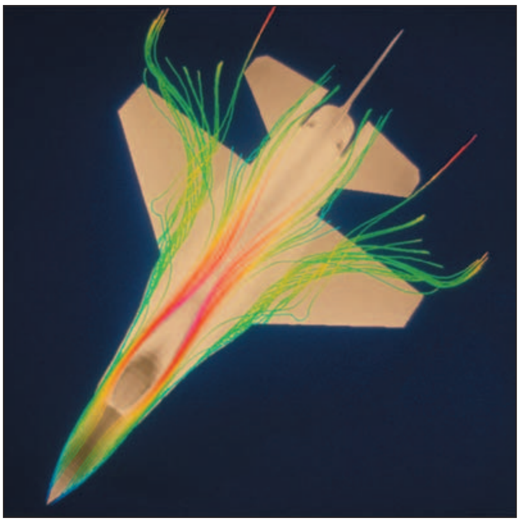
Vector Calculus

CHAPTER
14



The Volkswagen Beetle was one of the most beloved and recognizable cars of the 1950s, 1960s and 1970s. So, Volkswagen's decision to release a redesigned Beetle in 1998 created quite a stir in the automotive world. The new Beetle resembles the classic Beetle, but has been modernized to improve gas mileage, safety, handling and overall performance. The calculus that we introduce in this chapter will provide you with some of the basic tools necessary for designing and analyzing automobiles, aircraft and other types of complex machinery.

Think about how you might redesign an automobile to improve its aerodynamic performance. Engineers have identified many important principles of aerodynamics, but the design of a complicated structure like a car still has an element of trial and error. Before high-speed computers were available, engineers built small-scale or full-scale models of new designs and tested them in a wind tunnel. Unfortunately, such models don't always provide adequate information and can be prohibitively expensive to build, particularly if you have 20 or 30 new ideas you'd like to try.

The old Beetle

The new Beetle

With modern computers, wind tunnel tests can be accurately simulated by sophisticated programs. Mathematical models give engineers the ability to thoroughly test anything from minor modifications to radical changes.

The calculus that goes into a computer simulation of a wind tunnel is beyond what you've seen so far. Such simulations must keep track of the air velocity at each point on and around a car. A function assigning a vector (e.g., a velocity vector) to each point in space is called a vector field, which we introduce in section 14.1. To determine where vortices and turbulence occur in a fluid flow, you must compute line integrals, which are discussed in sections 14.2 and 14.3. The curl and divergence, introduced in section 14.5, allow you to analyze the rotational and linear properties of a fluid flow. Other properties of three-dimensional objects, such as mass and moments of inertia for a thin shell (such as a dome of a building), require the evaluation of surface integrals, which we develop in section 14.6. The relationships among line integrals, surface integrals, double integrals and triple integrals are explored in the remaining sections of the chapter.

In the case of the redesigned Volkswagen Beetle, computer simulations resulted in numerous improvements over the original. One measure of a vehicle's aerodynamic efficiency is its drag coefficient. Without getting into the technicalities, the lower its drag coefficient is, the less the velocity of the car is reduced by air resistance. The original Beetle has a drag coefficient of 0.46 (as reported by Robertson and Crowe in *Engineering Fluid Mechanics*). By comparison, a low-slung (and quite aerodynamic) 1985 Chevrolet Corvette has a drag coefficient of 0.34. Volkswagen's specification sheet for the new Beetle lists a drag coefficient of 0.38, representing a considerable reduction in air drag from the original Beetle. Through careful mathematical analysis, Volkswagen improved the performance of the Beetle while retaining the distinctive shape of the original car.

14.1 VECTOR FIELDS

To analyze the flight characteristics of an airplane, engineers use wind tunnel tests to provide information about the flow of air over the wings and around the fuselage. As you can imagine, to model such a test mathematically, we need to be able to describe the velocity of the air at various points throughout the tunnel. So, we need to define a function that assigns a vector to each point in space. Such a function would have both a multidimensional domain (like the functions of Chapters 12 and 13) and a multidimensional range (like the vector-valued functions introduced in Chapter 11). We call such a function a *vector field.* Although vector fields in higher dimensions can be very useful, we will focus here on vector fields in two and three dimensions.

DEFINITION 1.1

A **vector field** in the plane is a function $\mathbf{F}(x, y)$ mapping points in $\mathbb{R}^2$ into the set of two-dimensional vectors V_2. We write

$$\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y)\rangle = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j},$$

for scalar functions $f_1(x, y)$ and $f_2(x, y)$. In space, a **vector field** is a function $\mathbf{F}(x, y, z)$ mapping points in $\mathbb{R}^3$ into the set of three-dimensional vectors V_3. In this case, we write

$$\begin{aligned}\mathbf{F}(x, y, z) &= \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)\rangle \\ &= f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k},\end{aligned}$$

for scalar functions $f_1(x, y, z)$, $f_2(x, y, z)$ and $f_3(x, y, z)$.

To describe a two-dimensional vector field graphically, we draw a collection of the vectors $\mathbf{F}(x, y)$ for various points (x, y) in the domain, in each case drawing the vector so that its initial point is located at (x, y). We illustrate this in example 1.1.

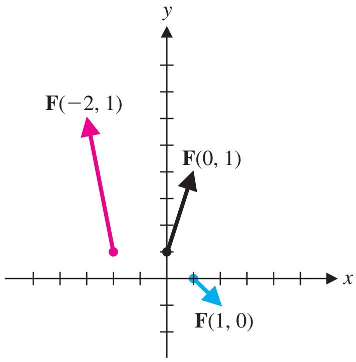


FIGURE 14.1
Values of $\mathbf{F}(x, y)$

EXAMPLE 1.1 Plotting a Vector Field

For the vector field $\mathbf{F}(x, y) = \langle x + y, 3y - x \rangle$, evaluate (a) $\mathbf{F}(1, 0)$, (b) $\mathbf{F}(0, 1)$ and (c) $\mathbf{F}(-2, 1)$. Plot each vector $\mathbf{F}(x, y)$ using the point (x, y) as the initial point.

Solution (a) Taking $x = 1$ and $y = 0$, we have $\mathbf{F}(1, 0) = \langle 1 + 0, 0 - 1 \rangle = \langle 1, -1 \rangle$. In Figure 14.1, we have plotted the vector $\langle 1, -1 \rangle$ with its initial point located at the point $(1, 0)$, so that its terminal point is located at $(2, -1)$.

(b) Taking $x = 0$ and $y = 1$, we have $\mathbf{F}(0, 1) = \langle 0 + 1, 3 - 0 \rangle = \langle 1, 3 \rangle$. In Figure 14.1, we have also indicated the vector $\langle 1, 3 \rangle$, taking the point $(0, 1)$ as its initial point, so that its terminal point is located at $(1, 4)$.

(c) With $x = -2$ and $y = 1$, we have $\mathbf{F}(-2, 1) = \langle -2 + 1, 3 + 2 \rangle = \langle -1, 5 \rangle$. In Figure 14.1, the vector $\langle -1, 5 \rangle$ is plotted by placing its initial point at $(-2, 1)$ and its terminal point at $(-3, 6)$. ■

Graphing vector fields poses something of a problem. Notice that the graph of a two-dimensional vector field would be *four*-dimensional (i.e., two independent variables plus two dimensions for the vectors). Likewise, the graph of a three-dimensional vector field would be *six*-dimensional. Despite this, we can visualize many of the important properties of a vector field by plotting a number of values of the vector field as we had started to do in Figure 14.1. In general, by the **graph of the vector field** $\mathbf{F}(x, y)$, we mean a two-dimensional graph with vectors $\mathbf{F}(x, y)$, plotted with their initial point located at (x, y), for a variety of points (x, y). Many graphing calculators and computer algebra systems have commands to graph vector fields. Notice in example 1.2 that the vectors in the computer-generated graphs are not drawn to the correct length. Instead, some software packages automatically shrink or stretch all of the vectors proportionally to a size that avoids cluttering up the overall graph.

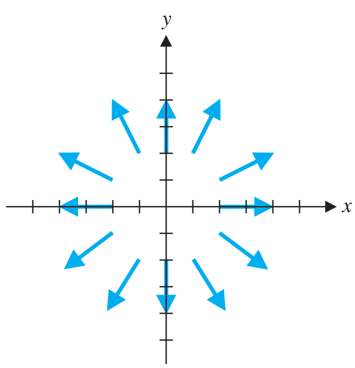


FIGURE 14.2a
$\mathbf{F}(x, y) = \langle x, y \rangle$

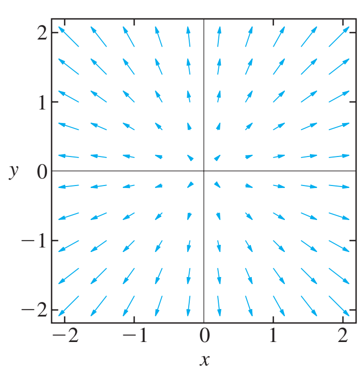


FIGURE 14.2b
$\mathbf{F}(x, y) = \langle x, y \rangle$

EXAMPLE 1.2 Graphing Vector Fields

Graph the vector fields $\mathbf{F}(x, y) = \langle x, y \rangle$, $\mathbf{G}(x, y) = \dfrac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$ and $\mathbf{H}(x, y) = \langle y, -x \rangle$ and identify any patterns.

Solution First choose a variety of points (x, y), evaluate the vector field at these points and plot the vectors using (x, y) as the initial point. Notice that in the following table we have chosen points on the axes and in each of the four quadrants.

(x, y)	$\langle x, y \rangle$		(x, y)	$\langle x, y \rangle$
$(2, 0)$	$\langle 2, 0 \rangle$		$(-2, 1)$	$\langle -2, 1 \rangle$
$(1, 2)$	$\langle 1, 2 \rangle$		$(-2, 0)$	$\langle -2, 0 \rangle$
$(2, 1)$	$\langle 2, 1 \rangle$		$(-1, -2)$	$\langle -1, -2 \rangle$
$(0, 2)$	$\langle 0, 2 \rangle$		$(0, -2)$	$\langle 0, -2 \rangle$
$(-1, 2)$	$\langle -1, 2 \rangle$		$(1, -2)$	$\langle 1, -2 \rangle$
$(-2, -1)$	$\langle -2, -1 \rangle$		$(2, -1)$	$\langle 2, -1 \rangle$

The vectors indicated in the table are plotted in Figure 14.2a. A computer-generated plot of the vector field is shown in Figure 14.2b. Notice that the vectors drawn here have not been drawn to scale, in order to improve the readability of the graph.

In both plots, notice that the vectors all point away from the origin and increase in length as the initial points get farther from the origin. In fact, the initial point (x, y) lies

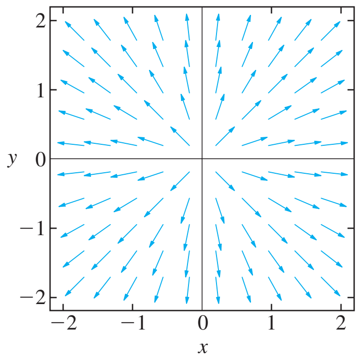


FIGURE 14.2c
$$\mathbf{G}(x, y) = \frac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$$

a distance $\sqrt{x^2+y^2}$ from the origin and the vector $\langle x, y\rangle$ has length $\sqrt{x^2+y^2}$. So, the length of each vector corresponds to the distance from its initial point to the origin. This gives us an important clue about the graph of $\mathbf{G}(x, y)$. Although the formula may look messy, notice that $\mathbf{G}(x, y)$ is the same as $\mathbf{F}(x, y)$ except for the division by $\sqrt{x^2+y^2}$, which is the magnitude of the vector $\langle x, y\rangle$. Recall that dividing a vector by its magnitude yields a unit vector in the same direction. Thus, for each (x, y), $\mathbf{G}(x, y)$ has the same direction as $\mathbf{F}(x, y)$, but is a unit vector. A computer-generated plot of $\mathbf{G}(x, y)$ is shown in Figure 14.2c.

We compute some sample vectors for $\mathbf{H}(x, y)$ in the following table and plot these in Figure 14.3a.

(x, y)	$\langle y, -x\rangle$		(x, y)	$\langle y, -x\rangle$
$(2, 0)$	$\langle 0, -2\rangle$		$(-2, 1)$	$\langle 1, 2\rangle$
$(1, 2)$	$\langle 2, -1\rangle$		$(-2, 0)$	$\langle 0, 2\rangle$
$(2, 1)$	$\langle 1, -2\rangle$		$(-1, -2)$	$\langle -2, 1\rangle$
$(0, 2)$	$\langle 2, 0\rangle$		$(0, -2)$	$\langle -2, 0\rangle$
$(-1, 2)$	$\langle 2, 1\rangle$		$(1, -2)$	$\langle -2, -1\rangle$
$(-2, -1)$	$\langle -1, 2\rangle$		$(2, -1)$	$\langle -1, -2\rangle$

A computer-generated plot of $\mathbf{H}(x, y)$ is shown in Figure 14.3b. If you think of $\mathbf{H}(x, y)$ as representing the velocity field for a fluid in motion, the vectors suggest a circular rotation of the fluid. Recall that tangent lines to a circle are perpendicular to radius lines. The radius vector from the origin to the point (x, y) is $\langle x, y\rangle$, which is perpendicular to the vector $\langle y, -x\rangle$, since $\langle x, y\rangle \cdot \langle y, -x\rangle = 0$. Also, notice that the vectors are not of constant size. As for $\mathbf{F}(x, y)$, the length of the vector $\langle y, -x\rangle$ is $\sqrt{x^2+y^2}$, which is the distance from the origin to the initial point (x, y).

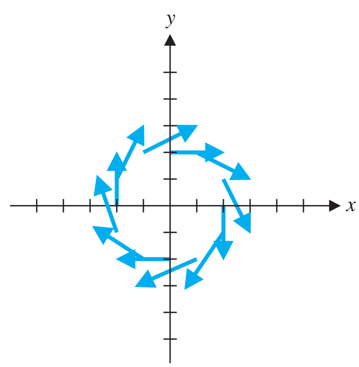


FIGURE 14.3a
$\mathbf{H}(x, y) = \langle y, -x\rangle$

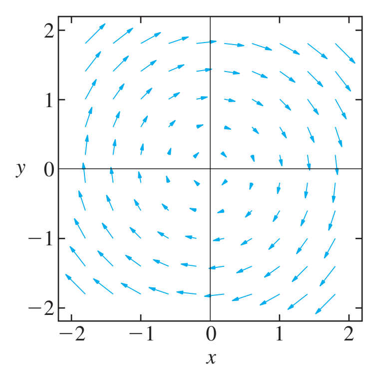


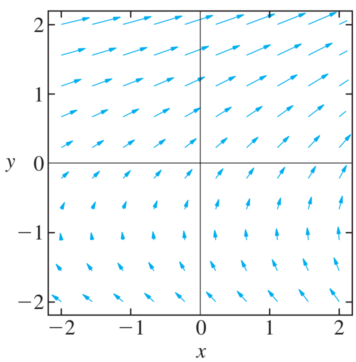
FIGURE 14.3b
$\mathbf{H}(x, y) = \langle y, -x\rangle$

Although the ideas in example 1.2 are very important, most vector fields are too complicated to effectively draw by hand. Example 1.3 illustrates how to relate the component functions of a vector field to its graph.

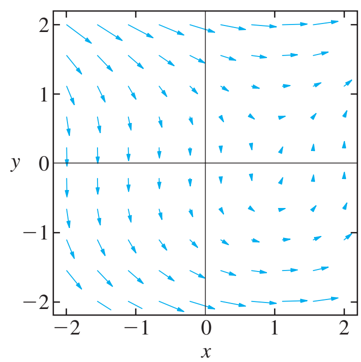
EXAMPLE 1.3 Matching Vector Fields to Graphs

Match the vector fields $\mathbf{F}(x, y) = \langle y^2, x - 1\rangle$, $\mathbf{G}(x, y) = \langle y + 1, e^{x/6}\rangle$ and $\mathbf{H}(x, y) = \langle y^3, x^2 - 1\rangle$ to the graphs shown.

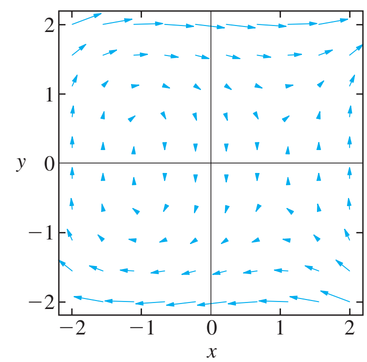
Solution While there is no general procedure for matching vector fields to their graphs, you should look for special features of the components of the vector fields and try to locate these in the graphs. For instance, the first component of $\mathbf{F}(x, y)$ is $y^2 \geq 0$, so the vectors $\mathbf{F}(x, y)$ will never point to the left. Graphs A and C both have vectors with negative first components (in the fourth quadrant), so Graph B must be the graph of $\mathbf{F}(x, y)$. The vectors in Graph B also have small vertical components near $x = 1$, where the second component of $\mathbf{F}(x, y)$ equals zero. Similarly, the second component of $\mathbf{G}(x, y)$ is $e^{x/6} > 0$, so the vectors $\mathbf{G}(x, y)$ will always point upward. Graph A is the only one of these graphs with this property. Further, the vectors in Graph A are almost vertical near $y = -1$, where the first component of $\mathbf{G}(x, y)$ equals zero. That leaves Graph C for $\mathbf{H}(x, y)$, but let's check to be sure this is reasonable. Observe that the first component of $\mathbf{H}(x, y)$ is y^3, which is negative for $y < 0$ and positive for $y > 0$. The vectors then point to the left for $y < 0$ and to the right for $y > 0$, as seen in Graph C. Finally, the vectors in Graph C have small vertical components near $x = 1$ and $x = -1$, where the second component of $\mathbf{H}(x, y)$ equals zero.



GRAPH A



GRAPH B



GRAPH C

As you might imagine, vector fields in space are typically more difficult to sketch than vector fields in the plane, but the idea is the same. That is, pick a variety of representative points and plot the vector $\mathbf{F}(x, y, z)$ with its initial point located at (x, y, z). Unfortunately, the difficulties associated with representing three-dimensional vectors on two-dimensional paper reduce the usefulness of these graphs.

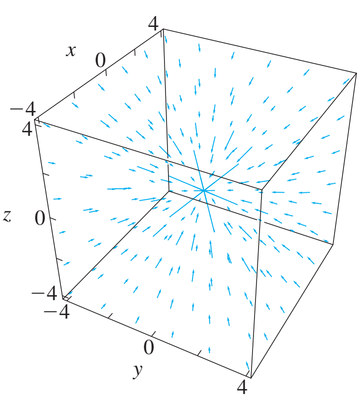


FIGURE 14.4
Gravitational force field

EXAMPLE 1.4 Graphing a Vector Field in Space

Use a CAS to graph the vector field $\mathbf{F}(x, y, z) = \dfrac{\langle -x, -y, -z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$.

Solution In Figure 14.4, we show a computer-generated plot of the vector field $\mathbf{F}(x, y, z)$.

Notice that the vectors all point toward the origin, getting larger near the origin (where the field is undefined). You should get the sense of an attraction to the origin that gets stronger the closer you get. In fact, you might have recognized that $\mathbf{F}(x, y, z)$ describes the gravitational force field for an object located at the origin or the electrical field for a charge located at the origin. ■

If the vector field graphed in Figure 14.4 represents a force field, then the graph indicates that an object acted on by this force field will be drawn toward the origin. However, this does

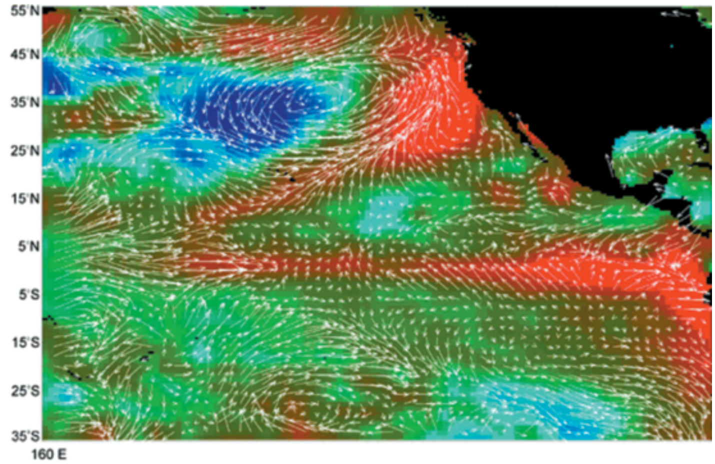


FIGURE 14.5
Velocity field for Pacific Ocean currents (March 1998)

not mean that a given object must move in a straight path toward the origin. For instance, an object with initial position (2, 0, 0) and initial velocity $\langle 0, 2, 0\rangle$ will spiral in toward the origin. To more accurately sketch the *path* followed by an object, we need additional information. Notice that in many cases, we can think of velocity as not explicitly depending on time, but instead depending on location. For instance, imagine watching a mountain stream with waterfalls and whirlpools that don't change (significantly) over time. In this case, the motion of a leaf dropped into the stream would depend on *where* you drop the leaf, rather than *when* you drop the leaf. This says that the velocity of the stream is a function of location. That is, the velocity of any particle located at the point (x, y) in the stream can be described by a vector field $\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y)\rangle$, called the **velocity field.** The path of any given particle in the flow starting at the point (x_0, y_0) is then the curve traced out by $\langle x(t), y(t)\rangle$, where $x(t)$ and $y(t)$ are the solutions of the differential equations $x'(t) = f_1(x(t), y(t))$ and $y'(t) = f_2(x(t), y(t))$, with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$. In these cases, we can use the velocity field to construct **flow lines,** which indicate the path followed by a particle starting at a given point in the flow.

In practice, one way to visualize the velocity field for a given process is to plot a number of velocity vectors at a single instant in time. Figure 14.5 shows the velocity field of Pacific Ocean currents in March 1998. The picture is color-coded for temperature, with a band of water swinging up from South America to the Pacific northwest representing "El Niño." The velocity field provides information about how the warmer and cooler areas of ocean water are likely to change. Since El Niño is associated with significant climate changes, an understanding of its movement is critically important.

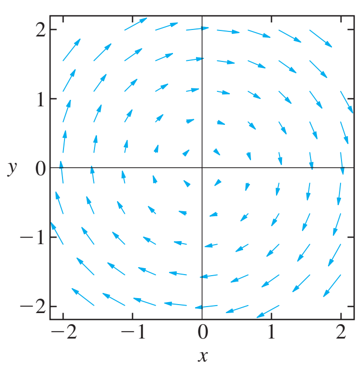


FIGURE 14.6a
$\langle y, -x\rangle$

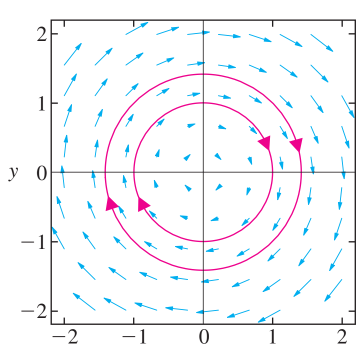


FIGURE 14.6b
Flow lines: $\langle y, -x\rangle$

EXAMPLE 1.5 Graphing Vector Fields and Flow Lines

Graph the vector fields $\langle y, -x\rangle$ and $\langle 2, 1 + 2xy\rangle$ and for each, sketch-in approximate flow lines through the points (0, 1), (0, −1) and (1, 1).

Solution We have previously graphed the vector field $\langle y, -x\rangle$ in example 1.2 and show a computer-generated graph of the vector field in Figure 14.6a. Notice that the plotted vectors nearly join together as concentric circles. In Figure 14.6b, we have superimposed circular paths that stay tangent to the velocity field and pass through the

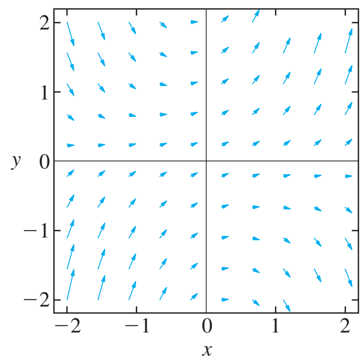


FIGURE 14.7a
$\langle 2, 1 + 2xy\rangle$

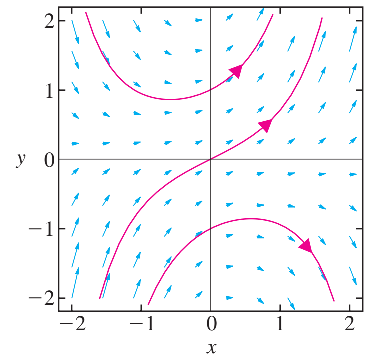


FIGURE 14.7b
Flow lines: $\langle 2, 1 + 2xy\rangle$

points (0, 1), (0, −1) and (1, 1). (Notice that the first two of these paths are the same.) It isn't difficult to verify that the flow lines are indeed circles, as follows. Observe that a circle of radius a centered at the origin with a clockwise orientation (as indicated) can be described by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle a \sin t, a \cos t\rangle$. The velocity vector $\mathbf{r}'(t) = \langle a \cos t, -a \sin t\rangle$ gives a tangent vector to the curve for each t. If we eliminate the parameter, the velocity field for the position vector $\mathbf{r} = \langle x, y\rangle$ is given by $\mathbf{T} = \langle y, -x\rangle$, which is the vector field we are presently plotting.

We show a computer-generated graph of the vector field $\langle 2, 1 + 2xy\rangle$ in Figure 14.7a, which suggests some parabolic-like paths. In Figure 14.7b, we sketch two of these paths through the points (0, 1) and (0, −1). However, the vectors in Figure 14.7a also indicate some paths that look more like cubics, such as the path through (0, 0) sketched in Figure 14.7b. In this case, though, it's more difficult to determine equations for the flow lines. As it turns out, these are neither parabolic nor cubic. We'll explore this further in the exercises. ■

A good sketch of a vector field allows us to visualize at least some of the flow lines. However, even a great sketch can't replace the information available from an exact equation for the flow lines. We can solve for an equation of a flow line by noting that if $\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y)\rangle$ is a velocity field and $\langle x(t), y(t)\rangle$ is the position function, then $x'(t) = f_1(x, y)$ and $y'(t) = f_2(x, y)$. By the chain rule, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} = \frac{f_2(x, y)}{f_1(x, y)}. \tag{1.1}$$

Equation (1.1) is a first-order differential equation for the unknown function $y(x)$. We refer you to section 7.2, where we developed a technique for solving one group of differential equations, called *separable equations*. In section 7.3, we presented a method (Euler's method) for approximating the solution of any first-order differential equation passing through a given point.

EXAMPLE 1.6 Using a Differential Equation to Construct Flow Lines

Construct the flow lines for the vector field $\langle y, -x\rangle$.

Solution From (1. 1), the flow lines are solutions of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

From our discussion in section 7.2, this differential equation is separable and can be solved as follows. We first rewrite the equation as

$$y\frac{dy}{dx} = -x.$$

Integrating both sides with respect to x gives us

$$\int y\frac{dy}{dx}\,dx = -\int x\,dx,$$

so that

$$\frac{y^2}{2} = -\frac{x^2}{2} + k.$$

Multiplying both sides by 2 and replacing the constant $2k$ by c, we have

$$y^2 = -x^2 + c$$

or

$$x^2 + y^2 = c.$$

That is, for any choice of the constant $c > 0$, the solution corresponds to a circle centered at the origin. The vector field and the flow lines are then exactly as plotted in Figures 14.6a and 14.6b. ■

In example 1.7, we illustrate the use of Euler's method for constructing an approximate flow line.

EXAMPLE 1.7 Using Euler's Method to Approximate Flow Lines

Use Euler's method with $h = 0.05$ to approximate the flow line for the vector field $\langle 2, 1 + 2xy\rangle$ passing through the point $(0, 1)$, for $0 \le x \le 1$.

Solution Recall that for the differential equation $y' = f(x, y)$ and for any given value of h, Euler's method produces a sequence of approximate values of the solution function $y = y(x)$ corresponding to the points $x_i = x_0 + ih$, for $i = 1, 2, \ldots$. Specifically, starting from an initial point (x_0, y_0), where $y_0 = y(x_0)$, we construct the approximate values $y_i \approx y(x_i)$, where the y_i's are determined iteratively from the equation

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, 2, \ldots.$$

Since the flow line must pass through the point $(0, 1)$, we start with $x_0 = 0$ and $y_0 = 1$. Further, here we have the differential equation

$$\frac{dy}{dx} = \frac{1 + 2xy}{2} = \frac{1}{2} + xy = f(x, y).$$

In this case (unlike example 1.6), the differential equation is not separable and you do not know how to solve it exactly. For Euler's method, we then have

$$y_{i+1} = y_i + hf(x_i, y_i) = y_i + 0.05\left(\frac{1}{2} + x_i y_i\right),$$

with $x_0 = 0$, $y_0 = 1$. For the first two steps, we have

$$y_1 = y_0 + 0.05\left(\frac{1}{2} + x_0 y_0\right) = 1 + 0.05(0.5) = 1.025,$$

$x_1 = 0.05$,

$$y_2 = y_1 + 0.05\left(\frac{1}{2} + x_1 y_1\right) = 1.025 + 0.05(0.5 + 0.05125) = 1.0525625$$

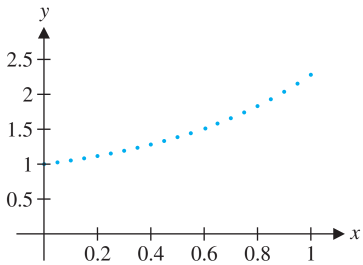


FIGURE 14.8
Approximate flow line through (0, 1)

and $x_2 = 0.1$. Continuing in this fashion, we get the sequence of approximate values indicated in the following table.

x_i	y_i		x_i	y_i		x_i	y_i
0	1		0.35	1.2344		0.70	1.6577
0.05	1.025		0.40	1.2810		0.75	1.7407
0.10	1.0526		0.45	1.3316		0.80	1.8310
0.15	1.0828		0.50	1.3866		0.85	1.9293
0.20	1.1159		0.55	1.4462		0.90	2.0363
0.25	1.1521		0.60	1.5110		0.95	2.1529
0.30	1.1915		0.65	1.5813		1.00	2.2801

A plot of these points is shown in Figure 14.8. Compare this path to the top curve (also through the point (0, 1)) shown in Figure 14.7b. ■

An important type of vector field with which we already have some experience is the *gradient field,* where the vector field is the gradient of some scalar function. Because of the importance of gradient fields, there are a number of terms associated with them. In Definition 1.2, we do not specify the number of independent variables, since the terms can be applied to functions of two, three or more variables.

DEFINITION 1.2

For any scalar function f, the vector field $\mathbf{F} = \nabla f$ is called the **gradient field** for the function f. We call f a **potential function** for $\mathbf{F}$. Whenever $\mathbf{F} = \nabla f$, for some scalar function f, we refer to $\mathbf{F}$ as a **conservative vector field.**

It's worth noting that if you read about conservative vector fields and potentials in some applied areas (such as physics and engineering), you will sometimes see the function $-f$ referred to as the potential function. This is a minor difference in terminology, only. In this text (as is traditional in mathematics), we will consistently refer to f as the potential function. Rest assured that everything we say here about conservative vector fields is also true in these applications areas. The only slight difference may be that we call f the potential function, while others may refer to $-f$ as the potential function.

Finding the gradient field corresponding to a given scalar function is a simple matter.

EXAMPLE 1.8 Finding Gradient Fields

Find the gradient fields corresponding to the functions (a) $f(x, y) = x^2y - e^y$ and (b) $g(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$, and use a CAS to sketch the fields.

Solution (a) We first compute the partial derivatives $\dfrac{\partial f}{\partial x} = 2xy$ and $\dfrac{\partial f}{\partial y} = x^2 - e^y$, so that

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2xy, x^2 - e^y \rangle.$$

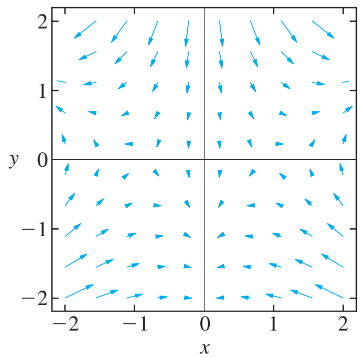


FIGURE 14.9a
$\nabla(x^2y - e^y)$

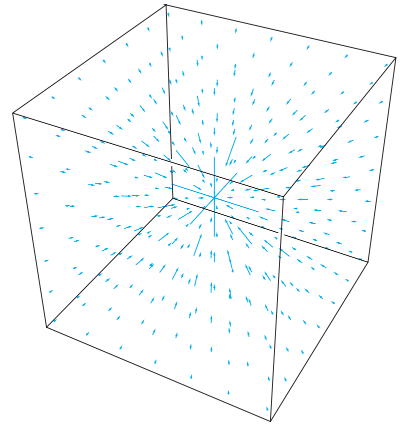


FIGURE 14.9b
$\nabla\left(\dfrac{1}{x^2+y^2+z^2}\right)$

A computer-generated graph of $\nabla f(x, y)$ is shown in Figure 14.9a.

(b) For $g(x, y, z) = (x^2 + y^2 + z^2)^{-1}$, we have

$$\frac{\partial g}{\partial x} = -(x^2 + y^2 + z^2)^{-2}(2x) = -\frac{2x}{x^2 + y^2 + z^2},$$

and by symmetry (think about this!), we conclude that $\dfrac{\partial g}{\partial y} = -\dfrac{2y}{x^2 + y^2 + z^2}$ and $\dfrac{\partial g}{\partial z} = -\dfrac{2z}{x^2 + y^2 + z^2}$. This gives us

$$\nabla g(x, y, z) = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = -\frac{2\langle x, y, z\rangle}{x^2 + y^2 + z^2}.$$

A computer-generated graph of $\nabla g(x, y, z)$ is shown in Figure 14.9b. ■

You will discover that many calculations involving vector fields simplify dramatically if the vector field is a gradient field (i.e., if the vector field is conservative). To take full advantage of these simplifications, you will need to be able to construct a potential function that generates a given conservative field. The technique introduced in example 1.9 will work for most of the examples in this chapter.

EXAMPLE 1.9 Finding Potential Functions

Determine whether each of the following vector fields is conservative. If it is, find a corresponding potential function $f(x, y)$: (a) $\mathbf{F}(x, y) = \langle 2xy - 3, x^2 + \cos y\rangle$ and (b) $\mathbf{G}(x, y) = \langle 3x^2y^2 - 2y, x^2y - 2x\rangle$.

Solution The idea here is to try to construct a potential function. In the process of trying to do so, we may instead recognize that there is no potential function for the given vector field. For (a), if $f(x, y)$ is a potential function for $\mathbf{F}(x, y)$, we have that

$$\nabla f(x, y) = \mathbf{F}(x, y) = \langle 2xy - 3, x^2 + \cos y\rangle,$$

so that
$$\frac{\partial f}{\partial x} = 2xy - 3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + \cos y. \tag{1.2}$$

Integrating the first of these two equations with respect to x and treating y as a constant, we get

$$f(x, y) = \int (2xy - 3)\,dx = x^2y - 3x + g(y). \tag{1.3}$$

Here, we have added an arbitrary function of y alone, $g(y)$, rather than a *constant* of integration, because any function of y is treated as a constant when integrating with respect to x. Differentiating the expression for $f(x, y)$ with respect to y gives us

$$\frac{\partial f}{\partial y}(x, y) = x^2 + g'(y) = x^2 + \cos y,$$

from (1.2). This gives us $g'(y) = \cos y$, so that

$$g(y) = \int \cos\ y\,dy = \sin\ y + c.$$

From (1.3), we now have

$$f(x, y) = x^2y - 3x + \sin y + c,$$

where c is an arbitrary constant. Since we have been able to construct a potential function, the vector field $\mathbf{F}(x, y)$ is conservative.

(b) Again, we assume that there is a potential function g for $\mathbf{G}(x, y)$ and try to construct it. In this case, we have

$$\nabla g(x, y) = \mathbf{G}(x, y) = \langle 3x^2y^2 - 2y, x^2y - 2x\rangle,$$

so that
$$\frac{\partial g}{\partial x} = 3x^2y^2 - 2y \quad \text{and} \quad \frac{\partial g}{\partial y} = x^2y - 2x. \tag{1.4}$$

REMARK 1.1

To find a potential function, you can either integrate $\frac{\partial f}{\partial x}$ with respect to x or integrate $\frac{\partial f}{\partial y}$ with respect to y. Before choosing which one to integrate, think about which integral will be easier to compute. In section 14.3, we introduce a simple method for determining whether or not a vector field is conservative.

Integrating the first equation in (1.4) with respect to x, we have

$$g(x, y) = \int (3x^2y^2 - 2y)\,dx = x^3y^2 - 2yx + h(y),$$

where h is an arbitrary function of y. Differentiating this with respect to y, we have

$$\frac{\partial f}{\partial y}(x, y) = 2x^3y - 2x + h'(y) = x^2y - 2x,$$

from (1.4). Solving for $h'(y)$, we get

$$h'(y) = x^2y - 2x - 2x^3y + 2x = x^2y - 2x^3y,$$

which is impossible, since $h(y)$ is a function of y alone. We then conclude that there is no potential function for $\mathbf{G}(x, y)$ and so, the vector field is not conservative. ■

Coulomb's law states that the electrostatic force on a charge q_0 due to a charge q is given by $\mathbf{F} = \frac{qq_0}{r^2}\mathbf{u}$, where r is the distance (in cm) between the charges and $\mathbf{u}$ is a unit vector from q to q_0. The unit of charge is esu and $\mathbf{F}$ is measured in dynes. The **electrostatic**

field E is defined as the force per unit charge, so that

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} = \frac{q}{r^2}\mathbf{u}.$$

In example 1.10, we compute the electrostatic field for an electric dipole.

EXAMPLE 1.10 Electrostatic Field of a Dipole

Find the electrostatic field due to a charge of $+1$ esu at $(1, 0)$ and a charge of -1 esu at $(-1, 0)$.

Solution The distance r from $(1, 0)$ to an arbitrary point (x, y) is $\sqrt{(x-1)^2 + y^2}$ and a unit vector from $(1, 0)$ to (x, y) is $\frac{1}{r}\langle x-1, y\rangle$. The contribution to **E** from $(1, 0)$ is then $\frac{1}{r^2}\frac{\langle x-1, y\rangle}{r} = \frac{\langle x-1, y\rangle}{[(x-1)^2 + y^2]^{3/2}}$. Similarly, the contribution to **E** from the negative charge at $(-1, 0)$ is $\frac{-1}{r^2}\frac{\langle x+1, y\rangle}{r} = \frac{-\langle x+1, y\rangle}{[(x+1)^2 + y^2]^{3/2}}$. Adding the two terms, we get the electrostatic field

$$\mathbf{E} = \frac{\langle x-1, y\rangle}{[(x-1)^2 + y^2]^{3/2}} - \frac{\langle x+1, y\rangle}{[(x+1)^2 + y^2]^{3/2}}.$$

A computer-generated graph of this vector field is shown in Figures 14.10a to 14.10c.

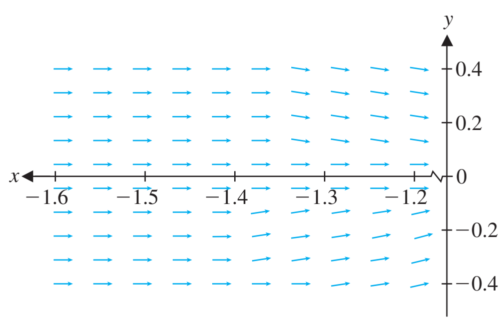


FIGURE 14.10a

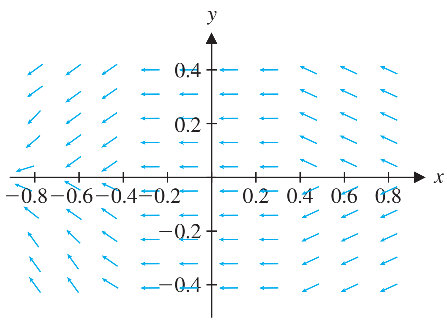


FIGURE 14.10b

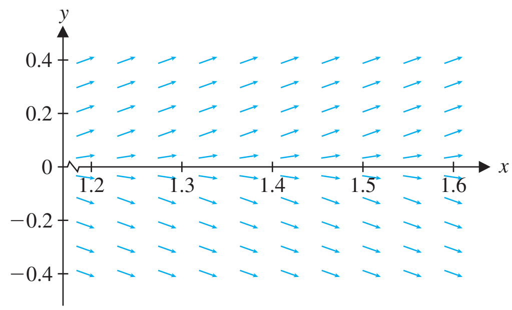


FIGURE 14.10c

Notice that the field lines point away from the positive charge at $(1, 0)$ and toward the negative charge at $(-1, 0)$. ■

EXERCISES 14.1

WRITING EXERCISES

1. Compare hand-drawn sketches of the vector fields $\langle y, -x\rangle$ and $\langle 10y, -10x\rangle$. In particular, describe which graph is easier to interpret. Computer-generated graphs of these vector fields are identical when the software "scales" the vector field by dividing out the 10. It may seem odd that computers don't draw accurate graphs, but explain why the software programmers chose to scale the vector fields.

2. The gravitational force field is an example of an "inverse square law." That is, the magnitude of the gravitational force is inversely proportional to the square of the distance from the origin. Explain why the $\frac{3}{2}$ exponent in the denominator of example 1.4 is correct for an inverse square law.

3. Explain why each vector in a vector field graph is tangent to a flow line. Explain why this means that a flow line can be visualized by joining together a large number of small (scaled) vector field vectors.

4. In example 1.9(b), explain why the presence of the x's in the expression for $g'(y)$ proves that there is no potential function.

In exercises 1–10, sketch several vectors in the vector field by hand and verify your sketch with a CAS.

1. $\mathbf{F}(x, y) = \langle -y, x\rangle$

2. $\mathbf{F}(x, y) = \dfrac{\langle -y, x\rangle}{\sqrt{x^2 + y^2}}$

3. $\mathbf{F}(x, y) = \langle 0, x^2\rangle$

4. $\mathbf{F}(x, y) = \langle 2x, 0\rangle$

5. $\mathbf{F}(x, y) = 2y\mathbf{i} + \mathbf{j}$

6. $\mathbf{F}(x, y) = -\mathbf{i} + y^2\mathbf{j}$

7. $\mathbf{F}(x, y, z) = \langle 0, z, 1\rangle$

8. $\mathbf{F}(x, y, z) = \langle 2, 0, 0\rangle$

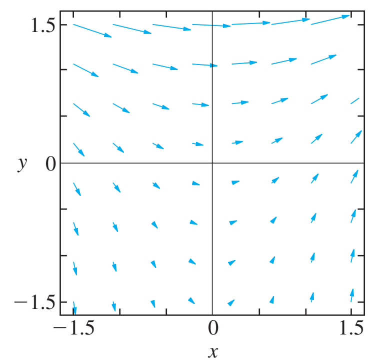
9. $\mathbf{F}(x, y, z) = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2 + y^2 + z^2}}$

10. $\mathbf{F}(x, y, z) = \dfrac{\langle x, y, z\rangle}{x^2 + y^2 + z^2}$

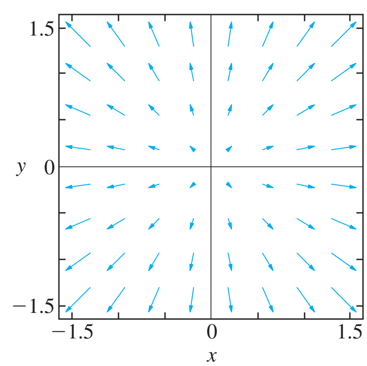
11. Match the vector fields with their graphs.

$$\mathbf{F}_1(x, y) = \frac{\langle x, y\rangle}{\sqrt{x^2 + y^2}}, \quad \mathbf{F}_2(x, y) = \langle x, y\rangle,$$

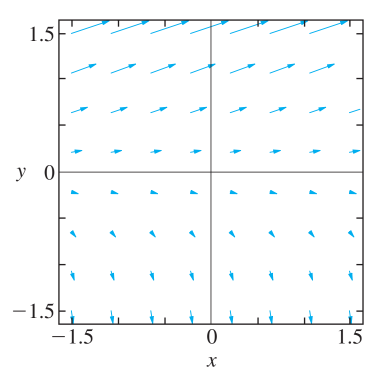
$$\mathbf{F}_3(x, y) = \langle e^y, x\rangle, \qquad \mathbf{F}_4(x, y) = \langle e^y, y\rangle$$



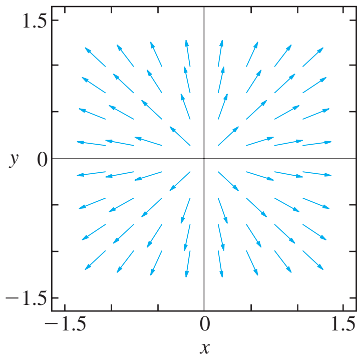
GRAPH A



GRAPH B

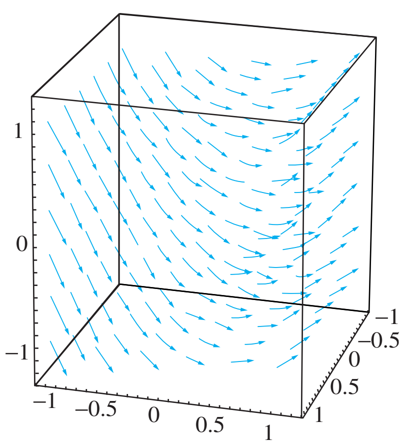


GRAPH C

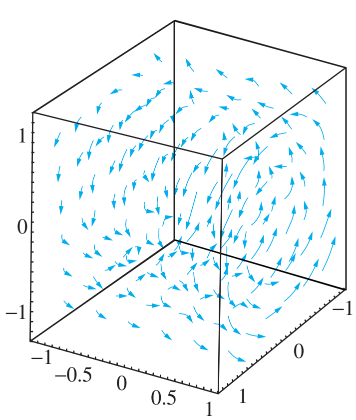


GRAPH D

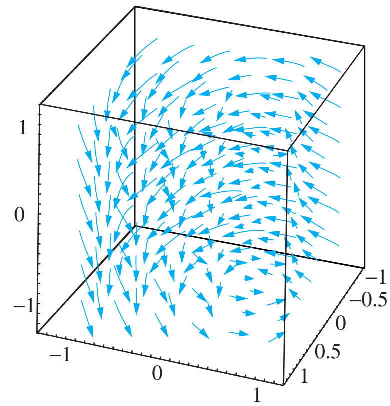
12. Match the vector fields with their graphs. $\mathbf{F}_1(x, y, z) = \langle 1, x, y \rangle$, $\mathbf{F}_2(x, y, z) = \langle 1, 1, y \rangle$, $\mathbf{F}_3(x, y, z) = \dfrac{\langle y, -x, 0 \rangle}{2 - z}$, $\mathbf{F}_4(x, y, z) = \dfrac{\langle z, 0, -x \rangle}{2 - y}$.



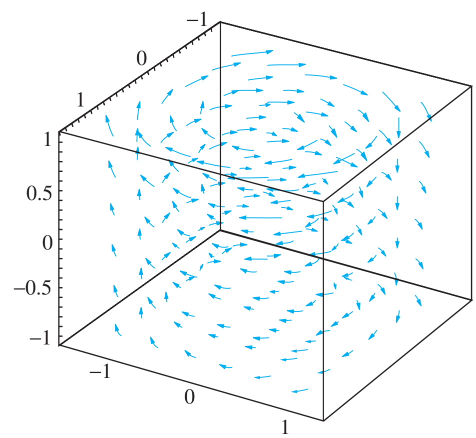
GRAPH A



GRAPH B



GRAPH C



GRAPH D

In exercises 13–22, find the gradient field corresponding to f. Use a CAS to graph it.

13. $f(x, y) = x^2 + y^2$

14. $f(x, y) = x^2 - y^2$

15. $f(x, y) = \sqrt{x^2 + y^2}$

16. $f(x, y) = \sin(x^2 + y^2)$

17. $f(x, y) = xe^{-y}$

18. $f(x, y) = y \sin x$

19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

20. $f(x, y, z) = xyz$

21. $f(x, y, z) = x^2 y + yz$

22. $f(x, y, z) = (x - y)^2 + z$

In exercises 23–34, determine whether or not the vector field is conservative. If it is, find a potential function.

23. $\langle y, x \rangle$

24. $\langle 2, y \rangle$

25. $\langle y, -x \rangle$

26. $\langle y, 1 \rangle$

27. $(x - 2xy)\mathbf{i} + (y^2 - x^2)\mathbf{j}$

28. $(x^2 - y)\mathbf{i} + (x - y)\mathbf{j}$

29. $\langle y \sin xy, x \sin xy \rangle$

30. $\langle y \cos x, \sin x - y \rangle$

31. $\langle 4x - z, 3y + z, y - x \rangle$

32. $\langle z^2 + 2xy, x^2 - z, 2xz - 1 \rangle$

33. $\langle y^2z^2 - 1, 2xyz^2, 4z^3 \rangle$

34. $\langle z^2 + 2xy, x^2 + 1, 2xz - 3 \rangle$

In exercises 35–42, find equations for the flow lines.

35. $\langle 2, \cos x \rangle$

36. $\langle x^2, 2 \rangle$

37. $\langle 2y, 3x^2 \rangle$

38. $\langle \frac{1}{y}, 2x \rangle$

39. $y\mathbf{i} + xe^y\mathbf{j}$

40. $e^{-x}\mathbf{i} + 2x\mathbf{j}$

41. $\langle y, y^2 + 1 \rangle$

42. $\langle 2, y^2 + 1 \rangle$

43. Suppose that $f(x)$, $g(y)$ and $h(z)$ are continuous functions. Show that $\langle f(x), g(y), h(z) \rangle$ is conservative, by finding a potential function.

44. Show that $\langle k_1, k_2 \rangle$ is conservative, for constants k_1 and k_2.

In exercises 45–52, use the notation $\mathbf{r} = \langle x, y \rangle$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2}$.

45. Show that $\nabla(r) = \dfrac{\mathbf{r}}{r}$.

46. Show that $\nabla(r^2) = 2\mathbf{r}$.

47. Find $\nabla(r^3)$.

48. Use exercises 45–47 to conjecture the value of $\nabla(r^n)$, for any positive integer n. Prove that your answer is correct.

49. Show that $\dfrac{\langle 1, 1 \rangle}{r}$ is *not* conservative.

50. Show that $\dfrac{\langle -y, x \rangle}{r^2}$ is conservative on the domain $y > 0$ by finding a potential function. Show that the potential function can be thought of as the polar angle θ.

51. The current in a wire produces a magnetic field $\mathbf{B} = \dfrac{k\langle -y, x \rangle}{r^2}$. Draw a sketch showing a wire and its magnetic field.

52. Show that $\dfrac{\mathbf{r}}{r^n} = \dfrac{\langle x, y \rangle}{(x^2 + y^2)^{n/2}}$ is conservative, for any integer n.

53. A two-dimensional force acts radially away from the origin with magnitude 3. Write the force as a vector field.

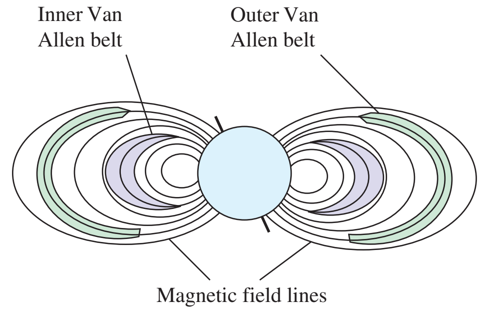
54. A two-dimensional force acts radially toward the origin with magnitude equal to the square of the distance from the origin. Write the force as a vector field.

55. A three-dimensional force acts radially toward the origin with magnitude equal to the square of the distance from the origin. Write the force as a vector field.

56. A three-dimensional force acts radially away from the z-axis (parallel to the xy-plane) with magnitude equal to the cube of the distance from the z-axis. Write the force as a vector field.

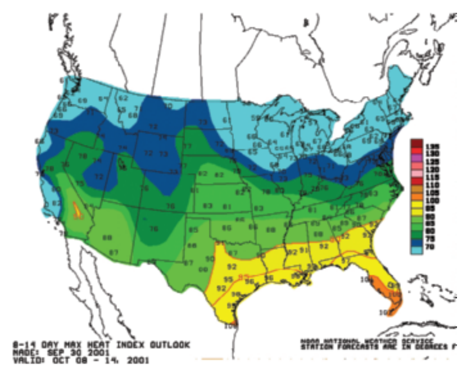
57. Derive the electrostatic field for positive charges q at $(-1, 0)$ and $(1, 0)$ and negative charge $-q$ at $(0, 0)$.

58. The figure shows the magnetic field of the earth. Compare this to the electrostatic field of a dipole shown in example 1.10. In what way is a bar magnet similar to an electric dipole?



59. If $T(x, y, z)$ gives the temperature at position (x, y, z) in space, the velocity field for heat flow is given by $\mathbf{F} = -k\nabla T$ for a constant $k > 0$. This is known as **Fourier's law.** Use this vector field to determine whether heat flows from hot to cold or vice versa. Would anything change if the law were $\mathbf{F} = k\nabla T$?

60. An isotherm is a curve on a map indicating areas of constant temperature. Given Fourier's law (exercise 59), determine the angle between the velocity field for heat flow and an isotherm.



EXPLORATORY EXERCISES

1. Show that the vector field $\mathbf{F}(x, y) = \langle y, x \rangle$ has potential function $f(x, y) = xy$. The curves $f(x, y) = c$ for constants c are called **equipotential curves.** Sketch equipotential curves for several constants (positive and negative). Find the flow lines for this vector field and show that the flow lines and equipotential curves intersect at right angles. This situation is common. To further develop these relationships, show that the potential function and the flow function $g(x, y) = \frac{1}{2}(y^2 - x^2)$ are both solutions of **Laplace's equation** $\nabla^2 u = 0$ where $\nabla^2 u = u_{xx} + u_{yy}$.

2. In example 1.5, we graphed the flow lines for the vector field $\langle 2, 1 + 2xy \rangle$ and mentioned that finding equations for the flow lines was beyond what's been presented in the text. We develop a method for finding the flow lines here by solving **linear ordinary differential equations.** We will illustrate this for an

easier vector field, $\langle x, 2x - y\rangle$. First, note that if $x'(t) = x$ and $y'(t) = 2x - y$, then

$$\frac{dy}{dx} = \frac{2x - y}{x} = 2 - \frac{y}{x}.$$

The flow lines will be the graphs of functions $y(x)$ such that $y'(x) = 2 - \frac{y}{x}$, or $y' + \frac{1}{x}y = 2$. The left-hand side of the equation should look a little like a product rule. Our main goal is to multiply by a term called an **integrating factor,** to make the left-hand side exactly a product rule derivative.

It turns out that for the equation $y' + f(x)y = g(x)$, an integrating factor is $e^{\int f(x)\,dx}$. In the present case, for $x > 0$, we have $e^{\int 1/x dx} = e^{\ln x} = x$. (We have chosen the integration constant to be 0 to keep the integrating factor simple.) Multiply both sides of the equation by x and show that $xy' + y = 2x$. Show that $xy' + y = (xy)'$. From $(xy)' = 2x$, integrate to get $xy = x^2 + c$, or $y = x + \frac{c}{x}$. To find a flow line passing through the point $(1, 2)$, show that $c = 1$ and thus, $y = x + \frac{1}{x}$. To find a flow line passing through the point $(1, 1)$, show that $c = 0$ and thus, $y = x$. Sketch the vector field and highlight the curves $y = x + \frac{1}{x}$ and $y = x$.

14.2 LINE INTEGRALS

In section 5.6, we used integration to find the mass of a thin rod with variable mass density. There, we had observed that if the rod extends from $x = a$ to $x = b$ and has mass density function $\rho(x)$, then the mass of the rod is given by $\int_a^b \rho(x)\,dx$. This definition is fine for objects that are essentially one-dimensional, but what if we wanted to find the mass of a helical spring (see Figure 14.11)? Calculus is remarkable in that the same technique can solve a wide variety of problems. As you should expect by now, we will derive a solution by first approximating the curve with line segments and then taking a limit.

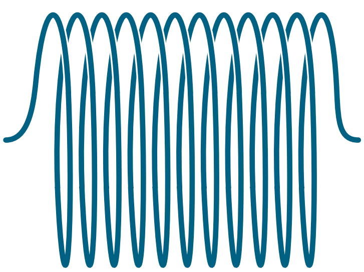


FIGURE 14.11
A helical spring

In this three-dimensional setting, the density function has the form $\rho(x, y, z)$ (where ρ is measured in units of mass per unit length). We assume that the object is in the shape of a curve C in three dimensions with endpoints (a, b, c) and (d, e, f). Further, we assume that the curve is **oriented,** which means that there is a direction to the curve. For example, the curve C might start at (a, b, c) and end at (d, e, f). We first partition the curve into n pieces with endpoints $(a, b, c) = (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n) = (d, e, f)$, as indicated in Figure 14.12. We will use the shorthand P_i to denote the point (x_i, y_i, z_i), and C_i for the section of the curve C extending from P_{i-1} to P_i, for each $i = 1, 2, \ldots, n$. Our initial objective is to approximate the mass of the portion of the object along C_i. Note that if the segment C_i is small enough, we can consider the density to be constant on C_i. In this case, the mass of this segment would simply be the product of the density and the length of C_i. For some point (x_i^*, y_i^*, z_i^*) on C_i, we approximate the density on C_i by $\rho(x_i^*, y_i^*, z_i^*)$. The mass of the section C_i is then approximately

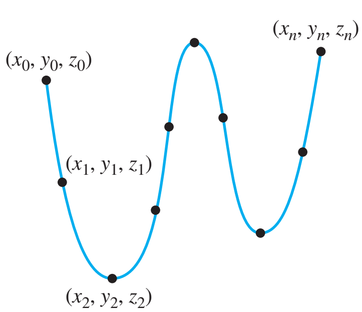


FIGURE 14.12
Partitioned curve

$$\rho(x_i^*, y_i^*, z_i^*)\Delta s_i,$$

where Δs_i represents the arc length of C_i. The mass m of the entire object is then approximately the sum of the masses of the n segments,

$$m \approx \sum_{i=1}^{n} \rho(x_i^*, y_i^*, z_i^*)\Delta s_i.$$

You should expect that this approximation will improve as we divide the curve into more and more segments that are shorter and shorter in length. Finally, taking the norm of the partition $\|P\|$ to be the maximum of the arc lengths $\Delta s_i (i = 1, 2, \ldots, n)$, we have

$$m = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \rho(x_i^*, y_i^*, z_i^*)\Delta s_i, \tag{2.1}$$

provided the limit exists and is the same for every choice of the evaluation points $(x_i^*, y_i^*, z_i^*)\,(i = 1, 2, \ldots, n)$.

You might recognize that (2.1) looks like the limit of a Riemann sum (an integral). As it turns out, this limit arises naturally in numerous applications. We pause now to give this limit a name and identify some useful properties.

DEFINITION 2.1

The **line integral of $f(x, y, z)$ with respect to arc length** along the oriented curve C in three-dimensional space, written $\int_C f(x, y, z)\,ds$, is defined by

$$\int_C f(x, y, z)\,ds = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_1^*, y_i^*, z_i^*)\Delta s_i,$$

provided the limit exists and is the same for all choices of evaluation points.

We define line integrals of functions $f(x, y)$ of two variables along an oriented curve C in the xy-plane in a similar way. Often, the curve C is specified by parametric equations or you can use your skills developed in Chapters 9 and 11 to construct parametric equations for the curve. In these cases, Theorem 2.1 allows us to evaluate the line integral as a definite integral of a function of one variable.

THEOREM 2.1 (Evaluation Theorem)

Suppose that $f(x, y, z)$ is continuous in a region D containing the curve C and that C is described parametrically by $(x(t), y(t), z(t))$, for $a \le t \le b$, where $x(t)$, $y(t)$ and $z(t)$ have continuous first derivatives. Then,

$$\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt.$$

Suppose that $f(x, y)$ is continuous in a region D containing the curve C and that C is described parametrically by $(x(t), y(t))$, for $a \le t \le b$, where $x(t)$ and $y(t)$ have continuous first derivatives. Then

$$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

PROOF

We prove the result for the case of a curve in two dimensions and leave the three-dimensional case as an exercise. From Definition 2.1 (adjusted for the two-dimensional case), we have

$$\int_C f(x, y)\,ds = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i, \tag{2.2}$$

where Δs_i represents the arc length of the section of the curve C between (x_{i-1}, y_{i-1}) and (x_i, y_i). Choose $t_0, t_1, \ldots, t_n$ so that $x(t_i) = x_i$ and $y(t_i) = y_i$, for $i = 0, 1, \ldots, n$. We approximate the arc length of such a small section of the curve by the straight-line distance:

$$\Delta s_i \approx \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}.$$

Further, since $x(t)$ and $y(t)$ have continuous first derivatives, we have by the Mean Value Theorem (as in the derivation of the arc length formula in section 9.3), that

$$\Delta s_i \approx \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \approx \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\,\Delta t_i,$$

for some $t_i^* \in (t_{i-1}, t_i)$. Together with (2.2), this gives us

$$\begin{aligned}\int_C f(x, y)\,ds &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*))\sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2}\,\Delta t_i \\ &= \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt. \quad \blacksquare\end{aligned}$$

We refer to any curve C satisfying the hypotheses of Theorem 2.1 as **smooth.** That is, C is smooth if it can be described parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$, for $a \le t \le b$, where $x(t)$, $y(t)$ and $z(t)$ all have continuous first derivatives on the interval $[a, b]$. Similarly, a plane curve is smooth if it can be parameterized by $x = x(t)$ and $y = y(t)$, for $a \le t \le b$, where $x(t)$ and $y(t)$ have continuous first derivatives on the interval $[a, b]$.

Notice that for curves in space, Theorem 2.1 says essentially that the arc length element ds can be replaced by

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt. \tag{2.3}$$

The term $\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$ in the integral should be very familiar, having been present in our integral representations of both arc length and surface area. Likewise, for curves in the plane, the arc length element is

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt. \tag{2.4}$$

EXAMPLE 2.1 Finding the Mass of a Helical Spring

Find the mass of a spring in the shape of the helix defined parametrically by $x = 2\cos t$, $y = t$, $z = 2\sin t$, for $0 \le t \le 6\pi$, with density $\rho(x, y, z) = 2y$.

Solution A graph of the helix is shown in Figure 14.13. The density is

$$\rho(x, y, z) = 2y = 2t,$$

and from (2.3), the arc length element ds is given by

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \sqrt{(-2\sin t)^2 + (1)^2 + (2\cos t)^2}\,dt = \sqrt{5}\,dt,$$

where we have used the identity $4\sin^2 t + 4\cos^2 t = 4$. By Theorem 2.1, we have

$$\begin{aligned}\text{mass} &= \int_C \rho(x, y, z)\,ds = \int_0^{6\pi} \underbrace{2t}_{\rho(x,y,z)}\,\underbrace{\sqrt{5}\,dt}_{ds} = 2\sqrt{5}\int_0^{6\pi} t\,dt \\ &= 2\sqrt{5}\,\frac{(6\pi)^2}{2} = 36\pi^2\sqrt{5}. \quad \blacksquare\end{aligned}$$

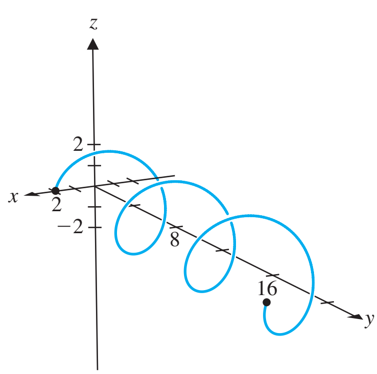


FIGURE 14.13
The helix $x = 2\cos t$, $y = t$, $z = 2\sin t$, $0 \le t \le 6\pi$

We should point out that example 2.1 is unusual in at least one respect: we were able to compute the integral exactly. Most line integrals of the type defined in Definition 2.1 are too complicated to evaluate exactly and will need to be approximated with some numerical method, as in example 2.2.

EXAMPLE 2.2 Evaluating a Line Integral with Respect to Arc Length

Evaluate the line integral $\int_C (2x^2 - 3yz)\,ds$, where C is the curve defined parametrically by $x = \cos t$, $y = \sin t$, $z = \cos t$ with $0 \le t \le 2\pi$.

Solution A graph of C is shown in Figure 14.14. The integrand is

$$f(x, y, z) = 2x^2 - 3yz = 2\cos^2 t - 3\sin t\cos t.$$

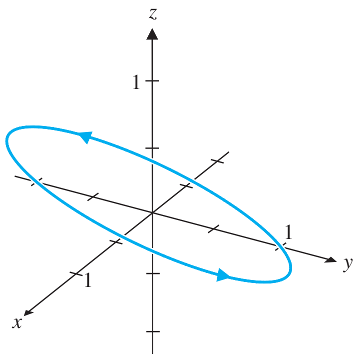


FIGURE 14.14
The curve $x = \cos t$, $y = \sin t$, $z = \cos t$ with $0 \le t \le 2\pi$

From (2.3), the arc length element is given by

$$\begin{aligned} ds &= \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt \\ &= \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\sin t)^2}\,dt = \sqrt{1 + \sin^2 t}\,dt, \end{aligned}$$

where we have used the identity $\sin^2 t + \cos^2 t = 1$. By Theorem 2.1, we now have

$$\int_C (2x^2 - 3yz)\,ds = \int_0^{2\pi} \underbrace{(2\cos^2 t - 3\sin t\cos t)}_{2x^2 - 3yz} \underbrace{\sqrt{1 + \sin^2 t}\,dt}_{ds} \approx 6.9922,$$

where we approximated the last integral numerically. ■

In example 2.3, you must find parameteric equations for the (two-dimensional) curve before evaluating the line integral. Also, you will discover an important fact about the orientation of the curve C.

EXAMPLE 2.3 Evaluating a Line Integral with Respect to Arc Length

Evaluate the line integral $\int_C 2x^2 y\,ds$, where C is (a) the portion of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$ and (b) the portion of the parabola $y = x^2$ from $(2, 4)$ to $(-1, 1)$.

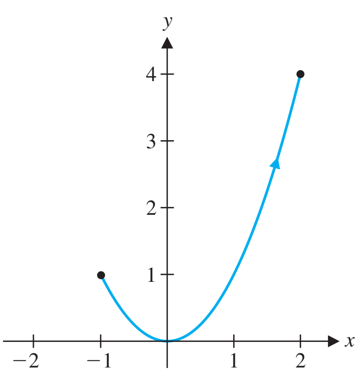


FIGURE 14.15a
$y = x^2$ from $(-1, 1)$ to $(2, 4)$

Solution (a) A sketch of the curve is shown in Figure 14.15a. Taking $x = t$ as the parameter (since the curve is already written explicitly in terms of x), we can write parametric equations for the curve as $x = t$ and $y = t^2$, for $-1 \le t \le 2$. Using this, the integrand becomes $2x^2 y = 2t^2 t^2 = 2t^4$ and from (2.4), the arc length element is

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \sqrt{1 + 4t^2}\,dt.$$

The integral is now written as

$$\int_C 2x^2 y\,ds = \int_{-1}^{2} \underbrace{2t^4}_{2x^2 y} \underbrace{\sqrt{1 + 4t^2}\,dt}_{ds} \approx 45.391,$$

where we have again evaluated the integral numerically (although in this case a good CAS can give you an exact answer).

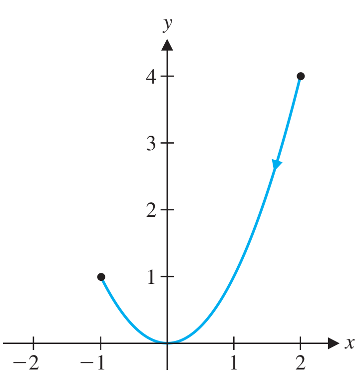


FIGURE 14.15b
$y = x^2$ from $(2, 4)$ to $(-1, 1)$

(b) The curve is the same as in part (a), except that the orientation is backward (see Figure 14.15b). In this case, we represent the curve with the parametric equations $x = -t$ and $y = t^2$, for $-2 \le t \le 1$. Observe that everything else in the integral remains the same and we have

$$\int_C 2x^2 y\,ds = \int_{-2}^{1} 2t^4 \sqrt{1 + 4t^2}\,dt \approx 45.391,$$

as before. ■

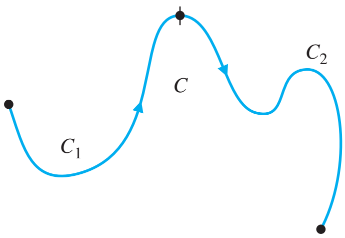


FIGURE 14.16
$C = C_1 \cup C_2$

Notice that in example 2.3, the line integral was the same no matter which orientation we took for the curve. It turns out that this is true in general for all line integrals defined by Definition 2.1 (i.e., line integrals with respect to arc length).

Notice that you can use Theorem 2.1 to rewrite a line integral only when the curve C is smooth. Fortunately, many curves of interest are not smooth, although we can extend the result of Theorem 2.1 to the case where C is a union of a finite number of smooth curves:

$$C = C_1 \cup C_2 \cup \cdots \cup C_n,$$

where each of $C_1, C_2, \ldots, C_n$ is smooth and where the terminal point of C_i is the same as the initial point of C_{i+1}, for $i = 1, 2, \ldots, n-1$. We call such a curve C **piecewise-smooth.** Notice that if C_1 and C_2 are oriented curves and the endpoint of C_1 is the same as the initial point of C_2, then the curve $C_1 \cup C_2$ is an oriented curve with the same initial point as C_1 and the same endpoint as C_2. (See Figure 14.16.) The results in Theorem 2.2 should not seem surprising. Here, for an oriented curve C in two or three dimensions, the curve $-C$ denotes the same curve as C, but with the opposite orientation.

THEOREM 2.2

Suppose that $f(x, y, z)$ is a continuous function in some region D containing the oriented curve C. Then, if C is piecewise-smooth, with $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where $C_1, C_2, \ldots, C_n$ are all smooth and where the terminal point of C_i is the same as the initial point of C_{i+1}, for $i = 1, 2, \ldots, n-1$, we have

(i)

$$\int_{-C} f(x, y, z)\, ds = \int_C f(x, y, z)\, ds$$

and (ii)

$$\int_C f(x, y, z)\, ds = \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds + \cdots + \int_{C_n} f(x, y, z)\, ds.$$

We leave the proof of the theorem as an exercise. Notice that the corresponding result will also be true in two dimensions. We use part (ii) of Theorem 2.2 in example 2.4.

EXAMPLE 2.4 Evaluating a Line Integral over a Piecewise-Smooth Curve

Evaluate the line integral $\int_C (3x - y)\, ds$, where C is the line segment from (1, 2) to (3, 3), followed by the portion of the circle $x^2 + y^2 = 18$ traversed from (3, 3) clockwise around to (3, −3).

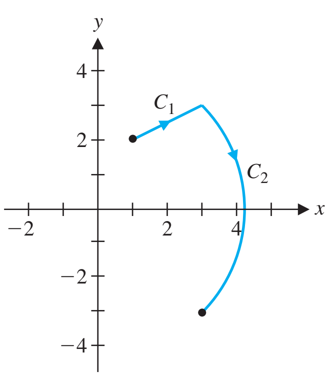


FIGURE 14.17
Piecewise-smooth curve

Solution A graph of the curve is shown in Figure 14.17. Notice that we'll need to evaluate the line integral separately over the line segment C_1 and the quarter-circle C_2. Further, although C is not smooth, it is piecewise-smooth, since C_1 and C_2 are both smooth. We can write C_1 parametrically as $x = 1 + (3-1)t = 1 + 2t$ and $y = 2 + (3-2)t = 2 + t$, for $0 \le t \le 1$. Also, on C_1, the integrand is given by

$$3x - y = 3(1 + 2t) - (2 + t) = 1 + 5t$$

and from (2.4), the arc length element is

$$ds = \sqrt{(2)^2 + (1)^2}\, dt = \sqrt{5}\, dt.$$

Putting this together, we have

$$\int_{C_1} f(x, y)\,ds = \int_0^1 \underbrace{(1+5t)}_{f(x,y)}\underbrace{\sqrt{5}\,dt}_{ds} = \frac{7}{2}\sqrt{5}. \tag{2.5}$$

Next, for C_2, the usual parametric equations for a circle of radius r oriented counterclockwise are $x(t) = r\cos t$ and $y(t) = r\sin t$. In the present case, the radius is $\sqrt{18}$ and the curve is oriented clockwise, which means that $y(t)$ has the opposite sign from the usual orientation. So, parametric equations for C_2 are $x(t) = \sqrt{18}\cos t$ and $y(t) = -\sqrt{18}\sin t$. Notice, too, that the initial point (3, 3) corresponds to the angle $-\frac{\pi}{4}$ and the endpoint $(3, -3)$ corresponds to the angle $\frac{\pi}{4}$. Finally, on C_2, the integrand is given by

$$3x - y = 3\sqrt{18}\cos t + \sqrt{18}\sin t$$

and the arc length element is

$$ds = \sqrt{\left(-\sqrt{18}\sin t\right)^2 + \left(-\sqrt{18}\cos t\right)^2}\,dt = \sqrt{18}\,dt,$$

where we have again used the fact that $\sin^2 t + \cos^2 t = 1$. This gives us

$$\int_{C_2} f(x, y)\,ds = \int_{-\pi/4}^{\pi/4} \underbrace{\left(3\sqrt{18}\cos t + \sqrt{18}\sin t\right)}_{f(x,y)}\underbrace{\sqrt{18}\,dt}_{ds} = 54\sqrt{2}. \tag{2.6}$$

Combining the integrals over the two curves, we have from (2.5) and (2.6) that

$$\int_C f(x, y)\,ds = \int_{C_1} f(x, y)\,ds + \int_{C_2} f(x, y)\,ds = \frac{7}{2}\sqrt{5} + 54\sqrt{2}.$$

So far, we have discussed how to calculate line integrals for curves described parametrically and we have given one application of line integrals (calculation of mass). In the exercises, we discuss further applications. We now develop a geometric interpretation of the line integral. Whereas $\int_a^b f(x)\,dx$ corresponds to a limit of sums of the heights of the function $f(x)$ above or below the x-axis for an interval $[a, b]$ of the x-axis, the line integral $\int_C f(x, y)\,ds$ corresponds to a limit of the sums of the heights of the function $f(x, y)$ above or below the xy-plane for a curve C lying in the xy-plane. We depict this in Figures 14.18a and 14.18b. Recall that for $f(x) \geq 0$, $\int_a^b f(x)\,dx$ measures the area under the curve $y = f(x)$

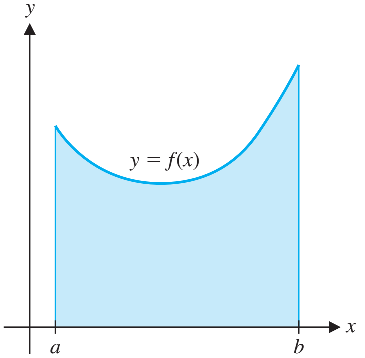


FIGURE 14.18a
$\int_a^b f(x)\,dx$

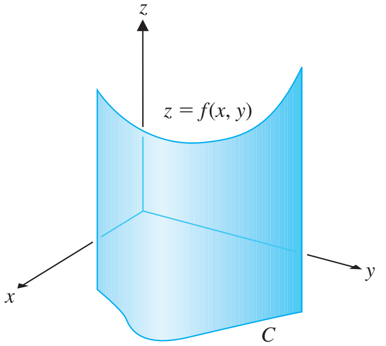


FIGURE 14.18b
$\int_C f(x, y)\,ds$

on the interval $[a, b]$, shaded in Figure 14.18a. Likewise, if $f(x, y) \geq 0$, $\int_C f(x, y)\,ds$ measures the surface area of the shaded surface indicated in Figure 14.18b. In general, $\int_a^b f(x)\,dx$ measures signed area [positive if $f(x) > 0$ and negative if $f(x) < 0$] and the line integral $\int_C f(x, y)\,ds$ measures the (signed) surface area of the surface formed by vertical segments from the xy-plane to the graph of $z = f(x, y)$.

Theorem 2.3, whose proof we leave as a straightforward exercise, gives some geometric significance to line integrals in both two and three dimensions.

THEOREM 2.3

For any piecewise-smooth curve C (in two or three dimensions), $\int_C 1\,ds$ gives the arc length of the curve C.

We have now extended the geometry and certain applications of definite integrals to line integrals of the form $\int_C f(x, y)\,ds$. However, not all of the properties of definite integrals can be extended to line integrals as we have defined them thus far. As we will see, a careful consideration of the calculation of work will force us to define several alternative versions of the line integral.

Recall that if a constant force f is exerted to move an object a distance d in a straight line, the work done is given by $W = f \cdot d$. In section 5.6, we extended this to a variable force $f(x)$ applied to an object as it moves in a straight line from $x = a$ to $x = b$, where the work done by the force is given by

$$W = \int_a^b f(x)\,dx.$$

We now extend this idea to find the work done as an object moves along a curve in three dimensions. Here, force vectors are given by the values of vector fields (force fields) and we want to compute the work done on an object by a force field $\mathbf{F}(x, y, z)$, as the object moves along a curve C. Unfortunately, our present notion of line integral (the line integral with respect to arc length) does not help in this case. As we did for finding mass, we need to start from scratch and so, partition the curve C into n segments $C_1, C_2, \ldots, C_n$. Notice that on each segment C_i ($i = 1, 2, \ldots, n$), if the segment is small and $\mathbf{F}$ is continuous, then $\mathbf{F}$ will be nearly constant on C_i and so, we can approximate $\mathbf{F}$ by its value at some point (x_i^*, y_i^*, z_i^*) on C_i. The work done along C_i (call it W_i) is then approximately the same as the product of the component of the force $\mathbf{F}(x_i^*, y_i^*, z_i^*)$ in the direction of the unit tangent vector $\mathbf{T}(x, y, z)$ to C at (x_i^*, y_i^*, z_i^*) and the distance traveled. That is,

$$W_i \approx \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)\,\Delta s_i,$$

where Δs_i is the arc length of the segment C_i. Now, if C_i can be represented parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$, for $a \leq t \leq b$, and assuming that C_i is smooth, we have

$$\begin{aligned} W_i &\approx \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)\,\Delta s_i \\ &= \frac{\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \langle x'(t_i^*), y'(t_i^*), z'(t_i^*)\rangle}{\sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2 + [z'(t_i^*)]^2}} \sqrt{[x'(t_i^*)]^2 + [y'(t_i^*)]^2 + [z'(t_i^*)]^2}\,\Delta t \\ &= \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \langle x'(t_i^*), y'(t_i^*), z'(t_i^*)\rangle\,\Delta t, \end{aligned}$$

where $(x_i^*, y_i^*, z_i^*) = (x(t_i^*), y(t_i^*), z(t_i^*))$. Next, if

$$\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle,$$

we have

$$W_i \approx \langle F_1(x_i^*, y_i^*, z_i^*), F_2(x_i^*, y_i^*, z_i^*), F_3(x_i^*, y_i^*, z_i^*)\rangle \cdot \langle x'(t_i^*), y'(t_i^*), z'(t_i^*)\rangle \,\Delta t.$$

Adding together the approximations of the work done over the various segments of C, we have that the total work done is approximately

$$W \approx \sum_{i=1}^{n} \langle F_1(x_i^*, y_i^*, z_i^*), F_2(x_i^*, y_i^*, z_i^*), F_3(x_i^*, y_i^*, z_i^*)\rangle \cdot \langle x'(t_i^*), y'(t_i^*), z'(t_i^*)\rangle \,\Delta t.$$

Finally, taking the limit as the norm of the partition of C approaches zero, we arrive at

$$\begin{aligned}
W &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} \mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \langle x'(t_i^*), y'(t_i^*), z'(t_i^*)\rangle \,\Delta t \\
&= \lim_{\|P\|\to 0} \sum_{i=1}^{n} [F_1(x_i^*, y_i^*, z_i^*)x'(t_i^*)\Delta t + F_2(x_i^*, y_i^*, z_i^*)y'(t_i^*)\,\Delta t \\
&\qquad + F_3(x_i^*, y_i^*, z_i^*)z'(t_i^*)\,\Delta t] \\
&= \int_a^b F_1(x(t), y(t), z(t))x'(t)\,dt + \int_a^b F_2(x(t), y(t), z(t))y'(t)\,dt \\
&\quad + \int_a^b F_3(x(t), y(t), z(t))z'(t)\,dt.
\end{aligned} \tag{2.7}$$

We now define line integrals corresponding to each of the three integrals in (2.7).

In Definition 2.2, the notation is the same as in Definition 2.1, with the added terms $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$ and $\Delta z_i = z_i - z_{i-1}$.

DEFINITION 2.2

The **line integral of $f(x, y, z)$ with respect to x** along the oriented curve C in three-dimensional space is written as $\int_C f(x, y, z)\,dx$ and is defined by

$$\int_C f(x, y, z)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta x_i,$$

provided the limit exists and is the same for all choices of evaluation points.

Likewise, we define the **line integral of $f(x, y, z)$ with respect to y** along C by

$$\int_C f(x, y, z)\,dy = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta y_i$$

and the **line integral of $f(x, y, z)$ with respect to z** along C by

$$\int_C f(x, y, z)\,dz = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i.$$

In each case, the line integral is defined whenever the corresponding limit exists and is the same for all choices of evaluation points.

If we have a parametric representation of the curve C, then we can rewrite each line integral as a definite integral. The proof of Theorem 2.4 is very similar to that of Theorem 2.1 and we leave it as an exercise.

THEOREM 2.4 (Evaluation Theorem)

Suppose that $f(x, y, z)$ is continuous in a region D containing the curve C and that C is described parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$, where t ranges from $t = a$ to $t = b$ and $x(t)$, $y(t)$ and $z(t)$ have continuous first derivatives. Then,

$$\int_C f(x, y, z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt,$$

$$\int_C f(x, y, z)\,dy = \int_a^b f(x(t), y(t), z(t))\,y'(t)\,dt \quad \text{and}$$

$$\int_C f(x, y, z)\,dz = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt.$$

Before returning to the calculation of work, we will examine some simpler examples. Recall that the line integral along a given curve with respect to arc length will not change if we traverse the curve in the opposite direction. On the other hand, as we see in example 2.5, line integrals with respect to *x, y* or *z* change sign when the orientation of the curve changes.

EXAMPLE 2.5 Calculating a Line Integral in Space

Compute the line integral $\int_C (4xz + 2y)\,dx$, where C is the line segment (a) from (2, 1, 0) to (4, 0, 2) and (b) from (4, 0, 2) to (2, 1, 0).

Solution First, parametric equations for C for part (a) are

$$\begin{aligned} x &= 2 + (4 - 2)t = 2 + 2t, \\ y &= 1 + (0 - 1)t = 1 - t \quad \text{and} \\ z &= 0 + (2 - 0)t = 2t, \end{aligned}$$

for $0 \le t \le 1$. The integrand is then

$$4xz + 2y = 4(2 + 2t)(2t) + 2(1 - t) = 16t^2 + 14t + 2$$

and the element dx is given by

$$dx = x'(t)\,dt = 2\,dt.$$

From the Evaluation Theorem, the line integral is now given by

$$\int_C (4xz + 2y)\,dx = \int_0^1 \underbrace{(16t^2 + 14t + 2)}_{4xz+2y}\,\underbrace{(2)\,dt}_{dx} = \frac{86}{3}.$$

For part (b), you can use the fact that the line segment connects the same two points as in part (a), but in the opposite direction. The same parametric equations will then work, with the single change that t will run from $t = 1$ to $t = 0$. This gives us

$$\int_C (4xz + 2y)\,dx = \int_1^0 (16t^2 + 14t + 2)(2)\,dt = -\frac{86}{3},$$

where you should recall that reversing the order of integration changes the sign of the integral. ■

Theorem 2.5 corresponds to Theorem 2.2 for line integrals with respect to arc length, but pay special attention to the minus sign in part (i). We state the result for line integrals with respect to x, with corresponding results holding true for line integrals with respect to y or z, as well. We leave the proof as an exercise.

THEOREM 2.5

Suppose that $f(x, y, z)$ is a continuous function in some region D containing the oriented curve C. Then, the following hold.

(i) If C is piecewise-smooth, then

$$\int_{-C} f(x, y, z)\, dx = -\int_C f(x, y, z)\, dx.$$

(ii) If $C = C_1 \cup C_2 \cup \cdots \cup C_n$, where $C_1, C_2, \ldots, C_n$ are all smooth and the terminal point of C_i is the same as the initial point of C_{i+1}, for $i = 1, 2, \ldots, n-1$, then

$$\int_C f(x, y, z)\, dx = \int_{C_1} f(x, y, z)\, dx + \int_{C_2} f(x, y, z)\, dx + \cdots + \int_{C_n} f(x, y, z)\, dx.$$

NOTES

As a convenience, we will usually write

$$\int_C f(x, y, z)\, dx + \int_C g(x, y, z)\, dy + \int_C h(x, y, z)\, dz = \int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz.$$

Line integrals with respect to x, y and z can be very simple when the curve C consists of line segments parallel to the coordinate axes, as we see in example 2.6.

EXAMPLE 2.6 Calculating a Line Integral in Space

Compute $\int_C 4x\, dy + 2y\, dz$, where C consists of the line segment from (0, 1, 0) to (0, 1, 1), followed by the line segment from (0, 1, 1) to (2, 1, 1) and followed by the line segment from (2, 1, 1) to (2, 4, 1).

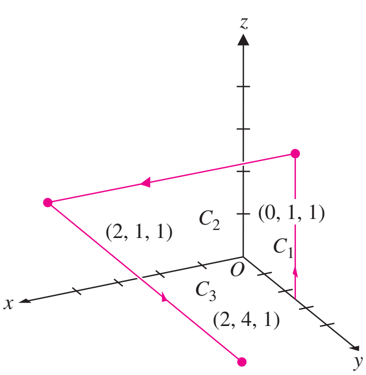


FIGURE 14.19
The path C

Solution We show a sketch of the curves in Figure 14.19. Parametric equations for the first segment C_1 are $x = 0$, $y = 1$ and $z = t$ with $0 \le t \le 1$. On this segment, we have $dy = 0\, dt$ and $dz = 1\, dt$. On the second segment C_2, parametric equations are $x = 2t$, $y = 1$ and $z = 1$ with $0 \le t \le 1$. On this segment, we have $dy = dz = 0\, dt$. On the third segment, parametric equations are $x = 2$, $y = 3t + 1$ and $z = 1$ with $0 \le t \le 1$. On this segment, we have $dy = 3\, dt$ and $dz = 0\, dt$. Putting this all together, we have

$$\begin{aligned}
\int_C 4x\, dy + 2y\, dz &= \int_{C_1} 4x\, dy + 2y\, dz + \int_{C_2} 4x\, dy + 2y\, dz + \int_{C_3} 4x\, dy + 2y\, dz \\
&= \int_0^1 [\underbrace{4(0)}_{4x}\underbrace{(0)}_{y'(t)} + \underbrace{2(1)}_{2y}\underbrace{(1)}_{z'(t)}]\, dt + \int_0^1 [\underbrace{4(2t)}_{4x}\underbrace{(0)}_{y'(t)} + \underbrace{2(1)}_{2y}\underbrace{(0)}_{z'(t)}]\, dt \\
&\quad + \int_0^1 [\underbrace{4(2)}_{4x}\underbrace{(3)}_{y'(t)} + \underbrace{2(3t+1)}_{2y}\underbrace{(0)}_{z'(t)}]\, dt \\
&= \int_0^1 26\, dt = 26.
\end{aligned}$$

Notice that a line integral will be zero if the integrand simplifies to 0 or if the variable of integration is constant along the curve. For instance, if z is constant on some curve, then the change in z (given by dz) will be 0 on that curve.

Recall that our motivation for introducing line integrals with respect to the three coordinate variables was to compute the work done by a force field while moving an object along a curve. From (2.7), the work performed by the force field $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ along the curve defined parametrically by $x = x(t)$, $y = y(t)$, $z = z(t)$, for $a \le t \le b$, is given by

$$W = \int_a^b F_1(x(t), y(t), z(t))x'(t)\,dt + \int_a^b F_2(x(t), y(t), z(t))y'(t)\,dt + \int_a^b F_3(x(t), y(t), z(t))z'(t)\,dt.$$

You should now recognize that we can rewrite each of the three terms in this expression for work using Theorem 2.4, to obtain

$$W = \int_C F_1(x, y, z)\,dx + \int_C F_2(x, y, z)\,dy + \int_C F_3(x, y, z)\,dz.$$

We now introduce some notation to write such a combination of line integrals in a simpler form.

Suppose that a vector field $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$. Since we use $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we define

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \quad \text{or} \quad d\mathbf{r} = \langle dx, dy, dz\rangle.$$

We now define the line integral

$$\begin{aligned}\int_C \mathbf{F}(x, y, z)\cdot d\mathbf{r} &= \int_C F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz \\ &= \int_C F_1(x, y, z)\,dx + \int_C F_2(x, y, z)\,dy + \int_C F_3(x, y, z)\,dz.\end{aligned}$$

In the case where $\mathbf{F}(x, y, z)$ is a force field, the work done by $\mathbf{F}$ in moving a particle along the curve C can be written simply as

$$\boxed{W = \int_C \mathbf{F}(x, y, z)\cdot d\mathbf{r}.} \tag{2.8}$$

Notice how the different parts of $\int_C \mathbf{F}(x, y, z)\cdot d\mathbf{r}$ correspond to our knowledge of work. The only way in which the x-component of force affects the work done is when the object moves in the x-direction (i.e., when $dx \ne 0$). Similarly, the y-component of force contributes to the work only when $dy \ne 0$ and the z-component of force contributes to the work only when $dz \ne 0$.

EXAMPLE 2.7 Computing Work

Compute the work done by the force field $\mathbf{F}(x, y, z) = \langle 4y, 2xz, 3y\rangle$ acting on an object as it moves along the helix defined parametrically by $x = 2\cos t$, $y = 2\sin t$ and $z = 3t$, from the point $(2, 0, 0)$ to the point $(-2, 0, 3\pi)$.

Solution From (2.8), the work is given by

$$W = \int_C \mathbf{F}(x, y, z)\cdot d\mathbf{r} = \int_C 4y\,dx + 2xz\,dy + 3y\,dz.$$

We have already provided parametric equations for the curve, but not the range of t-values. Notice that from $z = 3t$, you can determine that $(2, 0, 0)$ corresponds to $t = 0$

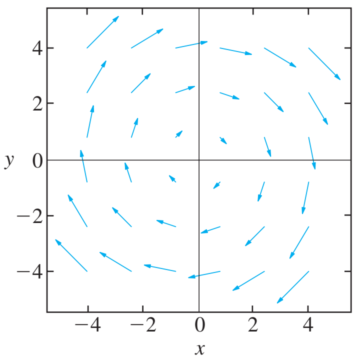


FIGURE 14.20
$\mathbf{F}(x, y) = \langle y, -x \rangle$

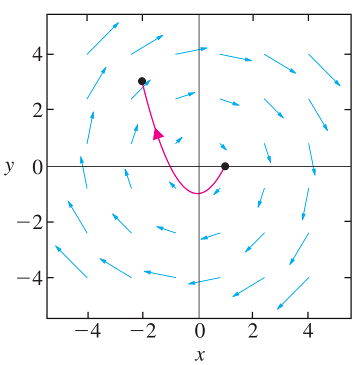


FIGURE 14.21
$\mathbf{F}(x, y) = \langle y, -x \rangle$ and
$x = t,\ y = t^2 - 1,\ -2 \le t \le 1$

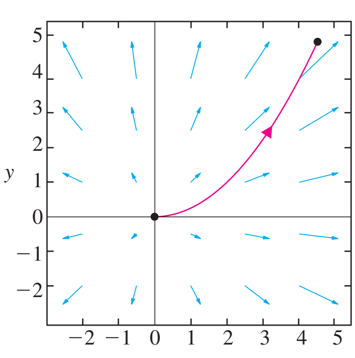


FIGURE A

and $(-2, 0, 3\pi)$ corresponds to $t = \pi$. Substituting in for x, y, z and $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$ and $dz = 3\,dt$, we have

$$W = \int_C 4y\,dx + 2xz\,dy + 3y\,dz$$

$$= \int_0^{\pi} [\underbrace{4(2\sin t)}_{4y}\underbrace{(-2\sin t)}_{x'(t)} + \underbrace{2(2\cos t)(3t)}_{2xz}\underbrace{(2\cos t)}_{y'(t)} + \underbrace{3(2\sin t)}_{3y}\underbrace{(3)}_{z'(t)}]dt$$

$$= \int_0^{\pi} (-16\sin^2 t + 24t\cos^2 t + 18\sin t)\,dt = 36 - 8\pi + 6\pi^2,$$

where we used a computer algebra system to evaluate the final integral. ■

We compute the work performed by a two-dimensional vector field in the same way as we did in three dimensions, as we illustrate in example 2.8.

EXAMPLE 2.8 Computing Work

Compute the work done by the force field $\mathbf{F}(x, y) = \langle y, -x \rangle$ acting on an object as it moves along the parabola $y = x^2 - 1$ from $(1, 0)$ to $(-2, 3)$.

Solution From (2.8), the work is given by

$$W = \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_C y\,dx - x\,dy.$$

Here, we use $x = t$ and $y = t^2 - 1$ as parametric equations for the curve, with t ranging from $t = 1$ to $t = -2$. In this case, $dx = 1\,dt$ and $dy = 2t\,dt$ and the work is

$$W = \int_C y\,dx - x\,dy = \int_1^{-2} [(t^2 - 1)(1) - (t)(2t)]\,dt = \int_1^{-2} (-t^2 - 1)\,dt = 6.$$

■

A careful look at example 2.8 graphically will show us an important geometric interpretation of the work line integral. A computer-generated graph of the vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$ is shown in Figure 14.20. Thinking of $\mathbf{F}(x, y)$ as describing the velocity field for a fluid in motion, notice that the fluid is rotating clockwise. In Figure 14.21, we superimpose the curve $x = t$, $y = t^2 - 1$, $-2 \le t \le 1$, onto the vector field $\mathbf{F}(x, y)$. Notice that an object moving along the curve from $(1, 0)$ to $(-2, 3)$ is generally moving in the same direction as that indicated by the vectors in the vector field. If $\mathbf{F}(x, y)$ represents a force field, then the force pushes an object moving along C, adding energy to it and therefore doing positive work. If the curve were oriented in the opposite direction, the force would oppose the motion of the object, thereby doing negative work.

EXAMPLE 2.9 Determining the Sign of a Line Integral Graphically

In each of the following graphs, an oriented curve is superimposed onto a vector field $\mathbf{F}(x, y)$. Determine whether $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is positive or negative.

Solution In Figure A, the curve is oriented in the same direction as the vectors, so the force is making a positive contribution to the object's motion. The work done by the force is then positive. In Figure B, the curve is oriented in the opposite direction as the vectors, so that the force is making a negative contribution to the object's motion. The work done by the force is then negative. In Figure C, the force field vectors are purely horizontal. Since both the curve and the force vectors point to the right, the work is positive. Finally, in Figure D, the force field is the same as in Figure C, but the curve is

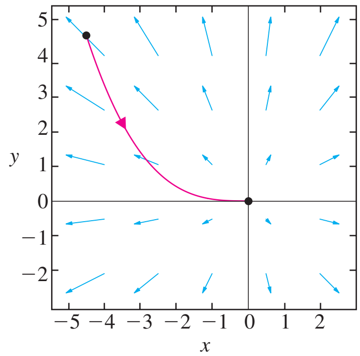


FIGURE B

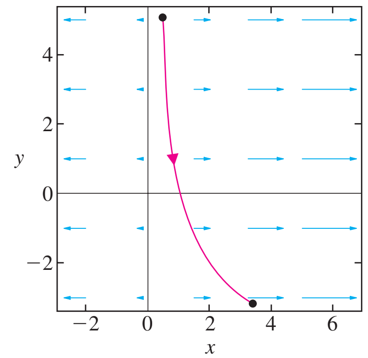


FIGURE C

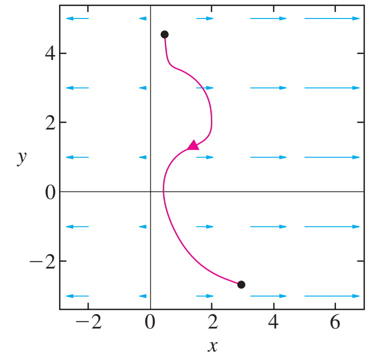


FIGURE D

more complicated. Since the force vectors are horizontal and do not depend on y, the work done as the object moves to the right is exactly canceled when the object doubles back to the left. Comparing the initial and terminal points, the object has made a net movement to the right (the same direction as the vector field), so that the work done is positive. ■

BEYOND FORMULAS

Several different line integrals are defined in this section. To keep them straight, always think of dx as an increment (small change) in x, dy as an increment in y and so on. In particular, ds represents an increment in arc length (distance) along the curve, which is why the arc length formula plays a role in the Evaluation Theorem. In applications, the dx, dy and dz line integrals are useful when the quantity being measured (such as a force) can be broken down into separate x, y and z components. By contrast, the ds line integral is applied to the measurement of a quantity as we move along the curve in three dimensions. What is an example of this situation?

EXERCISES 14.2

WRITING EXERCISES

1. It is important to understand why $\int_C f\,ds = \int_{-C} f\,ds$. Think of f as being a density function and the line integral as giving the mass of an object. Explain why the integrals must be equal.

2. For example 2.3, part (a), a different set of parametric equations is $x = -t$ and $y = t^2$, with t running from $t = 1$ to $t = -2$. In light of the Evaluation Theorem, explain why we couldn't use these parametric equations.

3. Explain in words why Theorem 2.5(i) is true. In particular, explain in terms of approximating sums why the integrals in Theorem 2.5(i) have opposite signs but the integrals in Theorem 2.2(i) are the same.

4. In example 2.9, we noted that the force vectors in Figure D are horizontal and independent of y. Explain why this allows us to ignore the vertical component of the curve. Also, explain why the work would be the same for *any* curve with the same initial and terminal points.

In exercises 1–24, evaluate the line integral.

1. $\int_C 2x\,ds$, where C is the line segment from (1, 2) to (3, 5)
2. $\int_C (x - y)\,ds$, where C is the line segment from (1, 0) to (3, 1)
3. $\int_C (3x + y)\,ds$, where C is the line segment from (5, 2) to (1, 1)
4. $\int_C 2xy\,ds$, where C is the line segment from (1, 2) to (−1, 0)
5. $\int_C 2x\,dx$, where C is the line segment from (0, 2) to (2, 6)
6. $\int_C 3y^2\,dy$, where C is the line segment from (2, 0) to (1, 3)

7. $\int_C 3x\,ds$, where C is the quarter-circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$

8. $\int_C (3x - y)\,ds$, where C is the quarter-circle $x^2 + y^2 = 9$ from $(0, 3)$ to $(3, 0)$

9. $\int_C 2x\,dx$, where C is the quarter-circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$

10. $\int_C 3y^2 dy$, where C is the quarter-circle $x^2 + y^2 = 4$ from $(0, 2)$ to $(-2, 0)$

11. $\int_C 3y\,dx$, where C is the half-ellipse $x^2 + 4y^2 = 4$ from $(0, 1)$ to $(0, -1)$ with $x \geq 0$

12. $\int_C x^2\,dy$, where C is the ellipse $4x^2 + y^2 = 4$ oriented counterclockwise

13. $\int_C 3y\,ds$, where C is the portion of $y = x^2$ from $(0, 0)$ to $(2, 4)$

14. $\int_C 2x\,ds$, where C is the portion of $y = x^2$ from $(-2, 4)$ to $(2, 4)$

15. $\int_C 2x\,dx$, where C is the portion of $y = x^2$ from $(2, 4)$ to $(0, 0)$

16. $\int_C 3y^2 dy$, where C is the portion of $y = x^2$ from $(2, 4)$ to $(0, 0)$

17. $\int_C 3y\,dx$, where C is the portion of $x = y^2$ from $(1, 1)$ to $(4, 2)$

18. $\int_C (x + y)\,dy$, where C is the portion of $x = y^2$ from $(1, 1)$ to $(1, -1)$

19. $\int_C 3x\,ds$, where C is the line segment from $(0, 0)$ to $(1, 0)$, followed by the quarter-circle to $(0, 1)$

20. $\int_C 2y\,ds$, where C is the portion of $y = x^2$ from $(0, 0)$ to $(2, 4)$, followed by the line segment to $(3, 0)$

21. $\int_C 4z\,ds$, where C is the line segment from $(1, 0, 1)$ to $(2, -2, 2)$

22. $\int_C xz\,ds$, where C is the line segment from $(2, 1, 0)$ to $(2, 0, 2)$

23. $\int_C 4(x - z)z\,dx$, where C is the portion of $y = x^2$ in the plane $z = 2$ from $(1, 1, 2)$ to $(2, 4, 2)$

24. $\int_C z\,ds$, where C is the intersection of $x^2 + y^2 = 4$ and $z = 0$ (oriented clockwise as viewed from above)

In exercises 25–36, compute the work done by the force field F along the curve C.

25. $\mathbf{F}(x, y) = \langle 2x, 2y \rangle$, C is the line segment from $(3, 1)$ to $(5, 4)$

26. $\mathbf{F}(x, y) = \langle 2y, -2x \rangle$, C is the line segment from $(4, 2)$ to $(0, 4)$

27. $\mathbf{F}(x, y) = \langle 2x, 2y \rangle$, C is the quarter-circle from $(4, 0)$ to $(0, 4)$

28. $\mathbf{F}(x, y) = \langle 2y, -2x \rangle$, C is the upper half-circle from $(-3, 0)$ to $(3, 0)$

29. $\mathbf{F}(x, y) = \langle 2, x \rangle$, C is the portion of $y = x^2$ from $(0, 0)$ to $(1, 1)$

30. $\mathbf{F}(x, y) = \langle 0, xy \rangle$, C is the portion of $y = x^3$ from $(0, 0)$ to $(1, 1)$

31. $\mathbf{F}(x, y) = \langle 3x, 2 \rangle$, C is the line segment from $(0, 0)$ to $(0, 1)$, followed by the line segment to $(4, 1)$

32. $\mathbf{F}(x, y) = \langle y, x \rangle$, C is the square from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

33. $\mathbf{F}(x, y, z) = \langle y, 0, z \rangle$, C is the triangle from $(0, 0, 0)$ to $(2, 1, 2)$ to $(2, 1, 0)$ to $(0, 0, 0)$

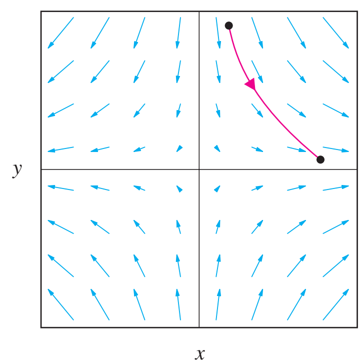
34. $\mathbf{F}(x, y, z) = \langle z, y, 0 \rangle$, C is the line segment from $(1, 0, 2)$ to $(2, 4, 2)$

35. $\mathbf{F}(x, y, z) = \langle xy, 3z, 1 \rangle$, C is the helix $x = \cos t$, $y = \sin t$, $z = 2t$ from $(1, 0, 0)$ to $(0, 1, \pi)$

36. $\mathbf{F}(x, y, z) = \langle z, 0, 3x^2 \rangle$, C is the quarter-ellipse $x = 2\cos t$, $y = 3\sin t$, $z = 1$ from $(2, 0, 1)$ to $(0, 3, 1)$

In exercises 37–42, use the graph to determine whether the work done is positive, negative or zero.

37.



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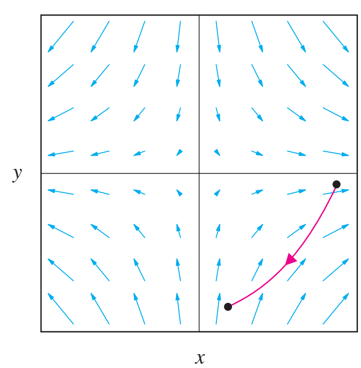

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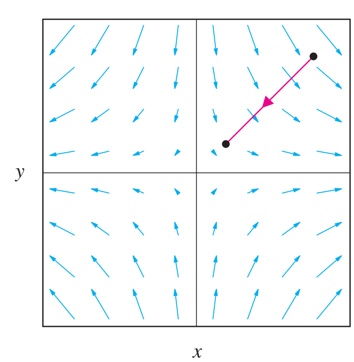

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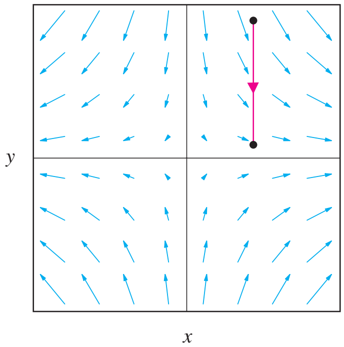

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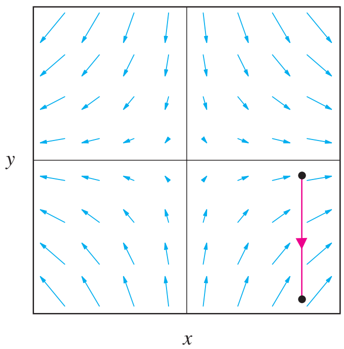

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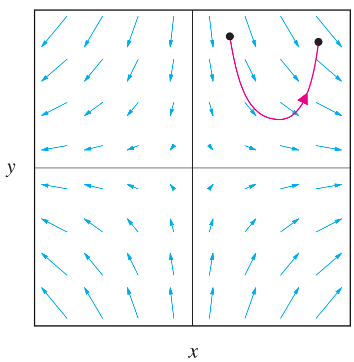


In exercises 43–52, use the formulas $m = \int_C \rho ds$, $\bar{x} = \frac{1}{m}\int_C x\rho ds$, $\bar{y} = \frac{1}{m}\int_C y\rho ds$, $I = \int_C w^2 \rho ds$.

43. Compute the mass m of a rod with density $\rho(x, y) = x$ in the shape of $y = x^2, 0 \le x \le 3$.

44. Compute the mass m of a rod with density $\rho(x, y) = y$ in the shape of $y = 4 - x^2, 0 \le x \le 2$.

45. Compute the center of mass $(\bar{x}, \bar{y})$ of the rod of exercise 43.

46. Compute the center of mass $(\bar{x}, \bar{y})$ of the rod of exercise 44.

47. Compute the moment of inertia I for rotating the rod of exercise 43 about the y-axis. Here, w is the distance from the point (x, y) to the y-axis.

48. Compute the moment of inertia I for rotating the rod of exercise 44 about the x-axis. Here, w is the distance from the point (x, y) to the x-axis.

49. Compute the moment of inertia I for rotating the rod of exercise 43 about the line $y = 9$. Here, w is the distance from the point (x, y) to $y = 9$.

50. Compute the moment of inertia I for rotating the rod of exercise 44 about the line $x = 2$. Here, w is the distance from the point (x, y) to $x = 2$.

51. Compute the mass m of the helical spring $x = \cos 2t$, $y = \sin 2t$, $z = t$, $0 \le t \le \pi$, with density $\rho = z^2$.

52. Compute the mass m of the ellipse $x = 4\cos t$, $y = 4\sin t$, $z = 4\cos t$, $0 \le t \le 2\pi$, with density $\rho = 4$.

53. Show that the center of mass in exercises 45 and 46 is not located at a point on the rod. Explain why this means that our previous interpretation of center of mass as a balance point is no longer valid. Instead, the center of mass is the point about which the object rotates when a torque is applied.

54. Suppose a torque is applied to the rod in exercise 43 such that the rod rotates but has no other motion. Find parametric equations for the position of the part of the rod that starts at the point (1, 1).

In exercises 55–60, find the surface area extending from the given curve in the *xy*-plane to the given surface.

55. Above the quarter-circle of radius 2 centered at the origin from (2, 0, 0) to (0, 2, 0) up to the surface $z = x^2 + y^2$

56. Above the portion of $y = x^2$ from (0, 0, 0) to (2, 4, 0) up to the surface $z = x^2 + y^2$

57. Above the line segment from (2, 0, 0) to (−2, 0, 0) up to the surface $z = 4 - x^2 - y^2$

58. Above the line segment from (1, 1, 0) to (−1, 1, 0) up to the surface $z = \sqrt{x^2 + y^2}$

59. Above the unit square $x \in [0, 1]$, $y \in [0, 1]$ up to the plane $z = 4 - x - y$

60. Above the ellipse $x^2 + 4y^2 = 4$ up to the plane $z = 4 - x$

In exercises 61 and 62, estimate the line integrals (a) $\int_C f\,ds$, (b) $\int_C f\,dx$ and (c) $\int_C f\,dy$.

61.

(x, y)	(0, 0)	(1, 0)	(1, 1)	(1.5, 1.5)
$f(x, y)$	2	3	3.6	4.4

(x, y)	(2, 2)	(3, 2)	(4, 1)
$f(x, y)$	5	4	4

62.

(x, y)	(0, 0)	(1, −1)	(2, 0)	(3, 1)
$f(x, y)$	1	0	−1.2	0.4

(x, y)	(4, 0)	(3, −1)	(2, −2)
$f(x, y)$	1.5	2.4	2

63. Prove Theorem 2.1 in the case of a curve in three dimensions.

64. Prove Theorem 2.2.

65. Prove Theorem 2.3.

66. Prove Theorem 2.4.

67. Prove Theorem 2.5.

68. If C has parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$, for differentiable functions x, y and z, show that $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b [F_1(x, y, z)\, x'(t) + F_2(x, y, z)\, y'(t) + F_3(x, y, z) z'(t)]\, dt$, which is the work line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

69. If the two-dimensional vector $\mathbf{n}$ is normal (perpendicular to the tangent) to the curve C at each point and $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y)\rangle$, show that $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C F_1\, dy - F_2\, dx$.

70. If $T(x, y)$ is the temperature function, the line integral $\int_C (-k\nabla T) \cdot \mathbf{n}\, ds$ gives the rate of heat loss across C. For $T(x, y) = 60e^{y/50}$ and C the rectangle with sides $x = -20$, $x = 20$, $y = -5$ and $y = 5$, compute the rate of heat loss. Explain in terms of the temperature function why the integral is 0 along two sides of C.

EXPLORATORY EXERCISES

1. Look carefully at the solutions to exercises 5–6, 9–10 and 15–16. Compare the solutions to integrals of the form $\int_a^b 2x\, dx$ and $\int_c^d 3y^2 dy$. Formulate a rule for evaluating line integrals of the form $\int_C f(x)\, dx$ and $\int_C g(y)\, dy$. If the curve C is a closed curve (e.g., a square or a circle), evaluate the line integrals $\int_C f(x)\, dx$ and $\int_C g(y)\, dy$.

14.3 INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS

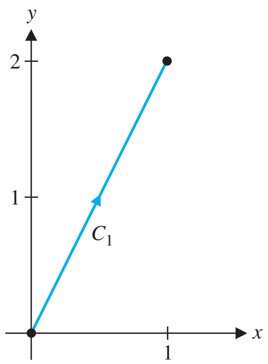


FIGURE 14.22a
The path C_1

As you've seen, there are a lot of steps needed to evaluate a line integral. First, you must parameterize the curve, rewrite the line integral as a definite integral and then evaluate the resulting definite integral. While this process is unavoidable for many line integrals, we will now consider a group of line integrals that are the same along every curve connecting the given endpoints. We'll determine the circumstances under which this occurs and see that when this does happen, there is a simple way to evaluate the integral.

We begin with a simple observation. Consider the line integral $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 2x, 3y^2\rangle$ and C_1 is the straight line segment joining the two points (0, 0) and (1, 2). (See Figure 14.22a.) To parameterize the curve, we take $x = t$ and $y = 2t$, for $0 \le t \le 1$. We then have

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \langle 2x, 3y^2\rangle \cdot \langle dx, dy\rangle \\ &= \int_{C_1} 2x\, dx + 3y^2\, dy \\ &= \int_0^1 [2t + 12t^2(2)]\, dt = 9,\end{aligned}$$

where we have left the details of the final (routine) calculation to you. For the same vector field $\mathbf{F}(x, y)$, consider now $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_2 is made up of the horizontal line segment from (0, 0) to (1, 0) followed by the vertical line segment from (1, 0) to (1, 2). (See Figure 14.22b.) In this case, we have

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_{C_2} \langle 2x, 3y^2\rangle \cdot \langle dx, dy\rangle \\ &= \int_0^1 2x\, dx + \int_0^2 3y^2\, dy = 9,\end{aligned}$$

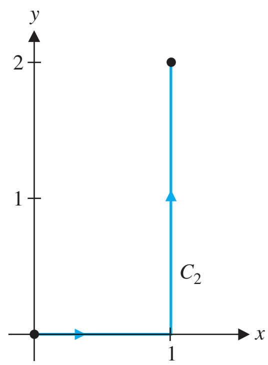


FIGURE 14.22b
The path C_2

where we have again left the final details to you. Look carefully at these two line integrals. Although the integrands are the same and the endpoints of the two curves are the same, the curves followed are quite different. You should try computing this line integral over several additional curves from (0, 0) to (1, 2). You will find that each line integral has the same value: 9. This integral is an example of one that is the same along any curve from (0, 0) to (1, 2).

Let C be any piecewise-smooth curve, traced out by the endpoint of the vector-valued function $\mathbf{r}(t)$, for $a \le t \le b$. In this context, we usually refer to a curve connecting two given points as a **path.** We say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** in the domain D if the integral is the same for every path contained in D that has the same beginning and ending points. Before we see when this happens, we need a definition.

DEFINITION 3.1

A region $D \subset \mathbb{R}^n$ (for $n \ge 2$) is called **connected** if every pair of points in D can be connected by a piecewise-smooth curve lying entirely in D.

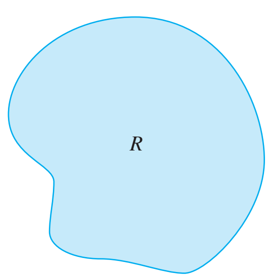


FIGURE 14.23a
Connected region

In Figure 14.23a, we show a region in $\mathbb{R}^2$ that is connected and in Figure 14.23b, we indicate a region that is not connected. We are now in a position to prove a result concerning integrals that are independent of path. While we state and prove the result for line integrals in the plane, the result is valid in any number of dimensions.

THEOREM 3.1

Suppose that the vector field $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ is continuous on the open, connected region $D \subset \mathbb{R}^2$. Then, the line integral $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path in D if and only if $\mathbf{F}$ is conservative on D.

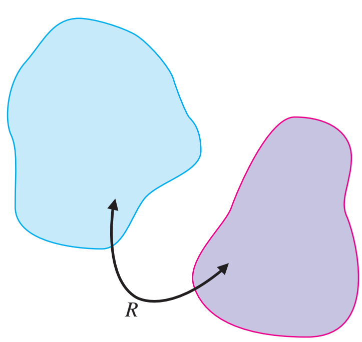


FIGURE 14.23b
Not connected

PROOF

Recall that a vector field $\mathbf{F}$ is conservative whenever $\mathbf{F} = \nabla f$, for some scalar function f (called a potential function for $\mathbf{F}$). You should recognize that there are several things to prove here.

First, suppose that $\mathbf{F}$ is conservative, with $\mathbf{F}(x, y) = \nabla f(x, y)$. Then

$$\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle = \nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

and so, we must have

$$M(x, y) = f_x(x, y) \quad \text{and} \quad N(x, y) = f_y(x, y).$$

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in D and let C be any smooth path from A to B, lying in D and defined parametrically by $C: x = g(t), y = h(t)$, where $t_1 \le t \le t_2$. (You can extend this proof to any piecewise-smooth path in the obvious way.) Then, we have

$$\begin{aligned}\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} &= \int_C M(x, y)\,dx + N(x, y)\,dy \\ &= \int_C f_x(x, y)\,dx + f_y(x, y)\,dy \\ &= \int_{t_1}^{t_2} [f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t)]\,dt. \qquad (3.1)\end{aligned}$$

Notice that since f_x and f_y were assumed to be continuous, we have by the chain rule that

$$\frac{d}{dt}[f(g(t), h(t))] = f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t),$$

which is the integrand in (3.1). By the Fundamental Theorem of Calculus, we now have

$$\begin{aligned}\int_C \mathbf{F}(x, y)\cdot d\mathbf{r} &= \int_{t_1}^{t_2} [f_x(g(t), h(t))g'(t) + f_y(g(t), h(t))h'(t)]\,dt \\ &= \int_{t_1}^{t_2} \frac{d}{dt}[f(g(t), h(t))]\,dt \\ &= f(g(t_2), h(t_2)) - f(g(t_1), h(t_1)) \\ &= f(x_2, y_2) - f(x_1, y_1).\end{aligned}$$

In particular, this says that the value of the integral depends only on the value of the potential function at the two endpoints of the curve and not on the particular path followed. That is, the line integral is independent of path, as desired.

Next, we need to prove that if the integral is independent of path, then the vector field must be conservative. So, suppose that $\int_C \mathbf{F}(x, y)\cdot d\mathbf{r}$ is independent of path in D. For any points (u, v) and $(x_0, y_0) \in D$, define the function

$$f(u, v) = \int_{(x_0, y_0)}^{(u,v)} \mathbf{F}(x, y)\cdot d\mathbf{r}.$$

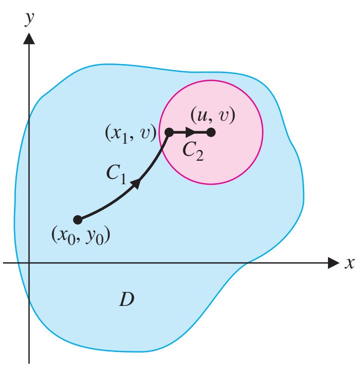


FIGURE 14.24
First path

(We are using the variables u and v, since the variables x and y inside the integral are dummy variables and cannot be used both inside and outside the line integral.) Notice that since the line integral is independent of path in D, we need not specify a path over which to integrate; it's the same over every path in D connecting these points. (Since D is connected, there is always a path lying in D that connects the points.) Further, since D is open, there is a disk centered at (u, v) and lying completely inside D. Pick any point (x_1, v) in the disk with $x_1 < u$ and let C_1 be any path from (x_0, y_0) to (x_1, v) lying in D. So, in particular, if we integrate over the path consisting of C_1 followed by the horizontal path C_2 indicated in Figure 14.24, we must have

$$f(u, v) = \int_{(x_0, y_0)}^{(x_1,v)} \mathbf{F}(x, y)\cdot d\mathbf{r} + \int_{(x_1, v)}^{(u,v)} \mathbf{F}(x, y)\cdot d\mathbf{r}. \tag{3.2}$$

Observe that the first integral in (3.2) is independent of u. So, taking the partial derivative of both sides of (3.2) with respect to u, we get

$$\begin{aligned}f_u(u, v) &= \frac{\partial}{\partial u}\int_{(x_0, y_0)}^{(x_1,v)} \mathbf{F}(x, y)\cdot d\mathbf{r} + \frac{\partial}{\partial u}\int_{(x_1, v)}^{(u,v)} \mathbf{F}(x, y)\cdot d\mathbf{r} \\ &= 0 + \frac{\partial}{\partial u}\int_{(x_1, v)}^{(u,v)} \mathbf{F}(x, y)\cdot d\mathbf{r} \\ &= \frac{\partial}{\partial u}\int_{(x_1, v)}^{(u,v)} M(x, y)\,dx + N(x, y)\,dy.\end{aligned}$$

Notice that on the second portion of the indicated path, y is a constant and so, $dy = 0$. This gives us

$$f_u(u, v) = \frac{\partial}{\partial u}\int_{(x_1, v)}^{(u,v)} M(x, y)\,dx + N(x, y)\,dy = \frac{\partial}{\partial u}\int_{(x_1, v)}^{(u,v)} M(x, y)\,dx.$$

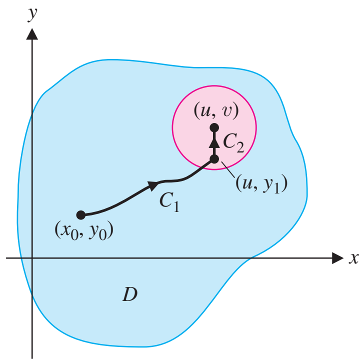


FIGURE 14.25
Second path

Finally, from the second form of the Fundamental Theorem of Calculus, we have

$$f_u(u, v) = \frac{\partial}{\partial u} \int_{(x_1,v)}^{(u,v)} M(x, y)\, dx = M(u, v). \tag{3.3}$$

Similarly, pick any point (u, y_1) in the disk centered at (u, v) with $y_1 < v$ and let C_1 be any path from (x_0, y_0) to (u, y_1) lying in D. Then, integrating over the path consisting of C_1 followed by the vertical path C_2 indicated in Figure 14.25, we find that

$$f(u, v) = \int_{(x_0,y_0)}^{(u,y_1)} \mathbf{F}(x, y) \cdot d\mathbf{r} + \int_{(u,y_1)}^{(u,v)} \mathbf{F}(x, y) \cdot d\mathbf{r}. \tag{3.4}$$

In this case, the first integral is independent of v. So, differentiating both sides of (3.4) with respect to v, we have

$$\begin{aligned} f_v(u, v) &= \frac{\partial}{\partial v} \int_{(x_0,y_0)}^{(u,y_1)} \mathbf{F}(x, y) \cdot d\mathbf{r} + \frac{\partial}{\partial v} \int_{(u,y_1)}^{(u,v)} \mathbf{F}(x, y) \cdot d\mathbf{r} \\ &= 0 + \frac{\partial}{\partial v} \int_{(u,y_1)}^{(u,v)} \mathbf{F}(x, y) \cdot d\mathbf{r} \\ &= \frac{\partial}{\partial v} \int_{(u,y_1)}^{(u,v)} M(x, y)\, dx + N(x, y)\, dy \\ &= \frac{\partial}{\partial v} \int_{(u,y_1)}^{(u,v)} N(x, y)\, dy = N(u, v), \end{aligned} \tag{3.5}$$

by the second form of the Fundamental Theorem of Calculus, where we have used the fact that on the second part of the indicated path, x is a constant, so that $dx = 0$. Replacing u and v by x and y, respectively, in (3.3) and (3.5) establishes that

$$\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle = \langle f_x(x, y), f_y(x, y)\rangle = \nabla f(x, y),$$

so that $\mathbf{F}$ is conservative in D. ■

Notice that in the course of the first part of the proof of Theorem 3.1, we also proved the following result, which corresponds to the Fundamental Theorem of Calculus for definite integrals.

THEOREM 3.2 (Fundamental Theorem for Line Integrals)

Suppose that $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ is continuous in the open, connected region $D \subset \mathbb{R}^2$ and that C is any piecewise-smooth curve lying in D, with initial point (x_1, y_1) and terminal point (x_2, y_2). Then, if $\mathbf{F}$ is conservative on D, with $\mathbf{F}(x, y) = \nabla f(x, y)$, we have

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = f(x, y)\Big|_{(x_1,y_1)}^{(x_2,y_2)} = f(x_2, y_2) - f(x_1, y_1).$$

You should quickly recognize the advantages presented by Theorem 3.2. For a conservative vector field, you don't need to parameterize the path to compute a line integral; you need only find a potential function and then simply evaluate the potential function between the endpoints of the curve. We illustrate this in example 3.1.

EXAMPLE 3.1 A Line Integral That Is Independent of Path

Show that for $\mathbf{F}(x, y) = \langle 2xy - 3, x^2 + 4y^3 + 5\rangle$, the line integral $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path. Then, evaluate the line integral for any curve C with initial point at $(-1, 2)$ and terminal point at $(2, 3)$.

Solution From Theorem 3.1, the line integral is independent of path if and only if the vector field $\mathbf{F}(x, y)$ is conservative. So, we look for a potential function for $\mathbf{F}$, that is, a function $f(x, y)$ for which

$$\mathbf{F}(x, y) = \langle 2xy - 3, x^2 + 4y^3 + 5\rangle = \nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle.$$

Of course, this occurs when

$$f_x = 2xy - 3 \quad \text{and} \quad f_y = x^2 + 4y^3 + 5. \tag{3.6}$$

Integrating the first of these two equations with respect to x (note that we might just as easily integrate the second one with respect to y), we get

$$f(x, y) = \int (2xy - 3)\, dx = x^2 y - 3x + g(y), \tag{3.7}$$

where $g(y)$ is some arbitrary function of y alone. (Recall that we get an arbitrary *function* of y instead of a *constant* of integration, since we are integrating a function of x and y with respect to x.) Differentiating with respect to y, we get

$$f_y(x, y) = x^2 + g'(y).$$

Notice that from (3.6), we already have an expression for f_y. Setting these two expressions equal, we get

$$x^2 + g'(y) = x^2 + 4y^3 + 5$$

and subtracting x^2 from both sides, we get

$$g'(y) = 4y^3 + 5.$$

Finally, integrating this last expression with respect to y gives us

$$g(y) = y^4 + 5y + c.$$

We now have from (3.7) that

$$f(x, y) = x^2 y - 3x + y^4 + 5y + c$$

is a potential function for $\mathbf{F}(x, y)$, for any constant c. Now that we have found a potential function, we have by Theorem 3.2 that for any path from $(-1, 2)$ to $(2, 3)$,

$$\begin{aligned} \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} &= f(x, y)\Big|_{(-1,2)}^{(2,3)} \\ &= [2^2(3) - 3(2) + 3^4 + 5(3) + c] - [2 + 3 + 2^4 + 5(2) + c] \\ &= 71. \blacksquare \end{aligned}$$

Notice that when we evaluated the line integral in example 3.1, the constant c in the expression for the potential function dropped out. For this reason, we usually take the constant to be zero when we write down a potential function.

We consider a curve C to be **closed** if its two endpoints are the same. That is, for a plane curve C defined parametrically by

$$C = \{(x, y) | x = g(t), y = h(t), a \le t \le b\},$$

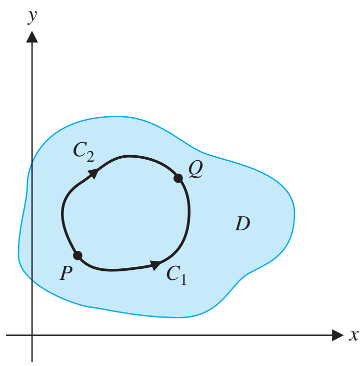


FIGURE 14.26a
Curves C_1 and C_2

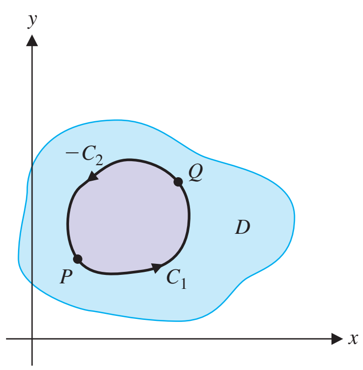


FIGURE 14.26b
The closed curve formed by $C_1 \cup (-C_2)$

C is closed if $(g(a), h(a)) = (g(b), h(b))$. Theorem 3.3 provides us with an important connection between conservative vector fields and line integrals along closed curves.

THEOREM 3.3

Suppose that $\mathbf{F}(x, y)$ is continuous in the open, connected region $D \subset \mathbb{R}^2$. Then $\mathbf{F}$ is conservative on D if and only if $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D.

PROOF

Suppose that $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D. Take any two points P and Q lying in D and let C_1 and C_2 be any two piecewise-smooth closed curves from P to Q that lie in D, as indicated in Figure 14.26a. (Note that since D is connected, there always exist such curves.) Then, the curve C consisting of C_1 followed by $-C_2$ is a piecewise-smooth closed curve lying in D, as indicated in Figure 14.26b. It now follows that

$$0 = \int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F}(x, y) \cdot d\mathbf{r}$$

$$= \int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} - \int_{C_2} \mathbf{F}(x, y) \cdot d\mathbf{r}, \qquad \text{From Theorem 2.5}$$

so that
$$\int_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_{C_2} \mathbf{F}(x, y) \cdot d\mathbf{r}.$$

Since C_1 and C_2 were any two curves from P to Q, we have that $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path and so, $\mathbf{F}$ is conservative by Theorem 3.1. The second half of the theorem (that $\mathbf{F}$ conservative implies $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D) is a simple consequence of Theorem 3.2 and is left as an exercise. ■

You have already seen that line integrals are not always independent of path. Said differently, not all vector fields are conservative. In view of this, it would be helpful to have a simple way of deciding whether or not a line integral is independent of path before going through the process of trying to construct a potential function.

Note that by Theorem 3.1, if $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ is continuous on the open, connected region D and the line integral $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ must be conservative. That is, there is a function $f(x, y)$ for which $\mathbf{F}(x, y) = \nabla f(x, y)$, so that

$$M(x, y) = f_x(x, y) \quad \text{and} \quad N(x, y) = f_y(x, y).$$

Differentiating the first equation with respect to y and the second equation with respect to x, we have

$$M_y(x, y) = f_{xy}(x, y) \quad \text{and} \quad N_x(x, y) = f_{yx}(x, y).$$

Notice now that if M_y and N_x are continuous in D, then the mixed second partial derivatives $f_{xy}(x, y)$ and $f_{yx}(x, y)$ must be equal in D, by Theorem 3.1 in Chapter 12. We must then have that

$$M_y(x, y) = N_x(x, y),$$

for all (x, y) in D. As it turns out, if we further assume that D is **simply-connected** (that is, that every closed curve in D encloses only points in D), then the converse of this result is

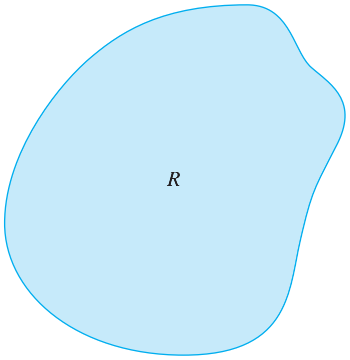


FIGURE 14.27a
Simply-connected

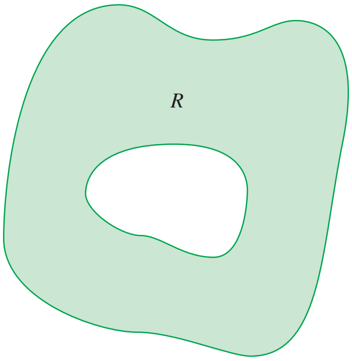


FIGURE 14.27b
Not simply-connected

also true [i.e., $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$ is independent of path whenever $M_y = N_x$ in D]. We illustrate a simply-connected region in Figure 14.27a and a region that is not simply-connected in Figure 14.27b. You can think about simply-connected regions as regions that have no holes. We can now state the following result.

THEOREM 3.4

Suppose that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on a simply-connected region D. Then, $\int_C M(x, y)\,dx + N(x, y)\,dy$ is independent of path in D if and only if $M_y(x, y) = N_x(x, y)$ for all (x, y) in D.

We have already proved that independence of path implies that $M_y(x, y) = N_x(x, y)$ for all (x, y) in D. We postpone the proof of the second half of the theorem until our presentation of Green's Theorem in section 14.4.

EXAMPLE 3.2 Testing a Line Integral for Independence of Path

Determine whether or not the line integral $\int_C (e^{2x} + x \sin y)\,dx + (x^2 \cos y)\,dy$ is independent of path.

Solution In this case, we have

$$M_y = \frac{\partial}{\partial y}(e^{2x} + x \sin y) = x \cos y$$

and

$$N_x = \frac{\partial}{\partial x}(x^2 \cos y) = 2x \cos y,$$

so that $M_y \neq N_x$. By Theorem 3.4, the line integral is thus not independent of path. ■

CONSERVATIVE VECTOR FIELDS

Before moving on to three-dimensional vector fields, we pause to summarize the results we have developed for two-dimensional vector fields $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$, where we assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on an open, simply-connected region $D \subset \mathbb{R}^2$. In this case, the following five statements are equivalent, meaning that for a given vector field, either all five statements are true or all five statements are false.

1. $\mathbf{F}(x, y)$ is conservative in D.
2. $\mathbf{F}(x, y)$ is a gradient field in D (*i.e.*, $\mathbf{F}(x, y) = \nabla f(x, y)$ for some potential function f, for all $(x, y) \in D$).
3. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.
4. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D.
5. $M_y(x, y) = N_x(x, y)$, for all $(x, y) \in D$.

All we have said about independence of path and conservative vector fields can be extended to higher dimensions, although the test for when a line integral is independent of path becomes slightly more complicated. For a three-dimensional vector field $\mathbf{F}(x, y, z)$, we say that **F** is **conservative** in a region D whenever there is a scalar function $f(x, y, z)$ for which

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z).$$

As in two dimensions, f is called a **potential function** for the vector field **F**. You can construct a potential function for a conservative vector field in three dimensions in much the same way as you did in two dimensions. We illustrate this in example 3.3.

EXAMPLE 3.3 Showing That a Three-Dimensional Vector Field Is Conservative

Show that the vector field $\mathbf{F}(x, y, z) = \langle 4xe^z, \cos y, 2x^2e^z \rangle$ is conservative, by finding a potential function f.

Solution We need to find a potential function $f(x, y, z)$ for which

$$\begin{aligned}\mathbf{F}(x, y, z) &= \langle 4xe^z, \cos y, 2x^2e^z \rangle = \nabla f(x, y, z)\\ &= \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle.\end{aligned}$$

This will occur if and only if

$$f_x = 4xe^z, \quad f_y = \cos y \quad \text{and} \quad f_z = 2x^2e^z. \tag{3.8}$$

Integrating the first of these equations with respect to x, we have

$$f(x, y, z) = \int 4xe^z\, dx = 2x^2e^z + g(y, z),$$

where $g(y, z)$ is an arbitrary function of y and z alone. Note that since y and z are treated as constants when integrating or differentiating with respect to x, we add an arbitrary function of y and z (instead of an arbitrary constant) after a partial integration with respect to x. Differentiating this expression with respect to y, we have

$$f_y(x, y, z) = g_y(y, z) = \cos y,$$

from the second equation in (3.8). Integrating $g_y(y, z)$ with respect to y now gives us

$$g(y, z) = \int \cos y\, dy = \sin y + h(z),$$

where $h(z)$ is an arbitrary function of z alone. Notice that here, we got an arbitrary function of z alone, since we were integrating $g(y, z)$ (a function of y and z alone) with respect to y. This now gives us

$$f(x, y, z) = 2x^2e^z + g(y, z) = 2x^2e^z + \sin y + h(z).$$

Differentiating this last equation with respect to z yields

$$f_z(x, y, z) = 2x^2e^z + h'(z) = 2x^2e^z,$$

from the third equation in (3.8). This gives us that $h'(z) = 0$, so that $h(z)$ is a constant. (We'll choose it to be 0.) We now have that a potential function for $\mathbf{F}(x, y, z)$ is

$$f(x, y, z) = 2x^2e^z + \sin y$$

and so, **F** is conservative. ■

We summarize the main results for line integrals for three-dimensional vector fields in Theorem 3.5.

THEOREM 3.5

Suppose that the vector field $\mathbf{F}(x, y, z)$ is continuous on the open, connected region $D \subset \mathbb{R}^3$. Then, the line integral $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ is independent of path in D if and only if the vector field $\mathbf{F}$ is conservative in D, that is, $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, for all (x, y, z) in D, for some scalar function f (a potential function for $\mathbf{F}$). Further, for any piecewise-smooth curve C lying in D, with initial point (x_1, y_1, z_1) and terminal point (x_2, y_2, z_2), we have

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(x, y, z)\Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

EXERCISES 14.3

WRITING EXERCISES

1. You have seen two different methods of determining whether a line integral is independent of path: one in example 3.1 and the other in example 3.2. If you have reason to believe that a line integral will be independent of path, explain which method you would prefer to use.

2. In the situation of exercise 1, if you doubt that a line integral is independent of path, explain which method you would prefer to use. If you have no evidence as to whether the line integral is or isn't independent of path, explain which method you would prefer to use.

3. In section 14.1, we introduced conservative vector fields and stated that some calculations simplified when the vector field is conservative. Discuss one important example of this.

4. Our definition of independence of path applies only to line integrals of the form $\int_C \mathbf{F} \cdot d\mathbf{r}$. Explain why an arc length line integral $\int_C f\, ds$ would not be independent of path (unless $f = 0$).

In exercises 1–12, determine whether F is conservative. If it is, find a potential function f.

1. $\mathbf{F}(x, y) = \langle 2xy - 1, x^2 \rangle$

2. $\mathbf{F}(x, y) = \langle 3x^2y^2, 2x^3y - y \rangle$

3. $\mathbf{F}(x, y) = \left\langle \frac{1}{y} - 2x, y - \frac{x}{y^2} \right\rangle$

4. $\mathbf{F}(x, y) = \langle \sin y - x, x \cos y \rangle$

5. $\mathbf{F}(x, y) = \langle e^{xy} - 1, xe^{xy} \rangle$

6. $\mathbf{F}(x, y) = \langle e^y - 2x, xe^y - x^2y \rangle$

7. $\mathbf{F}(x, y) = \langle ye^{xy}, xe^{xy} + \cos y \rangle$

8. $\mathbf{F}(x, y) = \langle y \cos xy - 2xy, x \cos xy - x^2 \rangle$

9. $\mathbf{F}(x, y, z) = \langle z^2 + 2xy, x^2 + 1, 2xz - 3 \rangle$

10. $\mathbf{F}(x, y, z) = \langle y^2 - x, 2xy + \sin z, y \cos z \rangle$

11. $\mathbf{F}(x, y, z) = \langle y^2z^2 + x, y + 2xyz^2, 2xy^2z \rangle$

12. $\mathbf{F}(x, y, z) = \langle 2xe^{yz} - 1, x^2 + e^{yz}, x^2ye^{yz} \rangle$

In exercises 13–18, show that the line integral is independent of path and use a potential function to evaluate the integral.

13. $\int_C 2xy\, dx + (x^2 - 1)\, dy$, where C runs from (1, 0) to (3, 1)

14. $\int_C 3x^2y^2 dx + (2x^3y - 4)\, dy$, where C runs from (1, 2) to (−1, 1)

15. $\int_C ye^{xy} dx + (xe^{xy} - 2y)\, dy$, where C runs from (1, 0) to (0, 4)

16. $\int_C (2xe^{x^2} - 2y)\, dx + (2y - 2x)\, dy$, where C runs from (1, 2) to (−1, 1)

17. $\int_C (z^2 + 2xy)\, dx + x^2 dy + 2xz\, dz$, where C runs from (2, 1, 3) to (4, −1, 0)

18. $\int_C (2x \cos z - x^2)\, dx + (z - 2y)\, dy + (y - x^2 \sin z)\, dz$, where C runs from (3, −2, 0) to $(1, 0, \pi)$

In exercises 19–30, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

19. $\mathbf{F}(x, y) = \langle x^2 + 1, y^3 - 3y + 2 \rangle$, C is the top half-circle from (−4, 0) to (4, 0)

20. $\mathbf{F}(x, y) = \langle xe^{x^2} - 2, \sin y \rangle$, C is the portion of the parabola $y = x^2$ from (−2, 4) to (2, 4)

21. $\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$, C is the top half-circle from (1, 4, −3) to (1, 4, 3)

22. $\mathbf{F}(x, y, z) = \langle \cos x, \sqrt{y} + 1, 4z^3 \rangle$, C is the quarter-circle from (2, 0, 3) to (2, 3, 0)

23. $\mathbf{F}(x, y, z) = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}}$, C runs from (1, 3, 2) to (2, 1, 5)

24. $\mathbf{F}(x, y, z) = \dfrac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$, C runs from (2, 0, 0) to (0, 1, −1)

25. $\mathbf{F}(x, y) = \langle 3x^2y + 1, 3xy^2 \rangle$, C is the bottom half-circle from $(1, 0)$ to $(-1, 0)$

26. $\mathbf{F}(x, y) = \langle 4xy - 2x, 2x^2 - x \rangle$, C is the portion of the parabola $y = x^2$ from $(-2, 4)$ to $(2, 4)$

27. $\mathbf{F}(x, y) = \langle y^2e^{xy^2} - y, 2xye^{xy^2} - x - 1 \rangle$, C is the line segment from $(2, 3)$ to $(3, 0)$

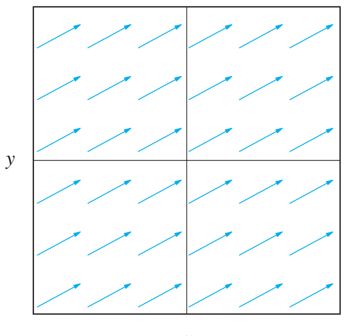
28. $\mathbf{F}(x, y) = \langle 2ye^{2x} + y^3, e^{2x} + 3xy^2 \rangle$, C is the line segment from $(4, 3)$ to $(1, -3)$

29. $\mathbf{F}(x, y) = \left\langle \frac{1}{y} - e^{2x}, 2y - \frac{x}{y^2} \right\rangle$, C is the circle $(x-5)^2 + (y+6)^2 = 16$, oriented counterclockwise

30. $\mathbf{F}(x, y) = \langle 3y - \sqrt{y/x}, 3x - \sqrt{x/y} \rangle$, C is the ellipse $4(x-4)^2 + 9(y-4)^2 = 36$, oriented counterclockwise

In exercises 31–36, use the graph to determine whether or not the vector field is conservative.

31.



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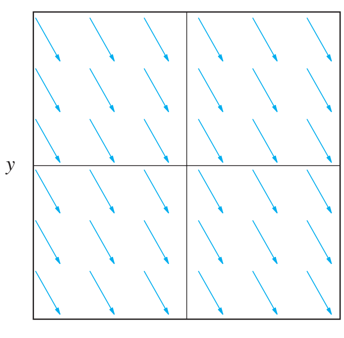

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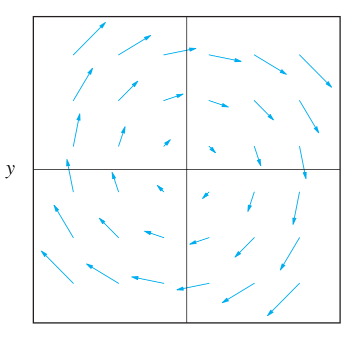

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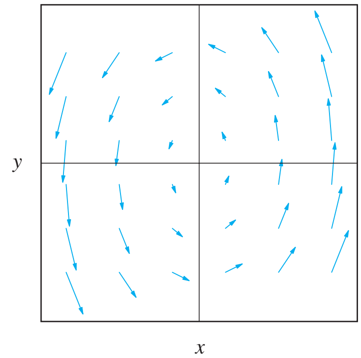

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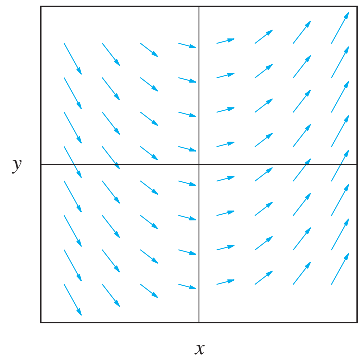
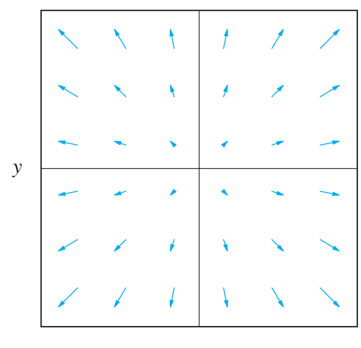


36.



In exercises 37–40, show that the line integral is not independent of path by finding two paths that give different values of the integral.

37. $\int_C y\,dx - x\,dy$, where C goes from $(-2, 0)$ to $(2, 0)$

38. $\int_C 2\,dx + x\,dy$, where C goes from $(1, 4)$ to $(2, -2)$

39. $\int_C y\,dx - 3\,dy$, where C goes from $(-2, 2)$ to $(0, 0)$

40. $\int_C y^2\,dx + x^2\,dy$, where C goes from $(0, 0)$ to $(1, 1)$

In exercises 41–44, label each statement as True or False and briefly explain.

41. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

42. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative.

43. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

44. If $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

45. Let $\mathbf{F}(x, y) = \dfrac{1}{x^2 + y^2}\langle -y, x\rangle$. Find a potential function f for $\mathbf{F}$ and carefully note any restrictions on the domain of f. Let C be the unit circle and show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$. Explain why the Fundamental Theorem for Line Integrals does not apply to this calculation. Quickly explain how to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the circle $(x-2)^2 + (y-3)^2 = 1$.

46. Finish the proof of Theorem 3.3 by showing that if $\mathbf{F}$ is conservative in an open, connected region $D \subset \mathbb{R}^2$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all piecewise-smooth closed curves C lying in D.

47. Determine whether or not each region is simply-connected. (a) $\{(x, y) : x^2 + y^2 < 2\}$ (b) $\{(x, y) : 1 < x^2 + y^2 < 2\}$

48. Determine whether or not each region is simply-connected. (a) $\{(x, y) : 1 < x < 2\}$ (b) $\{(x, y) : 1 < x^2 < 2\}$

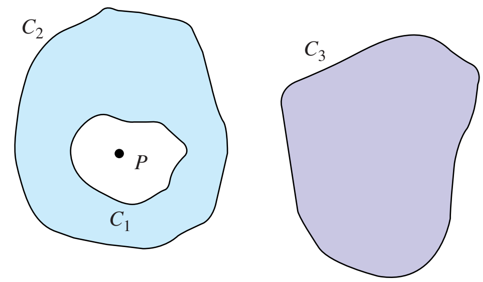
49. The Coulomb force for a unit charge at the origin and charge q at point $P_1 = (x_1, y_1, z_1)$ is $\mathbf{F} = \dfrac{kq}{r^2}\hat{\mathbf{r}}$, where $r = \sqrt{x^2 + y^2 + z^2}$ and $\hat{\mathbf{r}} = \dfrac{\langle x, y, z\rangle}{r}$. Show that the work done by $\mathbf{F}$ to move the charge q from P_1 to $P_2 = (x_2, y_2, z_2)$ is equal to $\dfrac{kq}{r_1} - \dfrac{kq}{r_2}$, where $r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$.

50. Interpret the result of exercise 49 in the case where (a) P_1 is closer to the origin than P_2. (Is the work positive or negative? Why does this make sense physically?) (b) P_2 is closer to the origin than P_1 and (c) P_1 and P_2 are the same distance from the origin.

51. The work done to increase the temperature of a gas from T_1 to T_2 and increase its pressure from P_1 to P_2 is given by $\displaystyle\int_C \left(\frac{RT}{P}\,dP - R\,dT\right)$. Here, R is a constant, T is temperature, P is pressure and C is the path of (P, T) values as the changes occur. Compare the work done along the following two paths. (a) C_1 consists of the line segment from (P_1, T_1) to (P_1, T_2), followed by the line segment to (P_2, T_2); (b) C_2 consists of the line segment from (P_1, T_1) to (P_2, T_1), followed by the line segment to (P_2, T_2).

52. Based on your answers in exercise 51, is the force field involved in changing the temperature and pressure of the gas conservative?

53. A vector field $\mathbf{F}$ satisfies $\mathbf{F} = \nabla\phi$ (where ϕ is continuous) at every point except P, where it is undefined. Suppose that C_1 is a small closed curve enclosing P, C_2 is a large closed curve enclosing C_1 and C_3 is a closed curve that does not enclose P. (See the figure.) Given that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$, explain why $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0$.



54. The circulation of a fluid with velocity field $\mathbf{v}$ around the closed path C is defined by $\Gamma = \int_C \mathbf{v} \cdot d\mathbf{r}$. For inviscid flow, $\dfrac{d}{dt}\Gamma = \displaystyle\int_C \mathbf{v} \cdot d\mathbf{v}$. Show that in this case $\dfrac{d}{dt}\Gamma = 0$. This is known as **Kelvin's Circulation Theorem** and explains why small whirlpools in a stream stay coherent and move for periods of time.

EXPLORATORY EXERCISES

1. For closed curves, we can take advantage of portions of a line integral that will equal zero. For example, if C is a closed curve, explain why you can simplify $\int_C (x + y^2)\,dx + (y^2 + x)\,dy$ to $\int_C y^2 dx + x\,dy$. In general, explain why the $f(x)$ and $g(y)$ terms can be dropped in the line integral $\int_C (f(x) + y^2)\,dx + (x + g(y))\,dy$. Describe which other terms can be dropped in the line integral over a closed curve. Use the example

$$\int_C (x^3 + y^2 + x^2y^2 + \cos y)\,dx + (y^2 + 2xy - x\sin y + x^3 y)\,dy$$

to help organize your thinking.

2. In this exercise, we explore a basic principle of physics called **conservation of energy.** Start with the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where the position function $\mathbf{r}(t)$ is a continuously differentiable function of time. Substitute Newton's second law: $\mathbf{F} = m\dfrac{d\mathbf{v}}{dt}$ and $d\mathbf{r} = \mathbf{r}'(t)\,dt = \mathbf{v}\,dt$ and show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \Delta K$. Here, K is **kinetic energy** defined by $K = \frac{1}{2}m\|\mathbf{v}\|^2$ and ΔK is the change of kinetic energy from the initial point of C to the terminal point of C. Next, assume that $\mathbf{F}$ is conservative with $\mathbf{F} = -\nabla f$, where the function f represents **potential energy.** Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = -\Delta f$ where Δf equals the change in potential energy from the initial point of C to the terminal point of C. Conclude that under these hypotheses (conservative force, continuous acceleration) the net change in energy $\Delta K + \Delta f$ equals 0. Therefore, $K + f$ is constant.

14.4 GREEN'S THEOREM

In this section, we develop a connection between certain line integrals around a closed curve in the plane and double integrals over the region enclosed by the curve. At first glance, you might think this a strange and abstract connection, one that only a mathematician could care about. Actually, the reverse is true; Green's Theorem is a significant result with far-reaching implications. It is of fundamental importance in the analysis of fluid flows and in the theories of electricity and magnetism.

Before stating the main result, we briefly define some terminology. Recall that for a plane curve C defined parametrically by

$$C = \{(x, y)|x = g(t),\ y = h(t),\ a \le t \le b\},$$

C is closed if its two endpoints are the same, i.e., $(g(a), h(a)) = (g(b), h(b))$. A curve C is **simple** if it does not intersect itself, except at the endpoints. We illustrate a simple closed curve in Figure 14.28a and a closed curve that is not simple in Figure 14.28b.

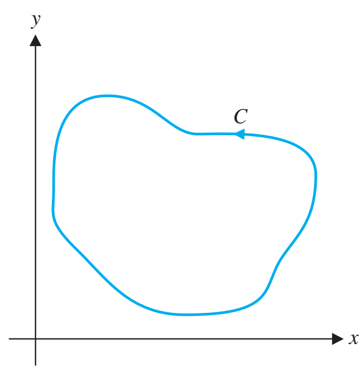


FIGURE 14.28a
Simple closed curve

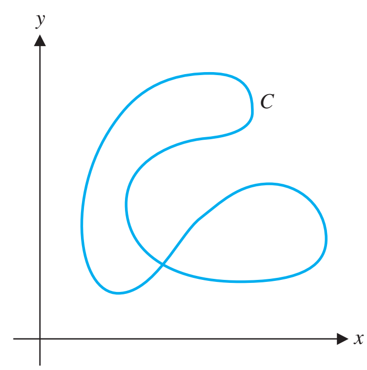


FIGURE 14.28b
Closed, but not simple curve

We say that a simple closed curve C has **positive orientation** if the region R enclosed by C stays to the left of C, as the curve is traversed; a curve has **negative orientation** if the region R stays to the right of C. In Figures 14.29a and 14.29b, we illustrate a simple closed curve with positive orientation and one with negative orientation, respectively.

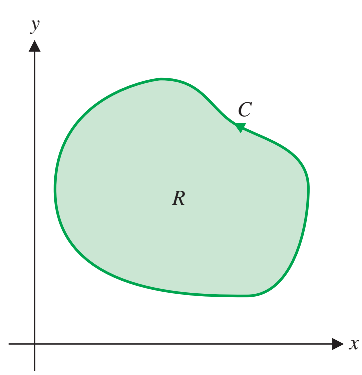


FIGURE 14.29a
Positive orientation

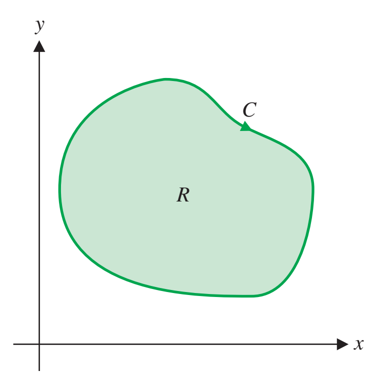


FIGURE 14.29b
Negative orientation

HISTORICAL NOTES

George Green (1793–1841) English mathematician who discovered Green's Theorem. Green was self-taught, receiving only two years of schooling before going to work in his father's bakery at age 9. He continued to work in and eventually took over the family mill while teaching himself mathematics. In 1828, he published an essay in which he gave potential functions their name and applied them to the study of electricity and magnetism. This little-read essay introduced Green's Theorem and the so-called Green's functions used in the study of partial differential equations. Green was admitted to Cambridge University at age 40 and published several papers before his early death from illness. The significance of his original essay remained unknown until shortly after his death.

We use the notation

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$

to denote a line integral along a simple closed curve C oriented in the positive direction.

We can now state the main result of the section.

THEOREM 4.1 (Green's Theorem)

Let C be a piecewise-smooth, simple closed curve in the plane with positive orientation and let R be the region enclosed by C, together with C. Suppose that $M(x, y)$ and $N(x, y)$ are continuous and have continuous first partial derivatives in some open region D, with $R \subset D$. Then,

$$\oint_C M(x, y)\, dx + N(x, y)\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

You can find a general proof of Green's Theorem in a more advanced text. We prove it here only for a special case.

PROOF

Here, we assume that the region R can be written in the form

$$R = \{(x, y) | a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\},$$

where $g_1(x) \le g_2(x)$, for all x in $[a, b]$, $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$, as illustrated in Figure 14.30a. Notice that we can divide C into the two pieces indicated in Figure 14.30a:

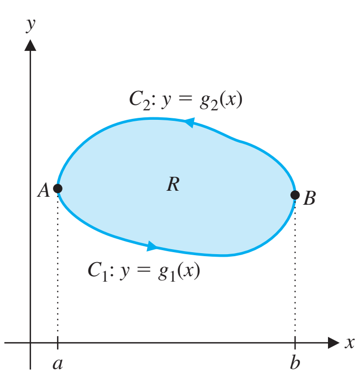


FIGURE 14.30a
The region R

$$C = C_1 \cup C_2,$$

where C_1 is the bottom portion of the curve, defined by

$$C_1 = \{(x, y) | a \le x \le b, y = g_1(x)\}$$

and C_2 is the top portion of the curve, defined by

$$C_2 = \{(x, y) | a \le x \le b, y = g_2(x)\},$$

where the orientation is as indicated in the figure. From the Evaluation Theorem for line integrals (Theorem 2.4), we then have

$$\begin{aligned} \oint_C M(x, y)\, dx &= \int_{C_1} M(x, y)\, dx + \int_{C_2} M(x, y)\, dx \\ &= \int_a^b M(x, g_1(x))\, dx - \int_a^b M(x, g_2(x))\, dx \\ &= \int_a^b [M(x, g_1(x)) - M(x, g_2(x))]\, dx, \qquad (4.1) \end{aligned}$$

where the minus sign in front of the second integral comes from our traversing C_2 "backward" (i.e., from right to left). On the other hand, notice that we can write

$$\iint_R \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \int_a^b M(x, y)\Big|_{y=g_1(x)}^{y=g_2(x)}\,dx \qquad \text{By the Fundamental Theorem of Calculus}$$

$$= \int_a^b [M(x, g_2(x)) - M(x, g_1(x))]\,dx.$$

Together with (4.1), this gives us

$$\oint_C M(x, y)\,dx = -\iint_R \frac{\partial M}{\partial y}\,dA. \tag{4.2}$$

We now assume that we can also write the region R in the form

$$R = \{(x, y) | c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\},$$

where $h_1(y) \le h_2(y)$ for all y in $[c, d]$, $h_1(c) = h_2(c)$ and $h_1(d) = h_2(d)$. Here, we write $C = C_3 \cup C_4$, as illustrated in Figure 14.30b. In this case, notice that we can write

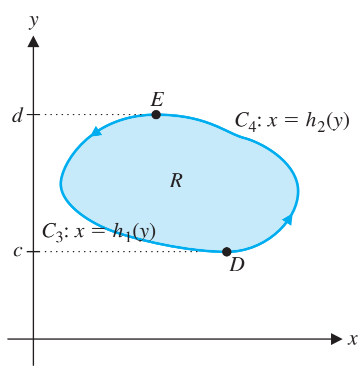


FIGURE 14.30b
The region R

$$\oint_C N(x, y)\,dy = \int_{C_3} N(x, y)\,dy + \int_{C_4} N(x, y)\,dy$$

$$= -\int_c^d N(h_1(y), y)\,dy + \int_c^d N(h_2(y), y)\,dy$$

$$= \int_c^d [N(h_2(y), y) - N(h_1(y), y)]\,dy, \tag{4.3}$$

where the minus sign in front of the first integral accounts for our traversing C_3 "backward" (in this case, from top to bottom). Further, notice that

$$\iint_R \frac{\partial N}{\partial x}\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \int_c^d [N(h_2(y), y) - N(h_1(y), y)]\,dy.$$

Together with (4.3), this gives us

$$\oint_C N(x, y)\,dy = \iint_R \frac{\partial N}{\partial x}\,dA. \tag{4.4}$$

Adding together (4.2) and (4.4), we have

$$\oint_C M(x, y)\,dx + N(x, y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA,$$

as desired. ■

Although the significance of Green's Theorem lies in the connection it provides between line integrals and double integrals in more theoretical settings, we illustrate the result in example 4.1 by using it to simplify the calculation of a line integral.

EXAMPLE 4.1 Using Green's Theorem

Use Green's Theorem to rewrite and evaluate $\oint_C (x^2 + y^3)\,dx + 3xy^2 dy$, where C consists of the portion of $y = x^2$ from (2, 4) to (0, 0), followed by the line segments from (0, 0) to (2, 0) and from (2, 0) to (2, 4).

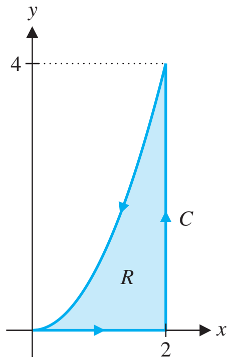


FIGURE 14.31
The region R

Solution We indicate the curve C and the enclosed region R in Figure 14.31. Notice that C is a piecewise-smooth, simple closed curve with positive orientation. Further, for $M(x, y) = x^2 + y^3$ and $N(x, y) = 3xy^2$, M and N are continuous and have continuous first partial derivatives everywhere. Green's Theorem then says that

$$\oint_C (x^2 + y^3)\,dx + 3xy^2 dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \iint_R (3y^2 - 3y^2)\,dA = 0.$$

Notice that in example 4.1, since the integrand of the double integral was zero, evaluating the double integral was far easier than evaluating the line integral directly. There is another simple way of thinking of the line integral in example 4.1. Notice that you can write this as $\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y^3, 3xy^2 \rangle$. Notice further that $\mathbf{F}$ is conservative [with potential function $f(x, y) = \frac{1}{3}x^3 + xy^3$] and so, by Theorem 3.3 in section 14.3, the line integral of $\mathbf{F}$ over any piecewise-smooth, closed curve must be zero.

EXAMPLE 4.2 Evaluating a Challenging Line Integral with Green's Theorem

Evaluate the line integral $\oint_C (7y - e^{\sin x})\,dx + [15x - \sin(y^3 + 8y)]\,dy$, where C is the circle of radius 3 centered at the point $(5, -7)$, as shown in Figure 14.32.

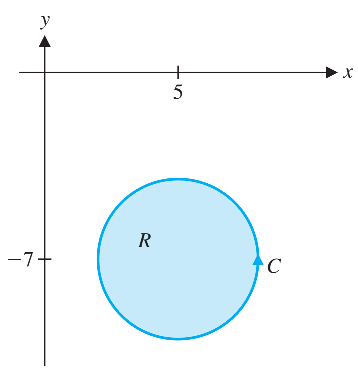


FIGURE 14.32
The region R

Solution First, notice that it will be virtually impossible to evaluate the line integral directly. (Think about this some, but don't spend too much time on it!) However, taking $M(x, y) = 7y - e^{\sin x}$ and $N(x, y) = 15x - \sin(y^3 + 8y)$, notice that M and N are continuous and have continuous first partial derivatives everywhere. So, we may apply Green's Theorem, which gives us

$$\oint_C (7y - e^{\sin x})\,dx + [15x - \sin(y^3 + 8y)]\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \iint_R (15 - 7)\,dA$$
$$= 8 \iint_R dA = 72\pi,$$

where $\iint_R dA$ is simply the area inside the region R, $\iint_R dA = \pi(3)^2 = 9\pi$.

If you look at example 4.2 critically, you might suspect that the integrand was chosen carefully so that the line integral was impossible to evaluate directly, but so that the integrand of the double integral was trivial. That's true: we did cook up the problem simply to illustrate the power of Green's Theorem. More significantly, Green's Theorem provides us with a wealth of interesting observations. One of these is as follows. Suppose that C is a piecewise-smooth, simple closed curve enclosing the region R. Then, taking $M(x, y) = 0$ and $N(x, y) = x$, we have

$$\oint_C x\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R dA,$$

which is simply the area of the region R. Alternatively, notice that if we take $M(x, y) = -y$ and $N(x, y) = 0$, we have

$$\oint_C -y\,dx = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R dA,$$

which is again the area of R. Putting these last two results together, we also have

$$\iint_R dA = \frac{1}{2}\oint_C x\,dy - y\,dx. \tag{4.5}$$

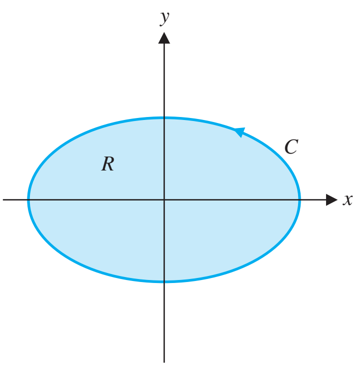


FIGURE 14.33
Elliptical region R

EXAMPLE 4.3 Using Green's Theorem to Find Area

Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution First, observe that the ellipse corresponds to the simple closed curve C defined parametrically by

$$C = \{(x, y) | x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi\},$$

where $a, b > 0$. You should also observe that C is smooth and positively oriented. (See Figure 14.33.) From (4.5), we have that the area A of the ellipse is given by

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)]\,dt \\ &= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \pi ab. \end{aligned}$$

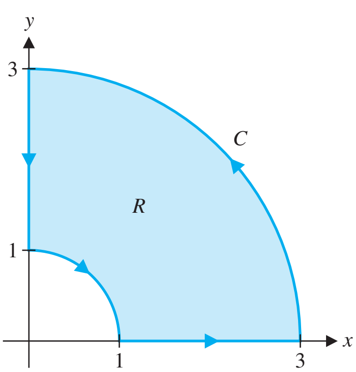


FIGURE 14.34
The region R

EXAMPLE 4.4 Using Green's Theorem to Evaluate a Line Integral

Evaluate the line integral $\oint_C (e^x + 6xy)\,dx + (8x^2 + \sin y^2)\,dy$, where C is the positively-oriented boundary of the region bounded by the circles of radii 1 and 3, centered at the origin and lying in the first quadrant, as indicated in Figure 14.34.

Solution Notice that since C consists of four distinct pieces, evaluating the line integral directly by parameterizing the curve is probably not a good choice. On the other hand, since C is a piecewise-smooth, simple closed curve, we have by Green's Theorem that

$$\begin{aligned} \oint_C (e^x + 6xy)\,dx + (8x^2 + \sin y^2)\,dy &= \iint_R \left[\frac{\partial}{\partial x}(8x^2 + \sin y^2) - \frac{\partial}{\partial y}(e^x + 6xy)\right] dA \\ &= \iint_R (16x - 6x)\,dA = \iint_R 10x\,dA, \end{aligned}$$

where R is the region between the two circles and lying in the first quadrant. Notice that this is easy to compute using polar coordinates, as follows:

$$\begin{aligned}\oint_C (e^x + 6xy)\,dx + (8x^2 + \sin y^2)\,dy &= \iint_R 10 \underbrace{x}_{r\cos\theta} \underbrace{dA}_{r\,dr\,d\theta} \\ &= \int_0^{\pi/2}\int_1^3 (10r\cos\theta)\,r\,dr\,d\theta \\ &= \int_0^{\pi/2} \cos\theta \left.\frac{10r^3}{3}\right|_{r=1}^{r=3} d\theta \\ &= \frac{10}{3}(3^3 - 1^3)\sin\theta\Big|_0^{\pi/2} \\ &= \frac{260}{3}.\end{aligned}$$

You should notice that in example 4.4, Green's Theorem is not a mere convenience; rather, it is a virtual necessity. Evaluating the line integral directly would prove to be a very significant challenge. (Go ahead and try it to see what we mean.)

For simplicity, we often will use the notation ∂R to refer to the boundary of the region R, oriented in the positive direction. Using this notation, the conclusion of Green's Theorem is written as

$$\oint_{\partial R} M(x, y)\,dx + N(x, y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

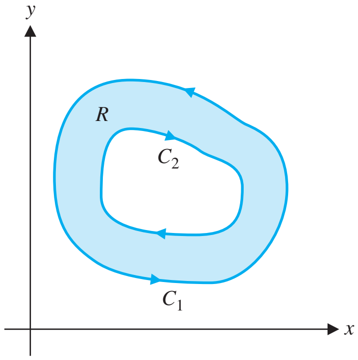


FIGURE 14.35a
Region with a hole

We can extend Green's Theorem to the case where a region is not simply-connected (i.e., where the region has one or more holes). We must emphasize that when dealing with such regions, the integration is taken over the *entire* boundary of the region (not just the outermost portion of the boundary!) and that the boundary curve is traversed in the positive direction, always keeping the region to the left. For instance, for the region R illustrated in Figure 14.35a with a single hole, notice that the boundary of R, ∂R, consists of two separate curves, C_1 and C_2, where C_2 is traversed clockwise, in order to keep the orientation positive on all of the boundary. Since the region is not simply-connected, we may not apply Green's Theorem directly. Rather, we first make two horizontal slits in the region, as indicated in Figure 14.35b, dividing R into the two simply-connected regions R_1 and R_2. Notice that we can then apply Green's Theorem in each of R_1 and R_2 separately. Adding the double integrals over R_1 and R_2 gives us the double integral over all of R. We have

$$\begin{aligned}\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA &= \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA + \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \oint_{\partial R_1} M(x, y)\,dx + N(x, y)\,dy \\ &\quad + \oint_{\partial R_2} M(x, y)\,dx + N(x, y)\,dy.\end{aligned}$$

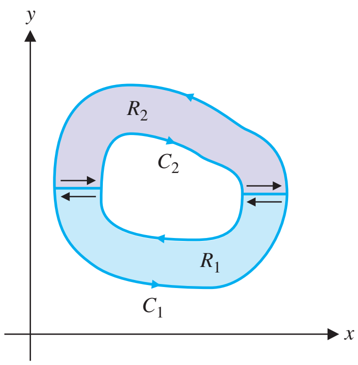


FIGURE 14.35b
$R = R_1 \cup R_2$

Further, since the line integrals over the common portions of ∂R_1 and ∂R_2 (i.e., the slits) are traversed in the opposite direction (one way on ∂R_1 and the other on ∂R_2), the line integrals over these portions will cancel out, leaving only the line integrals over C_1 and C_2.

This gives us

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_{\partial R_1} M(x, y)\,dx + N(x, y)\,dy + \oint_{\partial R_2} M(x, y)\,dx + N(x, y)\,dy$$
$$= \oint_{C_1} M(x, y)\,dx + N(x, y)\,dy + \oint_{C_2} M(x, y)\,dx + N(x, y)\,dy$$
$$= \oint_C M(x, y)\,dx + N(x, y)\,dy.$$

This says that Green's Theorem also holds for regions with a single hole. Of course, we can repeat the preceding argument to extend Green's Theorem to regions with any *finite* number of holes.

EXAMPLE 4.5 An Application of Green's Theorem

For $\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\langle -y, x\rangle$, show that $\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 2\pi$, for every simple closed curve C enclosing the origin.

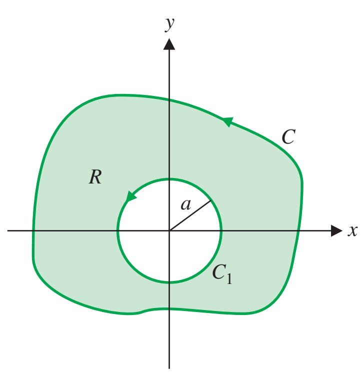


FIGURE 14.36
The region R

Solution Let C be any simple closed curve enclosing the origin and let C_1 be the circle of radius $a > 0$, centered at the origin (and positively oriented), where a is taken to be sufficiently small so that C_1 is completely enclosed by C, as illustrated in Figure 14.36. Further, let R be the region bounded between the curves C and C_1 (and including the curves themselves). Applying our extended version of Green's Theorem in R, we have

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} - \oint_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r}$$
$$= \oint_{\partial R} \mathbf{F}(x, y) \cdot d\mathbf{r}$$
$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \iint_R \left[\frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} - \frac{(-1)(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2}\right] dA$$
$$= \iint_R 0\,dA = 0.$$

This gives us

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F}(x, y) \cdot d\mathbf{r}.$$

Now, we chose C_1 to be a circle because we can easily parameterize a circle and then evaluate the line integral around C_1 explicitly. Notice that C_1 can be expressed parametrically by $x = a\cos t$, $y = a\sin t$, for $0 \le t \le 2\pi$. Noting that on C_1, $x^2 + y^2 = a^2$, this leaves us with an integral that we can easily evaluate, as follows:

$$\begin{aligned}\oint_C \mathbf{F}(x,y)\cdot d\mathbf{r} &= \oint_{C_1} \mathbf{F}(x,y)\cdot d\mathbf{r} = \oint_{C_1} \frac{1}{a^2}\langle -y, x\rangle \cdot d\mathbf{r} \\ &= \frac{1}{a^2}\oint_{C_1} -y\,dx + x\,dy \\ &= \frac{1}{a^2}\int_0^{2\pi} (-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)\,dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Notice that without Green's Theorem, proving a result such as that developed in example 4.5 would be elusive.

Now that we have Green's Theorem, we are in a position to prove the second half of Theorem 3.4. For convenience, we restate the theorem here.

THEOREM 4.2

Suppose that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on a simply-connected region D. Then, $\int_C M(x, y)\,dx + N(x, y)\,dy$ is independent of path if and only if $M_y(x, y) = N_x(x, y)$ for all (x, y) in D.

PROOF

Recall that in section 14.3, we proved the first part of the theorem: that if $\int_C M(x, y)\,dx + N(x, y)\,dy$ is independent of path, then it follows that $M_y(x, y) = N_x(x, y)$ for all (x, y) in D. We now prove that if $M_y(x, y) = N_x(x, y)$ for all (x, y) in D, then it follows that the line integral is independent of path. Let S be any piecewise-smooth closed curve lying in D. If S is simple and positively oriented, then since D is simply-connected, the region R enclosed by S is completely contained in D, so that $M_y(x, y) = N_x(x, y)$ for all (x, y) in R. From Green's Theorem, we now have that

$$\oint_S M(x,y)\,dx + N(x,y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

That is, for every piecewise-smooth, simple closed curve S lying in D, we have

$$\oint_S M(x,y)\,dx + N(x,y)\,dy = 0. \tag{4.6}$$

If S is not simple, then it intersects itself one or more times, creating two or more loops, each one of which is a simple closed curve. Since the line integral of $M(x, y)\,dx + N(x, y)\,dy$ over each of these is zero by (4. 6), it also follows that $\int_S M(x, y)\,dx + N(x, y)\,dy = 0$. It now follows from Theorem 3.3 that $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ must be conservative in D. Finally, it follows from Theorem 3.1 that $\int_C M(x, y)\,dx + N(x, y)\,dy$ is independent of path. ■

BEYOND FORMULAS

Green's Theorem is the first of three theorems in this chapter that relate different types of integrals. The alternatives given in these results can be helpful both computationally and theoretically. Example 4.2 shows how we can evaluate a difficult line integral by evaluating the equivalent (and simpler) double integral. Perhaps surprisingly, the use of Green's Theorem in example 4.5 is probably more important in applications. The theoretical result can be applied to any relevant problem and results like example 4.5 can sometimes provide important insight into general processes.

EXERCISES 14.4

WRITING EXERCISES

1. Given a line integral to evaluate, briefly describe the circumstances under which you should think about using Green's Theorem to replace the line integral with a double integral. Comment on the properties of the curve C and the functions involved.

2. In example 4.1, Green's Theorem allowed us to quickly show that the line integral equals 0. Following the example, we noted that this was the line integral for a conservative force field. Discuss which method (Green's Theorem, conservative field) you would recommend trying first to determine whether a line integral equals 0.

3. Equation (4.5) shows how to compute area as a line integral. Using example 4.3 as a guide, explain why we wrote the area as $\frac{1}{2}\oint_C x\,dy - y\,dx$ instead of $\oint_C x\,dy$ or $\oint_C -y\,dx$.

4. Suppose that you drive a car to a variety of places for shopping and then return home. If your path formed a simple closed curve, explain how you could use (4.5) to estimate the area enclosed by your path. (Hint: If $\langle x, y\rangle$ represents position, what does $\langle x', y'\rangle$ represent?)

In exercises 1–4, evaluate the indicated line integral (a) directly and (b) using Green's Theorem.

1. $\oint_C (x^2 - y)\,dx + y^2\,dy$, where C is the circle $x^2 + y^2 = 1$ oriented counterclockwise

2. $\oint_C (y^2 + x)\,dx + (3x + 2xy)\,dy$, where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise

3. $\oint_C x^2\,dx - x^3\,dy$, where C is the square from (0, 0) to (0, 2) to (2, 2) to (2, 0) to (0, 0)

4. $\oint_C (y^2 - 2x)\,dx + x^2\,dy$, where C is the square from (0, 0) to (1, 0) to (1, 1) to (0, 1) to (0, 0)

In exercises 5–20, use Green's Theorem to evaluate the indicated line integral.

5. $\oint_C xe^{2x}\,dx - 3x^2y\,dy$, where C is the rectangle from (0, 0) to (3, 0) to (3, 2) to (0, 2) to (0, 0)

6. $\oint_C ye^{2x}\,dx + x^2y^2\,dy$, where C is the rectangle from $(-2, 0)$ to (3, 0) to (3, 2) to $(-2, 2)$ to $(-2, 0)$

7. $\oint_C \left(\frac{x}{x^2+1} - y\right)dx + (3x - 4\tan y/2)\,dy$, where C is the portion of $y = x^2$ from $(-1, 1)$ to $(1, 1)$, followed by the portion of $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$

8. $\oint_C (xy - e^{2x})\,dx + (2x^2 - 4y^2)\,dy$, where C is formed by $y = x^2$ and $y = 8 - x^2$ oriented clockwise

9. $\oint_C (\tan x - y^3)\,dx + (x^3 - \sin y)\,dy$, where C is the circle $x^2 + y^2 = 2$

10. $\oint_C \left(\sqrt{x^2+1} - x^2y\right)dx + (xy^2 - y^{5/3})\,dy$, where C is the circle $x^2 + y^2 = 4$ oriented clockwise

11. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^3 - y, x + y^3\rangle$ and C is formed by $y = x^2$ and $y = x$

12. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle y^2 + 3x^2y, xy + x^3\rangle$ and C is formed by $y = x^2$ and $y = 2x$

13. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle e^{x^2} - y, e^{2x} + y\right\rangle$ and C is formed by $y = 1 - x^2$ and $y = 0$

14. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle xe^{xy} + y, ye^{xy} + 2x\rangle$ and C is formed by $y = x^2$ and $y = 4$

15. $\oint_C [y^3 - \ln(x+1)]\,dx + \left(\sqrt{y^2+1} + 3x\right)dy$, where C is formed by $x = y^2$ and $x = 4$

16. $\oint_C (y \sec^2 x - 2)\,dx + (\tan x - 4y^2)\,dy$, where C is formed by $x = 1 - y^2$ and $x = 0$

17. $\oint_C x^2\,dx + 2x\,dy + (z-2)\,dz$, where C is the triangle from (0, 0, 2) to (2, 0, 2) to (2, 2, 2) to (0, 0, 2)

18. $\oint_C 4y\,dx + y^3\,dy + z^4\,dz$, where C is $x^2 + y^2 = 4$ in the plane $z = 0$

19. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^3 - y^4, e^{x^2+z^2}, x^2 - 16y^2z^2\rangle$ and C is $x^2 + z^2 = 1$ in the plane $y = 0$

20. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^3 - y^2 z, \sqrt{x^2+z^2}, 4xy - z^4\rangle$ and C is formed by $z = 1 - x^2$ and $z = 0$ in the plane $y = 2$

In exercises 21–26, use a line integral to compute the area of the given region.

21. The ellipse $4x^2 + y^2 = 16$

22. The ellipse $4x^2 + y^2 = 4$

23. The region bounded by $x^{2/3} + y^{2/3} = 1$. (Hint: Let $x = \cos^3 t$ and $y = \sin^3 t$)

24. The region bounded by $x^{2/5} + y^{2/5} = 1$

25. The region bounded by $y = x^2$ and $y = 4$

26. The region bounded by $y = x^2$ and $y = 2x$

27. Use Green's Theorem to show that the center of mass of the region bounded by the positive curve C with constant density is given by $\bar{x} = \frac{1}{2A}\oint_C x^2\,dy$ and $\bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$, where A is the area of the region.

28. Use the result of exercise 27 to find the center of mass of the region in exercise 26, assuming constant density.

29. Use the result of exercise 27 to find the center of mass of the region bounded by the curve traced out by $\langle t^3 - t, 1 - t^2\rangle$, for $-1 \le t \le 1$, assuming constant density.

30. Use the result of exercise 27 to find the center of mass of the region bounded by the curve traced out by $\langle t^2 - t, t^3 - t\rangle$, for $0 \le t \le 1$, assuming constant density.

31. Use Green's Theorem to prove the change of variables formula

$$\iint_R dA = \iint_S \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv,$$

where $x = x(u, v)$ and $y = y(u, v)$ are functions with continuous partial derivatives.

32. For $\mathbf{F} = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$ and C any circle of radius $r > 0$ not containing the origin, show that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$.

In exercises 33–36, use the technique of example 4.5 to evaluate the line integral.

33. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle \dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right\rangle$ and C is any positively oriented simple closed curve containing the origin

34. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle \dfrac{y^2 - x^2}{(x^2+y^2)^2}, \dfrac{-2xy}{(x^2+y^2)^2}\right\rangle$ and C is any positively oriented simple closed curve containing the origin

35. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle \dfrac{x^3}{x^4+y^4}, \dfrac{y^3}{x^4+y^4}\right\rangle$ and C is any positively oriented simple closed curve containing the origin

36. $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \left\langle \dfrac{y^2 x}{x^4+y^4}, \dfrac{-x^2 y}{x^4+y^4}\right\rangle$ and C is any positively oriented simple closed curve containing the origin

37. Where is $\mathbf{F}(x, y) = \left\langle \dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}\right\rangle$ defined? Show that $M_y = N_x$ everywhere the partial derivatives are defined. If C is a simple closed curve enclosing the origin, does Green's Theorem guarantee that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$? Explain.

38. For the vector field of exercise 37, show that $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is the same for all closed curves enclosing the origin.

39. If $\mathbf{F}(x, y) = \left\langle \dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}\right\rangle$ and C is a simple closed curve in the fourth quadrant, does Green's Theorem guarantee that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$? Explain.

EXPLORATORY EXERCISES

1. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where

$$\mathbf{F} = \left\langle \frac{-y}{(x^2+y^2)^2}, \frac{x}{(x^2+y^2)^2}\right\rangle$$

and C is the circle $x^2 + y^2 = a^2$. Use the result and Green's Theorem to show that $\displaystyle\iint_R \frac{-2}{(x^2+y^2)^2}\,dA$ diverges, where R is the disk $x^2 + y^2 \le 1$.

14.5 CURL AND DIVERGENCE

We have seen how Green's Theorem relates the line integral of a function over the boundary of a plane region R to the double integral of a related function over the region R. In some cases, the line integral is easier to evaluate, while in other cases, the double integral is easier. More significantly, Green's Theorem provides us with a connection between physical quantities measured on the boundary of a plane region with related quantities in the interior of the region. The goal of the rest of the chapter is to extend Green's Theorem to results that relate triple integrals, double integrals and line integrals. The first step is to understand the vector operations of curl and divergence introduced in this section.

Both the curl and divergence are generalizations of the notion of derivative that are applied to vector fields. Both directly measure important physical quantities related to a vector field $\mathbf{F}(x, y, z)$.

DEFINITION 5.1

The **curl** of the vector field $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ is the vector field

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k},$$

defined at all points at which all the indicated partial derivatives exist.

An easy way to remember curl $\mathbf{F}$ is to use cross product notation, as follows. Notice that using a determinant, we can write

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \\
&= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle = \text{curl } \mathbf{F},
\end{aligned} \tag{5.1}$$

whenever all of the indicated partial derivatives are defined.

EXAMPLE 5.1 Computing the Curl of a Vector Field

Compute curl $\mathbf{F}$ for (a) $\mathbf{F}(x, y, z) = \langle x^2y, 3x - yz, z^3\rangle$ and (b) $\mathbf{F}(x, y, z) = \langle x^3 - y, y^5, e^z\rangle$.

Solution Using the cross product notation in (5.1), we have that for (a):

$$\begin{aligned}
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 3x - yz & z^3 \end{vmatrix} \\
&= \left(\frac{\partial(z^3)}{\partial y} - \frac{\partial(3x - yz)}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(z^3)}{\partial x} - \frac{\partial(x^2y)}{\partial z}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial(3x - yz)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right)\mathbf{k} \\
&= (0 + y)\mathbf{i} - (0 - 0)\mathbf{j} + (3 - x^2)\mathbf{k} = \langle y, 0, 3 - x^2\rangle.
\end{aligned}$$

Similarly, for part (b), we have

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3 - y & y^5 & e^z \end{vmatrix}$$

$$= \left(\frac{\partial(e^z)}{\partial y} - \frac{\partial(y^5)}{\partial z} \right) \mathbf{i} - \left(\frac{\partial(e^z)}{\partial x} - \frac{\partial(x^3 - y)}{\partial z} \right) \mathbf{j} + \left(\frac{\partial(y^5)}{\partial x} - \frac{\partial(x^3 - y)}{\partial y} \right) \mathbf{k}$$

$$= (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (0 + 1)\mathbf{k} = \langle 0, 0, 1 \rangle. \blacksquare$$

Notice that in part (b) of example 5.1, the only term that contributes to the curl is the term $-y$ in the $\mathbf{i}$-component of $\mathbf{F}(x, y, z)$. This illustrates an important property of the curl. Terms in the $\mathbf{i}$-component of the vector field involving only x will not contribute to the curl, nor will terms in the $\mathbf{j}$-component involving only y nor terms in the $\mathbf{k}$-component involving only z. You can use these observations to simplify some calculations of the curl. For instance, notice that

$$\begin{aligned} \operatorname{curl}\langle x^3, \sin^2 y, \sqrt{z^2 + 1} + x^2 \rangle &= \operatorname{curl}\langle 0, 0, x^2 \rangle \\ &= \nabla \times \langle 0, 0, x^2 \rangle = \langle 0, -2x, 0 \rangle. \end{aligned}$$

The simplification discussed above gives an important hint about what the curl measures, since the variables must get "mixed up" to produce a nonzero curl. Example 5.2 provides a clue as to the meaning of the curl of a vector field.

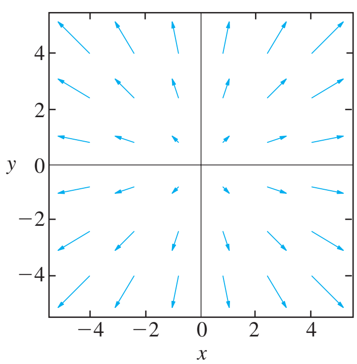


FIGURE 14.37a
Graph of $\langle x, y, 0 \rangle$

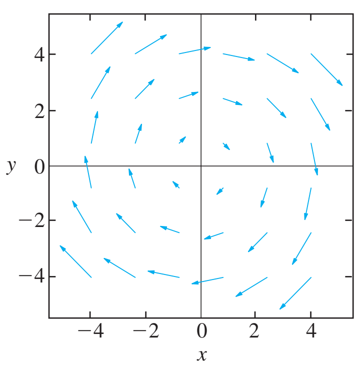


FIGURE 14.37b
Graph of $\langle y, -x, 0 \rangle$

EXAMPLE 5.2 Interpreting the Curl of a Vector Field

Compute the curl of (a) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$ and (b) $\mathbf{G}(x, y, z) = y\mathbf{i} - x\mathbf{j}$, and interpret each graphically.

Solution For (a), we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \langle 0 - 0, -(0 - 0), 0 - 0 \rangle = \langle 0, 0, 0 \rangle.$$

For (b), we have

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \langle 0 - 0, -(0 - 0), -1 - 1 \rangle = \langle 0, 0, -2 \rangle.$$

Graphs of the vector fields $\mathbf{F}$ and $\mathbf{G}$ in two dimensions are shown in Figures 14.37a and 14.37b, respectively. It is helpful to think of each of these vector fields as the velocity field for a fluid in motion across the xy-plane. In this case, the vectors indicated in the graph of the velocity field indicate the direction of flow of the fluid. For the vector field $\langle x, y, 0 \rangle$, observe that the fluid flows directly away from the origin, so that the fluid has no rotation and in (a) we found that curl $\mathbf{F} = \mathbf{0}$. By contrast, the vector field $\langle y, -x, 0 \rangle$ indicates a clockwise rotation of the fluid, while in (b) we computed a nonzero curl. In particular, notice that if you curl the fingers of your right hand so that your fingertips point in the direction of the flow, your thumb will point into the page, in the direction of $-\mathbf{k}$, which has the same direction as

$$\operatorname{curl}\langle y, -x, 0\rangle = \nabla \times \langle y, -x, 0\rangle = -2\mathbf{k}.$$

As we will see through our discussion of Stokes' Theorem in section 14.8, $\nabla \times \mathbf{F}(x, y, z)$ provides a measure of the tendency of the fluid flow to rotate about an axis parallel to $\nabla \times \mathbf{F}(x, y, z)$. If $\nabla \times \mathbf{F} = \mathbf{0}$, we say that the vector field is **irrotational** at that point. (That is, the fluid does not tend to rotate near the point.)

We noted earlier that there is no contribution to the curl of a vector field $\mathbf{F}(x, y, z)$ from terms in the $\mathbf{i}$-component of $\mathbf{F}$ that involve only x, nor from terms in the $\mathbf{j}$-component of $\mathbf{F}$ involving only y nor terms in the $\mathbf{k}$-component of $\mathbf{F}$ involving only z. By contrast, these terms make important contributions to the **divergence** of a vector field, the other major vector operation introduced in this section.

DEFINITION 5.2

The **divergence** of the vector field $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ is the scalar function

$$\operatorname{div} \mathbf{F}(x, y, z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

defined at all points at which all the indicated partial derivatives exist.

NOTES

Take care to note that, while the curl of a vector field is another vector field, the divergence of a vector field is a scalar function.

While we wrote the curl using cross product notation, note that we can write the divergence of a vector field using dot product notation, as follows:

$$\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3\rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \operatorname{div} \mathbf{F}(x, y, z). \tag{5.2}$$

EXAMPLE 5.3 Computing the Divergence of a Vector Field

Compute $\operatorname{div} \mathbf{F}$ for (a) $\mathbf{F}(x, y, z) = \langle x^2 y, 3x - yz, z^3\rangle$ and (b) $\mathbf{F}(x, y, z) = \langle x^3 - y, z^5, e^y\rangle$.

Solution For (a), we have from (5.2) that

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial (x^2 y)}{\partial x} + \frac{\partial (3x - yz)}{\partial y} + \frac{\partial (z^3)}{\partial z} = 2xy - z + 3z^2.$$

For (b), we have from (5.2) that

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial (x^3 - y)}{\partial x} + \frac{\partial (z^5)}{\partial y} + \frac{\partial (e^y)}{\partial z} = 3x^2 + 0 + 0 = 3x^2.$$

Notice that in part (b) of example 5.3 the only term contributing to the divergence is the x^3 term in the $\mathbf{i}$-component of $\mathbf{F}$. Further, observe that in general, the divergence of $\mathbf{F}(x, y, z)$ is not affected by terms in the $\mathbf{i}$-component of $\mathbf{F}$ that do not involve x, terms in the $\mathbf{j}$-component of $\mathbf{F}$ that do not involve y or terms in the $\mathbf{k}$-component of $\mathbf{F}$ that do not involve z. Returning to the two-dimensional vector fields of example 5.2, we can develop a graphical interpretation of the divergence.

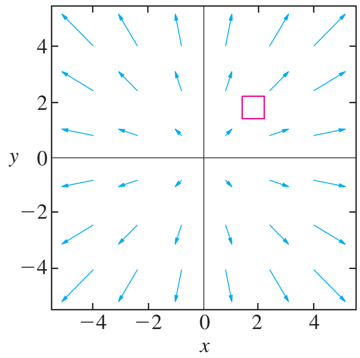


FIGURE 14.38a
Graph of $\langle x, y\rangle$

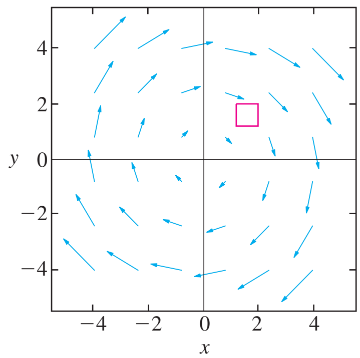


FIGURE 14.38b
Graph of $\langle y, -x\rangle$

EXAMPLE 5.4 Interpreting the Divergence of a Vector Field

Compute the divergence of (a) $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$ and (b) $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$ and interpret each graphically.

Solution For (a), we have $\nabla \cdot \mathbf{F} = \dfrac{\partial(x)}{\partial x} + \dfrac{\partial(y)}{\partial y} = 2$. For (b), we have $\nabla \cdot \mathbf{F} = \dfrac{\partial(y)}{\partial x} + \dfrac{\partial(-x)}{\partial y} = 0$. Graphs of the vector fields in (a) and (b) are shown in Figures 14.38a and 14.38b, respectively. Notice the boxes that we have superimposed on the graph of each vector field. If $\mathbf{F}(x, y)$ represents the velocity field of a fluid in motion, try to use the graphs to estimate the net flow of fluid into or out of each box. For $\langle y, -x\rangle$, the fluid is rotating in circular paths, so that the velocity of any particle on a given circle centered at the origin is a constant. This suggests that the flow into the box should equal the flow out of the box and the net flow is 0, which you'll notice is also the value of the divergence of this velocity field. By contrast, for the vector field $\langle x, y\rangle$, notice that the arrows coming into the box are shorter than the arrows exiting the box. This says that the net flow *out of* the box is positive (i.e., there is more fluid exiting the box than entering the box). Notice that in this case, the divergence is positive. ■

We'll show in section 14.7 (using the Divergence Theorem) that the divergence of a vector field at a point (x, y, z) corresponds to the net flow of fluid per unit volume out of a small box centered at (x, y, z). If $\nabla \cdot \mathbf{F}(x, y, z) > 0$, more fluid exits the box than enters (as illustrated in Figure 14.38a) and we call the point (x, y, z) a **source.** If $\nabla \cdot \mathbf{F}(x, y, z) < 0$, more fluid enters the box than exits and we call the point (x, y, z) a **sink.** If $\nabla \cdot \mathbf{F}(x, y, z) = 0$, throughout some region D, then we say that the vector field $\mathbf{F}$ is **source-free** or **incompressible.**

We have now used the "del" operator ∇ for three different derivative-like operations. The gradient of a scalar function f is the vector field ∇f, the curl of a vector field $\mathbf{F}$ is the vector field $\nabla \times \mathbf{F}$ and the divergence of a vector field $\mathbf{F}$ is the scalar function $\nabla \cdot \mathbf{F}$. Pay special attention to the different roles of scalar and vector functions in these operations. An analysis of the possible combinations of these operations will give us further insight into the properties of vector fields.

EXAMPLE 5.5 Vector Fields and Scalar Functions Involving the Gradient

If $f(x, y, z)$ is a scalar function and $\mathbf{F}(x, y, z)$ is a vector field, determine whether each operation is a scalar function, a vector field or undefined: (a) $\nabla \times (\nabla f)$, (b) $\nabla \times (\nabla \cdot \mathbf{F})$, (c) $\nabla \cdot (\nabla f)$.

Solution Examine each of these expressions one step at a time, working from the inside out. In (a), ∇f is a vector field, so the curl of ∇f is defined and gives a vector field. In (b), $\nabla \cdot \mathbf{F}$ is a scalar function, so the curl of $\nabla \cdot \mathbf{F}$ is undefined. In (c), ∇f is a vector field, so the divergence of ∇f is defined and gives a scalar function. ■

We can say more about the two operations defined in example 5.5 parts (a) and (c). If f has continuous second-order partial derivatives, then $\nabla f = \langle f_x, f_y, f_z\rangle$ and the divergence of the gradient is the scalar function

$$\nabla \cdot (\nabla f) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f_x, f_y, f_z\rangle = f_{xx} + f_{yy} + f_{zz}.$$

This combination of second partial derivatives arises in many important applications in physics and engineering. We call $\nabla \cdot (\nabla f)$ the **Laplacian** of f and typically use the shorthand notation

$$\nabla \cdot (\nabla f) = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$$

or $\Delta f = \nabla^2 f$.

Using the same notation, the curl of the gradient of a scalar function f is

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle,$$

assuming the mixed partial derivatives are equal. (We've seen that this occurs whenever all of the second-order partial derivatives are continuous in some open region.) Recall that if $\mathbf{F} = \nabla f$, then we call $\mathbf{F}$ a conservative field. The result $\nabla \times (\nabla f) = \mathbf{0}$ proves Theorem 5.1, which gives us a simple way for determining when a given three-dimensional vector field is not conservative.

THEOREM 5.1

Suppose that $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$ is a vector field whose components F_1, F_2 and F_3 have continuous first-order partial derivatives throughout an open region $D \subset \mathbb{R}^3$. If $\mathbf{F}$ is conservative, then $\nabla \times \mathbf{F} = \mathbf{0}$.

We can use Theorem 5.1 to determine that a given vector field is not conservative, as we illustrate in example 5.6.

EXAMPLE 5.6 Determining When a Vector Field Is Conservative

Use Theorem 5.1 to determine whether the following vector fields are conservative: (a) $\mathbf{F} = \langle \cos x - z, y^2, xz \rangle$ and (b) $\mathbf{F} = \langle 2xz, 3z^2, x^2 + 6yz \rangle$.

Solution For (a), we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x - z & y^2 & xz \end{vmatrix} = \langle 0 - 0, -1 - z, 0 - 0 \rangle \neq \mathbf{0}$$

and so, by Theorem 5.1, $\mathbf{F}$ is not conservative.

For (b), we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & 3z^2 & x^2 + 6yz \end{vmatrix} = \langle 6z - 6z, 2x - 2x, 0 - 0 \rangle = \mathbf{0}.$$

Notice that in this case, Theorem 5.1 does not tell us whether or not **F** is conservative. However, you might notice that

$$\mathbf{F}(x, y, z) = \langle 2xz, 3z^2, x^2 + 6yz\rangle = \nabla(x^2 z + 3yz^2).$$

Since we have found a potential function for **F**, we now see that it is indeed a conservative field. ■

Given example 5.6, you might be wondering whether or not the converse of Theorem 5.1 is true. That is, if $\nabla \times \mathbf{F} = \mathbf{0}$, must it follow that **F** is conservative? The answer to this is, "NO". We had an important clue to this in example 4.5. There, we saw that for the two-dimensional vector field $\mathbf{F}(x, y) = \dfrac{1}{x^2 + y^2}\langle -y, x\rangle$, $\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r} = 2\pi$, for every simple closed curve C enclosing the origin. We follow up on this idea in example 5.7.

EXAMPLE 5.7 An Irrotational Vector Field That Is Not Conservative

For $\mathbf{F}(x, y, z) = \dfrac{1}{x^2 + y^2}\langle -y, x, 0\rangle$, show that $\nabla \times \mathbf{F} = \mathbf{0}$ throughout the domain of **F**, but that **F** is not conservative.

Solution First, notice that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} & 0 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{x^2 + y^2} & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2 + y^2} & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{-y}{x^2 + y^2} & \dfrac{x}{x^2 + y^2} \end{vmatrix}$$

$$= \mathbf{k}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right)\right]$$

$$= \mathbf{k}\left[\frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} + \frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2}\right] = \mathbf{0},$$

so that **F** is irrotational at every point at which it's defined (i.e., everywhere but on the line $x = y = 0$, that is, the z-axis). However, in example 4.5, we already showed that $\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = 2\pi$, for every simple closed curve C lying in the xy-plane and enclosing the origin. Given this, it follows from Theorem 3.3 that **F** cannot be conservative, since if it were, we would need to have $\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in the domain of **F**. ■

Note that in example 5.7, the vector field in question had a singularity (i.e., a point where one or more of the components of the vector field blow up to ∞) at every point on the z-axis. Even though the curves we considered did not pass through any of these singularities, they in some sense "enclosed" the z-axis. This is enough to make the converse of Theorem 5.1 false. As it turns out, the converse is true if we add some additional hypotheses. Specifically, we can say the following.

THEOREM 5.2

Suppose that $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ is a vector field whose components F_1, F_2 and F_3 have continuous first partial derivatives throughout all of $\mathbb{R}^3$. Then, $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.

Notice that half of this theorem is already known from Theorem 5.1. Also, notice that we required that the components of $\mathbf{F}$ have continuous first partial derivatives throughout *all* of $\mathbb{R}^3$ (a requirement that was not satisfied by the vector field in example 5.7). The other half of the theorem requires the additional sophistication of Stokes' Theorem and we will prove a more general version of this in section 14.8.

CONSERVATIVE VECTOR FIELDS

We can now summarize a number of equivalent properties for three-dimensional vector fields. Suppose that $\mathbf{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$ is a vector field whose components F_1, F_2 and F_3 have continuous first partial derivatives throughout all of $\mathbb{R}^3$. Then the following are equivalent:

1. $\mathbf{F}(x, y, z)$ is conservative.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed curve C.
4. $\nabla \times \mathbf{F} = \mathbf{0}$.
5. $\mathbf{F}(x, y, z)$ is a gradient field ($\mathbf{F} = \nabla f$ for some potential function f).

We close this section by rewriting Green's Theorem in terms of the curl and divergence.

First, suppose that $\mathbf{F}(x, y) = \langle M(x, y), N(x, y), 0\rangle$ is a vector field, for some functions $M(x, y)$ and $N(x, y)$. Suppose that R is a region in the xy-plane whose boundary curve C is piecewise-smooth, positively oriented, simple and closed and that M and N are continuous and have continuous first partial derivatives in some open region D, where $R \subset D$. Then, from Green's Theorem, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy.$$

Notice that the integrand of the double integral, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$, is the $\mathbf{k}$ component of $\nabla \times \mathbf{F}$. Further, since $dz = 0$ on any curve lying in the xy-plane, we have

$$\oint_C M\,dx + N\,dy = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Thus, we can write Green's Theorem in the form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA.$$

We generalize this to Stokes' Theorem in section 14.8.

To take Green's Theorem in yet another direction, suppose that $\mathbf{F}$ and R are as just defined and suppose that C is traced out by the endpoint of the vector-valued function

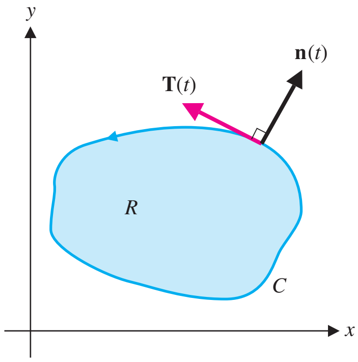


FIGURE 14.39
Unit tangent and exterior unit normal vectors to R

$\mathbf{r}(t) = \langle x(t), y(t) \rangle$, for $a \le t \le b$, where $x(t)$ and $y(t)$ have continuous first derivatives for $a \le t \le b$. Recall that the unit tangent vector to the curve is given by

$$\mathbf{T}(t) = \left\langle \frac{x'(t)}{\|\mathbf{r}'(t)\|}, \frac{y'(t)}{\|\mathbf{r}'(t)\|} \right\rangle.$$

It's then easy to verify that the exterior unit normal vector to C at any point (i.e., the unit normal vector that points out of R) is given by

$$\mathbf{n}(t) = \left\langle \frac{y'(t)}{\|\mathbf{r}'(t)\|}, \frac{-x'(t)}{\|\mathbf{r}'(t)\|} \right\rangle.$$

(See Figure 14.39.) Now, from Theorem 2.1, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t) \|\mathbf{r}'(t)\|\, dt \\
&= \int_a^b \left[\frac{M(x(t), y(t)) y'(t)}{\|\mathbf{r}'(t)\|} - \frac{N(x(t), y(t)) x'(t)}{\|\mathbf{r}'(t)\|} \right] \|\mathbf{r}'(t)\|\, dt \\
&= \int_a^b [M(x(t), y(t)) y'(t)\, dt - N(x(t), y(t)) x'(t)\, dt] \\
&= \oint_C M(x, y)\, dy - N(x, y)\, dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA,
\end{aligned}$$

from Green's Theorem. Finally, recognize that the integrand of the double integral is the divergence of $\mathbf{F}$ and this gives us another vector form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}(x, y)\, dA. \tag{5.3}$$

This form of Green's Theorem is generalized to the Divergence Theorem in section 14.7.

EXERCISES 14.5

WRITING EXERCISES

1. Suppose that $\nabla \times \mathbf{F} = \langle 2, 0, 0 \rangle$. Describe what the graph of the vector field $\mathbf{F}$ looks like. Explain how the graph of the vector field $\mathbf{G}$ with $\nabla \times \mathbf{G} = \langle 20, 0, 0 \rangle$ compares.
2. If $\nabla \cdot \mathbf{F} > 0$ at a point P and $\mathbf{F}$ is the velocity field of a fluid, explain why the word **source** is a good choice for what's happening at P. Explain why **sink** is a good word if $\nabla \cdot \mathbf{F} < 0$.
3. You now have two ways of determining whether or not a vector field is conservative: try to find the potential or see whether the curl equals **0**. If you have reason to believe that the vector field is conservative, explain which test you prefer.
4. In the text, we discussed geometrical interpretations of the divergence and curl. Discuss the extent to which the divergence and curl are analogous to tangential and normal components of acceleration.

In exercises 1–12, find the curl and divergence of the given vector field.

1. $x^2\mathbf{i} - 3xy\mathbf{j}$
2. $y^2\mathbf{i} + 4x^2y\mathbf{j}$
3. $2xz\mathbf{i} - 3y\mathbf{k}$
4. $x^2\mathbf{i} - 3xy\mathbf{j} + x\mathbf{k}$
5. $\langle xy, yz, x^2 \rangle$
6. $\langle xe^z, yz^2, x + y \rangle$
7. $\langle x^2, y - z, xe^y \rangle$
8. $\langle y, x^2y, 3z + y \rangle$
9. $\langle 3yz, x^2, x \cos y \rangle$
10. $\langle y^2, x^2e^z, \cos xy \rangle$
11. $\langle 2xz, y + z^2, zy^2 \rangle$
12. $\langle xy^2, 3y^2z^2, 2x - zy^3 \rangle$

In exercises 13–26, determine whether the given vector field is conservative and/or incompressible.

13. $\langle 2x, 2yz^2, 2y^2z \rangle$
14. $\langle 2xy, x^2 - 3y^2z^2, 1 - 2zy^3 \rangle$
15. $\langle 3yz, x^2, x \cos y \rangle$
16. $\langle y^2, x^2e^z, \cos xy \rangle$

17. $\langle \sin z, z^2 e^{yz^2}, x \cos z + 2yze^{yz^2} \rangle$

18. $\langle 2xy \cos z, x^2 \cos z - 3y^2 z, -x^2 y \sin z - y^3 \rangle$

19. $\left\langle z^2 - 3ye^{3x}, z^2 - e^{3x}, 2z\sqrt{xy} \right\rangle$

20. $\langle 2xz, 3y, x^2 - y \rangle$

21. $\langle xy^2, 3xz, 4 - zy^2 \rangle$

22. $\langle x, y, 1 - 3z \rangle$

23. $\langle 4x, 3y^3, e^z \rangle$

24. $\langle \sin x, 2y^2, \sqrt{z} \rangle$

25. $\langle -2xy, z^2 \cos yz^2 - x^2, 2yz \cos yz^2 \rangle$

26. $\langle e^y, xe^y + z^2, 2yz - 1 \rangle$

27. Label each expression as a scalar quantity, a vector quantity or undefined, if f is a scalar function and $\mathbf{F}$ is a vector field.

a. $\nabla \cdot (\nabla f)$ **b.** $\nabla \times (\nabla \cdot \mathbf{F})$ **c.** $\nabla(\nabla \times \mathbf{F})$
d. $\nabla(\nabla \cdot \mathbf{F})$ **e.** $\nabla \times (\nabla f)$

28. Label each expression as a scalar quantity, a vector quantity or undefined, if f is a scalar function and $\mathbf{F}$ is a vector field.

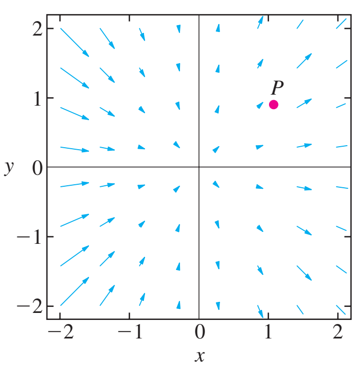
a. $\nabla(\nabla f)$ **b.** $\nabla \cdot (\nabla \cdot \mathbf{F})$ **c.** $\nabla \cdot (\nabla \times \mathbf{F})$
d. $\nabla \times (\nabla \mathbf{F})$ **e.** $\nabla \times (\nabla \times (\nabla \times \mathbf{F}))$

29. If $\mathbf{r} = \langle x, y, z \rangle$, prove that $\nabla \times \mathbf{r} = \mathbf{0}$ and $\nabla \cdot \mathbf{r} = 3$.

30. If $\mathbf{r} = \langle x, y, z \rangle$ and $r = \|\mathbf{r}\|$, prove that $\nabla \cdot (r\mathbf{r}) = 4r$.

In exercises 31–36, conjecture whether the divergence at point *P* is positive, negative or zero.

31.



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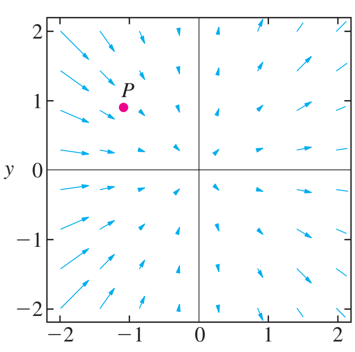

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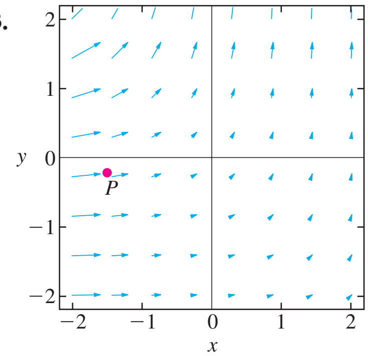

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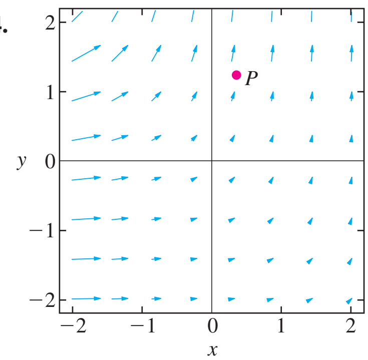

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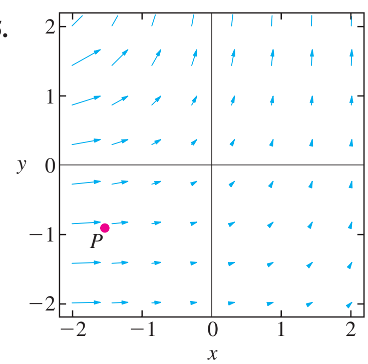

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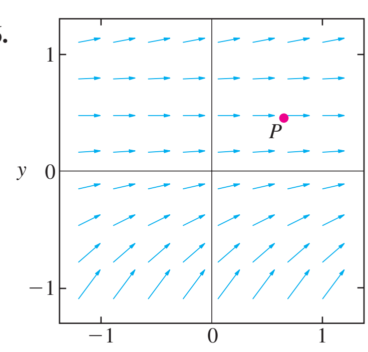


37. If **F** and **G** are vector fields, prove that

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

38. If **F** is a vector field, prove that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

39. If **F** is a vector field, prove that

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

40. If **A** is a constant vector and $\mathbf{r} = \langle x, y, z \rangle$, prove that

$$\nabla \times (\mathbf{A} \times \mathbf{r}) = 2\mathbf{A}.$$

41. If the **j**-component, $\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}$, of the curl of **F** is positive, show that there is a closed curve C such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

42. If the **k**-component, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, of the curl of **F** is positive, show that there is a closed curve C such that $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.

43. Prove **Green's first identity:** For $C = \partial R$,

$$\iint_R f \nabla^2 g \, dA = \int_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_R (\nabla f \cdot \nabla g) \, dA.$$

[Hint: Use the vector form of Green's Theorem in (5.3) applied to $\mathbf{F} = f \nabla g$.]

44. Prove **Green's second identity:** For $C = \partial R$,

$$\iint_R (f \nabla^2 g - g \nabla^2 f) \, dA = \int_C (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds.$$

(Hint: Use Green's first identity from exercise 43.)

45. For a vector field $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle$ and closed curve C with normal vector **n** (that is, **n** is perpendicular to the tangent vector to C at each point), show that $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \nabla \cdot \mathbf{F} \, dA = \oint_C F_1 \, dy - F_2 \, dx$.

46. If $T(x, y, t)$ is the temperature function at position (x, y) at time t, heat flows across a curve C at a rate given by $\oint_C (-k \nabla T) \cdot \mathbf{n} \, ds$, for some constant k. At steady-state, this rate is zero and the temperature function can be written as $T(x, y)$. In this case, use Green's Theorem to show that $\nabla^2 T = 0$.

47. If f is a scalar function and **F** a vector field, show that

$$\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F}).$$

48. If f is a scalar function and **F** a vector field, show that

$$\nabla \times (f\mathbf{F}) = \nabla f \times \mathbf{F} + f(\nabla \times \mathbf{F}).$$

49. If $\nabla \cdot \mathbf{F} = 0$, we say that **F** is **solenoidal.** If $\nabla^2 f = 0$, show that ∇f is both solenoidal and irrotational.

50. If **F** and **G** are irrotational, prove that $\mathbf{F} \times \mathbf{G}$ is solenoidal. (Refer to exercise 49.)

51. If f is a scalar function, $\mathbf{r} = \langle x, y \rangle$ and $r = \|\mathbf{r}\|$, show that

$$\nabla f(r) = f'(r) \frac{\mathbf{r}}{r}.$$

52. If f is a scalar function, $\mathbf{r} = \langle x, y \rangle$ and $r = \|\mathbf{r}\|$, show that

$$\nabla^2 f(r) = f''(r) + \frac{1}{r} f'(r).$$

53. Compute the Laplacian Δf for $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

54. Compute the Laplacian Δf for $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$.

55. Suppose that $\mathbf{F}(x, y) = \langle x^2, y^2 - 4x \rangle$ represents the velocity field of a fluid in motion. For a small box centered at (x, y), determine whether the flow into the box is greater than, less than or equal to the flow out of the box. (a) $(x, y) = (0, 0)$ and (b) $(x, y) = (1, 0)$.

56. Repeat exercise 55 for (a) $(x, y) = (1, 1)$ and (b) $(x, y) = (0, -1)$.

57. Give an example of a vector field **F** such that $\nabla \cdot \mathbf{F}$ is a positive function of y only.

58. Give an example of a vector field **F** such that $\nabla \times \mathbf{F}$ is a function of x only.

59. Gauss' law states that $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$. Here, **E** is an electrostatic field, ρ is the charge density and ϵ_0 is the permittivity. If **E** has a potential function $-\phi$, derive **Poisson's equation** $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$.

60. For two-dimensional fluid flow, if $\mathbf{v} = \langle v_x(x, y), v_y(x, y) \rangle$ is the velocity field, then **v** has a **stream function** g if $\frac{\partial g}{\partial x} = -v_y$ and $\frac{\partial g}{\partial y} = v_x$. Show that if **v** has a stream function and the components v_x and v_y have continuous partial derivatives, then $\nabla \cdot \mathbf{v} = 0$.

61. For $\mathbf{v} = \langle 2xy, -y^2 + x \rangle$, show that $\nabla \cdot \mathbf{v} = 0$ and find a stream function g.

62. For $\mathbf{v} = \langle xe^{xy} - 1, 2 - ye^{xy} \rangle$, show that $\nabla \cdot \mathbf{v} = 0$ and find a stream function g.

63. Sketch the function $f(x) = \dfrac{1}{1 + x^2}$ and use it to sketch the vector field $\mathbf{F} = \left\langle 0, \dfrac{1}{1 + x^2}, 0 \right\rangle$. If this represents the velocity field of a fluid and a paddle wheel is placed in the fluid at various points near the origin, explain why the paddle wheel would start spinning. Compute $\nabla \times \mathbf{F}$ and label the fluid flow as rotational or irrotational. How does this compare to the motion of the paddle wheel?

64. Sketch the vector field $\mathbf{F} = \left\langle \dfrac{1}{1 + x^2}, 0, 0 \right\rangle$. If this represents the velocity field of a fluid and a paddle wheel is placed in the fluid at various points near the origin, explain why the paddle wheel would not start spinning. Compute $\nabla \times \mathbf{F}$ and label the fluid flow as rotational or irrotational. How does this compare to the motion of the paddle wheel?

65. Show that if $\mathbf{G} = \nabla \times \mathbf{H}$, for some vector field $\mathbf{H}$ with continuous partial derivatives, then $\nabla \cdot \mathbf{G} = 0$.

66. Show the converse of exercise 65; that is, if $\nabla \cdot \mathbf{G} = 0$, then $\mathbf{G} = \nabla \times \mathbf{H}$ for some vector field $\mathbf{H}$. $\left[\text{Hint: Let } \mathbf{H}(x, y, z) = \left\langle 0, \int_0^x G_3(u, y, z)\,du, -\int_0^x G_2(u, y, z)\,du \right\rangle.\right]$

EXPLORATORY EXERCISES

1. In some calculus and engineering books, you will find the vector identity

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}) - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}).$$

Which two of the four terms on the right-hand side look like they should be undefined? Write out the left-hand side as completely as possible, group it into four terms, identify the two familiar terms on the right-hand side and then define the unusual terms on the right-hand side. (Hint: The notation makes sense as a generalization of the definitions in this section.)

2. Prove the vector formula

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

As in exercise 1, a major part of the problem is to decipher an unfamiliar notation.

3. Maxwell's laws relate an electric field $\mathbf{E}(t)$ to a magnetic field $\mathbf{H}(t)$. In a region with no charges and no current, the laws state that $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$, $\nabla \times \mathbf{E} = -\mu \mathbf{H}_t$ and $\nabla \times \mathbf{H} = \mu \mathbf{E}_t$. From these laws, prove that

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu^2 \mathbf{E}_{tt}$$

and

$$\nabla \times (\nabla \times \mathbf{H}) = -\mu^2 \mathbf{H}_{tt}.$$

14.6 SURFACE INTEGRALS

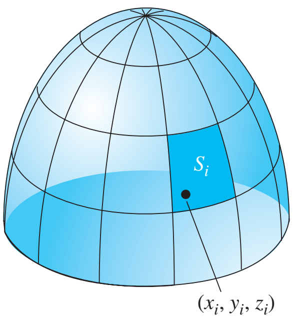


FIGURE 14.40
Partition of a surface

Whether it is the ceiling of the Sistine Chapel, the dome of a college library or the massive roof of the Toronto SkyDome, domes are impressive architectural structures, in part because of their lack of visible support. This feature of domes worries architects, who must be certain that the weight is properly supported. A critical part of an architect's calculation is the mass of the dome.

How would you compute the mass of a dome? You have already seen how to use double integrals to compute the mass of a two-dimensional lamina and triple integrals to find the mass of a three-dimensional solid. However, a dome is a three-dimensional structure more like a thin shell (a surface) than a solid. We hope you're one step ahead of us on this one: if you don't know how to find the mass of a dome exactly, you can try to approximate its mass by slicing it into a number of small sections and estimating the mass of each section. In Figure 14.40, we show a curved surface that has been divided into a number of pieces. If the pieces are small enough, notice that the density of each piece will be approximately constant.

So, first subdivide (partition) the surface into n smaller pieces, $S_1, S_2, \ldots, S_n$. Next, let $\rho(x, y, z)$ be the density function (measured in units of mass per unit area). Further, for each $i = 1, 2, \ldots, n$, let (x_i, y_i, z_i) be a point on the section S_i and let ΔS_i be the surface area of S_i. The mass of the section S_i is then given approximately by $\rho(x_i, y_i, z_i)\Delta S_i$. The total mass m of the surface is given approximately by the sum of these approximate masses,

$$m \approx \sum_{i=1}^{n} \rho(x_i, y_i, z_i)\,\Delta S_i.$$

You should expect that the exact mass is given by the limit of these sums as the size of the pieces gets smaller and smaller. We define the **diameter** of a section S_i to be the maximum distance between any two points on S_i and the norm of the partition $\|P\|$ as the maximum of the diameters of the S_i's. Then we have that

$$m = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \rho\,(x_i, y_i, z_i)\,\Delta S_i.$$

This limit is an example of a new type of integral, the **surface integral,** which is the focus of this section.

DEFINITION 6.1

The **surface integral** of a function $g(x, y, z)$ over a surface $S \subset \mathbb{R}^3$, written $\iint_S g(x, y, z)\, dS$, is given by

$$\iint_S g(x, y, z)\, dS = \lim_{\|P\| \to 0} \sum_{i=1}^{n} g\,(x_i, y_i, z_i)\, \Delta S_i,$$

provided the limit exists and is the same for all choices of the evaluation points (x_i, y_i, z_i).

Notice how our development of the surface integral parallels our development of the line integral. Whereas the line integral extended a single integral over an interval to an integral over a curve in three dimensions, the surface integral extends a double integral over a two-dimensional region to an integral over a two-dimensional surface in three dimensions. In both cases, we are "curving" our domain into three dimensions.

Now that we have defined the surface integral, how can we calculate one? The basic idea is to rewrite a surface integral as a double integral and then evaluate the double integral using existing techniques. To convert a given surface integral into a double integral, you will have two main tasks:

1. Write the integrand $g(x, y, z)$ as a function of two variables.
2. Write the surface area element dS in terms of the area element dA.

We will develop a general rule for step (2) before considering specific examples.

Consider a surface such as the one pictured in Figure 14.40. For the sake of simplicity, we assume that the surface is the graph of the equation $z = f(x, y)$, where f has continuous first partial derivatives in some region R in the xy-plane. Notice that for an inner partition $R_1, R_2, \ldots, R_n$ of R, if we take the point $(x_i, y_i, 0)$ as the point in R_i closest to the origin, then the portion of the surface S_i lying above R_i will differ very little from the portion T_i of the tangent plane to the surface at $(x_i, y_i, f(x_i, y_i))$ lying above R_i. More to the point, the surface area of S_i will be approximately the same as the area of the parallelogram T_i. In Figure 14.41, we have indicated the portion T_i of the tangent plane lying above R_i.

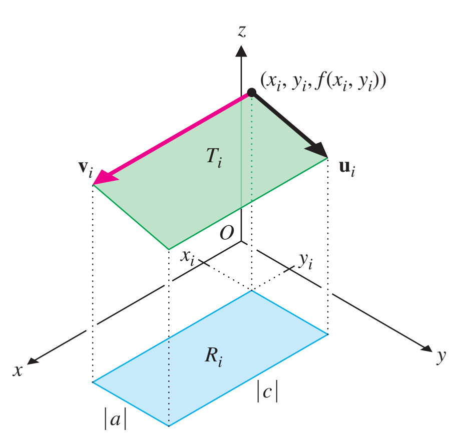


FIGURE 14.41
Portion of the tangent plane lying above R_i

Let the vectors $\mathbf{u}_i = \langle 0, a, b \rangle$ and $\mathbf{v}_i = \langle c, 0, d \rangle$ form two adjacent sides of the parallelogram T_i, as indicated in Figure 14.41. Notice that since $\mathbf{u}_i$ and $\mathbf{v}_i$ lie in the tangent plane, $\mathbf{n}_i = \mathbf{u}_i \times \mathbf{v}_i = \langle ad, bc, -ac \rangle$ is a normal vector to the tangent plane. We saw in section 10.4 that the area of the parallelogram can be written as

$$\Delta S_i = \|\mathbf{u}_i \times \mathbf{v}_i\| = \|\mathbf{n}_i\|.$$

We further observe that the area of R_i is given by $\Delta A_i = |ac|$ and $\mathbf{n}_i \cdot \mathbf{k} = -ac$, so that $|\mathbf{n}_i \cdot \mathbf{k}| = |ac|$. We can now write

$$\Delta S_i = \frac{|ac|\,\|\mathbf{n}_i\|}{|ac|} = \frac{\|\mathbf{n}_i\|}{|\mathbf{n}_i \cdot \mathbf{k}|} \Delta A_i,$$

since $ac \neq 0$. The corresponding expression relating the surface area element dS and the area element dA is then

$$\boxed{dS = \frac{\|\mathbf{n}\|}{|\mathbf{n} \cdot \mathbf{k}|}\, dA.}$$

In the exercises, we will ask you to derive similar formulas for the cases where the surface S is written as a function of x and z or as a function of y and z.

We will consider two main cases of surface integrals. In the first, the surface is defined by a function $z = f(x, y)$. In the second, the surface is defined by parametric equations $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$. In each case, your primary task will be to determine a normal vector to use in the general conversion formula for dS.

If S is the surface $z = f(x, y)$, recall from our discussion in section 12.4 that a normal vector to S is given by $\mathbf{n} = \langle f_x, f_y, -1 \rangle$. This is a convenient normal vector for our purposes, since $|\mathbf{n} \cdot \mathbf{k}| = 1$. With $\|\mathbf{n}\| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$, we have the following result.

THEOREM 6.1 (Evaluation Theorem)

If the surface S is given by $z = f(x, y)$ for (x, y) in the region $R \subset \mathbb{R}^2$, where f has continuous first partial derivatives, then

$$\iint_S g(x, y, z)\, dS = \iint_R g(x, y, f(x, y))\sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA.$$

PROOF

From the definition of surface integral in Definition 6.1, we have

$$\begin{aligned}
\iint_S g(x, y, z)\, dS &= \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, z_i)\, \Delta S_i \\
&= \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, z_i) \frac{\|\mathbf{n}_i\|}{|\mathbf{n}_i \cdot \mathbf{k}|}\, \Delta A_i \\
&= \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(x_i, y_i, f(x_i, y_i)) \sqrt{(f_x)^2 + (f_y)^2 + 1}\,\Big|_{(x_i, y_i)} \Delta A_i \\
&= \iint_R g(x, y, f(x, y))\sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA,
\end{aligned}$$

as desired. ■

Theorem 6.1 says that we can evaluate a surface integral by evaluating a related double integral. To convert the surface integral into a double integral, substitute $z = f(x, y)$ in the function $g(x, y, z)$ and replace the surface area element dS with $\|\mathbf{n}\|\, dA$, which for the surface $z = f(x, y)$ is given by

$$dS = \|\mathbf{n}\|\, dA = \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA. \tag{6.1}$$

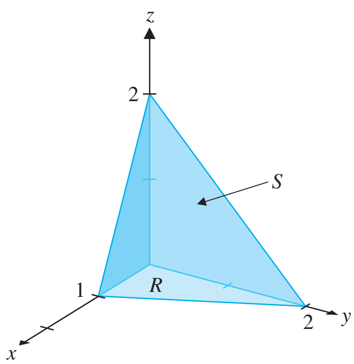


FIGURE 14.42a
$z = 2 - 2x - y$

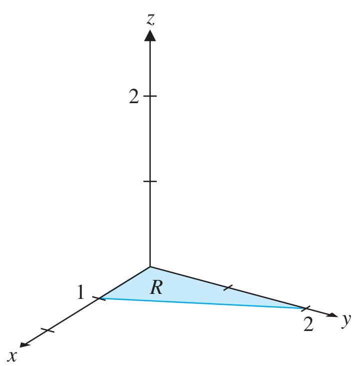


FIGURE 14.42b
The projection R of the surface S onto the xy-plane

EXAMPLE 6.1 Evaluating a Surface Integral

Evaluate $\iint_S 3z\, dS$, where the surface S is the portion of the plane $2x + y + z = 2$ lying in the first octant.

Solution On S, we have $z = 2 - 2x - y$, so we must evaluate $\iint_S 3(2 - 2x - y)\, dS$. Note that a normal vector to the plane $2x + y + z = 2$ is $\mathbf{n} = \langle 2, 1, 1\rangle$, so that in this case, the element of surface area is given by (6.1) to be

$$dS = \|\mathbf{n}\|\, dA = \sqrt{6}\, dA.$$

From Theorem 6.1, we then have

$$\iint_S 3(2 - 2x - y)\, dS = \iint_R 3(2 - 2x - y)\sqrt{6}\, dA,$$

where R is the projection of the surface onto the xy-plane. A graph of the surface S is shown in Figure 14.42a. In this case, notice that R is the triangle indicated in Figure 14.42b. The triangle is bounded by $x = 0$, $y = 0$ and the line $2x + y = 2$ (the intersection of the plane $2x + y + z = 2$ with the plane $z = 0$). If we integrate with respect to y first, the inside integration limits are $y = 0$ and $y = 2 - 2x$, with x ranging from 0 to 1. This gives us

$$\begin{aligned}\iint_S 3(2 - 2x - y)\, dS &= \iint_R 3(2 - 2x - y)\sqrt{6}\, dA \\ &= \int_0^1 \int_0^{2-2x} 3\sqrt{6}(2 - 2x - y)\, dy\, dx \\ &= 2\sqrt{6},\end{aligned}$$

where we leave the routine details of the integration as an exercise. ■

In example 6.2, we will need to rewrite the double integral using polar coordinates.

EXAMPLE 6.2 Evaluating a Surface Integral Using Polar Coordinates

Evaluate $\iint_S z\, dS$, where the surface S is the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane.

Solution Substituting $z = 4 - x^2 - y^2$, we have

$$\iint_S z\, dS = \iint_S (4 - x^2 - y^2)\, dS.$$

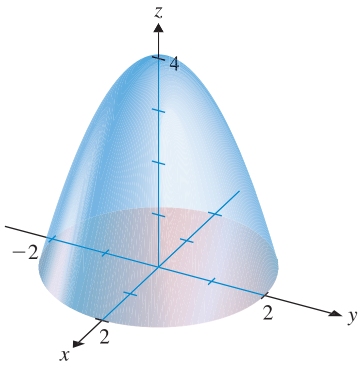


FIGURE 14.43
$z = 4 - x^2 - y^2$

In this case, a normal vector to the surface $z = 4 - x^2 - y^2$ is $\mathbf{n} = \langle -2x, -2y, -1\rangle$, so that

$$dS = \|\mathbf{n}\|\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

This gives us

$$\iint_S (4 - x^2 - y^2)\, dS = \iint_R (4 - x^2 - y^2)\sqrt{4x^2 + 4y^2 + 1}\, dA.$$

Here, the region R is enclosed by the intersection of the paraboloid with the xy-plane, which is the circle $x^2 + y^2 = 4$ (see Figure 14.43). With a circular region of integration and the term $x^2 + y^2$ appearing (twice!) in the integrand, you had better be thinking about polar coordinates. We have $4 - x^2 - y^2 = 4 - r^2$, $\sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}$ and $dA = r\, dr\, d\theta$. For the circle $x^2 + y^2 = 4$, r ranges from 0 to 2 and θ ranges from 0 to 2π. Then, we have

$$\begin{aligned}\iint_S (4 - x^2 - y^2)\, dS &= \iint_R (4 - x^2 - y^2)\sqrt{4x^2 + 4y^2 + 1}\, dA \\ &= \int_0^{2\pi}\int_0^2 (4 - r^2)\sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \frac{289}{60}\pi\sqrt{17} - \frac{41}{60}\pi,\end{aligned}$$

where we leave the details of the final integration to you. ■

Parametric Representation of Surfaces

In the remainder of this section, we study parametric representations of surface integrals. Before applying parametric equations to the computation of surface integrals, we need a better understanding of surfaces that have been defined parametrically. You have already seen parametric surfaces in section 11.6. For instance, you can describe the cone $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates by $z = r$, $0 \le \theta \le 2\pi$, which is a parametric representation with parameters r and θ. Similarly, the equation $\rho = 4$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$, is a parametric representation of the sphere $x^2 + y^2 + z^2 = 16$, with parameters θ and ϕ. It will be helpful to review these graphs as well as to look at some new ones.

Given a particular surface, we may need to find a convenient parametric representation of the surface. The general form for parametric equations representing a surface in three dimensions is $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ for $u_1 \le u \le u_2$ and $v_1 \le v \le v_2$. The parameters u and v can correspond to familiar coordinates (x and y, or r and θ, for instance), or less familiar expressions. Keep in mind that to fully describe a surface, you will need to define two parameters.

EXAMPLE 6.3 Finding Parametric Representations of a Surface

Find a simple parametric representation for (a) the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$ and (b) the portion of the sphere $x^2 + y^2 + z^2 = 16$ inside of the cone $z = \sqrt{x^2 + y^2}$.

Solution It is important to realize that both parts (a) and (b) have numerous solutions. (In fact, every surface can be represented parametrically in an infinite number of ways.) The solutions we show here are among the simplest and most useful, but they are *not* the only reasonable solutions. In (a), the repeated appearance of the term $x^2 + y^2$

suggests that cylindrical coordinates (r, θ, z) might be convenient. A sketch of the surface is shown in Figure 14.44a. Notice that the cone $z = \sqrt{x^2 + y^2}$ becomes $z = r$ in cylindrical coordinates. Recall that in cylindrical coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. Notice that the parameters r and θ have ranges determined by the cylinder $x^2 + y^2 = 4$, so that $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Parametric equations for the cone are then $x = r\cos\theta$, $y = r\sin\theta$ and $z = r$ with $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

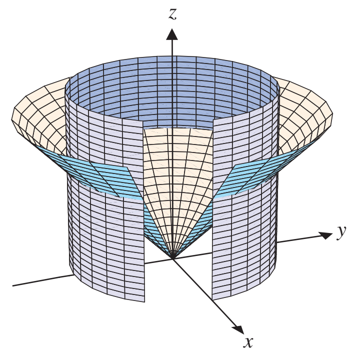


FIGURE 14.44a
The cone $z = \sqrt{x^2 + y^2}$ and the cylinder $x^2 + y^2 = 4$

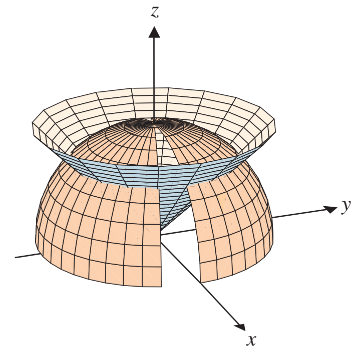


FIGURE 14.44b
The portion of the sphere inside the cone

The surface in part (b) is a portion of a sphere, which suggests (what else?) spherical coordinates: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$, where $\rho^2 = x^2 + y^2 + z^2$. The equation of the sphere $x^2 + y^2 + z^2 = 16$ is then $\rho = 4$. Using this, a parametric representation of the sphere is $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$ and $z = 4\cos\phi$, where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. To find the portion of the sphere inside the cone, observe that the cone can be described in spherical coordinates as $\phi = \frac{\pi}{4}$. Referring to Figure 14.44b, note that the portion of the sphere inside the cone is then described by $x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$ and $z = 4\cos\phi$, where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{4}$. ■

Suppose that we have a parametric representation for the surface S: $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, defined on the rectangle $R = \{(u, v) | a \le u \le b$ and $c \le v \le d\}$ in the uv-plane. It is often convenient to use parametric equations to evaluate the surface integral $\iint_S f(x, y, z)\, dS$. Of course, to do this, we must substitute for x, y and z to rewrite the integrand in terms of the parameters u and v, as

$$g(u, v) = f(x(u, v), y(u, v), z(u, v)).$$

We must also write the surface area element dS in terms of the area element dA for the uv-plane. Unfortunately, we can't use (6.1) here, since this holds only for the case of a surface written in the form $z = f(x, y)$. Instead, we'll need to back up just a bit.

First, notice that the position vector for points on the surface S is $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. We define the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ (the subscripts denote partial derivatives) by

$$\mathbf{r}_u(u, v) = \langle x_u(u, v), y_u(u, v), z_u(u, v)\rangle$$

and

$$\mathbf{r}_v(u, v) = \langle x_v(u, v), y_v(u, v), z_v(u, v)\rangle.$$

Notice that for any fixed (u, v), both of the vectors $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ lie in the tangent plane to S at the point $(x(u, v), y(u, v), z(u, v))$. So, unless these two vectors are parallel,

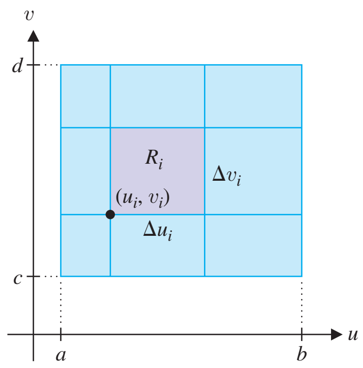


FIGURE 14.45a
Partition of parameter domain (uv-plane)

$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the surface at the point $(x(u, v), y(u, v), z(u, v))$. We say that the surface S is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$, for all $(u, v) \in R$. (This says that the surface will not have any corners.) We say that S is **piecewise-smooth** if we can write $S = S_1 \cup S_2 \cup \cdots \cup S_n$, for some smooth surfaces $S_1, S_2, \ldots, S_n$.

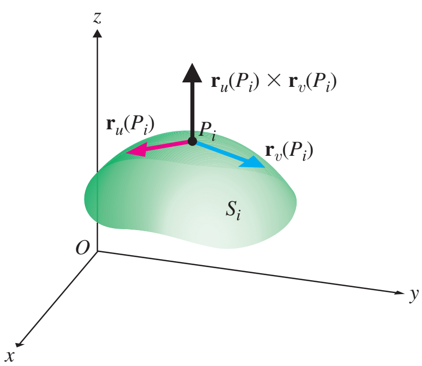


FIGURE 14.45b
Curvilinear region S_i

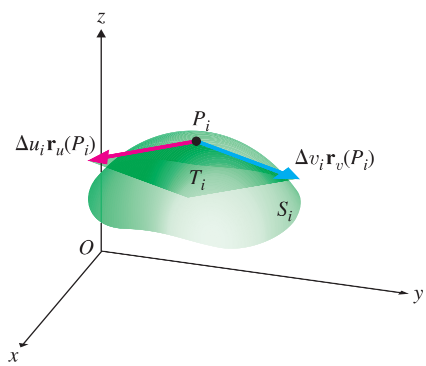


FIGURE 14.45c
The parallelogram T_i

As we have done many times now, we partition the rectangle R in the uv-plane. For each rectangle R_i in the partition, let (u_i, v_i) be the closest point in R_i to the origin, as indicated in Figure 14.45a. Notice that each of the sides of R_i gets mapped to a curve in xyz-space, so that R_i gets mapped to a curvilinear region S_i in xyz-space, as indicated in Figure 14.45b. Observe that if we locate their initial points at the point $P_i(x(u_i, v_i), y(u_i, v_i), z(u_i, v_i))$, the vectors $\mathbf{r}_u(u_i, v_i)$ and $\mathbf{r}_v(u_i, v_i)$ lie tangent to two adjacent curved sides of S_i. So, we can approximate the area ΔS_i of S_i by the area of the parallelogram T_i whose sides are formed by the vectors $\Delta u_i \mathbf{r}_u(u_i, v_i)$ and $\Delta v_i \mathbf{r}_v(u_i, v_i)$. (See Figure 14.45c.) As we know, the area of the parallelogram is given by the magnitude of the cross product

$$\begin{aligned}\|\Delta u_i \mathbf{r}_u(u_i, v_i) \times \Delta v_i \mathbf{r}_v(u_i, v_i)\| &= \|\mathbf{r}_u(u_i, v_i) \times \mathbf{r}_v(u_i, v_i)\| \, \Delta u_i \Delta v_i \\ &= \|\mathbf{r}_u(u_i, v_i) \times \mathbf{r}_v(u_i, v_i)\| \, \Delta A_i,\end{aligned}$$

where ΔA_i is the area of the rectangle R_i. We then have that

$$\Delta S_i \approx \|\mathbf{r}_u(u_i, v_i) \times \mathbf{r}_v(u_i, v_i)\| \Delta A_i$$

and it follows that the element of surface area can be written as

$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA. \tag{6.2}$$

Notice that this corresponds closely to (6.1), as $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to S. Finally, we developed (6.2) in the comparatively simple case where the parameter domain R (that is, the domain in the uv-plane) was a rectangle. If the parameter domain is not a rectangle, you should recognize that we can do the same thing by constructing an inner partition of the region. We can now evaluate surface integrals using parametric equations, as in example 6.4.

EXAMPLE 6.4 Evaluating a Surface Integral Using Spherical Coordinates

Evaluate $\iint_S (3x^2 + 3y^2 + 3z^2) \, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

Solution Since the surface is a sphere and the integrand contains the term $x^2 + y^2 + z^2$, spherical coordinates are indicated. Notice that the sphere is described by $\rho = 2$ and so, on the surface of the sphere, the integrand becomes $3(x^2 + y^2 + z^2) = 12$.

Further, we can describe the sphere $\rho = 2$ with the parametric equations $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$ and $z = 2\cos\phi$, for $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. This says that the sphere is traced out by the endpoint of the vector-valued function

$$\mathbf{r}(\phi,\ \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle.$$

We then have the partial derivatives

$$\mathbf{r}_\theta = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0\rangle$$

and

$$\mathbf{r}_\phi = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi\rangle.$$

We leave it as an exercise to show that a normal vector to the surface is given by

$$\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_\phi = \langle -4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi\rangle,$$

so that $\|\mathbf{n}\| = 4|\sin\phi|$. Equation (6.2) now gives us $dS = 4|\sin\phi|\,dA$, so that

$$\begin{aligned}\iint_S (3x^2 + 3y^2 + 3z^2)\,dS &= \iint_R (12)(4)\,|\sin\phi|\,dA \\ &= \int_0^{2\pi}\int_0^{\pi} 48\sin\phi\,d\phi\,d\theta \\ &= 192\pi,\end{aligned}$$

where we replaced $|\sin\phi|$ by $\sin\phi$ by using the fact that for $0 \le \phi \le \pi$, $\sin\phi \ge 0$. ■

NOTES

For the sphere $x^2+y^2+z^2 = R^2$, the surface area element in spherical coordinates is $dS = R^2\sin\phi\,d\phi\,d\theta$.

If you did the calculation of dS in example 6.4, you may not think that parametric equations lead to simple solutions. (That's why we didn't show all of the details!) However, recall that for changing a triple integral from rectangular to spherical coordinates, you replace $dx\,dy\,dz$ by $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$. In example 6.4, we have $\rho^2 = 4$ and $dS = 4\sin\phi\,dA$. Looks familiar now, doesn't it? This shortcut is valuable, since surface integrals over spheres are reasonably common.

So, when you evaluate a surface integral, what have you computed? We close the section with three examples. The first is familiar: observe that the surface integral of the function $f(x, y, z) = 1$ over the surface S is simply the surface area of S. That is,

$$\iint_S 1\,dS = \text{Surface area of } S.$$

The proof of this follows directly from the definition of the surface integral and is left as an exercise.

EXAMPLE 6.5 Using a Surface Integral to Compute Surface Area

Compute the surface area of the portion of the hyperboloid $x^2 + y^2 - z^2 = 4$ between $z = 0$ and $z = 2$.

Solution We need to evaluate $\iint_S 1\,dS$. Notice that we can write the hyperboloid parametrically as $x = 2\cos u\,\cosh v$, $y = 2\sin u\,\cosh v$ and $z = 2\sinh v$. (You can derive parametric equations in the following way. To get a circular cross section of radius 2 in the xy-plane, start with $x = 2\cos u$ and $y = 2\sin u$. To get a hyperbola in the xz- or yz-plane, multiply x and y by $\cosh v$ and set $z = \sinh v$.) We have $0 \le u \le 2\pi$ to get the circular cross sections and $0 \le v \le \sinh^{-1} 1\,(\approx 0.88)$. The hyperboloid is traced out by the endpoint of the vector-valued function

$$\mathbf{r}(u, v) = \langle 2\cos u\,\cosh v, 2\sin u\,\cosh v, 2\sinh v\rangle,$$

so that $$\mathbf{r}_u = \langle -2\sin u \cosh v, 2\cos u \cosh v, 0\rangle$$

and $$\mathbf{r}_v = \langle 2\cos u \sinh v, 2\sin u \sinh v, 2\cosh v\rangle.$$

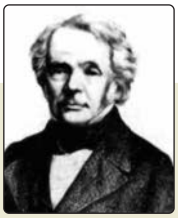
This gives us the normal vector

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \langle 4\cos u \cosh^2 v, 4\sin u \cosh^2 v, -4\cosh v \sinh v\rangle,$$

where $\|\mathbf{n}\| = 4\cosh v\sqrt{\cosh^2 v + \sinh^2 v}$. We now have

$$\begin{aligned}\iint_S 1\, dS &= \iint_R 4\cosh v\sqrt{\cosh^2 v + \sinh^2 v}\, dA \\ &= \int_0^{\sinh^{-1} 1}\int_0^{2\pi} 4\cosh v\sqrt{\cosh^2 v + \sinh^2 v}\, du\, dv \\ &\approx 31.95,\end{aligned}$$

where we evaluated the final integral numerically. ■



HISTORICAL NOTES

August Ferdinand Möbius (1790–1868) German astronomer and mathematician who gave one of the earliest descriptions of the one-sided surface that bears his name. Möbius' doctoral thesis was in astronomy, but his astronomy teachers included the great mathematician Carl Friedrich Gauss. Möbius did research in both fields. His mathematics publications, primarily in geometry and topology, were exceedingly clear and well developed.

Our next example of a surface integral requires some preliminary discussion. First, we say that a surface S is **orientable** (or **two-sided**) if it is possible to define a unit normal vector $\mathbf{n}$ at each point (x, y, z) not on the boundary of the surface and if $\mathbf{n}$ is a continuous function of (x, y, z). In this case, S has two identifiable sides (a top and a bottom or an inside and an outside). Once we choose a consistent direction for all normal vectors to point, we call the surface **oriented.** For instance, a sphere is a two-sided surface; the two sides of the surface are the inside and the outside. Notice that you can't get from the inside to the outside without passing through the sphere. The **positive orientation** for the sphere (or for any other *closed* surface) is to choose outward normal vectors (normal vectors pointing away from the interior).

All of the surfaces we have seen so far in this course are two-sided, but it's not difficult to construct a one-sided surface. Perhaps the most famous example of a one-sided surface is the **Möbius strip,** named after the German mathematician A. F. Möbius. You can easily construct a Möbius strip by taking a long rectangular strip of paper, giving it a half-twist and then taping the short edges together, as illustrated in Figures 14.46a through 14.46c. Notice that if you started painting the strip, you would eventually return to your starting point, having painted both "sides" of the strip, but without having crossed any edges. This says that the Möbius strip has no inside and no outside and is therefore not orientable.

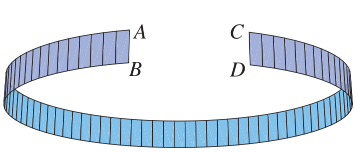


FIGURE 14.46a
A long, thin strip

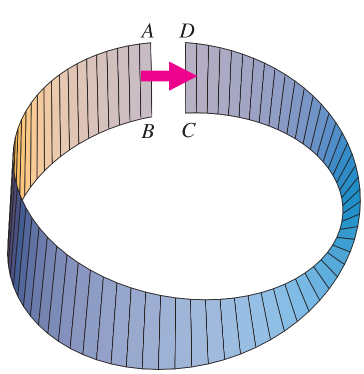


FIGURE 14.46b
Make one half-twist

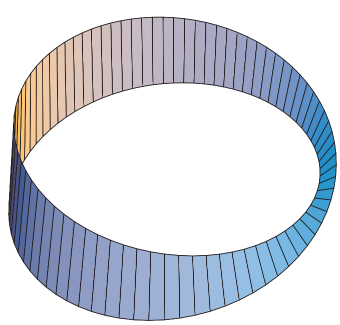


FIGURE 14.46c
A Möbius strip

One reason we need to be able to orient a surface is to compute the **flux** of a vector field. It's easiest to visualize the flux for a vector field representing the velocity field for a fluid in motion. In this context, the flux measures the net flow rate of the fluid across the surface in the direction of the specified normal vectors. (Notice that for this to make sense, the surface must have two identifiable sides. That is, the surface must be orientable.) The orientation of the surface lets us distinguish one direction from the other. In general, we have the following definition.

DEFINITION 6.2

Let $\mathbf{F}(x, y, z)$ be a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$. The **surface integral of F over** S (or the **flux of F over** S) is given by $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

Think carefully about the role of the unit normal vector in Definition 6.2. Notice that since $\mathbf{n}$ is a unit vector, the integrand $\mathbf{F} \cdot \mathbf{n}$ gives (at any given point on S) the component of $\mathbf{F}$ in the direction of $\mathbf{n}$. So, if $\mathbf{F}$ represents the velocity field for a fluid in motion, $\mathbf{F} \cdot \mathbf{n}$ corresponds to the component of the velocity that moves the fluid across the surface (from one side to the other). Also, note that $\mathbf{F} \cdot \mathbf{n}$ can be positive or negative, depending on which normal vector we have chosen. (Keep in mind that at each point on a surface, there are two unit normal vectors, one pointing toward each side of the surface.) You should recognize that this is why we need to have an oriented surface.

EXAMPLE 6.6 Computing the Flux of a Vector Field

Compute the flux of the vector field $\mathbf{F}(x, y, z) = \langle x, y, 0\rangle$ over the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$ (oriented with upward-pointing normal vectors).

Solution First, observe that at any given point, the normal vectors for the paraboloid $z = x^2 + y^2$ are $\pm\langle 2x, 2y, -1\rangle$. For the normal vector to point upward, we need a positive z-component. In this case,

$$\mathbf{m} = -\langle 2x, 2y, -1\rangle = \langle -2x, -2y, 1\rangle$$

is such a normal vector. A unit vector pointing in the same direction as $\mathbf{m}$ is then

$$\mathbf{n} = \frac{\langle -2x, -2y, 1\rangle}{\sqrt{4x^2 + 4y^2 + 1}}.$$

Before computing $\mathbf{F} \cdot \mathbf{n}$, we use the normal vector $\mathbf{m}$ to write the surface area increment dS in terms of dA. From (6.1), we have

$$dS = \|\mathbf{m}\|\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

Putting this all together gives us

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \langle x, y, 0\rangle \cdot \frac{\langle -2x, -2y, 1\rangle}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1}\, dA \\ &= \iint_R \langle x, y, 0\rangle \cdot \langle -2x, -2y, 1\rangle\, dA \\ &= \iint_R (-2x^2 - 2y^2)\, dA,\end{aligned}$$

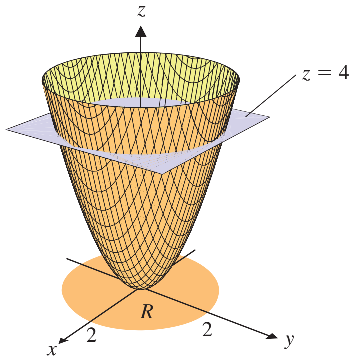


FIGURE 14.47
$z = x^2 + y^2$

where the region R is the projection of the portion of the paraboloid under consideration onto the xy-plane. Note how the square roots arising from the calculation of $\|\mathbf{n}\|$ and dS canceled out one another. Look at the graph in Figure 14.47 and recognize that this projection is bounded by the circle $x^2 + y^2 = 4$. You should quickly realize that the double integral should be set up in polar coordinates. We have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R (-2x^2 - 2y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^2 (-2r^2) r\, dr\, d\theta = -16\pi.$$

Flux integrals enable engineers and physicists to compute the flow of a variety of quantities in three dimensions. If $\mathbf{F}$ represents the velocity field of a fluid, then the flux gives the net amount of fluid crossing the surface S. In example 6.7, we compute heat flow across a surface. If $T(x, y, z)$ gives the temperature at (x, y, z), then the net heat flow across the surface S is the flux of $\mathbf{F} = -k\nabla T$, where the constant k is called the **heat conductivity** of the material.

EXAMPLE 6.7 Computing the Heat Flow Out of a Sphere

For $T(x, y, z) = 30 - \frac{1}{18}z^2$ and $k = 2$, compute the heat flow out of the region bounded by $x^2 + y^2 + z^2 = 9$.

Solution We compute the flux of $-2\nabla(30 - \frac{1}{18}z^2) = -2\langle 0, 0, -\frac{1}{9}z\rangle = \langle 0, 0, \frac{2}{9}z\rangle$. Since we want the flow out of the sphere, we need to find an outward unit normal to the sphere. An outward normal is $\nabla(x^2 + y^2 + z^2) = \langle 2x, 2y, 2z\rangle$, so we take $\mathbf{n} = \dfrac{\langle 2x, 2y, 2z\rangle}{\sqrt{4x^2 + 4y^2 + 4z^2}}$. On the sphere, we have $4x^2 + 4y^2 + 4z^2 = 36$, so that the denominator simplifies to $\sqrt{36} = 6$. This gives us $\mathbf{n} = \frac{1}{3}\langle x, y, z\rangle$ and $\mathbf{F} \cdot \mathbf{n} = \frac{2}{27}z^2$. Since the surface is a sphere, we will use spherical coordinates for the surface integral. On the sphere, $z = \rho\cos\phi = 3\cos\phi$, so that $\mathbf{F} \cdot \mathbf{n} = \frac{2}{27}(3\cos\phi)^2 = \frac{2}{3}\cos^2\phi$. Further, $dS = \rho^2 \sin\phi\, dA = 9\sin\phi\, dA$. The flux is then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{2}{3}\cos^2\phi\right) 9\sin\phi\, dA = \int_0^{2\pi} \int_0^{\pi} 6\cos^2\phi\, \sin\phi\, d\phi\, d\theta = 8\pi.$$

Since the flux is positive, the heat is flowing out of the sphere. Finally, observe that since the temperature decreases as $|z|$ increases, this result should make sense.

BEYOND FORMULAS

Surface integrals complete the set of integrals introduced in this book. (There are many more types of integrals used in more advanced mathematics courses and applications areas.) Although the different types of integrals sometimes require different methods of evaluation, the underlying concepts are the same. In each case, the integral is a limit of approximating sums and in some cases can be evaluated directly with some form of an antiderivative. What is being summed in a surface integral? What is being summed in a flux integral?

EXERCISES 14.6

WRITING EXERCISES

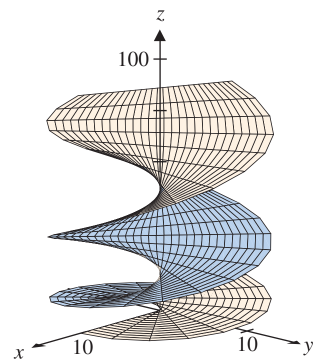
1. For definition 6.1, we defined the partition of a surface and took the limit as the norm of the partition tends to 0. Explain why it would not be sufficient to have the number of segments in the partition tend to ∞. (Hint: The pieces of the partition don't have to be the same size.)
2. In example 6.2, you could alternatively start with cylindrical coordinates and use a parametric representation as we did in example 6.4. Discuss which method you think would be simpler.
3. Explain in words why $\iint\limits_S 1\,dS$ equals the surface area of S. (Hint: Although you are supposed to explain in words, you will need to refer to Riemann sums.)
4. For example 6.6, sketch a graph showing the surface S and several normal vectors to the surface. Also, show several vectors in the graph of the vector field $\mathbf{F}$. Explain why the flux is negative.

In exercises 1–8, find a parametric representation of the surface.

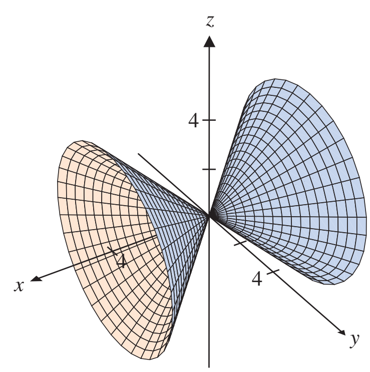
1. $z = 3x + 4y$
2. $x^2 + y^2 + z^2 = 4$
3. $x^2 + y^2 - z^2 = 1$
4. $x^2 - y^2 + z^2 = 4$
5. The portion of $x^2 + y^2 = 4$ from $z = 0$ to $z = 2$
6. The portion of $y^2 + z^2 = 9$ from $x = -1$ to $x = 1$
7. The portion of $z = 4 - x^2 - y^2$ above the xy-plane
8. The portion of $z = x^2 + y^2$ below $z = 4$

In exercises 9–16, sketch a graph of the parametric surface.

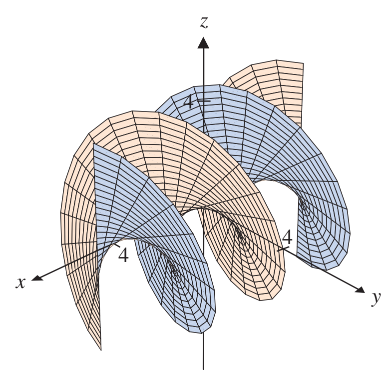
9. $x = u,\ y = v,\ z = u^2 + 2v^2$
10. $x = u,\ y = v,\ z = 4 - u^2 - v^2$
11. $x = u\cos v,\ y = u\sin v,\ z = u^2$
12. $x = u\cos v,\ y = u\sin v,\ z = u$
13. $x = 2\sin u\cos v,\ y = 2\sin u\sin v,\ z = 2\cos u$
14. $x = 2\cos v,\ y = 2\sin v,\ z = u$
15. $x = u,\ y = \sin u\cos v,\ z = \sin u\sin v$
16. $x = \cos u\cos v,\ y = u,\ z = \cos u\sin v$
17. Match the parametric equations with the surface.
   a. $x = u\cos v,\ y = u\sin v,\ z = v^2$
   b. $x = v,\ y = u\cos v,\ z = u\sin v$
   c. $x = u,\ y = u\cos v,\ z = u\sin v$



SURFACE A



SURFACE B



SURFACE C

18. In example 6.4, show that

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = \langle -4\sin^2\phi\cos\theta, -4\sin^2\phi\sin\theta, -4\sin\phi\cos\phi\rangle$$

and then show that $\|\mathbf{n}\| = 4|\sin\phi|$.

In exercises 19–26, find the surface area of the given surface.

19. The portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 4$

20. The portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$

21. The portion of the plane $3x + y + 2z = 6$ inside the cylinder $x^2 + y^2 = 4$

22. The portion of the plane $x + 2y + z = 4$ above the region bounded by $y = x^2$ and $y = 1$

23. The portion of the cone $z = \sqrt{x^2 + y^2}$ above the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$

24. The portion of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$

25. The portion of the hemisphere $z = \sqrt{4 - x^2 - y^2}$ above the plane $z = 1$

26. The portion of $y = 4 - x^2$ with $y \geq 0$ and between $z = 0$ and $z = 2$

In exercises 27–36, set up a double integral and evaluate the surface integral $\iint_S g(x, y, z)\, dS$.

27. $\iint_S xz\, dS$, S is the portion of the plane $z = 2x + 3y$ above the rectangle $1 \leq x \leq 2$, $1 \leq y \leq 3$

28. $\iint_S (z - y^2)\, dS$, S is the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$

29. $\iint_S (x^2 + y^2 + z^2)^{3/2}\, dS$, S is the lower hemisphere $z = -\sqrt{9 - x^2 - y^2}$

30. $\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$, S is the sphere $x^2 + y^2 + z^2 = 9$

31. $\iint_S (x^2 + y^2 - z)\, dS$, S is the portion of the paraboloid $z = 4 - x^2 - y^2$ between $z = 1$ and $z = 2$

32. $\iint_S \sqrt{z}\, dS$, S is the hemisphere $z = -\sqrt{9 - x^2 - y^2}$

33. $\iint_S z^2\, dS$, S is the portion of the cone $z^2 = x^2 + y^2$ between $z = -4$ and $z = 4$

34. $\iint_S z^2\, dS$, S is the portion of the cone $z = \sqrt{x^2 + y^2}$ above the rectangle $0 \leq x \leq 2$, $-1 \leq y \leq 2$

35. $\iint_S x\, dS$, S is the portion of $x^2 + y^2 - z^2 = 1$ between $z = 0$ and $z = 1$

36. $\iint_S \sqrt{x^2 + y^2 + z^2}\, dS$, S is the portion of $x = -\sqrt{4 - y^2 - z^2}$ between $y = 0$ and $y = x$

In exercises 37–48, evaluate the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

37. $\mathbf{F} = \langle x, y, z \rangle$, S is the portion of $z = 4 - x^2 - y^2$ above the xy-plane ($\mathbf{n}$ upward)

38. $\mathbf{F} = \langle y, -x, 1 \rangle$, S is the portion of $z = x^2 + y^2$ below $z = 4$ ($\mathbf{n}$ downward)

39. $\mathbf{F} = \langle y, -x, z \rangle$, S is the portion of $z = \sqrt{x^2 + y^2}$ below $z = 3$ ($\mathbf{n}$ downward)

40. $\mathbf{F} = \langle 0, 1, y \rangle$, S is the portion of $z = -\sqrt{x^2 + y^2}$ inside $x^2 + y^2 = 4$ ($\mathbf{n}$ upward)

41. $\mathbf{F} = \langle xy, y^2, z \rangle$, S is the boundary of the unit cube with $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$ ($\mathbf{n}$ outward)

42. $\mathbf{F} = \langle y, z, 0 \rangle$, S is the boundary of the box with $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 1$ ($\mathbf{n}$ outward)

43. $\mathbf{F} = \langle 1, 0, z \rangle$, S is the boundary of the region bounded above by $z = 4 - x^2 - y^2$ and below by $z = 1$ ($\mathbf{n}$ outward)

44. $\mathbf{F} = \langle x, y, z \rangle$, S is the boundary of the region between $z = 0$ and $z = -\sqrt{4 - x^2 - y^2}$

45. $\mathbf{F} = \langle yx, 1, x \rangle$, S is the portion of $z = 2 - x - y$ above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ ($\mathbf{n}$ upward)

46. $\mathbf{F} = \langle y, 3, z \rangle$, S is the portion of $z = x^2 + y^2$ above the triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ ($\mathbf{n}$ downward)

47. $\mathbf{F} = \langle y, 0, 2 \rangle$, S is the boundary of the region bounded above by $z = \sqrt{8 - x^2 - y^2}$ and below by $z = \sqrt{x^2 + y^2}$ ($\mathbf{n}$ outward)

48. $\mathbf{F} = \langle 3, z, y \rangle$, S is the boundary of the region between $z = 8 - 2x - y$ and $z = \sqrt{x^2 + y^2}$ and inside $x^2 + y^2 = 1$ ($\mathbf{n}$ outward)

In exercises 49–52, find the mass and center of mass of the region.

49. The portion of the plane $3x + 2y + z = 6$ inside the cylinder $x^2 + y^2 = 4$, $\rho(x, y, z) = x^2 + 1$

50. The portion of the plane $x + 2y + z = 4$ above the region bounded by $y = x^2$ and $y = 1$, $\rho(x, y, z) = y$

51. The hemisphere $z = \sqrt{1 - x^2 - y^2}$, $\rho(x, y, z) = 1 + x$

52. The portion of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$, $\rho(x, y, z) = z$

53. State the formula converting a surface integral into a double integral for a projection into the yz-plane.

54. State the formula converting a surface integral into a double integral for a projection into the xz-plane.

In exercises 55–62, use the formulas of exercises 53 and 54 to evaluate the surface integral.

55. $\iint_S z\, dS$, where S is the portion of $x^2 + y^2 = 1$ with $x \geq 0$ and z between $z = 1$ and $z = 2$

56. $\iint_S yz\,dS$, where S is the portion of $x^2+y^2=1$ with $x \ge 0$ and z between $z=1$ and $z=4-y$

57. $\iint_S (y^2+z^2)\,dS$, where S is the portion of the paraboloid $x=9-y^2-z^2$ in front of the yz-plane

58. $\iint_S (y^2+z^2)\,dS$, where S is the hemisphere $x=\sqrt{4-y^2-z^2}$

59. $\iint_S x^2\,dS$, where S is the portion of the paraboloid $y=x^2+z^2$ to the left of the plane $y=1$

60. $\iint_S (x^2+z^2)\,dS$, where S is the hemisphere $y=\sqrt{4-x^2-z^2}$

61. $\iint_S 4dS$, where S is the portion of $y=1-x^2$ with $y \ge 0$ and between $z=0$ and $z=2$

62. $\iint_S (x^2+z^2)\,dS$, where S is the portion of $y=\sqrt{4-x^2}$ between $z=1$ and $z=4$

63. Explain the following result geometrically. The flux integral of $\mathbf{F}(x,y,z)=\langle x,y,z\rangle$ across the cone $z=\sqrt{x^2+y^2}$ is 0.

64. In geometric terms, determine whether the flux integral of $\mathbf{F}(x,y,z)=\langle x,y,z\rangle$ across the hemisphere $z=\sqrt{1-x^2-y^2}$ is 0.

65. For the cone $z=c\sqrt{x^2+y^2}$ (where $c>0$), show that in spherical coordinates $\tan\phi=\frac{1}{c}$. Then show that parametric equations are $x=\dfrac{u\cos v}{\sqrt{c^2+1}}$, $y=\dfrac{u\sin v}{\sqrt{c^2+1}}$ and $z=\dfrac{cu}{\sqrt{c^2+1}}$.

66. Find the surface area of the portion of $z=c\sqrt{x^2+y^2}$ below $z=1$, using the parametric equations in exercise 65.

67. Find the flux of $\langle x,y,z\rangle$ across the portion of $z=c\sqrt{x^2+y^2}$ below $z=1$. Explain in physical terms why this answer makes sense.

68. Find the flux of $\langle x,y,z\rangle$ across the entire cone $z^2=c^2(x^2+y^2)$.

69. Find the flux of $\langle x,y,0\rangle$ across the portion of $z=c\sqrt{x^2+y^2}$ below $z=1$. Explain in physical terms why this answer makes sense.

70. Find the limit as c approaches 0 of the flux in exercise 69. Explain in physical terms why this answer makes sense.

EXPLORATORY EXERCISES

1. If $x=3\sin u\cos v$, $y=3\cos u$ and $z=3\sin u\sin v$, show that $x^2+y^2+z^2=9$. Explain why this equation doesn't guarantee that the parametric surface defined is the entire sphere, but it does guarantee that all points on the surface are also on the sphere. In this case, the parametric surface is the entire sphere. To verify this in graphical terms, sketch a picture showing geometric interpretations of the "spherical coordinates" u and v. To see what problems can occur, sketch the surface defined by $x=3\sin\dfrac{u^2}{u^2+1}\cos v$, $y=3\cos\dfrac{u^2}{u^2+1}$ and $z=3\sin\dfrac{u^2}{u^2+1}\sin v$. Explain why you do not get the entire sphere. To see a more subtle example of the same problem, sketch the surface $x=\cos u\cosh v$, $y=\sinh v$, $z=\sin u\cosh v$. Use identities to show that $x^2-y^2+z^2=1$ and identify the surface. Then sketch the surface $x=\cos u\cosh v$, $y=\cos u\sinh v$, $z=\sin u$ and use identities to show that $x^2-y^2+z^2=1$. Explain why the second surface is not the entire hyperboloid. Explain in words and pictures exactly what the second surface is.

14.7 THE DIVERGENCE THEOREM

Recall that at the end of section 14.5, we had rewritten Green's Theorem in terms of the divergence of a two-dimensional vector field. We had found there (see equation 5.3) that

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \nabla\cdot\mathbf{F}(x,y)\,dA.$$

Here, R is a region in the xy-plane enclosed by a piecewise-smooth, positively oriented, simple closed curve C. Further, $\mathbf{F}(x,y)=\langle M(x,y),N(x,y),0\rangle$, where $M(x,y)$ and $N(x,y)$ are continuous and have continuous first partial derivatives in some open region D in the xy-plane, with $R\subset D$.

We can extend this two-dimensional result to three dimensions in exactly the way you might expect. That is, for a solid region $Q \subset \mathbb{R}^3$ bounded by the surface ∂Q, we have

$$\iint_{\partial Q} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_Q \nabla \cdot \mathbf{F}(x, y, z)\, dV.$$

This result (referred to as the **Divergence Theorem** or **Gauss' Theorem**) has great significance in a variety of settings. If **F** represents the velocity field of a fluid in motion, the Divergence Theorem says that the total flux of the velocity field across the boundary of the solid is equal to the triple integral of the divergence of the velocity field over the solid.

In Figure 14.48, the velocity field **F** of a fluid is shown superimposed on a solid Q bounded by the closed surface ∂Q. Observe that there are two ways to compute the rate of change of the amount of fluid inside of Q. One way is to calculate the fluid flow into or out of Q across its boundary, which is given by the flux integral $\iint_{\partial Q} \mathbf{F} \cdot \mathbf{n}\, dS$. On the other hand, instead of focusing on the boundary, we can consider the accumulation or dispersal of fluid at each point in Q. As we'll see, this is given by $\nabla \cdot \mathbf{F}$, whose value at a given point measures the extent to which that point acts as a source or sink of fluid. To obtain the total change in the amount of the fluid in Q, we "add up" all of the values of $\nabla \cdot \mathbf{F}$ in Q, giving us the triple integral of $\nabla \cdot \mathbf{F}$ over Q. Since the flux integral and the triple integral both give the net rate of change of the amount of fluid in Q, they must be equal.

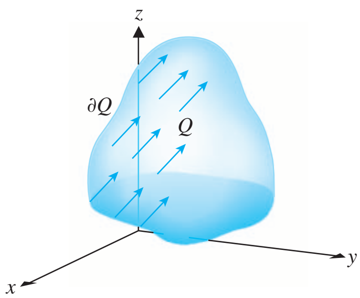


FIGURE 14.48
Flow of fluid across ∂Q

We now state and prove the result.

> **THEOREM 7.1** (Divergence Theorem)
>
> Suppose that $Q \subset \mathbb{R}^3$ is bounded by the closed surface ∂Q and that $\mathbf{n}(x, y, z)$ denotes the exterior unit normal vector to ∂Q. Then, if the components of $\mathbf{F}\,(x, y, z)$ have continuous first partial derivatives in Q, we have
>
> $$\iint_{\partial Q} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_Q \nabla \cdot \mathbf{F}(x, y, z)\, dV.$$

Although we have stated the theorem in the general case, we prove the result only for the case where the solid Q is fairly simple. A proof for the general case can be found in a more advanced text.

PROOF

For $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$, the divergence of **F** is

$$\nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

We then have that

$$\iiint_Q \nabla \cdot \mathbf{F}(x, y, z)\, dV = \iiint_Q \frac{\partial M}{\partial x}\, dV + \iiint_Q \frac{\partial N}{\partial y}\, dV + \iiint_Q \frac{\partial P}{\partial z}\, dV. \qquad (7.1)$$

Further, we can write the flux integral as

$$\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\partial Q} M(x,y,z)\mathbf{i}\cdot\mathbf{n}\,dS + \iint_{\partial Q} N(x,y,z)\mathbf{j}\cdot\mathbf{n}\,dS + \iint_{\partial Q} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS. \tag{7.2}$$

Looking carefully at (7.1) and (7.2), observe that the theorem will follow if we can show that

$$\iiint_Q \frac{\partial M}{\partial x}\,dV = \iint_{\partial Q} M(x,y,z)\mathbf{i}\cdot\mathbf{n}\,dS, \tag{7.3}$$

$$\iiint_Q \frac{\partial N}{\partial y}\,dV = \iint_{\partial Q} N(x,y,z)\mathbf{j}\cdot\mathbf{n}\,dS \tag{7.4}$$

and

$$\iiint_Q \frac{\partial P}{\partial z}\,dV = \iint_{\partial Q} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS. \tag{7.5}$$

As you might imagine, the proofs of (7.3), (7.4) and (7.5) are all virtually identical (and all fairly long). Consequently, we prove only one of these three equations here. In order to prove (7.5), we assume that we can describe the solid Q as follows:

$$Q = \{(x,y,z) \mid g(x,y) \le z \le h(x,y),\ \text{for } (x,y) \in R\},$$

where R is some region in the xy-plane, as illustrated in Figure 14.49a. (We can prove (7.3) and (7.4) by making corresponding assumptions regarding Q.) Now, notice from Figure 14.49a that there are three distinct surfaces that make up the boundary of Q. In Figure 14.49b, we have labeled these surfaces S_1 (the bottom surface), S_2 (the top surface) and S_3 (the lateral surface), where we have also indicated exterior normal vectors to each of the surfaces.

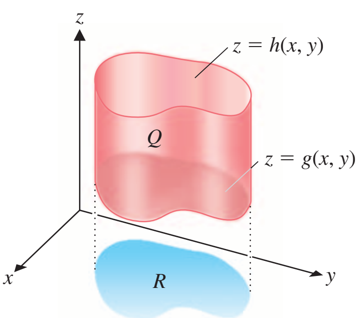


FIGURE 14.49a
The solid Q

Notice that on the lateral surface S_3, the $\mathbf{k}$ component of the exterior unit normal $\mathbf{n}$ is zero and so, the flux integral of $P(x,y,z)\mathbf{k}$ over S_3 is zero. This gives us

$$\iint_{\partial Q} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS = \iint_{S_1} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_2} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS. \tag{7.6}$$

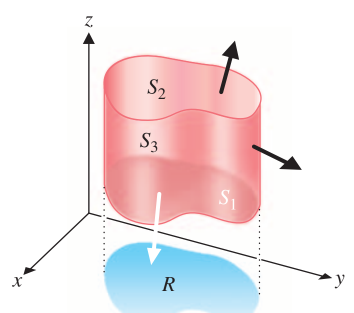


FIGURE 14.49b
The surfaces S_1, S_2 and S_3 and several exterior normal vectors

In order to prove the result, we need to rewrite the two integrals on the right side of (7.6) as double integrals over the region R in the xy-plane. First, you must notice that on the surface S_1 (the bottom surface), the exterior unit normal $\mathbf{n}$ points downward (i.e., it has a negative $\mathbf{k}$ component). Now, S_1 is defined by

$$S_1 = \{(x,y,z) \mid z = g(x,y),\ \text{for } (x,y) \in R\}.$$

If we define $k_1(x,y,z) = z - g(x,y)$, then the exterior unit normal on S_1 is given by

$$\mathbf{n} = \frac{-\nabla k_1}{\|\nabla k_1\|} = \frac{g_x(x,y)\mathbf{i} + g_y(x,y)\mathbf{j} - \mathbf{k}}{\sqrt{[g_x(x,y)]^2 + [g_y(x,y)]^2 + 1}}$$

and

$$\mathbf{k}\cdot\mathbf{n} = \frac{-1}{\sqrt{[g_x(x,y)]^2 + [g_y(x,y)]^2 + 1}},$$

since the unit vectors **i**, **j** and **k** are all mutually orthogonal. We now have

$$\begin{aligned}\iint_{S_1} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS &= -\iint_{S_1} \frac{P(x,y,z)}{\sqrt{[g_x(x,y)]^2+[g_y(x,y)]^2+1}}\,dS \\ &= -\iint_R \frac{P(x,y,g(x,y))}{\sqrt{[g_x(x,y)]^2+[g_y(x,y)]^2+1}} \\ &\qquad \cdot \sqrt{[g_x(x,y)]^2+[g_y(x,y)]^2+1}\,dA \\ &= -\iint_R P(x,y,g(x,y))\,dA, \end{aligned} \tag{7.7}$$

thanks to the two square roots canceling out one another. In a similar way, notice that on S_2 (the top surface), the exterior unit normal **n** points upward (i.e., it has a positive **k** component). Since S_2 corresponds to the portion of the surface $z = h(x, y)$ for $(x, y) \in R$, if we take $k_2(x, y) = z - h(x, y)$, we have that on S_2,

$$\mathbf{n} = \frac{\nabla k_2}{\|\nabla k_2\|} = \frac{-h_x(x,y)\mathbf{i} - h_y(x,y)\mathbf{j} + \mathbf{k}}{\sqrt{[h_x(x,y)]^2+[h_y(x,y)]^2+1}}$$

and so,

$$\mathbf{k}\cdot\mathbf{n} = \frac{1}{\sqrt{[h_x(x,y)]^2+[h_y(x,y)]^2+1}}.$$

We now have

$$\begin{aligned}\iint_{S_2} P(x,y,z)\,\mathbf{k}\cdot\mathbf{n}\,dS &= \iint_{S_2} \frac{P(x,y,z)}{\sqrt{[h_x(x,y)]^2+[h_y(x,y)]^2+1}}\,dS \\ &= \iint_R \frac{P(x,y,h(x,y))}{\sqrt{[h_x(x,y)]^2+[h_y(x,y)]^2+1}} \\ &\qquad \cdot \sqrt{[h_x(x,y)]^2+[h_y(x,y)]^2+1}\,dA \\ &= \iint_R P(x,y,h(x,y))\,dA. \end{aligned} \tag{7.8}$$

Putting together (7.6), (7.7) and (7.8) gives us

$$\begin{aligned}\iint_{\partial Q} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS &= \iint_{S_1} P(x,y,z)\,\mathbf{k}\cdot\mathbf{n}\,dS + \iint_{S_2} P(x,y,z)\mathbf{k}\cdot\mathbf{n}\,dS \\ &= \iint_R P(x,y,h(x,y))\,dA - \iint_R P(x,y,g(x,y))\,dA \\ &= \iint_R [P(x,y,h(x,y)) - P(x,y,g(x,y))]\,dA \\ &= \iint_R \int_{g(x,y)}^{h(x,y)} \frac{\partial P}{\partial z}\,dz\,dA \qquad \text{By the Fundamental Theorem of Calculus} \\ &= \iiint_Q \frac{\partial P}{\partial z}\,dV, \end{aligned}$$

which proves (7.5). With appropriate assumptions on Q, we can similarly prove (7.3) and (7.4). This proves the theorem for the special case where the solid Q can be described as indicated. ■

EXAMPLE 7.1 Applying the Divergence Theorem

Let Q be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Find the flux of the vector field $\mathbf{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$ over the surface ∂Q.

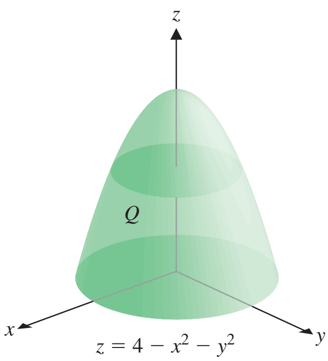


FIGURE 14.50
The solid Q

Solution We show a sketch of the solid in Figure 14.50. Notice that to compute the flux directly, we must consider the two different portions of ∂Q (the surface of the paraboloid and its base in the xy-plane) separately. Alternatively, observe that the divergence of $\mathbf{F}$ is given by

$$\nabla \cdot \mathbf{F}(x, y, z) = \nabla \cdot \langle x^3, y^3, z^3 \rangle = 3x^2 + 3y^2 + 3z^2.$$

From the Divergence Theorem, we now have that the flux of $\mathbf{F}$ over ∂Q is given by

$$\begin{aligned}\iint_{\partial Q} \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_Q \nabla \cdot \mathbf{F}(x, y, z)\, dV \\ &= \iiint_Q (3x^2 + 3y^2 + 3z^2)\, dV.\end{aligned}$$

If we rewrite the triple integral in cylindrical coordinates, we get

$$\begin{aligned}\iint_{\partial Q} \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_Q (3x^2 + 3y^2 + 3z^2)\, dV \\ &= 3\int_0^{2\pi}\int_0^2\int_0^{4-r^2} (r^2 + z^2)\, r\, dz\, dr\, d\theta \\ &= 3\int_0^{2\pi}\int_0^2 \left(r^2 z + \frac{z^3}{3}\right)\Big|_{z=0}^{z=4-r^2} r\, dr\, d\theta \\ &= 3\int_0^{2\pi}\int_0^2 \left[r^3(4 - r^2) + \frac{1}{3}(4 - r^2)^3 r\right] dr\, d\theta \\ &= 96\pi,\end{aligned}$$

where we have left the details of the final integrations as a straightforward exercise. ■

Notice that in example 7.1, we used the Divergence Theorem to replace a very messy surface integral calculation by a comparatively simple triple integral. In example 7.2, we prove a general result regarding the flux of a certain vector field over any surface, something we would be unable to do without the Divergence Theorem.

EXAMPLE 7.2 Proving a General Result with the Divergence Theorem

Prove that the flux of the vector field $\mathbf{F}(x, y, z) = \langle 3y \cos z, x^2 e^z, x \sin y \rangle$ is zero over any closed surface ∂Q enclosing a solid region Q.

Solution Notice that in this case, the divergence of $\mathbf{F}$ is

$$\begin{aligned}\nabla \cdot \mathbf{F}(x, y, z) &= \nabla \cdot \langle 3y \cos z, x^2 e^z, x \sin y \rangle \\ &= \frac{\partial}{\partial x}(3y \cos z) + \frac{\partial}{\partial y}(x^2 e^z) + \frac{\partial}{\partial z}(x \sin y) = 0.\end{aligned}$$

From the Divergence Theorem, we then have that the flux of $\mathbf{F}$ over ∂Q is given by

$$\begin{aligned}\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_{Q} \nabla\cdot\mathbf{F}(x,y,z)\,dV \\ &= \iiint_{Q} 0\,dV = 0,\end{aligned}$$

for any solid region $Q \subset \mathbb{R}^3$. ■

In section 4.4, we saw that for a function $f(x)$ of a single variable, if f is continuous on the interval $[a, b]$ then the average value of f on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Similarly, when $f(x, y, z)$ is a continuous function on the region $Q \subset \mathbb{R}^3$ (bounded by the surface ∂Q), the **average value** of f on Q is given by

$$f_{\text{ave}} = \frac{1}{V}\iiint_{Q} f(x,y,z)\,dV,$$

where V is the volume of Q. Further, by continuity, there must be a point $P(a, b, c) \in Q$ at which f equals its average value, that is, where

$$f(P) = \frac{1}{V}\iiint_{Q} f(x,y,z)\,dV.$$

This says that if $\mathbf{F}(x, y, z)$ has continuous first partial derivatives on Q, then div $\mathbf{F}$ is continuous on Q and so, there is a point $P(a, b, c) \in Q$ for which

$$\begin{aligned}(\nabla\cdot\mathbf{F})|_P &= \frac{1}{V}\iiint_{Q} \nabla\cdot\mathbf{F}(x,y,z)\,dV \\ &= \frac{1}{V}\iint_{\partial Q} \mathbf{F}(x,y,z)\cdot\mathbf{n}\,dS,\end{aligned}$$

by the Divergence Theorem. Finally, observe that since the surface integral represents the flux of $\mathbf{F}$ over the surface ∂Q, then $(\nabla\cdot\mathbf{F})|_P$ represents the flux per unit volume over ∂Q.

In particular, for any point $P_0(x_0, y_0, z_0)$ in the interior of Q (i.e., in Q, but not on ∂Q), let S_a be the sphere of radius a, centered at P_0, where a is sufficiently small so that S_a lies completely inside Q. From the preceding discussion, there must be some point P_a in the interior of S_a for which

$$(\nabla\cdot\mathbf{F})|_{P_a} = \frac{1}{V_a}\iint_{S_a} \mathbf{F}(x,y,z)\cdot\mathbf{n}\,dS,$$

where V_a is the volume of the sphere ($V_a = \frac{4}{3}\pi a^3$). Finally, taking the limit as $a \to 0$, we have by the continuity of $\nabla \cdot \mathbf{F}$ that

$$(\nabla \cdot \mathbf{F})\big|_{P_0} = \lim_{a\to 0} \frac{1}{V_a} \iint_{S_a} \mathbf{F}(x, y, z) \cdot \mathbf{n}\, dS$$

or

$$\boxed{\operatorname{div} \mathbf{F}(P_0) = \lim_{a\to 0} \frac{1}{V_a} \iint_{S_a} \mathbf{F}(x, y, z) \cdot \mathbf{n}\, dS.} \tag{7.9}$$

In other words, the divergence of a vector field at a point P_0 is the limiting value of the flux per unit volume over a sphere centered at P_0, as the radius of the sphere tends to zero.

In the case where $\mathbf{F}$ represents the velocity field for a fluid in motion, (7.9) provides us with an important interpretation of the divergence of a vector field. In this case, if $\operatorname{div} \mathbf{F}(P_0) > 0$, then the flux per unit volume at P_0 is positive. From (7.9), this means that for a sphere S_a of sufficiently small radius centered at P_0, the net (outward) flux through the surface of S_a is positive. For an incompressible fluid (such as a liquid), this says that more fluid is passing out through the surface of S_a than is passing in through the surface, which can happen only if there is a source somewhere in S_a, where additional fluid is coming into the flow. Likewise, if $\operatorname{div} \mathbf{F}(P_0) < 0$, there must be a sphere S_a for which the net (outward) flux through the surface of S_a is negative. This says that more fluid is passing in through the surface than is flowing out. Once again, for an incompressible fluid, this can occur only if there is a sink somewhere in S_a, where fluid is leaving the flow. For this reason, in incompressible fluid flow, a point where $\operatorname{div} \mathbf{F}(P) > 0$ is called a **source** and a point where $\operatorname{div} \mathbf{F}(P) < 0$ is called a **sink.** Notice that for an incompressible fluid flow with no sources or sinks, we must have that $\operatorname{div} \mathbf{F}(P) = 0$ throughout the flow.

EXAMPLE 7.3 Finding the Flux of an Inverse Square Field

Show that the flux of an inverse square field over every closed surface enclosing the origin is a constant.

Solution Suppose that S is a closed surface forming the boundary of the solid region Q, where the origin lies in the interior of Q and suppose that $\mathbf{F}$ is an inverse square field. That is,

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r},$$

where $\mathbf{r} = \langle x, y, z\rangle$, $\|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$ and c is a constant. Before you rush to apply the Divergence Theorem, notice that $\mathbf{F}$ is *not* continuous in Q, since $\mathbf{F}$ is undefined at the origin and so, we cannot apply the theorem in Q. Notice, though, that if we could somehow exclude the origin from the region, then we could apply the theorem. A very common method of doing this is to "punch out" a sphere S_a of radius a centered at the origin, where a is sufficiently small that S_a is completely contained in the interior of Q (see Figure 14.51). That is, if we define Q_a to be the set of all points inside Q, but outside of S_a (so that Q_a corresponds to Q, where the sphere S_a has been "punched out"), we can now apply the Divergence Theorem on Q_a. Before we do that, notice that

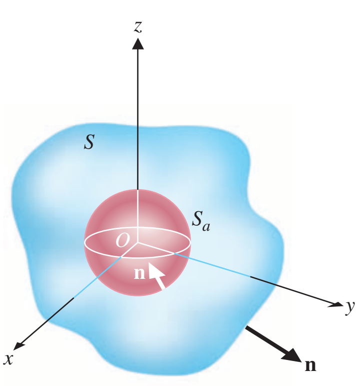


FIGURE 14.51
The region Q_a

the boundary of Q_a consists of the two surfaces S and S_a. We now have

$$\iiint_{Q_a} \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_a} \mathbf{F}\cdot\mathbf{n}\, dS.$$

We leave it as an exercise to show that for any inverse square field, $\nabla \cdot \mathbf{F} = 0$. This now gives us

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = -\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\, dS. \tag{7.10}$$

Since the integral on the right side of (7.10) is taken over a sphere centered at the origin, we should be able to easily calculate it. You need to be careful, though, to note that the exterior normals here point *out* of Q_a and so, the normal on the right side of (7.10) must point *toward* the origin. That is,

$$\mathbf{n} = -\frac{1}{\|\mathbf{r}\|}\mathbf{r} = -\frac{1}{a}\mathbf{r},$$

since $\|\mathbf{r}\| = a$ on S_a. We now have from (7.10) that

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\, dS &= -\iint_{S_a} \frac{c}{a^3}\mathbf{r}\cdot\left(-\frac{1}{a}\mathbf{r}\right) dS \\ &= \frac{c}{a^4}\iint_{S_a} \mathbf{r}\cdot\mathbf{r}\, dS \\ &= \frac{c}{a^4}\iint_{S_a} \|\mathbf{r}\|^2\, dS \\ &= \frac{c}{a^2}\iint_{S_a} dS = \frac{c}{a^2}(4\pi a^2) = 4\pi c, \qquad \text{Since } \|\mathbf{r}\| = a\end{aligned}$$

since $\iint_{S_a} dS$ simply gives the surface area of the sphere, $4\pi a^2$. Notice that this says that over any closed surface enclosing the origin, the flux of an inverse square field is a constant: $4\pi c$. ■

The principle derived in example 7.3 is called **Gauss' Law** for inverse square fields and has many important applications, notably in the theory of electricity and magnetism. The method we used to derive Gauss' Law, where we punched out a sphere surrounding the discontinuity of the integrand, is a common technique used in applying the Divergence Theorem to a variety of important cases where the integrand is discontinuous. In particular, such applications to discontinuous vector fields are quite important in the field of differential equations.

We close the section with a straightforward application of the Divergence Theorem to show that the flux of a magnetic field across a closed surface is always zero.

EXAMPLE 7.4 Finding the Flux of a Magnetic Field

Use the Divergence Theorem and Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ to show that $\iint_S \mathbf{B}\cdot\mathbf{n}\, dS = 0$ for any closed surface S.

Solution Applying the Divergence Theorem to $\iint_S \mathbf{B}\cdot\mathbf{n}\,dS$ and using $\nabla\cdot\mathbf{B}=0$, we have

$$\iint_S \mathbf{B}\cdot\mathbf{n}\,dS = \iiint_Q \nabla\cdot\mathbf{B}\,dV = 0.$$

Observe that this result says that the flux of a magnetic field over any closed surface is zero. ■

EXERCISES 14.7

WRITING EXERCISES

1. If $\mathbf{F}$ is the velocity field of a fluid, explain what $\mathbf{F}\cdot\mathbf{n}$ represents and then what $\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\,dS$ represents.
2. If $\mathbf{F}$ is the velocity field of a fluid, explain what $\nabla\cdot\mathbf{F}$ represents and then what $\iiint_Q \nabla\cdot\mathbf{F}\,dV$ represents.
3. Use your answers to exercises 1 and 2 to explain in physical terms why the Divergence Theorem makes sense.
4. For fluid flowing through a pipe, give one example each of a source and a sink.

In exercises 1–4, verify the Divergence Theorem by computing both integrals.

1. $\mathbf{F}=\langle 2xz, y^2, -xz\rangle$, Q is the cube $0\le x\le 1, 0\le y\le 1, 0\le z\le 1$
2. $\mathbf{F}=\langle x, y, z\rangle$, Q is the ball $x^2+y^2+z^2\le 1$
3. $\mathbf{F}=\langle xz, zy, 2z^2\rangle$, Q is bounded by $z=1-x^2-y^2$ and $z=0$
4. $\mathbf{F}=\langle x^2, 2y, -x^2\rangle$, Q is the tetrahedron bounded by $x+2y+z=4$ and the coordinate planes

In exercises 5–16, use the Divergence Theorem to compute $\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\,dS$.

5. Q is bounded by $x+y+2z=2$ (first octant) and the coordinate planes, $\mathbf{F}=\langle 2x-y^2, 4xz-2y, xy^3\rangle$.
6. Q is bounded by $4x+2y-z=4$ $(z\le 0)$ and the coordinate planes, $\mathbf{F}=\langle x^2-y^2z, x\sin z, 4y^2\rangle$.
7. Q is the cube $-1\le x\le 1, -1\le y\le 1, -1\le z\le 1$, $\mathbf{F}=\langle 4y^2, 3z-\cos x, z^3-x\rangle$.
8. Q is the rectangular box $0\le x\le 2, 1\le y\le 2, -1\le z\le 2$, $\mathbf{F}=\langle y^3-2x, e^{xz}, 4z\rangle$.
9. Q is bounded by $z=x^2+y^2$ and $z=4$, $\mathbf{F}=\langle x^3, y^3-z, xy^2\rangle$.
10. Q is bounded by $z=\sqrt{x^2+y^2}$ and $z=4$, $\mathbf{F}=\langle y^3, x+z^2, z+y^2\rangle$.
11. Q is bounded by $z=4-x^2-y^2, z=1$ and $z=0$, $\mathbf{F}=\langle z^3, x^2y, y^2z\rangle$.
12. Q is bounded by $z=\sqrt{x^2+y^2}, z=1$ and $z=2$, $\mathbf{F}=\langle x^3, x^2z^2, 3y^2z\rangle$.
13. Q is bounded by $x^2+y^2=1, z=0$ and $z=1$, $\mathbf{F}=\langle x-y^3, x^2\sin z, 3z\rangle$.
14. Q is bounded by $x^2+y^2=4, z=1$ and $z=8-y$, $\mathbf{F}=\langle y^2z, 2y-e^z, \sin x\rangle$.
15. Q is bounded by $z=\sqrt{1-x^2-y^2}$ and $z=0$, $\mathbf{F}=\langle x^3, y^3, z^3\rangle$.
16. Q is bounded by $z=-\sqrt{4-x^2-y^2}$ and $z=0$, $\mathbf{F}=\langle x^3, y^3, z^3\rangle$.

In exercises 17–28, find the flux of F over ∂Q.

17. Q is bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{2-x^2-y^2}$, $\mathbf{F}=\langle x^2, z^2-x, y^3\rangle$.
18. Q is bounded by $z=\sqrt{x^2+y^2}$ and $z=\sqrt{8-x^2-y^2}$, $\mathbf{F}=\langle 3xz^2, y^3, 3zx^2\rangle$.
19. Q is bounded by $z=\sqrt{x^2+y^2}, x^2+y^2=1$ and $z=0$, $\mathbf{F}=\langle y^2, x^2z, z^2\rangle$.
20. Q is bounded by $z=x^2+y^2$ and $z=8-x^2-y^2$, $\mathbf{F}=\langle 3y^2, 4x^3, 2z-x^2\rangle$.
21. Q is bounded by $x^2+z^2=1, y=0$ and $y=1$, $\mathbf{F}=\langle z-y^3, 2y-\sin z, x^2-z\rangle$.

22. Q is bounded by $y^2 + z^2 = 4$, $x = 1$ and $x = 8 - y$, $\mathbf{F} = \langle x^2 z, 2y - e^z, \sin x\rangle$.

23. Q is bounded by $x = y^2 + z^2$ and $x = 4$, $\mathbf{F} = \langle x^3, y^3 - z, z^3 - y^2\rangle$.

24. Q is bounded by $y = 4 - x^2 - z^2$ and the xz-plane, $\mathbf{F} = \langle z^2 x, x^2 y, y^2 x\rangle$.

25. Q is bounded by $3x + 2y + z = 6$ and the coordinate planes, $\mathbf{F} = \langle y^2 x, 4x^2 \sin z, 3\rangle$.

26. Q is bounded by $x + 2y + 3z = 12$ and the coordinate planes, $\mathbf{F} = \langle x^2 y, 3x, 4y - x^2\rangle$.

27. Q is bounded by $z = 1 - x^2$, $z = -3$, $y = -2$ and $y = 2$, $\mathbf{F} = \langle x^2, y^3, x^3 y^2\rangle$.

28. Q is bounded by $z = 1 - x^2$, $z = 0$, $y = 0$ and $x + y = 4$, $\mathbf{F} = \langle y^3, x^2 - z, z^2\rangle$.

29. Coulomb's law for an electrostatic field applied to a point charge q at the origin gives us $\mathbf{E}(\mathbf{r}) = q\dfrac{\mathbf{r}}{r^3}$, where $r = \|\mathbf{r}\|$. Let Q be bounded by the sphere $x^2 + y^2 + z^2 = a^2$ for some constant $a > 0$. Show that the flux of $\mathbf{E}$ over ∂Q equals $4\pi q$. Discuss the fact that the flux does not depend on the value of a.

30. Show that for any inverse square field (see example 7.3), the divergence is 0.

31. Prove Green's first identity in three dimensions (see exercise 43 in section 14.5 for Green's first identity in two dimensions):

$$\iiint_Q f\nabla^2 g\, dV = \iint_{\partial Q} f(\nabla g)\cdot \mathbf{n}\, dS - \iiint_Q (\nabla f \cdot \nabla g)\, dV.$$

(Hint: Use the Divergence Theorem applied to $\mathbf{F} = f\nabla g$.)

32. Prove Green's second identity in three dimensions (see exercise 44 in section 14.5 for Green's second identity in two dimensions):

$$\iiint_Q (f\nabla^2 g - g\nabla^2 f)\, dV = \iint_{\partial Q} (f\nabla g - g\nabla f)\cdot \mathbf{n}\, dS.$$

(Hint: Use Green's first identity from exercise 31.)

Exercises 33–36 use Gauss' Law $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ for an electric field E, charge density ρ and permittivity ϵ_0.

33. If S is a closed surface, show that the total charge q enclosed by S satisfies $q = \epsilon_0 \iint_S \mathbf{E}\cdot \mathbf{n}\, dS$.

34. Let $\mathbf{E}$ be the electric field for an infinite line charge on the z-axis. Assume that $\mathbf{E}$ has the form $\mathbf{E} = c\hat{\mathbf{r}} = c\dfrac{\langle x, y, 0\rangle}{x^2 + y^2}$, for some constant c and let ρ be constant density with respect to length on the z-axis.

a. If S is a portion of the cylinder $x^2 + y^2 = 1$ with height h, argue that $q = \rho h$.

b. Use the results of exercise 33 and part (a) to find c in terms of ρ and ϵ_0.

35. Let $\mathbf{E}$ be the electric field for an infinite plane of charge density ρ. Assume that $\mathbf{E}$ has the form $\mathbf{E} = \langle 0, 0, |z|\rangle$, for some constant $c > 0$.

a. If S is a portion of the cylinder $x^2 + y^2 = 1$ with height h extending above and below the xy-plane, argue that $q = 2\pi\rho$.

b. Use the technique of exercise 34 to determine the constant c.

36. The integral form of Gauss' Law is $\iint_S \mathbf{E}\cdot\mathbf{n}\, dS = \dfrac{q}{\epsilon_0}$, where $\mathbf{E}$ is an electric field, q is the total charge enclosed by S and ϵ_0 is the permittivity constant. Use equation (7.1) to derive the differential form of Gauss' Law: $\nabla\cdot\mathbf{E} = \dfrac{\rho}{\epsilon_0}$, where ρ is the charge density.

EXPLORATORY EXERCISES

1. In this exercise, we develop the **continuity equation,** one of the most important results in vector calculus. Suppose that a fluid has density ρ (a scalar function of space and time) and velocity $\mathbf{v}$. Argue that the rate of change of the mass m of the fluid contained in a region Q can be written as $\dfrac{dm}{dt} = \iiint_Q \dfrac{\partial\rho}{\partial t}\, dV$. Next, explain why the only way that the mass can change is for fluid to cross the boundary of Q (∂Q). Argue that $\dfrac{dm}{dt} = -\iint_{\partial Q} (\rho\mathbf{v})\cdot\mathbf{n}\, dS$. In particular, explain why the minus sign in front of the surface integral is needed. Use the Divergence Theorem to rewrite this expression as a triple integral over Q. Explain why the two triple integrals must be equal. Since the integration is taken over arbitrary solids Q, the integrands must be equal to each other. Conclude that the continuity equation holds:

$$\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t} = 0.$$

14.8 STOKES' THEOREM

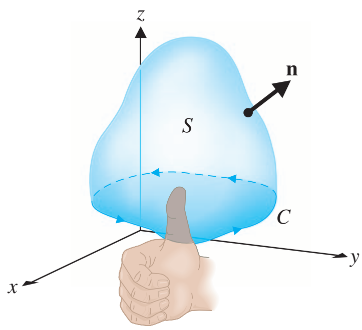


FIGURE 14.52a
Positive orientation

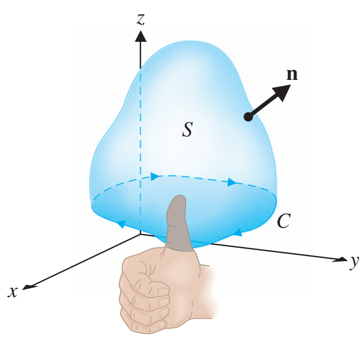


FIGURE 14.52b
Negative orientation

Recall that, after introducing the curl in section 14.5, we observed that for a piecewise, smooth, positively oriented, simple closed curve C in the xy-plane enclosing the region R, we could rewrite Green's Theorem in the vector form

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA, \tag{8.1}$$

where $\mathbf{F}(x, y)$ is a vector field of the form $\mathbf{F}(x, y) = \langle M(x, y), N(x, y), 0\rangle$. In this section, we generalize this result to the case of a vector field defined on a surface in three dimensions. Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$. If S is bounded by the simple closed curve C, we determine the orientation of C using a right-hand rule like the one used to determine the direction of a cross product of two vectors. Align the thumb of your right hand so that it points in the direction of one of the unit normals to S. Then if you curl your fingers, they will indicate the **positive orientation** on C, as indicated in Figure 14.52a. If the orientation of C is opposite that indicated by the curling of the fingers on your right hand, as shown in Figure 14.52b, we say that C has **negative orientation.** The vector form of Green's Theorem in (8.1) generalizes as follows.

THEOREM 8.1 (Stokes' Theorem)

Suppose that S is an oriented, piecewise-smooth surface with unit normal vector $\mathbf{n}$, bounded by the simple closed, piecewise-smooth boundary curve ∂S having positive orientation. Let $\mathbf{F}(x, y, z)$ be a vector field whose components have continuous first partial derivatives in some open region containing S. Then,

$$\int_{\partial S} \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS. \tag{8.2}$$

Notice right away that the vector form of Green's Theorem (8.1) is a special case of (8.2), as follows. If S is simply a region in the xy-plane, then a unit normal to the surface at every point on S is the vector $\mathbf{n} = \mathbf{k}$. Further, $dS = dA$ (i.e., the surface area of the plane region is simply the area) and (8.2) simplifies to (8.1). The proof of Stokes' Theorem for the special case considered below hinges on Green's Theorem and the chain rule.

One important interpretation of Stokes' Theorem arises in the case where $\mathbf{F}$ represents a force field. Note that in this case, the integral on the left side of (8.2) corresponds to the work done by the force field $\mathbf{F}$ as the point of application moves along the boundary of S. Likewise, the right side of (8.2) represents the net flux of the curl of $\mathbf{F}$ over the surface S. A general proof of Stokes' Theorem can be found in more advanced texts. We prove it here only for a special case of the surface S.

PROOF (Special Case)

We consider here the special case where S is a surface of the form

$$S = \{(x, y, z) | z = f(x, y),\ \text{for } (x, y) \in R\},$$

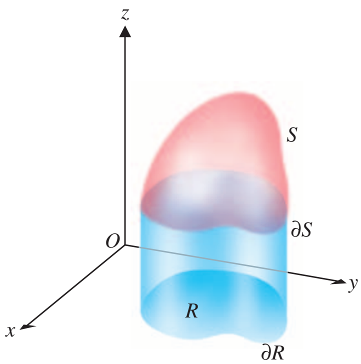


FIGURE 14.53
The surface S and its projection R onto the xy-plane

HISTORICAL NOTES

George Gabriel Stokes (1819–1903) English mathematician who published important results in the field of hydrodynamics. Many of Stokes' results, including what are now known as the Navier-Stokes equations, were independently developed but duplicated previously published European results. (Due to the lasting bitterness over the Newton-Leibniz calculus dispute, the communication of results between Europe and England was minimal.) Stokes had a long list of publications, including a paper in which he named and explained fluorescence.

where R is a region in the xy-plane with piecewise-smooth boundary ∂R, where $f(x, y)$ has continuous first partial derivatives and for which ∂R is the projection of the boundary of the surface ∂S onto the xy-plane (see Figure 14.53).

Let $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$. We then have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

Note that a normal vector at any point on S is given by

$$\mathbf{m} = \langle -f_x(x, y), -f_y(x, y), 1\rangle$$

and so, we orient S with the unit normal vector

$$\mathbf{n} = \frac{\langle -f_x(x, y), -f_y(x, y), 1\rangle}{\sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}}.$$

Since $dS = \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$, we now have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \iint_R \left[-\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) f_x - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) f_y + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right]_{z=f(x,y)} dA.$$

Equation (8.2) is now equivalent to

$$\int_{\partial S} M\, dx + N\, dy + P\, dz$$

$$= \iint_R \left[-\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) f_x - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) f_y + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right]_{z=f(x,y)} dA. \quad (8.3)$$

We will now show that

$$\int_{\partial S} M(x, y, z)\, dx = -\iint_R \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} f_y\right)_{z=f(x,y)} dA. \quad (8.4)$$

Suppose that the boundary of R is described parametrically by

$$\partial R = \{(x, y) | x = x(t), y = y(t), a \le t \le b\}.$$

Then, the boundary of S is described parametrically by

$$\partial S = \{(x, y, z) | x = x(t), y = y(t), z = f(x(t), y(t)), a \le t \le b\}$$

and we have

$$\int_{\partial S} M(x, y, z)\, dx = \int_a^b M(x(t), y(t), f(x(t), y(t)))\, x'(t)\, dt.$$

Now, notice that for $m(x, y) = M(x, y, f(x, y))$, this gives us

$$\int_{\partial S} M(x, y, z)\, dx = \int_a^b m(x(t), y(t))\, x'(t)\, dt = \int_{\partial R} m(x, y)\, dx. \quad (8.5)$$

From Green's Theorem, we know that

$$\int_{\partial R} m(x, y)\,dx = -\iint_R \frac{\partial m}{\partial y} dA. \tag{8.6}$$

However, from the chain rule,

$$\frac{\partial m}{\partial y} = \frac{\partial}{\partial y} M(x, y, f(x, y)) = \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} f_y\right)_{z=f(x,y)}.$$

Putting this together with (8.5) and (8.6) gives us

$$\int_{\partial S} M(x, y, z)\,dx = -\iint_R \frac{\partial m}{\partial y}\,dA = -\iint_R \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} f_y\right)_{z=f(x,y)} dA,$$

which is (8.4). Similarly, you can show that

$$\int_{\partial S} N(x, y, z)\,dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial N}{\partial z} f_x\right)_{z=f(x,y)} dA \tag{8.7}$$

and

$$\int_{\partial S} P(x, y, z)\,dz = \iint_R \left(\frac{\partial P}{\partial x} f_y - \frac{\partial P}{\partial y} f_x\right)_{z=f(x,y)} dA. \tag{8.8}$$

Putting together (8.4), (8.7) and (8.8) now gives us (8.3), which proves Stokes' Theorem for this special case of the surface. ■

EXAMPLE 8.1 Using Stokes' Theorem to Evaluate a Line Integral

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, for $\mathbf{F}(x, y, z) = \langle -y, x^2, z^3 \rangle$, where C is the intersection of the circular cylinder $x^2 + y^2 = 4$ and the plane $x + z = 3$, oriented so that it is traversed counterclockwise when viewed from high up on the positive z-axis.

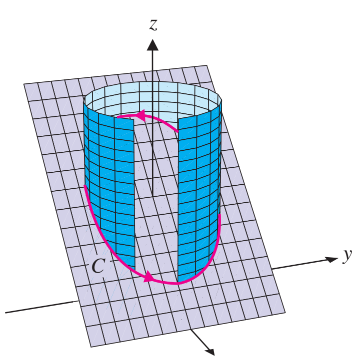


FIGURE 14.54
Intersection of the plane and the cylinder producing the curve C

Solution First, notice that C is an ellipse, as indicated in Figure 14.54. Unfortunately, C is rather difficult to parameterize, which makes the direct evaluation of the line integral somewhat difficult. Instead, we can use Stokes' Theorem to evaluate the integral. First, we calculate the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x^2 & z^3 \end{vmatrix} = (2x + 1)\,\mathbf{k}.$$

Notice that on the surface S, consisting of the portion of the plane $x + z = 3$ enclosed by C, we have the unit normal vector

$$\mathbf{n} = \frac{1}{\sqrt{2}} \langle 1, 0, 1 \rangle.$$

From Stokes' Theorem, we now have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\,dS = \iint_R \underbrace{\frac{1}{\sqrt{2}}(2x + 1)}_{(\nabla \times \mathbf{F}) \cdot \mathbf{n}} \underbrace{\sqrt{2}\,dA}_{dS},$$

where R is the disk of radius 2, centered at the origin (i.e., the projection of S onto the xy-plane). Introducing polar coordinates, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R (2x+1)\,dA = \int_0^{2\pi}\int_0^2 (2r\cos\theta+1)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^2 (2r^2\cos\theta + r)\,dr\,d\theta\\ &= \int_0^{2\pi}\left(2\frac{r^3}{3}\cos\theta + \frac{r^2}{2}\right)\Bigg|_{r=0}^{r=2} d\theta\\ &= \int_0^{2\pi}\left(\frac{16}{3}\cos\theta + 2\right)d\theta = 4\pi,\end{aligned}$$

where we have left the final details of the calculation to you. ■

EXAMPLE 8.2 Using Stokes' Theorem to Evaluate a Surface Integral

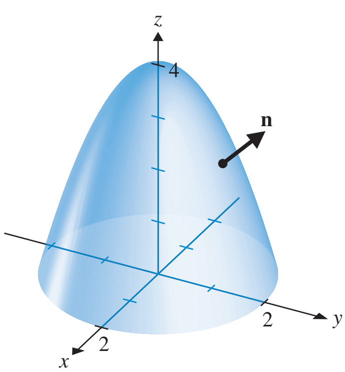


FIGURE 14.55
$z = 4 - x^2 - y^2$

Evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = \langle e^{z^2}, 4z - y, 8x\sin y\rangle$ and where S is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane, oriented so that the unit normal vectors point to the outside of the paraboloid, as indicated in Figure 14.55.

Solution Notice that the boundary curve is simply the circle $x^2 + y^2 = 4$ lying in the xy-plane. By Stokes' Theorem, we then have

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \int_{\partial S}\mathbf{F}(x,y,z)\cdot d\mathbf{r}\\ &= \int_{\partial S} e^{z^2}\,dx + (4z-y)\,dy + 8x\sin y\,dz.\end{aligned}$$

Now, we can parameterize ∂S by $x = 2\cos t$, $y = 2\sin t$, $z = 0$, $0 \le t \le 2\pi$. This says that on ∂S, we have $dx = -2\sin t$, $dy = 2\cos t$ and $dz = 0$. In view of this, we have

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \int_{\partial S} e^{z^2}\,dx + (4z-y)\,dy + 8x\sin y\,dz\\ &= \int_0^{2\pi}\{e^0(-2\sin t) + [4(0) - 2\sin t](2\cos t)\}\,dt = 0,\end{aligned}$$

where we leave the (straightforward) details of the calculation to you. ■

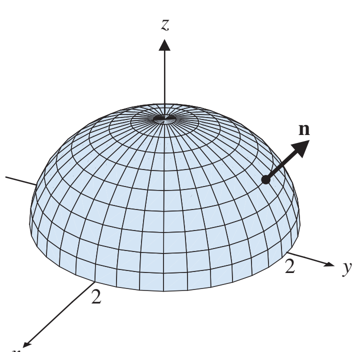


FIGURE 14.56
$z = \sqrt{4 - x^2 - y^2}$

In example 8.3, we consider the same surface integral as in example 8.2, but over a different surface. Although the surfaces are different, they have the same boundary curve, so that they must have the same value.

EXAMPLE 8.3 Using Stokes' Theorem to Evaluate a Surface Integral

Evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = \langle e^{z^2}, 4z - y, 8x\sin y\rangle$ and where S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$, oriented so that the unit normal vectors point to the outside of the hemisphere, as indicated in Figure 14.56.

Solution Notice that although this is not the same surface as in example 8.2, the two surfaces have the same boundary curve, the circle $x^2 + y^2 = 4$ lying in the xy-plane, and the same orientation. Just as in example 8.2, we then have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 0.$$

Much as we used the Divergence Theorem in section 14.7 to give an interpretation of the meaning of the divergence of a vector field, we can use Stokes' Theorem to give some meaning to the curl of a vector field. Suppose once again that $\mathbf{F}(x, y, z)$ represents the velocity field for a fluid in motion and let C be an oriented closed curve in the domain of $\mathbf{F}$, traced out by the endpoint of the vector-valued function $\mathbf{r}(t)$ for $a \le t \le b$. Notice that the closer the direction of $\mathbf{F}$ is to the direction of $\frac{d\mathbf{r}}{dt}$, the larger its component is in the direction of $\frac{d\mathbf{r}}{dt}$ (see Figure 14.57). In other words, the closer the direction of $\mathbf{F}$ is to the direction of $\frac{d\mathbf{r}}{dt}$, the larger $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ will be. Now, recall that $\frac{d\mathbf{r}}{dt}$ points in the direction of the unit tangent vector along C. Then, since

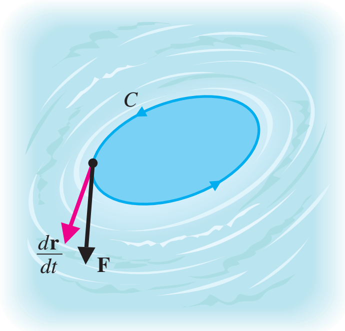


FIGURE 14.57
The surface S in a fluid flow

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt,$$

it follows that the closer the direction of $\mathbf{F}$ is to the direction of $\frac{d\mathbf{r}}{dt}$ along C, the larger $\int_C \mathbf{F} \cdot d\mathbf{r}$ will be. This says that $\int_C \mathbf{F} \cdot d\mathbf{r}$ measures the tendency of the fluid to flow around or *circulate* around C. For this reason, we refer to $\int_C \mathbf{F} \cdot d\mathbf{r}$ as the **circulation** of $\mathbf{F}$ around C.

For any point (x_0, y_0, z_0) in the fluid flow, let S_a be a disk of radius a centered at (x_0, y_0, z_0), with unit normal vector $\mathbf{n}$, as indicated in Figure 14.58 and let C_a be the (positively oriented) boundary of S_a. Then, by Stokes' Theorem, we have

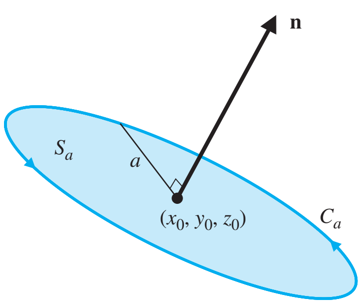


FIGURE 14.58
The disk S_a

$$\int_{C_a} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS. \tag{8.9}$$

Notice that the average value of a function f on the surface S_a is given by

$$f_{\text{ave}} = \frac{1}{\pi a^2} \iint_{S_a} f(x, y, z)\, dS.$$

Further, if f is continuous on S_a, there must be some point P_a on S_a at which f equals its average value, that is, where

$$f(P_a) = \frac{1}{\pi a^2} \iint_{S_a} f(x, y, z)\, dS.$$

In particular, if the velocity field $\mathbf{F}$ has continuous first partial derivatives throughout S_a, then it follows from equation (8.9) that for some point P_a on S_a,

$$(\nabla \times \mathbf{F})(P_a) \cdot \mathbf{n} = \frac{1}{\pi a^2} \iint_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \frac{1}{\pi a^2} \int_{C_a} \mathbf{F} \cdot d\mathbf{r}. \tag{8.10}$$

TODAY IN MATHEMATICS

Cathleen Synge Morawetz (1923–) a Canadian mathematician whose work on transonic flows greatly influenced the design of air foils. With similar methods, she made fundamental contributions to a variety of physically important questions about wave propagation. Her mother was a mathematics teacher for a few years and her father, John Lighton Synge, was Ireland's most distinguished mathematician of the twentieth century. However, her mathematics career was almost derailed by World War II and a culture in which she felt that, "It really was considered very bad form for a woman to be overly ambitious." Fortunately, she overcame both social and mathematical obstacles with a work ethic that would not let a problem go until it was solved. She has served as Director of the Courant Institute and President of the American Mathematical Society, and is a winner of the United States' National Medal of Science.

Notice that the expression on the far right of (8.10) corresponds to the circulation of $\mathbf{F}$ around C_a per unit area. Taking the limit as $a \to 0$, we have by the continuity of curl $\mathbf{F}$ that

$$(\nabla \times \mathbf{F})(x_0, y_0, z_0) \cdot \mathbf{n} = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{F} \cdot d\mathbf{r}. \tag{8.11}$$

Read equation (8.11) very carefully. Notice that it says that at any given point, the component of curl $\mathbf{F}$ in the direction of $\mathbf{n}$ is the limiting value of the circulation per unit area around circles of radius a centered at that point (and normal to $\mathbf{n}$), as the radius a tends to zero. In this sense, $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ measures the tendency of the fluid to rotate about an axis aligned with the vector $\mathbf{n}$. You can visualize this by thinking of a small paddle wheel with axis parallel to $\mathbf{n}$, which is immersed in the fluid flow. (See Figure 14.59.) Notice that the circulation per unit area is greatest (so that the paddle wheel moves fastest) when $\mathbf{n}$ points in the direction of $\nabla \times \mathbf{F}$.

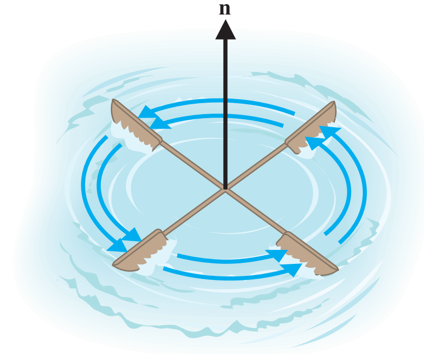


FIGURE 14.59
Paddle wheel

If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in a fluid flow, we say that the flow is **irrotational,** since the circulation about every point is zero. In particular, notice that if the velocity field $\mathbf{F}$ is a constant vector throughout the fluid flow, then

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0},$$

everywhere in the fluid flow and so, the flow is irrotational. Physically, this says that there are no eddies in such a flow.

Notice, too, that by Stokes' Theorem, if curl $\mathbf{F} = \mathbf{0}$ at every point in some open region D, then we must have that for every simple closed curve C that is the boundary of an oriented surface contained in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

In other words, the circulation is zero around every such curve C lying in the region D. It turns out that by suitably restricting the type of regions $D \subset \mathbb{R}^3$ we consider, we can show that the circulation is zero around every simple closed curve contained in D. (The converse of this is also true. That is, if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for every simple closed curve C contained in the region D, then we must have that curl $\mathbf{F} = \mathbf{0}$ at every point in D.) To obtain this result, we consider regions in space that are simply-connected. Recall that in the plane a region is said to be simply-connected whenever every closed curve contained in the region encloses only points in the region (that is, the region contains no holes). In three dimensions, the situation is slightly more complicated. A region D in $\mathbb{R}^3$ is called **simply-connected** whenever every simple closed curve C lying in D can be continuously shrunk to a point without crossing the boundary of D. For instance, notice that the interior of a sphere or a rectangular box is simply-connected, but a region with a hole drilled through it is not simply-connected. Be careful not to confuse connected with simply-connected. Recall that a connected region is one where every two points contained in the region can be connected with a path that is completely contained in the region. We illustrate connected and simply-connected two-dimensional regions in Figures 14.60a to 14.60c. We can now state the complete theorem.

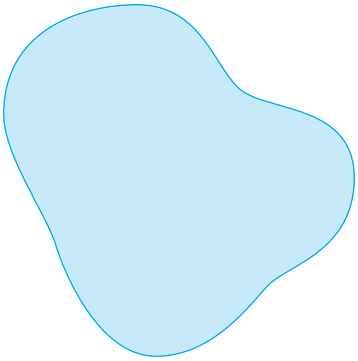


FIGURE 14.60a
Connected and simply-connected

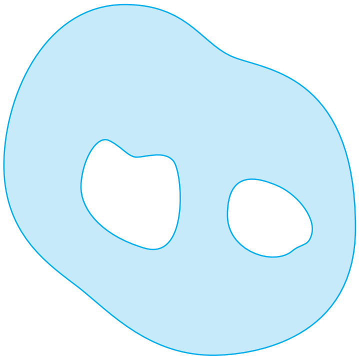


FIGURE 14.60b
Connected but not simply-connected

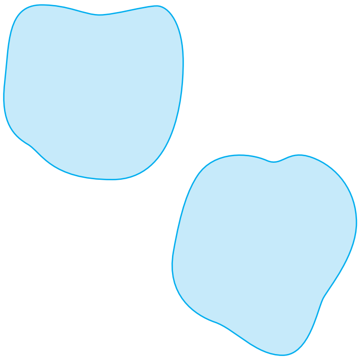


FIGURE 14.60c
Simply-connected but not connected

THEOREM 8.2

Suppose that $\mathbf{F}(x, y, z)$ is a vector field whose components have continuous first partial derivatives throughout the simply-connected open region $D \subset \mathbb{R}^3$. Then, curl $\mathbf{F} = \mathbf{0}$ in D if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for every simple closed curve C contained in the region D.

PROOF (Necessity)

We have already suggested that when curl $\mathbf{F} = \mathbf{0}$ in an open, simply-connected region D, it can be shown that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for every simple closed curve C contained in the region D (although the proof is beyond the level of this text). Conversely, suppose now that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed curve C contained in the region D and assume that curl $\mathbf{F} \neq \mathbf{0}$ at some point $(x_0, y_0, z_0) \in D$. Since the components of $\mathbf{F}$ have continuous first partial derivatives, curl $\mathbf{F}$ must be continuous in D and so, there must be a sphere of radius $a_0 > 0$, contained in D and centered at (x_0, y_0, z_0), throughout whose interior S, curl $\mathbf{F} \neq \mathbf{0}$ and curl $\mathbf{F}\ (x, y, z) \cdot$ curl $\mathbf{F}(x_0, y_0, z_0) > 0$. (Note that this is possible since curl $\mathbf{F}$ is continuous and curl $\mathbf{F}\ (x_0, y_0, z_0) \cdot$ curl $\mathbf{F}(x_0, y_0, z_0) > 0$.) Let S_a be the disk of radius $a < a_0$ centered at (x_0, y_0, z_0) and oriented by the unit normal vector $\mathbf{n}$ having the same direction as curl $\mathbf{F}\ (x_0, y_0, z_0)$. Notice that since $a < a_0$, S_a will be completely contained in S as illustrated in Figure 14.61. If C_a is the boundary of S_a, then we have by Stokes' Theorem that

$$\int_{C_a} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_a} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS > 0,$$

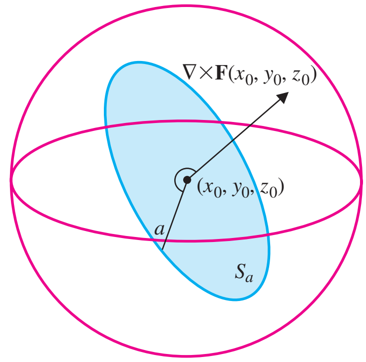


FIGURE 14.61
The disk S_a

since $\mathbf{n}$ was chosen to be parallel to $\nabla \times \mathbf{F}(x_0, y_0, z_0)$. This contradicts the assumption that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for every simple closed curve C contained in the region D. It now follows that curl $\mathbf{F} = \mathbf{0}$ throughout D. ■

Recall that we had observed earlier that a vector field is conservative in a given region if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for every simple closed curve C contained in the region. Theorem 8.2 has then established the following results.

THEOREM 8.3

Suppose that $\mathbf{F}(x, y, z)$ has continuous first partial derivatives in a simply-connected region D. Then, the following statements are equivalent.

(i) $\mathbf{F}$ is conservative in D. That is, for some scalar function $f(x, y, z)$, $\mathbf{F} = \nabla f$;
(ii) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D;
(iii) $\mathbf{F}$ is irrotational (i.e., curl $\mathbf{F} = \mathbf{0}$) in D; and
(iv) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for every simple closed curve C contained in D.

We close this section with a simple application of Stokes' Theorem.

EXAMPLE 8.4 Finding the Flux of a Magnetic Field

Use Stokes' Theorem and Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ to show that the flux of a magnetic field $\mathbf{B}$ across a surface S satisfying the hypotheses of Stokes' Theorem equals the circulation of $\mathbf{A}$ around ∂S, where $\mathbf{B} = \nabla \times \mathbf{A}$.

Solution The flux of $\mathbf{B}$ across S is given by $\iint_S \mathbf{B} \cdot \mathbf{n}\, dS$. Since $\nabla \cdot \mathbf{B} = 0$, it follows from exercise 66 in section 14.5 that there exists a vector field $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. We can now rewrite the flux of $\mathbf{B}$ across S as $\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS$. Applying Stokes' Theorem gives us

$$\iint_S \mathbf{B} \cdot \mathbf{n}\, dS = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{r}.$$

You should recognize the line integral on the right side as the circulation of $\mathbf{A}$ around ∂S, as desired. ■

EXERCISES 14.8

WRITING EXERCISES

1. Describe circumstances (e.g., example 8.1) in which the surface integral of Stokes' Theorem will be simpler than the line integral.

2. Describe circumstances (e.g., example 8.2) in which the line integral of Stokes' Theorem will be simpler than the surface integral.

3. The surfaces in example 8.2 and 8.3 have the same boundary curve C. Explain why all surfaces above the xy-plane with the boundary C will share the same value of

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

What would change if the surface were below the xy-plane?

4. Explain why part (iv) of Theorem 8.3 follows immediately from part (ii). Explain why parts (ii) and (iii) follow immediately from part (i).

In exercises 1–4, verify Stokes' Theorem by computing both integrals.

1. S is the portion of $z = 4 - x^2 - y^2$ above the xy-plane, $\mathbf{F} = \langle zx, 2y, z^3 \rangle$.

2. S is the portion of $z = 1 - x^2 - y^2$ above the xy-plane, $\mathbf{F} = \langle x^2 z, xy, xz^2 \rangle$.

3. S is the portion of $z = \sqrt{4 - x^2 - y^2}$ above the xy-plane, $\mathbf{F} = \langle 2x - y, yz^2, y^2 z \rangle$.

4. S is the portion of $z = \sqrt{1 - x^2 - y^2}$ above the xy-plane, $\mathbf{F} = \langle 2x, z^2 - x, xz^2 \rangle$.

In exercises 5–14, use Stokes' Theorem to compute $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

5. S is the portion of the tetrahedron bounded by $x + y + 2z = 2$ and the coordinate planes with $z > 0$, $\mathbf{n}$ upward, $\mathbf{F} = \langle zy^4 - y^2, y - x^3, z^2 \rangle$.

6. S is the portion of the tetrahedron bounded by $x + y + 4z = 8$ and the coordinate planes with $z > 0$, $\mathbf{n}$ upward, $\mathbf{F} = \langle y^2, y + 2x, z^2 \rangle$.

7. S is the portion of $z = 1 - x^2 - y^2$ above the xy-plane with $\mathbf{n}$ upward, $\mathbf{F} = \left\langle zx^2, ze^{xy^2} - x, x \ln y^2 \right\rangle$.

8. S is the portion of $z = \sqrt{4 - x^2 - y^2}$ above the xy-plane with $\mathbf{n}$ upward, $\mathbf{F} = \left\langle zx^2, ze^{xy^2} - x, x \ln y^2 \right\rangle$.

9. S is the portion of the tetrahedron in exercise 5 with $y > 0$, $\mathbf{n}$ to the right, $\mathbf{F} = \langle zy^4 - y^2, y - x^3, z^2 \rangle$.

10. S is the portion of $y = x^2 + z^2$ with $y \le 2$, $\mathbf{n}$ to the left, $\mathbf{F} = \left\langle xy, 4xe^{z^2}, yz + 1 \right\rangle$.

11. S is the portion of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ with $z < 1$, $\mathbf{n}$ upward, $\mathbf{F} = \langle xyz, 4x^2y^3 - z, 8\cos xz^2 \rangle$.

12. S is the portion of the unit cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ with $z < 1$, $\mathbf{n}$ downward, $\mathbf{F} = \langle xyz, 4x^2y^3 - z, 8\cos xz^2 \rangle$.

13. S is the portion of the cone $z = \sqrt{x^2 + y^2}$ below the sphere $x^2 + y^2 + z^2 = 2$, $\mathbf{n}$ downward, $\mathbf{F} = \left\langle x^2 + y^2, ze^{x^2+y^2}, e^{x^2+z^2} \right\rangle$.

14. S is the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 2$, $\mathbf{n}$ downward, $\mathbf{F} = \left\langle zx, x^2 + y^2, z^2 - y^2 \right\rangle$.

In exercises 15–24, use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

15. C is the boundary of the portion of the paraboloid $y = 4 - x^2 - z^2$ with $y > 0$, $\mathbf{n}$ to the right, $\mathbf{F} = \langle x^2 z, 3\cos y, 4z^3 \rangle$.

16. C is the boundary of the portion of the paraboloid $x = y^2 + z^2$ with $x \le 4$, $\mathbf{n}$ to the back, $\mathbf{F} = \langle yz, y - 4, 2xy \rangle$.

17. C is the boundary of the portion of $z = 4 - x^2 - y^2$ above the xy-plane, oriented upward, $\mathbf{F} = \left\langle x^2 e^x - y, \sqrt{y^2 + 1}, z^3 \right\rangle$.

18. C is the boundary of the portion of $z = x^2 + y^2$ below $z = 4$, oriented downward, $\mathbf{F} = \langle x^2, y^4 - x, z^2 \sin z \rangle$.

19. C is the intersection of $z = x^2 + y^2$ and $z = 8 - y$, oriented clockwise from above, $\mathbf{F} = \left\langle 2x^2, 4y^2, e^{8z^2} \right\rangle$.

20. C is the intersection of $x^2 + y^2 = 1$ and $z = x - y$, oriented clockwise from above, $\mathbf{F} = \langle \cos x^2, \sin y^2, \tan z^2 \rangle$.

21. C is the triangle from $(0, 1, 0)$ to $(0, 0, 4)$ to $(2, 0, 0)$, $\mathbf{F} = \langle x^2 + 2xy^3 z, 3x^2y^2z - y, x^2y^3 \rangle$.

22. C is the square from $(0, 2, 2)$ to $(2, 2, 2)$ to $(2, 2, 0)$ to $(0, 2, 0)$, $\mathbf{F} = \langle x^2, y^3 + x, 3y^2 \cos z \rangle$.

23. C is the intersection of $z = 4 - x^2 - y^2$ and $x^2 + z^2 = 1$ with $y > 0$, oriented clockwise as viewed from the right, $\mathbf{F} = \left\langle x^2 + 3y, \cos y^2, z^3 \right\rangle$.

24. C is the intersection of $z = x^2 + y^2 - 4$ and $z = y - 1$, oriented clockwise as viewed from above, $\mathbf{F} = \langle \sin x^2, y^3, z \ln z - x \rangle$.

25. Show that $\oint_C (f\,\nabla f)\cdot d\mathbf{r} = 0$ for any simple closed curve C and differentiable function f.

26. Show that $\oint_C (f\,\nabla g + g\nabla f)\cdot d\mathbf{r} = 0$ for any simple closed curve C and differentiable functions f and g.

27. Let $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$ be a vector field whose components M and N have continuous first partial derivatives in all of $\mathbb{R}^2$. Show that $\nabla\cdot\mathbf{F} = 0$ if and only if $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$ for all simple closed curves C. (Hint: Use a vector form of Green's Theorem.)

28. Under the assumptions of exercise 27, show that $\int_C \mathbf{F}\cdot\mathbf{n}\,ds = 0$ for all simple closed curves C if and only if $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ is path-independent.

29. Under the assumptions of exercise 27, show that $\nabla\cdot\mathbf{F} = 0$ if and only if $\mathbf{F}$ has a **stream function** $g(x, y)$ such that $M(x, y) = g_y(x, y)$ and $N(x, y) = -g_x(x, y)$.

30. Combine the results of exercises 27–29 to state a two-variable theorem analogous to Theorem 8.3.

31. If S_1 and S_2 are surfaces that satisfy the hypotheses of Stokes' Theorem and that share the same boundary curve, under what circumstances can you conclude that

$$\iint_{S_1} (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \iint_{S_2} (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS?$$

32. Give an example where the two surface integrals of exercise 31 are not equal.

33. Use Stokes' Theorem to verify that

$$\oint_C (f\nabla g)\cdot d\mathbf{r} = \iint_S (\nabla f\times\nabla g)\cdot\mathbf{n}\,dS,$$

where C is the positively oriented boundary of the surface S.

34. Use Stokes' Theorem to verify that $\oint_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = 0$, where C is the positively oriented boundary of some surface S.

EXPLORATORY EXERCISES

1. The **circulation** of a vector field $\mathbf{F}$ around the curve C is defined by $\int_C \mathbf{F}\cdot d\mathbf{r}$. Show that the curl $\nabla\times\mathbf{F}(0, 0, 0)$ is in the same direction as the normal to the plane in which the circulation per unit area around the origin is a maximum as the area around the origin goes to 0. Relate this to the interpretation of the curl given in section 14.5.

2. The Fundamental Theorem of Calculus can be viewed as relating the values of the function on the boundary of a region (interval) to the sum of the derivative values of the function within the region. Explain what this statement means and then explain why the same statement can be applied to Theorem 3.2, Green's Theorem, the Divergence Theorem and Stokes' Theorem. In each case, carefully state what the "region" is, what its boundary is and what type derivative is involved.

14.9 APPLICATIONS OF VECTOR CALCULUS

Through the past eight sections, we have developed a powerful set of tools for analyzing vector quantities. You can now compute flux integrals and line integrals for work and circulation, and you have the Divergence Theorem and Stokes' Theorem to relate these quantities to one another. To this point in the text, we have emphasized the conceptual and computational aspects of vector analysis. In this section, we present a small selection of applications from fluid mechanics and electricity and magnetism. As you work through the examples in this section, notice that we are using vector calculus to derive general results that can be applied to any specific vector field that you may run across in an application.

Our first example is similar to example 7.4, which concerns magnetic fields. Here, we also apply Stokes' Theorem to derive a second result.

EXAMPLE 9.1 Finding the Flux of a Velocity Field

Suppose that the velocity field $\mathbf{v}$ of a fluid has a vector potential $\mathbf{w}$, that is, $\mathbf{v} = \nabla\times\mathbf{w}$. Show that $\mathbf{v}$ is incompressible and that the flux of $\mathbf{v}$ across any closed surface is 0. Also, show that if a closed surface S is partitioned into surfaces S_1 and S_2 (that is, $S = S_1\cup S_2$ and $S_1\cap S_2 = \emptyset$), then the flux of $\mathbf{v}$ across S_1 is the additive inverse of the flux of $\mathbf{v}$ across S_2.

Solution To show that $\mathbf{v}$ is incompressible, compute $\nabla \cdot \mathbf{v} = \nabla \cdot (\nabla \times \mathbf{w}) = 0$, since the divergence of the curl of *any* vector field is zero. Next, suppose that the closed surface S is the boundary of the solid Q. Then from the Divergence Theorem, we have

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dS = \iiint_Q \nabla \cdot \mathbf{v}\, dV = \iiint_Q 0\, dV = 0.$$

Finally, since $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, we have

$$\iint_{S_1} \mathbf{v} \cdot \mathbf{n}\, dS + \iint_{S_2} \mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \mathbf{v} \cdot \mathbf{n}\, dS = 0,$$

so that

$$\iint_{S_1} \mathbf{v} \cdot \mathbf{n}\, dS = -\iint_{S_2} \mathbf{v} \cdot \mathbf{n}\, dS.$$

The general result shown in example 9.1 also has practical implications for computing integrals. One use of this result is given in example 9.2.

EXAMPLE 9.2 Computing a Surface Integral Using the Complement of the Surface

Find the flux of the vector field $\nabla \times \mathbf{F}$ across S, where $\mathbf{F} = \langle e^{x^2} - 2xy, \sin y^2, 3yz - 2x \rangle$ and S is the portion of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ above the xy-plane.

Solution We have several options for computing $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. Notice that the surface S consists of five faces of the cube, so five separate surface integrals would be required to compute it directly. We could use Stokes' Theorem and rewrite it as $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the square boundary of the open face of S. However, this would require four line integrals involving a complicated vector field $\mathbf{F}$. Example 9.1 gives us a third option: the flux over the entire cube is zero, so that the flux over S is the additive inverse of the flux over the missing side of the cube. Notice that the (outward) normal vector for this side is $\mathbf{n} = -\mathbf{k}$, and the curl of $\mathbf{F}$ is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{x^2} - 2xy & \sin y^2 & 3yz - 2x \end{vmatrix}$$
$$= \mathbf{i}(3z - 0) - \mathbf{j}(-2 - 0) + \mathbf{k}(0 + 2x) = 3z\mathbf{i} + 2\mathbf{j} + 2x\mathbf{k}.$$

So,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (3z\mathbf{i} + 2\mathbf{j} + 2x\mathbf{k}) \cdot (-\mathbf{k}) = -2x$$

and $dS = dA$. Taking S_2 as the bottom face of the cube, we now have that the flux is given by

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = -\iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = -\iint_R -2x\, dA = \int_0^1 \int_0^1 2x\, dx\, dy = 1.$$

One very important use of the Divergence Theorem and Stokes' Theorem is in deriving certain fundamental equations in physics and engineering. The technique we use here to derive the **heat equation** is typical of the use of these theorems. In this technique, we start with two different descriptions of the same quantity and use the vector calculus to draw conclusions about the functions involved.

For the heat equation, we analyze the amount of heat per unit time leaving a solid Q. Recall from example 6.7 that the net heat flow out of Q is given by $\iint_S (-k\nabla T)\cdot \mathbf{n}\, dS$, where S is a closed surface bounding Q, T is the temperature function, $\mathbf{n}$ is the outward unit normal and k is a constant (called the heat conductivity). Alternatively, physics tells us that the total heat within Q equals $\iiint_Q \rho\sigma T\, dV$, where ρ is the (constant) density and σ is the **specific heat** of the solid. From this, it follows that the heat flow out of Q is given by $-\frac{d}{dt}\left[\iiint_Q \rho\sigma T\, dV\right]$. Notice that the negative sign is needed to give us the heat flow *out of* the region Q. If the temperature function T has a continuous partial derivative with respect to t, we can bring the derivative inside the integral and write this as $-\iiint_Q \rho\sigma \frac{\partial T}{\partial t} dV$. Equating these two expressions for the heat flow out of Q, we have

$$\iint_S (-k\nabla T)\cdot \mathbf{n}\, dS = -\iiint_Q \rho\sigma \frac{\partial T}{\partial t}\, dV. \tag{9.1}$$

EXAMPLE 9.3 Deriving the Heat Equation

Use the Divergence Theorem and equation (9.1) to derive the heat equation $\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T$, where $\alpha^2 = \frac{k}{\rho\sigma}$ and $\nabla^2 T = \nabla\cdot(\nabla T)$ is the Laplacian of T.

Solution Applying the Divergence Theorem to the left-hand side of equation (9.1), we have

$$\iiint_Q \nabla\cdot(-k\nabla T)\, dV = -\iiint_Q \rho\sigma \frac{\partial T}{\partial t}\, dV.$$

Combining the preceding two integrals, we get

$$\begin{aligned} 0 &= \iiint_Q -k\nabla\cdot(\nabla T)\, dV + \iiint_Q \rho\sigma \frac{\partial T}{\partial t}\, dV \\ &= \iiint_Q \left(-k\nabla^2 T + \rho\sigma \frac{\partial T}{\partial t}\right) dV. \end{aligned} \tag{9.2}$$

Observe that the only way for the integral in (9.2) to be zero for *every* solid Q is for the integrand to be zero. (Think about this carefully; you can let Q be a small sphere around any point you like.) That is,

$$0 = -k\nabla^2 T + \rho\sigma \frac{\partial T}{\partial t}$$

or

$$\rho\sigma \frac{\partial T}{\partial t} = k\nabla^2 T.$$

Finally, dividing both sides by $\rho\sigma$ gives us

$$\frac{\partial T}{\partial t} = \frac{k}{\rho\sigma}\nabla^2 T = \alpha^2\nabla^2 T,$$

as desired. ■

We next derive a fundamental result in the study of fluid dynamics, diffusion theory and electricity and magnetism. We consider a fluid that has density function ρ (in general, ρ is a scalar function of space and time). We also assume that the fluid has velocity field $\mathbf{v}$ and that there are no sources or sinks. Since the total mass of fluid contained in a given region Q is given by the triple integral $m = \iiint_Q \rho(x, y, z, t)\, dV$, the rate of change of the mass is given by

$$\frac{dm}{dt} = \frac{d}{dt}\left[\iiint_Q \rho(x, y, z, t)\, dV\right] = \iiint_Q \frac{\partial\rho}{\partial t}(x, y, z, t)\, dV, \tag{9.3}$$

assuming that the density function has a continuous partial derivative with respect to t, so that we can bring the derivative inside the integral. Now, look at the same problem in a different way. In the absence of sources or sinks, the only way for the mass inside Q to change is for fluid to cross the boundary ∂Q. That is, the rate of change of mass is the additive inverse of the flux of the velocity field across the boundary of Q. (You will be asked in the exercises to explain the negative sign. Think about why it needs to be there!) So, we also have

$$\frac{dm}{dt} = -\iint_{\partial Q} (\rho\mathbf{v})\cdot\mathbf{n}\, dS. \tag{9.4}$$

Given these alternative representations of the rate of change of mass, we derive the *continuity equation* in example 9.4.

EXAMPLE 9.4 Deriving the Continuity Equation

Use the Divergence Theorem and equations (9.3) and (9.4) to derive the ***continuity equation:*** $\nabla\cdot(\rho\mathbf{v}) + \dfrac{\partial\rho}{\partial t} = 0.$

Solution We start with equal expressions for the rate of change of mass in a generic solid Q, given in (9.3) and (9.4). We have

$$\iiint_Q \frac{\partial\rho}{\partial t}(x, y, z, t)\, dV = -\iint_{\partial Q} (\rho\mathbf{v})\cdot\mathbf{n}\, dS.$$

Applying the Divergence Theorem to the right-hand side gives us

$$\iiint_Q \frac{\partial\rho}{\partial t}(x, y, z, t)\, dV = -\iiint_Q \nabla\cdot(\rho\mathbf{v})\, dV.$$

Combining the two integrals, we have

$$0 = \iiint_Q \nabla \cdot (\rho \mathbf{v})\, dV + \iiint_Q \frac{\partial \rho}{\partial t}(x, y, z, t)\, dV$$
$$= \iiint_Q \left[\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} \right] dV.$$

Since this equation must hold for *all* solids Q, the integrand must be zero. That is,

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0,$$

which is the continuity equation, as desired. ■

Bernoulli's Theorem is often used to explain the lift force of a curved airplane wing. This result relates the speed and pressure in a steady fluid flow. (Here, steady means that the fluid's velocity, pressure etc., do not change with time.) The starting point for our derivation is Euler's equation for steady flow. In this case, a fluid moves with velocity $\mathbf{u}$ and vorticity $\mathbf{w}$ through a medium with density ρ and the speed is given by $u = \|\mathbf{u}\|$. We consider the case where there is an external force, such as gravity, with a potential function ϕ and where the fluid pressure is given by the scalar function p. Since the flow is steady, all quantities are functions of position (x, y, z), but not time. In this case, **Euler's equation** states that

$$\mathbf{w} \times \mathbf{u} + \frac{1}{2}\nabla u^2 = -\frac{1}{\rho}\nabla p - \nabla \phi. \tag{9.5}$$

Bernoulli's Theorem then says that $\frac{1}{2}u^2 + \phi + \frac{p}{\rho}$ is constant along flow lines. A more precise formula is given in the derivation in example 9.5.

EXAMPLE 9.5 Deriving Bernoulli's Theorem

Use Euler's equation (9.5) to derive Bernoulli's Theorem.

Solution Recall that the flow lines are tangent to the velocity field. So, to compute the component of a vector function along a flow line, you start by finding the dot product of the function with velocity. In this case, we take Euler's equation and find the dot product of each term with $\mathbf{u}$. We get

$$\mathbf{u} \cdot (\mathbf{w} \times \mathbf{u}) + \mathbf{u} \cdot \left(\frac{1}{2}\nabla u^2\right) = -\mathbf{u} \cdot \left(\frac{1}{\rho}\nabla p\right) - \mathbf{u} \cdot \nabla \phi$$

or

$$\mathbf{u} \cdot (\mathbf{w} \times \mathbf{u}) + \mathbf{u} \cdot \left(\frac{1}{2}\nabla u^2\right) + \mathbf{u} \cdot (\nabla \phi) + \mathbf{u} \cdot \frac{1}{\rho}\nabla p = 0.$$

Notice that $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{u}) = 0$, since the cross product $\mathbf{w} \times \mathbf{u}$ is perpendicular to $\mathbf{u}$. All three remaining terms are gradients, so factoring out the scalar functions involved, we have

$$\mathbf{u} \cdot \nabla \left(\frac{1}{2}u^2 + \phi + \frac{p}{\rho}\right) = 0.$$

This says that the component of $\nabla \left(\frac{1}{2}u^2 + \phi + \frac{p}{\rho}\right)$ along $\mathbf{u}$ is zero. So, the directional derivative of $\frac{1}{2}u^2 + \phi + \frac{p}{\rho}$ is zero in the direction of the tangent to the flow lines. This gives us Bernoulli's Theorem, that $\frac{1}{2}u^2 + \phi + \frac{p}{\rho}$ is constant along flow lines. ■

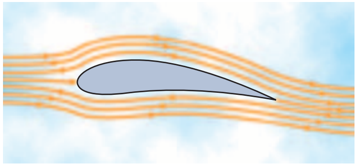


FIGURE 14.62
Cross section of a wing

Consider now what Bernoulli's Theorem means in the case of steady airflow around an airplane wing. Since the wing is curved on top (see Figure 14.62), the air flowing across the top must have a greater speed. From Bernoulli's Theorem, the quantity $\frac{1}{2}u^2 + \phi + \frac{p}{\rho}$ is constant along flow lines, so an increase in speed must be compensated for by a decrease in pressure. Due to the lower pressure on top, the wing experiences a lift force. Of course, airflow around an airplane wing is more complicated than this. The interaction of the air with the wing itself (the boundary layer) is quite complicated and determines many of the flight characteristics of a wing. Still, Bernoulli's Theorem gives us some insight into why a curved wing produces a lift force.

Maxwell's equations are a set of four equations relating the fundamental vector fields of electricity and magnetism. From these equations, you can derive many more important relationships. Taken together, Maxwell's equations give a concise statement of the fundamentals of electricity and magnetism. The equations can be written in different ways, depending on whether the integral or differential form is given and whether magnetic or polarizable media are included. Also, you may find that different texts refer to these equations by different names. Listed below are Maxwell's equations in differential form in the absence of magnetic or polarizable media.

MAXWELL'S EQUATIONS

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss' Law for electricity)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss' Law for magnetism)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law of induction)}$$

$$\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampere's Law)}$$

In these equations, **E** represents an electrostatic field, **B** is the corresponding magnetic field, ϵ_0 is the permittivity, ρ is the charge density, c is the speed of light and **J** is the current density. In example 9.6, we derive a simplified version of the differential form of Ampere's law. The hypothesis in this case is a common form for Ampere's law: the line integral of a magnetic field around a closed path is proportional to the current enclosed by the path.

EXAMPLE 9.6 Deriving Ampere's Law

In the case where **E** is constant and I represents current, use the relationship $\oint_C \mathbf{B} \cdot d\mathbf{r} = \frac{1}{\epsilon_0 c^2} I$ to derive Ampere's law: $\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J}$.

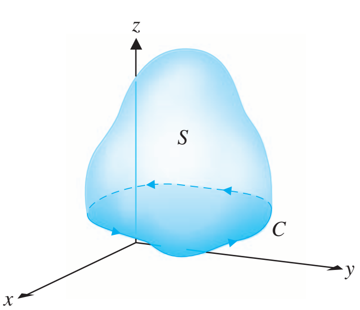


FIGURE 14.63
Positive orientation

Solution Let S be any capping surface for C, that is, any positively oriented two-sided surface bounded by C (see Figure 14.63). The enclosed current I is then related to the current density by $I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$. By Stokes' Theorem, we can rewrite the line integral of **B** as

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{B}) \cdot \mathbf{n}\, dS.$$

Equating the two expressions, we now have

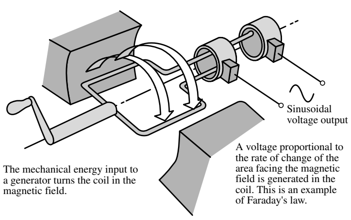
$$\iint_S (\nabla \times \mathbf{B}) \cdot \mathbf{n}\, dS = \frac{1}{\epsilon_0 c^2} \iint_S \mathbf{J} \cdot \mathbf{n}\, dS,$$

from which it follows that

$$\iint_S \left(\nabla \times \mathbf{B} - \frac{1}{\epsilon_0 c^2}\mathbf{J}\right) \cdot \mathbf{n}\, dS = 0.$$

Since this holds for all capping surfaces S, it must be that $\nabla \times \mathbf{B} - \frac{1}{\epsilon_0 c^2}\mathbf{J} = \mathbf{0}$ or $\nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J}$, as desired. ■

In our final example, we illustrate one of the uses of Faraday's law. In an AC generator, the turning of a coil in a magnetic field produces a voltage. In terms of the electric field $\mathbf{E}$, the voltage generated is given by $\oint_C \mathbf{E} \cdot d\mathbf{r}$, where C is a closed curve. As we see in example 9.7, Faraday's law relates this to the magnetic flux function $\phi = \iint_S \mathbf{B} \cdot \mathbf{n}\, dS$.



EXAMPLE 9.7 Using Faraday's Law to Analyze the Output of a Generator

An AC generator produces a voltage of $120 \sin(120\pi t)$ volts. Determine the magnetic flux ϕ.

Solution The voltage is given by

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = 120 \sin(120\pi t).$$

Applying Stokes' Theorem to the left-hand side, we have

$$\iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{E} \cdot d\mathbf{r} = 120 \sin(120\pi t).$$

Applying Faraday's law to the left-hand side, we have

$$\iint_S \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \cdot \mathbf{n}\, dS = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n}\, dS = 120 \sin(120\pi t).$$

Assuming that the integrand is continuous and we can bring the derivative outside, we get

$$-\frac{d}{dt} \iint_S (\mathbf{B} \cdot \mathbf{n})\, dS = 120 \sin(120\pi t).$$

Writing this in terms of the magnetic flux ϕ, we have

$$-\frac{d}{dt}\phi = 120 \sin(120\pi t)$$

or

$$\phi'(t) = -120 \sin(120\pi t).$$

Integrating this gives us

$$\phi(t) = \frac{1}{\pi} \cos(120\pi t) + c,$$

for some constant c. ■

EXERCISES 14.9

WRITING EXERCISES

1. Give an example of a fluid with velocity field with zero flux as in example 9.1.
2. Give an example of a fluid with velocity field with nonzero flux.
3. In the derivation of the continuity equation, explain why it is important to assume no sources or sinks.
4. From Bernoulli's Theorem, if all other things are equal and the density ρ increases, in what way does velocity change?

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1. Rework example 9.2 by computing $\oint_C \mathbf{F} \cdot d\mathbf{r}$.
2. Rework example 9.2 by directly computing $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

In exercises 3–8, use Gauss' Law for electricity and the relationship $q = \iiint_Q \rho\, dV$.

3. For $\mathbf{E} = \langle yz, xz, xy \rangle$, find the total charge in the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.
4. For $\mathbf{E} = \langle 2xy, y^2, 5x \rangle$, find the total charge in the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.
5. For $\mathbf{E} = \langle 4x - y, 2y + z, 3xy \rangle$, find the total charge in the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.
6. For $\mathbf{E} = \langle 2xz^2, 2yx^2, 2zy^2 \rangle$, find the total charge in the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.
7. For $\mathbf{E} = \langle 2xy, y^2, 5xy \rangle$, find the total charge in the cone $y = \sqrt{x^2 + z^2}$.
8. For $\mathbf{E} = \langle 4x - y, 2y + z, 3xy \rangle$, find the total charge in the solid bounded by $z = R - x^2 - y^2$ and $z = 0$.
9. Faraday showed that $\oint_C \mathbf{E} \cdot d\mathbf{r} = -\dfrac{d\phi}{dt}$, where $\phi = \iint_S \mathbf{B} \cdot \mathbf{n}\, dS$, for any capping surface S (that is, any positively oriented open surface with boundary C). Use this to show that $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$. What mathematical assumption must be made?
10. If an electric field $\mathbf{E}$ is conservative with potential function—ϕ, use Gauss' Law of electricity to show that **Poisson's equation** must hold: $\nabla^2\phi = -\dfrac{\rho}{\epsilon_0}$.
11. Use Maxwell's equation and $\mathbf{J} = \rho\mathbf{v}$ to derive the continuity equation. (Hint: Start by computing $\nabla \cdot \mathbf{J}$.) What mathematical assumption must be made?
12. For a magnetic field $\mathbf{B}$, Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ implies that $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector field $\mathbf{A}$. Show that the flux of $\mathbf{B}$ across an open surface S equals the circulation of $\mathbf{A}$ around the closed curve C, where C is the positively oriented boundary of S.
13. Let I be the current crossing an open surface S, so that $I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$. Given that $I = \oint_C \mathbf{B} \cdot d\mathbf{r}$ (where C is the positively oriented boundary of S), show that $\mathbf{J} = \nabla \times \mathbf{B}$.
14. Using the same notation as in exercise 13, start with $I = \iint_S \mathbf{J} \cdot \mathbf{n}\, dS$ and $\mathbf{J} = \nabla \times \mathbf{B}$ and show that $I = \oint_C \mathbf{B} \cdot d\mathbf{r}$.

In exercises 15–18, use the electrostatic force $\mathbf{E} = \dfrac{q}{4\pi\epsilon_0 r^3}\mathbf{r}$ for a charge q at the origin, where $\mathbf{r} = \langle x, y, z \rangle$ and $r = \sqrt{x^2 + y^2 + z^2}$.

15. If S is a closed surface not enclosing the origin, show that $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = 0$.
16. If S is the sphere $x^2 + y^2 + z^2 = 1$, show that $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \dfrac{q}{\epsilon_0}$.
17. If S is the sphere $x^2 + y^2 + z^2 = R^2$, show directly that $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \dfrac{q}{\epsilon_0}$.
18. Use exercises 15 and 16 to show that $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \dfrac{q}{\epsilon_0}$, for any closed surface S enclosing the origin.
19. Assume that $\iint_S \mathbf{D} \cdot \mathbf{n}\, dS = q$, for any closed surface S, where $\mathbf{D} = \epsilon_0\mathbf{E}$ is the electric flux density and q is the charge enclosed by S. Show that $\nabla \cdot \mathbf{D} = Q$, where Q is the charge density satisfying $q = \iiint_R Q\, dV$.
20. The moment of inertia about the z-axis of a solid Q with constant density ρ is $I_z = \iiint_Q (x^2 + y^2)\rho\, dV$. Express this as a surface integral.
21. Let u be a scalar function with continuous second partial derivatives. Define the **normal derivative** $\dfrac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$. Show that $\iint_S \dfrac{\partial u}{\partial n}\, dS = \iiint_Q \nabla^2 u\, dV$.
22. Suppose that u is a harmonic function (that is, $\nabla^2 u = 0$). Show that $\iint_S \dfrac{\partial u}{\partial n}\, dS = 0$.
23. If the heat conductivity k is not constant, our derivation of the heat equation is no longer valid. If $k = K(x, y, z)$, show that the heat equation becomes $K\nabla^2 T + \nabla K \cdot \nabla T = \sigma\rho\dfrac{\partial T}{\partial t}$.

24. If h has continuous partial derivatives and S is a closed surface enclosing a solid Q, show that $\iint_S (h\nabla h)\cdot \mathbf{n}\, dS = \iiint_Q (h\nabla^2 h + \nabla h \cdot \nabla h)\, dV$.

25. Suppose that f and g are both harmonic (that is, $\nabla^2 f = \nabla^2 g = 0$) and $f = g$ on a closed surface S, where S encloses a solid Q. Use the result of exercise 24, with $h = f - g$, to show that $f = g$ in Q.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Vector field	Velocity field	Flow lines
Gradient field	Potential function	Conservative field
Curl	Divergence	Laplacian
Line integral	Work line integral	Path independence
Green's Theorem	Surface integral	Flux integral
Divergence Theorem	Stokes' Theorem	Heat equation
Continuity equation	Bernoulli's Theorem	Maxwell's equations

TRUE OR FALSE

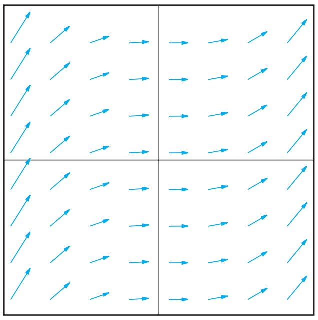
State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. The graph of a vector field shows vectors $\mathbf{F}(x, y)$ for all points (x, y).
2. The antiderivative of a vector field is called the potential function.
3. If the flow lines of $\mathbf{F}(x, y)$ are straight lines, then $\nabla \times \mathbf{F}(x, y) = \mathbf{0}$.
4. $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.
5. The line integral $\int_C f\, ds$ equals the amount of work done by f along C.
6. If $\nabla \times \mathbf{F} = \mathbf{0}$, then the work done by $\mathbf{F}$ along any path is 0.
7. If the curve C is split into pieces C_1 and C_2, then $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = -\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$.
8. Green's Theorem cannot be applied to a region with a hole.
9. When using Green's Theorem, positive orientation means counterclockwise.
10. When converting a surface integral to a double integral, you must replace z with a function of x and y.
11. A flux integral is always positive.
12. The Divergence Theorem applies only to three-dimensional solids without holes.
13. By Stokes' Theorem, the flux of $\nabla \times \mathbf{F}$ across two nonclosed surfaces sharing the same boundary is the same.

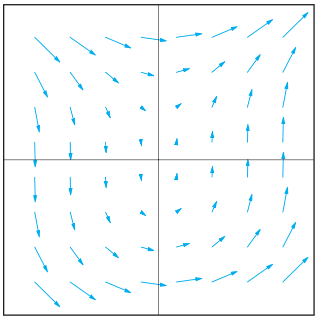
In exercises 1 and 2, sketch several vectors in the velocity field by hand and verify your sketch with a CAS.

1. $\langle x, -y\rangle$ **2.** $\langle 0, 2y\rangle$

3. Match the vector fields with their graphs. $\mathbf{F}_1(x, y) = \langle \sin x, y\rangle$, $\mathbf{F}_2(x, y) = \langle \sin y, x\rangle$, $\mathbf{F}_3(x, y) = \langle y^2, 2x\rangle$, $\mathbf{F}_4(x, y) = \langle 3, x^2\rangle$

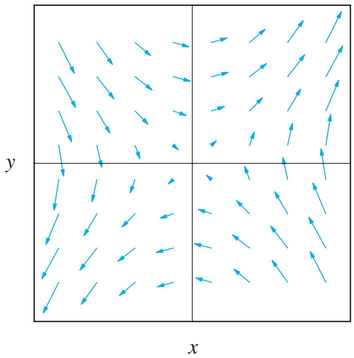


GRAPH A

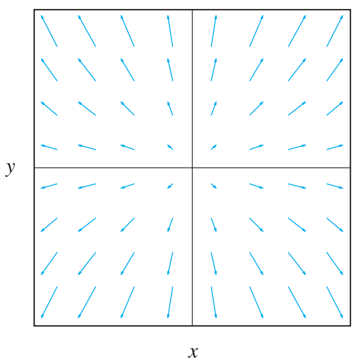


GRAPH B

Review Exercises



GRAPH C



GRAPH D

4. Find the gradient field corresponding to f. Use a CAS to graph it.

a. $f(x, y) = \ln\sqrt{x^2 + y^2}$ **b.** $f(x, y) = e^{-x^2-y^2}$

In exercises 5–8, determine whether or not the vector field is conservative. If it is, find a potential function.

5. $\langle y - 2xy^2, x - 2yx^2 + 1\rangle$ **6.** $\langle y^2 + 2e^{2y}, 2xy + 4xe^{2y}\rangle$

7. $\langle 2xy - 1, x^2 + 2xy\rangle$ **8.** $\langle y\cos xy - y, x\cos xy - x\rangle$

In exercises 9 and 10, find equations for the flow lines.

9. $\left\langle y, \dfrac{2x}{y}\right\rangle$ **10.** $\left\langle \dfrac{3}{x}, y\right\rangle$

In exercises 11 and 12, use the notation $\mathbf{r} = \langle x, y\rangle$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2}$.

11. Show that $\nabla(\ln r) = \dfrac{\mathbf{r}}{r^2}$. **12.** Show that $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{r}}{r^3}$.

In exercises 13–18, evaluate the line integral.

13. $\int_C 3y\,dx$, where C is the line segment from (2, 3) to (4, 3)

14. $\int_C (x^2 + y^2)\,ds$, where C is the half-circle $x^2 + y^2 = 16$ from (4, 0) to (−4, 0) with $y \geq 0$

15. $\int_C \sqrt{x^2 + y^2}\,ds$, where C is the circle $x^2 + y^2 = 9$, oriented clockwise

16. $\int_C (x - y)\,ds$, where C is the portion of $y = x^3$ from (1, 1) to (−1, −1)

17. $\int_C 2x\,dx$, where C is the upper half-circle from (2, 0) to (−2, 0), followed by the line segment to (2, 0)

18. $\int_C 3y^2 dy$, where C is the portion of $y = x^2$ from (−1, 1) to (1, 1), followed by the line segment to (−1, 1)

In exercises 19–22, compute the work done by the force F along the curve *C*.

19. $\mathbf{F}(x, y) = \langle x, -y\rangle$, C is the circle $x^2 + y^2 = 4$ oriented counterclockwise

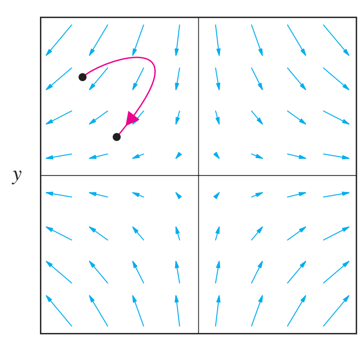
20. $\mathbf{F}(x, y) = \langle y, -x\rangle$, C is the circle $x^2 + y^2 = 4$ oriented counterclockwise

21. $\mathbf{F}(x, y) = \langle 2, 3x\rangle$, C is the quarter-circle from (2, 0) to (0, 2), followed by the line segment to (0, 0)

22. $\mathbf{F}(x, y) = \langle y, -x\rangle$, C is the square from (−2, 0) to (2, 0) to (2, 4) to (−2, 4) to (−2, 0)

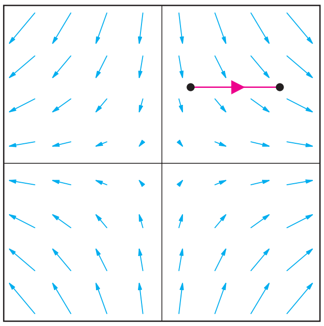
In exercises 23 and 24, use the graph to determine whether the work done is positive, negative or zero.

23.



Review Exercises

24.



In exercises 25 and 26, find the mass of the indicated object.

25. A spring in the shape of $\langle \cos 3t, \sin 3t, 4t \rangle$, $0 \le t \le 2\pi$, $\rho(x, y, z) = 4$

26. The portion of $z = x^2 + y^2$ under $z = 4$ with $\rho(x, y) = 12$

In exercises 27 and 28, show that the integral is independent of path and evaluate the integral.

27. $\int_C (3x^2y - x)\,dx + x^3\,dy$, where C runs from $(2, -1)$ to $(4, 1)$

28. $\int_C (y^2 - x^2)\,dx + (2xy + 1)\,dy$, where C runs from $(3, 2)$ to $(1, 3)$

In exercises 29–32, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

29. $\mathbf{F}(x, y) = \langle 2xy + y\sin x + e^{x+y}, e^{x+y} - \cos x + x^2 \rangle$, C is the quarter-circle from $(0, 3)$ to $(3, 0)$

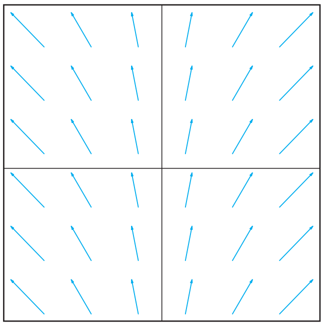
30. $\mathbf{F}(x, y) = \langle 2y + y^3 + \frac{1}{2}\sqrt{y/x}, 3xy^2 + \frac{1}{2}\sqrt{x/y} \rangle$, C is the top half-circle from $(1, 3)$ to $(3, 3)$

31. $\mathbf{F}(x, y, z) = \langle 2xy, x^2 - y, 2z \rangle$, C runs from $(1, 3, 2)$ to $(2, 1, -3)$

32. $\mathbf{F}(x, y, z) = \langle yz - x, xz - y, xy - z \rangle$, C runs from $(2, 0, 0)$ to $(0, 1, -1)$

In exercises 33 and 34, use the graph to determine whether or not the vector field is conservative.

33.



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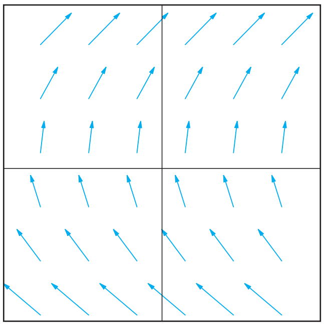


In exercises 35–40, use Green's Theorem to evaluate the indicated line integral.

35. $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^3 - y, x + y^3 \rangle$ and C is formed by $y = x^2$ and $y = x$, oriented positively.

36. $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle y^2 + 3x^2y, xy + x^3 \rangle$ and C is formed by $y = x^2$ and $y = 2x$, oriented positively.

37. $\oint_C \tan x^2\,dx + x^2\,dy$, where C is the triangle from $(0, 0)$ to $(1, 1)$ to $(2, 0)$ to $(0, 0)$.

38. $\oint_C x^2y\,dx + \ln\sqrt{1 + y^2}\,dy$, where C is the triangle from $(0, 0)$ to $(2, 2)$ to $(0, 2)$ to $(0, 0)$.

39. $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 3x^2, 4y^3 - z, z^2 \rangle$ and C is formed by $z = y^2$ and $z = 4$, oriented positively in the yz-plane.

40. $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle 4y^2, 3x^2, 8z \rangle$ and C is $x^2 + y^2 = 4$, oriented positively in the plane $z = 3$.

Review Exercises

In exercises 41 and 42, use a line integral to compute the area of the given region.

41. The ellipse $4x^2 + 9y^2 = 36$

42. The region bounded by $y = \sin x$ and the x-axis for $0 \le x \le \pi$

In exercises 43–46, find the curl and divergence of the given vector field.

43. $x^3\mathbf{i} - y^3\mathbf{j}$

44. $y^3\mathbf{i} - x^3\mathbf{j}$

45. $\langle 2x, 2yz^2, 2y^2z \rangle$

46. $\langle 2xy, x^2 - 3y^2z^2, 1 - 2zy^3 \rangle$

In exercises 47–50, determine whether the given vector field is conservative and/or incompressible.

47. $\langle 2x - y^2, z^2 - 2xy, xy^2 \rangle$

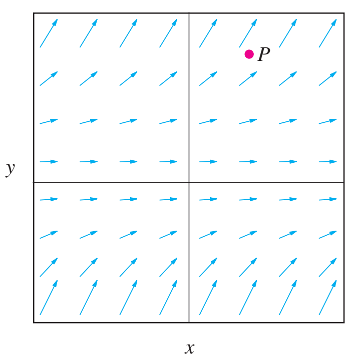
48. $\langle y^2z, x^2 - 3z^2y, z^3 - y \rangle$

49. $\langle 4x - y, 3 - x, 2 - 4z \rangle$

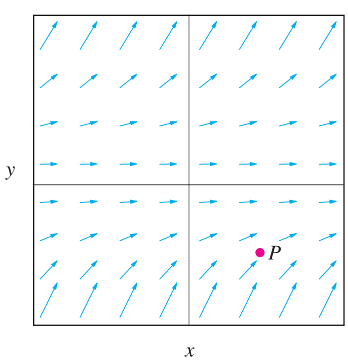
50. $\langle 4, 2xy^3, z^4 - x \rangle$

In exercises 51 and 52, conjecture whether the divergence at point P is positive, negative or zero.

51.



52.



In exercises 53 and 54, sketch a graph of the parametric surface.

53. $x = u^2, y = v^2, z = u + 2v$

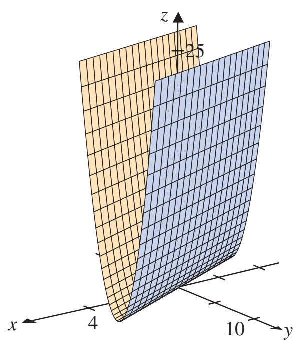
54. $x = (3 + 2\cos u)\cos v, y = (3 + 2\cos u)\sin v, z = 2\cos v$

55. Match the parametric equations with the surfaces.

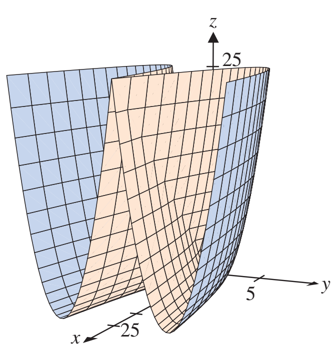
a. $x = u^2, y = u + v, z = v^2$

b. $x = u^2, y = u + v, z = v$

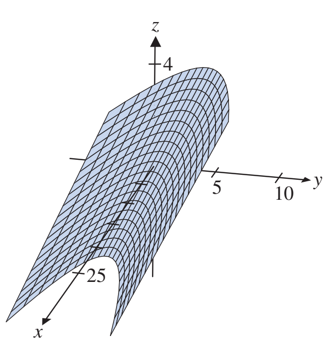
c. $x = u, y = u + v, z = v^2$



SURFACE A



SURFACE B



SURFACE C

Review Exercises

56. Find a parametric representation of $x^2 + y^2 + z^2 = 9$.

In exercises 57 and 58, find the surface area.

57. The portion of the paraboloid $z = x^2 + y^2$ between the cylinder $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

58. The portion of the paraboloid $z = 9 - x^2 - y^2$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

In exercises 59–64, evaluate the surface integral $\iint_S f(x, y, z)\, dS$.

59. $\iint_S (x - y)\, dS$, where S is the portion of the plane $3x + 2y + z = 12$ in the first octant

60. $\iint_S (x^2 + y^2)\, dS$ where S is the portion of $y = 4 - x^2$ above the xy-plane, $y \geq 0$ and below $z = 2$

61. $\iint_S (4x + y + 3z)\, dS$, where S is the portion of the plane $4x + y + 3z = 12$ inside $x^2 + y^2 = 1$

62. $\iint_S (x - z)\, dS$, where S is the portion of the cylinder $x^2 + z^2 = 1$ above the xy-plane between $y = 1$ and $y = 2$

63. $\iint_S yz\, dS$, where S is the portion of the cone $y = \sqrt{x^2 + z^2}$ to the left of $y = 3$

64. $\iint_S (x^2 + z^2)\, dS$, where S is the portion of the paraboloid $x = y^2 + z^2$ behind the plane $x = 4$

In exercises 65 and 66, find the mass and center of mass of the solid.

65. The portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$, $\rho(x, y, z) = 2$

66. The portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 4$, $\rho(x, y, z) = z$

In exercises 67–70, use the Divergence Theorem to compute $\iint_{\partial Q} \mathbf{F} \cdot \mathbf{n}\, dS$.

67. Q is bounded by $x + 2y + z = 4$ (first octant) and the coordinate planes, $\mathbf{F} = \langle y^2 z, y^2 - \sin z, 4y^2 \rangle$.

68. Q is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$, $\mathbf{F} = \langle 4x, 3z, 4y^2 - x \rangle$.

69. Q is bounded by $z = 1 - y^2$, $z = 0$, $x = 0$ and $x + z = 4$, $\mathbf{F} = \langle 2xy, z^3 + 7yx, 4xy^2 \rangle$.

70. Q is bounded by $z = \sqrt{4 - x^2}$, $z = 0$, $y = 0$ and $y + z = 6$, $\mathbf{F} = \langle y^2, 4yz, 2xy \rangle$.

In exercises 71 and 72, find the flux of F over ∂Q.

71. Q is bounded by $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 = 4$ and $z = 0$, $\mathbf{F} = \langle xz, yz, x^2 - z \rangle$.

72. Q is bounded by $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$, $\mathbf{F} = \langle 4x, x^2 - 2y, 3z + x^2 \rangle$.

In exercises 73–76, use Stokes' Theorem, if appropriate, to compute $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

73. S is the portion of the tetrahedron bounded by $x + y + 2z = 2$ and the coordinate planes in front of the yz-plane, $\mathbf{F} = \langle zy^4 - y^2, y - x^3, z^2 \rangle$.

74. S is the portion of $z = x^2 + y^2$ below $z = 4$, $\mathbf{F} = \langle z^2 - x, 2y, z^3 xy \rangle$.

75. S is the portion of the cone $z = \sqrt{x^2 + y^2}$ below $x + 2y + 3z = 24$, $\mathbf{F} = \langle 4x^2, 2ye^{2y}, \sqrt{z^2 + 1} \rangle$.

76. S is the portion of the paraboloid $y = x^2 + 4z^2$ to the left of $y = 8 - z$, $\mathbf{F} = \langle xe^{3x}, 4y^{2/3}, z^2 + 2 \rangle$.

In exercises 77 and 78, use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

77. C is the triangle from $(0, 1, 0)$ to $(1, 0, 0)$ to $(0, 0, 20)$, $\mathbf{F} = \langle 2xy \cos z, y^2 + x^2 \cos z, z - x^2 y \sin z \rangle$.

78. C is the square from $(0, 0, 2)$ to $(1, 0, 2)$ to $(1, 1, 2)$ to $(0, 1, 2)$, $\mathbf{F} = \langle x^3 + yz, y^2, z^2 \rangle$.

EXPLORATORY EXERCISES

1. In exploratory exercise 2 of section 14.1, we developed a technique for finding equations for flow lines of certain vector fields. The field $\langle 2, 1 + 2xy \rangle$ from example 1.5 is such a vector field, but the calculus is more difficult. First, show that the differential equation is $y' - xy = \frac{1}{2}$ and show that an integrating factor is $e^{-x^2/2}$. The flow lines come from equations of the form $y = e^{x^2/2} \int \frac{1}{2} e^{-x^2/2} dx + ce^{x^2/2}$. Unfortunately, there is no elementary function equal to $\int \frac{1}{2} e^{-x^2/2} dx$. It can help to write this in the form $y = e^{x^2/2} \int_0^x \frac{1}{2} e^{-u^2/2} du + ce^{x^2/2}$. In this form, show that $c = y(0)$. In example 1.5, the curve passing through $(0, 1)$ is $y = e^{x^2/2} \int_0^x \frac{1}{2} e^{-u^2/2}\, du + e^{x^2/2}$. Graph this function and compare it to the path shown in Figure 14.7b. Find an equation for and plot the curve through $(0, -1)$. To find the curve through $(1, 1)$, change the limits of integration and rewrite the solution. Plot this curve and compare to Figure 14.7b.