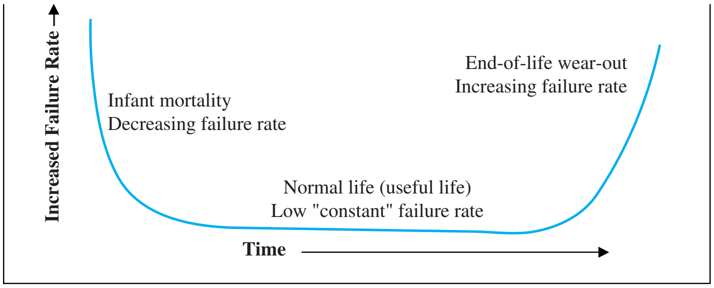
Integration Techniques

CHAPTER 6

Electronics companies constantly test their products for reliability. A small change in the reliability of a component can make or break the sales of a product. The lifetime of an electronics component is often viewed as having three stages, as illustrated by the so-called *bathtub curve* shown in the figure.

This curve indicates the average failure rate of the product as a function of age. In the first stage (called the **infant mortality phase**), the failure rate drops rapidly as faulty components quickly fail. If the component survives this initial phase, it enters a lengthy second phase (the **useful life phase**) of constant failure rate. The third phase shows an increase in failure rate as the components reach the physical limit of their lifespan.

The constant failure rate of the useful life phase has several interesting consequences. First, the failures are "memoryless" in the sense that the probability that the component lasts another hour is independent of the age of the component. A component that is 40 hours old may be as likely to last another hour as a component that is only 10 hours old. This unusual property holds for electronics components such as lightbulbs, during the useful life phase.



A constant failure rate also implies that component failures follow what is called an exponential distribution. (See exercise 73 of section 6.6.) The computation of statistics for the exponential distribution requires more sophisticated integration techniques than those discussed so far. For instance, the mean (average) lifetime of certain electronics components is given by an integral of the form $\int_0^\infty cxe^{-cx}\,dx$, for some constant $c > 0$. Before we evaluate this, we will need to extend our notion of integral to include *improper integrals* such as this, where one or both of the limits of integration are infinite. We do this in section 6.6. Another difficulty with this integral is that we do not presently know an antiderivative for $f(x) = xe^{-cx}$.

In section 6.2, we introduce a powerful technique called *integration by parts* that can be used to find antiderivatives of many such functions.

The new techniques of integration introduced in this chapter provide us with a broad range of tools used to solve countless problems of interest to engineers, mathematicians and scientists.

6.1 REVIEW OF FORMULAS AND TECHNIQUES

In this brief section, we draw together all of the integration formulas and the one integration technique (integration by substitution) that we have developed so far. We use these to develop some more general formulas, as well as to solve more complicated integration problems. First, look over the following table of the basic integration formulas developed in Chapter 4.

$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$, for $r \neq -1$ (power rule)	$\int \frac{1}{x}\,dx = \ln\|x\| + c$, for $x \neq 0$
$\int \sin x\,dx = -\cos x + c$	$\int \cos x\,dx = \sin x + c$
$\int \sec^2 x\,dx = \tan x + c$	$\int \sec x \tan x\,dx = \sec x + c$
$\int \csc^2 x\,dx = -\cot x + c$	$\int \csc x \cot x\,dx = -\csc x + c$
$\int e^x\,dx = e^x + c$	$\int e^{-x}\,dx = -e^{-x} + c$
$\int \tan x\,dx = -\ln\|\cos x\| + c$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$	$\int \frac{1}{\|x\|\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

Recall that each of these follows from a corresponding differentiation rule. So far, we have expanded this list slightly by using the method of substitution, as in example 1.1.

EXAMPLE 1.1 A Simple Substitution

Evaluate $\int \sin(ax)\,dx$, for $a \neq 0$.

Solution The obvious choice here is to let $u = ax$, so that $du = a\,dx$. This gives us

$$\int \sin(ax)\,dx = \frac{1}{a}\int \underbrace{\sin(ax)}_{\sin u}\,\underbrace{a\,dx}_{du} = \frac{1}{a}\int \sin u\,du$$

$$= -\frac{1}{a}\cos u + c = -\frac{1}{a}\cos(ax) + c.$$

There is no need to memorize general rules like the ones given in examples 1.1 and 1.2, although it is often convenient to do so. You can reproduce such general rules any time you need them using substitution.

EXAMPLE 1.2 Generalizing a Basic Integration Rule

Evaluate $\int \frac{1}{a^2+x^2}\,dx$, for $a \neq 0$.

Solution Notice that this is nearly the same as $\int \frac{1}{1+x^2}\,dx$ and we can write

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx.$$

Now, letting $u = \frac{x}{a}$, we have $du = \frac{1}{a}\,dx$ and so,

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx = \frac{1}{a}\int \frac{1}{\underbrace{1+\left(\frac{x}{a}\right)^2}_{1+u^2}}\underbrace{\left(\frac{1}{a}\right)dx}_{du}$$

$$= \frac{1}{a}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\tan^{-1} u + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

Substitution will not resolve all of your integration difficulties, as we see in example 1.3.

EXAMPLE 1.3 An Integrand That Must Be Expanded

Evaluate $\int (x^2-5)^2\,dx$.

Solution Your first impulse might be to substitute $u = x^2 - 5$. However, this fails, as we don't have $du = 2x\,dx$ in the integral. (We can force the constant 2 into the integral, but we can't get the x in there.) On the other hand, you can always multiply out the binomial to obtain

$$\int (x^2-5)^2\,dx = \int (x^4 - 10x^2 + 25)\,dx = \frac{x^5}{5} - 10\frac{x^3}{3} + 25x + c.$$

The moral of example 1.3 is to make certain you don't overlook simpler methods. The most general rule in integration is to *keep trying*. Sometimes, you will need to do some algebra before you can recognize the form of the integrand.

EXAMPLE 1.4 An Integral Where We Must Complete the Square

Evaluate $\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx$.

Solution Not much may come to mind here. Substitution for either the entire denominator or the quantity under the square root does not work. (Why not?) So, what's left? Recall that there are essentially only two things you can do to a quadratic polynomial: either factor it or complete the square. Here, doing the latter sheds some light on the integral. We have

$$\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx = \int \frac{1}{\sqrt{-5-(x^2-6x+9)+9}}\,dx = \int \frac{1}{\sqrt{4-(x-3)^2}}\,dx.$$

Notice how much this looks like $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$. If we factor out the 4 in the square root, we get

$$\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-3)^2}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x-3}{2}\right)^2}}\,\frac{1}{2}\,dx.$$

Now, let $u = \frac{x-3}{2}$, so that $du = \frac{1}{2}\,dx$. This gives us

$$\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx = \int \frac{1}{\underbrace{\sqrt{1-\left(\frac{x-3}{2}\right)^2}}_{\sqrt{1-u^2}}}\,\underbrace{\frac{1}{2}\,dx}_{du} = \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1} u + c = \sin^{-1}\left(\frac{x-3}{2}\right) + c.$$

Example 1.5 illustrates the value of perseverance.

EXAMPLE 1.5 An Integral Requiring Some Imagination

Evaluate $\int \frac{4x+1}{2x^2+4x+10}\,dx$.

Solution As with most integrals, you cannot evaluate this as it stands. Notice that the numerator is very nearly the derivative of the denominator (but not quite). Recognize that you can complete the square in the denominator, to obtain

$$\int \frac{4x+1}{2x^2+4x+10}\,dx = \int \frac{4x+1}{2(x^2+2x+1)-2+10}\,dx = \int \frac{4x+1}{2(x+1)^2+8}\,dx.$$

Now, the denominator nearly looks like the denominator in $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$. If we factor out an 8, it will look even more like this, as follows.

$$\int \frac{4x+1}{2x^2+4x+10}\,dx = \int \frac{4x+1}{2(x+1)^2+8}\,dx$$

$$= \frac{1}{8}\int \frac{4x+1}{\frac{1}{4}(x+1)^2+1}\,dx$$

$$= \frac{1}{8}\int \frac{4x+1}{\left(\frac{x+1}{2}\right)^2+1}\,dx.$$

Now, taking $u = \frac{x+1}{2}$, we have $du = \frac{1}{2}\,dx$ and $x = 2u - 1$ and so,

$$\int \frac{4x+1}{2x^2+4x+10}\,dx = \frac{1}{8}\int \frac{4x+1}{\left(\frac{x+1}{2}\right)^2+1}\,dx = \frac{1}{4}\int \frac{\overbrace{4x+1}^{4(2u-1)+1}}{\underbrace{\left(\frac{x+1}{2}\right)^2+1}_{u^2+1}}\,\underbrace{\frac{1}{2}\,dx}_{du}$$

$$= \frac{1}{4}\int \frac{4(2u-1)+1}{u^2+1}\,du = \frac{1}{4}\int \frac{8u-3}{u^2+1}\,du$$

$$= \frac{4}{4}\int \frac{2u}{u^2+1}\,du - \frac{3}{4}\int \frac{1}{u^2+1}\,du$$
$$= \ln(u^2+1) - \frac{3}{4}\tan^{-1} u + c$$
$$= \ln\left[\left(\frac{x+1}{2}\right)^2 + 1\right] - \frac{3}{4}\tan^{-1}\left(\frac{x+1}{2}\right) + c.$$

Example 1.5 was tedious, but reasonably straightforward. The issue in integration is to recognize what pieces are present in a given integral and to see how you might rewrite the integral in a more familiar form.

EXERCISES 6.1

WRITING EXERCISES

1. In example 1.2, explain how you should know to write the denominator as $a^2\left[1+\left(\frac{x}{a}\right)^2\right]$. Would this still be a good first step if the numerator were x instead of 1? What would you do if the denominator were $\sqrt{a^2-x^2}$?
2. In both examples 1.4 and 1.5, we completed the square and found antiderivatives involving $\sin^{-1} x$, $\tan^{-1} x$ and $\ln(x^2+1)$. Briefly describe how the presence of an x in the numerator or a square root in the denominator affects which of these functions will be in the antiderivative.

In exercises 1–40, evaluate the integral.

1. $\int \sin 6x\,dx$
2. $\int 3\cos 4x\,dx$
3. $\int \sec 2x \tan 2x\,dx$
4. $\int x\sec x^2 \tan x^2\,dx$
5. $\int e^{3-2x}\,dx$
6. $\int \frac{3}{e^{6x}}\,dx$
7. $\int \frac{4}{x^{1/3}(1+x^{2/3})}\,dx$
8. $\int \frac{2}{x^{1/4}+x}\,dx$
9. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
10. $\int \frac{\cos(1/x)}{x^2}\,dx$
11. $\int_0^{\pi} \cos x\, e^{\sin x}\,dx$
12. $\int_0^{\pi/4} \sec^2 x\, e^{\tan x}\,dx$
13. $\int_{-\pi/4}^{0} \frac{\sin x}{\cos^2 x}\,dx$
14. $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\,dx$
15. $\int \frac{3}{16+x^2}\,dx$
16. $\int \frac{2}{4+4x^2}\,dx$
17. $\int \frac{x^2}{1+x^6}\,dx$
18. $\int \frac{x^5}{1+x^6}\,dx$
19. $\int \frac{1}{\sqrt{4-x^2}}\,dx$
20. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$
21. $\int \frac{x}{\sqrt{1-x^4}}\,dx$
22. $\int \frac{2x^3}{\sqrt{1-x^4}}\,dx$
23. $\int \frac{4}{5+2x+x^2}\,dx$
24. $\int \frac{4x+4}{5+2x+x^2}\,dx$
25. $\int \frac{4x}{5+2x+x^2}\,dx$
26. $\int \frac{x+1}{x^2+2x+4}\,dx$
27. $\int (x^2+4)^2\,dx$
28. $\int x(x^2+4)^2\,dx$
29. $\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$
30. $\int \frac{x+1}{\sqrt{3-2x-x^2}}\,dx$
31. $\int \frac{1+x}{1+x^2}\,dx$
32. $\int \frac{1}{\sqrt{x}+x}\,dx$
33. $\int_{-2}^{-1} e^{\ln(x^2+1)}\,dx$
34. $\int_1^3 e^{2\ln x}\,dx$
35. $\int_3^4 x\sqrt{x-3}\,dx$
36. $\int_0^1 x(x-3)^2\,dx$
37. $\int_0^2 \frac{e^x}{1+e^{2x}}\,dx$
38. $\int_{-1}^{0} e^x\cot(e^x)\csc(e^x)\,dx$
39. $\int_1^4 \frac{x^2+1}{\sqrt{x}}\,dx$
40. $\int_{-2}^{0} xe^{-x^2}\,dx$

In exercises 41–46, you are given a pair of integrals. Evaluate the integral that can be worked using the techniques covered so far (the other cannot).

41. $\int \frac{5}{3+x^2}\,dx$ and $\int \frac{5}{3+x^3}\,dx$
42. $\int \sin 2x\,dx$ and $\int \sin^2 x\,dx$
43. $\int \ln x\,dx$ and $\int \frac{\ln x}{2x}\,dx$
44. $\int \frac{x^3}{1+x^8}\,dx$ and $\int \frac{x^4}{1+x^8}\,dx$
45. $\int e^{-x^2}\,dx$ and $\int xe^{-x^2}\,dx$

46. $\displaystyle\int \sec x\,dx$ and $\displaystyle\int \sec^2 x\,dx$

47. Find $\displaystyle\int_0^2 f(x)\,dx$, where $f(x) = \begin{cases} x/(x^2+1) & \text{if } x \le 1 \\ x^2/(x^2+1) & \text{if } x > 1 \end{cases}$

48. Find $\displaystyle\int_{-2}^2 f(x)\,dx$, where $f(x) = \begin{cases} xe^{x^2} & \text{if } x < 0 \\ x^2 e^{x^3} & \text{if } x \ge 0 \end{cases}$

49. Rework example 1.5 by rewriting the integral as $\displaystyle\int \frac{4x+4}{2x^2+4x+10}\,dx - \int \frac{3}{2x^2+4x+10}\,dx$ and completing the square in the second integral.

EXPLORATORY EXERCISES

1. Find $\displaystyle\int \frac{1}{1+x^2}\,dx$, $\displaystyle\int \frac{x}{1+x^2}\,dx$, $\displaystyle\int \frac{x^2}{1+x^2}\,dx$ and $\displaystyle\int \frac{x^3}{1+x^2}\,dx$. Generalize to give the form of $\displaystyle\int \frac{x^n}{1+x^2}\,dx$ for any positive integer n, as completely as you can.

2. Find $\displaystyle\int \frac{x}{1+x^4}\,dx$, $\displaystyle\int \frac{x^3}{1+x^4}\,dx$ and $\displaystyle\int \frac{x^5}{1+x^4}\,dx$. Generalize to give the form of $\displaystyle\int \frac{x^n}{1+x^4}\,dx$ for any odd positive integer n.

3. Use a CAS to find $\int xe^{-x^2}dx$, $\int x^3e^{-x^2}dx$ and $\int x^5e^{-x^2}dx$. Verify that each antiderivative is correct. Generalize to give the form of $\int x^n e^{-x^2}dx$ for any odd positive integer n.

4. In many situations, the integral as we've defined it must be extended to the **Riemann-Stieltjes integral** considered in this exercise. For functions f and g, let P be a regular partition of $[a, b]$ and define the sums $R(f, g, P) = \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})]$. The integral $\int_a^b f(x)\,dg(x)$ equals the limit of the sums $R(f, g, P)$ as $n \to \infty$, if the limit exists and equals the same number for all evaluation points c_i. (a) Show that if g' exists, then $\int_a^b f(x)\,dg(x) = \int_a^b f(x)g'(x)\,dx$. (b) If $g(x) = \begin{cases} 1 & a \le x \le c \\ 2 & c < x \le b \end{cases}$ for some constant c with $a < c < b$, evaluate $\int_a^b f(x)\,dg(x)$. (c) Find a function $g(x)$ such that $\int_0^1 \frac{1}{x}\,dg(x)$ exists.

6.2 INTEGRATION BY PARTS

HISTORICAL NOTES

Brook Taylor (1685–1731)
An English mathematician who is credited with devising integration by parts. Taylor made important contributions to probability, the theory of magnetism and the use of vanishing lines in linear perspective. However, he is best known for Taylor's Theorem (see section 8.7), in which he generalized results of Newton, Halley, the Bernoullis and others. Personal tragedy (both his wives died during childbirth) and poor health limited the mathematical output of this brilliant mathematician.

At this point, you will have recognized that there are many integrals that cannot be evaluated using our basic formulas or integration by substitution. For instance,

$$\int x \sin x\,dx$$

cannot be evaluated with what you presently know. We improve this situation in the current section by introducing a powerful tool called *integration by parts*.

We have observed that every differentiation rule gives rise to a corresponding integration rule. So, for the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x),$$

integrating both sides of this equation gives us

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Ignoring the constant of integration, the integral on the left-hand side is simply $f(x)g(x)$. Solving for the second integral on the right-hand side then yields

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This rule is called **integration by parts.** You're probably wondering about the significance of this new rule. In short, it lets us replace a given integral with an easier one. We'll let the examples convince you of the power of this technique. First, it's usually convenient to write this using the notation $u = f(x)$ and $v = g(x)$. Then,

$$du = f'(x)\,dx \quad \text{and} \quad dv = g'(x)\,dx,$$

so that the integration by parts algorithm becomes

INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du. \tag{2.1}$$

To apply integration by parts, you need to make a judicious choice of u and dv so that the integral on the right-hand side of (2.1) is one that you know how to evaluate.

EXAMPLE 2.1 Integration by Parts

Evaluate $\int x \sin x\,dx$.

Solution First, observe that this is not one of our basic integrals and there's no obvious substitution that will help. To use integration by parts, you will need to choose u (something to differentiate) and dv (something to integrate). If we let

$$u = x \quad \text{and} \quad dv = \sin x\,dx,$$

then $du = dx$ and integrating dv, we have

$$v = \int \sin x\,dx = -\cos x + k.$$

In performing integration by parts, we drop this constant of integration. (Think about why it makes sense to do this.) Also, we usually write this information as the block:

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x \end{aligned}$$

This gives us

$$\begin{aligned} \int \underbrace{x}_{u}\,\underbrace{\sin x\,dx}_{dv} &= \int u\,dv = uv - \int v\,du \\ &= -x\cos x - \int (-\cos x)\,dx \\ &= -x\cos x + \sin x + c. \end{aligned} \tag{2.2}$$

It's a simple matter to differentiate the expression on the right-hand side of (2.2) and verify directly that you have indeed found an antiderivative of $x \sin x$. ■

You should quickly realize that the choice of u and dv is critical. Observe what happens if we switch the choice of u and dv made in example 2.1.

EXAMPLE 2.2 A Poor Choice of *u* and *dv*

Consider $\int x \sin x\,dx$ as in example 2.1, but this time, reverse the choice of u and dv.

Solution Here, we let

$$\begin{aligned} u &= \sin x & dv &= x\,dx \\ du &= \cos x\,dx & v &= \tfrac{1}{2}x^2 \end{aligned}$$

This gives us

$$\int \underbrace{\sin x}_{u}\,\underbrace{x\,dx}_{dv} = uv - \int v\,du = \frac{1}{2}x^2 \sin x - \frac{1}{2}\int x^2 \cos x\,dx.$$

REMARK 2.1

When using integration by parts, keep in mind that you are splitting up the integrand into two pieces. One of these pieces, corresponding to u, will be differentiated and the other, corresponding to dv, will be integrated. Since you can differentiate virtually every function you run across, you should choose a dv for which you know an antiderivative and make a choice of both that will result in an easier integral. If possible, a choice of $u = x$ results in the simple $du = dx$. You will learn what works best by working through lots of problems. Even if you don't see how the problem is going to end up, *try something!*

Notice that the last integral is one that we do *not* know how to calculate any better than the original one. In fact, we have made the situation worse in that the power of x in the new integral is higher than in the original integral. ■

EXAMPLE 2.3 An Integrand with a Single Term

Evaluate $\int \ln x \, dx$.

Solution This may look like it should be simple, but it's not one of our basic integrals and there's no obvious substitution that will simplify it. That leaves us with integration by parts. Remember that you must pick u (to be differentiated) and dv (to be integrated). You obviously can't pick $dv = \ln x \, dx$, since the problem here is to find a way to integrate this very term. So, try

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \tfrac{1}{x}\,dx & v &= x \end{aligned}$$

Integration by parts now gives us

$$\begin{aligned} \int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv} &= uv - \int v\,du = x\ln x - \int x\left(\frac{1}{x}\right)dx \\ &= x\ln x - \int 1\,dx = x\ln x - x + c. \end{aligned}$$ ■

Frequently, an integration by parts results in an integral that we cannot evaluate directly, but instead, one that we can evaluate only by repeating integration by parts one or more times.

EXAMPLE 2.4 Repeated Integration by Parts

Evaluate $\int x^2 \sin x \, dx$.

Solution Certainly, you cannot evaluate this as it stands and there is no simplification or obvious substitution that will help. We choose

$$\begin{aligned} u &= x^2 & dv &= \sin x\,dx \\ du &= 2x\,dx & v &= -\cos x \end{aligned}$$

With this choice, integration by parts yields

$$\int \underbrace{x^2}_{u}\, \underbrace{\sin x\,dx}_{dv} = -x^2\cos x + 2\int x\cos x\,dx.$$

Of course, this last integral cannot be evaluated as it stands, but we could do it using a further integration by parts. We now choose

$$\begin{aligned} u &= x & dv &= \cos x\,dx \\ du &= dx & v &= \sin x \end{aligned}$$

Applying integration by parts to the last integral, we now have

$$\begin{aligned} \int x^2 \sin x\,dx &= -x^2\cos x + 2\int \underbrace{x}_{u}\, \underbrace{\cos x\,dx}_{dv} \\ &= -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) \\ &= -x^2\cos x + 2x\sin x + 2\cos x + c. \end{aligned}$$ ■

REMARK 2.2

In the second integration by parts in example 2.4, if you choose $u = \cos x$ and $dv = x\,dx$, then integration by parts will fail and leave you with the less than astounding conclusion that the integral that you started with equals itself. (Try this as an exercise.)

Based on our work in example 2.4, try to figure out how many integrations by parts would be required to evaluate $\int x^n \sin x\, dx$, for a positive integer n. (There will be more on this including a shortcut, in the exercises.)

Repeated integration by parts sometimes takes you back to the integral you started with. This can be bad news (see Remark 2.2), or this can give us a clever way of evaluating an integral, as in example 2.5.

EXAMPLE 2.5 Repeated Integration by Parts with a Twist

Evaluate $\int e^{2x} \sin x\, dx$.

Solution None of our elementary methods works on this integral. For integration by parts, there are two viable choices for u and dv. We take

$$\begin{aligned} u &= e^{2x} & dv &= \sin x\, dx \\ du &= 2e^{2x}\, dx & v &= -\cos x \end{aligned}$$

(The opposite choice also works. Try this as an exercise.) Integration by parts yields

$$\int \underbrace{e^{2x}}_{u} \underbrace{\sin x\, dx}_{dv} = -e^{2x} \cos x + 2 \int e^{2x} \cos x\, dx.$$

The remaining integral again requires integration by parts. We choose

$$\begin{aligned} u &= e^{2x} & dv &= \cos x\, dx \\ du &= 2e^{2x}\, dx & v &= \sin x \end{aligned}$$

It now follows that

$$\begin{aligned} \int e^{2x} \sin x\, dx &= -e^{2x} \cos x + 2 \int \underbrace{e^{2x}}_{u} \underbrace{\cos x\, dx}_{dv} \\ &= -e^{2x} \cos x + 2 \left(e^{2x} \sin x - 2 \int e^{2x} \sin x\, dx \right) \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x\, dx. \end{aligned} \tag{2.3}$$

Observe that the last line includes the integral that we started with. Treating the integral $\int e^{2x} \sin x\, dx$ as the unknown, we can add $4 \int e^{2x} \sin x\, dx$ to both sides of equation (2.3), leaving

$$5 \int e^{2x} \sin x\, dx = -e^{2x} \cos x + 2e^{2x} \sin x + K,$$

where we have added the constant of integration K on the right side. Dividing both sides by 5 then gives us

$$\int e^{2x} \sin x\, dx = -\frac{1}{5} e^{2x} \cos x + \frac{2}{5} e^{2x} \sin x + c,$$

where we have replaced the arbitrary constant of integration $\frac{K}{5}$ by c. ■

REMARK 2.3

For integrals like $\int e^{2x} \sin x\, dx$ (or related integrals like $\int e^{-3x} \cos 2x\, dx$), repeated integration by parts as in example 2.5 will produce an antiderivative. The first choice of u and dv is up to you (either choice will work) but your choice of u and dv in the second integration by parts must be consistent with your first choice. For instance, in example 2.5, our initial choice of $u = e^{2x}$ commits us to using $u = e^{2x}$ for the second integration by parts, as well. To see why, rework the second integral taking $u = \cos x$ and observe what happens!

Observe that for any positive integer n, the integral $\int x^n e^x\, dx$ will require integration by parts. At this point, it should be no surprise that we take

$$\begin{aligned} u &= x^n & dv &= e^x\, dx \\ du &= nx^{n-1}\, dx & v &= e^x \end{aligned}$$

Applying integration by parts gives us

$$\int \underbrace{x^n}_{u} \underbrace{e^x\,dx}_{dv} = x^n e^x - n \int x^{n-1} e^x\,dx. \tag{2.4}$$

Notice that if $n - 1 > 0$, we will need to perform another integration by parts. In fact, we'll need to perform a total of n integrations by parts to complete the process. An alternative is to apply formula (2.4) (called a **reduction formula**) repeatedly to evaluate a given integral. We illustrate this in example 2.6.

EXAMPLE 2.6 Using a Reduction Formula

Evaluate the integral $\int x^4 e^x\,dx$.

Solution The prospect of performing four integrations by parts may not particularly appeal to you. However, we can use the reduction formula (2.4) repeatedly to evaluate the integral with relative ease, as follows. From (2.4), with $n = 4$, we have

$$\int x^4 e^x\,dx = x^4 e^x - 4\int x^{4-1} e^x\,dx = x^4 e^x - 4\int x^3 e^x\,dx.$$

Applying (2.4) again, this time with $n = 3$, gives us

$$\int x^4 e^x\,dx = x^4 e^x - 4\left(x^3 e^x - 3\int x^2 e^x\,dx\right).$$

By now, you should see that we can resolve this by applying the reduction formula two more times. By doing so, we get

$$\int x^4 e^x\,dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + c,$$

where we leave the details of the remaining calculations to you. ■

Note that to evaluate a definite integral, it is always possible to apply integration by parts to the corresponding indefinite integral and then simply evaluate the resulting antiderivative between the limits of integration. Whenever possible, however (i.e., when the integration is not too involved), you should apply integration by parts directly to the definite integral. Observe that the integration by parts algorithm for definite integrals is simply

Integration by Parts for a definite integral

$$\boxed{\int_{x=a}^{x=b} u\,dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du,}$$

where we have written the limits of integration as we have to remind you that these refer to the values of x. (Recall that we derived the integration by parts formula by taking u and v both to be functions of x.)

EXAMPLE 2.7 Integration by Parts for a Definite Integral

Evaluate $\int_1^2 x^3 \ln x\,dx$.

Solution Again, since more elementary methods are fruitless, we try integration by parts. Since we do not know how to integrate $\ln x$ (except via integration by parts), we choose

$$\begin{aligned} u &= \ln x & dv &= x^3\,dx \\ du &= \tfrac{1}{x}\,dx & v &= \tfrac{1}{4}x^4 \end{aligned}$$

and hence, we have

$$\int_1^2 \underbrace{\ln x}_{u}\, \underbrace{x^3 dx}_{dv} = uv\Big|_1^2 - \int_1^2 v\,du = \frac{1}{4}x^4 \ln x\Big|_1^2 - \frac{1}{4}\int_1^2 x^4\left(\frac{1}{x}\right)dx$$
$$= \frac{1}{4}(2^4 \ln 2 - 1^4 \ln 1) - \frac{1}{4}\int_1^2 x^3 dx$$
$$= \frac{16\ln 2}{4} - 0 - \frac{1}{16}x^4\Big|_1^2 = 4\ln 2 - \frac{1}{16}(2^4 - 1^4)$$
$$= 4\ln 2 - \frac{1}{16}(16-1) = 4\ln 2 - \frac{15}{16}.$$

Integration by parts is the most powerful tool in our integration arsenal. In order to master its use, you will need to work through many problems. We provide a wide assortment of these in the exercise set that follows.

EXERCISES 6.2

WRITING EXERCISES

1. Discuss your best strategy for determining which part of the integrand should be u and which part should be dv.
2. Integration by parts comes from the product rule for derivatives. Which integration technique comes from the chain rule? Briefly discuss why there is no commonly used integration technique derived from the quotient rule.

In exercises 1–26, evaluate the integrals.

1. $\int x\cos x\,dx$
2. $\int x\sin 4x\,dx$
3. $\int xe^{2x}dx$
4. $\int x\ln x\,dx$
5. $\int x^2\ln x\,dx$
6. $\int \frac{\ln x}{x}\,dx$
7. $\int x^2e^{-3x}dx$
8. $\int x^2e^{x^3}\,dx$
9. $\int e^x\sin 4x\,dx$
10. $\int e^{2x}\cos x\,dx$
11. $\int \cos x\cos 2x\,dx$
12. $\int \sin x\sin 2x\,dx$
13. $\int x\sec^2 x\,dx$
14. $\int x^3e^{x^2}dx$
15. $\int (\ln x)^2\,dx$
16. $\int x^2e^{3x}dx$
17. $\int \cos x\ln(\sin x)\,dx$
18. $\int x\sin x^2 dx$
19. $\int_0^1 x\sin 2x\,dx$
20. $\int_0^\pi 2x\cos x\,dx$
21. $\int_0^1 x\cos \pi x\,dx$
22. $\int_0^1 xe^{3x}dx$
23. $\int_0^1 x\sin \pi x\,dx$
24. $\int_0^1 x\cos 2\pi x\,dx$
25. $\int_1^{10} \ln x\,dx$
26. $\int_1^2 x\ln x\,dx$

In exercises 27–36, evaluate the integral using integration by parts and substitution. (As we recommended in the text, "Try something!")

27. $\int \cos^{-1} x\,dx$
28. $\int \tan^{-1} x\,dx$
29. $\int \sin\sqrt{x}\,dx$
30. $\int e^{\sqrt{x}}\,dx$
31. $\int \sin(\ln x)\,dx$
32. $\int x\ln(4+x^2)\,dx$
33. $\int e^{6x}\sin(e^{2x})\,dx$
34. $\int \cos\sqrt[3]{x}\,dx$
35. $\int_0^8 e^{\sqrt[3]{x}}dx$
36. $\int_0^1 x\tan^{-1} x\,dx$
37. How many times would integration by parts need to be performed to evaluate $\int x^n \sin x\,dx$ (where n is a positive integer)?
38. How many times would integration by parts need to be performed to evaluate $\int x^n \ln x\,dx$ (where n is a positive integer)?
39. Several useful integration formulas (called *reduction formulas*) are used to automate the process of performing multiple integrations by parts. Prove that for any positive integer n,

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

(Use integration by parts with $u = \cos^{n-1} x$ and $dv = \cos x\, dx$.)

40. Use integration by parts to prove that for any positive integer n,

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx.$$

In exercises 41–48, evaluate the integral using the reduction formulas from exercises 39 and 40 and (2.4).

41. $\int x^3 e^x\, dx$

42. $\int \cos^5 x\, dx$

43. $\int \cos^3 x\, dx$

44. $\int \sin^4 x\, dx$

45. $\int_0^1 x^4 e^x\, dx$

46. $\int_0^{\pi/2} \sin^4 x\, dx$

47. $\int_0^{\pi/2} \sin^5 x\, dx$

48. $\int_0^{\pi/2} \sin^6 x\, dx$

49. Based on exercises 46–48, conjecture a formula for $\int_0^{\pi/2} \sin^m x\ dx$. (Note: You will need different formulas for m odd and for m even.)

50. Conjecture a formula for $\int_0^{\pi/2} \cos^m x\ dx$.

51. The excellent movie *Stand and Deliver* tells the story of mathematics teacher Jaime Escalante, who developed a remarkable AP calculus program in inner-city Los Angeles. In one scene, Escalante shows a student how to evaluate the integral $\int x^2 \sin x\, dx$. He forms a chart like the following:

	$\sin x$	
x^2	$-\cos x$	+
$2x$	$-\sin x$	−
2	$\cos x$	+

Multiplying across each full row, the antiderivative is $-x^2 \cos x + 2x \sin x + 2\cos x + c$. Explain where each column comes from and why the method works on this problem.

In exercises 52–57, use the method of exercise 51 to evaluate the integral.

52. $\int x^4 \sin x\, dx$

53. $\int x^4 \cos x\, dx$

54. $\int x^4 e^x\, dx$

55. $\int x^4 e^{2x} dx$

56. $\int x^5 \cos 2x\, dx$

57. $\int x^3 e^{-3x} dx$

58. You should be aware that the method of exercise 51 doesn't always work, especially if both the derivative and antiderivative columns have powers of x. Show that the method doesn't work on $\int x^2 \ln x\, dx$.

59. Show that $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 0$ and $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = 0$ for positive integers $m \neq n$.

60. Show that $\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\, dx = 0$ for positive integers m and n and $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$, for any positive integer n.

61. Find all mistakes in the following (invalid) attempted proof that $0 = -1$. Start with $\int e^x e^{-x} dx$ and apply integration by parts with $u = e^x$ and $dv = e^{-x} dx$. This gives $\int e^x e^{-x} dx = -1 + \int e^x e^{-x} dx$. Then subtract $\int e^x e^{-x} dx$ to get $0 = -1$.

62. Find the volume of the solid formed by revolving the region bounded by $y = x\sqrt{\sin x}$ and $y = 0$ $(0 \le x \le \pi)$ about the x-axis.

63. Evaluate $\int e^x \left(\ln x + \frac{1}{x}\right) dx$ by using integration by parts on $\int e^x \ln x\, dx$.

64. Generalize the technique of exercise 63 to any integral of the form $\int e^x [f(x) + f'(x)]\, dx$. Prove your result without using integration by parts.

65. Use the quotient rule to show that $\int \frac{f'(x)}{g(x)} dx = \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{[g(x)]^2} dx$.

66. Derive the formula of exercise 65 using integration by parts with $u = \frac{1}{g(x)}$.

EXPLORATORY EXERCISES

1. Integration by parts can be used to compute coefficients for important functions called **Fourier series.** We cover Fourier series in detail in Chapter 8. Here, you will discover what some of the fuss is about. Start by computing $a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$ for an unspecified positive integer n. Write out the specific values for a_1, a_2, a_3 and a_4 and then form the function

$$f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x.$$

Compare the graphs of $y = x$ and $y = f(x)$ on the interval $[-\pi, \pi]$. From writing out a_1 through a_4, you should notice a nice pattern. Use it to form the function

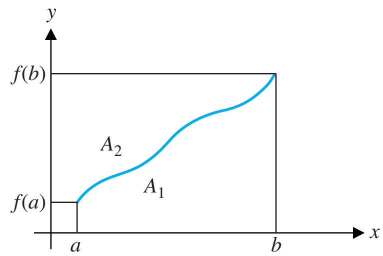
$$g(x) = f(x) + a_5 \sin 5x + a_6 \sin 6x + a_7 \sin 7x + a_8 \sin 8x.$$

Compare the graphs of $y = x$ and $y = g(x)$ on the interval $[-\pi, \pi]$. Is it surprising that you can add sine functions together and get something close to a straight line? It turns out that Fourier series can be used to find cosine and sine approximations to nearly any continuous function on a closed interval.

2. Along with giving us a technique to compute antiderivatives, integration by parts is very important theoretically. In this context, it can be thought of as a technique for moving derivatives off of one function and onto another. To see what we mean, suppose that $f(x)$ and $g(x)$ are functions with $f(0) = g(0) = 0$, $f(1) = g(1) = 0$ and with continuous second derivatives $f''(x)$ and $g''(x)$. Use integration by parts twice to show that

$$\int_0^1 f''(x)g(x)\,dx = \int_0^1 f(x)g''(x)\,dx.$$

3. Assume that f is an increasing continuous function on $[a, b]$ with $0 \le a < b$ and $f(x) \ge 0$. Let A_1 be the area under $y = f(x)$ from $x = a$ to $x = b$ and let A_2 be the area to the left of $y = f(x)$ from $f(a)$ to $f(b)$. Show that $A_1 + A_2 = bf(b) - af(a)$ and $\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$. Use this result to evaluate $\int_0^{\pi/4} \tan^{-1} x\,dx$.



4. Assume that f is a function with a continuous second derivative. Show that $f(b) = f(a) + f'(a)(b - a) + \int_a^b f''(x)(b - x)\,dx$. Use this result to show that the error in the approximation $\sin x \approx x$ is at most $\frac{1}{2}x^2$.

6.3 TRIGONOMETRIC TECHNIQUES OF INTEGRATION

Integrals Involving Powers of Trigonometric Functions

Evaluating an integral whose integrand contains powers of one or more trigonometric functions often involves making a clever substitution. These integrals are sufficiently common that we present them here as a group.

We first consider integrals of the form

$$\int \sin^m x \cos^n x\,dx,$$

where m and n are positive integers.

Case 1: *m* or *n* Is an Odd Positive Integer

If m is odd, first isolate one factor of $\sin x$. (You'll need this for du.) Then, replace any factors of $\sin^2 x$ with $1 - \cos^2 x$ and make the substitution $u = \cos x$. Likewise, if n is odd, first isolate one factor of $\cos x$. (You'll need this for du.) Then, replace any factors of $\cos^2 x$ with $1 - \sin^2 x$ and make the substitution $u = \sin x$.

We illustrate this for the case where m is odd in example 3.1.

EXAMPLE 3.1 A Typical Substitution

Evaluate $\int \cos^4 x \sin x\,dx$.

Solution Since you cannot evaluate this integral as it stands, you should consider substitution. (Hint: Look for terms that are derivatives of other terms.) Here, letting $u = \cos x$, so that $du = -\sin x\,dx$, gives us

$$\int \cos^4 x \sin x\,dx = -\int \underbrace{\cos^4 x}_{u^4}\,\underbrace{(-\sin x)\,dx}_{du} = -\int u^4\,du$$

$$= -\frac{u^5}{5} + c = -\frac{\cos^5 x}{5} + c. \quad \text{Since } u = \cos x.$$

While this first example was not particularly challenging, it should give you an idea of what to do with example 3.2.

EXAMPLE 3.2 An Integrand with an Odd Power of Sine

Evaluate $\int \cos^4 x \sin^3 x \, dx$.

Solution If you're looking for terms that are derivatives of other terms, you should see both sine and cosine terms, but for which do you substitute? Here, with $u = \cos x$, we have $du = -\sin x \, dx$, so that

$$\begin{aligned}
\int \cos^4 x \sin^3 x \, dx &= \int \cos^4 x \sin^2 x \sin x \, dx = -\int \cos^4 x \sin^2 x (-\sin x) \, dx \\
&= -\int \underbrace{\cos^4 x (1 - \cos^2 x)}_{u^4(1-u^2)} \underbrace{(-\sin x) \, dx}_{du} = -\int u^4(1-u^2) \, du \\
&= -\int (u^4 - u^6) \, du = -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + c \\
&= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c. \quad \text{Since } u = \cos x.
\end{aligned}$$

Pay close attention to how we did this. We took the odd power (in this case, $\sin^3 x$) and factored out one power of $\sin x$ (to use for du). The remaining (even) powers of $\sin x$ were rewritten in terms of $\cos x$ using the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1.$$

The ideas used in example 3.2 can be applied to any integral of the specified form.

EXAMPLE 3.3 An Integrand with an Odd Power of Cosine

Evaluate $\int \sqrt{\sin x} \cos^5 x \, dx$.

Solution Observe that we can rewrite this as

$$\int \sqrt{\sin x} \cos^5 x \, dx = \int \sqrt{\sin x} \cos^4 x \cos x \, dx = \int \sqrt{\sin x} (1 - \sin^2 x)^2 \cos x \, dx.$$

Substituting $u = \sin x$, so that $du = \cos x \, dx$, we have

$$\begin{aligned}
\int \sqrt{\sin x} \cos^5 x \, dx &= \int \underbrace{\sqrt{\sin x} (1 - \sin^2 x)^2}_{\sqrt{u}(1-u^2)^2} \underbrace{\cos x \, dx}_{du} \\
&= \int \sqrt{u}(1-u^2)^2 \, du = \int u^{1/2}(1 - 2u^2 + u^4) \, du \\
&= \int (u^{1/2} - 2u^{5/2} + u^{9/2}) \, du \\
&= \frac{2}{3}u^{3/2} - 2\left(\frac{2}{7}\right)u^{7/2} + \frac{2}{11}u^{11/2} + c \\
&= \frac{2}{3}\sin^{3/2} x - \frac{4}{7}\sin^{7/2} x + \frac{2}{11}\sin^{11/2} x + c. \quad \text{Since } u = \sin x.
\end{aligned}$$

Looking beyond the details of calculation here, you should see the main point: that all integrals of this form are calculated in essentially the same way.

NOTES

Half-angle formulas

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

Case 2: *m* and *n* Are Both Even Positive Integers

In this case, we can use the half-angle formulas for sine and cosine (shown in the margin) to reduce the powers in the integrand.

We illustrate this case in example 3.4.

EXAMPLE 3.4 An Integrand with an Even Power of Sine

Evaluate $\int \sin^2 x\,dx$.

Solution Using the half-angle formula, we can rewrite the integral as

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1 - \cos 2x)\,dx.$$

We can evaluate this last integral by using the substitution $u = 2x$, so that $du = 2\,dx$. This gives us

$$\int \sin^2 x\,dx = \frac{1}{2}\left(\frac{1}{2}\right)\int \underbrace{(1 - \cos 2x)}_{1-\cos u}\underbrace{2\,dx}_{du} = \frac{1}{4}\int (1 - \cos u)\,du$$
$$= \frac{1}{4}(u - \sin u) + c = \frac{1}{4}(2x - \sin 2x) + c. \quad \text{Since } u = 2x. \blacksquare$$

With some integrals, you may need to apply the half-angle formulas several times, as in example 3.5.

EXAMPLE 3.5 An Integrand with an Even Power of Cosine

Evaluate $\int \cos^4 x\,dx$.

Solution Using the half-angle formula for cosine, we have

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx = \frac{1}{4}\int (1 + \cos 2x)^2\,dx$$
$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx.$$

Using the half-angle formula again, on the last term in the integrand, we get

$$\int \cos^4 x\,dx = \frac{1}{4}\int \left[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]dx$$
$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c,$$

where we leave the details of the final integration as an exercise. ■

Our next aim is to devise a strategy for evaluating integrals of the form

$$\int \tan^m x \sec^n x\,dx,$$

where m and n are integers.

Case 1: *m* Is an Odd Positive Integer

First, isolate one factor of $\sec x \tan x$. (You'll need this for du.) Then, replace any factors of $\tan^2 x$ with $\sec^2 x - 1$ and make the substitution $u = \sec x$.

We illustrate this in example 3.6.

EXAMPLE 3.6 An Integrand with an Odd Power of Tangent

Evaluate $\int \tan^3 x \sec^3 x \, dx$.

Solution Looking for terms that are derivatives of other terms, we rewrite the integral as

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx$$
$$= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) \, dx,$$

where we have used the Pythagorean identity

$$\tan^2 x = \sec^2 x - 1.$$

You should see the substitution now. We let $u = \sec x$, so that $du = \sec x \tan x \, dx$ and hence,

$$\int \tan^3 x \sec^3 x \, dx = \int \underbrace{(\sec^2 x - 1) \sec^2 x}_{(u^2-1)u^2} \underbrace{(\sec x \tan x) \, dx}_{du}$$
$$= \int (u^2 - 1) u^2 du = \int (u^4 - u^2) \, du$$
$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + c = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + c. \quad \text{Since } u = \sec x.$$

Case 2: *n* Is an Even Positive Integer

First, isolate one factor of $\sec^2 x$. (You'll need this for du.) Then, replace any remaining factors of $\sec^2 x$ with $1 + \tan^2 x$ and make the substitution $u = \tan x$.

We illustrate this in example 3.7.

EXAMPLE 3.7 An Integrand with an Even Power of Secant

Evaluate $\int \tan^2 x \sec^4 x \, dx$.

Solution Since $\frac{d}{dx} \tan x = \sec^2 x$, we rewrite the integral as

$$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx.$$

Now, we let $u = \tan x$, so that $du = \sec^2 x \, dx$ and

$$\int \tan^2 x \sec^4 x \, dx = \int \underbrace{\tan^2 x (1 + \tan^2 x)}_{u^2(1+u^2)} \underbrace{\sec^2 x \, dx}_{du}$$
$$= \int u^2 (1 + u^2) \, du = \int (u^2 + u^4) \, du$$
$$= \frac{1}{3} u^3 + \frac{1}{5} u^5 + c$$
$$= \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + c. \quad \text{Since } u = \tan x.$$

Case 3: *m* Is an Even Positive Integer and *n* Is an Odd Positive Integer
Replace any factors of $\tan^2 x$ with $\sec^2 x - 1$ and then use a special *reduction formula* (given in the exercises) to evaluate integrals of the form $\int \sec^n x\, dx$. This complicated case will be covered briefly in the exercises. Much of this depends on example 3.8.

EXAMPLE 3.8 An Unusual Integral

Evaluate the integral $\int \sec x\, dx$.

Solution Finding an antiderivative here depends on an unusual observation. Notice that if we multiply the integrand by the fraction $\dfrac{\sec x + \tan x}{\sec x + \tan x}$ (which is of course equal to 1), we get

$$\begin{aligned}\int \sec x\, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.\end{aligned}$$

Now, observe that the numerator is exactly the derivative of the denominator. That is,

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x,$$

so that taking $u = \sec x + \tan x$ gives us

$$\begin{aligned}\int \sec x\, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{1}{u}\, du = \ln|u| + c \\ &= \ln|\sec x + \tan x| + c. \qquad \text{Since } u = \sec x + \tan x.\end{aligned}$$

Trigonometric Substitution

If an integral contains a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$, for some $a > 0$, you can often evaluate the integral by making a substitution involving a trig function (hence, the name *trigonometric substitution*).

First, suppose that an integrand contains a term of the form $\sqrt{a^2 - x^2}$, for some $a > 0$. Letting $x = a \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we can eliminate the square root, as follows:

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} = \sqrt{a^2 - a^2\sin^2\theta} \\ &= a\sqrt{1 - \sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta,\end{aligned}$$

since for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$. Example 3.9 is typical of how these substitutions are used.

NOTE

Terms of the form $\sqrt{a^2 - x^2}$ can also be simplified using the substitution $x = a\cos\theta$, using a different restriction for θ.

EXAMPLE 3.9 An Integral Involving $\sqrt{a^2 - x^2}$

Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{4 - x^2}}\, dx$.

Solution You should always first consider whether an integral can be done directly, by substitution or by parts. Since none of these methods help here, we consider trigonometric substitution. Keep in mind that the immediate objective here is to eliminate the square root. A substitution that will accomplish this is

$$x = 2\sin\theta, \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(Why do we need *strict* inequalities here?) This gives us

$$dx = 2\cos\theta\, d\theta$$

and hence,

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4-x^2}}\,dx &= \int \frac{1}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}\,2\cos\theta\,d\theta \\ &= \int \frac{2\cos\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}\,d\theta \\ &= \int \frac{\cos\theta}{(2\sin^2\theta)\,2\sqrt{1-\sin^2\theta}}\,d\theta \\ &= \int \frac{\cos\theta}{4\sin^2\theta\cos\theta}\,d\theta \qquad \text{Since } 1-\sin^2\theta = \cos^2\theta. \\ &= \frac{1}{4}\int \csc^2\theta\,d\theta = -\frac{1}{4}\cot\theta + c. \qquad \text{Since } \frac{1}{\sin^2\theta} = \csc^2\theta.\end{aligned}$$

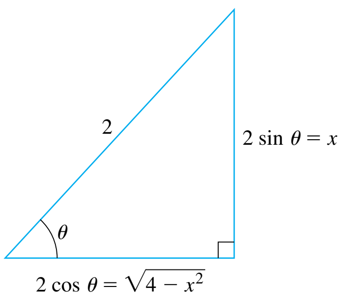


FIGURE 6.1

The only remaining problem is that the antiderivative is presently written in terms of the variable θ. When converting back to the original variable $x = 2\sin\theta$, we urge you to draw a diagram, as in Figure 6.1. Since the substitution was $x = 2\sin\theta$, we have $\sin\theta = \frac{x}{2} = \frac{\text{opposite}}{\text{hypotenuse}}$ and so we label the hypotenuse as 2. The side opposite the angle θ is then $2\sin\theta$. By the Pythagorean Theorem, we get that the adjacent side is $\sqrt{4-x^2}$, as indicated. So, we have

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{4-x^2}}{x}.$$

It now follows that

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\cot\theta + c = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + c.$$

Next, suppose that an integrand contains a term of the form $\sqrt{a^2+x^2}$, for some $a > 0$. Taking $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we eliminate the square root, as follows:

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2+a^2\tan^2\theta} \\ &= a\sqrt{1+\tan^2\theta} = a\sqrt{\sec^2\theta} = a\sec\theta,\end{aligned}$$

since for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\sec\theta \ge 0$. Example 3.10 is typical of how these substitutions are used.

EXAMPLE 3.10 An Integral Involving $\sqrt{a^2+x^2}$

Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$.

Solution You can eliminate the square root by letting $x = 3\tan\theta$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This gives us $dx = 3\sec^2\theta d\theta$, so that

$$\begin{aligned}\int \frac{1}{\sqrt{9+x^2}}\,dx &= \int \frac{1}{\sqrt{9+(3\tan\theta)^2}}\,3\sec^2\theta\,d\theta \\ &= \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta \\ &= \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,d\theta \\ &= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta \qquad \text{Since } 1+\tan^2\theta = \sec^2\theta. \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + c,\end{aligned}$$

from example 3.8. We're not done here, though, since we must still express the integral in terms of the original variable x. Observe that we had $x = 3\tan\theta$, so that $\tan\theta = \frac{x}{3}$. It remains only to solve for $\sec\theta$. Although you can do this with a triangle, as in example 3.9, the simplest way to do this is to recognize that for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{3}\right)^2}.$$

This leaves us with
$$\begin{aligned}\int \frac{1}{\sqrt{9+x^2}}\,dx &= \ln|\sec\theta + \tan\theta| + c \\ &= \ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2} + \frac{x}{3}\right| + c.\end{aligned}$$

Finally, suppose that an integrand contains a term of the form $\sqrt{x^2 - a^2}$, for some $a > 0$. Taking $x = a\sec\theta$, where $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, we eliminate the square root, as follows:

$$\begin{aligned}\sqrt{x^2-a^2} &= \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} \\ &= a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a\,|\tan\theta|.\end{aligned}$$

Notice that the absolute values are needed, as $\tan\theta$ can be both positive and negative on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Example 3.11 is typical of how these substitutions are used.

EXAMPLE 3.11 An Integral Involving $\sqrt{x^2 - a^2}$

Evaluate the integral $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$, for $x > 5$.

Solution Here, we let $x = 5\sec\theta$, for $\theta \in \left[0, \frac{\pi}{2}\right)$, where we chose the first half of the domain $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, so that $x = 5\sec\theta > 5$. (If we had $x < -5$, we would have chosen $\theta \in (\frac{\pi}{2}, \pi]$.) This gives us $dx = 5\sec\theta\tan\theta\, d\theta$ and the integral then becomes:

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{(5\sec\theta)^2-25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta \\ &= \int \sqrt{25\sec^2\theta-25}\tan\theta\,d\theta \\ &= \int 5\sqrt{\sec^2\theta-1}\tan\theta\,d\theta \\ &= 5\int \tan^2\theta\,d\theta \qquad \text{Since } \sec^2\theta - 1 = \tan^2\theta. \\ &= 5\int (\sec^2\theta-1)\,d\theta \\ &= 5(\tan\theta-\theta)+c.\end{aligned}$$

Finally, observe that since $x = 5\sec\theta$, for $\theta \in [0, \frac{\pi}{2})$, we have that

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{5}\right)^2 - 1} = \frac{1}{5}\sqrt{x^2-25}$$

and $\theta = \sec^{-1}\left(\frac{x}{5}\right)$. We now have

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x}\,dx &= 5(\tan\theta-\theta)+c \\ &= \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)+c.\end{aligned}$$

You will find a number of additional integrals requiring trigonometric substitution in the exercises. The principal idea here is to see that you can eliminate certain square root terms in an integrand by making use of a carefully chosen trigonometric substitution.

We summarize the three trigonometric substitutions presented here in the following table.

Expression	*Trigonometric Substitution*	*Interval*	*Identity*
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$\sec^2\theta - 1 = \tan^2\theta$

EXERCISES 6.3

WRITING EXERCISES

1. Suppose a friend in your calculus class tells you that this section just has too many rules to memorize. (By the way, the authors would agree.) Help your friend out by making it clear that each rule indicates when certain substitutions will work. In turn, a substitution $u(x)$ works if the expression $u'(x)$ appears in the integrand and the resulting integral is easier to integrate. For each of the rules covered in the text, identify $u'(x)$ and point out why n has to be odd (or whatever the rule says) for the remaining integrand to be workable. Without memorizing rules, you can remember a small number of potential substitutions and see which one works for a given problem.

2. In the text, we suggested that when the integrand contains a term of the form $\sqrt{4-x^2}$, you might try the trigonometric substitution $x = 2\sin\theta$. We should admit now that this does not always work. How can you tell whether this substitution will work?

In exercises 1–30, evaluate the integrals.

1. $\int \cos x \sin^4 x\, dx$
2. $\int \cos^3 x \sin^4 x\, dx$
3. $\int_0^{\pi/4} \cos x \sin^3 x\, dx$
4. $\int_{\pi/4}^{\pi/3} \cos^3 x \sin^3 x\, dx$
5. $\int_0^{\pi/2} \cos^2 x \sin x\, dx$
6. $\int_{-\pi/2}^{0} \cos^3 x \sin x\, dx$
7. $\int \cos^2 x\, dx$
8. $\int \sin^4 x\, dx$
9. $\int \tan x \sec^3 x\, dx$
10. $\int \cot x \csc^4 x\, dx$
11. $\int_0^{\pi/4} \tan^4 x \sec^4 x\, dx$
12. $\int_{-\pi/4}^{\pi/4} \tan^4 x \sec^2 x\, dx$
13. $\int \cos^2 x \sin^2 x\, dx$
14. $\int (\cos^2 x + \sin^2 x)\, dx$
15. $\int_{-\pi/3}^{0} \sqrt{\cos x} \sin^3 x\, dx$
16. $\int_{\pi/4}^{\pi/2} \cot^2 x \csc^4 x\, dx$
17. $\int \frac{1}{x^2\sqrt{9-x^2}}\, dx$
18. $\int \frac{1}{x^2\sqrt{16-x^2}}\, dx$
19. $\int_0^2 \sqrt{4-x^2}\, dx$
20. $\int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx$
21. $\int \frac{x^2}{\sqrt{x^2-9}}\, dx$
22. $\int x^3\sqrt{x^2-1}\, dx$
23. $\int \frac{2}{\sqrt{x^2-4}}\, dx$
24. $\int \frac{x}{\sqrt{x^2-4}}\, dx$
25. $\int \frac{x^2}{\sqrt{x^2+9}}\, dx$
26. $\int x^3\sqrt{x^2+8}\, dx$
27. $\int \sqrt{x^2+16}\, dx$
28. $\int \frac{1}{\sqrt{x^2+4}}\, dx$
29. $\int_0^1 x\sqrt{x^2+8}\, dx$
30. $\int_0^2 x^2\sqrt{x^2+9}\, dx$

In exercises 31 and 32, evaluate the integral using both substitutions $u = \tan x$ and $u = \sec x$ and compare the results.

31. $\int \tan x \sec^4 x\, dx$
32. $\int \tan^3 x \sec^4 x\, dx$

33. Show that for any integer $n > 1$, we have the reduction formula
$$\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx.$$

34. Evaluate (a) $\int \sec^3 x\, dx$, (b) $\int \sec^4 x\, dx$ and (c) $\int \sec^5 x\, dx$. Hint: Use the result of exercise 33.

35. In an AC circuit, the current has the form $i(t) = I\cos(\omega t)$ for constants I and ω. The power is defined as Ri^2 for a constant R. Find the average value of the power by integrating over the interval $[0, 2\pi/\omega]$.

36. The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $\frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$. Compute this integral.

37. Evaluate the antiderivatives in examples 3.2, 3.3, 3.5, 3.6 and 3.7 using your CAS. Based on these examples, speculate whether or not your CAS uses the same techniques that we do. In the cases where your CAS gives a different antiderivative than we do, comment on which antiderivative looks simpler.

38. Repeat exercise 37 for examples 3.9, 3.10 and 3.11.

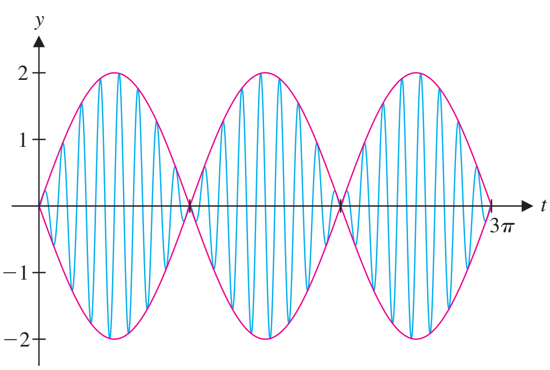
39. One CAS produces $-\frac{1}{7}\sin^2 x\cos^5 x - \frac{2}{35}\cos^5 x$ as an antiderivative in example 3.2. Find c such that this equals our antiderivative of $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c$.

40. One CAS produces $-\frac{2}{15}\tan x - \frac{1}{15}\sec^2 x\tan x + \frac{1}{5}\sec^4 x\tan x$ as an antiderivative in example 3.7. Find c such that this equals our antiderivative of $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$.

EXPLORATORY EXERCISES

1. In section 6.2, you were asked to show that for positive integers m and n with $m \neq n$, $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0$, $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0$. Also, $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$. Finally, $\int_{-\pi}^{\pi} \cos mx \sin nx\, dx = 0$, for any positive integers m and n. We will use these formulas to explain how a radio can tune in an AM station.

Amplitude modulation (or AM) radio sends a signal (e.g., music) that modulates the carrier frequency. For example, if the signal is $2\sin t$ and the carrier frequency is 16, then the radio sends out the modulated signal $2\sin t \sin 16t$. The graphs of $y = 2\sin t$, $y = -2\sin t$ and $y = 2\sin t \sin 16t$ are shown in the figure.



The graph of $y = 2\sin t \sin 16t$ oscillates as rapidly as the carrier $\sin 16t$, but the amplitude varies between $2\sin t$ and $-2\sin t$ (hence the term amplitude modulation). The radio's problem is to tune in the frequency 16 and recover the signal $2\sin t$. The difficulty is that other radio stations are broadcasting simultaneously. A radio receives all the signals mixed together. To see how this works, suppose a second station broadcasts the signal $3\sin t$ at frequency 32. The combined signal that the radio receives is $2\sin t\sin 16t + 3\sin t\sin 32t$. We will decompose this signal. The first step is to rewrite the signal using the identity

$$\sin A\sin B = \frac{1}{2}\cos(B-A) - \frac{1}{2}\cos(B+A).$$

The signal then equals

$$f(t) = \cos 15t - \cos 17t + \frac{3}{2}\cos 31t - \frac{3}{2}\cos 33t.$$

If the radio "knows" that the signal has the form $c\sin t$, for some constant c, it can determine the constant c at frequency 16 by computing the integral $\int_{-\pi}^{\pi} f(t)\cos 15t\,dt$ and multiplying by $2/\pi$. Show that $\int_{-\pi}^{\pi} f(t)\cos 15t\,dt = \pi$, so that the correct constant is $c = \pi(2/\pi) = 2$. The signal is then $2\sin t$. To recover the signal sent out by the second station, compute $\int_{-\pi}^{\pi} f(t)\cos 31t\,dt$ and multiply by $2/\pi$. Show that you correctly recover the signal $3\sin t$.

2. In this exercise, we derive an important result called **Wallis' product.** Define the integral $I_n = \int_0^{\pi/2}\sin^n x\,dx$ for a positive integer n. (a) Show that $I_n = \frac{n}{n-1}I_{n-2}$. (b) Show that $\frac{I_{2n+1}}{I_{2n}} = \frac{2^2 4^2\cdots(2n)^2 2}{3^2 5^2\cdots(2n-1)^2(2n+1)\pi}$. (c) Conclude that $\frac{\pi}{2} = \lim_{n\to\infty}\frac{2^2 4^2\cdots(2n)^2}{3^2 5^2\cdots(2n-1)^2(2n+1)}$.

6.4 INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

In this section we introduce a method for rewriting certain rational functions that is very useful in integration as well as in other applications. We begin with a simple observation. Note that

$$\frac{3}{x+2} - \frac{2}{x-5} = \frac{3(x-5)-2(x+2)}{(x+2)(x-5)} = \frac{x-19}{x^2-3x-10}. \tag{4.1}$$

So, suppose that you wanted to evaluate the integral of the function on the right-hand side of (4.1). While it's not clear how to evaluate this integral, the integral of the (equivalent) function on the left-hand side of (4.1) is easy to evaluate. From (4.1), we now have

$$\int \frac{x-19}{x^2-3x-10}\,dx = \int\left(\frac{3}{x+2} - \frac{2}{x-5}\right)dx = 3\ln|x+2| - 2\ln|x-5| + c.$$

The second integrand,

$$\frac{3}{x+2} - \frac{2}{x-5}$$

is called a **partial fractions decomposition** of the first integrand. More generally, if the three factors a_1x+b_1, a_2x+b_2 and a_3x+b_3 are all distinct (i.e., none is a constant multiple of another), then we can write

$$\boxed{\frac{a_1x+b_1}{(a_2x+b_2)(a_3x+b_3)} = \frac{A}{a_2x+b_2} + \frac{B}{a_3x+b_3},}$$

for some choice of constants A and B to be determined. Notice that if you wanted to integrate this expression, the partial fractions on the right-hand side are very easy to integrate, just as they were in the introductory example just presented.

EXAMPLE 4.1 Partial Fractions: Distinct Linear Factors

Evaluate $\displaystyle\int \frac{1}{x^2+x-2}\,dx.$

Solution First, note that you can't evaluate this as it stands and all of our earlier methods fail to help. (Consider each of these for this problem.) However, we can make a partial fractions decomposition, as follows.

$$\frac{1}{x^2+x-2}=\frac{1}{(x-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+2}.$$

Multiplying both sides of this equation by the common denominator $(x-1)(x+2)$, we get

$$1=A(x+2)+B(x-1). \tag{4.2}$$

We would like to solve this equation for A and B. The key is to realize that this equation must hold for all x, including $x=1$ and $x=-2$. [We single out these two values because they will make one or the other of the terms in (4.2) zero and thereby allow us to easily solve for the unknowns A and B.] In particular, for $x=1$, notice that from (4.2), we have

$$1=A(1+2)+B(1-1)=3A,$$

so that $A=\frac{1}{3}$. Likewise, taking $x=-2$, we have

$$1=A(-2+2)+B(-2-1)=-3B,$$

so that $B=-\frac{1}{3}$. Thus, we have

$$\begin{aligned}\int\frac{1}{x^2+x-2}\,dx&=\int\left[\frac{1}{3}\left(\frac{1}{x-1}\right)-\frac{1}{3}\left(\frac{1}{x+2}\right)\right]dx\\&=\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln|x+2|+c.\end{aligned}$$

We can do the same as we did in example 4.1 whenever a rational expression has a denominator that factors into n distinct linear factors, as follows. If the degree of $P(x)<n$ and the factors (a_ix+b_i), for $i=1,2,\ldots,n$ are all distinct, then we can write

Partial fractions: distinct linear factors

$$\boxed{\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)}=\frac{c_1}{a_1x+b_1}+\frac{c_2}{a_2x+b_2}+\cdots+\frac{c_n}{a_nx+b_n},}$$

for some constants $c_1, c_2, \ldots, c_n$.

EXAMPLE 4.2 Partial Fractions: Three Distinct Linear Factors

Evaluate $\displaystyle\int\frac{3x^2-7x-2}{x^3-x}\,dx$.

Solution Once again, our earlier methods fail us, but we can rewrite the integrand using partial fractions. We have

$$\frac{3x^2-7x-2}{x^3-x}=\frac{3x^2-7x-2}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}.$$

Multiplying by the common denominator $x(x-1)(x+1)$, we get

$$3x^2-7x-2=A(x-1)(x+1)+Bx(x+1)+Cx(x-1). \tag{4.3}$$

In this case, notice that taking $x=0$, $x=1$ or $x=-1$ will make two of the three terms on the right side of (4.3) zero. Specifically, for $x=0$, we get

$$-2=A(-1)(1)=-A,$$

so that $A = 2$. Likewise, taking $x = 1$, we find $B = -3$ and taking $x = -1$, we find $C = 4$. Thus, we have

$$\int \frac{3x^2 - 7x - 2}{x^3 - x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}\right) dx$$
$$= 2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + c. \blacksquare$$

EXAMPLE 4.3 Partial Fractions Where Long Division Is Required

Find the indefinite integral of $f(x) = \dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}$ using a partial fractions decomposition.

Solution Since the degree of the numerator exceeds that of the denominator, first divide. We show the long division below. (You should perform the division however you are most comfortable.)

$$\begin{array}{r} 2x \phantom{{}-4x^2 - 15x + 5} \\ x^2 - 2x - 8 \overline{\big) 2x^3 - 4x^2 - 15x + 5} \\ \underline{2x^3 - 4x^2 - 16x \phantom{{}+5}} \\ x + 5 \end{array}$$

Thus, we have $\quad f(x) = \dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \dfrac{x+5}{x^2 - 2x - 8}.$

The remaining proper fraction can be expanded as

$$\frac{x+5}{x^2 - 2x - 8} = \frac{x+5}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}.$$

It is a simple matter to solve for the constants: $A = \frac{3}{2}$ and $B = -\frac{1}{2}$. (This is left as an exercise.) We now have

$$\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx = \int \left[2x + \frac{3}{2}\left(\frac{1}{x-4}\right) - \frac{1}{2}\left(\frac{1}{x+2}\right)\right] dx$$
$$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + c. \blacksquare$$

REMARK 4.1

If the numerator of a rational expression has the same or higher degree than the denominator, you must first perform a long division and follow this with a partial fractions decomposition of the remaining (proper) fraction.

You may already have begun to wonder what happens when the denominator of a rational expression contains repeated linear factors, such as

$$\frac{2x+3}{(x-1)^2}.$$

In this case, the decomposition looks like the following. If the degree of $P(x)$ is less than n, then we can write

Partial fractions: repeated linear factors

$$\boxed{\begin{array}{l} \dfrac{P(x)}{(ax+b)^n} = \dfrac{c_1}{ax+b} + \dfrac{c_2}{(ax+b)^2} + \cdots + \dfrac{c_n}{(ax+b)^n}, \\ \text{for constants } c_1, c_2, \ldots, c_n \text{ to be determined.} \end{array}}$$

Example 4.4 is typical.

EXAMPLE 4.4 Partial Fractions with a Repeated Linear Factor

Use a partial fractions decomposition to find an antiderivative of

$$f(x) = \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Solution First, note that there is a repeated linear factor in the denominator. We have

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying by the common denominator $x(x+1)^2$, we have

$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx.$$

Taking $x = 0$, we find $A = 6$. Likewise, taking $x = -1$, we find that $C = 9$. To determine B, substitute any convenient value for x, say $x = 1$. (Unfortunately, notice that there is no choice of x that will make the two terms containing A and C both zero, without also making the term containing B zero.) You should find that $B = -1$. So, we have

$$\begin{aligned}\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx &= \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right] dx \\ &= 6\ln|x| - \ln|x+1| - 9(x+1)^{-1} + c.\end{aligned}$$ ■

We can extend the notion of partial fractions decomposition to rational expressions with denominators containing irreducible quadratic factors (i.e., quadratic factors that have no real factorization). If the degree of $P(x)$ is less than $2n$ (the degree of the denominator) and all of the factors in the denominator are distinct, then we can write

Partial fractions: irreducible quadratic factors

$$\boxed{\begin{aligned}&\frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_nx^2 + b_nx + c_n)} \\ &\quad = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n}.\end{aligned}} \tag{4.4}$$

Think of this in terms of irreducible quadratic denominators in a partial fractions decomposition getting linear numerators, while linear denominators get constant numerators. If you think this looks messy, you're right, but only the algebra is messy (and you can always use a CAS to do the algebra for you). You should note that the partial fractions on the right-hand side of (4.4) are integrated comparatively easily using substitution together with possibly completing the square.

EXAMPLE 4.5 Partial Fractions with a Quadratic Factor

Use a partial fractions decomposition to find an antiderivative of $f(x) = \dfrac{2x^2 - 5x + 2}{x^3 + x}$.

Solution First, note that

$$\frac{2x^2 - 5x + 2}{x^3 + x} = \frac{2x^2 - 5x + 2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Multiplying through by the common denominator $x(x^2+1)$ gives us

$$\begin{aligned} 2x^2 - 5x + 2 &= A(x^2+1) + (Bx+C)x \\ &= (A+B)x^2 + Cx + A. \end{aligned}$$

Rather than substitute numbers for x (notice that there are no convenient values to plug in, except for $x = 0$), we instead match up the coefficients of like powers of x:

$$\begin{aligned} 2 &= A + B \\ -5 &= C \\ 2 &= A. \end{aligned}$$

This leaves us with $B = 0$ and so,

$$\int \frac{2x^2 - 5x + 2}{x^3 + x}\,dx = \int \left(\frac{2}{x} - \frac{5}{x^2+1}\right) dx = 2\ln|x| - 5\tan^{-1} x + c.$$

Partial fractions decompositions involving irreducible quadratic terms often lead to expressions that require further massaging (such as completing the square) before we can find an antiderivative. We illustrate this in example 4.6.

EXAMPLE 4.6 Partial Fractions with a Quadratic Factor

Use a partial fractions decomposition to find an antiderivative for

$$f(x) = \frac{5x^2 + 6x + 2}{(x+2)(x^2+2x+5)}.$$

Solution First, notice that the quadratic factor in the denominator does not factor and so, the correct decomposition is

$$\frac{5x^2 + 6x + 2}{(x+2)(x^2+2x+5)} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 2x + 5}.$$

Multiplying through by $(x+2)(x^2+2x+5)$, we get

$$5x^2 + 6x + 2 = A(x^2 + 2x + 5) + (Bx + C)(x + 2).$$

Matching up the coefficients of like powers of x, we get

$$\begin{aligned} 5 &= A + B \\ 6 &= 2A + 2B + C \\ 2 &= 5A + 2C. \end{aligned}$$

You'll need to solve this by elimination. We leave it as an exercise to show that $A = 2$, $B = 3$ and $C = -4$. Integrating, we have

$$\int \frac{5x^2 + 6x + 2}{(x+2)(x^2+2x+5)}\,dx = \int \left(\frac{2}{x+2} + \frac{3x - 4}{x^2 + 2x + 5}\right) dx. \tag{4.5}$$

The integral of the first term is easy, but what about the second term? Since the denominator doesn't factor, you have very few choices. Try substituting for the denominator: let $u = x^2 + 2x + 5$, so that $du = (2x + 2)\,dx$. Notice that we can now write the integral of the second term as

$$\int \frac{3x-4}{x^2+2x+5}\,dx = \int \frac{3(x+1)-7}{x^2+2x+5}\,dx = \int \left[\left(\frac{3}{2}\right)\frac{2(x+1)}{x^2+2x+5} - \frac{7}{x^2+2x+5}\right]dx$$
$$= \frac{3}{2}\int \frac{2(x+1)}{x^2+2x+5}\,dx - \int \frac{7}{x^2+2x+5}\,dx$$
$$= \frac{3}{2}\ln(x^2+2x+5) - \int \frac{7}{x^2+2x+5}\,dx. \quad (4.6)$$

Completing the square in the denominator of the remaining integral, we get

$$\int \frac{7}{x^2+2x+5}\,dx = \int \frac{7}{(x+1)^2+4}\,dx = \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c.$$

(We leave the details of this last integration as an exercise.) Putting this together with (4.5) and (4.6), we now have

$$\int \frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx = 2\ln|x+2| + \frac{3}{2}\ln(x^2+2x+5) - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c.$$

REMARK 4.2

Most CASs include commands for performing partial fractions decomposition. Even so, we urge you to work through the exercises in this section by hand. Once you have the idea of how these decompositions work, by all means, use your CAS to do the drudge work for you. Until that time, be patient and work carefully by hand.

Rational expressions with repeated irreducible quadratic factors in the denominator are explored in the exercises. The idea of these is the same as the preceding decompositions, but the algebra (without a CAS) is even messier.

After mastering decompositions involving repeated irreducible quadratic factors, you will be able to find the partial fractions decomposition of *any* rational function. Theoretically, the denominator of such a function (a polynomial) can always be factored into linear and quadratic factors, some of which may be repeated. Then, use the techniques covered in this section. You should recognize that while this observation is certainly true, in practice you may require a CAS to accurately complete the calculations.

Brief Summary of Integration Techniques

At this point, we pause to briefly summarize what we have learned about techniques of integration. As you certainly recognize by now, integration is far less straightforward than differentiation. You can differentiate virtually any function that you can write down, simply by applying the formulas. We are not nearly so fortunate with integrals. Many cannot be evaluated at all exactly, while others can be evaluated, but only by recognizing which technique might lead to a solution. With these things in mind, we present now a few hints for evaluating integrals.

Integration by Substitution: $\displaystyle\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$

What to look for:

1. Compositions of the form $f(u(x))$, where the integrand also contains $u'(x)$; for example,

$$\int 2x\cos(x^2)\,dx = \int \underbrace{\cos(x^2)}_{\cos u}\,\underbrace{2x\,dx}_{du} = \int \cos u\,du.$$

2. Compositions of the form $f(ax+b)$; for example,

$$\int \frac{\overbrace{x}^{u-1}}{\underbrace{\sqrt{x+1}}_{\sqrt{u}}} \underbrace{dx}_{du} = \int \frac{u-1}{u}\, du.$$

Integration by Parts: $\int u\, dv = uv - \int v\, du$

What to look for: products of different types of functions: x^n, $\cos x$, e^x; for example,

$$\int 2x \cos x\, dx \qquad \begin{cases} u = x & dv = \cos x\, dx \\ du = dx & v = \sin x \end{cases}$$

$$= x \sin x - \int \sin x\, dx.$$

Trigonometric Substitution:

What to look for:

1. Terms like $\sqrt{a^2 - x^2}$: Let $x = a \sin\theta$ $(-\pi/2 \le \theta \le \pi/2)$, so that $dx = a\cos\theta\, d\theta$ and $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$; for example,

$$\int \frac{\overbrace{x^2}^{\sin^2\theta}}{\underbrace{\sqrt{1-x^2}}_{\cos\theta}} \underbrace{dx}_{\cos\theta\, d\theta} = \int \sin^2\theta\, d\theta.$$

2. Terms like $\sqrt{x^2 + a^2}$: Let $x = a\tan\theta$ $(-\pi/2 < \theta < \pi/2)$, so that $dx = a\sec^2\theta\, d\theta$ and $\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta$; for example,

$$\int \frac{\overbrace{x^3}^{27\tan^3\theta}}{\underbrace{\sqrt{x^2+9}}_{3\sec\theta}} \underbrace{dx}_{3\sec^2\theta\, d\theta} = 27\int \tan^3\theta \sec\theta\, d\theta.$$

3. Terms like $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$, for $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$, so that $dx = a\sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$; for example,

$$\int \underbrace{x^3}_{8\sec^3\theta} \underbrace{\sqrt{x^2-4}}_{2\tan\theta} \underbrace{dx}_{2\sec\theta\tan\theta\, d\theta} = 32\int \sec^4\theta \tan^2\theta\, d\theta.$$

Partial Fractions:

What to look for: rational functions; for example,

$$\int \frac{x+2}{x^2-4x+3}\, dx = \int \frac{x+2}{(x-1)(x-3)}\, dx = \int \left(\frac{A}{x-1} + \frac{B}{x-3}\right) dx.$$

EXERCISES 6.4

WRITING EXERCISES

1. There is a shortcut for determining the constants for linear terms in a partial fractions decomposition. For example, take

$$\frac{x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}.$$

To compute A, take the original fraction on the left, cover up the $x+1$ in the denominator and replace x with -1: $A = \frac{-1-1}{-1-2} = \frac{2}{3}$. Similarly, to solve for B, cover up the $x-2$ and replace x with 2: $B = \frac{2-1}{2+1} = \frac{1}{3}$. Explain why this works and practice it for yourself.

2. For partial fractions, there is a big distinction between quadratic functions that factor into linear terms and quadratic functions that are irreducible. Recall that a quadratic function factors as $(x-a)(x-b)$ if and only if a and b are zeros of the function. Explain how you can use the quadratic formula to determine whether a given quadratic function is irreducible.

In exercises 1–30, find the partial fractions decomposition and an antiderivative. If you have a CAS available, use it to check your answer.

1. $\frac{x-5}{x^2-1}$
2. $\frac{5x-2}{x^2-4}$
3. $\frac{6x}{x^2-x-2}$
4. $\frac{3x}{x^2-3x-4}$
5. $\frac{-x+5}{x^3-x^2-2x}$
6. $\frac{3x+8}{x^3+5x^2+6x}$
7. $\frac{x^3+x+2}{x^2+2x-8}$
8. $\frac{x^2+1}{x^2-5x-6}$
9. $\frac{5x-23}{6x^2-11x-7}$
10. $\frac{3x+5}{5x^2-4x-1}$
11. $\frac{x-1}{x^3+4x^2+4x}$
12. $\frac{4x-5}{x^3-3x^2}$
13. $\frac{x+4}{x^3+3x^2+2x}$
14. $\frac{-2x^2+4}{x^3+3x^2+2x}$
15. $\frac{x+2}{x^3+x}$
16. $\frac{1}{x^3+4x}$
17. $\frac{4x-2}{16x^4-1}$
18. $\frac{3x+7}{x^4-16}$
19. $\frac{4x^2-7x-17}{6x^2-11x-10}$
20. $\frac{x^3+x}{x^2-1}$
21. $\frac{2x+3}{x^2+2x+1}$
22. $\frac{2x}{x^2-6x+9}$
23. $\frac{x^3-4}{x^3+2x^2+2x}$
24. $\frac{4}{x^3-2x^2+4x}$
25. $\frac{x^3+x}{3x^2+2x+1}$
26. $\frac{x^3-2x}{2x^2-3x+2}$
27. $\frac{4x^2+3}{x^3+x^2+x}$
28. $\frac{4x+4}{x^4+x^3+2x^2}$
29. $\frac{3x^3+1}{x^3-x^2+x-1}$
30. $\frac{2x^4+9x^2+x-4}{x^3+4x}$

31. In this exercise, we find the partial fractions decomposition of $\frac{4x^2+2}{(x^2+1)^2}$. Consistent with the form for repeated linear factors, the form for the decomposition is $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$. We set

$$\frac{4x^2+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Multiplying through by $(x^2+1)^2$, we get

$$\begin{aligned} 4x^2+2 &= (Ax+B)(x^2+1)+Cx+D \\ &= Ax^3+Bx^2+Ax+B+Cx+D \end{aligned}$$

As in example 4.5, we match up coefficients of like powers of x. For x^3, we have $0 = A$. For x^2, we have $4 = B$. Match the coefficients of x and the constants to finish the decomposition.

In exercises 32–36, find the partial fractions decomposition. (Refer to exercise 31.)

32. $\frac{x^3+2}{(x^2+1)^2}$
33. $\frac{2x^2+4}{(x^2+4)^2}$
34. $\frac{2x^3-x^2}{(x^2+1)^2}$
35. $\frac{4x^2+3}{(x^2+x+1)^2}$
36. $\frac{x^4+x^3}{(x^2+4)^2}$

37. Often, more than one integration technique can be applied. Evaluate $\int \frac{3}{x^4+x}\,dx$ in each of the following ways. First, use the substitution $u = x^3+1$ and partial fractions. Second, use the substitution $u = \frac{1}{x}$ and evaluate the resulting integral. Show that the two answers are equivalent.

38. Evaluate $\int \frac{2}{x^3+x}\,dx$ in each of the following ways. First, use the substitution $u = x^2+1$ and partial fractions. Second, use the substitution $u = \frac{1}{x}$ and evaluate the resulting integral. Show that the two answers are equivalent.

EXPLORATORY EXERCISES

1. In developing the definite integral, we looked at sums such as $\sum_{i=1}^{n} \frac{2}{i^2+i}$. As with Riemann sums, we are especially interested in the limit as $n \to \infty$. Write out several terms of the sum and try to guess what the limit is. It turns out that this is one of the few sums for which a precise formula exists, because this is a **telescoping sum.** To find out what this means, write out the partial fractions decomposition for $\frac{2}{i^2+i}$. Using the partial fractions form, write out several terms of the sum and notice how much cancellation there is. Briefly describe why the term *telescoping* is appropriate, and determine $\sum_{i=1}^{n} \frac{2}{i^2+i}$. Then find the limit as $n \to \infty$. Repeat this process for the telescoping sum $\sum_{i=2}^{n} \frac{4}{i^2-1}$.

2. Use the substitution $u = x^{1/4}$ to evaluate $\int \frac{1}{x^{5/4}+x}\,dx$. Use similar substitutions to evaluate $\int \frac{1}{x^{1/4}+x^{1/3}}\,dx$, $\int \frac{1}{x^{1/5}+x^{1/7}}\,dx$ and $\int \frac{1}{x^{1/4}+x^{1/6}}\,dx$. Find the form of the substitution for the general integral $\int \frac{1}{x^p+x^q}\,dx$.

6.5 INTEGRATION TABLES AND COMPUTER ALGEBRA SYSTEMS

Ask anyone who has ever needed to evaluate a large number of integrals as part of their work (this includes engineers, mathematicians, physicists and others) and they will tell you that they have made extensive use of integral tables and/or a computer algebra system. These are extremely powerful tools for the professional user of mathematics. However, they do *not* take the place of learning all the basic techniques of integration. To use a table, you often must first rewrite the integral in the form of one of the integrals in the table. This may require you to perform some algebraic manipulation or to make a substitution. While a CAS will report an antiderivative, it will occasionally report it in an inconvenient form. More significantly, a CAS will from time to time report an answer that is (at least technically) incorrect. We will point out some of these shortcomings in the examples that follow.

Using Tables of Integrals

We include a small table of indefinite integrals at the back of the book. A larger table can be found in the *CRC Standard Mathematical Tables*. An amazingly extensive table is found in the book *Table of Integrals, Series and Products*, compiled by Gradshteyn and Ryzhik.

EXAMPLE 5.1 Using an Integral Table

Use a table to evaluate $\int \frac{\sqrt{3+4x^2}}{x}\,dx$.

Solution Certainly, you could evaluate this integral using trigonometric substitution. However, if you look in our integral table, you will find

$$\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + c. \tag{5.1}$$

Unfortunately, the integral in question is not quite in the form of (5.1). However, we can fix this with the substitution $u = 2x$, so that $du = 2\,dx$. This gives us

$$\begin{aligned}\int \frac{\sqrt{3+4x^2}}{x}\,dx &= \int \frac{\sqrt{3+(2x)^2}}{2x}(2)\,dx = \int \frac{\sqrt{3+u^2}}{u}\,du \\ &= \sqrt{3+u^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{3+u^2}}{u}\right| + c \\ &= \sqrt{3+4x^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{3+4x^2}}{2x}\right| + c.\end{aligned}$$

A number of the formulas in the table are called **reduction formulas.** These are of the form

$$\int f(u)\,du = g(u) + \int h(u)\,du,$$

where the second integral is simpler than the first. These are often applied repeatedly, as in example 5.2.

EXAMPLE 5.2 Using a Reduction Formula

Use a reduction formula to evaluate $\int \sin^6 x\,dx$.

Solution You should recognize that this integral can be evaluated using techniques you already know. (How?) However, for any integer $n \geq 1$, we have the reduction formula

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du. \tag{5.2}$$

(See number 59 in the table of integrals found inside the back cover of the book.) If we apply (5.2) with $n = 6$, we get

$$\int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx.$$

We can apply the same reduction formula (this time with $n = 4$) to evaluate $\int \sin^4 x\,dx$. We have

$$\begin{aligned}\int \sin^6 x\,dx &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx \\ &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\right).\end{aligned}$$

Finally, for $\int \sin^2 x\,dx$, we can use (5.2) once again (with $n = 2$), or evaluate the integral using a half-angle formula. We choose the former here and obtain

$$\begin{aligned}\int \sin^6 x\,dx &= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\right) \\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}\int dx\right) \\ &= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + c.\end{aligned}$$

We should remind you at this point that there are many different ways to find an antiderivative. Antiderivatives found through different means may look quite different, even though they are equivalent. For instance, notice that if an antiderivative has the form

$\sin^2 x + c$, then an equivalent antiderivative is $-\cos^2 x + c$, since we can write

$$\sin^2 x + c = 1 - \cos^2 x + c = -\cos^2 x + (1 + c).$$

Finally, since c is an arbitrary constant, so is $1 + c$. In example 5.2, observe that the first three terms all have factors of $\sin x \cos x$, which equals $\frac{1}{2}\sin 2x$. Using this and other identities, you can show that our solution in example 5.2 is equivalent to the following solution obtained from a popular CAS:

$$\int \sin^6 x\, dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x + c.$$

So, do not panic if your answer differs from the one in the back of the book. Both answers may be correct. If you're unsure, find the derivative of your answer. If you get the integrand, you're right.

You will sometimes want to apply different reduction formulas at different points in a given problem.

EXAMPLE 5.3 Making a Substitution Before Using a Reduction Formula

Evaluate $\int x^3 \sin 2x\, dx$.

Solution From our table (see number 63), we have the reduction formula

$$\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1}\cos u\, du. \tag{5.3}$$

In order to use (5.3), we must first make the substitution $u = 2x$, so that $du = 2\,dx$, which gives us

$$\begin{aligned}\int x^3 \sin 2x\, dx &= \frac{1}{2}\int \frac{(2x)^3}{2^3}\sin 2x(2)\, dx = \frac{1}{16}\int u^3 \sin u\, du\\ &= \frac{1}{16}\left(-u^3\cos u + 3\int u^2\cos u\, du\right),\end{aligned}$$

where we have used the reduction formula (5.3) with $n = 3$. Now, to evaluate this last integral, we use the reduction formula (see number 64 in our table)

$$\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1}\sin u\, du,$$

with $n = 2$ to get

$$\begin{aligned}\int x^3 \sin 2x\, dx &= -\frac{1}{16}u^3\cos u + \frac{3}{16}\int u^2\cos u\, du\\ &= -\frac{1}{16}u^3\cos u + \frac{3}{16}\left(u^2\sin u - 2\int u\sin u\, du\right).\end{aligned}$$

Applying the first reduction formula (5.3) one more time (this time, with $n = 1$), we get

$$\begin{aligned}\int x^3 \sin 2x\, dx &= -\frac{1}{16}u^3\cos u + \frac{3}{16}u^2\sin u - \frac{3}{8}\int u\sin u\, du\\ &= -\frac{1}{16}u^3\cos u + \frac{3}{16}u^2\sin u - \frac{3}{8}\left(-u\cos u + \int u^0\cos u\, du\right)\\ &= -\frac{1}{16}u^3\cos u + \frac{3}{16}u^2\sin u + \frac{3}{8}u\cos u - \frac{3}{8}\sin u + c\\ &= -\frac{1}{16}(2x)^3\cos 2x + \frac{3}{16}(2x)^2\sin 2x + \frac{3}{8}(2x)\cos 2x - \frac{3}{8}\sin 2x + c\\ &= -\frac{1}{2}x^3\cos 2x + \frac{3}{4}x^2\sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + c.\end{aligned}$$

As we'll see in example 5.4, some integrals require some insight before using an integral table.

EXAMPLE 5.4 Making a Substitution Before Using an Integral Table

Evaluate $\displaystyle\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx$.

Solution You won't find this integral or anything particularly close to it in our integral table. However, with a little fiddling, we can rewrite this in a simpler form. First, use the double-angle formula to rewrite the numerator of the integrand. We get

$$\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx = 2\int \frac{\sin x \cos x}{\sqrt{4\cos x - 1}}\,dx.$$

Remember to always be on the lookout for terms that are derivatives of other terms. Here, taking $u = \cos x$, we have $du = -\sin x\,dx$ and so,

$$\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx = 2\int \frac{\sin x \cos x}{\sqrt{4\cos x - 1}}\,dx = -2\int \frac{u}{\sqrt{4u - 1}}\,du.$$

From our table (see number 18), notice that

$$\int \frac{u}{\sqrt{a + bu}}\,du = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + c. \qquad (5.4)$$

Taking $a = -1$ and $b = 4$ in (5.4), we have

$$\begin{aligned}\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx &= -2\int \frac{u}{\sqrt{4u - 1}}\,du = (-2)\frac{2}{3(4^2)}(4u + 2)\sqrt{4u - 1} + c\\ &= -\frac{1}{12}(4\cos x + 2)\sqrt{4\cos x - 1} + c.\end{aligned}$$

Integration Using a Computer Algebra System

Computer algebra systems are some of the most powerful new tools to arrive on the mathematical scene in the last 20 years. They run the gamut from handheld calculators (like the TI-89 and the HP-48) to powerful software systems (like Mathematica and Maple), which will run on nearly any personal computer.

The examples that follow focus on some of the rare problems you may encounter using a CAS. We admit that we intentionally searched for CAS mistakes. The good news is that the mistakes were very uncommon and the CAS you're using won't necessarily make any of them. The bottom line here is that a CAS is "taught" many rules by its programmers. If it applies the wrong rule to the problem at hand, you get an incorrect answer. Be aware that these are software bugs and the next version of your CAS may be given a more complete set of rules. As an intelligent user of technology, you need to be aware of common errors and have the calculus skills to catch mistakes when they occur.

The first thing you notice when using a CAS to evaluate an indefinite integral is that it typically supplies *an* antiderivative, instead of the most general one (the indefinite integral) by leaving off the constant of integration (a minor shortcoming of this very powerful software).

EXAMPLE 5.5 A Shortcoming of Some Computer Algebra Systems

Use a computer algebra system to evaluate $\displaystyle\int \frac{1}{x}\,dx$.

Solution Many CASs evaluate

$$\int \frac{1}{x}\,dx = \ln x.$$

(Actually, one CAS reports the integral as $\log x$, where it is using the notation $\log x$ to denote the natural logarithm.) Not only is this missing the constant of integration, but notice that this antiderivative is valid only for $x > 0$. A popular calculator returns the more general antiderivative

$$\int \frac{1}{x}\,dx = \ln|x|,$$

which, while still missing the constant of integration, at least is valid for all $x \neq 0$. On the other hand, all of the CASs we tested correctly evaluate

$$\int_{-2}^{-1} \frac{1}{x}\,dx = -\ln 2,$$

even though the reported antiderivative $\ln x$ is not defined at the limits of integration. ■

Sometimes the antiderivative reported by a CAS is not valid, as written, for *any* real values of x, as in example 5.6. (In some cases, CASs give an antiderivative that is correct for the more advanced case of a function of a complex variable.)

EXAMPLE 5.6 An Incorrect Antiderivative

Use a computer algebra system to evaluate $\displaystyle\int \frac{\cos x}{\sin x - 2}\,dx$.

Solution One CAS reports the incorrect antiderivative

$$\int \frac{\cos x}{\sin x - 2}\,dx = \ln(\sin x - 2).$$

At first glance, this may not appear to be wrong, especially since the chain rule seems to indicate that it's correct:

$$\frac{d}{dx}\ln(\sin x - 2) = \frac{\cos x}{\sin x - 2}. \quad \text{This is incorrect!}$$

The error is more fundamental (and subtle) than a misuse of the chain rule. Notice that the expression $\ln(\sin x - 2)$ is undefined for *all* real values of x, as $\sin x - 2 < 0$ for all x. A general antiderivative rule that applies here is

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c,$$

where the absolute value is important. The correct antiderivative is $\ln|\sin x - 2| + c$, which can also be written as $\ln(2 - \sin x) + c$ since $2 - \sin x > 0$ for all x. ■

Probably the most common errors you will run into are actually your own. If you give your CAS a problem in the wrong form, it may solve a different problem than you intended. One simple, but common, mistake is shown in example 5.7.

EXAMPLE 5.7 A Problem Where the CAS Misinterprets What You Enter

Use a computer algebra system to evaluate $\int 4x\ 8x\,dx$.

Solution After entering the integrand as $4x8x$, one CAS returned the odd answer

$$\int 4x8x\,dx = 4x8xx.$$

You can easily evaluate the integral (first, rewrite the integrand as $32x^2$) to show this is incorrect, but what was the error? Because of the way in which we wrote the integrand, the CAS interpreted it as four times a variable named $x8x$, which is unrelated to the variable of integration, x. Its answer is of the form $\int 4c\,dx = 4cx$. ■

The form of the antiderivative reported by a CAS will not always be the most convenient.

EXAMPLE 5.8 An Inconvenient Form of an Antiderivative

Use a computer algebra system to evaluate $\int x(x^2+3)^5\,dx$.

Solution Several CASs evaluate

$$\int x(x^2+3)^5\,dx = \frac{1}{12}x^{12} + \frac{3}{2}x^{10} + \frac{45}{4}x^8 + 45x^6 + \frac{405}{4}x^4 + \frac{243}{2}x^2,$$

while others return the much simpler expression

$$\int x(x^2+3)^5\,dx = \frac{(x^2+3)^6}{12}.$$

The two answers are equivalent, although they differ by a constant. ■

CASs will often correctly evaluate an integral, but report it in terms of a function or functions with which you are not especially familiar, as in example 5.9.

TODAY IN MATHEMATICS

Jean-Christophe Yoccoz (1957–) A French mathematician who earned a Fields Medal for his contributions to dynamical systems. His citation for the Fields Medal stated, "He combines an extremely acute geometric intuition, an impressive command of analysis, and a penetrating combinatorial sense to play the chess game at which he excels. He occasionally spends half a day on mathematical 'experiments' by hand or by computer. 'When I make such an experiment,' he says, 'it is not just the results that interest me, but the manner in which it unfolds, which sheds light on what is really going on.'"

EXAMPLE 5.9 A Less Familiar Antiderivative

Use a computer algebra system to evaluate $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$.

Solution Recall that we have already evaluated this integral in example 3.10. There, we found through trigonometric substitution that

$$\int \frac{1}{\sqrt{9+x^2}}\,dx = \ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2} + \frac{x}{3}\right| + c.$$

One CAS reports an antiderivative of $\operatorname{arcsinh}\frac{1}{3}x$. While this is equivalent to what we had obtained in example 3.10, it is likely less familiar to most students. On the other hand, it is certainly a simpler form of the antiderivative. ■

Typically, a CAS will perform even lengthy integrations with ease.

EXAMPLE 5.10 Some Good Integrals for Using a CAS

Use a computer algebra system to evaluate $\int x^3 \sin 2x\,dx$ and $\int x^{10}\sin 2x\,dx$.

Solution Using a CAS, you can get in one step

$$\int x^3\sin 2x\,dx = -\frac{1}{2}x^3\cos 2x + \frac{3}{4}x^2\sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + c.$$

With the same effort, you can obtain

$$\int x^{10}\sin 2x\,dx = -\frac{1}{2}x^{10}\cos 2x + \frac{5}{2}x^9\sin 2x + \frac{45}{4}x^8\cos 2x - 45x^7\sin 2x$$
$$-\frac{315}{2}x^6\cos 2x + \frac{945}{2}x^5\sin 2x + \frac{4725}{4}x^4\cos 2x$$
$$-\frac{4725}{2}x^3\sin 2x - \frac{14{,}175}{4}x^2\cos 2x + \frac{14{,}175}{4}x\sin 2x$$
$$+\frac{14{,}175}{8}\cos 2x + c.$$

If you wanted to, you could even evaluate

$$\int x^{100}\sin 2x\,dx,$$

although the large number of terms makes displaying the result prohibitive. Think about doing this by hand, using a staggering 100 integrations by parts or by applying a reduction formula 100 times. ■

You should get the idea by now: a CAS can perform repetitive calculations (numerical or symbolic) that you could never dream of doing by hand. It is difficult to find a function that has an elementary antiderivative that your CAS cannot find. Consider the following example of a hard integral.

EXAMPLE 5.11 A Very Hard Integral

Evaluate $\int x^7 e^x \sin x\,dx$.

Solution Consider what you would need to do to evaluate this integral by hand and then use a computer algebra system. For instance, one CAS reports the antiderivative

$$\int x^7 e^x\sin x\,dx = \left(-\frac{1}{2}x^7 + \frac{7}{2}x^6 - \frac{21}{2}x^5 + 105x^3 - 315x^2 + 315x\right)e^x\cos x$$
$$+\left(\frac{1}{2}x^7 - \frac{21}{2}x^5 + \frac{105}{2}x^4 - 105x^3 + 315x - 315\right)e^x\sin x.$$

Don't try this by hand unless you have plenty of time and patience. However, based on your experience, observe that this antiderivative is plausible. ■

BEYOND FORMULAS

You may ask why we've spent so much time on integration techniques when you can always let a CAS do the work for you. No, it's not to prepare you in the event that you are shipwrecked on a desert island without a CAS. Your CAS can solve virtually all of the computational problems that arise in this text. On rare occasions, however, a CAS-generated answer may be incorrect or misleading and you need to be prepared for these. More importantly, many of the insights at the heart of science and engineering are arrived at through a precise use of several integration techniques. As a result, you need to understand *how* the integration techniques transform one set of symbols into another. Computers are faster and more accurate at symbol manipulation than humans will ever be. However, our special ability as humans is to understand the *intent* as well as the logic of the techniques and to apply the right technique at the right time to make a surprising connection or important discovery. In what other areas of computer technology is the same human input needed?

EXERCISES 6.5

WRITING EXERCISES

1. Suppose that you are hired by a company to develop a new CAS. Outline a strategy for symbolic integration. Include provisions for formulas in the Table of Integrals at the back of the book and the various techniques you have studied.

2. In the text, we discussed the importance of knowing general rules for integration. Consider the integral in example 5.4, $\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx$. Can your CAS evaluate this integral? For many integrals like this that *do* show up in applications (there are harder ones in the exploratory exercises), you have to do some work before the technology can finish the task. For this purpose, discuss the importance of recognizing basic forms and understanding how substitution works.

In exercises 1–28, use the Table of Integrals at the back of the book to find an antiderivative. Note: When checking the back of the book or a CAS for answers, beware of functions that look very different but that are equivalent (through a trig identity, for instance).

1. $\int \frac{x}{(2+4x)^2}\,dx$

2. $\int \frac{x^2}{(2+4x)^2}\,dx$

3. $\int e^{2x}\sqrt{1+e^x}\,dx$

4. $\int e^{3x}\sqrt{1+e^{2x}}\,dx$

5. $\int \frac{x^2}{\sqrt{1+4x^2}}\,dx$

6. $\int \frac{\cos x}{\sin^2 x(3+2\sin x)}\,dx$

7. $\int_0^1 x^8\sqrt{4-x^6}\,dx$

8. $\int_0^{\ln 4} \sqrt{16-e^{2x}}\,dx$

9. $\int_0^{\ln 2} \frac{e^x}{\sqrt{e^{2x}+4}}\,dx$

10. $\int_{\sqrt{3}}^{2} \frac{x\sqrt{x^4-9}}{x^2}\,dx$

11. $\int \frac{\sqrt{6x-x^2}}{(x-3)^2}\,dx$

12. $\int \frac{\sec^2 x}{\tan x\sqrt{8\tan x - \tan^2 x}}\,dx$

13. $\int \tan^6 x\,dx$

14. $\int \csc^4 x\,dx$

15. $\int \frac{\cos x}{\sin x\sqrt{4+\sin x}}\,dx$

16. $\int \frac{x^5}{\sqrt{4+x^2}}\,dx$

17. $\int x^3\cos x^2\,dx$

18. $\int x\sin 3x^2\cos 4x^2\,dx$

19. $\int \frac{\sin x\cos x}{\sqrt{1+\cos x}}\,dx$

20. $\int \frac{x\sqrt{1+4x^2}}{x^4}\,dx$

21. $\int \frac{\sin^2 x\cos x}{\sqrt{\sin^2 x+4}}\,dx$

22. $\int \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$

23. $\int \frac{e^{-2/x^2}}{x^3}\,dx$

24. $\int x^3 e^{2x^2}\,dx$

25. $\int \frac{x}{\sqrt{4x-x^2}}\,dx$

26. $\int e^{5x}\cos 3x\,dx$

27. $\int e^x\tan^{-1}(e^x)\,dx$

28. $\int (\ln 4x)^3\,dx$

29. Check your CAS against all examples in this section. Discuss which errors, if any, your CAS makes.

30. Find out how your CAS evaluates $\int x\sin x\,dx$ if you fail to leave a space between x and $\sin x$.

31. Have your CAS evaluate $\int(\sqrt{1-x}+\sqrt{x-1})\,dx$. If you get an answer, explain why it's wrong.

32. To find out if your CAS "knows" integration by parts, try $\int x^3\cos 3x\,dx$ and $\int x^3e^{5x}\cos 3x\,dx$. To see if it "knows" reduction formulas, try $\int \sec^5 x\,dx$.

33. To find out how many trigonometric techniques your CAS "knows," try $\int \sin^6 x\,dx$, $\int \sin^4 x\cos^3 x\,dx$ and $\int \tan^4 x\sec^3 x\,dx$.

34. Find out if your CAS has a special command (e.g., APART in Mathematica) to do partial fractions decompositions. Also, try $\int \frac{x^2+2x-1}{(x-1)^2(x^2+4)}\,dx$ and $\int \frac{3x}{(x^2+x+2)^2}\,dx$.

35. To find out if your CAS "knows" how to do substitution, try $\int \frac{1}{x^2(3+2x)}\,dx$ and $\int \frac{\cos x}{\sin^2 x(3+2\sin x)}\,dx$. Try to find one that your CAS can't do: start with a basic formula like $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$ and substitute your favorite function. With $x = e^u$, the preceding integral becomes $\int \frac{e^u}{e^u\sqrt{e^{2u}-1}}\,du$, which you can use to test your CAS.

36. To compute the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, note that the upper-right quarter of the ellipse is given by

$$y = b\sqrt{1-\frac{x^2}{a^2}}$$

for $0 \le x \le a$. Thus, the area of the ellipse is $4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$. Try this integral on your CAS. The (implicit) assumption we usually make is that $a > 0$, but your CAS should not make this assumption for you. Does your CAS give you πab or $\pi b|a|$?

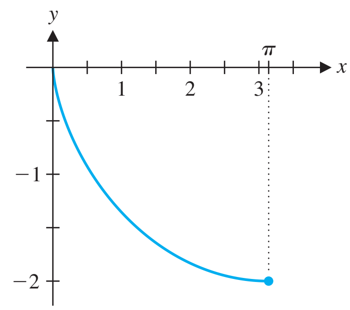
EXPLORATORY EXERCISES

1. This exercise explores two aspects of a very famous problem (solved in the 1600s; when you finish the problem, think about solving this problem before calculators, computers or even much of calculus was invented). The idea is to imagine a bead sliding down a thin wire that extends in some shape from the point (0, 0) to the point $(\pi, -2)$. Assume that gravity pulls the bead down but that there is no friction or other force acting on the bead. This situation is easiest to analyze using **parametric equations** where we have functions $x(u)$ and $y(u)$ giving the horizontal and vertical position of the bead in terms of the variable u. Examples of paths the bead might follow are $\begin{cases} x(u) = \pi u \\ y(u) = -2u \end{cases}$ and $\begin{cases} x(u) = \pi u \\ y(u) = 2(u-1)^2 - 2 \end{cases}$ and $\begin{cases} x(u) = \pi u - \sin \pi u \\ y(u) = \cos \pi u - 1 \end{cases}$. In each case, the bead starts at (0, 0) for $u = 0$ and finishes at $(\pi, -2)$ for $u = 1$. You should graph each path on your graphing calculator. The first path is a line, the second is a parabola and the third is a special curve called a **brachistochrone.** For a given path, the time it takes the bead to travel the path is given by

$$T = \frac{1}{\sqrt{g}}\int_0^1 \sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{-2y(u)}}\,du,$$

where g is the gravitational constant. Compute this quantity for the line and the parabola. Explain why the parabola would be a faster path for the bead to slide down, even though the line is shorter in distance. (Think of which would be a faster hill to ski down.) It can be shown that the brachistochrone is the fastest path possible. Try to get your CAS to compute the optimal time. Comparing the graphs of the parabola and brachistochrone, what important advantage does the brachistochrone have at the start of the path?

2. It turns out that the brachistochrone in exploratory exercise 1 has an amazing property, along with providing the fastest time (which is essentially what the term *brachistochrone* means). The path is shown in the figure.



Suppose that instead of starting the bead at the point (0, 0), you start the bead partway down the path at $x = c$. How would the time to reach the bottom from $x = c$ compare to the total time from $x = 0$? Note that the answer is *not* obvious, since the farther down you start, the less speed the bead will gain. If $x = c$ corresponds to $u = a$, the time to reach the bottom is given by $\frac{\pi}{\sqrt{g}}\int_a^1 \sqrt{\frac{1-\cos \pi u}{\cos a\pi - \cos \pi u}}\,du$. If $a = 0$ (that is, the bead starts at the top), the time is $\pi/\sqrt{g}$ (the integral equals 1). If you have a very good CAS, try to evaluate the integral for various values of a between 0 and 1. If your CAS can't handle it, approximate the integral numerically. You should discover the amazing fact that *the integral always equals 1*. The brachistochrone is also the **tautochrone,** a curve for which the time to reach the bottom is the same regardless of where you start.

6.6 IMPROPER INTEGRALS

Improper Integrals with a Discontinuous Integrand

We're willing to bet that you have heard the saying "familiarity breeds contempt" more than once. This phrase has particular relevance for us in this section. You have been using the Fundamental Theorem of Calculus for quite some time now. Do you always check to see that the hypotheses of the theorem are met before applying it? What hypotheses, you ask? We won't make you look back, but before we give you the answer, try to see what is wrong with the following *erroneous* calculation.

$$\int_{-1}^2 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg|_{-1}^2 = -\frac{3}{2}. \quad \text{This is incorrect!}$$

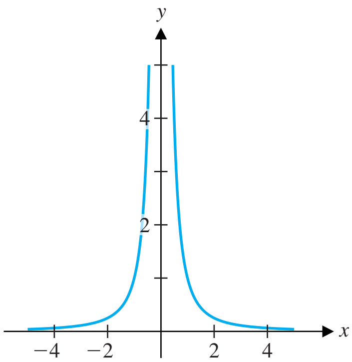


FIGURE 6.2
$y = \frac{1}{x^2}$

There is something fundamentally wrong with this "calculation." Note that $f(x) = 1/x^2$ is not continuous over the interval of integration. (See Figure 6.2.) Since the Fundamental Theorem assumes a continuous integrand, our use of the theorem is invalid and our answer is *incorrect*. Further, note that an answer of $-\frac{3}{2}$ is especially suspicious given that the integrand $\frac{1}{x^2}$ is always positive.

Recall that in Chapter 4, we defined the definite integral by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i was taken to be any point in the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$ and where the limit had to be the same for any choice of these c_i's. So, if $f(x) \to \infty$ [or $f(x) \to -\infty$] at some point in $[a, b]$, then the limit defining $\int_a^b f(x)\,dx$ is meaningless. [How would we add $f(c_i)$ to the sum, if $f(x) \to \infty$ as $x \to c_i$?] In this case, we call this integral an **improper integral** and we will need to carefully define what we mean by such an integral. First, we examine a somewhat simpler case.

Consider $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$. Observe that this is *not* a proper definite integral, as the integrand is discontinuous at $x = 1$. In Figure 6.3a, note that the integrand blows up to ∞ as $x \to 1^-$. Despite this, can we find the area under the curve on the interval [0, 1]? Assuming the area is finite, notice from Figure 6.3b that for $0 < R < 1$, we can approximate it by $\int_0^R \frac{1}{\sqrt{1-x}}\,dx$. This is a proper definite integral, since for $0 \le x \le R < 1$, $f(x)$ is continuous. Further, the closer R is to 1, the better the approximation should be. In the accompanying table, we compute some approximate values of $\int_0^R \frac{1}{\sqrt{1-x}}\,dx$, for a sequence of values of R approaching 1.

R	$\int_0^R \frac{1}{\sqrt{1-x}}\,dx$
0.9	1.367544
0.99	1.8
0.999	1.936754
0.9999	1.98
0.99999	1.993675
0.999999	1.998
0.9999999	1.999368
0.99999999	1.9998

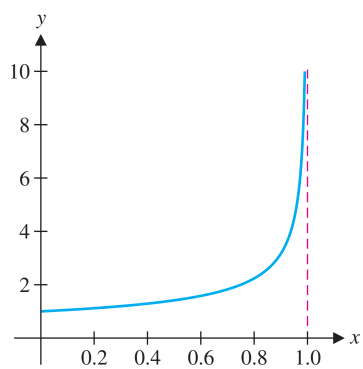


FIGURE 6.3a
$y = \frac{1}{\sqrt{1-x}}$

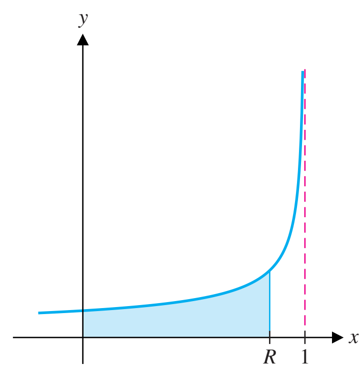


FIGURE 6.3b
$\int_0^R f(x)\,dx$

From the table, the sequence of integrals seems to be approaching 2, as $R \to 1^-$. Notice that since we know how to compute $\int_0^R \frac{1}{\sqrt{1-x}}\,dx$, for any $0 < R < 1$, we can compute this limiting value exactly. We have

$$\begin{aligned}\lim_{R\to 1^-} \int_0^R \frac{1}{\sqrt{1-x}}\,dx &= \lim_{R\to 1^-} \left[-2(1-x)^{1/2}\right]_0^R \\ &= \lim_{R\to 1^-} \left[-2(1-R)^{1/2} + 2(1-0)^{1/2}\right] = 2.\end{aligned}$$

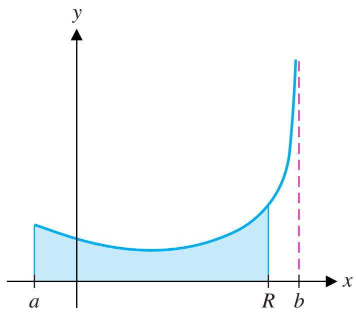


FIGURE 6.4
$\int_a^R f(x)\,dx$

From this computation, we can see that the area under the curve is the limiting value, 2.

In general, suppose that f is continuous on the interval $[a, b)$ and $|f(x)| \to \infty$, as $x \to b^-$ (i.e., as x approaches b from the left). Then we can approximate $\int_a^b f(x)\,dx$ by $\int_a^R f(x)\,dx$, for some $R < b$, but close to b. [Recall that since f is continuous on $[a, R]$, for any $a < R < b$, $\int_a^R f(x)\,dx$ is defined.] Further, as in our introductory example, the closer R is to b, the better the approximation should be. See Figure 6.4 for a graphical representation of this approximation.

Finally, let $R \to b^-$; if $\int_a^R f(x)\,dx$ approaches some value, L, then we define the **improper integral** $\int_a^b f(x)\,dx$ to be this limiting value. We have the following definition.

DEFINITION 6.1

If f is continuous on the interval $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$, we define the improper integral of f on $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist, we say that the improper integral **diverges.**

EXAMPLE 6.1 An Integrand That Blows Up at the Right Endpoint

Determine whether $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ converges or diverges.

Solution Based on the work we just completed,

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{R \to 1^-} \int_0^R \frac{1}{\sqrt{1-x}}\,dx = 2$$

and so, the improper integral converges to 2. ■

In example 6.2, we illustrate a divergent improper integral closely related to this section's introductory example.

EXAMPLE 6.2 A Divergent Improper Integral

Determine whether the improper integral $\int_{-1}^0 \frac{1}{x^2}\,dx$ converges or diverges.

Solution From Definition 6.1, we have

$$\int_{-1}^0 \frac{1}{x^2}\,dx = \lim_{R \to 0^-} \int_{-1}^R \frac{1}{x^2}\,dx = \lim_{R \to 0^-} \left(\frac{x^{-1}}{-1}\right)\Bigg|_{-1}^R$$

$$= \lim_{R \to 0^-} \left(-\frac{1}{R} - \frac{1}{1}\right) = \infty.$$

Since the defining limit does not exist, the improper integral diverges. ■

In examples 6.3 and 6.4, the integrand is discontinuous at the lower limit of integration.

EXAMPLE 6.3 A Convergent Improper Integral

Determine whether the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

Solution We show a graph of the integrand on the interval in question in Figure 6.5. Notice that in this case $f(x) = \frac{1}{\sqrt{x}}$ is continuous on $(0, 1]$ and $f(x) \to \infty$ as $x \to 0^+$.

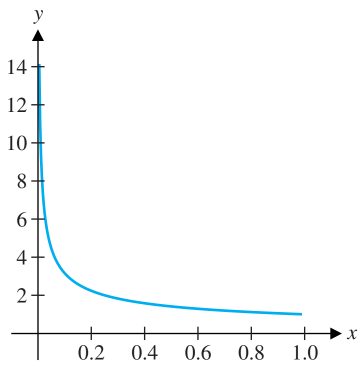


FIGURE 6.5

$y = \frac{1}{\sqrt{x}}$

R	$\int_R^1 \frac{1}{\sqrt{x}}\,dx$
0.1	1.367544
0.01	1.8
0.001	1.936754
0.0001	1.98
0.00001	1.993675
0.000001	1.998
0.0000001	1.999368
0.00000001	1.9998

From the computed values shown in the table, it appears that the integrals are approaching 2 as $R \to 0^+$. Since we know an antiderivative for the integrand, we can compute these integrals exactly, for any fixed $0 < R < 1$. We have from Definition 6.1 that

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+} \int_R^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+} \left.\frac{x^{1/2}}{\frac{1}{2}}\right|_R^1 = \lim_{R\to 0^+} 2(1^{1/2} - R^{1/2}) = 2$$

and so, the improper integral converges to 2. ■

HISTORICAL NOTES

Pierre Simon Laplace (1749–1827) A French mathematician who utilized improper integrals to develop the Laplace transform and other important mathematical techniques. Laplace made numerous contributions in probability, celestial mechanics, the theory of heat and a variety of other mathematical topics. Adept at political intrigue, Laplace worked on a new calendar for the French Revolution, served as an advisor to Napoleon and was named a marquis by the Bourbons.

EXAMPLE 6.4 A Divergent Improper Integral

Determine whether the improper integral $\int_1^2 \frac{1}{x-1}\,dx$ converges or diverges.

Solution From Definition 6.1, we have

$$\int_1^2 \frac{1}{x-1}\,dx = \lim_{R\to 1^+} \int_R^2 \frac{1}{x-1}\,dx = \lim_{R\to 1^+} \ln|x-1|\Big|_R^2$$

$$= \lim_{R\to 1^+} (\ln|2-1| - \ln|R-1|) = \infty,$$

so that the improper integral diverges. ■

The introductory example in this section represents a third type of improper integral, one where the integrand blows up at a point in the interior of the interval (a, b). We can define such an integral as follows.

DEFINITION 6.2

Suppose that f is continuous on the interval $[a, b]$, except at some $c \in (a, b)$, and $|f(x)| \to \infty$ as $x \to c$. Again, the integral is improper and we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If *both* $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge (to L_1 and L_2, respectively), we say that the improper integral $\int_a^b f(x)\,dx$ **converges,** also, (to $L_1 + L_2$). If *either* of the improper integrals $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ diverges, then we say that the improper integral $\int_a^b f(x)\,dx$ **diverges,** also.

We can now return to our introductory example.

EXAMPLE 6.5 An Integrand That Blows Up in the Middle of an Interval

Determine whether the improper integral $\displaystyle\int_{-1}^2 \frac{1}{x^2}\,dx$ converges or diverges.

Solution From Definition 6.2, we have

$$\int_{-1}^2 \frac{1}{x^2}\,dx = \int_{-1}^0 \frac{1}{x^2}\,dx + \int_0^2 \frac{1}{x^2}\,dx.$$

In example 6.2, we determined that $\int_{-1}^0 \frac{1}{x^2}\,dx$ diverges. Thus, $\int_{-1}^2 \frac{1}{x^2}\,dx$ also diverges. Note that you do *not* need to consider $\int_0^2 \frac{1}{x^2}\,dx$ (although it's an easy exercise to show that this, too, diverges). Keep in mind that if either of the two improper integrals defining this type of improper integral diverges, then the original integral diverges, too. ■

Improper Integrals with an Infinite Limit of Integration

Another type of improper integral that is frequently encountered in applications is one where one or both of the limits of integration is infinite. For instance, $\int_0^\infty e^{-x^2}\,dx$ is of fundamental importance in probability and statistics.

So, given a continuous function f defined on $[a, \infty)$, what could we mean by $\int_a^\infty f(x)\,dx$? Notice that the usual definition of the definite integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(c_i)\,\Delta x,$$

where $\Delta x = \dfrac{b-a}{n}$, makes no sense when $b = \infty$. We should define $\int_a^\infty f(x)\,dx$ in some way consistent with what we already know about integrals.

Since $f(x) = \frac{1}{x^2}$ is positive and continuous on the interval $[1, \infty)$, $\int_1^\infty \frac{1}{x^2}\,dx$ should correspond to area under the curve (assuming this area is, in fact, finite). From the graph of $y = \frac{1}{x^2}$ shown in Figure 6.6, it should appear at least plausible that the area under this curve is finite.

Assuming the area is finite, you could approximate it by $\int_1^R \frac{1}{x^2}\,dx$, for some large value R. (Notice that this is a proper definite integral, as long as R is finite.) A sequence of values of this integral for increasingly large values of R is displayed in the table.

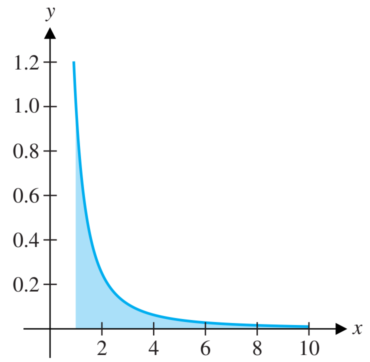


FIGURE 6.6
$y = \frac{1}{x^2}$

R	$\int_1^R \frac{1}{x^2}\,dx$
10	0.9
100	0.99
1000	0.999
10,000	0.9999
100,000	0.99999
1,000,000	0.999999

The sequence of approximating definite integrals seems to be approaching 1, as $R \to \infty$. As it turns out, we can compute this limit exactly. We have

$$\lim_{R\to\infty} \int_1^R x^{-2}\,dx = \lim_{R\to\infty} \left.\frac{x^{-1}}{-1}\right|_1^R = \lim_{R\to\infty}\left(-\frac{1}{R}+1\right) = 1.$$

Thus, the area under the curve on the interval $[1, \infty)$ is seen to be 1, even though the interval is infinite.

More generally, we have Definition 6.3.

DEFINITION 6.3

If f is continuous on the interval $[a, \infty)$, we define the **improper integral** $\int_a^\infty f(x)\,dx$ to be

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(-\infty, a]$, we define

$$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist, we say that the improper integral **diverges.**

EXAMPLE 6.6 An Integral with an Infinite Limit of Integration

Determine whether the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges or diverges.

Solution From our work above, observe that the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R x^{-2}\,dx = 1,$$

so that the improper integral converges to 1. ■

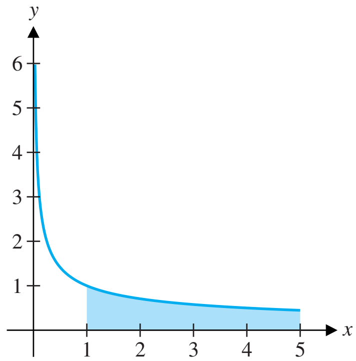


FIGURE 6.7
$y = \frac{1}{\sqrt{x}}$

You may have already observed that for a decreasing function f, in order for $\int_a^\infty f(x)\,dx$ to converge, it must be the case that $f(x) \to 0$ as $x \to \infty$. (Think about this in terms of area.) However, the reverse need *not* be true. That is, even though $f(x) \to 0$ as $x \to \infty$, the improper integral may diverge, as we see in example 6.7.

EXAMPLE 6.7 A Divergent Improper Integral

Determine whether $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

Solution Note that $\frac{1}{\sqrt{x}} \to 0$ as $x \to \infty$. Further, from the graph in Figure 6.7, it should seem at least plausible that the area under the curve is finite. However, from Definition 6.3, we have that

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{R\to\infty}\int_1^R x^{-1/2}\,dx = \lim_{R\to\infty}\left.\frac{x^{1/2}}{\frac{1}{2}}\right|_1^R = \lim_{R\to\infty}(2R^{1/2} - 2) = \infty.$$

This says that the improper integral diverges. ■

Note that examples 6.6 and 6.7 are special cases of $\int_1^\infty \frac{1}{x^p}\,dx$, corresponding to $p = 2$ and $p = 1/2$, respectively. In the exercises, you will show that this integral converges whenever $p > 1$ and diverges for $p \le 1$.

You may need to utilize l'Hôpital's Rule to evaluate the defining limit, as in example 6.8.

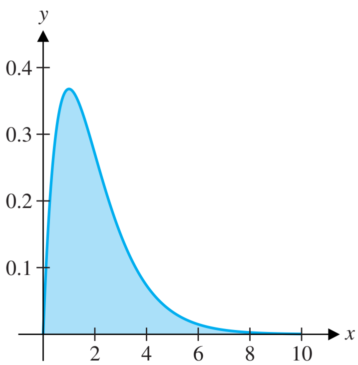


FIGURE 6.8
$y = xe^{-x}$

EXAMPLE 6.8 A Convergent Improper Integral

Determine whether $\int_0^\infty xe^{-x}\,dx$ converges or diverges.

Solution The graph of $y = xe^{-x}$ in Figure 6.8 makes it appear plausible that there could be a finite area under the graph. From Definition 6.3, we have

$$\int_0^\infty xe^{-x}\,dx = \lim_{R\to\infty}\int_0^R xe^{-x}\,dx.$$

To evaluate the last integral, you will need integration by parts. Let

$$\begin{aligned} u &= x & dv &= e^{-x}\,dx \\ du &= dx & v &= -e^{-x} \end{aligned}$$

We then have

$$\begin{aligned} \int_0^\infty xe^{-x}\,dx &= \lim_{R\to\infty}\int_0^R xe^{-x}\,dx = \lim_{R\to\infty}\left(-xe^{-x}\Big|_0^R + \int_0^R e^{-x}\,dx\right) \\ &= \lim_{R\to\infty}\left[(-Re^{-R} + 0) - e^{-x}\Big|_0^R\right] = \lim_{R\to\infty}(-Re^{-R} - e^{-R} + e^0). \end{aligned}$$

Note that the limit $\lim\limits_{R\to\infty} Re^{-R}$ has the indeterminate form $\infty \cdot 0$. We resolve this with l'Hôpital's Rule, as follows:

$$\begin{aligned} \lim_{R\to\infty} Re^{-R} &= \lim_{R\to\infty}\frac{R}{e^R} \quad \left(\frac{\infty}{\infty}\right) \\ &= \lim_{R\to\infty}\frac{\frac{d}{dR}R}{\frac{d}{dR}e^R} = \lim_{R\to\infty}\frac{1}{e^R} = 0. \quad \text{By l'Hôpital's Rule.} \end{aligned}$$

Returning to the improper integral, we now have

$$\int_0^{\infty} xe^{-x}\,dx = \lim_{R\to\infty}(-Re^{-R} - e^{-R} + e^0) = 0 - 0 + 1 = 1.$$

In examples 6.9 and 6.10, the lower limit of integration is $-\infty$.

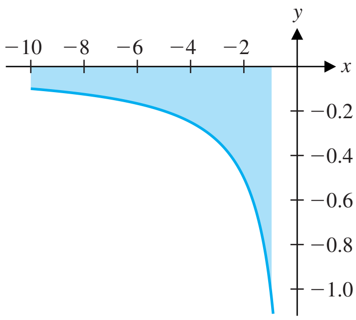


FIGURE 6.9
$y = \frac{1}{x}$

EXAMPLE 6.9 An Integral with an Infinite Lower Limit of Integration

Determine whether $\displaystyle\int_{-\infty}^{-1} \frac{1}{x}\,dx$ converges or diverges.

Solution In Figure 6.9, it appears plausible that there might be a finite area bounded between the graph of $y = \frac{1}{x}$ and the x-axis, on the interval $(-\infty, -1]$. However, from Definition 6.3, we have

$$\int_{-\infty}^{-1} \frac{1}{x}\,dx = \lim_{R\to-\infty}\int_R^{-1}\frac{1}{x}\,dx = \lim_{R\to-\infty} \ln|x|\Big|_R^{-1} = \lim_{R\to-\infty}[\ln|-1| - \ln|R|] = -\infty$$

and hence, the improper integral diverges.

EXAMPLE 6.10 A Convergent Improper Integral

Determine whether $\displaystyle\int_{-\infty}^{0} \frac{1}{(x-1)^2}\,dx$ converges or diverges.

Solution The graph in Figure 6.10 gives us hope to believe that the improper integral might converge. From Definition 6.3, we have

$$\int_{-\infty}^{0} \frac{1}{(x-1)^2}\,dx = \lim_{R\to-\infty}\int_R^0 (x-1)^{-2}\,dx = \lim_{R\to-\infty}\frac{(x-1)^{-1}}{-1}\Bigg|_R^0$$

$$= \lim_{R\to-\infty}\left[1 + \frac{1}{R-1}\right] = 1$$

and hence, the improper integral converges.

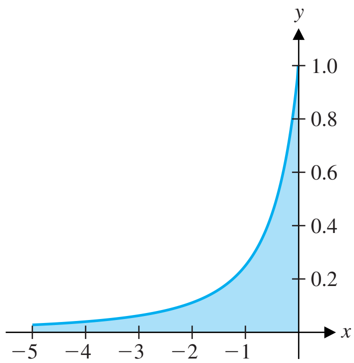


FIGURE 6.10
$y = \frac{1}{(x-1)^2}$

A final type of improper integral is $\int_{-\infty}^{\infty} f(x)\,dx$, defined as follows.

DEFINITION 6.4

If f is continuous on $(-\infty, \infty)$, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx, \quad \text{for any constant } a,$$

where $\int_{-\infty}^{\infty} f(x)\,dx$ converges if and only if *both* $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$ converge. If either one diverges, the original improper integral also diverges.

In Definition 6.4, note that you can choose a to be any real number. So, choose it to be something convenient (usually 0).

EXAMPLE 6.11 An Integral with Two Infinite Limits of Integration

Determine whether $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges or diverges.

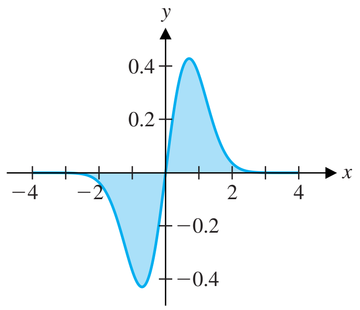


FIGURE 6.11
$y = xe^{-x^2}$

Solution Notice from the graph of the integrand in Figure 6.11 that, since the function tends to 0 relatively quickly (both as $x \to \infty$ and as $x \to -\infty$), it appears plausible that there is a finite area bounded by the graph of the function and the x-axis. From Definition 6.4 we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx. \tag{6.1}$$

You must evaluate each of the improper integrals on the right side of (6.1) separately. First, we have

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{R\to-\infty}\int_{R}^{0} xe^{-x^2}\,dx.$$

Letting $u = -x^2$, we have $du = -2x\,dx$ and so, being careful to change the limits of integration to match the new variable, we have

$$\begin{aligned}\int_{-\infty}^{0} xe^{-x^2}\,dx &= -\frac{1}{2}\lim_{R\to-\infty}\int_{R}^{0} e^{-x^2}(-2x)\,dx = -\frac{1}{2}\lim_{R\to-\infty}\int_{-R^2}^{0} e^{u}\,du \\ &= -\frac{1}{2}\lim_{R\to-\infty} e^{u}\Big|_{-R^2}^{0} = -\frac{1}{2}\lim_{R\to-\infty}\left(e^0 - e^{-R^2}\right) = -\frac{1}{2}.\end{aligned}$$

Similarly, we get (you should fill in the details)

$$\int_{0}^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty}\int_{0}^{R} xe^{-x^2}\,dx = -\frac{1}{2}\lim_{R\to\infty} e^{u}\Big|_{0}^{-R^2} = -\frac{1}{2}\lim_{R\to\infty}\left(e^{-R^2} - e^0\right) = \frac{1}{2}.$$

Since both of the preceding improper integrals converge, we get from (6.1) that the original integral also converges, to

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0. \blacksquare$$

EXAMPLE 6.12 An Integral with Two Infinite Limits of Integration

Determine whether $\int_{-\infty}^{\infty} e^{-x}\,dx$ converges or diverges.

Solution From Definition 6.4, we write

$$\int_{-\infty}^{\infty} e^{-x}\,dx = \int_{-\infty}^{0} e^{-x}\,dx + \int_{0}^{\infty} e^{-x}\,dx.$$

It's easy to show that $\int_0^{\infty} e^{-x}\,dx$ converges. (This is left as an exercise.) However,

$$\int_{-\infty}^{0} e^{-x}\,dx = \lim_{R\to-\infty}\int_{R}^{0} e^{-x}\,dx = \lim_{R\to-\infty} -e^{-x}\Big|_{R}^{0} = \lim_{R\to-\infty}\left(-e^0 + e^{-R}\right) = \infty.$$

This says that $\int_{-\infty}^{0} e^{-x}\,dx$ diverges and hence, $\int_{-\infty}^{\infty} e^{-x}\,dx$ diverges, also, even though $\int_0^{\infty} e^{-x}\,dx$ converges. $\blacksquare$

CAUTION

Do *not* write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

It's certainly tempting to write this, especially since this will often give a correct answer, with about half of the work. Unfortunately, this will often give incorrect answers, too, as the limit on the right-hand side frequently exists for divergent integrals. We explore this issue further in the exercises.

We can't emphasize enough that you should verify the continuity of the integrand for every single integral you evaluate. In example 6.13, we see another reminder of why you must do this.

EXAMPLE 6.13 An Integral That Is Improper for Two Reasons

Determine the convergence or divergence of the improper integral $\int_0^\infty \frac{1}{(x-1)^2}\,dx$.

Solution First try to see what is wrong with the following erroneous calculation:

$$\int_0^\infty \frac{1}{(x-1)^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{(x-1)^2}\,dx. \quad \text{This is incorrect!}$$

Look carefully at the integrand and observe that it is *not* continuous on $[0, \infty)$. In fact, the integrand blows up at $x = 1$, which is in the interval over which you're trying to integrate. Thus, this integral is improper for several different reasons. In order to deal with the discontinuity at $x = 1$, we must break up the integral into several pieces, as in Definition 6.2. We write

$$\int_0^\infty \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^\infty \frac{1}{(x-1)^2}\,dx. \tag{6.2}$$

The second integral on the right side of (6.2) must be further broken into two pieces, since it is improper, both at the left endpoint and by virtue of having an infinite limit of integration. You can pick any point on $(1, \infty)$ at which to break up the interval. We'll simply choose $x = 2$. We now have

$$\int_0^\infty \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx + \int_2^\infty \frac{1}{(x-1)^2}\,dx.$$

Each of these three improper integrals must be evaluated separately, using the appropriate limit definitions. We leave it as an exercise to show that the first two integrals diverge, while the third one converges. This says that the original improper integral diverges (a conclusion you would miss if you did not notice that the integrand blows up at $x = 1$). ■

A Comparison Test

We have now defined several different types of improper integrals, each as a limit of a proper definite integral. In order to compute such a limit, we first need to find an antiderivative. However, since no antiderivative is available for e^{-x^2}, how would you establish the convergence or divergence of $\int_0^\infty e^{-x^2}\,dx$? An answer lies in the following result.

Given two functions f and g that are continuous on the interval $[a, \infty)$, suppose that

$$0 \le f(x) \le g(x), \quad \text{for all } x \ge a.$$

We illustrate this situation in Figure 6.12. In this case, $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ correspond to the areas under the respective curves. Notice that if $\int_a^\infty g(x)\,dx$ (corresponding to the larger area) converges, then this says that there is a finite area under the curve $y = g(x)$ on the interval $[a, \infty)$. Since $y = f(x)$ lies *below* $y = g(x)$, there can be only a finite area under the curve $y = f(x)$, as well. Thus, $\int_a^\infty f(x)\,dx$ converges also.

On the other hand, if $\int_a^\infty f(x)\,dx$ (corresponding to the smaller area) diverges, the area under the curve $y = f(x)$ is infinite. Since $y = g(x)$ lies *above* $y = f(x)$, there must be an infinite area under the curve $y = g(x)$, also, so that $\int_a^\infty g(x)\,dx$ diverges, as well. This comparison of improper integrals based on the relative size of their integrands is called a **comparison test** (one of several) and is spelled out in Theorem 6.1.

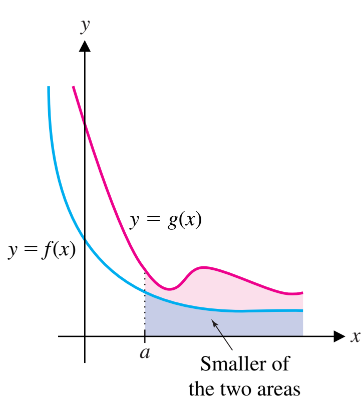


FIGURE 6.12
The Comparison Test

THEOREM 6.1 (Comparison Test)

Suppose that f and g are continuous on $[a, \infty)$ and $0 \le f(x) \le g(x)$, for all $x \in [a, \infty)$.

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges, also.
(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges, also.

REMARK 6.1

We can state corresponding comparison tests for improper integrals of the form $\int_{-\infty}^a f(x)\,dx$, where f is continuous on $(-\infty, a]$, as well as for integrals that are improper owing to a discontinuity in the integrand.

We omit the proof of Theorem 6.1, leaving it to stand on the intuitive argument already made.

The idea of the Comparison Test is to compare a given improper integral to another improper integral whose convergence or divergence is already known (or can be more easily determined). If you use a comparison to establish that an improper integral converges, then you can always approximate its value numerically. If you use a comparison to establish that an improper integral diverges, then there's nothing more to do.

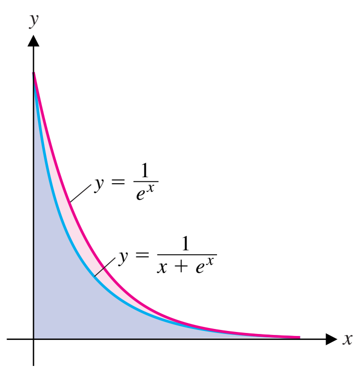


FIGURE 6.13
Comparing $y = \frac{1}{e^x}$ and $y = \frac{1}{x + e^x}$

R	$\int_0^R \frac{1}{x + e^x}\,dx$
10	0.8063502
20	0.8063956
30	0.8063956
40	0.8063956

EXAMPLE 6.14 Using the Comparison Test for an Improper Integral

Determine the convergence or divergence of $\int_0^\infty \frac{1}{x + e^x}\,dx$.

Solution First, note that you do not know an antiderivative for $\frac{1}{x + e^x}$ and so, there is no way to compute the improper integral directly. However, notice that for $x \ge 0$,

$$0 \le \frac{1}{x + e^x} \le \frac{1}{e^x}.$$

(See Figure 6.13.) It's an easy exercise to show that $\int_0^\infty \frac{1}{e^x}\,dx$ converges (to 1). From Theorem 6.1, it now follows that $\int_0^\infty \frac{1}{x + e^x}\,dx$ converges, also. So, we know that the integral is convergent, but to what value does it converge? The Comparison Test only helps us to determine whether or not the integral converges. Notice that it does *not* help to find the value of the integral. We can, however, use numerical integration (e.g., Simpson's Rule) to approximate $\int_0^R \frac{1}{x + e^x}\,dx$, for a sequence of values of R. The accompanying table illustrates some approximate values of $\int_0^R \frac{1}{x + e^x}\,dx$, produced using the numerical integration package built into our CAS. [If you use Simpson's Rule for this, note that you will need to increase the value of n (the number of subintervals in the partition) as R increases.] Notice that as R gets larger and larger, the approximate values for the corresponding integrals seem to be approaching 0.8063956, so we take this as an approximate value for the improper integral.

$$\int_0^\infty \frac{1}{x + e^x}\,dx \approx 0.8063956.$$

You should calculate approximate values for even larger values of R to convince yourself that this estimate is accurate. ■

In example 6.15, we examine an integral that has important applications in probability and statistics.

EXAMPLE 6.15 Using the Comparison Test for an Improper Integral

Determine the convergence or divergence of $\int_0^\infty e^{-x^2}\,dx$.

Solution Once again, notice that you do not know an antiderivative for the integrand e^{-x^2}. However, observe that for $x > 1$, $e^{-x^2} < e^{-x}$. (See Figure 6.14.) We can rewrite the integral as

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

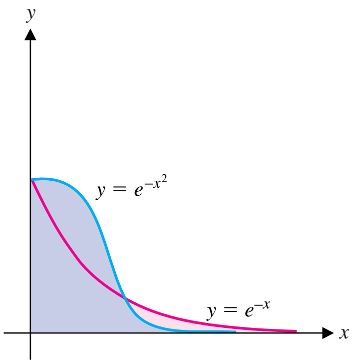


FIGURE 6.14
$y = e^{-x^2}$ and $y = e^{-x}$

Since the first integral on the right-hand side is a proper definite integral, only the second integral is improper. It's an easy matter to show that $\int_1^\infty e^{-x}\,dx$ converges. By the Comparison Test, it then follows that $\int_1^\infty e^{-x^2}\,dx$ also converges. We leave it as an exercise to show that

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx \approx 0.8862269.$$

Using more advanced techniques of integration, it is possible to prove the surprising result that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. ■

The Comparison Test can be used with equal ease to show that an improper integral is divergent.

EXAMPLE 6.16 Using the Comparison Test: A Divergent Integral

Determine the convergence or divergence of $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\,dx$.

Solution As in examples 6.14 and 6.15, you do not know an antiderivative for the integrand and so, your only hope for determining whether or not the integral converges is to use a comparison. First, recall that

$$-1 \le \sin x \le 1, \quad \text{for all } x.$$

We then have that

$$\frac{1}{\sqrt{x}} = \frac{2-1}{\sqrt{x}} \le \frac{2+\sin x}{\sqrt{x}}, \quad \text{for } 1 \le x < \infty.$$

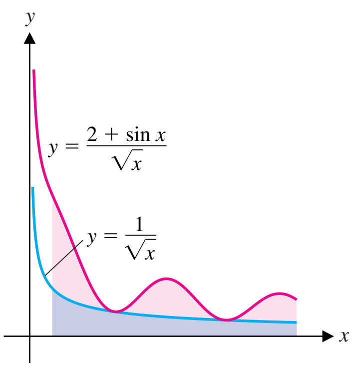


FIGURE 6.15
Comparing $y = \frac{1}{\sqrt{x}}$ and $y = \frac{2+\sin x}{\sqrt{x}}$

(See Figure 6.15 for a graph of the two functions.) Recall that we showed in example 6.7 that $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges. The Comparison Test now tells us that $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\,dx$ must diverge, also. ■

The big question, of course, is how to find an improper integral to compare to a given integral. Look carefully at the integrand to see if it resembles any functions whose antiderivative you might know (or at least have a hope of finding using our various techniques of integration). Beyond this, our best answer is that this comes with experience. Comparisons are typically done by the seat of your pants. We provide ample exercises on this topic to give you some experience with finding appropriate comparisons. Look hard for comparisons and don't give up too easily.

BEYOND FORMULAS

It may seem that this section introduces an overwhelming number of new formulas to memorize. Actually, you can look at this entire section as a warning with a familiar outcome. The warning is that not all functions and intervals satisfy the hypotheses of the Fundamental Theorem. Watch out for this! The solution is one we've seen over and over again. Approximate the integral and compute a limit as the approximate interval approaches the desired interval. Answer the following for yourself. How do each of the examples in this section fit this pattern?

EXERCISES 6.6

WRITING EXERCISES

1. For many students, our emphasis on working through the limit process for an improper integral may seem unnecessarily careful. Explain, using examples from this section, why it is important to have and use precise definitions.

2. Identify the following statement as true or false (meaning not always true) and explain why: If the integrand $f(x) \to \infty$ as $x \to a^+$ or as $x \to b^-$, then the area $\int_a^b f(x)\,dx$ is infinite; that is, $\int_a^b f(x)\,dx$ diverges.

In exercises 1–6, determine whether or not the integral is improper.

1. $\int_0^2 x^{-2/5}\,dx$
2. $\int_1^2 x^{-2/5}\,dx$
3. $\int_0^2 x^{2/5}\,dx$
4. $\int_0^\infty x^{2/5}\,dx$
5. $\int_{-2}^2 \frac{3}{x}\,dx$
6. $\int_2^\infty \frac{3}{x}\,dx$

In exercises 7–34, determine whether the integral converges or diverges. Find the value of the integral if it converges.

7. $\int_0^1 x^{-1/3}\,dx$
8. $\int_0^1 x^{-4/3}\,dx$
9. $\int_1^\infty x^{-4/5}\,dx$
10. $\int_1^\infty x^{-6/5}\,dx$
11. $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$
12. $\int_1^5 \frac{2}{\sqrt{5-x}}\,dx$
13. $\int_0^1 \ln x\,dx$
14. $\int_0^{\pi/2} \tan x\,dx$
15. $\int_0^3 \frac{2}{x^2-1}\,dx$
16. $\int_{-4}^4 \frac{2x}{x^2-1}\,dx$
17. $\int_0^\infty xe^x\,dx$
18. $\int_1^\infty x^2e^{-2x}\,dx$
19. $\int_{-\infty}^1 x^2e^{3x}\,dx$
20. $\int_{-\infty}^0 xe^{-4x}\,dx$
21. $\int_{-\infty}^\infty \frac{1}{x^2}\,dx$
22. $\int_{-\infty}^\infty \frac{1}{\sqrt[3]{x}}\,dx$
23. $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$
24. $\int_{-\infty}^\infty \frac{1}{x^2-1}\,dx$
25. $\int_0^\pi \cot x\,dx$
26. $\int_0^\pi \sec^2 x\,dx$
27. $\int_0^2 \frac{x}{x^2-1}\,dx$
28. $\int_0^\infty \frac{1}{(x-2)^2}\,dx$
29. $\int_0^1 \frac{2}{\sqrt{1-x^2}}\,dx$
30. $\int_0^1 \frac{2}{x\sqrt{1-x^2}}\,dx$
31. $\int_0^\infty \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx$
32. $\int_0^\infty \tan x\,dx$
33. $\int_0^\infty \cos x\,dx$
34. $\int_0^\infty \cos x e^{-\sin x}\,dx$

35. Based on exercises 7–8 and similar integrals, conjecture a value of r for which $\int_0^1 x^{-n}\,dx$ converges if and only if $n < r$.

36. Based on exercises 9–10 and similar integrals, conjecture a value of r for which $\int_1^\infty x^{-n}\,dx$ converges if and only if $n > r$.

37. Based on exercises 17–20, conjecture that the exponential term controls the convergence or divergence of $\int_0^\infty xe^{cx}\,dx$ and $\int_{-\infty}^0 xe^{cx}\,dx$. For which values of c do these integrals converge?

38. At the beginning of this section, we indicated that the calculation $\int_{-1}^2 \frac{1}{x^2}\,dx = -\frac{3}{2}$ was incorrect. Without any calculations, explain how you should immediately recognize that *negative* 3/2 is not a correct value.

In exercises 39–48, use a comparison to determine whether the integral converges or diverges.

39. $\int_1^\infty \frac{x}{1+x^3}\,dx$
40. $\int_1^\infty \frac{x^2-2}{x^4+3}\,dx$
41. $\int_2^\infty \frac{x}{x^{3/2}-1}\,dx$
42. $\int_1^\infty \frac{2+\sec^2 x}{x}\,dx$

43. $\displaystyle\int_0^{\infty} \frac{3}{x+e^x}\,dx$ **44.** $\displaystyle\int_1^{\infty} e^{-x^3}\,dx$

45. $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{1+e^x}\,dx$ **46.** $\displaystyle\int_2^{\infty} \frac{\ln x}{e^x+1}\,dx$

47. $\displaystyle\int_2^{\infty} \frac{x^2 e^x}{\ln x}\,dx$ **48.** $\displaystyle\int_1^{\infty} e^{x^2+x+1}\,dx$

In exercises 49 and 50, use integration by parts and l'Hôpital's Rule.

49. $\displaystyle\int_0^{1} x\ln 4x\,dx$ **50.** $\displaystyle\int_0^{\infty} xe^{-2x}\,dx$

51. In this exercise, you will look at an interesting pair of calculations known as Gabriel's horn. The horn is formed by taking the curve $y = 1/x$ for $x \geq 1$ and revolving it about the x-axis. Show that the volume is finite (i.e., the integral converges), but that the surface area is infinite (i.e., the integral diverges). The paradox is that this would seem to indicate that the horn could be filled with a finite amount of paint but that the outside of the horn could not be covered with any finite amount of paint.

52. Show that $\int_{-\infty}^{\infty} x^3\,dx$ diverges but $\lim\limits_{R\to\infty} \int_{-R}^{R} x^3\,dx = 0$.

In exercises 53–56, determine whether the statement is true or false (not always true).

53. If $\lim\limits_{x\to\infty} f(x) = 1$, then $\int_0^{\infty} f(x)\,dx$ diverges.

54. If $\lim\limits_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.

55. If $\lim\limits_{x\to 0} f(x) = \infty$, then $\int_0^{1} f(x)\,dx$ diverges.

56. If $f(-x) = -f(x)$ for all x, then $\int_{-\infty}^{\infty} f(x)\,dx = 0$.

57. Find all values of p for which $\int_0^1 \frac{1}{x^p}\,dx$ converges. For these values of p, show that $\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{(1-x)^p}\,dx$.

58. Show that $\int_{-\infty}^{\infty} x^p\,dx$ diverges for every p.

59. Given that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, evaluate $\int_{-\infty}^{\infty} e^{-kx^2}\,dx$ for $k > 0$.

60. Given that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, evaluate $\int_{-\infty}^{\infty} x^2 e^{-kx^2}\,dx$ for $k > 0$.

61. A function $f(x) \geq 0$ is a probability density function (pdf) on the interval $[0,\infty)$ if $\int_0^{\infty} f(x)\,dx = 1$. Find the value of the constant k to make each of the following pdf's on the interval $[0,\infty)$.

(a) ke^{-2x} (b) ke^{-4x} (c) ke^{-rx}

62. Find the value of the constant k to make each of the following pdf's on the interval $[0,\infty)$. (See exercise 61.)

(a) kxe^{-2x} (b) kxe^{-4x} (c) kxe^{-rx}

63. The **mean** μ (one measure of average) of a random variable with pdf $f(x)$ on the interval $[0,\infty)$ is $\mu = \int_0^{\infty} xf(x)\,dx$. Find the mean if $f(x) = re^{-rx}$.

64. Find the mean of a random variable with pdf $f(x) = r^2 xe^{-rx}$. (See exercise 63.)

65. For the mean μ found in exercise 63, compute the probability that the random variable is greater than μ. This probability is given by $\int_{\mu}^{\infty} re^{-rx}\,dx$. Do you think that it is odd that the probability is *not* equal to $1/2$?

66. Find the **median** (another measure of average) for a random variable with pdf re^{-rx} on $x \geq 0$. The median is the 50% mark on probability, that is, the value m for which

$$\int_m^{\infty} re^{-rx}\,dx = \frac{1}{2}.$$

67. Explain why $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$ is an improper integral. Assuming that it converges, explain why it is equal to $\displaystyle\int_0^{\pi/2} f(x)\,dx$, where $f(x) = \begin{cases} \dfrac{1}{1+\tan x} & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } x = \frac{\pi}{2} \end{cases}$. Similarly, find a function $g(x)$ such that the improper integral $\displaystyle\int_0^{\pi/2} \frac{\tan x}{1+\tan x}\,dx$ equals the proper integral $\displaystyle\int_0^{\pi/2} g(x)\,dx$.

68. Interpreting $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$ and $\displaystyle\int_0^{\pi/2} \frac{\tan x}{1+\tan x}\,dx$ as in exercise 67, use the substitution $u = x - \frac{\pi}{2}$ to show that $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} \frac{\tan x}{1+\tan x}\,dx$. Adding the first integral to both sides of the equation, evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$.

69. Being careful to use limits for the improper integrals, use the substitution $u = \frac{\pi}{2} - x$ to show that (a) $\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx$. Add $\int_0^{\pi/2} \ln(\sin x)\,dx$ to both sides of this equation and simplify the right-hand side with the identity $\sin 2x = 2\sin x\cos x$. (b) Use this result to show that $2\int_0^{\pi/2} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi} \ln(\sin x)\,dx$. (c) Show that $\int_0^{\pi} \ln(\sin x)\,dx = 2\int_0^{\pi/2} \ln(\sin x)\,dx$. (d) Use parts (b) and (c) to evaluate $\int_0^{\pi/2} \ln(\sin x)\,dx$.

70. Determine whether $\int_0^{\pi/2} \ln x\,dx$ converges or diverges. Given the result of exercise 69 and the approximation $\sin x \approx x$ for small x, explain why this result is not surprising.

71. Assuming that all integrals converge, use integration by parts to write $\int_{-\infty}^{\infty} x^4 e^{-x^2}\,dx$ in terms of $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$ and then in terms of $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. By induction, show that $\displaystyle\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^{n-1}}\sqrt{\pi}$, for any positive integer n.

72. Show that $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$, for any positive constant a. Formally (that is, ignore issues of convergence) compute n derivatives with respect to a of this equation, set $a = 1$ and compare the result to that of exercise 71.

73. As discussed in the chapter introduction, the mean (average) lifetime of a lightbulb might have the form

$\int_0^\infty 0.001xe^{-0.001x}\,dx$. To determine the mean, compute $\lim_{b\to\infty}\int_0^b 0.001xe^{-0.001x}\,dx$.

74. As in exercise 73, find the mean of any exponential distribution with pdf $f(x)=\lambda e^{-\lambda x}$, $0\le x<\infty$.

75. Many probability questions involve **conditional probabilities.** For example, if you know that a lightbulb has already burned for 30 hours, what is the probability that it will last at least 5 more hours? This is the "probability that $x>35$ given that $x>30$" and is written as $P(x>35|x>30)$. In general, for events A and B, $P(A|B)=\dfrac{P(A \text{ and } B)}{P(B)}$, which in this case reduces to $P(x>35|x>30)=\dfrac{P(x>35)}{P(x>30)}$. For the pdf $f(x)=\frac{1}{40}e^{-x/40}$ (in hours), compute $P(x>35|x>30)$. Also, compute $P(x>40|x>35)$ and $P(x>45|x>40)$. (Hint: $P(x>35)=1-P(x\le 35)=1-\int_0^{35} f(x)\,dx$.)

76. Exercise 75 illustrates the "memoryless property" of exponential distributions. The probability that a lightbulb last m more hours given that it has already lasted n hours depends only on m and not on n. Prove this for the pdf $f(x)=\frac{1}{40}e^{-x/40}$.

77. Show that any exponential pdf $f(x)=ce^{-cx}$ has the memoryless property of exercise 76.

78. The **reliability function** $R(t)$ gives the probability that $x>t$. For the pdf of a lightbulb, this is the probability that the bulb lasts at least t hours. Compute $R(t)$ for a general exponential pdf $f(x)=ce^{-cx}$.

79. The **Omega function** is used for risk/reward analysis of financial investments. Suppose that $f(x)$ is a pdf on $(-\infty,\infty)$ and gives the distribution of returns on an investment. (Then $\displaystyle\int_a^b f(x)\,dx$ is the probability that the investment returns between \$$a$ and \$$b$.) Let $F(x)=\displaystyle\int_{-\infty}^x f(t)\,dt$ be the **cumulative distribution function** for returns. Then $\Omega(r)=\dfrac{\int_r^\infty [1-F(x)]\,dx}{\int_{-\infty}^r F(x)\,dx}$ is the Omega function for the investment.

(a) Compute $\Omega_1(r)$ for the exponential distribution $f_1(x)=2e^{-2x}$, $0\le x<\infty$. Note that $\Omega_1(r)$ will be undefined (∞) for $r\le 0$.

(b) Compute $\Omega_2(r)$ for $f_2(x)=1$, $0\le x\le 1$.

(c) Show that the means of $f_1(x)$ and $f_2(x)$ are the same and that $\Omega(r)=1$ when r equals the mean.

(d) Even though the means are the same, investments with distributions $f_1(x)$ and $f_2(x)$ are not equivalent. Use the graphs of $f_1(x)$ and $f_2(x)$ to explain why $f_1(x)$ corresponds to a riskier investment than $f_2(x)$.

(e) Show that for some value c, $\Omega_2(r)>\Omega_1(r)$ for $r<c$ and $\Omega_2(r)<\Omega_1(r)$ for $r>c$. In general, the larger $\Omega(r)$ is, the better the investment is. Explain this in terms of this example.

EXPLORATORY EXERCISES

1. The so-called **Boltzmann integral**

$$I(p)=\int_0^1 p(x)\ln p(x)\,dx$$

is important in the mathematical field of **information theory.** Here, $p(x)$ is a pdf on the interval [0, 1]. Graph the pdf's $p_1(x)=1$ and

$$p_2(x)=\begin{cases}4x & \text{if } 0\le x\le 1/2\\ 4-4x & \text{if } 1/2\le x\le 1\end{cases}$$

and compute the integrals $\int_0^1 p_1(x)\,dx$ and $\int_0^1 p_2(x)\,dx$ to verify that they are pdf's. Then compute the Boltzmann integrals $I(p_1)$ and $I(p_2)$. Suppose that you are trying to determine the value of a quantity that you know is between 0 and 1. If the pdf for this quantity is $p_1(x)$, then all values are equally likely. What would a pdf of $p_2(x)$ indicate? Noting that $I(p_2)>I(p_1)$, explain why it is fair to say that the Boltzmann integral measures the amount of *information* available. Given this interpretation, sketch a pdf $p_3(x)$ that would have a larger Boltzmann integral than $p_2(x)$.

2. The **Laplace transform** is an invaluable tool in many engineering disciplines. As the name suggests, the transform turns a function $f(t)$ into a different function $F(s)$. By definition, the Laplace transform of the function $f(t)$ is

$$F(s)=\int_0^\infty f(t)e^{-st}\,dt.$$

To find the Laplace transform of $f(t)=1$, compute

$$\int_0^\infty (1)e^{-st}\,dt=\int_0^\infty e^{-st}\,dt.$$

Show that the integral equals $1/s$, for $s>0$. We write $L\{1\}=1/s$. Show that

$$L\{t\}=\int_0^\infty te^{-st}\,dt=\frac{1}{s^2},$$

for $s>0$. Compute $L\{t^2\}$ and $L\{t^3\}$ and conjecture the general formula for $L\{t^n\}$. Then, find $L\{e^{at}\}$ for $s>a$.

3. The **gamma function** is defined by $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$, if the integral converges. For such a complicated-looking function, the gamma function has some surprising properties. First, show that $\Gamma(1)=1$. Then use integration by parts and l'Hôpital's Rule to show that $\Gamma(n+1)=n\Gamma(n)$, for any $n>0$. Use this property and mathematical induction to show that $\Gamma(n+1)=n!$, for any positive integer n. (Notice that this includes the value $0!=1$.) Numerically approximate $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{5}{2}\right)$. Is it reasonable to define these as $\left(\frac{1}{2}\right)!$ and $\left(\frac{3}{2}\right)!$, respectively? In this sense, show that $\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$. Finally, for $x < 1$, the defining integral for $\Gamma(x)$ is improper in two ways. Use a comparison test to show the convergence of $\int_1^\infty t^{x-1}e^{-t}\,dt$. This leaves $\int_0^1 t^{x-1}e^{-t}\,dt$. Determine the range of p-values for which $\int_0^1 t^p e^{-t}\,dt$ converges and then determine the set of x's for which $\Gamma(x)$ is defined.

4. Generalize exercise 68 to evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan^k x}\,dx$ for any real number k.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Integration by parts	Reduction formula
Partial fractions decomposition	CAS
Improper integral	Integral converges
Integral diverges	Comparison Test

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. Integration by parts works only for integrals of the form $\int f(x)g(x)\,dx$.
2. For an integral of the form $\int xf(x)\,dx$, always use integration by parts with $u = x$.
3. The trigonometric techniques in section 6.3 are all versions of substitution.
4. If an integrand contains a factor of $\sqrt{1-x^2}$, you should substitute $x = \sin\theta$.
5. If p and q are polynomials, then any integral of the form $\displaystyle\int \frac{p(x)}{q(x)}\,dx$ can be evaluated.
6. With an extensive integral table, you don't need to know any integration techniques.
7. If $f(x)$ has a vertical asymptote at $x = a$, then $\int_a^b f(x)\,dx$ diverges for any b.
8. If $\lim\limits_{x\to\infty} f(x) = L \neq 0$, then $\int_1^\infty f(x)\,dx$ diverges.

In exercises 1–44, evaluate the integral.

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
2. $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx$
3. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$
4. $\displaystyle\int \frac{2}{\sqrt{9-x^2}}\,dx$
5. $\displaystyle\int x^2e^{-3x}\,dx$
6. $\displaystyle\int x^2e^{-x^3}\,dx$
7. $\displaystyle\int \frac{x}{1+x^4}\,dx$
8. $\displaystyle\int \frac{x^3}{1+x^4}\,dx$
9. $\displaystyle\int \frac{x^3}{4+x^4}\,dx$
10. $\displaystyle\int \frac{x}{4+x^4}\,dx$
11. $\displaystyle\int e^{2\ln x}\,dx$
12. $\displaystyle\int \cos 4x\,dx$
13. $\displaystyle\int_0^1 x\sin 3x\,dx$
14. $\displaystyle\int_0^1 x\sin 4x^2\,dx$
15. $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx$
16. $\displaystyle\int_0^{\pi/2} \cos^3 x\,dx$
17. $\displaystyle\int_{-1}^1 x\sin\pi x\,dx$
18. $\displaystyle\int_0^1 x^2\cos\pi x\,dx$
19. $\displaystyle\int_1^2 x^3\ln x\,dx$
20. $\displaystyle\int_0^{\pi/4} \sin x\cos x\,dx$
21. $\displaystyle\int \cos x\sin^2 x\,dx$
22. $\displaystyle\int \cos x\sin^3 x\,dx$
23. $\displaystyle\int \cos^3 x\sin^3 x\,dx$
24. $\displaystyle\int \cos^4 x\sin^3 x\,dx$
25. $\displaystyle\int \tan^2 x\sec^4 x\,dx$
26. $\displaystyle\int \tan^3 x\sec^2 x\,dx$
27. $\displaystyle\int \sqrt{\sin x}\cos^3 x\,dx$
28. $\displaystyle\int \tan^3 x\sec^3 x\,dx$
29. $\displaystyle\int \frac{2}{8+4x+x^2}\,dx$
30. $\displaystyle\int \frac{3}{\sqrt{-2x-x^2}}\,dx$

Review Exercises

31. $\int \frac{2}{x^2\sqrt{4-x^2}}\,dx$

32. $\int \frac{x}{\sqrt{9-x^2}}\,dx$

33. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

34. $\int \frac{x^3}{\sqrt{x^2-9}}\,dx$

35. $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

36. $\int \frac{4}{\sqrt{x+9}}\,dx$

37. $\int \frac{x+4}{x^2+3x+2}\,dx$

38. $\int \frac{5x+6}{x^2+x-12}\,dx$

39. $\int \frac{4x^2+6x-12}{x^3-4x}\,dx$

40. $\int \frac{5x^2+2}{x^3+x}\,dx$

41. $\int e^x \cos 2x\,dx$

42. $\int x^3 \sin x^2\,dx$

43. $\int x\sqrt{x^2+1}\,dx$

44. $\int \sqrt{1-x^2}\,dx$

In exercises 45–50, find the partial fractions decomposition.

45. $\frac{4}{x^2-3x-4}$

46. $\frac{2x}{x^2+x-6}$

47. $\frac{-6}{x^3+x^2-2x}$

48. $\frac{x^2-2x-2}{x^3+x}$

49. $\frac{x-2}{x^2+4x+4}$

50. $\frac{x^2-2}{(x^2+1)^2}$

In exercises 51–60, use the Table of Integrals to find the integral.

51. $\int e^{3x}\sqrt{4+e^{2x}}\,dx$

52. $\int x\sqrt{x^4-4}\,dx$

53. $\int \sec^4 x\,dx$

54. $\int \tan^5 x\,dx$

55. $\int \frac{4}{x(3-x)^2}\,dx$

56. $\int \frac{\cos x}{\sin^2 x(3+4\sin x)}\,dx$

57. $\int \frac{\sqrt{9+4x^2}}{x^2}\,dx$

58. $\int \frac{x^2}{\sqrt{4-9x^2}}\,dx$

59. $\int \frac{\sqrt{4-x^2}}{x}\,dx$

60. $\int \frac{x^2}{(x^6-4)^{3/2}}\,dx$

In exercises 61–68, determine whether the integral converges or diverges. If it converges, find the limit.

61. $\int_0^1 \frac{x}{x^2-1}\,dx$

62. $\int_4^{10} \frac{2}{\sqrt{x-4}}\,dx$

63. $\int_1^{\infty} \frac{3}{x^2}\,dx$

64. $\int_1^{\infty} xe^{-3x}\,dx$

65. $\int_0^{\infty} \frac{4}{4+x^2}\,dx$

66. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

67. $\int_{-2}^{2} \frac{3}{x^2}\,dx$

68. $\int_{-2}^{2} \frac{x}{1-x^2}\,dx$

69. Cardiologists test heart efficiency by injecting a dye at a constant rate R into a vein near the heart and measuring the concentration of the dye in the bloodstream over a period of T seconds. If all of the dye is pumped through, the concentration is $c(t) = R$. Compute the total amount of dye $\int_0^T c(t)\,dt$. For a general concentration, the cardiac output is defined by $\frac{RT}{\int_0^T c(t)\,dt}$. Interpret this quantity. Compute the cardiac output if $c(t) = 3te^{2Tt}$.

70. For $\int \ln(x+1)\,dx$, you can use integration by parts with $u = \ln(x+1)$ and $dv = 1$. Compare your answers using $v = x$ versus using $v = x+1$.

71. Show that the average value of $\ln x$ on the interval $(0, e^n)$ equals $n-1$ for any positive integer n.

72. Many probability questions involve **conditional probabilities.** For example, if you know that a lightbulb has already burned for 30 hours, what is the probability that it will last at least 5 more hours? This is the "probability that $x > 35$ given that $x > 30$" and is written as $P(x > 35|x > 30)$. In general, for events A and B, $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$. The **failure rate function** is given as the limit of $\frac{P(x < t + \Delta t|x > t)}{\Delta t}$ as $\Delta t \to 0$. For the pdf $f(x)$ of the lifetime of a lightbulb, the numerator is the probability that the bulb burns out between times t and $t + \Delta t$. Use $R(t) = P(x > t)$ to show that the failure rate function can be written as $\frac{f(t)}{R(t)}$.

73. Show that the failure rate function (see exercise 72) of an exponential pdf $f(x) = ce^{-cx}$ is constant.

74. For the gamma distribution $f(x) = xe^{-x}$, (a) use a CAS to show that $P(x > s+t|x > s) = e^{-t} + \frac{t}{1+s}e^{-t}$. (b) Show that this is a decreasing function of s (for a fixed t). (c) If this is the pdf for annual rainfall amounts in a certain city, interpret the result of part (b).

75. Scores on IQ tests are intended to follow the distribution $f(x) = \frac{1}{\sqrt{450\pi}}e^{-(x-100)^2/450}$. Based on this distribution, what percentage of people are supposed to have IQs between 90 and 100? If the top 1% of scores are to be given the title of "genius," how high do you have to score to get this title?

Review Exercises

EXPLORATORY EXERCISES

1. In this exercise, you will try to determine whether or not $\int_0^1 \sin(1/x)\,dx$ converges. Since $|\sin(1/x)| \le 1$, the integral does not diverge to ∞, but that does not necessarily mean it converges. Explain why the integral $\int_0^\infty \sin x\,dx$ diverges (not to ∞, but by oscillating indefinitely). You need to determine whether a similar oscillation occurs for $\int_0^1 \sin(1/x)\,dx$. First, estimate $\int_R^1 \sin(1/x)\,dx$ numerically for $R = 1/\pi, 1/(2\pi), 1/(3\pi)$ and so on. Note that once you have $\int_{1/\pi}^1 \sin(1/x)\,dx$, you can get $\int_{1/(2\pi)}^1 \sin(1/x)\,dx$ by "adding" $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx$. We put this in quotes because this new integral is negative. Verify that the integrals $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx$, $\int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\,dx$ and so on, are alternately negative and positive, so that the sum $\int_R^1 \sin(1/x)\,dx$ does seem to converge as $R \to 0^+$. It turns out that the limit does converge if the additional integrals $\int_{1/((n+1)\pi)}^{1/(n\pi)} \sin(1/x)\,dx$ tend to 0 as $n \to \infty$. Show that this is true.

2. Suppose that $f(x)$ is a function such that both $\int_{-\infty}^\infty f(x)\,dx$ and $\int_{-\infty}^\infty f(x-1/x)\,dx$ converge. Start with $\int_{-\infty}^\infty f(x-1/x)\,dx$ and make the substitution $u = -\frac{1}{x}$. Show that $2\int_{-\infty}^\infty f(x-1/x)\,dx = \int_{-\infty}^\infty \frac{1}{u^2} f(u-1/u)\,du$. Then let $y = u - 1/u$. Show that $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^\infty f(x-1/x)\,dx$. Use this result to evaluate $\int_{-\infty}^\infty \frac{x^2}{x^2+(x^2-1)^2}\,dx$ and $\int_{-\infty}^\infty e^{-x^2+2-1/x^2}\,dx$.

3. Evaluate $\int_0^{\pi/2} \frac{ab}{(a\cos x + b\sin x)^2}\,dx$, by dividing all terms by $\cos^2 x$, using the substitution $u = ab\tan x$ and evaluating the improper integral $\int_0^\infty \frac{a^2}{(u+a^2)^2}\,dx$.