when it is likely to be convenient to do so.

In exercises 1–14, evaluate the triple integral $\iiint\limits_Q f(x, y, z)\,dV$.

1. $f(x, y, z) = 2x + y - z$, $Q = \{(x, y, z) \mid 0 \le x \le 2, -2 \le y \le 2, 0 \le z \le 2\}$
2. $f(x, y, z) = 2x^2 + y^3$, $Q = \{(x, y, z) \mid 0 \le x \le 3, -2 \le y \le 1, 1 \le z \le 2\}$
3. $f(x, y, z) = \sqrt{y} - 3z^2$, $Q = \{(x, y, z) \mid 2 \le x \le 3, 0 \le y \le 1, -1 \le z \le 1\}$
4. $f(x, y, z) = 2xy - 3xz^2$, $Q = \{(x, y, z) \mid 0 \le x \le 2, -1 \le y \le 1, 0 \le z \le 2\}$
5. $f(x, y, z) = 4yz$, Q is the tetrahedron bounded by $x + 2y + z = 2$ and the coordinate planes
6. $f(x, y, z) = 3x - 2y$, Q is the tetrahedron bounded by $4x + y + 3z = 12$ and the coordinate planes
7. $f(x, y, z) = 3y^2 - 2z$, Q is the tetrahedron bounded by $3x + 2y - z = 6$ and the coordinate planes
8. $f(x, y, z) = 6xz^2$, Q is the tetrahedron bounded by $-2x + y + z = 4$ and the coordinate planes
9. $f(x, y, z) = 2xy$, Q is bounded by $z = 1 - x^2 - y^2$ and $z = 0$
10. $f(x, y, z) = x - y$, Q is bounded by $z = x^2 + y^2$ and $z = 4$
11. $f(x, y, z) = 2yz$, Q is bounded by $z + x = 2$, $z - x = 2$, $z = 1$, $y = -2$ and $y = 2$
12. $f(x, y, z) = x^3 y$, Q is bounded by $z = 1 - y^2$, $z = 0$, $x = -1$ and $x = 1$
13. $f(x, y, z) = 15$, Q is bounded by $2x + y + z = 4$, $z = 0$, $x = 1 - y^2$ and $x = 0$
14. $f(x, y, z) = 2x + y$, Q is bounded by $z = 6 - x - y$, $z = 0$, $y = 2 - x$, $y = 0$ and $x = 0$
15. Sketch the region Q in exercise 9 and explain why the triple integral equals 0. Would the integral equal 0 for $f(x, y, z) = 2x^2 y$? For $f(x, y, z) = 2x^2 y^2$?
16. Show that $\iiint\limits_Q (z - x)\,dV = 0$, where Q is bounded by $z = 6 - x - y$ and the coordinate planes. Explain geometrically why this is correct.

In exercises 17–28, compute the volume of the solid bounded by the given surfaces.

17. $z = x^2$, $z = 1$, $y = 0$ and $y = 2$
18. $z = 1 - y^2$, $z = 0$, $x = 2$ and $x = 4$
19. $z = 1 - y^2$, $z = 0$, $z = 4 - 2x$ and $x = 4$
20. $z = x^2$, $z = x + 2$, $y + z = 5$ and $y = -1$
21. $y = 4 - x^2$, $z = 0$ and $z - y = 6$
22. $x = y^2$, $x = 4$, $x + z = 6$ and $x + z = 8$
23. $y = 3 - x$, $y = 0$, $z = x^2$ and $z = 1$
24. $x = y^2$, $x = 4$, $z = 2 + x$ and $z = 0$
25. $z = 1 + x$, $z = 1 - x$, $z = 1 + y$, $z = 1 - y$ and $z = 0$ (a pyramid)
26. $z = 5 - y^2$, $z = 6 - x$, $z = 6 + x$ and $z = 1$
27. $z = 4 - x^2 - y^2$ and the xy-plane
28. $z = 6 - x - y$, $x^2 + y^2 = 1$ and $z = -1$

In exercises 29–32, find the mass and center of mass of the solid with density $\rho(x, y, z)$ and the given shape.

29. $\rho(x, y, z) = 4$, solid bounded by $z = x^2 + y^2$ and $z = 4$
30. $\rho(x, y, z) = 2 + x$, solid bounded by $z = x^2 + y^2$ and $z = 4$
31. $\rho(x, y, z) = 10 + x$, tetrahedron bounded by $x + 3y + z = 6$ and the coordinate planes
32. $\rho(x, y, z) = 1 + x$, tetrahedron bound by $2x + y + 4z = 4$ and the coordinate planes
33. Explain why the x-coordinate of the center of mass in exercise 29 is zero, but the x-coordinate in exercise 30 is not zero.

34. In exercise 29, if $\rho(x, y, z) = 2 + x^2$, is the x-coordinate of the center of mass zero? Explain.

35. In exercise 5, evaluate the integral in three different ways, using each variable as the innermost variable once.

36. In exercise 6, evaluate the integral in three different ways, using each variable as the innermost variable once.

In exercises 37–42, sketch the solid whose volume is given and rewrite the iterated integral using a different innermost variable.

37. $\int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} dx\,dz\,dy$

38. $\int_0^1 \int_0^{2-2y} \int_0^{2-x-2y} dz\,dx\,dy$

39. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

40. $\int_0^1 \int_0^{1-x^2} \int_0^{2-x} dy\,dz\,dx$

41. $\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} dy\,dx\,dz$

42. $\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{\sqrt{y^2+z^2}}^{2} dx\,dy\,dz$

43. Suppose that the density of an airborne pollutant in a room is given by $f(x, y, z) = xyze^{-x^2-2y^2-4z^2}$ grams per cubic foot for $0 \le x \le 12$, $0 \le y \le 12$ and $0 \le z \le 8$. Find the total amount of pollutant in the room. Divide by the volume of the room to get the average density of pollutant in the room.

44. If the danger level for the pollutant in exercise 43 is 1 gram per 1000 cubic feet, show that the room on the whole is below the danger level, but there is a portion of the room that is well above the danger level.

Exercises 45–48 involve probability.

45. A function $f(x, y, z)$ is a pdf on the three-dimensional region Q if $f(x, y, z) \ge 0$ for all (x, y, z) in Q and $\iiint_Q f(x, y, z)\,dV = 1$. Find c such that $f(x, y, z) = c$ is a pdf on the tetrahedron bounded by $x + 2y + z = 2$ and the coordinate planes.

46. If a point is chosen at random from the tetrahedron in exercise 45, find the probability that $z < 1$.

47. Find the value of k such that the probability that $z < k$ in exercise 45 equals $\frac{1}{2}$.

48. Compare your answer to exercise 47 to the z-coordinate of the center of mass of the tetrahedron Q with constant density.

49. Write $\int_a^b \int_c^d \int_r^s f(x)g(y)h(z)\,dz\,dy\,dx$ as a product of three single integrals. In general, can any triple integral with integrand $f(x)g(y)h(z)$ be factored as the product of three single integrals?

50. Compute $\iiint_Q f(x, y, z)\,dV$, where Q is the tetrahedron bounded by $2x + y + 3z = 6$ and the coordinate planes, and $f(x, y, z) = \max\{x, y, z\}$.

51. Let T be the tetrahedron in the first octant with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, for positive constants a, b and c. Let C be the parallelepiped in the first octant with the same vertices. Show that the volume of T is one-sixth the volume of C.

EXPLORATORY EXERCISES

1. In this exercise, you will examine several models of baseball bats. Sketch the region extending from $y = 0$ to $y = 32$ with distance from the y-axis given by $r = \frac{1}{2} + \frac{3}{128}y$. This should look vaguely like a baseball bat, with 32 representing the 32-inch length of a typical bat. Assume a constant **weight density** of $\rho = 0.39$ ounce per cubic inch. Compute the weight of the bat and the center of mass of the bat. (Hint: Compute the y-coordinate and argue that the x- and z-coordinates are zero.) Sketch each of the following regions, explain what the name means and compute the mass and center of mass. (a) **Long bat:** same as the original except y extends from $y = 0$ to $y = 34$. (b) **Choked up:** y goes from -2 to 30 with $r = \frac{35}{64} + \frac{3}{128}y$. (c) **Corked bat:** same as the original with the cylinder $26 \le y \le 32$ and $0 \le r \le \frac{1}{4}$ removed. (d) **Aluminum bat:** same as the original with the section from $r = 0$ to $r = \frac{3}{8} + \frac{3}{128}y$, $0 \le y \le 32$ removed and density $\rho = 1.56$. Explain why it makes sense that the choked-up bat has the center of mass 2 inches to the left of the original bat. Part of the "folklore" of baseball is that batters with aluminum bats can hit "inside" pitches better than batters with traditional wood bats. If "inside" means smaller values of y and the center of mass represents the "sweet spot" of the bat (the best place to hit the ball), discuss whether your calculations support baseball's folk wisdom.

2. In this exercise, we continue with the baseball bats of exercise 1. This time, we want to compute the moment of inertia $\iiint_Q y^2\rho\,dV$ for each of the bats. The smaller the moment of inertia is, the easier it is to swing the bat. Use your calculations to answer the following questions. How much harder is it to swing a slightly longer bat? How much easier is it to swing a bat that has been choked up 2 inches? Does corking really make a noticeable difference in the ease with which a bat can be swung? How much easier is it to swing a hollow aluminum bat, even if it weighs the same as a regular bat?

13.6 CYLINDRICAL COORDINATES

In example 5.6 of section 13.5, we found it convenient to introduce polar coordinates in order to evaluate the outer double integral in a triple integral problem. Sometimes, this is more than a mere convenience, as we see in example 6.1.

EXAMPLE 6.1 A Triple Integral Requiring Polar Coordinates

Evaluate $\iiint_Q e^{x^2+y^2}\,dV$, where Q is the solid bounded by the cylinder $x^2+y^2=9$, the xy-plane and the plane $z=5$.

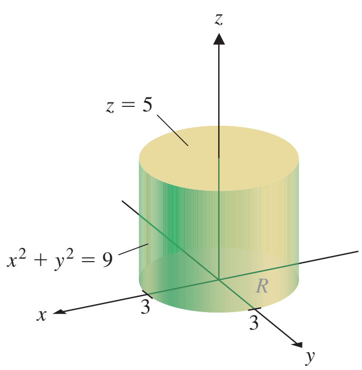


FIGURE 13.49a
The solid Q

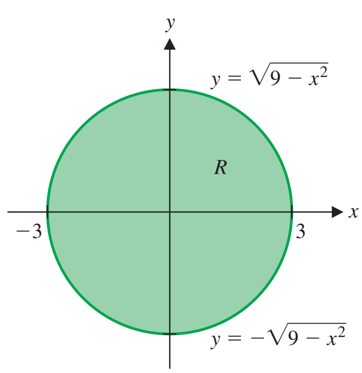


FIGURE 13.49b
The region R

Solution We show a sketch of the solid in Figure 13.49a. This might seem simple enough; certainly the solid is not particularly complicated. Unfortunately, the integral is rather troublesome. Notice that the base of the solid is the circle R of radius 3 centered at the origin and lying in the xy-plane. Further, for each (x, y) in R, z ranges from 0 up to 5. So, we have

$$\iiint_Q e^{x^2+y^2}\,dV = \iint_R \int_0^5 e^{x^2+y^2}\,dz\,dA$$
$$= 5\iint_R e^{x^2+y^2}\,dA.$$

The challenge lies in evaluating the remaining double integral. From Figure 13.49b, observe that for each fixed $x \in [-3, 3]$, y ranges from $-\sqrt{9-x^2}$ (the bottom semicircle) up to $\sqrt{9-x^2}$ (the top semicircle). We now have

$$\iiint_Q e^{x^2+y^2}\,dV = 5\iint_R e^{x^2+y^2}\,dA = 5\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} e^{x^2+y^2}\,dy\,dx.$$

Without polar coordinates, we're at a dead end, since we don't know an antiderivative for $e^{x^2+y^2}$. Even the authors' computer algebra system has difficulty with this, giving a nearly indecipherable answer in terms of an integral of the *error* function, which you have likely never seen before. Even so, our computer algebra system could not handle the second integration, except approximately. On the other hand, if we introduce polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$, we get that for each $\theta \in [0, 2\pi]$, r ranges from 0 up to 3. We now have an integral requiring only a simple substitution:

$$\iiint_Q e^{x^2+y^2}\,dV = 5\iint_R \underbrace{e^{x^2+y^2}}_{e^{r^2}}\,\underbrace{dA}_{r\,dr\,d\theta}$$
$$= 5\int_0^{2\pi}\int_0^3 e^{r^2}\,r\,dr\,d\theta$$
$$= \frac{5}{2}\int_0^{2\pi} e^{r^2}\Big|_{r=0}^{r=3}\,d\theta$$
$$= 5\pi(e^9 - 1)$$
$$\approx 1.27 \times 10^5,$$

which is a much more acceptable answer. ■

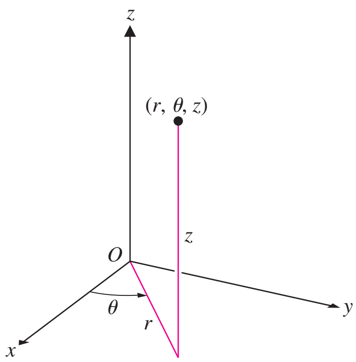


FIGURE 13.50
Cylindrical coordinates

The process of replacing two of the variables in a three-dimensional coordinate system by polar coordinates, as we illustrated in example 6.1, is so common that we give this coordinate system a name: cylindrical coordinates.

To be precise, we specify a point $P(x, y, z) \in \mathbb{R}^3$ by identifying polar coordinates for the point $(x, y) \in \mathbb{R}^2$: $x = r\cos\theta$ and $y = r\sin\theta$, where $r^2 = x^2 + y^2$ and θ is the angle made by the line segment connecting the origin and the point $(x, y, 0)$ with the positive x-axis, as indicated in Figure 13.50. Then, $\tan\theta = \frac{y}{x}$. We refer to (r, θ, z) as **cylindrical coordinates** for the point P.

EXAMPLE 6.2 Equation of a Cylinder in Cylindrical Coordinates

Write the equation for the cylinder $x^2 + y^2 = 16$ (see Figure 13.51) in cylindrical coordinates.

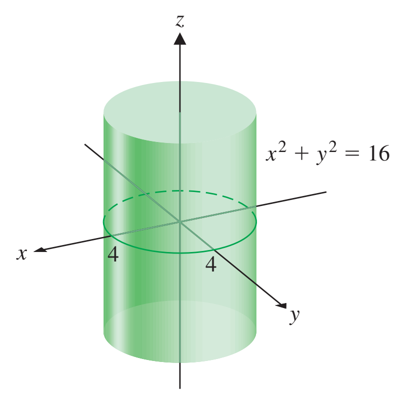


FIGURE 13.51
The cylinder $r = 4$

Solution In cylindrical coordinates $r^2 = x^2 + y^2$, so the cylinder becomes $r^2 = 16$ or $r = \pm 4$. But note that since θ is not specified, the equation $r = 4$ describes the same cylinder. ■

EXAMPLE 6.3 Equation of a Cone in Cylindrical Coordinates

Write the equation for the cone $z^2 = x^2 + y^2$ (see Figure 13.52) in cylindrical coordinates.

Solution Since $x^2 + y^2 = r^2$, the cone becomes $z^2 = r^2$ or $z = \pm r$. In cases where we need r to be positive, we write separate equations for the top cone $z = r$ and the bottom cone $z = -r$. ■

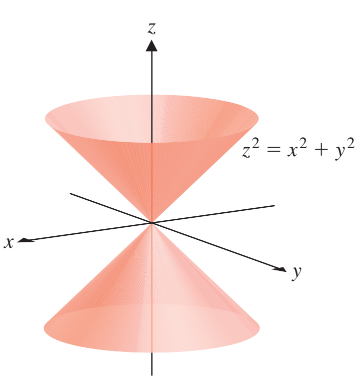


FIGURE 13.52
The cone $z = r$

As we did on a case-by-case basis in several examples in section 13.5 and in example 6.1, we can use cylindrical coordinates to simplify certain triple integrals. For instance, suppose that we can write the solid Q as

$$Q = \{(r, \theta, z) \,|\, (r, \theta) \in R \text{ and } k_1(r, \theta) \le z \le k_2(r, \theta)\},$$

where $k_1(r, \theta) \le k_2(r, \theta)$, for all (r, θ) in R and R is the region of the xy-plane defined by

$$R = \{(r, \theta) | \alpha \le \theta \le \beta \text{ and } g_1(\theta) \le r \le g_2(\theta)\},$$

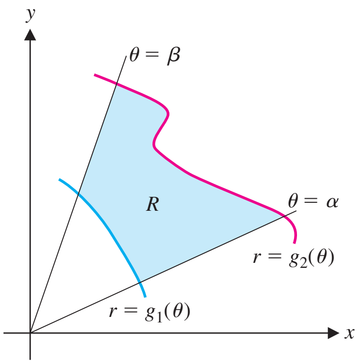


FIGURE 13.53
The region R

where $0 \le g_1(\theta) \le g_2(\theta)$, for all θ in $[\alpha, \beta]$, as shown in Figure 13.53. Then, notice that from (5.4), we can write

$$\iiint_Q f(r,\theta,z)\,dV = \iint_R \left[\int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\,dz\right] dA.$$

Since the outer double integral is a double integral in polar coordinates, we already know how to write it as an iterated integral. We have

$$\iiint_Q f(r,\theta,z)\,dV = \iint_R \left[\int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\,dz\right] \underbrace{dA}_{r\,dr\,d\theta}$$

$$= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \left[\int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\,dz\right] r\,dr\,d\theta.$$

This gives us an evaluation formula for triple integrals in cylindrical coordinates:

$$\boxed{\iiint_Q f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta.} \qquad (6.1)$$

In setting up triple integrals in cylindrical coordinates, it often helps to visualize the volume element $dV = r\,dz\,dr\,d\theta$. (See Figure 13.54.)

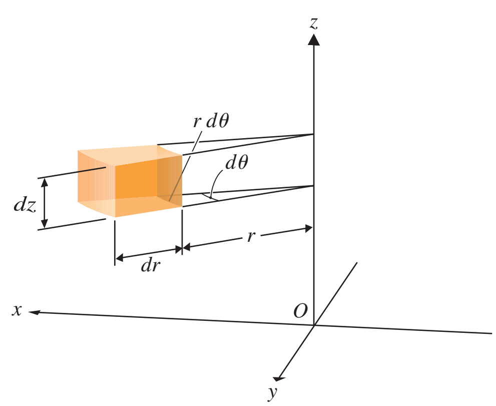


FIGURE 13.54
Volume element for cylindrical coordinates

EXAMPLE 6.4 A Triple Integral in Cylindrical Coordinates

Write $\iiint_Q f(r,\theta,z)\,dV$ as a triple iterated integral in cylindrical coordinates if

$$Q = \left\{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}\right\}.$$

Solution The first task in setting up any iterated multiple integral is to draw a sketch of the region over which you are integrating. Here, $z = \sqrt{x^2+y^2}$ is the top half of a right circular cone, with vertex at the origin and axis lying along the z-axis, and $z = \sqrt{18 - x^2 - y^2}$ is the top hemisphere of radius $\sqrt{18}$ centered at the origin. So, we are looking for the set of all points lying above the cone and below the hemisphere. (See Figure 13.55a.) Recognize that in cylindrical coordinates, the cone is written $z = r$ and the hemisphere becomes $z = \sqrt{18 - r^2}$, since $x^2 + y^2 = r^2$. This says that for each r and θ, z ranges from r up to $\sqrt{18 - r^2}$. Notice that the cone and the hemisphere intersect when

$$\sqrt{18 - r^2} = r$$

or

$$18 - r^2 = r^2,$$

so that

$$18 = 2r^2 \quad \text{or} \quad r = 3.$$

That is, the two surfaces intersect in a circle of radius 3 lying in the plane $z = 3$. The projection of the solid down onto the xy-plane is then the circle of radius 3 centered at the origin (see Figure 13.55b) and we have

$$\iiint_Q f(r, \theta, z)\, dV = \int_0^{2\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} f(r, \theta, z)\, r\, dz\, dr\, d\theta.$$

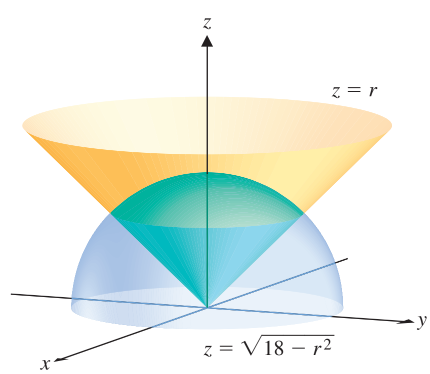


FIGURE 13.55a
The solid Q

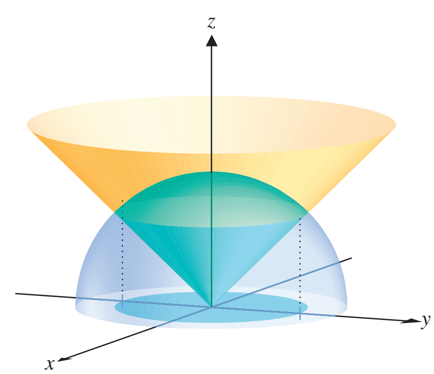


FIGURE 13.55b
Projection of Q onto the xy-plane

Very often, a triple integral in rectangular coordinates would be simpler in cylindrical coordinates. You must then recognize how to write the solid in cylindrical coordinates, as well as how to rewrite the integral.

EXAMPLE 6.5 Changing from Rectangular to Cylindrical Coordinates

Evaluate the triple iterated integral $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2}\, dz\, dy\, dx.$

Solution As written, the integral is virtually impossible to evaluate exactly. (Even our computer algebra system had trouble with it.) Notice that the integrand involves $x^2 + y^2$, which is simply r^2 in cylindrical coordinates. You should also try to visualize the region over which you are integrating. First, from the innermost limits of integration, notice that $z = 2 - x^2 - y^2$ is a paraboloid opening downward, with vertex at the point $(0, 0, 2)$ and $z = x^2 + y^2$ is a paraboloid opening upward with vertex at the origin. So, the solid is some portion of the solid bounded by the two paraboloids. The two paraboloids intersect when

$$2 - x^2 - y^2 = x^2 + y^2$$

or

$$1 = x^2 + y^2.$$

So, the intersection forms a circle of radius 1 lying in the plane $z = 1$ and centered at the point $(0, 0, 1)$. Looking at the outer two integrals, note that for each $x \in [-1, 1]$, y ranges from $-\sqrt{1 - x^2}$ (the bottom semicircle of radius 1 centered at the origin) to $\sqrt{1 - x^2}$ (the top semicircle of radius 1 centered at the origin). Since this corresponds to the projection of the circle of intersection onto the xy-plane, the triple integral is over the entire solid below the one paraboloid and above the other. (See Figure 13.56.)

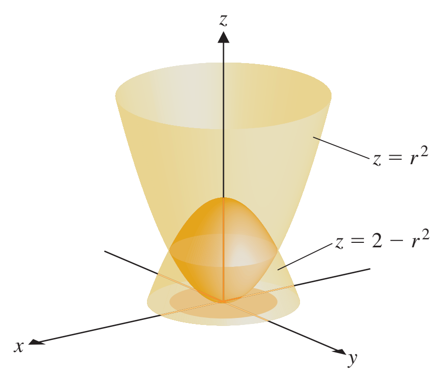


FIGURE 13.56
The solid Q

In cylindrical coordinates, the top paraboloid becomes $z = 2 - x^2 - y^2 = 2 - r^2$ and the bottom paraboloid becomes $z = x^2 + y^2 = r^2$. So, for each fixed value of r and θ, z varies from r^2 up to $2 - r^2$. Further, since the projection of the solid onto the xy-plane is the circle of radius 1 centered at the origin, r varies from 0 to 1 and θ varies from 0 to 2π. We can now write the triple integral in cylindrical coordinates as

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{3/2}\,dz\,dy\,dx &= \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} (r^2)^{3/2}\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^4\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 r^4(2-2r^2)\,dr\,d\theta \\ &= 2\int_0^{2\pi}\left(\frac{r^5}{5}-\frac{r^7}{7}\right)\Bigg|_{r=0}^{r=1} d\theta = \frac{8\pi}{35}.\end{aligned}$$

Evaluating the triple integral in cylindrical coordinates was easy, compared to evaluating the original integral directly. ■

When converting an iterated integral from rectangular to cylindrical coordinates, it's important to carefully visualize the solid over which you are integrating. While we have so far defined cylindrical coordinates by replacing x and y by their polar coordinate representations, we can do this with any two of the three variables, as we see in example 6.6.

TODAY IN MATHEMATICS

Enrico Bombieri (1940–)
An Italian mathematician who proved what is now known as Bombieri's Mean Value Theorem on the distribution of prime numbers. Bombieri is known as a solver of "deep" problems, which are fundamental questions that baffle the world's best mathematicians. This is a risky enterprise, because many such problems are unsolvable. One of Bombieri's special talents is a seemingly instinctual sense of which problems are both solvable and interesting. To see why this is a true talent, think about how hard it would be to write double and triple integrals that are at the same time interesting, challenging and solvable (antiderivatives found and evaluated) and then imagine performing an equivalent task for problems that nobody has ever solved.

EXAMPLE 6.6 Using a Triple Integral to Find Volume

Use a triple integral to find the volume of the solid Q bounded by the graph of $y = 4 - x^2 - z^2$ and the xz-plane.

Solution Notice that the graph of $y = 4 - x^2 - z^2$ is a paraboloid with vertex at $(0, 4, 0)$, whose axis is the y-axis and that opens toward the negative y-axis. We show the solid in Figure 13.57a. Without thinking too much about it, we might consider integration with respect to z first. In this case, the projection of the solid onto the xy-plane is the parabola formed by the intersection of the paraboloid with the xy-plane (see Figure 13.57b). Notice from Figure 13.57b that for each fixed x and y, the line through the point $(x, y, 0)$ and perpendicular to the xy-plane enters the solid on the bottom half of the paraboloid ($z = -\sqrt{4 - x^2 - y}$) and exits the solid on the top surface of the paraboloid ($z = \sqrt{4 - x^2 - y}$). This gives you the innermost limits of integration. We get the rest from looking at the projection of the paraboloid onto the xy-plane. (See Figure 13.57c.)

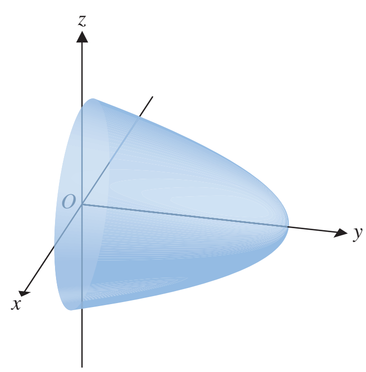


FIGURE 13.57a
The solid Q

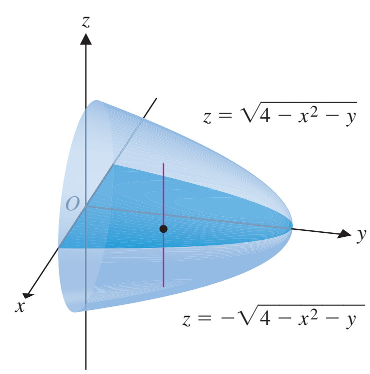


FIGURE 13.57b
Solid, showing projection onto xy-plane

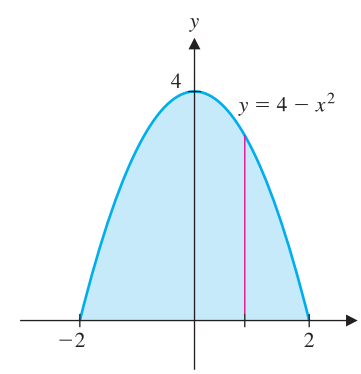


FIGURE 13.57c
Projection onto the xy-plane

Reading the outer limits of integration from Figure 13.57c and using (5.5), we get

$$\begin{aligned} V &= \iiint_Q dV = \int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}} dz\,dy\,dx \\ &= \int_{-2}^{2}\int_{0}^{4-x^2} z\Big|_{z=-\sqrt{4-x^2-y}}^{z=\sqrt{4-x^2-y}} dy\,dx \\ &= \int_{-2}^{2}\int_{0}^{4-x^2} 2\sqrt{4-x^2-y}\,dy\,dx \\ &= \int_{-2}^{2}(-2)\left(\frac{2}{3}\right)(4-x^2-y)^{3/2}\Big|_{y=0}^{y=4-x^2} dx \\ &= \frac{4}{3}\int_{-2}^{2}(4-x^2)^{3/2}dx = 8\pi. \end{aligned}$$

Notice that the last integration here is challenging. (We used a CAS to carry it out.) Alternatively, if you look at Figure 13.57a and turn your head to the side, you should see a paraboloid with a circular base in the xz-plane. This suggests that you might want

NOTES

Example 6.6 suggests that polar coordinates may be used with any two of x, y and z, to produce a cylindrical coordinate representation of a solid.

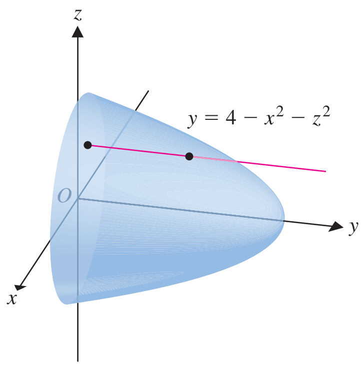


FIGURE 13.57d
Paraboloid with base in the xz-plane

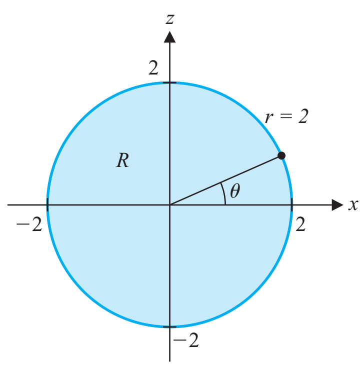


FIGURE 13.57e
Base of the solid

to integrate first with respect to y. Referring to Figure 13.57d, observe that for each point in the base of the solid in the xz-plane, y ranges from 0 to $4 - x^2 - z^2$. Notice that the base in this case is formed by the intersection of the paraboloid with the xz-plane $(y = 0)$: $0 = 4 - x^2 - z^2$ or $x^2 + z^2 = 4$ (i.e., the circle of radius 2 centered at the origin; see Figure 13.57e).

We can now write the volume as

$$V = \iiint_Q dV = \iint_R \int_0^{4-x^2-z^2} dy\, dA$$

$$= \iint_R (4 - x^2 - z^2)\, dA,$$

where R is the disk indicated in Figure 13.57e. Since the region R is a circle and the integrand contains the combination of variables $x^2 + z^2$, we define polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$. From Figure 13.57e, notice that for each fixed angle $\theta \in [0, 2\pi]$, r runs from 0 to 2. This gives us

$$V = \iint_R \underbrace{(4 - x^2 - z^2)}_{4-r^2} \underbrace{dA}_{r\,dr\,d\theta}$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2)\, r\, dr\, d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \frac{(4 - r^2)^2}{2} \bigg|_{r=0}^{r=2} d\theta$$

$$= 4 \int_0^{2\pi} d\theta = 8\pi.$$

Notice that in some sense, viewing the solid as having its base in the xz-plane is more natural than our first approach to the problem and the integrations are much simpler. ■

EXERCISES 13.6

WRITING EXERCISES

1. Using the examples in this section as a guide, make a short list of figures that are easily described in cylindrical coordinates.
2. The three-dimensional solid bounded by $x = a, x = b$, $y = c, y = d, z = e$ and $z = f$ is a rectangular box. Given this, speculate on how **cylindrical coordinates** got its name. Specifically, identify what type of cylinder is the basic figure of cylindrical coordinates.
3. In example 6.4, explain why the outer double integration limits are determined by the intersection of the cone and the hemisphere (and not, for example, by the trace of the hemisphere in the xy-plane).
4. Carefully examine the rectangular and cylindrical limits of integration in example 6.5. Note that in both integrals, z is the innermost variable of integration. In this case, explain why the innermost limits of integration for the triple integral in cylindrical coordinates can be obtained by substituting polar coordinates into the innermost limits of integration of the triple integral in rectangular coordinates. Explain why this would not be the case if the order of integration had been changed.

In exercises 1–8, write the given equation in cylindrical coordinates.

1. $x^2 + y^2 = 16$
2. $x^2 + y^2 = 1$
3. $(x - 2)^2 + y^2 = 4$
4. $x^2 + (y - 3)^2 = 9$
5. $z = x^2 + y^2$
6. $z = \sqrt{x^2 + y^2}$
7. $y = 2x$
8. $z = e^{-x^2-y^2}$

In exercises 9–20, set up the triple integral $\iiint_Q f(x,y,z)\,dV$ in cylindrical coordinates.

9. Q is the region above $z = \sqrt{x^2+y^2}$ and below $z = \sqrt{8-x^2-y^2}$.
10. Q is the region above $z = -\sqrt{x^2+y^2}$ and inside $x^2+y^2=4$.
11. Q is the region above the xy-plane and below $z = 9-x^2-y^2$.
12. Q is the region above the xy-plane and below $z = 4-x^2-y^2$ in the first octant.
13. Q is the region above $z = x^2+y^2-1$, below $z=8$ and between $x^2+y^2=3$ and $x^2+y^2=8$.
14. Q is the region above $z = x^2+y^2-4$ and below $z = -x^2-y^2$.
15. Q is the region bounded by $y = 4-x^2-z^2$ and $y=0$.
16. Q is the region bounded by $y = \sqrt{x^2+z^2}$ and $y=9$.
17. Q is the region bounded by $x = y^2+z^2$ and $x = 2-y^2-z^2$.
18. Q is the region bounded by $x = \sqrt{y^2+z^2}$ and $x=4$.
19. Q is the frustum of a cone bounded by $z=2$, $z=3$ and $z = \sqrt{x^2+y^2}$.
20. Q is the region bounded by $z=1$ and $z=3$ and under $z = 4-x^2-y^2$.

In exercises 21–32, set up and evaluate the indicated triple integral in the appropriate coordinate system.

21. $\iiint_Q e^{x^2+y^2}\,dV$, where Q is the region inside $x^2+y^2=4$ and between $z=1$ and $z=2$.
22. $\iiint_Q ze^{\sqrt{x^2+y^2}}\,dV$, where Q is the region inside $x^2+y^2=4$, outside $x^2+y^2=1$ and between $z=0$ and $z=3$.
23. $\iiint_Q (x+z)\,dV$, where Q is the region below $x+2y+3z=6$ in the first octant.
24. $\iiint_Q (y+2)\,dV$, where Q is the region below $x+z=4$ in the first octant between $y=1$ and $y=2$.
25. $\iiint_Q z\,dV$, where Q is the region between $z = \sqrt{x^2+y^2}$ and $z = \sqrt{4-x^2-y^2}$.
26. $\iiint_Q \sqrt{x^2+y^2}\,dV$, where Q is the region between $z = \sqrt{x^2+y^2}$ and $z=0$ and inside $x^2+y^2=4$.
27. $\iiint_Q (x+y)\,dV$, where Q is the tetrahedron bounded by $x+2y+z=4$ and the coordinate planes.
28. $\iiint_Q (2x-y)\,dV$, where Q is the tetrahedron bounded by $3x+y+2z=6$ and the coordinate planes.
29. $\iiint_Q e^z\,dV$, where Q is the region above $z = -\sqrt{4-x^2-y^2}$, below the xy-plane and outside $x^2+y^2=3$.
30. $\iiint_Q \sqrt{x^2+y^2}\,e^z\,dV$, where Q is the region inside $x^2+y^2=1$ and between $z = (x^2+y^2)^{3/2}$ and $z=0$.
31. $\iiint_Q 2x\,dV$, where Q is the region between $z = \sqrt{x^2+y^2}$ and $z=0$ and inside $x^2+(y-1)^2=1$.
32. $\iiint_Q y\,dV$, where Q is the region between $z = x^2+y^2$ and $z=0$ and inside $(x-2)^2+y^2=4$.

In exercises 33–38, evaluate the iterated integral after changing coordinate systems.

33. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} 3z^2\,dz\,dy\,dx$
34. $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{2-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$
35. $\displaystyle\int_{0}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} 2\,dz\,dx\,dy$
36. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{1-x^2-y^2}^{4} \sqrt{x^2+y^2}\,dz\,dy\,dx$
37. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_0^{x^2+z^2} (x^2+z^2)\,dy\,dz\,dx$
38. $\displaystyle\int_{-2}^{0}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{y^2+z^2}^{4} (y^2+z^2)^{3/2}\,dx\,dy\,dz$

In exercises 39–46, sketch graphs of the cylindrical equations.

39. $z = r$
40. $z = r^2$
41. $z = 4-r^2$
42. $z = \sqrt{4-r^2}$
43. $r = 2\sec\theta$
44. $r = 2\sin\theta$
45. $\theta = \pi/4$
46. $r = 4$

In exercises 47–50, find the mass and center of mass of the solid with the given density and bounded by the graphs of the indicated equations.

47. $\rho(x,y,z) = \sqrt{x^2+y^2}$, bounded by $z = \sqrt{x^2+y^2}$ and $z=4$.
48. $\rho(x,y,z) = e^{-x^2-y^2}$, bounded by $z = \sqrt{4-x^2-y^2}$ and the xy-plane.
49. $\rho(x,y,z) = 4$, between $z = x^2+y^2$ and $z=4$ and inside $x^2+(y-1)^2=1$.
50. $\rho(x,y,z) = \sqrt{x^2+z^2}$, bounded by $y = \sqrt{x^2+z^2}$ and $y = \sqrt{8-x^2-z^2}$.

Exercises 51–60, relate to unit basis vectors in cylindrical coordinates.

51. For the position vector $\mathbf{r} = \langle x, y, 0\rangle = \langle r\cos\theta, r\sin\theta, 0\rangle$ in cylindrical coordinates, compute the unit vector $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{r}$, where $r = \|\mathbf{r}\| \neq 0$.

52. Referring to exercise 51, for the unit vector $\hat{\boldsymbol{\theta}} = \langle -\sin\theta, \cos\theta, 0\rangle$, show that $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ and $\mathbf{k}$ are mutually orthogonal.

53. The unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in exercises 51 and 52 are not constant vectors. This changes many of our calculations and interpretations. For an object in motion (that is, where r, θ and z are functions of time), compute the derivatives of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of each other.

54. For the vector $\mathbf{v}$ from $(0, 0, 0)$ to $(2, 2, 0)$, show that $\mathbf{v} = r\hat{\mathbf{r}}$.

55. For the vector $\mathbf{v}$ from $(1, 1, 0)$ to $(3, 3, 0)$, find a constant c such that $\mathbf{v} = c\hat{\mathbf{r}}$. Compare to exercise 54.

56. For the vector $\mathbf{v}$ from $(-1, -1, 0)$ to $(1, 1, 0)$, find a constant c such that $\mathbf{v} = c\hat{\mathbf{r}}$. Compare to exercise 55.

57. For the vector $\mathbf{v}$ from $(1, -1, 0)$ to $(1, 1, 0)$, find a constant c such that $\mathbf{v} = c\int_{-\pi/4}^{\pi/4} \hat{\boldsymbol{\theta}} d\theta$.

58. For the vector $\mathbf{v}$ from $(-1, -1, 0)$ to $(1, 1, 0)$, write $\mathbf{v}$ in the form $c\int_a^b \hat{\boldsymbol{\theta}}\, d\theta$. Compare to exercise 56.

59. For the point $(-1, -1, 0)$, sketch the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. Illustrate graphically how the vector $\mathbf{v}$ from $(-1, -1, 0)$ to $(1, 1, 0)$ can be represented both in terms of $\hat{\mathbf{r}}$ and in terms of $\hat{\boldsymbol{\theta}}$.

60. For the vector $\mathbf{v}$ from $(-1, -1, 0)$ to $(1, \sqrt{3}, 1)$, find constants a, b, θ_1, θ_2 and c such that $\mathbf{v} = a\hat{\mathbf{r}} + b\int_{\theta_1}^{\theta_2} \hat{\boldsymbol{\theta}}\, d\theta + c\mathbf{k}$.

EXPLORATORY EXERCISES

1. Many computer graphing packages will sketch graphs in cylindrical coordinates, with one option being to have r as a function of z and θ. In some cases, the graphs are very familiar. Sketch the following and solve for z to write the equation in the notation of this section: (a) $r = \sqrt{z}$, (b) $r = z^2$, (c) $r = \ln z$, (d) $r = \sqrt{4 - z^2}$, (e) $r = z^2\cos\theta$. By leaving z out altogether, some old polar curves get an interesting three-dimensional extension: (f) $r = \sin^2\theta, 0 \le z \le 4$, (g) $r = 2 - 2\cos\theta$, $0 \le z \le 4$. Many graphs are simply new. Explore the following graphs and others of your own creation: (h) $r = \cos\theta - \ln z$, (i) $r = z^2\ln(\theta + 1)$, (j) $r = ze^{\theta/8}$, (k) $r = \theta e^{-z}$.

2. In this exercise, you will explore a class of surfaces known as **Plücker's conoids.** In parametric equations, the conoid with n folds is given by $x = r\cos\theta$, $y = r\sin\theta$ and $z = \sin(n\theta)$. Use a CAS to sketch the conoid with 2 folds. Show that on this surface $z = \frac{2xy}{x^2+y^2}$. In vector notation, the parametric equations can be written as $\langle 0, 0, \sin(n\theta)\rangle + \langle r\cos\theta, r\sin\theta, 0\rangle$, with the interpretation that the conoid is generated by moving a line around and perpendicular to the circle $\langle\cos\theta, \sin\theta, 0\rangle$. For $n = 2$, sketch a parametric graph with $1 \le r \le 2$ and $0 \le \theta \le 2\pi$ and compare the surface to a Möbius strip. Explain how the line segment moving around the circle rotates according to the function $\sin 2\theta$. Sketch similar graphs for $n = 3$, $n = 4$ and $n = 5$, and explain why n is referred to as the number of folds.

13.7 SPHERICAL COORDINATES

We introduce here another common coordinate system that is frequently more convenient than either rectangular or cylindrical coordinates. In particular, some triple integrals that cannot be calculated exactly in either rectangular or cylindrical coordinates can be dealt with easily in spherical coordinates.

We can specify a point P with rectangular coordinates (x, y, z) by the corresponding **spherical coordinates** (ρ, ϕ, θ). Here, ρ is defined to be the distance from the origin,

$$\rho = \sqrt{x^2 + y^2 + z^2}. \tag{7.1}$$

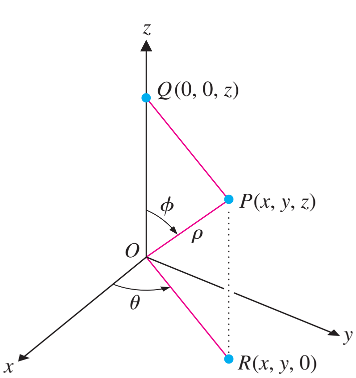


FIGURE 13.58
Spherical coordinates

Note that specifying the distance a point lies away from the origin specifies a sphere on which the point must lie (i.e., the equation $\rho = \rho_0 > 0$ represents the sphere of radius ρ_0 centered at the origin). To name a specific point on the sphere, we further specify two angles, ϕ and θ, as indicated in Figure 13.58. Notice that ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$ and θ is the angle from the positive x-axis to the vector $\overrightarrow{OR}$, where R is the point lying in the xy-plane with rectangular coordinates $(x, y, 0)$ (i.e., R is the projection of P onto the xy-plane). You should observe from this description that

$$\rho \ge 0 \quad \text{and} \quad 0 \le \phi \le \pi.$$

If you look closely at Figure 13.58, you can see how to relate rectangular and spherical coordinates. Notice that

$$x = \|\overrightarrow{OR}\| \cos\theta = \|\overrightarrow{QP}\| \cos\theta.$$

Looking at the triangle OQP, we find that $\|\overrightarrow{QP}\| = \rho \sin\phi$, so that

$$x = \rho \sin\phi \cos\theta. \tag{7.2}$$

Similarly, we have

$$y = \|\overrightarrow{OR}\| \sin\theta = \rho \sin\phi \sin\theta. \tag{7.3}$$

Finally, focusing again on triangle OQP, we have

$$z = \rho \cos\phi. \tag{7.4}$$

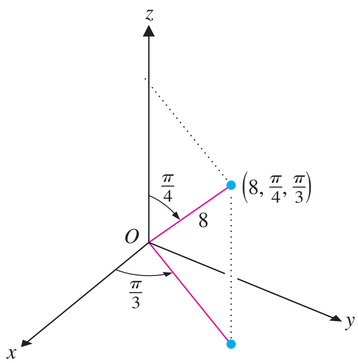


FIGURE 13.59
The point $(8, \frac{\pi}{4}, \frac{\pi}{3})$

EXAMPLE 7.1 Converting from Spherical to Rectangular Coordinates

Find rectangular coordinates for the point described by $(8, \pi/4, \pi/3)$ in spherical coordinates.

Solution We show a sketch of the point in Figure 13.59. From (7.2), (7.3) and (7.4), we have

$$x = 8 \sin\frac{\pi}{4} \cos\frac{\pi}{3} = 8\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = 2\sqrt{2},$$

$$y = 8 \sin\frac{\pi}{4} \sin\frac{\pi}{3} = 8\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{6}$$

and

$$z = 8 \cos\frac{\pi}{4} = 8\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}.$$

It's often very helpful (especially when dealing with triple integrals) to represent common surfaces in spherical coordinates.

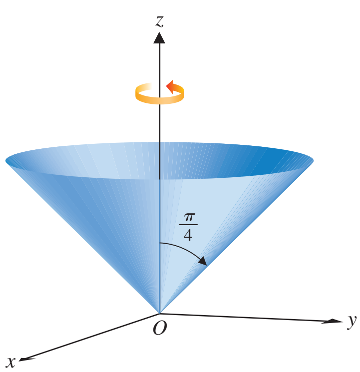


FIGURE 13.60a
Top half-cone $\phi = \frac{\pi}{4}$

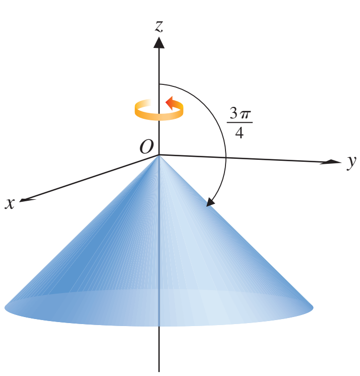


FIGURE 13.60b
Bottom half-cone $\phi = \frac{3\pi}{4}$

EXAMPLE 7.2 Equation of a Cone in Spherical Coordinates

Rewrite the equation of the cone $z^2 = x^2 + y^2$ in spherical coordinates.

Solution Using (7.2), (7.3) and (7.4), the equation of the cone becomes

$$\begin{aligned} \rho^2 \cos^2\phi &= \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta \\ &= \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2 \sin^2\phi. \end{aligned}$$

Since $\cos^2\theta + \sin^2\theta = 1$.

Notice that in order to have $\rho^2 \cos^2\phi = \rho^2 \sin^2\phi$, we must either have $\rho = 0$ (which corresponds to the origin) or $\cos^2\phi = \sin^2\phi$. For the latter to occur, we must have $\phi = \frac{\pi}{4}$ or $\phi = \frac{3\pi}{4}$. (Recall that $0 \le \phi \le \pi$.) Observe that taking $\phi = \frac{\pi}{4}$ (and allowing ρ and θ to be anything) describes the top half of the cone, as shown in Figure 13.60a.
You can think of this as taking a single ray (say in the yz-plane) with $\phi = \frac{\pi}{4}$ and revolving this around the z-axis. (This is the effect of letting θ run from 0 to 2π.) Similarly, $\phi = \frac{3\pi}{4}$ describes the bottom half cone, as seen in Figure 13.60b.

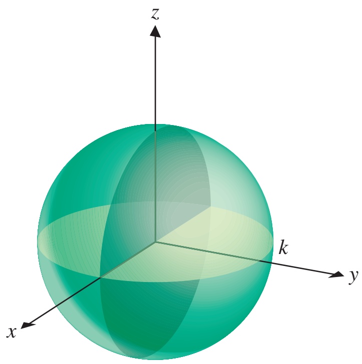


FIGURE 13.61a
The sphere $\rho = k$

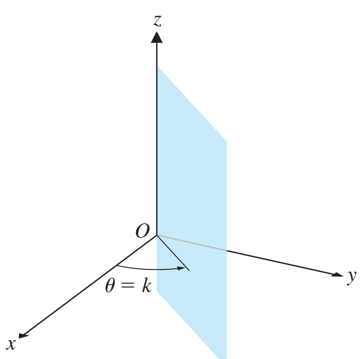


FIGURE 13.61b
The half-plane $\theta = k$

Notice that, in general, the equation $\rho = k$ (for any constant $k > 0$) represents the sphere of radius k, centered at the origin. (See Figure 13.61a.) The equation $\theta = k$ (for any constant k) represents a vertical half-plane, with its edge along the z-axis. (See Figure 13.61b.) Further, the equation $\phi = k$ (for any constant k) represents the top half of a cone if $0 < k < \frac{\pi}{2}$ (see Figure 13.62a) and represents the bottom half of a cone if $\frac{\pi}{2} < k < \pi$ (see Figure 13.62b). Finally, note that $\phi = \frac{\pi}{2}$ describes the xy-plane. Can you think of what the equations $\phi = 0$ and $\phi = \pi$ represent?

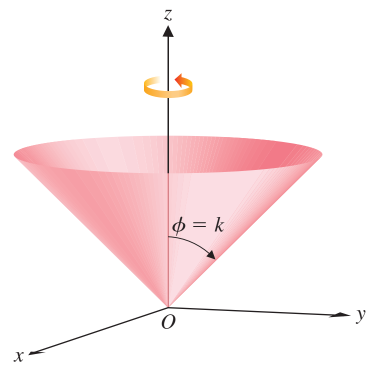


FIGURE 13.62a
Top half-cone $\phi = k$, where $0 < k < \frac{\pi}{2}$

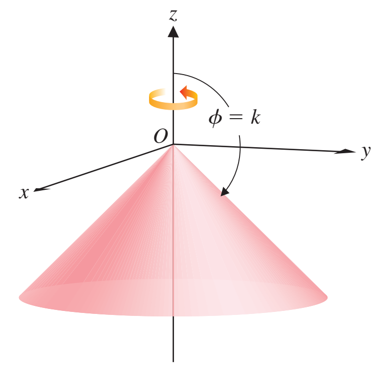


FIGURE 13.62b
Bottom half-cone $\phi = k$, where $\frac{\pi}{2} < k < \pi$

Triple Integrals in Spherical Coordinates

Just as polar coordinates are indispensable in calculating double integrals over circular regions, especially when the integrand involves the particular combination of variables $x^2 + y^2$, spherical coordinates are an indispensable aid in dealing with triple integrals over spherical regions, particularly with those where the integrand involves the combination $x^2 + y^2 + z^2$. Integrals of this type are encountered frequently in applications. For instance, consider the triple integral

$$\iiint_Q \cos(x^2 + y^2 + z^2)^{3/2}\, dV,$$

where Q is the **unit ball:** $x^2 + y^2 + z^2 \le 1$. No matter which order you choose for the integrations, you will arrive at a triple iterated integral that looks like

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \cos(x^2 + y^2 + z^2)^{3/2}\, dz\, dy\, dx.$$

In rectangular coordinates (or cylindrical coordinates, for that matter), you have little hope of calculating this integral exactly. In spherical coordinates, however, this integral is a snap. First, we need to see how to write triple integrals in spherical coordinates.

For the integral $\iiint_Q f(\rho, \phi, \theta)\, dV$, we begin, as we have many times before, by constructing an inner partition of the solid Q. But, rather than slicing up Q using planes parallel to the three coordinate planes, we divide Q by slicing it with spheres of the form $\rho = \rho_k$, half-planes of the form $\theta = \theta_k$ and half-cones of the form $\phi = \phi_k$. Notice that instead of subdividing Q into a number of rectangular boxes, this divides Q into a number of

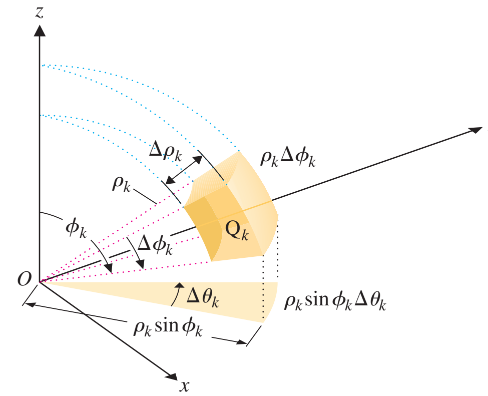


FIGURE 13.63
The spherical wedge Q_k

spherical wedges of the form:

$$Q_k = \{(\rho, \phi, \theta) | \rho_{k-1} \le \rho \le \rho_k, \phi_{k-1} \le \phi \le \phi_k, \theta_{k-1} \le \theta \le \theta_k\},$$

as depicted in Figure 13.63. Here, we have $\Delta\rho_k = \rho_k - \rho_{k-1}$, $\Delta\phi_k = \phi_k - \phi_{k-1}$ and $\Delta\theta_k = \theta_k - \theta_{k-1}$. Notice that Q_k is nearly a rectangular box and so, its volume ΔV_k is approximately the same as that of a rectangular box with the same dimensions:

$$\begin{aligned}\Delta V_k &\approx \Delta\rho_k(\rho_k\,\Delta\phi_k)(\rho_k \sin\phi_k\,\Delta\theta_k)\\ &= \rho_k^2 \sin\phi_k\,\Delta\rho_k \Delta\phi_k \Delta\theta_k.\end{aligned}$$

We consider only those wedges that lie completely inside Q, to form an inner partition $Q_1, Q_2, \ldots, Q_n$ of the solid Q. Summing over the inner partition and letting the norm of the partition $\|P\|$ (here, the longest diagonal of any of the wedges in the inner partition) approach zero, we get

$$\begin{aligned}\iiint_Q f(\rho, \phi, \theta)\,dV &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\Delta V_k\\ &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\rho_k^2 \sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k\\ &= \iiint_Q f(\rho, \phi, \theta)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta, \end{aligned} \tag{7.5}$$

where the limits of integration for each of the three iterated integrals are found in much the same way as we have done for other coordinate systems. From (7.5), notice that the volume element in spherical coordinates is given by

$$\boxed{dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.}$$

We can now return to our introductory example.

EXAMPLE 7.3 A Triple Integral in Spherical Coordinates

Evaluate the triple integral $\iiint_Q \cos(x^2 + y^2 + z^2)^{3/2}\,dV$, where Q is the unit ball: $x^2 + y^2 + z^2 \le 1$.

Solution Notice that since Q is the unit ball, ρ (the radial distance from the origin) ranges from 0 to 1. Further, the angle ϕ ranges from 0 to π (where $\phi = 0$ starts us at the top of the sphere, $\phi \in [0, \pi/2]$ corresponds to the top hemisphere and $\phi \in [\pi/2, \pi]$ corresponds to the bottom hemisphere). Finally (to get all the way around the sphere), the angle θ ranges from 0 to 2π. From (7.5), we have that since $x^2 + y^2 + z^2 = \rho^2$,

$$\begin{aligned}
\iiint_Q \cos(\underbrace{x^2+y^2+z^2}_{\rho^2})^{3/2} \underbrace{dV}_{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \cos(\rho^2)^{3/2}\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \cos(\rho^3)\,(3\rho^2)\sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} \sin(\rho^3)\Big|_{\rho=0}^{\rho=1} \sin\phi \, d\phi \, d\theta \\
&= \frac{\sin 1}{3}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta \\
&= -\frac{\sin 1}{3}\int_0^{2\pi} \cos\phi\Big|_{\phi=0}^{\phi=\pi} d\theta \\
&= -\frac{\sin 1}{3}\int_0^{2\pi} (\cos\pi - \cos 0)\, d\theta \\
&= \frac{2}{3}(\sin 1)(2\pi) \approx 3.525.
\end{aligned}$$

NOTES

Since the integrand $\rho^2\cos(\rho^3)\sin\phi$ is in the factored form $f(\rho)g(\phi)h(\theta)$ and the limits of integration are all constant (which is frequently the case for integrals in spherical coordinates), the triple integral may be rewritten as:

$$\left(\int_0^1 \rho^2\cos(\rho^3)\,d\rho\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{2\pi} 1\,d\theta\right).$$

Generally, spherical coordinates are useful in triple integrals when the solid over which you are integrating is in some way spherical and particularly when the integrand contains the term $x^2 + y^2 + z^2$. In example 7.4, we use spherical coordinates to simplify the calculation of a volume.

EXAMPLE 7.4 Finding a Volume Using Spherical Coordinates

Find the volume lying inside the sphere $x^2 + y^2 + z^2 = 2z$ and inside the cone $z^2 = x^2 + y^2$.

Solution Notice that by completing the square in the equation of the sphere, we get

$$x^2 + y^2 + (z^2 - 2z + 1) = 1$$

or

$$x^2 + y^2 + (z-1)^2 = 1,$$

the sphere of radius 1, centered at the point (0, 0, 1). Notice, too, that since the sphere sits completely above the xy-plane, only the top half of the cone, $z = \sqrt{x^2 + y^2}$ intersects the sphere. See Figure 13.64 for a sketch of the solid. You might try to find the volume using rectangular coordinates (but, don't spend much time on it). Because of the spherical geometry, we consider the problem in spherical coordinates. (Keep in mind that cones have a very simple representation in spherical coordinates.) From (7.1), (7.4) and the original equation of the sphere, we get

$$\underbrace{x^2 + y^2 + z^2}_{\rho^2} = 2\underbrace{z}_{\rho\cos\phi}$$

or

$$\rho^2 = 2\rho\cos\phi.$$

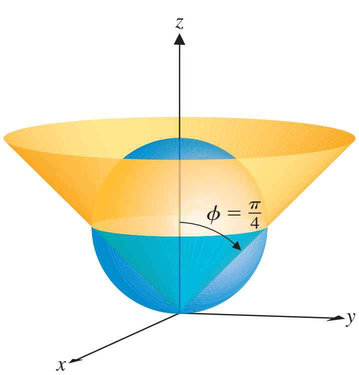


FIGURE 13.64
The cone $\phi = \frac{\pi}{4}$ and the sphere $\rho = 2\cos\phi$

This equation is satisfied when $\rho = 0$ (corresponding to the origin) or when $\rho = 2\cos\phi$ (the equation of the sphere in spherical coordinates). For the top half of the cone, we have $z = \sqrt{x^2 + y^2}$, or in spherical coordinates $\phi = \frac{\pi}{4}$, as discussed in example 7.2.

Referring again to Figure 13.64, notice that to stay inside the cone and inside the sphere, we have that for each fixed ϕ and θ, ρ can range from 0 up to $2\cos\phi$. For each fixed θ, to stay inside the cone, ϕ must range from 0 to $\frac{\pi}{4}$. Finally, to get all the way around the solid, θ ranges from 0 to 2π. The volume of the solid is then given by

$$\begin{aligned}
V &= \iiint_Q \underbrace{dV}_{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta} \\
&= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3}\rho^3 \Big|_{\rho=0}^{\rho=2\cos\phi} \sin\phi\, d\phi\, d\theta \\
&= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \cos^3\phi \sin\phi\, d\phi\, d\theta \\
&= -\frac{8}{3} \int_0^{2\pi} \frac{\cos^4\phi}{4} \Big|_{\phi=0}^{\phi=\pi/4} d\theta = -\frac{2}{3} \int_0^{2\pi} \left(\cos^4\frac{\pi}{4} - 1\right) d\theta \\
&= -\frac{4\pi}{3}\left(\cos^4\frac{\pi}{4} - 1\right) = -\frac{4\pi}{3}\left(\frac{1}{4} - 1\right) = \pi.
\end{aligned}$$

EXAMPLE 7.5 Changing an Integral from Rectangular to Spherical Coordinates

Evaluate the triple iterated integral $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dy\, dx$.

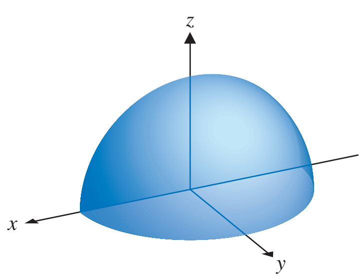


FIGURE 13.65
The solid Q

Solution Although the integrand is simply a polynomial, the limits of integration make the second and third integrations very messy. Notice that the integrand contains the combination of variables $x^2 + y^2 + z^2$, which equals ρ^2 in spherical coordinates. Further, the solid over which we are integrating is a portion of a sphere, as follows. Notice that for each x in the interval $[-2, 2]$ indicated by the outermost limits of integration, y varies from 0 (corresponding to the x-axis) to $y = \sqrt{4 - x^2}$ (the top semicircle of radius 2 centered at the origin). Finally, z varies from 0 (corresponding to the xy-plane) up to $z = \sqrt{4 - x^2 - y^2}$ (the top hemisphere of radius 2 centered at the origin). The solid Q over which we are integrating is then the half of the hemisphere that lies above the first and second quadrants of the xy-plane, as illustrated in Figure 13.65. In spherical coordinates, this portion of the sphere is obtained if we let ρ range from 0 up to 2, ϕ range from 0 up to $\frac{\pi}{2}$ and θ range from 0 to π. The integral then becomes

$$\begin{aligned}
&\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dy\, dx \\
&\quad = \iiint_Q \underbrace{(x^2 + y^2 + z^2)}_{\rho^2} \underbrace{dV}_{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta} \\
&\quad = \int_0^{\pi} \int_0^{\pi/2} \int_0^{2} \rho^2(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \frac{32}{5}\pi,
\end{aligned}$$

where we leave the details of this relatively simple integration to you.

BEYOND FORMULAS

Spherical coordinates are used in a variety of applications, usually to take advantage of certain symmetries present in the structure or force being analyzed. In particular, if the value of a function $f(x, y, z)$ depends only on the distance of the point (x, y, z) from the origin, then spherical coordinates can be convenient to use. This is analogous to the use of polar coordinates in two dimensions to take advantage of radial symmetry. In what way is $r = c$ in two dimensions analogous to $\rho = c$ in three dimensions?

EXERCISES 13.7

WRITING EXERCISES

1. Discuss the relationship between the spherical coordinates angles ϕ and θ and the longitude and latitude angles on a map of the earth. Satellites in geosynchronous orbit remain at a constant distance above a fixed point on the earth. Discuss how spherical coordinates could be used to represent the position of the satellite.
2. Explain why any point in $\mathbb{R}^3$ can be represented in spherical coordinates with $\rho \geq 0, 0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. In particular, explain why it is not necessary to allow $\rho < 0$ or $\pi < \phi \leq 2\pi$. Discuss whether the ranges $\rho \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ would suffice to describe all points.
3. Explain why in spherical coordinates the equation $\theta = k$ represents a half-plane (see Figure 13.61b) and not a whole plane.
4. Using the examples in this section as a guide, make a short list of surfaces that are simple to describe in spherical coordinates.

In exercises 1–8, convert the spherical point (ρ, ϕ, θ) into rectangular coordinates.

1. $(4, 0, \pi)$
2. $(4, \frac{\pi}{2}, \pi)$
3. $(4, \frac{\pi}{2}, 0)$
4. $(4, \pi, \frac{\pi}{2})$
5. $(2, \frac{\pi}{4}, 0)$
6. $(2, \frac{\pi}{4}, \frac{2\pi}{3})$
7. $(\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{3})$
8. $(\sqrt{2}, \frac{\pi}{6}, \frac{2\pi}{3})$

In exercises 9–16, convert the equation into spherical coordinates.

9. $x^2 + y^2 + z^2 = 9$
10. $x^2 + y^2 + z^2 = 6$
11. $y = x$
12. $z = 0$
13. $z = 2$
14. $x^2 + y^2 + (z - 1)^2 = 1$
15. $z = \sqrt{3(x^2 + y^2)}$
16. $z = -\sqrt{x^2 + y^2}$

In exercises 17–24, sketch the graph of the spherical equation and give a corresponding *xy*-equation.

17. $\rho = 2$
18. $\rho = 4$
19. $\phi = \frac{\pi}{4}$
20. $\phi = \frac{\pi}{2}$
21. $\theta = 0$
22. $\theta = \frac{\pi}{4}$
23. $\phi = \frac{\pi}{3}$
24. $\theta = \frac{\pi}{10}$

In exercises 25–32, sketch the region defined by the given ranges.

25. $0 \leq \rho \leq 4, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \theta \leq \pi$
26. $0 \leq \rho \leq 4, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi$
27. $0 \leq \rho \leq 2, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq \pi$
28. $0 \leq \rho \leq 2, 0 \leq \phi \leq \frac{\pi}{4}, \pi \leq \theta \leq 2\pi$
29. $0 \leq \rho \leq 3, 0 \leq \phi \leq \pi, 0 \leq \theta \leq \pi$
30. $0 \leq \rho \leq 3, 0 \leq \phi \leq \frac{3\pi}{4}, 0 \leq \theta \leq 2\pi$
31. $2 \leq \rho \leq 3, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$
32. $2 \leq \rho \leq 3, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi$

In exercises 33–42, set up and evaluate the indicated triple integral in an appropriate coordinate system.

33. $\iiint\limits_Q e^{(x^2+y^2+z^2)^{3/2}} dV$, where Q is bounded by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the *xy*-plane.
34. $\iiint\limits_Q \sqrt{x^2 + y^2 + z^2}\, dV$, where Q is bounded by the hemisphere $z = -\sqrt{9 - x^2 - y^2}$ and the *xy*-plane.
35. $\iiint\limits_Q (x^2 + y^2 + z^2)^{5/2} dV$, where Q is inside $x^2 + y^2 + z^2 = 2$ and outside $x^2 + y^2 = 1$.
36. $\iiint\limits_Q e^{\sqrt{x^2+y^2+z^2}} dV$, where Q is bounded by $y = \sqrt{4 - x^2 - z^2}$ and $y = 0$.
37. $\iiint\limits_Q (x^2 + y^2 + z^2)\, dV$, where Q is the cube with $0 \leq x \leq 1$, $1 \leq y \leq 2$ and $3 \leq z \leq 4$.
38. $\iiint\limits_Q (x + y + z)\, dV$, where Q is the tetrahedron bounded by $x + 2y + z = 4$ and the coordinate planes.

39. $\iiint_Q (x^2+y^2)\,dV$, where Q is bounded by $z=4-x^2-y^2$ and the xy-plane.

40. $\iiint_Q e^{x^2+y^2}\,dV$, where Q is bounded by $x^2+y^2=4$, $z=0$ and $z=2$.

41. $\iiint_Q \sqrt{x^2+y^2+z^2}\,dV$, where Q is bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{2-x^2-y^2}$.

42. $\iiint_Q (x^2+y^2+z^2)^{3/2}\,dV$, where Q is the region below $z=-\sqrt{x^2+y^2}$ and inside $z=-\sqrt{4-x^2-y^2}$.

In exercises 43–52, use an appropriate coordinate system to find the volume of the given solid.

43. The region below $x^2+y^2+z^2=4z$ and above $z=\sqrt{x^2+y^2}$

44. The region above $z=\sqrt{x^2+y^2}$ and below $x^2+y^2+z^2=4$

45. The region inside $z=\sqrt{2x^2+2y^2}$ and between $z=2$ and $z=4$

46. The region bounded by $z=4x^2+4y^2$, $z=0$, $x^2+y^2=1$ and $x^2+y^2=2$

47. The region under $z=\sqrt{x^2+y^2}$ and above the square $-1\le x\le 1$, $-1\le y\le 1$

48. The region bounded by $x+2y+z=4$ and the coordinate planes

49. The region below $x^2+y^2+z^2=4$, above $z=\sqrt{x^2+y^2}$ in the first octant

50. The region below $x^2+y^2+z^2=4$, above $z=\sqrt{x^2+y^2}$, between $y=x$ and $x=0$ with $y\ge 0$

51. The region below $z=\sqrt{x^2+y^2}$, above the xy-plane and inside $x^2+y^2=4$

52. The region between $z=4-x^2-y^2$ and the xy-plane

In exercises 53–56, evaluate the iterated integral by changing coordinate systems.

53. $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

54. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}}(x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$

55. $\displaystyle\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}}(x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx$

56. $\displaystyle\int_0^4\int_0^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{4}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$

57. Find the center of mass of the solid with constant density and bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{4-x^2-y^2}$.

58. Find the center of mass of the solid with constant density in the first quadrant and bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{4-x^2-y^2}$.

Exercises 59–64 relate to unit basis vectors in spherical coordinates.

59. For the position vector $\mathbf{r}=\langle x,y,z\rangle=\langle\rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi\rangle$ in spherical coordinates, compute the unit vector $\hat{\boldsymbol\rho}=\dfrac{\mathbf{r}}{r}$, where $r=\|\mathbf{r}\|\ne 0$.

60. For the unit vectors $\hat{\boldsymbol\theta}=\langle-\sin\theta,\cos\theta,0\rangle$ and $\hat{\boldsymbol\phi}=\langle\cos\theta\cos\phi,\sin\theta\cos\phi,-\sin\phi\rangle$, show that $\hat{\boldsymbol\rho}$, $\hat{\boldsymbol\theta}$ and $\hat{\boldsymbol\phi}$ are mutually orthogonal.

61. For the vector $\mathbf{v}$ from $(1,1,\sqrt2)$ to $(2,2,2\sqrt2)$, find a constant c such that $\mathbf{v}=c\hat{\boldsymbol\rho}$.

62. For the vector $\mathbf{v}$ from $(1,1,\sqrt2)$ to $(-1,1,\sqrt2)$, find a constant c such that $\mathbf{v}=c\int_{\pi/4}^{3\pi/4}\hat{\boldsymbol\theta}\,d\theta$. Compare to exercise 61.

63. For the vector $\mathbf{v}$ from $(1,1,\sqrt2)$ to $(\sqrt2,\sqrt2,0)$, find a constant c such that $\mathbf{v}=c\int_{\pi/4}^{\pi/2}\hat{\boldsymbol\phi}\,d\phi$.

64. From the point $(1,1,\sqrt2)$, sketch the vectors $\hat{\boldsymbol\rho}$, $\hat{\boldsymbol\theta}$ and $\hat{\boldsymbol\phi}$. Illustrate graphically the vectors $\mathbf{v}$ in exercises 61–63.

65. Sketch the cardioid $r=2-2\sin\theta$ in the xy-plane. Define $\tilde r$ and $\tilde\theta$ to be polar coordinates in the yz-plane (that is, $y=\tilde r\cos\tilde\theta$ and $z=\tilde r\sin\tilde\theta$). Show that $\tilde\theta=\frac{\pi}{2}-\phi$ and $\cos\phi=\sin\tilde\theta$. Use this information to graph $\rho=2-2\cos\phi$.

66. As in exercise 65, relate the graph of $\rho=\cos^2 3\phi$ to the two-dimensional graph of $r=\sin^2 3\theta$ and use this information to sketch the three-dimensional surface.

67. As in exercise 65, sketch the surface $\rho=\sin^2\phi$ by relating it to a two-dimensional polar curve.

68. As in exercise 65, sketch the surface $\rho=1+\sin 3\phi$ by relating it to a two-dimensional polar curve.

EXPLORATORY EXERCISES

1. If you have a graphing utility that will graph surfaces of the form $\rho=f(\phi,\theta)$, graph $\rho=2\phi$ and $\rho=\left(\phi-\frac{\pi}{2}\right)^2$. Discuss the symmetry that results from the variable θ not appearing in the equation. Discuss the changes in ρ as you move down from $\phi=0$ to $\phi=\pi$. Using what you have learned, try graphing the following by hand and then compare your sketches to those of your graphing utility: (a) $\rho=e^{-\phi}$, (b) $\rho=e^{\phi}$, (c) $\rho=\sin^2\phi$ and (d) $\rho=\sin^2\left(\phi-\frac{\pi}{2}\right)$.

2. Use a graphing utility to graph $\rho=5\cos\theta$ and $\rho=\sqrt{\cos\theta}$. Discuss the symmetry that results from the variable ϕ not appearing in the equation. Discuss the changes in ρ as you move around from $\theta=0$ to $\theta=2\pi$. Using what you have learned, try graphing the following by hand and then compare your sketches to those of your graphing utility: (a) $\rho = \sin^2\theta$, (b) $\rho = \sin^2\left(\theta - \frac{\pi}{2}\right)$, (c) $\rho = e^{\theta}$ and (d) $\rho = e^{-\theta}$.

3. Use a graphing utility to graph each of the following. Adjust the graphing window as needed to get a good idea of what the graph looks like. (a) $\rho = \sin(\phi + \theta)$, (b) $\rho = \phi\sin\theta$, (c) $\rho = \sin^2\theta\cos\phi$, (d) $\rho = 4\cos^2\theta + 2\sin\phi - 3\sin\theta$ and (e) $\rho = 4\cos\theta\sin 5\phi + 3\cos^2\phi$. There are innumerable interesting and unusual graphs in spherical coordinates. Experiment and find your own!

13.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

One of our most basic tools for evaluating a definite integral is substitution. For instance, to evaluate the integral $\int_0^2 2xe^{x^2+3}dx$, you would make the substitution $u = x^2 + 3$. While you can probably do this in your head, let's consider the details one more time. Here, $du = 2x\,dx$ and don't forget: when you change variables in a definite integral, you must also change the limits of integration to suit the new variable. In this case, when $x = 0$, we have $u = 0^2 + 3 = 3$ and when $x = 2$, $u = 2^2 + 3 = 7$. This leaves us with

$$\int_0^2 2xe^{x^2+3}dx = \int_0^2 \underbrace{e^{x^2+3}}_{e^u}\underbrace{(2x)\,dx}_{du}$$
$$= \int_3^7 e^u\,du = e^u\Big|_3^7 = e^7 - e^3.$$

The primary reason for making the preceding change of variable was to simplify the integrand, so that it was easier to find an antiderivative. Notice too that we not only transformed the integrand, but we also changed the interval over which we were integrating.

You should recognize that we have already implemented changes of variables in multiple integrals in the very special cases of polar coordinates (for double integrals) and cylindrical and spherical coordinates (for triple integrals). There were several reasons for doing this. In the case of double integrals in rectangular coordinates, if the integrand contains the term $x^2 + y^2$ or if the region over which you are integrating is in some way circular, then polar coordinates may be indicated. For instance, consider the iterated integral

$$\int_0^3\int_0^{\sqrt{9-x^2}} \cos(x^2+y^2)\,dy\,dx.$$

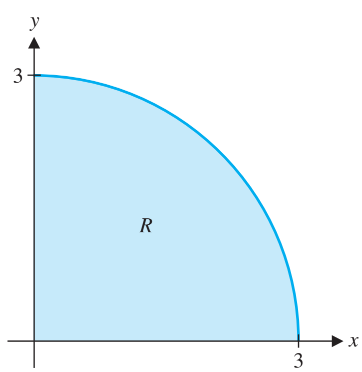


FIGURE 13.66a
The region of integration in the xy-plane

There is really no way to evaluate this integral as it is written in rectangular coordinates. (Try it!) However, recognizing that the region of integration R is the portion of the circle of radius 3 centered at the origin that lies in the first quadrant (see Figure 13.66a), and since the integrand includes the term $x^2 + y^2$, it's a good bet that polar coordinates will help. In fact, we have

$$\int_0^3\int_0^{\sqrt{9-x^2}} \cos(x^2+y^2)\,dy\,dx = \iint_R \cos\underbrace{(x^2+y^2)}_{r^2}\,\underbrace{dA}_{r\,dr\,d\theta}$$
$$= \int_0^{\pi/2}\int_0^3 \cos(r^2)\,r\,dr\,d\theta,$$

which is now an easy integral to evaluate. Notice that there are two things that happened here. First, we simplified the integrand (into one with a known antiderivative) and second, we transformed the region over which we integrated, as follows. In the xy-plane, we integrated over the circular sector indicated in Figure 13.66a. In the $r\theta$-plane, we integrated over

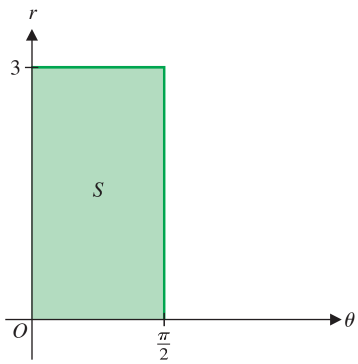


FIGURE 13.66b
The region of integration in the $r\theta$-plane

a (simpler) region: the rectangle S defined by $S = \{(r, \theta)|0 \le r \le 3 \text{ and } 0 \le \theta \le \frac{\pi}{2}\}$, as indicated in Figure 13.66b.

Recall that we changed to cylindrical or spherical coordinates in triple integrals for similar reasons. In each case, we ended up simplifying the integrand and transforming the region of integration. More generally, how do we change variables in a multiple integral? Before we answer this question, we must first explore the concept of transformation in several variables.

A **transformation** T from the uv-plane to the xy-plane is a function that maps points in the uv-plane to points in the xy-plane, so that

$$T(u, v) = (x, y),$$

where $$x = g(u, v) \quad \text{and} \quad y = h(u, v),$$

for some functions g and h. We consider changes of variables in double integrals as defined by a transformation T from a region S in the uv-plane onto a region R in the xy-plane (see Figure 13.67). We refer to R as the **image** of S under the transformation T. We say that T is **one-to-one** on S if for every point (x, y) in R there is exactly one point (u, v) in S such that $T(u, v) = (x, y)$. Notice that this says that (at least in principle), we can solve for u and v in terms of x and y. Further, we consider only transformations for which g and h have continuous first partial derivatives in the region S.

The primary reason for introducing a change of variables in a multiple integral is to simplify the calculation of the integral. This is accomplished by simplifying the integrand, the region over which you are integrating or both. Before exploring the effect of a transformation on a multiple integral, we examine several examples of how a transformation can simplify a region in two dimensions.

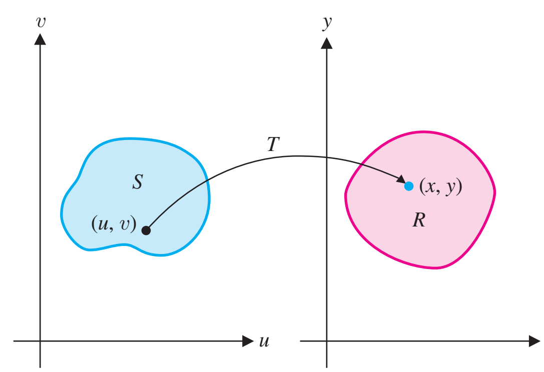


FIGURE 13.67
The transformation T mapping S onto R

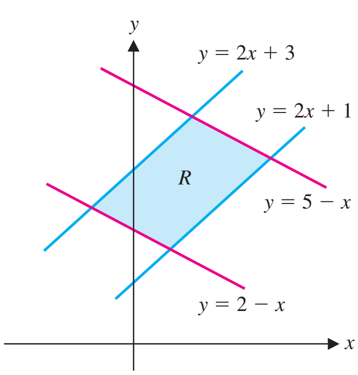


FIGURE 13.68a
The region R in the xy-plane

EXAMPLE 8.1 The Transformation of a Simple Region

Let R be the region bounded by the straight lines $y = 2x + 3$, $y = 2x + 1$, $y = 5 - x$ and $y = 2 - x$. Find a transformation T mapping a region S in the uv-plane onto R, where S is a rectangular region, with sides parallel to the u- and v-axes.

Solution First, notice that the region R is a parallelogram in the xy-plane. (See Figure 13.68a.) We can rewrite the equations for the lines forming the boundaries of R as $y - 2x = 3$, $y - 2x = 1$, $y + x = 5$ and $y + x = 2$. This suggests the

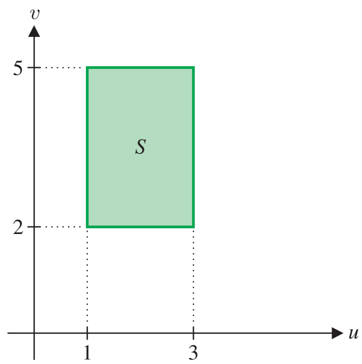


FIGURE 13.68b
The region S in the uv-plane

change of variables

$$u = y - 2x \quad \text{and} \quad v = y + x. \tag{8.1}$$

Observe that the lines forming the boundaries of R then correspond to the lines $u = 3$, $u = 1$, $v = 5$ and $v = 2$, respectively, forming the boundaries of the corresponding region S in the uv-plane. (See Figure 13.68b.) Solving equations (8.1) for x and y, we have the transformation T defined by

$$x = \frac{1}{3}(v - u) \quad \text{and} \quad y = \frac{1}{3}(2v + u).$$

Note that the transformation maps the four corners of the rectangles S to the vertices of the parallelogram R, as follows:

$$T(1, 2) = \left(\frac{1}{3}(2 - 1), \frac{1}{3}[2(2) + 1]\right) = \left(\frac{1}{3}, \frac{5}{3}\right),$$

$$T(3, 2) = \left(\frac{1}{3}(2 - 3), \frac{1}{3}[2(2) + 3]\right) = \left(-\frac{1}{3}, \frac{7}{3}\right),$$

$$T(1, 5) = \left(\frac{1}{3}(5 - 1), \frac{1}{3}[2(5) + 1]\right) = \left(\frac{4}{3}, \frac{11}{3}\right)$$

and

$$T(3, 5) = \left(\frac{1}{3}(5 - 3), \frac{1}{3}[2(5) + 3]\right) = \left(\frac{2}{3}, \frac{13}{3}\right).$$

We leave it as an exercise to verify that the above four points are indeed the vertices of the parallelogram R. (To do this, simply solve the system of equations for the points of intersection.) ■

In example 8.2, we see how polar coordinates can be used to transform a rectangle in the $r\theta$-plane into a sector of a circular annulus.

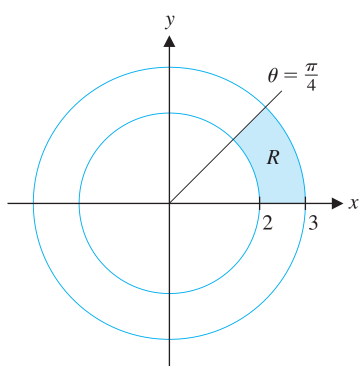


FIGURE 13.69a
The region R in the xy-plane

EXAMPLE 8.2 A Transformation Involving Polar Coordinates

Let R be the region inside the circle $x^2 + y^2 = 9$ and outside the circle $x^2 + y^2 = 4$ and lying in the first quadrant between the lines $y = 0$ and $y = x$. Find a transformation T from a rectangular region S in the $r\theta$-plane to the region R.

Solution First, we picture the region R (a sector of a circular annulus) in Figure 13.69a. The obvious transformation is accomplished with polar coordinates. We let $x = r\cos\theta$ and $y = r\sin\theta$, so that $x^2 + y^2 = r^2$. The inner and outer circles forming a portion of the boundary of R then correspond to $r = 2$ and $r = 3$, respectively. Further, the line $y = x$ corresponds to the line $\theta = \frac{\pi}{4}$ and the line $y = 0$ corresponds to the line $\theta = 0$. We show the region S in Figure 13.69b. ■

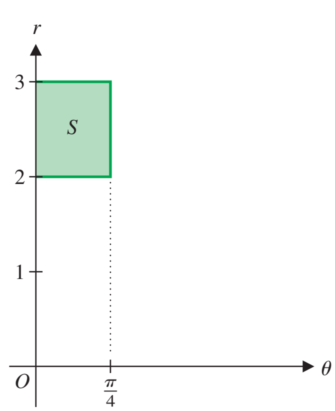


FIGURE 13.69b
The region S in the $r\theta$-plane

Now that we have introduced transformations, we consider our primary goal for this section: to determine how a change of variables in a multiple integral will affect the integral. We consider the double integral

$$\iint_R f(x, y)\, dA,$$

where f is continuous on R. Further, we assume that R is the image of a region S in the uv-plane under the one-to-one transformation T. Recall that we originally constructed the double integral by forming an inner partition of R and taking a limit of the corresponding Riemann sums. We now consider an inner partition of the region S in the uv-plane, consisting of the n rectangles $S_1, S_2, \ldots, S_n$, as depicted in Figure 13.70a. We denote the lower left

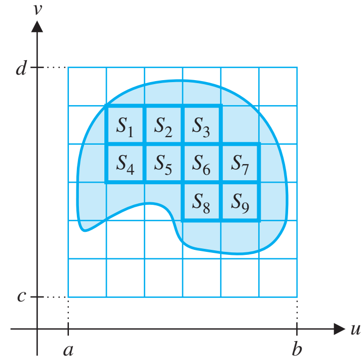


FIGURE 13.70a
An inner partition of the region S in the uv-plane

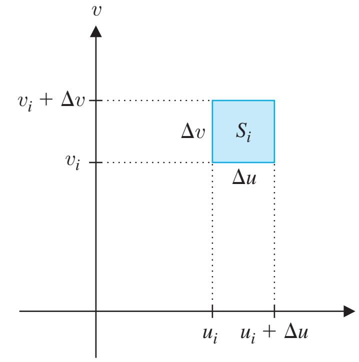


FIGURE 13.70b
The rectangle S_i

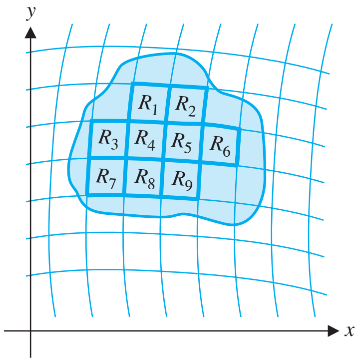


FIGURE 13.71
Curvilinear inner partition of the region R in the xy-plane

corner of each rectangle S_i by $(u_i, v_i)\,(i = 1, 2, \ldots, n)$ and take all of the rectangles to have the same dimensions Δu by Δv, as indicated in Figure 13.70b. Let $R_1, R_2, \ldots, R_n$ be the images of $S_1, S_2, \ldots, S_n$, respectively, under the transformation T and let the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the images of $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$, respectively. Notice that $R_1, R_2, \ldots, R_n$ will then form an inner partition of the region R in the xy-plane (although it will not generally consist of rectangles), as indicated in Figure 13.71. In particular, the image of the rectangle S_i under T is the curvilinear region R_i. From our development of the double integral, we know that

$$\iint_R f(x, y)\,dA \approx \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i, \tag{8.2}$$

where ΔA_i is the area of R_i, for $i = 1, 2, \ldots, n$. The only problem with this approximation is that we don't know how to find ΔA_i, since the regions R_i are not generally rectangles. We can, however, find a reasonable approximation, as follows.

Notice that T maps the four corners of S_i: (u_i, v_i), $(u_i + \Delta u, v_i)$, $(u_i + \Delta u, v_i + \Delta v)$ and $(u_i, v_i + \Delta v)$ to four points denoted A, B, C and D, respectively, on the boundary of R_i, as indicated below:

$$(u_i, v_i) \xrightarrow{T} A(g(u_i, v_i), h(u_i, v_i)) = A(x_i, y_i),$$

$$(u_i + \Delta u, v_i) \xrightarrow{T} B(g(u_i + \Delta u, v_i), h(u_i + \Delta u, v_i)),$$

$$(u_i + \Delta u, v_i + \Delta v) \xrightarrow{T} C(g(u_i + \Delta u, v_i + \Delta v), h(u_i + \Delta u, v_i + \Delta v))$$

and $$(u_i, v_i + \Delta v) \xrightarrow{T} D(g(u_i, v_i + \Delta v), h(u_i, v_i + \Delta v)).$$

We indicate these four points and a typical curvilinear region R_i in Figure 13.72a. Notice that as long as Δu and Δv are small, we can approximate the area of R_i by the area of the parallelogram determined by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$, as indicated in Figure 13.72b. If we consider $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as three-dimensional vectors (with zero $\mathbf{k}$ components), recall from our discussion in section 10.4 that the area of the parallelogram is simply $\|\overrightarrow{AB} \times \overrightarrow{AD}\|$. We will take this as an approximation of the area ΔA_i. First, notice that

$$\overrightarrow{AB} = \langle g(u_i + \Delta u, v_i) - g(u_i, v_i), h(u_i + \Delta u, v_i) - h(u_i, v_i)\rangle \tag{8.3}$$

and $$\overrightarrow{AD} = \langle g(u_i, v_i + \Delta v) - g(u_i, v_i), h(u_i, v_i + \Delta v) - h(u_i, v_i)\rangle. \tag{8.4}$$

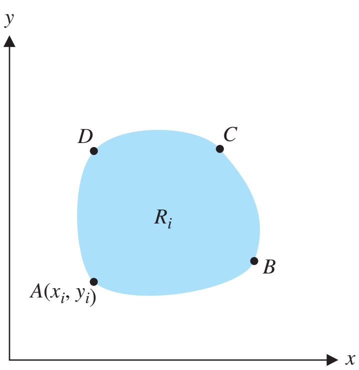


FIGURE 13.72a
The region R_i

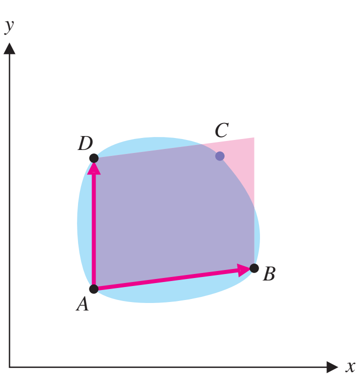


FIGURE 13.72b
The parallelogram determined by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$

From the definition of partial derivative, we have

$$g_u(u_i, v_i) = \lim_{\Delta u \to 0} \frac{g(u_i + \Delta u, v_i) - g(u_i, v_i)}{\Delta u}.$$

This tells us that for Δu small,

$$g(u_i + \Delta u, v_i) - g(u_i, v_i) \approx g_u(u_i, v_i)\,\Delta u.$$

Likewise, we have $\quad h(u_i + \Delta u, v_i) - h(u_i, v_i) \approx h_u(u_i, v_i)\,\Delta u.$

Similarly, for Δv small, we have

$$g(u_i, v_i + \Delta v) - g(u_i, v_i) \approx g_v(u_i, v_i)\,\Delta v$$

and

$$h(u_i, v_i + \Delta v) - h(u_i, v_i) \approx h_v(u_i, v_i)\,\Delta v.$$

Together with (8.3) and (8.4), these give us

$$\overrightarrow{AB} \approx \langle g_u(u_i, v_i)\,\Delta u, h_u(u_i, v_i)\,\Delta u\rangle = \Delta u \langle g_u(u_i, v_i), h_u(u_i, v_i)\rangle$$

and

$$\overrightarrow{AD} \approx \langle g_v(u_i, v_i)\,\Delta v, h_v(u_i, v_i)\,\Delta v\rangle = \Delta v \langle g_v(u_i, v_i), h_v(u_i, v_i)\rangle.$$

An approximation of the area of R_i is then given by

$$\Delta A_i \approx \|\overrightarrow{AB} \times \overrightarrow{AD}\|, \tag{8.5}$$

where

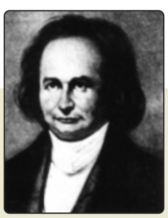
$$\overrightarrow{AB} \times \overrightarrow{AD} \approx \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta u g_u(u_i, v_i) & \Delta u h_u(u_i, v_i) & 0 \\ \Delta v g_v(u_i, v_i) & \Delta v h_v(u_i, v_i) & 0 \end{vmatrix}$$

$$= \begin{vmatrix} g_u(u_i, v_i) & h_u(u_i, v_i) \\ g_v(u_i, v_i) & h_v(u_i, v_i) \end{vmatrix} \Delta u\,\Delta v\,\mathbf{k}. \tag{8.6}$$

For simplicity, we write the determinant as

$$\begin{vmatrix} g_u(u_i, v_i) & h_u(u_i, v_i) \\ g_v(u_i, v_i) & h_v(u_i, v_i) \end{vmatrix} = \begin{vmatrix} g_u(u_i, v_i) & g_v(u_i, v_i) \\ h_u(u_i, v_i) & h_v(u_i, v_i) \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} (u_i, v_i).$$

We give this determinant a name and introduce some new notation in Definition 8.1.



HISTORICAL NOTES

Carl Gustav Jacobi (1804–1851) German mathematician who made important advances in the functional determinant named for him. As a child, Jacobi was academically advanced beyond what the German educational system could offer and did much independent research. He obtained a university teaching post after converting from Judaism to Christianity. He was considered the world's top researcher on elliptic functions and made contributions to number theory and partial differential equations. An inspiring teacher, Jacobi and his students revived German mathematics.

DEFINITION 8.1

The determinant $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is referred to as the **Jacobian** of the transformation T and is written using the notation $\dfrac{\partial(x, y)}{\partial(u, v)}$.

From (8.5) and (8.6), we now have (since $\mathbf{k}$ is a unit vector) that

$$\Delta A_i \approx \|\overrightarrow{AB} \times \overrightarrow{AD}\| = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\,\Delta v,$$

where the determinant is evaluated at the point (u_i, v_i). From (8.2), we now have

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i \approx \sum_{i=1}^{n} f(x_i, y_i)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\,\Delta v$$

$$= \sum_{i=1}^{n} f(g(u_i, v_i), h(u_i, v_i))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\,\Delta v.$$

You should recognize this last expression as a Riemann sum for the double integral

$$\iint_S f(g(u,v), h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

The preceding analysis is a sketch of the more extensive proof of Theorem 8.1.

THEOREM 8.1 (Change of Variables in Double Integrals)

Suppose that the region S in the uv-plane is mapped onto the region R in the xy-plane by the one-to-one transformation T defined by $x = g(u, v)$ and $y = h(u, v)$, where g and h have continuous first partial derivatives on S. If f is continuous on R and the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ is nonzero on S, then

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

We first observe that the change of variables to polar coordinates in a double integral is just a special case of Theorem 8.1.

EXAMPLE 8.3 Changing Variables to Polar Coordinates

Use Theorem 8.1 to derive the evaluation formula for polar coordinates ($r > 0$):

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Solution First, recognize that a change of variables to polar coordinates consists of the transformation from the $r\theta$-plane to the xy-plane, defined by $x = r\cos\theta$ and $y = r\sin\theta$. This gives us the Jacobian

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

By Theorem 8.1, we now have the familiar formula

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$ ■

In example 8.4, we show how a change of variables can be used to simplify the region of integration (thereby also simplifying the integral).

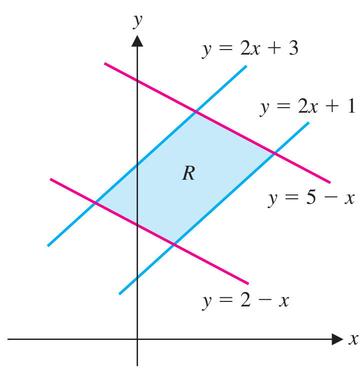


FIGURE 13.73
The region R

EXAMPLE 8.4 Changing Variables to Transform a Region

Evaluate the integral $\iint_R (x^2 + 2xy)\, dA$, where R is the region bounded by the lines $y = 2x + 3$, $y = 2x + 1$, $y = 5 - x$ and $y = 2 - x$.

Solution The difficulty in evaluating this integral is that the region of integration (see Figure 13.73) requires us to break the integral into three pieces. (Think about this some!) An alternative is to find a change of variables corresponding to a transformation from a rectangle in the uv-plane to R in the xy-plane. Recall that we did just this in example 8.1. There, we had found that the change of variables

$$x = \frac{1}{3}(v - u) \quad \text{and} \quad y = \frac{1}{3}(2v + u)$$

maps the rectangle $S = \{(u, v) | 1 \le u \le 3 \text{ and } 2 \le v \le 5\}$ to R. Notice that the Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = -\frac{1}{3}.$$

By Theorem 8.1, we now have:

$$\begin{aligned} \iint_R (x^2 + 2xy)\, dA &= \iint_S \left[\frac{1}{9}(v - u)^2 + \frac{2}{9}(v - u)(2v + u)\right] \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv \\ &= \frac{1}{27} \int_2^5 \int_1^3 [(v - u)^2 + 2(2v^2 - uv - u^2)]\, du\, dv \\ &= \frac{196}{27}, \end{aligned}$$

where we leave the calculation of the final (routine) iterated integral to you. ■

Recall that for single definite integrals, we often must introduce a change of variable in order to find an antiderivative for the integrand. This is also the case in double integrals, as we see in example 8.5.

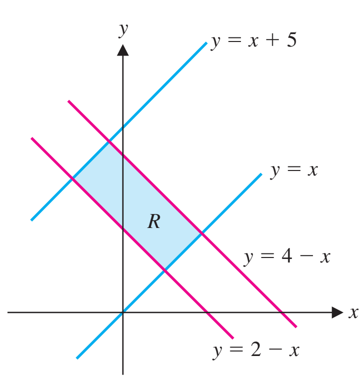


FIGURE 13.74a
The region R

EXAMPLE 8.5 A Change of Variables Required to Find an Antiderivative

Evaluate the double integral $\iint_R \frac{e^{x-y}}{x+y}\, dA$, where R is the rectangle bounded by the lines $y = x$, $y = x + 5$, $y = 2 - x$ and $y = 4 - x$.

Solution First, notice that although the region over which you are to integrate is simply a rectangle in the xy-plane, its sides are not parallel to the x- and y-axes. (See Figure 13.74a.) This is the least of your problems right now, though. If you look carefully at the integrand, you'll recognize that you do not know an antiderivative for this integrand (no matter which variable you integrate with respect to first). A straightforward change of variables is to let $u = x - y$ and $v = x + y$. Solving these equations for x and y gives us

$$x = \frac{1}{2}(u+v) \quad \text{and} \quad y = \frac{1}{2}(v-u). \tag{8.7}$$

The Jacobian of this transformation is then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}.$$

The next issue is to find the region S in the uv-plane that is mapped onto the region R in the xy-plane by this transformation. Remember that the boundary curves of the region S are mapped to the boundary curves of R. From (8.7), we have that $y = x$ corresponds to

$$\frac{1}{2}(v-u) = \frac{1}{2}(u+v) \quad \text{or} \quad u = 0.$$

Likewise, $y = x + 5$ corresponds to

$$\frac{1}{2}(v-u) = \frac{1}{2}(u+v) + 5 \quad \text{or} \quad u = -5,$$

$y = 2 - x$ corresponds to

$$\frac{1}{2}(v-u) = 2 - \frac{1}{2}(u+v) \quad \text{or} \quad v = 2$$

and $y = 4 - x$ corresponds to

$$\frac{1}{2}(v-u) = 4 - \frac{1}{2}(u+v) \quad \text{or} \quad v = 4.$$

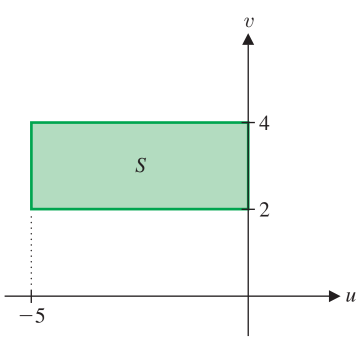


FIGURE 13.74b
The region S

This says that the region S in the uv-plane corresponding to the region R in the xy-plane is the rectangle

$$S = \{(u,v) \mid -5 \le u \le 0 \text{ and } 2 \le v \le 4\},$$

as indicated in Figure 13.74b. You can now easily read off the limits of integration in the uv-plane. By Theorem 8.1, we have

$$\begin{aligned}\iint_R \frac{e^{x-y}}{x+y}\, dA &= \iint_S \frac{e^u}{v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \frac{1}{2} \int_2^4 \int_{-5}^0 \frac{e^u}{v}\, du\, dv \\ &= \frac{1}{2} \int_2^4 \frac{1}{v} e^u \Big|_{u=-5}^{u=0} dv = \frac{1}{2}(e^0 - e^{-5}) \int_2^4 \frac{1}{v}\, dv \\ &= \frac{1}{2}(1 - e^{-5}) \ln|v| \Big|_{v=2}^{v=4} = \frac{1}{2}(1 - e^{-5})(\ln 4 - \ln 2) \\ &\approx 0.34424.\end{aligned}$$

Much as we have now done in two dimensions, we can develop a change of variables formula for triple integrals. The proof of Theorem 8.2 can be found in most texts on advanced calculus. We first define the Jacobian of a transformation in three dimensions.

For a transformation T from a region S of uvw-space onto a region R in xyz-space, defined by $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = l(u, v, w)$, the **Jacobian** of the transformation is the determinant $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ defined by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Theorem 8.2 presents a result for triple integrals that corresponds to Theorem 8.1.

THEOREM 8.2 (Change of Variables in Triple Integrals)

Suppose that the region S in uvw-space is mapped onto the region R in xyz-space by the one-to-one transformation T defined by $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = l(u, v, w)$, where g, h and l have continuous first partial derivatives in S. If f is continuous in R and the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is nonzero in S, then

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(g(u,v,w), h(u,v,w), l(u,v,w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw.$$

We introduce a change of variables in a triple integral for precisely the same reasons as we do in double integrals: in order to simplify the integrand or the region of integration or both. In example 8.6, we use Theorem 8.2 to derive the change of variables formula for the conversion from rectangular to spherical coordinates and see that this is simply a special case of the general change of variables process given in Theorem 8.2.

EXAMPLE 8.6 Deriving the Evaluation Formula for Spherical Coordinates

Use Theorem 8.2 to derive the evaluation formula for triple integrals in spherical coordinates:

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Solution Suppose that the region R in xyz-space is the image of the region S in $\rho\phi\theta$-space under the transformation T defined by the change to spherical coordinates. Recall that we have

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta \quad \text{and} \quad z = \rho \cos\phi.$$

The Jacobian of this transformation is then

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}.$$

For the sake of convenience, we expand this determinant along to the third row, rather than the first row. This gives us

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \cos\phi\begin{vmatrix}\rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\end{vmatrix}\\ &\quad+\rho\sin\phi\begin{vmatrix}\sin\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta\end{vmatrix}\\ &= \cos\phi\,(\rho^2\sin\phi\cos\phi\cos^2\theta+\rho^2\sin\phi\cos\phi\sin^2\theta)\\ &\quad+\rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta+\rho\sin^2\phi\sin^2\theta)\\ &= \rho^2\sin\phi\cos^2\phi+\rho^2\sin^3\phi=\rho^2\sin\phi\,(\cos^2\phi+\sin^2\phi)\\ &= \rho^2\sin\phi.\end{aligned}$$

From Theorem 8.2, we now have that

$$\begin{aligned}\iiint_R f(x,y,z)\,dV &= \iiint_S f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right|d\rho\,d\phi\,d\theta\\ &= \iiint_S f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta,\end{aligned}$$

where we have used the fact that $0\le\phi\le\pi$ to write $|\sin\phi|=\sin\phi$. Notice that this is the same evaluation formula that we developed in section 13.7. ■

EXERCISES 13.8

WRITING EXERCISES

1. Explain what is meant by a "rectangular region" in the uv-plane. In particular, explain what is rectangular about the polar region $1\le r\le 2$ and $0\le\theta\le\pi$.
2. The order of variables in the Jacobian is not important in the sense that $\left|\frac{\partial(x,y)}{\partial(v,u)}\right|=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ but the order is very important in the sense that $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ne\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$. Give a geometric explanation of why $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\left|\frac{\partial(u,v)}{\partial(x,y)}\right|=1$.

In exercises 1–12, find a transformation from a rectangular region S in the uv-plane to the region R.

1. R is bounded by $y=4x+2$, $y=4x+5$, $y=3-2x$ and $y=1-2x$
2. R is bounded by $y=2x-1$, $y=2x+5$, $y=1-3x$ and $y=-1-3x$
3. R is bounded by $y=1-3x$, $y=3-3x$, $y=x-1$ and $y=x-3$
4. R is bounded by $y=2x-1$, $y=2x+1$, $y=3$ and $y=1$
5. R is inside $x^2+y^2=4$, outside $x^2+y^2=1$ and in the first quadrant
6. R is inside $x^2+y^2=4$, outside $x^2+y^2=1$ and in the first quadrant between $y=x$ and $x=0$
7. R is inside $x^2+y^2=9$, outside $x^2+y^2=4$ and between $y=x$ and $y=-x$ with $y\ge 0$
8. R is inside $x^2+y^2=9$ with $x\ge 0$
9. R is bounded by $y=x^2$, $y=x^2+2$, $y=4-x^2$ and $y=2-x^2$ with $x\ge 0$
10. R is bounded by $y=x^2$, $y=x^2+2$, $y=3-x^2$ and $y=2-x^2$ with $x\le 0$
11. R is bounded by $y=e^x$, $y=e^x+1$, $y=3-e^x$ and $y=5-e^x$
12. R is bounded by $y=2x^2+1$, $y=2x^2+3$, $y=2-x^2$ and $y=4-x^2$ with $x\ge 0$

In exercises 13–22, evaluate the double integral.

13. $\iint_R (y-4x)\,dA$, where R is given in exercise 1.
14. $\iint_R (y+3x)\,dA$, where R is given in exercise 2.

15. $\iint_R (y+3x)^2\,dA$, where R is given in exercise 3.

16. $\iint_R e^{y-x}\,dA$, where R is given in exercise 4.

17. $\iint_R x\,dA$, where R is given in exercise 5.

18. $\iint_R e^{y-e^x}\,dA$, where R is given in exercise 11.

19. $\displaystyle\iint_R \frac{e^{y-4x}}{y+2x}\,dA$, where R is given in exercise 1.

20. $\displaystyle\iint_R \frac{e^{y+3x}}{y-2x}\,dA$, where R is given in exercise 2.

21. $\iint_R (x+y)\,dA$, where R is given in exercise 1.

22. $\iint_R (x+2y)\,dA$, where R is given in exercise 2.

In exercises 23–26, find the Jacobian of the given transformation.

23. $x = ue^v,\ y = ue^{-v}$

24. $x = 2uv,\ y = 3u - v$

25. $x = u/v,\ y = v^2$

26. $x = 4u + v^2,\ y = 2uv$

In exercises 27 and 28, find a transformation from a (three-dimensional) rectangular region S in uvw-space to the solid Q.

27. Q is bounded by $x+y+z=1, x+y+z=2, x+2y=0, x+2y=1, y+z=2$ and $y+z=4$.

28. Q is bounded by $x+z=1, x+z=2, 2y+3z=0, 2y+3z=1, y+2z=2$ and $y+2z=4$.

In exercises 29 and 30, find the volume of the given solid.

29. Q in exercise 27

30. Q in exercise 28

31. In Theorem 8.1, we required that the Jacobian be nonzero. To see why this is necessary, consider a transformation where $x = u - v$ and $y = 2v - 2u$. Show that the Jacobian is zero. Then try solving for u and v.

32. Compute the Jacobian for the spherical-like transformation $x = \rho\sin\phi$, $y = \rho\cos\phi\cos\theta$ and $z = \rho\cos\phi\sin\theta$.

33. The integral $\displaystyle\int_0^1\int_0^1 \frac{1}{1-(xy)^2}\,dx\,dy$ arises in the study of the Riemann-zeta function. Use the transformation $x = \dfrac{\sin u}{\cos v}$ and $y = \dfrac{\sin v}{\cos u}$ to write this integral in the form $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2-v} f(u,v)\,du\,dv$ and then evaluate the integral.

34. Show that the transformation $x = \dfrac{\sin u}{\cos v}$ and $y = \dfrac{\sin v}{\cos u}$ in exercise 33 transforms the square $0 \le x \le 1, 0 \le y \le 1$ into the triangle $0 \le u \le \frac{\pi}{2} - v, 0 \le v \le \frac{\pi}{2}$. (Hint: Transform each side of the square separately.)

EXPLORATORY EXERCISES

1. Transformations are involved in many important applications of mathematics. The **direct linear transformation** discussed in this exercise was used by Titleist golf researchers Gobush, Pelletier and Days to study the motion of golf balls (see *Science and Golf II,* 1996). Bright dots are drawn onto golf balls. The dots are tracked by a pair of cameras as the ball is hit. The challenge is to use this information to reconstruct the exact position of the ball at various times, allowing the researchers to estimate the speed, spin rate and launch angle of the ball. In the direct linear transformation model developed by Abdel-Aziz and Karara, a dot at actual position (x, y, z) will appear at pixel (u_1, v_1) of camera 1's digitized image where

$$u_1 = \frac{c_{11}x + c_{21}y + c_{31}z + c_{41}}{d_{11}x + d_{21}y + d_{31}z + 1} \quad \text{and}$$
$$v_1 = \frac{c_{51}x + c_{61}y + c_{71}z + c_{81}}{d_{11}x + d_{21}y + d_{31}z + 1},$$

for constants $c_{11}, c_{21}, \ldots, c_{81}$ and d_{11}, d_{21} and d_{31}. Similarly, camera 2 "sees" this dot at pixel (u_2, v_2) where

$$u_2 = \frac{c_{12}x + c_{22}y + c_{32}z + c_{42}}{d_{12}x + d_{22}y + d_{32}z + 1} \quad \text{and}$$
$$v_2 = \frac{c_{52}x + c_{62}y + c_{72}z + c_{82}}{d_{12}x + d_{22}y + d_{32}z + 1},$$

for a different set of constants $c_{12}, c_{22}, \ldots, c_{82}$ and d_{12}, d_{22} and d_{32}. The constants are determined by taking a series of measurements of motionless balls to calibrate the model. Given that the model for each camera consists of eleven constants, explain why in theory, six different measurements would more than suffice to determine the constants. In reality, more measurements are taken and a least-squares criterion is used to find the best fit of the model to the data. Suppose that this procedure gives us the model

$$u_1 = \frac{2x + y + z + 1}{x + y + 2z + 1}, \quad v_1 = \frac{3x + z}{x + y + 2z + 1},$$
$$u_2 = \frac{x + z + 6}{2x + 3z + 1}, \quad v_2 = \frac{4x + y + 3}{2x + 3z + 1}.$$

If the screen coordinates of a dot are $(u_1, v_1) = (0, -3)$ and $(u_2, v_2) = (5, 0)$, solve for the actual position (x, y, z) of the dot. Actually, a dot would not show up as a single pixel, but as a somewhat blurred image over several pixels. The dot is officially located at the pixel nearest the center of mass of the pixels involved. Suppose that a dot's image activates the following pixels: (34, 42), (35, 42), (32, 41), (33, 41), (34, 41), (35, 41), (36, 41), (34, 40), (35, 40), (36, 40) and (36, 39). Find the center of mass of these pixels and round off to determine the "location" of the dot.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Irregular partition	Definite integral	Double integral
Fubini's Theorem	Double Riemann sum	Volume
Center of mass	First moment	Moment of inertia
Surface area	Triple integral	Mass
Cylindrical coordinates	Spherical coordinates	Rectangular coordinates
Jacobian	Transformation	

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. When using a double integral to compute volume, the choice of integration variables and order is determined by the geometry of the region.
2. $\iint_R f(x, y)\,dA$ gives the volume between $z = f(x, y)$ and the xy-plane.
3. When using a double integral to compute area, the choice of integration variables and order is determined by the geometry of the region.
4. A line through the center of mass of a region divides the region into subregions of equal area.
5. If R is bounded by a circle, $\iint_R f(x, y)\,dA$ should be computed using polar coordinates.
6. The surface area of a region is approximately equal to the area of the projection of the region into the xy-plane.
7. A triple integral in rectangular coordinates has three possible orders of integration.
8. The choice of coordinate systems for a triple integral is determined by the function being integrated.
9. If a region or a function involves $x^2 + y^2$, you should use cylindrical coordinates.
10. For a triple integral in spherical coordinates, the order of integration does not matter.
11. Transforming a double integral in xy-coordinates to one in uv-coordinates, you need formulas for u and v in terms of x and y.

In exercises 1 and 2, compute the Riemann sum for the given function and region, a partition with n equal-sized rectangles and the given evaluation rule.

1. $f(x, y) = 5x - 2y$, $1 \le x \le 3$, $0 \le y \le 1$, $n = 4$, evaluate at midpoint
2. $f(x, y) = 4x^2 + y$, $0 \le x \le 1$, $1 \le y \le 3$, $n = 4$, evaluate at midpoint

In exercises 3–10, evaluate the double integral.

3. $\iint_R (4x + 9x^2y^2)\,dA$, where $R = \{(x, y) \mid 0 \le x \le 3, 1 \le y \le 2\}$
4. $\iint_R 2e^{4x+2y}\,dA$, where $R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$
5. $\iint_R e^{-x^2-y^2}\,dA$, where $R = \{(x, y) \mid 1 \le x^2 + y^2 \le 4\}$
6. $\iint_R 2xy\,dA$, where R is bounded by $y = x$, $y = 2 - x$ and $y = 0$
7. $\int_{-1}^{1} \int_{x^2}^{2x} (2xy - 1)\,dy\,dx$
8. $\int_{0}^{1} \int_{2x}^{2} (3y^2x + 4)\,dy\,dx$

Review Exercises

9. $\iint_R xy\,dA$, where R is bounded by $r = 2\cos\theta$

10. $\iint_R \sin(x^2 + y^2)\,dA$, where R is bounded by $x^2 + y^2 = 4$

In exercises 11 and 12, approximate the double integral.

11. $\iint_R 4xy\,dA$, where R is bounded by $y = x^2 - 4$ and $y = \ln x$

12. $\iint_R 6x^2y\,dA$, where R is bounded by $y = \cos x$ and $y = x^2 - 1$

In exercises 13–24, compute the volume of the solid.

13. Bounded by $z = 1 - x^2$, $z = 0$, $y = 0$ and $y = 1$

14. Bounded by $z = 4 - x^2 - y^2$, $z = 0$, $x = 0$, $x + y = 1$ and $y = 0$

15. Between $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$

16. Under $z = e^{\sqrt{x^2+y^2}}$ and inside $x^2 + y^2 = 4$

17. Bounded by $x + 2y + z = 8$ and the coordinate planes

18. Bounded by $x + 5y + 7z = 1$ and the coordinate planes

19. Bounded by $z = \sqrt{x^2 + y^2}$ and $z = 4$

20. Bounded by $x = \sqrt{y^2 + z^2}$ and $x = 2$

21. Between $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 4$

22. Inside $x^2 + y^2 + z^2 = 4z$ and below $z = 1$

23. Under $z = 6 - x^2 - y^2$ and inside $x^2 + y^2 = 1$

24. Under $z = x$ and inside $r = \cos\theta$

In exercises 25 and 26, change the order of integration.

25. $\int_0^2 \int_0^{x^2} f(x, y)\,dy\,dx$

26. $\int_0^2 \int_{x^2}^4 f(x, y)\,dy\,dx$

In exercises 27 and 28, convert to polar coordinates and evaluate the integral.

27. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x\,dy\,dx$

28. $\int_0^2 \int_0^{\sqrt{4-x^2}} 2\sqrt{x^2 + y^2}\,dy\,dx$

In exercises 29–32, find the mass and center of mass.

29. The lamina bounded by $y = 2x$, $y = x$ and $x = 2$, $\rho(x, y) = 2x$

30. The lamina bounded by $y = x$, $y = 4 - x$ and $y = 0$, $\rho(x, y) = 2y$

31. The solid bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, $y + z = 2$, $\rho(x, y, z) = 2$

32. The solid bounded by $x = \sqrt{y^2 + z^2}$, $x = 2$, $\rho(x, y, z) = 3x$

In exercises 33 and 34, use a double integral to find the area.

33. Bounded by $y = x^2$, $y = 2 - x$ and $y = 0$

34. One leaf of $r = \sin 4\theta$

In exercises 35 and 36, find the average value of the function on the indicated region.

35. $f(x, y) = x^2$, region bounded by $y = 2x$, $y = x$ and $x = 1$

36. $f(x, y) = \sqrt{x^2 + y^2}$, region bounded by $x^2 + y^2 = 1$, $x = 0$, $y = 0$

In exercises 37–42, evaluate or estimate the surface area.

37. The portion of $z = 2x + 4y$ between $y = x$, $y = 2$ and $x = 0$

38. The portion of $z = x^2 + 6y$ between $y = x^2$ and $y = 4$

39. The portion of $z = xy$ inside $x^2 + y^2 = 8$, in the first octant

40. The portion of $z = \sin(x^2 + y^2)$ inside $x^2 + y^2 = \pi$

41. The portion of $z = \sqrt{x^2 + y^2}$ below $z = 4$

42. The portion of $x + 2y + 3z = 6$ in the first octant

In exercises 43–50, set up the triple integral $\iiint_Q f(x, y, z)\,dV$ in an appropriate coordinate system. If $f(x, y, z)$ is given, evaluate the integral.

43. $f(x, y, z) = z(x + y)$,
$Q = \{(x, y, z) | 0 \le x \le 2, -1 \le y \le 1, -1 \le z \le 1\}$

44. $f(x, y, z) = 2xye^{yz}$,
$Q = \{(x, y, z) | 0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 1\}$

45. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, Q is above $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 4$.

46. $f(x, y, z) = 3x$, Q is the region below $z = \sqrt{x^2 + y^2}$, above $z = 0$ and inside $x^2 + y^2 = 4$.

Review Exercises

47. Q is bounded by $x + y + z = 6, z = 0, y = x, y = 2$ and $x = 0$.

48. Q is the region below $z = \sqrt{4 - x^2 - y^2}$, above $z = 0$ and inside $x^2 + y^2 = 1$.

49. Q is the region below $z = \sqrt{4 - x^2 - y^2}$ and above $z = 0$.

50. Q is the region below $z = 6 - x - y$, above $z = 0$ and inside $x^2 + y^2 = 8$.

In exercises 51–54, evaluate the integral after changing coordinate systems.

51. $\int_0^1 \int_x^{\sqrt{2-x^2}} \int_0^{\sqrt{x^2+y^2}} e^z \, dz \, dy \, dx$

52. $\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^2 4z \, dz \, dx \, dy$

53. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx$

54. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} dz \, dx \, dy$

In exercises 55–60, write the given equation in (a) cylindrical and (b) spherical coordinates.

55. $y = 3$

56. $x^2 + y^2 = 9$

57. $x^2 + y^2 + z^2 = 4$

58. $y = x$

59. $z = \sqrt{x^2 + y^2}$

60. $z = 4$

In exercises 61–66, sketch the graph.

61. $r = 4$

62. $\rho = 4$

63. $\theta = \frac{\pi}{4}$

64. $\phi = \frac{\pi}{4}$

65. $r = 2\cos\theta$

66. $\rho = 2\sec\phi$

In exercises 67 and 68, find a transformation from a rectangular region S in the uv-plane to the region R.

67. R bounded by $y = 2x - 1, y = 2x + 1, y = 2 - 2x$ and $y = 4 - 2x$

68. R inside $x^2 + y^2 = 9$, outside $x^2 + y^2 = 4$ and in the second quadrant

In exercises 69 and 70, evaluate the double integral.

69. $\iint_R e^{y-2x} \, dA$, where R is given in exercise 67

70. $\iint_R (y + 2x)^3 \, dA$, where R is given in exercise 67

In exercises 71 and 72, find the Jacobian of the given transformation.

71. $x = u^2 v, y = 4u + v^2$

72. $x = 4u - 5v, y = 2u + 3v$

EXPLORATORY EXERCISES

1. Let S be a sphere of radius R centered at the origin. Different types of symmetry produce different simplifications in integration. If $f(x, y, z) = -f(-x, y, z)$, show that $\iiint_S f \, dV = 0$. If $f(x, y, z) = -f(x, -y, z)$, show that $\iiint_S f \, dV = 0$. If $f(x, y, z) = -f(-x, -y, z)$, show that $\iiint_S f \, dV = 0$. If $f(x, y, z) = -f(-x, -y, -z)$, what, if anything, can be said about $\iiint_S f \, dV$? Next, suppose that $f(a, b, c) = f(x, y, z)$ whenever $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$. Show that $\iiint_S f(x, y, z) \, dV = 4\pi \int_0^R \rho^2 g(\rho) \, d\rho$ for some function g. (State the relationship between g and f.) If f can be written in cylindrical coordinates as $f(r, \theta, z) = g(r)$, for some continous function g, simplify $\iiint_S f \, dV$ as much as possible.

2. Let S be the solid bounded above by $z = \sqrt{R^2 - x^2 - y^2}$ and below by $z = \sqrt{x^2 + y^2}$. Set up and simplify as much as possible $\iiint_S f \, dV$ in each of the following cases: (a) $f(x, y, z) = -f(-x, y, z)$; (b) $f(x, y, z) = -f(x, -y, z)$; (c) $f(x, y, z) = -f(-x, -y, z)$; (d) $f(x, y, z) = -f(-x, -y, -z)$; (e) $f(r, \theta, z) = g(r)$, for some function g and (f) $f(\rho, \phi, \theta) = g(\rho)$ for some function g.