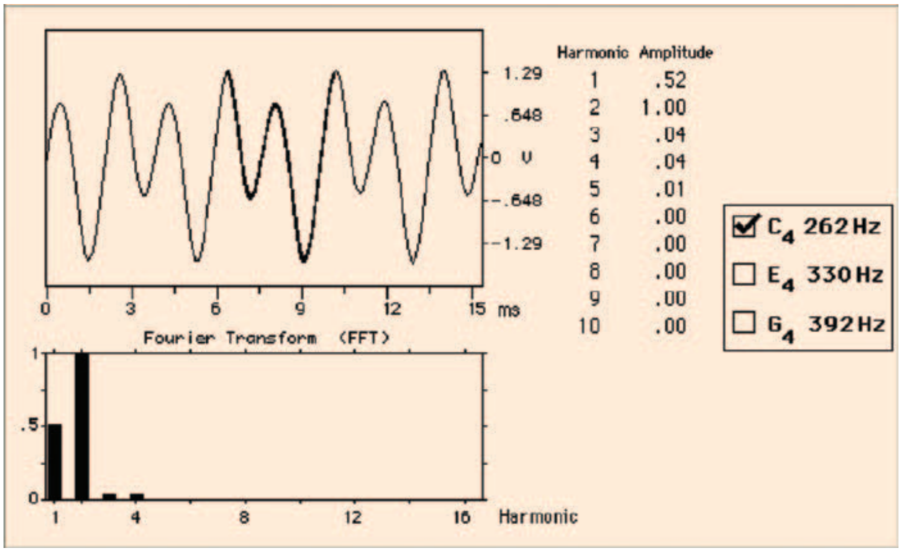
CHAPTER

8

Infinite Series

In our daily lives, we are increasingly seeing the impact of digital technologies. For instance, the dominant media for the entertainment industry are now CDs and DVDs; we have digital video and still cameras, and the Internet gives us easy access to a virtual world of digital information. An essential ingredient in this digital revolution is the use of Fourier analysis, a mathematical idea that is introduced in this chapter.

In this digital age, we have learned to represent information in a variety of ways. The ability to easily transform one representation into another gives us tremendous problem-solving powers. As an example, consider the music made by a saxophone. The music is initially represented as a series of notes on sheet music, but the musician brings her own special interpretation to the music. Such an individual performance can then be recorded, to be copied and replayed later. While this is easily accomplished with conventional analog technology, the advent of digital technology has allowed us to record the performance with a previously unknown fidelity. The key to this is that the music is broken down into its component parts, which are individually recorded and then reassembled on demand to recreate the original sound. Think for a moment how spectacular this feat really is. The complex rhythms and intonations



generated by the saxophone reed and body are somehow converted into a relatively small number of digital bits (zeros and ones). The bits are then turned back into music by a CD player.

The basic idea behind any digital technology is to break down a complex whole into a set of component pieces. To digitally capture a saxophone note, all of the significant features of the saxophone waveform must be captured. Done properly, the components can then be recombined to reproduce each original note.

In this chapter, we learn how series of numbers combine and how functions can be broken down into a series of component functions. As part of this discussion, we will explore how music synthesizers work, but we will also see how calculators can quickly approximate a quantity like sin 1.234567 and how equations can be solved using functions for which we don't even have names. This chapter opens up a new world of important applications.

8.1 SEQUENCES OF REAL NUMBERS

The mathematical notion of sequence is not much different from the common English usage of the word. For instance, to describe the sequence of events that led up to a traffic accident, you'd not only need to list the events, but you'd need to do so in the correct *order*. In mathematics, the term *sequence* refers to an infinite collection of real numbers, written in a specific order.

We have already seen sequences several times now. For instance, to find approximate solutions to nonlinear equations such as $\tan x - x = 0$, we began by first making an initial guess, x_0 and then using Newton's method to compute a sequence of successively improved approximations, $x_1, x_2, \ldots, x_n, \ldots$.

DEFINITION OF SEQUENCE

A **sequence** is any function whose domain is the set of integers starting with some integer n_0 (often 0 or 1). For instance, the function $a(n) = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$, defines the sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Here, $\frac{1}{1}$ is called the first **term,** $\frac{1}{2}$ is the second term and so on. We call $a(n) = \frac{1}{n}$ the **general term,** since it gives a (general) formula for computing all the terms of the sequence. Further, we use subscript notation instead of function notation and write a_n instead of $a(n)$.

EXAMPLE 1.1 The Terms of a Sequence

Write out the first four terms of the sequence whose general term is given by $a_n = \dfrac{n+1}{n}$, for $n = 1, 2, 3, \ldots$.

Solution We have the sequence

$$a_1 = \frac{1+1}{1} = \frac{2}{1}, \quad a_2 = \frac{2+1}{2} = \frac{3}{2}, \quad a_3 = \frac{4}{3}, \quad a_4 = \frac{5}{4}, \ldots .$$

We often use set notation to denote a sequence. For instance, the sequence with general term $a_n = \frac{1}{n^2}$, for $n = 1, 2, 3, \ldots$, is denoted by

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty},$$

or equivalently, by listing the terms of the sequence:

$$\left\{\frac{1}{1}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots\right\}.$$

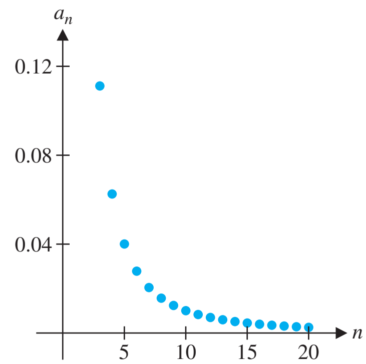


FIGURE 8.1
$a_n = \frac{1}{n^2}$

To graph this sequence, we plot a number of discrete points, since a sequence is a function defined only on the integers (see Figure 8.1). You have likely already noticed something about the sequence $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$. As n gets larger and larger, the terms of the sequence, $a_n = \frac{1}{n^2}$, get closer and closer to zero. In this case, we say that the sequence **converges** to 0 and write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

In general, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L (i.e., $\lim_{n\to\infty} a_n = L$) if we can make a_n as close to L as desired, simply by making n sufficiently large. Notice that this language parallels that used in the definition of the limit $\lim_{x\to\infty} f(x) = L$ for a function of a real variable x (given in section 1.6). The only difference is that n can take on only integer values, while x can take on any real value (integer, rational or irrational).

When we say that we can make a_n as close to L as desired (i.e., arbitrarily close), just how close must we be able to make a_n to L? Well, if you pick any (small) real number, $\varepsilon > 0$, you must be able to make a_n within a distance ε of L, simply by making n sufficiently large. That is, we need $|a_n - L| < \varepsilon$.

We summarize this in Definition 1.1.

DEFINITION 1.1

The sequence $\{a_n\}_{n=n_0}^{\infty}$ **converges** to L if and only if given any number $\varepsilon > 0$, there is an integer N for which

$$|a_n - L| < \varepsilon, \quad \text{for every } n > N.$$

If there is no such number L, then we say that the sequence **diverges.**

We illustrate Definition 1.1 in Figure 8.2. Notice that the definition says that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if given any number $\varepsilon > 0$, we can find an integer N so that the terms of the sequence stay between $L - \varepsilon$ and $L + \varepsilon$ for all values of $n > N$.

In example 1.2, we show how to use Definition 1.1 to prove that a sequence converges.

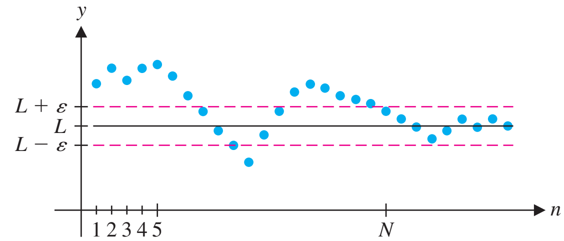


FIGURE 8.2
Convergence of a sequence

EXAMPLE 1.2 Proving That a Sequence Converges

Use Definition 1.1 to show that the sequence $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ converges to 0.

Solution Here, we must show that we can make $\frac{1}{n^2}$ as close to 0 as desired, just by making n sufficiently large. So, given any $\varepsilon > 0$, we must find N sufficiently large so that for every $n > N$,

$$\left|\frac{1}{n^2} - 0\right| < \varepsilon \quad \text{or} \quad \frac{1}{n^2} < \varepsilon. \tag{1.1}$$

Since n^2 and ε are positive, we can divide both sides of (1.1) by ε and multiply by n^2, to obtain

$$\frac{1}{\varepsilon} < n^2.$$

Taking square roots gives us

$$\sqrt{\frac{1}{\varepsilon}} < n.$$

Working backwards now, observe that if we choose N to be an integer with $N \geq \sqrt{\frac{1}{\varepsilon}}$, then $n > N$ implies that $\frac{1}{n^2} < \varepsilon$, as desired. ■

Most of the usual rules for computing limits of functions of a real variable also apply to computing the limit of a sequence, as we see in Theorem 1.1.

THEOREM 1.1

Suppose that $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ both converge. Then

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$,

(ii) $\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$,

(iii) $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$ and

(iv) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ (assuming $\lim\limits_{n\to\infty} b_n \neq 0$).

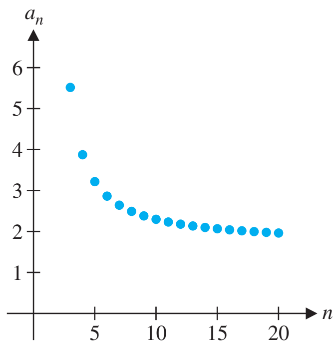


FIGURE 8.3
$a_n = \frac{5n+7}{3n-5}$

The proof of Theorem 1.1 is virtually identical to the proof of the corresponding theorem about limits of a function of a real variable (see Theorem 3.1 in section 1.3 and Appendix A) and is omitted.

To find the limit of a sequence, you should work largely the same as when computing the limit of a function of a real variable, but keep in mind that sequences are defined *only* for integer values of the variable.

EXAMPLE 1.3 Finding the Limit of a Sequence

Evaluate $\lim\limits_{n\to\infty} \frac{5n+7}{3n-5}$.

Solution This has the indeterminate form $\frac{\infty}{\infty}$. The graph in Figure 8.3 suggests that the sequence tends to some limit around 2. Note that we cannot apply l'Hôpital's Rule here, since the functions in the numerator and the denominator are defined only for integer values of n and hence, are not differentiable. Instead, simply divide numerator and denominator by the highest power of n in the denominator. We have

$$\lim_{n\to\infty} \frac{5n+7}{3n-5} = \lim_{n\to\infty} \frac{(5n+7)\left(\frac{1}{n}\right)}{(3n-5)\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{5+\frac{7}{n}}{3-\frac{5}{n}} = \frac{5}{3}.$$ ■

REMARK 1.1

If you (incorrectly) apply l'Hôpital's Rule in example 1.3, you get the right answer. (Go ahead and try it; nobody's looking.) Unfortunately, you will not always be so lucky. It's a lot like trying to cross a busy highway: while there are times when you can successfully cross with your eyes closed, it's not generally recommended. Theorem 1.2 describes how you can safely use l'Hôpital's Rule.

In example 1.4, we see a sequence that diverges by virtue of its terms tending to $+\infty$.

EXAMPLE 1.4 A Divergent Sequence

Evaluate $\lim\limits_{n\to\infty} \frac{n^2+1}{2n-3}$.

Solution Again, this has the indeterminate form $\frac{\infty}{\infty}$, but from the graph in Figure 8.4, the sequence appears to be increasing without bound. Dividing top and bottom by n (the highest power of n in the denominator), we have

$$\lim_{n\to\infty} \frac{n^2+1}{2n-3} = \lim_{n\to\infty} \frac{(n^2+1)\left(\frac{1}{n}\right)}{(2n-3)\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n+\frac{1}{n}}{2-\frac{3}{n}} = \infty$$

and so, the sequence $\left\{\frac{n^2+1}{2n-3}\right\}_{n=1}^{\infty}$ diverges. ■

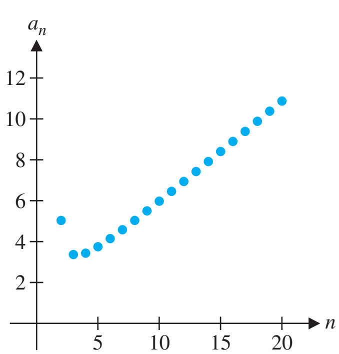


FIGURE 8.4
$a_n = \frac{n^2+1}{2n-3}$

In example 1.5, we see that a sequence doesn't need to tend to $\pm\infty$ in order to diverge.

EXAMPLE 1.5 A Divergent Sequence Whose Terms Do Not Tend to ∞

Determine the convergence or divergence of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

Solution If we write out the terms of the sequence, we have

$$\{-1, 1, -1, 1, -1, 1, \ldots\}.$$

That is, the terms of the sequence alternate back and forth between -1 and 1 and so, the sequence diverges. To see this graphically, we plot the first few terms of the sequence in Figure 8.5 (on the following page). Notice that the points do not approach any limit (a horizontal line). ■

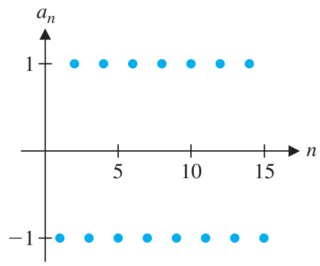


FIGURE 8.5
$a_n = (-1)^n$

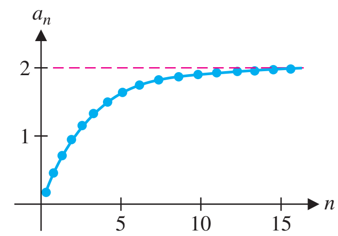


FIGURE 8.6
$a_n = f(n)$, where $f(x) \to 2$, as $x \to \infty$

You can use an advanced tool like l'Hôpital's Rule to find the limit of a sequence, but you must be careful. Theorem 1.2 says that if $f(x) \to L$ as $x \to \infty$ through all real values, then $f(n)$ must approach L, too, as $n \to \infty$ through integer values. (See Figure 8.6 for a graphical representation of this.)

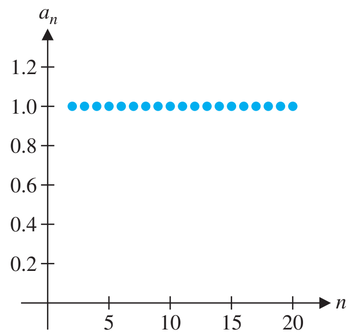


FIGURE 8.7a
$a_n = \cos(2\pi n)$

THEOREM 1.2

Suppose that $\lim\limits_{x\to\infty} f(x) = L$. Then, $\lim\limits_{n\to\infty} f(n) = L$, also.

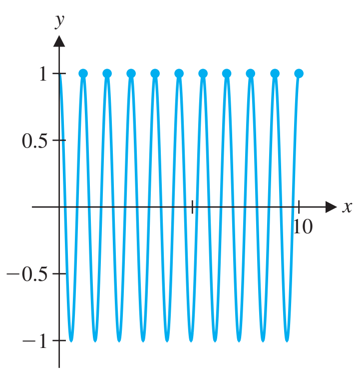


FIGURE 8.7b
$y = \cos(2\pi x)$

REMARK 1.2

The converse of Theorem 1.2 is false. That is, if $\lim\limits_{n\to\infty} f(n) = L$, it need *not* be true that $\lim\limits_{x\to\infty} f(x) = L$. This is clear from the following observation. Note that

$$\lim_{n\to\infty} \cos(2\pi n) = 1,$$

since $\cos(2\pi n) = 1$ for every *integer* n (see Figure 8.7a). However,

$$\lim_{x\to\infty} \cos(2\pi x) \text{ does not exist,}$$

since as $x \to \infty$, $\cos(2\pi x)$ oscillates between -1 and 1 (see Figure 8.7b).

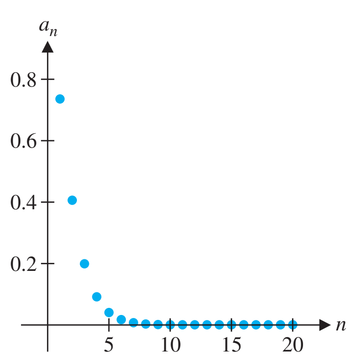


FIGURE 8.8
$a_n = \dfrac{n+1}{e^n}$

EXAMPLE 1.6 Applying l'Hôpital's Rule to a Related Function

Evaluate $\lim\limits_{n\to\infty} \dfrac{n+1}{e^n}$.

Solution This has the indeterminate form $\frac{\infty}{\infty}$, but the graph in Figure 8.8 suggests that the sequence converges to 0. However, there is no obvious way to resolve this, except by l'Hôpital's Rule (which does *not* apply to limits of sequences). So, we instead consider the limit of the corresponding function of a real variable to which we may apply l'Hôpital's Rule. (Be sure you check the hypotheses.) We have

$$\lim_{x\to\infty} \frac{x+1}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

From Theorem 1.2, we now have

$$\lim_{n\to\infty} \frac{n+1}{e^n} = 0, \text{ also.}$$

Although we now have a few tools for computing the limit of a sequence, for many sequences (including infinite series, which we study in the remainder of this chapter), we don't even have an explicit formula for the general term. In such cases, we must test for convergence in some indirect way. Our first indirect tool corresponds to the result (of the same name) for limits of functions of a real variable presented in section 1.3.

THEOREM 1.3 (Squeeze Theorem)

Suppose $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ are convergent sequences, both converging to the limit L. If there is an integer $n_1 \geq n_0$ such that for all $n \geq n_1$, $a_n \leq c_n \leq b_n$, then $\{c_n\}_{n=n_0}^{\infty}$ converges to L, too.

In example 1.7, we demonstrate how to apply the Squeeze Theorem to a sequence. Observe that the trick here is to find two sequences, one on each side of the given sequence (i.e., one larger and one smaller) that have the same limit.

EXAMPLE 1.7 Applying the Squeeze Theorem to a Sequence

Determine the convergence or divergence of $\left\{\dfrac{\sin n}{n^2}\right\}_{n=1}^{\infty}$.

Solution From the graph in Figure 8.9, the sequence appears to converge to 0, despite the oscillation. Further, note that you cannot compute this limit using the rules we have established so far. (Try it!) However, since

$$-1 \leq \sin n \leq 1, \text{ for all } n,$$

dividing through by n^2 gives us

$$\frac{-1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}, \text{ for all } n \geq 1.$$

Finally, since

$$\lim_{n\to\infty} \frac{-1}{n^2} = 0 = \lim_{n\to\infty} \frac{1}{n^2},$$

the Squeeze Theorem gives us that

$$\lim_{n\to\infty} \frac{\sin n}{n^2} = 0,$$

also.

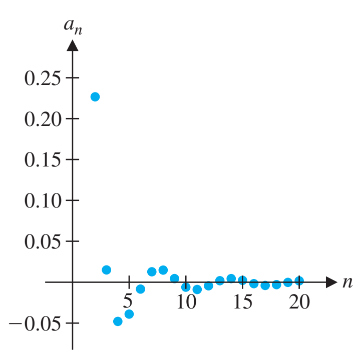


FIGURE 8.9

$a_n = \dfrac{\sin n}{n^2}$

The following useful result follows immediately from Theorem 1.3.

COROLLARY 1.1

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$, also.

PROOF

Notice that for all n, $$-|a_n| \le a_n \le |a_n|.$$

Further, $$\lim_{n\to\infty} |a_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} (-|a_n|) = -\lim_{n\to\infty} |a_n| = 0.$$

So, from the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$, too. ■

Corollary 1.1 is particularly useful for sequences with both positive and negative terms, as in example 1.8.

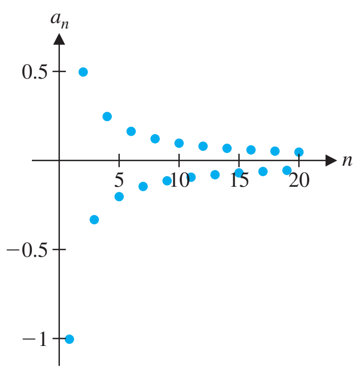


FIGURE 8.10
$a_n = \dfrac{(-1)^n}{n}$

EXAMPLE 1.8 A Sequence with Terms of Alternating Signs

Determine the convergence or divergence of $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

Solution The graph of the sequence in Figure 8.10 suggests that although the sequence oscillates, it still may be converging to 0. Since $(-1)^n$ oscillates back and forth between -1 and 1, we cannot compute $\lim_{n\to\infty} \dfrac{(-1)^n}{n}$ directly. However, notice that

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

and $$\lim_{n\to\infty} \frac{1}{n} = 0.$$

From Corollary 1.1, we get that $\lim_{n\to\infty} \dfrac{(-1)^n}{n} = 0$, too. ■

We remind you of the following definition, which we use throughout the chapter.

DEFINITION 1.2

For any integer $n \ge 1$, the **factorial,** $n!$ is defined as the product of the first n positive integers,

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n.$$

We define $0! = 1$.

Example 1.9 shows a sequence whose limit would be extremely difficult to find without the Squeeze Theorem.

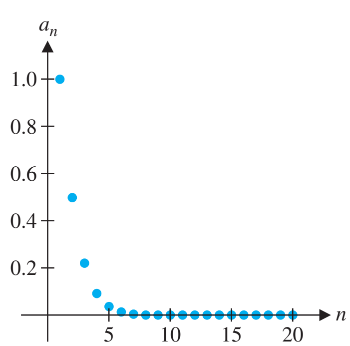


FIGURE 8.11
$a_n = \dfrac{n!}{n^n}$

EXAMPLE 1.9 An Indirect Proof of Convergence

Investigate the convergence of $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$.

Solution First, notice that we have no means of computing $\lim_{n\to\infty} \dfrac{n!}{n^n}$ directly. (Try this!) From the graph of the sequence in Figure 8.11, it appears that the sequence is converging to 0. Notice that the general term of the sequence satisfies

$$0 < \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{\underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{n \text{ factors}}}$$

$$= \left(\frac{1}{n}\right) \frac{2 \cdot 3 \cdot \cdots \cdot n}{\underbrace{n \cdot n \cdot \cdots \cdot n}_{n-1 \text{ factors}}} \le \left(\frac{1}{n}\right)(1) = \frac{1}{n}. \tag{1.2}$$

From the Squeeze Theorem and (1.2), we have that since

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} 0 = 0,$$

then $$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \text{ also.}$$

Just as we did with functions of a real variable, we need to distinguish between sequences that are increasing and decreasing. The definitions are straightforward.

DEFINITION 1.3

(i) The sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if

$$a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots.$$

(ii) The sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if

$$a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge \cdots.$$

If a sequence is either increasing or decreasing, it is called **monotonic.**

There are several ways to show that a sequence is monotonic. Regardless of which method you use, you will need to show that either $a_n \le a_{n+1}$ for all n (increasing) or $a_{n+1} \le a_n$ for all n (decreasing). We illustrate two very useful methods in examples 1.10 and 1.11.

EXAMPLE 1.10 An Increasing Sequence

Investigate whether the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ is increasing, decreasing or neither.

Solution From the graph in Figure 8.12, it appears that the sequence is increasing. However, you should not be deceived by looking at the first few terms of a sequence. More generally, we look at the ratio of two successive terms. Defining $a_n = \dfrac{n}{n+1}$, we have $a_{n+1} = \dfrac{n+1}{n+2}$ and so,

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{n+1}{n+2}\right)}{\left(\frac{n}{n+1}\right)} = \left(\frac{n+1}{n+2}\right)\left(\frac{n+1}{n}\right)$$

$$= \frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1. \tag{1.3}$$

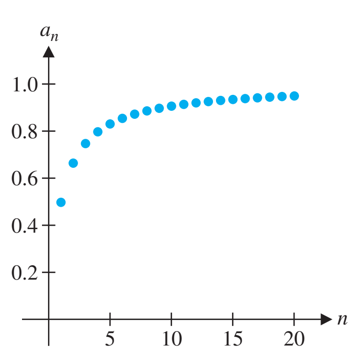


FIGURE 8.12
$a_n = \dfrac{n}{n+1}$

Multiplying both sides of (1.3) by $a_n > 0$, we obtain

$$a_{n+1} > a_n,$$

for all n and so, the sequence is increasing. Alternatively, consider the function $f(x) = \dfrac{x}{x+1}$ (of the real variable x) corresponding to the sequence. Observe that

$$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0,$$

which says that the function $f(x)$ is increasing. From this, it follows that the corresponding sequence $a_n = \dfrac{n}{n+1}$ is also increasing. ■

EXAMPLE 1.11 A Sequence That Is Increasing for $n \geq 2$

Investigate whether the sequence $\left\{\dfrac{n!}{e^n}\right\}_{n=1}^{\infty}$ is increasing, decreasing or neither.

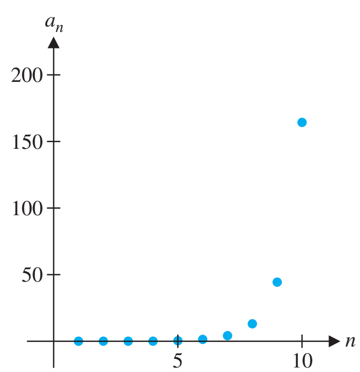


FIGURE 8.13

$a_n = \dfrac{n!}{e^n}$

Solution From the graph of the sequence in Figure 8.13, it appears that the sequence is increasing (and rather rapidly, at that). Here, for $a_n = \dfrac{n!}{e^n}$, we have $a_{n+1} = \dfrac{(n+1)!}{e^{n+1}}$, so that

$$\frac{a_{n+1}}{a_n} = \frac{\left[\dfrac{(n+1)!}{e^{n+1}}\right]}{\left(\dfrac{n!}{e^n}\right)} = \frac{(n+1)!}{e^{n+1}}\,\frac{e^n}{n!}$$

$$= \frac{(n+1)n!e^n}{e(e^n)n!} = \frac{n+1}{e} > 1, \text{ for } n \geq 2. \tag{1.4}$$

Since $(n+1)! = (n+1)\cdot n!$ and $e^{n+1} = e \cdot e^n$.

Multiplying both sides of (1.4) by $a_n > 0$, we get

$$a_{n+1} > a_n, \text{ for } n \geq 2.$$

Notice that in this case, although the sequence is not increasing for all n, it is increasing for $n \geq 2$. Keep in mind that it doesn't really matter what the first few terms do, anyway. We are only concerned with the behavior of a sequence as $n \to \infty$. ■

We need to define one additional property of sequences.

DEFINITION 1.4

We say that the sequence $\{a_n\}_{n=n_0}^{\infty}$ is **bounded** if there is a number $M > 0$ (called a **bound**) for which $|a_n| \leq M$, for all n.

It is important to realize that a given sequence may have any number of bounds (for instance, if $|a_n| \leq 10$ for all n, then $|a_n| \leq 20$, for all n, too).

EXAMPLE 1.12 A Bounded Sequence

Show that the sequence $\left\{\dfrac{3-4n^2}{n^2+1}\right\}_{n=1}^{\infty}$ is bounded.

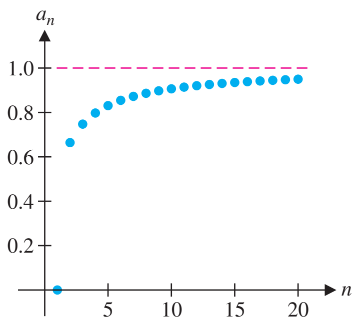


FIGURE 8.14a
A bounded and increasing sequence

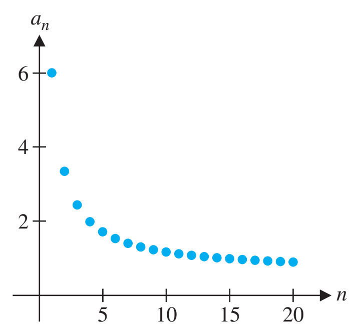


FIGURE 8.14b
A bounded and decreasing sequence

Solution We use the fact that $4n^2 - 3 > 0$, for all $n \geq 1$, to get

$$|a_n| = \left|\frac{3 - 4n^2}{n^2 + 1}\right| = \frac{4n^2 - 3}{n^2 + 1} < \frac{4n^2}{n^2 + 1} < \frac{4n^2}{n^2} = 4.$$

So, this sequence is bounded by 4. (We might also say in this case that the sequence is bounded between -4 and 4.) Further, note that we could also use any number greater than 4 as a bound. ■

Theorem 1.4 provides a powerful tool for the investigation of sequences.

THEOREM 1.4

Every bounded, monotonic sequence converges.

A typical bounded and increasing sequence is illustrated in Figure 8.14a, while a bounded and decreasing sequence is illustrated in Figure 8.14b. In both figures, notice that a bounded and monotonic sequence has nowhere to go and consequently, must converge. The proof of Theorem 1.4 is rather involved and we leave it to the end of the section.

Theorem 1.4 says that if we can show that a sequence is bounded and monotonic, then it must also be convergent, although we may have little idea of what its limit might be. Once we establish that a sequence converges, we can approximate its limit by computing a sufficient number of terms, as in example 1.13.

EXAMPLE 1.13 An Indirect Proof of Convergence

Investigate the convergence of the sequence $\left\{\dfrac{2^n}{n!}\right\}_{n=1}^{\infty}$.

Solution First, note that we do not know how to compute $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$. This has the indeterminate form $\frac{\infty}{\infty}$, but we cannot use l'Hôpital's Rule here directly or indirectly. (Why not?) The graph in Figure 8.15 suggests that the sequence converges to 0. To confirm this suspicion, we first show that the sequence is monotonic. We have

$$\frac{a_{n+1}}{a_n} = \frac{\left[\dfrac{2^{n+1}}{(n+1)!}\right]}{\left(\dfrac{2^n}{n!}\right)} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n}$$

$$= \frac{2(2^n)n!}{(n+1)n!2^n} = \frac{2}{n+1} \leq 1, \text{ for } n \geq 1. \qquad (1.5)$$

Since $2^{n+1} = 2 \cdot 2^n$ and $(n+1)! = (n+1) \cdot n!$.

Multiplying both sides of (1.5) by $a_n > 0$ gives us $a_{n+1} \leq a_n$ for all n and so, the sequence is decreasing. Next, since the sequence is decreasing, we have that

$$|a_n| = \frac{2^n}{n!} \leq \frac{2^1}{1!} = 2,$$

for $n \geq 1$ (i.e., the sequence is bounded by 2). Since the sequence is both bounded and monotonic, it must be convergent, by Theorem 1.4. We display a number of terms of the sequence in the accompanying table, from which it appears that the sequence is converging to approximately 0. We can make a slightly stronger statement, though.

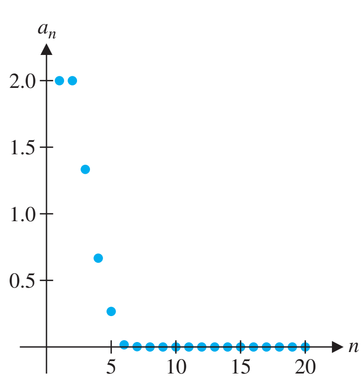


FIGURE 8.15
$a_n = \dfrac{2^n}{n!}$

n	$a_n = \frac{2^n}{n!}$
2	2
4	0.666667
6	0.088889
8	0.006349
10	0.000282
12	0.0000086
14	1.88×10^{-7}
16	3.13×10^{-9}
18	4.09×10^{-11}
20	4.31×10^{-13}

Since we have established that the sequence is *decreasing* and convergent, we have from our computations that

$$0 \le a_n \le a_{20} \approx 4.31 \times 10^{-13}, \quad \text{for } n \ge 20.$$

Further, the limit L must also satisfy the inequality

$$0 \le L \le 4.31 \times 10^{-13}.$$

We can confirm that the limit is 0, as follows. From (1.5),

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{2}{n+1}\right) a_n,$$

so that

$$L = \left(\lim_{n\to\infty} \frac{2}{n+1}\right)\left(\lim_{n\to\infty} a_n\right) = 0 \cdot L = 0.$$

REMARK 1.3

Do not understimate the importance of Theorem 1.4. This indirect way of testing a sequence for convergence takes on additional significance when we study infinite series (a special type of sequence that is the topic of the remainder of this chapter).

Proof of Theorem 1.4

Before we can prove Theorem 1.4, we need to state one of the properties of the real number system.

THE COMPLETENESS AXIOM

If a nonempty set S of real numbers has a lower bound, then it has a *greatest lower bound*. Equivalently, if it has an upper bound, it has a *least upper bound*.

This axiom says that if a nonempty set S has an upper bound, that is, a number M such that

$$x \le M, \quad \text{for all } x \in S,$$

then there is an upper bound L, for which

$$L \le M \text{ for } all \text{ upper bounds, } M,$$

with a corresponding statement holding for lower bounds.

The Completeness Axiom enables us to prove Theorem 1.4.

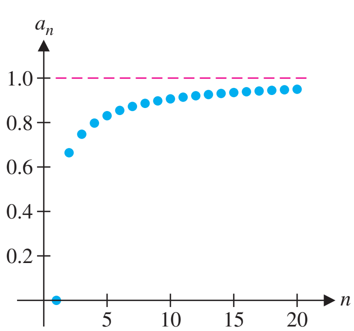


FIGURE 8.16
Bounded and increasing

PROOF

(Increasing sequence) Suppose that $\{a_n\}_{n=n_0}^{\infty}$ is increasing and bounded. This is illustrated in Figure 8.16, where you can see an increasing sequence bounded by 1. We have

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le a_{n+1} \le \cdots$$

and for some number $M > 0$, $|a_n| \le M$ for all n. This is the same as saying that

$$-M \le a_n \le M, \quad \text{for all } n.$$

Now, let S be the set containing all of the terms of the sequence, $S = \{a_1, a_2, \ldots, a_n, \ldots\}$. Notice that M is an upper bound for the set S. From the Completeness Axiom, S must have a least upper bound, L. That is, L is the *smallest* number for which

$$a_n \le L, \quad \text{for all } n. \tag{1.6}$$

Notice that for any number $\varepsilon > 0$, $L - \varepsilon < L$ and so, $L - \varepsilon$ is *not* an upper bound, since L is the *least* upper bound. Since $L - \varepsilon$ is not an upper bound for S, there is some element, a_N, of S for which

$$L - \varepsilon < a_N.$$

Since $\{a_n\}$ is increasing, we have that for $n \ge N$, $a_N \le a_n$. Finally, from (1.6) and the fact that L is an upper bound for S and since $\varepsilon > 0$, we have

$$L - \varepsilon < a_N \le a_n \le L < L + \varepsilon,$$

or more simply

$$L - \varepsilon < a_n < L + \varepsilon,$$

for $n \ge N$. This is equivalent to

$$|a_n - L| < \varepsilon, \quad \text{for } n \ge N.$$

This says that $\{a_n\}$ converges to L. The proof for the case of a decreasing sequence is similar and is left as an exercise. ■

BEYOND FORMULAS

The essential logic behind sequences is the same as that behind much of the calculus. When evaluating limits (including limits of sequences and those that define derivatives and integrals), we are frequently able to compute an exact answer directly, as in example 1.3. However, some limits are more difficult to determine and can be found only by using an indirect method, as in example 1.13. Such indirect methods prove to be extremely important (and increasingly common) as we expand our study of sequences to those defining infinite series in the rest of this chapter.

EXERCISES 8.1

WRITING EXERCISES

1. Compare and contrast $\lim_{x\to\infty} \sin \pi x$ and $\lim_{n\to\infty} \sin \pi n$. Indicate the domains of the two functions and how they affect the limits.

2. Explain why Theorem 1.2 should be true, taking into account the respective domains and their effect on the limits.

3. In words, explain why a decreasing bounded sequence must converge.

4. A sequence is said to diverge if it does not converge. The word "diverge" is well chosen for sequences that diverge to ∞, but is less descriptive of sequences such as $\{1, 2, 1, 2, 1, 2, \ldots\}$ and $\{1, 2, 3, 1, 2, 3, \ldots\}$. Briefly describe the limiting behavior of these sequences and discuss other possible limiting behaviors of divergent sequences.

In exercises 1–4, write out the terms $a_1, a_2, \ldots, a_6$ of the given sequence.

1. $a_n = \dfrac{2n-1}{n^2}$
2. $a_n = \dfrac{3}{n+4}$
3. $a_n = \dfrac{4}{n!}$
4. $a_n = (-1)^n \dfrac{n}{n+1}$

In exercises 5–10, (a) find the limit of each sequence, (b) use the definition to show that the sequence converges and (c) plot the sequence on a calculator or CAS.

5. $a_n = \dfrac{1}{n^3}$
6. $a_n = \dfrac{2}{n^2}$
7. $a_n = \dfrac{n}{n+1}$
8. $a_n = \dfrac{2n+1}{n}$
9. $a_n = \dfrac{2}{\sqrt{n}}$
10. $a_n = \dfrac{4}{\sqrt{n+1}}$

In exercises 11–28, determine whether the sequence converges or diverges.

11. $a_n = \dfrac{3n^2+1}{2n^2-1}$

12. $a_n = \dfrac{5n^3-1}{2n^3+1}$

13. $a_n = \dfrac{n^2+1}{n+1}$

14. $a_n = \dfrac{n^2+1}{n^3+1}$

15. $a_n = (-1)^n \dfrac{n+2}{3n-1}$

16. $a_n = (-1)^n \dfrac{n+4}{n+1}$

17. $a_n = (-1)^n \dfrac{n+2}{n^2+4}$

18. $a_n = \cos \pi n$

19. $a_n = ne^{-n}$

20. $a_n = \dfrac{\cos n}{e^n}$

21. $a_n = \dfrac{e^n+2}{e^{2n}-1}$

22. $a_n = \dfrac{3^n}{e^n+1}$

23. $a_n = \dfrac{n2^n}{3^n}$

24. $a_n = \dfrac{\cos n}{n!}$

25. $a_n = \dfrac{n!}{2^n}$

26. $a_n = \sqrt{n^2+n} - n$

27. $a_n = \ln(2n+1) - \ln(n)$

28. $a_n = \left|\cos \dfrac{n\pi}{2} + \sin \dfrac{n\pi}{2}\right| \dfrac{2n-1}{n+2}$

In exercises 29–32, use the Squeeze Theorem and Corollary 1.1 to prove that the sequence converges to 0 (given that $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = 0$).

29. $a_n = \dfrac{\cos n}{n^2}$

30. $a_n = \dfrac{\cos n\pi}{n^2}$

31. $a_n = (-1)^n \dfrac{e^{-n}}{n}$

32. $a_n = (-1)^n \dfrac{\ln n}{n^2}$

In exercises 33–38, determine whether the sequence is increasing, decreasing or neither.

33. $a_n = \dfrac{n+3}{n+2}$

34. $a_n = \dfrac{n-1}{n+1}$

35. $a_n = \dfrac{e^n}{n}$

36. $a_n = \dfrac{n!}{5^n}$

37. $a_n = \dfrac{2^n}{(n+1)!}$

38. $a_n = \dfrac{3^n}{(n+2)!}$

In exercises 39–42, show that the sequence is bounded.

39. $a_n = \dfrac{3n^2-2}{n^2+1}$

40. $a_n = \dfrac{6n-1}{n+3}$

41. $a_n = \dfrac{\sin(n^2)}{n+1}$

42. $a_n = e^{1/n}$

43. Numerically estimate the limits of the sequences $a_n = \left(1+\frac{2}{n}\right)^n$ and $b_n = \left(1-\frac{2}{n}\right)^n$. Compare the answers to e^2 and e^{-2}.

44. Given that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, show that $\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$ for any constant r. (Hint: Make the substitution $n = rm$.)

45. A packing company works with 12″ square boxes. Show that for $n = 1, 2, 3, \ldots$, a total of n^2 disks of diameter $\frac{12''}{n}$ fit into a box. Let a_n be the wasted area in a box with n^2 disks. Compute a_n.

46. The pattern of a sequence can't always be determined from the first few terms. Start with a circle, pick two points on the circle and connect them with a line segment. The circle is divided into $a_1 = 2$ regions. Add a third point, connect all points and show that there are now $a_2 = 4$ regions. Add a fourth point, connect all points and show that there are $a_3 = 8$ regions. Is the pattern clear? Show that $a_4 = 16$ and then compute a_5 for a surprise!

47. You have heard about the "population explosion." The following dramatic warning is adapted from the article "Doomsday: Friday 13 November 2026" by Foerster, Mora and Amiot in *Science* (Nov. 1960). Start with $a_0 = 3.049$ to indicate that the world population in 1960 was approximately 3.049 billion. Then compute $a_1 = a_0 + 0.005a_0^{2.01}$ to estimate the population in 1961. Compute $a_2 = a_1 + 0.005a_1^{2.01}$ to estimate the population in 1962, then $a_3 = a_2 + 0.005a_2^{2.01}$ for 1963 and so on. Continue iterating and compare your calculations to the actual populations in 1970 (3.721 billion), 1980 (4.473 billion) and 1990 (5.333 billion). Then project ahead to the year 2035. Frightening, isn't it?

48. The so-called **hailstone sequence** is defined by

$$x_k = \begin{cases} 3x_{k-1}+1 & \text{if } x_{k-1} \text{ is odd} \\ \frac{1}{2}x_{k-1} & \text{if } x_{k-1} \text{ is even} \end{cases}.$$

If you start with $x_1 = 2^n$ for some positive integer n, show that $x_{n+1} = 1$. The question (an unsolved research problem) is whether you eventually reach 1 from *any* starting value. Try several odd values for x_1 and show that you always reach 1.

49. A different population model was studied by Fibonacci, an Italian mathematician of the thirteenth century. He imagined a population of rabbits starting with a pair of newborns. For one month, they grow and mature. The second month, they have a pair of newborn baby rabbits. We count the number of pairs of rabbits. Thus far, $a_0 = 1$, $a_1 = 1$ and $a_2 = 2$. The rules are: adult rabbit pairs give birth to a pair of newborns every month, newborns take one month to mature and no rabbits die. Show that $a_3 = 3$, $a_4 = 5$ and in general $a_n = a_{n-1} + a_{n-2}$. This sequence of numbers, known as the **Fibonacci sequence,** occurs in an amazing number of applications.

50. In this exercise, we visualize the Fibonacci sequence (see exercise 49). Start with two squares of side 1 placed next to each other (see Figure A). Place a square on the long side of the resulting rectangle (see Figure B); this square has side 2. Continue placing squares on the long sides of the rectangles: a square of side 3 is added in Figure C, then a square of side 5 is added to the bottom of Figure C, and so on.

FIGURE A

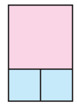


FIGURE B

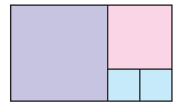


FIGURE C

Argue that the sides of the squares are determined by the Fibonacci sequence of exercise 49.

51. Suppose that $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right)$. Show numerically that the sequence converges to 2. To find this limit analytically, let $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ and solve the equation $L = \frac{1}{2}\left(L + \frac{4}{L}\right)$.

52. As in exercise 51, determine the limit of the sequence defined by $a_1 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $c > 0$ and $a_n > 0$.

53. Define the sequence a_n with $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + \sqrt{a_{n-1}}}$ for $n \ge 2$. Show that $\{a_n\}$ is increasing and bounded by 2. Evaluate the limit of the sequence by estimating the appropriate solution of $x = \sqrt{2 + \sqrt{x}}$.

54. Define the sequence a_n with $a_1 = \sqrt{3}$ and $a_n = \sqrt{3 + 2a_{n-1}}$ for $n \ge 2$. Show that $\{a_n\}$ converges and estimate the limit of the sequence.

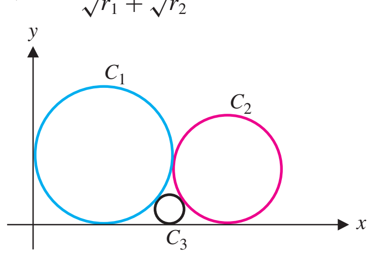
55. Find all values of p such that the sequence $a_n = \frac{1}{p^n}$ converges.

56. Find all values of p such that the sequence $a_n = \frac{1}{n^p}$ converges.

57. Define $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$. Evaluate the sum using a formula from section 4.2 and show that the sequence converges. By thinking of a_n as a Riemann sum, identify the definite integral to which the sequence converges.

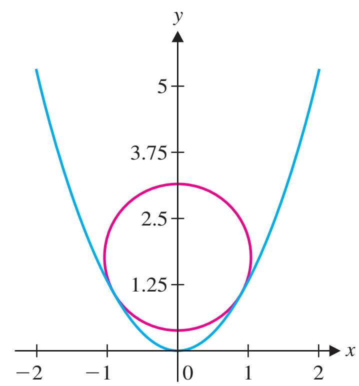
58. Define $a_n = \sum_{k=1}^{n} \dfrac{1}{n+k}$. By thinking of a_n as a Riemann sum, identify the definite integral to which the sequence converges.

59. Start with two circles C_1 and C_2 of radius r_1 and r_2, respectively, that are tangent to each other and each tangent to the x-axis. Construct the circle C_3 that is tangent to C_1, C_2 and the x-axis. (See the accompanying figure.) If the centers of C_1 and C_2 are (c_1, r_1) and (c_2, r_2), respectively, show that $(c_2 - c_1)^2 + (r_2 - r_1)^2 = (r_1 + r_2)^2$ and then $|c_2 - c_1| = 2\sqrt{r_1 r_2}$. Find similar relationships for circles C_1 and C_3 and for circles C_2 and C_3. Show that the radius r_3 of C_3 is given by $\sqrt{r_3} = \dfrac{\sqrt{r_1 r_2}}{\sqrt{r_1} + \sqrt{r_2}}$.



60. In exercise 59, construct a sequence of circles where C_4 is tangent to C_2, C_3 and the x-axis; then C_5 is tangent to C_3, C_4 and the x-axis. If you start with unit circles $r_1 = r_2 = 1$, find a formula for the radius r_n in terms of F_n, the nth term in the Fibonacci sequence of exercises 49 and 50.

61. Let C be the circle of radius r inscribed in the parabola $y = x^2$. (See the figure.) Show that the y-coordinate c of the center of the circle equals $c = \frac{1}{4} + r^2$.



62. In exercise 61, let C_1 be the circle of radius $r_1 = 1$ inscribed in $y = x^2$. Construct a sequence of circles C_2, C_3 and so on, where each circle C_n rests on top of the previous circle C_{n-1} (that is, C_n is tangent to C_{n-1}) and is inscribed in the parabola. If r_n is the radius of circle C_n, find a (simple) formula for r_n.

EXPLORATORY EXERCISES

1. Suppose that a ball is launched from the ground with initial velocity v. Ignoring air resistance, it will rise to a height of $v^2/(2g)$ and fall back to the ground at time $t = 2v/g$. Depending on how "lively" the ball is, the next bounce will rise to only a fraction of the previous height. The **coefficient of restitution** r, defined as the ratio of landing velocity to rebound velocity, measures the liveliness of the ball. The second bounce has launch velocity rv, the third bounce has launch velocity $r^2 v$ and so on. It might seem that the ball will bounce forever. To see that it does not, argue that the time to complete two bounces is $a_2 = \frac{2v}{g}(1 + r)$, the time to complete three bounces is $a_3 = \frac{2v}{g}(1 + r + r^2)$, etc. Take $r = 0.5$ and numerically determine the limit of this sequence. (We study this type of sequence in detail in section 8.2.) In particular, show that $(1 + 0.5) = \frac{3}{2}$, $(1 + 0.5 + 0.5^2) = \frac{7}{4}$ and $(1 + 0.5 + 0.5^2 + 0.5^3) = \frac{15}{8}$, find a general expression for a_n and determine the limit of the sequence. Argue that at the end of this amount of time, the ball has stopped bouncing.

2. A surprising follow-up to the bouncing ball problem of exercise 1 is found in *An Experimental Approach to Nonlinear Dynamics and Chaos* by Tufillaro, Abbott and Reilly. Suppose the ball is bouncing on a moving table that oscillates up and down according to the equation $A\cos\omega t$ for some amplitude A and frequency ω. Without the motion of the table, the ball will quickly reach a height of 0 as in exercise 1. For different values of A and ω, however, the ball can settle into an amazing variety of patterns. To understand this, explain why the collision between table and ball could subtract or add velocity to the ball (what happens if the table is going up? down?). A simplified model of the velocity of the ball at successive collisions with the table is $v_{n+1} = 0.8v_n - 10\cos(v_0 + v_1 + \cdots + v_n)$. Starting with $v_0 = 5$, compute $v_1, v_2, \ldots, v_{15}$. In this case, the ball never settles into a pattern; its motion is chaotic.

3. (a) If $a_1 = 3$ and $a_{n+1} = a_n + \sin a_n$ for $n \geq 2$, show numerically that $\{a_n\}$ converges to π. With the same relation $a_{n+1} = a_n + \sin a_n$, try other starting values a_1 (Hint: Try $a_1 = -3$, $a_1 = 9$, $a_1 = 15$ and other values.) and state a general rule for the limit of the sequence as a function of the starting value. (b) If $a_1 = 6$ and $a_{n+1} = a_n - \sin a_n$ for $n \geq 2$, numerically estimate the limit of $\{a_n\}$ in terms of π. Then try other starting values and state a general rule for the limit of the sequence as a function of the starting value. (c) State a general rule for the limit of the sequence with $a_{n+1} = a_n + \cos a_n$ as a function of the starting value a_1.

8.2 INFINITE SERIES

Recall that we write the decimal expansion of $\frac{1}{3}$ as the repeating decimal $\frac{1}{3} = 0.3333333\bar{3}$, where we understand that the 3's in this expansion go on forever. Alternatively, we can think of this as

$$\begin{aligned}\frac{1}{3} &= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \\ &= 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + 3(0.1)^4 + \cdots + 3(0.1)^k + \cdots. \end{aligned} \tag{2.1}$$

For convenience, we write (2.1) using summation notation as

$$\frac{1}{3} = \sum_{k=1}^{\infty} 3(0.1)^k. \tag{2.2}$$

Since we can't add together infinitely many terms, we need to carefully define the *infinite sum* indicated in (2.2). Equation (2.2) means that as you add together more and more terms, the sum gets closer and closer to $\frac{1}{3}$.

In general, for any sequence $\{a_k\}_{k=1}^{\infty}$, suppose we start adding the terms together. We define the *partial sums* $S_1, S_2, \ldots, S_n, \ldots$ by

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2 = S_1 + a_2, \\ S_3 &= \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3, \\ S_4 &= \underbrace{a_1 + a_2 + a_3}_{S_3} + a_4 = S_3 + a_4, \\ &\vdots \\ S_n &= \underbrace{a_1 + a_2 + \cdots + a_{n-1}}_{S_{n-1}} + a_n = S_{n-1} + a_n \end{aligned} \tag{2.3}$$

and so on. Note that each partial sum S_n is the sum of two numbers: the nth term, a_n, and the previous partial sum, S_{n-1}, as indicated in (2.3).

For instance, for the sequence $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$, consider the partial sums

$$S_1 = \frac{1}{2}, \qquad S_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4},$$

$$S_3 = \frac{3}{4} + \frac{1}{2^3} = \frac{7}{8}, \qquad S_4 = \frac{7}{8} + \frac{1}{2^4} = \frac{15}{16}$$

and so on. Look at these carefully and you might notice that $S_2 = \frac{3}{4} = 1 - \frac{1}{2^2}$, $S_3 = \frac{7}{8} = 1 - \frac{1}{2^3}$, $S_4 = \frac{15}{16} = 1 - \frac{1}{2^4}$ and so on, so that $S_n = 1 - \frac{1}{2^n}$, for each $n = 1, 2, \ldots$. Observe that the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums converges, since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

This says that as we add together more and more terms of the sequence $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$, the partial sums are drawing closer and closer to 1. In view of this, we write

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \tag{2.4}$$

It's very important to understand what's going on here. This new mathematical object, $\sum_{k=1}^{\infty} \frac{1}{2^k}$, is called a **series** (or **infinite series**). It is *not a sum* in the usual sense of the word, but rather, the *limit* of the sequence of partial sums. Equation (2.4) says that as we add together more and more terms, the sums are approaching the limit of 1.

In general, for any sequence, $\{a_k\}_{k=1}^{\infty}$, we can write down the series

$$a_1 + a_2 + \cdots + a_k + \cdots = \sum_{k=1}^{\infty} a_k.$$

If the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges (to some number S), then we say that the series $\sum_{k=1}^{\infty} a_k$ **converges** (to S) and write

DEFINITION OF INFINITE SERIES

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = S. \tag{2.5}$$

In this case, we call S the **sum** of the series. Alternatively, if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ diverges (i.e., $\lim_{n\to\infty} S_n$ does not exist), then we say that the series **diverges.**

EXAMPLE 2.1 A Convergent Series

Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges or diverges.

Solution From our work on the introductory example, observe that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

In this case, we say that the series converges to 1. ■

In example 2.2, we examine a simple divergent series.

EXAMPLE 2.2 A Divergent Series

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} k^2$.

Solution Here, we have the nth partial sum.

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1^2 + 2^2 + \cdots + n^2) = \infty.$$

Since the sequence of partial sums diverges, the series diverges also. ■

Determining the convergence or divergence of a series is only rarely as simple as it was in examples 2.1 and 2.2.

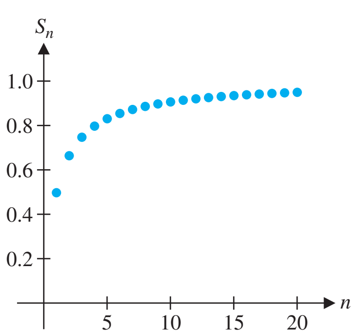


FIGURE 8.17

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

n	$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$
10	0.90909091
100	0.99009901
1000	0.999001
10,000	0.99990001
100,000	0.99999
1×10^6	0.999999
1×10^7	0.9999999

EXAMPLE 2.3 A Series with a Simple Expression for the Partial Sums

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

Solution In Figure 8.17, we have plotted the first 20 partial sums. In the accompanying table, we list a number of partial sums of the series.

From both the graph and the table, it appears that the partial sums are approaching 1, as $n \to \infty$. However, we must urge caution. It is extremely difficult to look at a graph or a table of any partial sums and decide whether a given series converges or diverges. In the present case, we can find a simple expression for the partial sums. The partial fractions decomposition of the general term of the series is

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}. \tag{2.6}$$

Now, consider the nth partial sum. From (2.6), we have

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Notice how nearly every term in the partial sum is canceled by another term in the sum (the next term). For this reason, such a sum is referred to as a **telescoping sum** (or **collapsing sum**). We now have

$$S_n = 1 - \frac{1}{n+1}$$

and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

This says that the series converges to 1, as suggested by the graph and the table. ■

It is relatively rare that we can find the sum of a convergent series exactly. Usually, we must test a series for convergence using some indirect method and then approximate the sum by calculating some partial sums. The series we considered in example 2.1, $\sum_{k=1}^{\infty} \frac{1}{2^k}$, is an example of a *geometric series,* whose sum is known exactly. We have the following result.

NOTE

A geometric series is any series that can be written in the form $\sum_{k=0}^{\infty} ar^k$ for nonzero constants a and r. In this case, each term in the series equals the constant r times the previous term.

THEOREM 2.1

For $a \neq 0$, the **geometric series** $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$. (Here, r is referred to as the **ratio.**)

PROOF

The proof relies on a clever observation. Since the first term of the series corresponds to $k = 0$, the nth partial sum (the sum of the first n terms) is

$$S_n = a + ar^1 + ar^2 + \cdots + ar^{n-1}. \tag{2.7}$$

Multiplying (2.7) by r, we get

$$rS_n = ar^1 + ar^2 + ar^3 + \cdots + ar^n. \tag{2.8}$$

Subtracting (2.8) from (2.7), we get

$$\begin{aligned}(1-r)S_n &= (a + ar^1 + ar^2 + \cdots + ar^{n-1}) - (ar^1 + ar^2 + ar^3 + \cdots + ar^n)\\ &= a - ar^n = a(1 - r^n).\end{aligned}$$

Dividing both sides by $(1 - r)$ gives us

$$S_n = \frac{a(1-r^n)}{1-r}.$$

If $|r| < 1$, notice that $r^n \to 0$ as $n \to \infty$ and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

We leave it as an exercise to show that if $|r| \geq 1$, $\lim_{n\to\infty} S_n$ does not exist. ■

EXAMPLE 2.4 A Convergent Geometric Series

Investigate the convergence or divergence of the series $\sum_{k=2}^{\infty} 5\left(\frac{1}{3}\right)^k$.

Solution The first 20 partial sums are plotted in Figure 8.18. It appears from the graph that the sequence of partial sums is converging to some number around 0.8. Further evidence is found in the following table of partial sums.

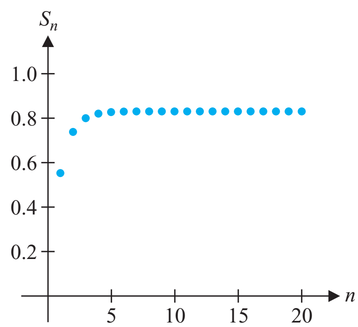


FIGURE 8.18
$S_n = \sum_{k=2}^{n+1} 5 \cdot \left(\frac{1}{3}\right)^k$

n	$S_n = \sum_{k=2}^{n+1} 5\left(\frac{1}{3}\right)^k$
6	0.83219021
8	0.83320632
10	0.83331922
12	0.83333177
14	0.83333316
16	0.83333331
18	0.83333333
20	0.83333333

The table suggests that the series converges to approximately 0.83333333. Again, we must urge caution. Some sequences and series converge (or diverge) far too slowly to observe graphically or numerically. You must *always* confirm your suspicions with careful mathematical analysis. In the present case, note that the series is a geometric series, as follows:

$$\begin{aligned}\sum_{k=2}^{\infty} 5\left(\frac{1}{3}\right)^k &= 5\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^4 + \cdots + 5\left(\frac{1}{3}\right)^n + \cdots \\ &= 5\left(\frac{1}{3}\right)^2\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right] \\ &= \sum_{k=0}^{\infty}\left\{5\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^k\right\}.\end{aligned}$$

You can now see that this is a geometric series with ratio $r = \frac{1}{3}$ and $a = 5\left(\frac{1}{3}\right)^2$. Further, since

$$|r| = \frac{1}{3} < 1,$$

we have from Theorem 2.1 that the series converges to

$$\frac{a}{1-r} = \frac{5\left(\frac{1}{3}\right)^2}{1 - \left(\frac{1}{3}\right)} = \frac{\left(\frac{5}{9}\right)}{\left(\frac{2}{3}\right)} = \frac{5}{6} = 0.8333333\bar{3},$$

which is consistent with the graph and the table of partial sums. ■

EXAMPLE 2.5 A Divergent Geometric Series

Investigate the convergence or divergence of the series $\sum_{k=0}^{\infty} 6\left(-\frac{7}{2}\right)^k$.

Solution A graph showing the first 20 partial sums (see Figure 8.19) is not particularly helpful, until you look at the vertical scale. The following table showing a number of partial sums is more revealing.

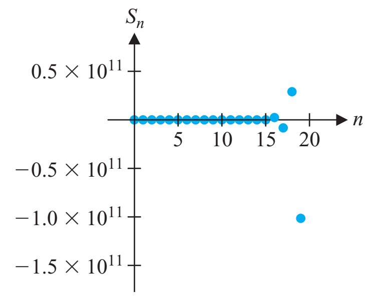


FIGURE 8.19
$S_n = \sum_{k=0}^{n-1} 6 \cdot \left(-\frac{7}{2}\right)^k$

n	$S_n = \sum_{k=0}^{n-1} 6\left(-\frac{7}{2}\right)^k$
11	1.29×10^6
12	-4.5×10^6
13	1.6×10^7
14	-5.5×10^7
15	1.9×10^8
16	-6.8×10^8
17	2.4×10^9
18	-8.3×10^9
19	2.9×10^{10}
20	-1×10^{11}

Note that while the partial sums are oscillating back and forth between positive and negative values, they are growing larger and larger in absolute value. We can confirm our suspicions by observing that this is a geometric series with ratio $r = -\frac{7}{2}$. Since

$$|r| = \left|-\frac{7}{2}\right| = \frac{7}{2} \geq 1,$$

the series is divergent, as we suspected. ■

You will find that determining whether a series is convergent or divergent usually involves a lot of hard work. The following simple observation provides us with a very useful test.

THEOREM 2.2

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

PROOF

Suppose that $\sum_{k=1}^{\infty} a_k$ converges to some number L. This means that the sequence of partial sums defined by $S_n = \sum_{k=1}^{n} a_k$ also converges to L. Notice that

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = S_{n-1} + a_n.$$

Subtracting S_{n-1} from both sides, we have

$$a_n = S_n - S_{n-1}.$$

This gives us

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0,$$

as desired. ■

The following very useful test follows directly from Theorem 2.2.

*k*th-Term Test for Divergence

If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

The kth-term test is so simple, you should use it to test every series you run into. It says that if the terms don't tend to zero, the series is divergent and there's nothing more to do. However, as we'll soon see, if the terms *do* tend to zero, the series may or may not converge and additional testing is needed.

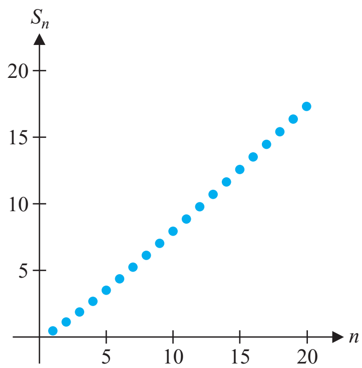


FIGURE 8.20

$S_n = \sum_{k=1}^{n} \frac{k}{k+1}$

EXAMPLE 2.6 A Series Whose Terms Do Not Tend to Zero

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$.

Solution A graph showing the first 20 partial sums is shown in Figure 8.20. The partial sums appear to be increasing without bound as n increases. We can resolve the question of convergence quickly by observing that

$$\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0.$$

From the kth-term test for divergence, the series must diverge. ■

REMARK 2.1

The converse of Theorem 2.2 is *false*. That is, having $\lim_{k\to\infty} a_k = 0$ does *not* guarantee that the series $\sum_{k=1}^{\infty} a_k$ converges. *Be very clear about this point. This is a very common misconception.*

Example 2.7 shows an important series whose terms tend to 0 as $k \to \infty$, but that diverges, nonetheless.

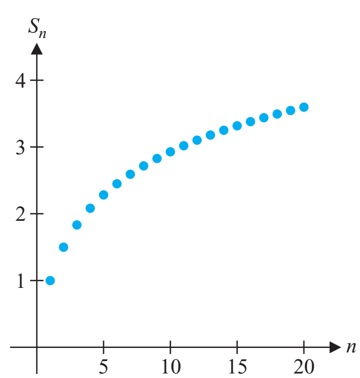


FIGURE 8.21

$S_n = \sum_{k=1}^{n} \frac{1}{k}$

EXAMPLE 2.7 The Harmonic Series

Investigate the convergence or divergence of the **harmonic series:** $\sum_{k=1}^{\infty} \frac{1}{k}$.

Solution In Figure 8.21, we see the first 20 partial sums of the series. In the table, we display several partial sums. The table and the graph suggest that the series might converge to a number around 3.6. As always with sequences and series, we need to confirm this suspicion. First, note that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Be careful: once again, this does *not* say that the series converges. If the limit had been nonzero, we would have concluded that the series diverges. In the present case, where the limit is 0, we can conclude only that the series *may* converge, but we will need to investigate further.

n	$S_n = \sum_{k=1}^{n} \frac{1}{k}$
11	3.01988
12	3.10321
13	3.18013
14	3.25156
15	3.31823
16	3.38073
17	3.43955
18	3.49511
19	3.54774
20	3.59774

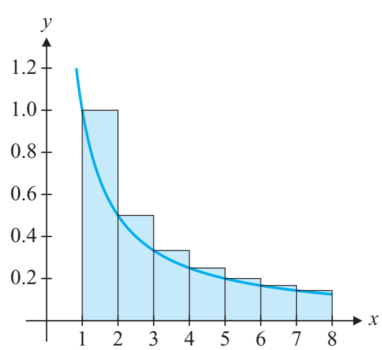


FIGURE 8.22
$y = \frac{1}{x}$

The following clever proof provides a preview of things to come. Consider the nth partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Note that S_n corresponds to the sum of the areas of the n rectangles superimposed on the graph of $y = \frac{1}{x}$, as shown in Figure 8.22 for the case where $n = 7$.

Since each of the indicated rectangles lies partly above the curve, we have

$$\begin{aligned} S_n &= \text{Sum of areas of } n \text{ rectangles} \\ &\geq \text{Area under the curve} = \int_1^{n+1} \frac{1}{x}\,dx \\ &= \ln|x|\Big|_1^{n+1} = \ln(n+1). \end{aligned} \tag{2.9}$$

However, the sequence $\{\ln(n+1)\}_{n=1}^{\infty}$ diverges, since

$$\lim_{n\to\infty} \ln(n+1) = \infty.$$

Since $S_n \geq \ln(n+1)$, for all n [from (2.9)], we must also have that $\lim_{n\to\infty} S_n = \infty$. From the definition of convergence of a series, we now have that $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, too, even though $\lim_{k\to\infty} \frac{1}{k} = 0$. ■

We conclude this section with several unsurprising results.

THEOREM 2.3

(i) If $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B, then the series $\sum_{k=1}^{\infty} (a_k \pm b_k)$ converges to $A \pm B$ and $\sum_{k=1}^{\infty} (ca_k)$ converges to cA, for any constant, c.

(ii) If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} (a_k \pm b_k)$ diverges.

The proof of the theorem is left as an exercise.

BEYOND FORMULAS

The harmonic series illustrates one of the most counterintuitive facts in calculus. A full understanding of this particular infinite series will help you recognize many of the subtle issues that arise in later mathematics courses. The general result may be stated this way: in the case where $\lim_{k\to\infty} a_k = 0$, the series $\sum_{k=1}^{\infty} a_k$ might diverge and might converge, depending on *how fast* the sequence a_k approaches zero. Keep thinking about why the harmonic series diverges and you will develop a deeper understanding of how infinite series in particular and calculus in general work.

EXERCISES 8.2

WRITING EXERCISES

1. Suppose that your friend is confused about the difference between the convergence of a sequence and the convergence of a series. Carefully explain the difference between convergence or divergence of the sequence $a_k = \frac{k}{k+1}$ and the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$.

2. Explain in words why the kth-term test for divergence is valid. Explain why it is *not* true that if $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ necessarily converges. In your explanation, include an important example that proves that this is not true and comment on the fact that the convergence of a_k to 0 can be slow or fast.

3. In Theorems 2.2 and 2.3, the series start at $k = 1$, as in $\sum_{k=1}^{\infty} a_k$. Explain why the conclusions of the theorems hold if the series start at $k = 2$, $k = 3$ or at any positive integer.

4. We emphasized in the text that numerical and graphical evidence for the convergence of a series can be misleading. Suppose your calculator carries 14 digits in its calculations. Explain why for large enough values of n, the term $\frac{1}{n}$ will be too small to change the partial sum $\sum_{k=1}^{n} \frac{1}{k}$. Thus, the calculator would incorrectly indicate that the harmonic series converges.

In exercises 1–22, determine whether the series converges or diverges. For convergent series, find the sum of the series.

1. $\sum_{k=0}^{\infty} 3\left(\frac{1}{5}\right)^k$

2. $\sum_{k=0}^{\infty} \frac{1}{3}(5)^k$

3. $\sum_{k=0}^{\infty} \frac{1}{2}\left(-\frac{1}{3}\right)^k$

4. $\sum_{k=0}^{\infty} 4\left(\frac{1}{2}\right)^k$

5. $\sum_{k=0}^{\infty} \frac{1}{2}(3)^k$

6. $\sum_{k=0}^{\infty} 5\left(-\frac{1}{3}\right)^k$

7. $\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$

8. $\sum_{k=1}^{\infty} \frac{4k}{k+2}$

9. $\sum_{k=1}^{\infty} \frac{3k}{k+4}$

10. $\sum_{k=1}^{\infty} \frac{9}{k(k+3)}$

11. $\sum_{k=1}^{\infty} \frac{2}{k}$

12. $\sum_{k=0}^{\infty} \frac{4}{k+1}$

13. $\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2}$

14. $\sum_{k=1}^{\infty} \frac{4}{k(k+1)(k+3)(k+4)}$

15. $\sum_{k=2}^{\infty} 3^{-k}$

16. $\sum_{k=3}^{\infty} (-1)^k \frac{3}{2^k}$

17. $\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{1}{k+1}\right)$

18. $\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k}\right)$

19. $\sum_{k=2}^{\infty} \left(\frac{2}{3^k} + \frac{1}{2^k}\right)$

20. $\sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{4^k}\right)$

21. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{3k}{k+1}$

22. $\sum_{k=0}^{\infty} (-1)^k \frac{k^3}{k^2+1}$

In exercises 23–26, use graphical and numerical evidence to conjecture the convergence or divergence of the series.

23. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

24. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

25. $\sum_{k=1}^{\infty} \frac{3}{k!}$

26. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

27. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=m}^{\infty} a_k$ converges for any positive integer m. In particular, if $\sum_{k=1}^{\infty} a_k$ converges to L, what does $\sum_{k=m}^{\infty} a_k$ converge to?

28. Prove that if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=m}^{\infty} a_k$ diverges for any positive integer m.

29. Prove Theorem 2.3 (i).

30. Prove Theorem 2.3 (ii).

31. The harmonic series is probably the single most important series to understand. In this exercise, we guide you through another proof of the divergence of this series. Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$. Show that $S_1 = 1$ and $S_2 = \frac{3}{2}$. Since $\frac{1}{3} > \frac{1}{4}$, we have $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Therefore, $S_4 > \frac{3}{2} + \frac{1}{2} = 2$. Similarly, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$, so $S_8 > \frac{5}{2}$. Show that $S_{16} > 3$ and $S_{32} > \frac{7}{2}$. For which n can you guarantee that $S_n > 4$? $S_n > 5$? For any positive integer m, determine n such that $S_n > m$. Conclude that the harmonic series diverges.

32. Compute several partial sums of the series $1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$. Argue that the limit of the sequence of partial sums does not exist, so that the series diverges. Also, write this series as a geometric series and use Theorem 2.1 to conclude that the series diverges. Finally, use the kth-term test for divergence to conclude that the series diverges.

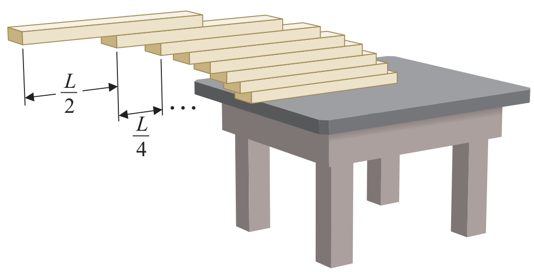
33. Write $0.9999\bar{9} = 0.9 + 0.09 + 0.009 + \cdots$ and sum the geometric series to prove that $0.9999\bar{9} = 1$.

34. As in exercise 33, prove that $0.199999\bar{9} = 0.2$.

35. Write $0.181\overline{818}$ as a geometric series and then write the sum of the geometric series as a fraction.

36. As in exercise 35, write $2.134\overline{134}$ as a fraction.

37. Suppose you have n boards of length L. Place the first board with length $\frac{L}{2n}$ hanging over the edge of the table. Place the next board with length $\frac{L}{2(n-1)}$ hanging over the edge of the first board. The next board should hang $\frac{L}{2(n-2)}$ over the edge of the second board. Continue on until the last board hangs $\frac{L}{2}$ over the edge of the $(n-1)$st board. Theoretically, this stack will balance (in practice, don't use quite as much overhang). With $n = 8$, compute the total overhang of the stack. Determine the number of boards n such that the total overhang is greater than L. This means that the last board is entirely beyond the edge of the table. What is the limit of the total overhang as $n \to \infty$?



38. Have you ever felt that the line you're standing in moves more slowly than the other lines? In *An Introduction to Probability Theory and Its Applications,* William Feller proved just how bad your luck is. Let N be the number of people who get in line until someone waits longer than you do (you're the first, so $N \geq 2$). The probability that $N = k$ is given by $p(k) = \dfrac{1}{k(k-1)}$. Prove that the total probability equals 1; that is, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1$. From probability theory, the average (mean) number of people who must get in line before someone has waited longer than you is given by $\displaystyle\sum_{k=2}^{\infty} k \frac{1}{k(k-1)}$. Prove that this diverges to ∞. Talk about bad luck!

39. If $0 < r < \frac{1}{2}$, show that $1 + 2r + 4r^2 + \cdots + (2r)^n + \cdots = \dfrac{1}{1-2r}$. Replace r with $\dfrac{1}{1000}$ and discuss what's interesting about the decimal representation of $\dfrac{500}{499}$.

40. In exploratory exercise 1 of section 8.1, you showed that a particular bouncing ball takes 2 seconds to complete its infinite number of bounces. In general, the total time it takes for a ball to complete its bounces is $\dfrac{2v}{g}\displaystyle\sum_{k=0}^{\infty} r^k$ and the total distance the ball moves is $\dfrac{v^2}{g}\displaystyle\sum_{k=0}^{\infty} r^{2k}$. Assuming $0 < r < 1$, find the sums of these geometric series.

41. To win a deuce tennis game, one player or the other must win the next two points. If each player wins one point, the deuce starts over. If you win each point with probability p, the probability that you win the next two points is p^2. The probability that you win one of the next two points is $2p(1-p)$. The probability that you win a deuce game is then $p^2 + 2p(1-p)p^2 + [2p(1-p)]^2p^2 + [2p(1-p)]^3p^2 + \cdots$. Explain what each term represents, explain why the geometric series converges and find the sum of the series. If $p = 0.6$, you're a better player than your opponent. Show that you are more likely to win a deuce game than you are a single point. The slightly strange scoring rules in tennis make it more likely that the better player wins.

42. On an analog clock, at 1:00, the minute hand points to 12 and the hour hand points to 1. When the minute hand reaches 1, the hour hand has progressed to $1 + \frac{1}{12}$. When the minute hand reaches $1 + \frac{1}{12}$, the hour hand has moved to $1 + \frac{1}{12} + \frac{1}{12^2}$. Find the sum of a geometric series to determine the time at which the minute hand and hour hand are in the same location.

43. A dosage d of a drug is given at times $t = 0, 1, 2, \ldots$. The drug decays exponentially with rate r in the bloodstream. The amount in the bloodstream after $n+1$ doses is $d + de^{-r} + de^{-2r} + \cdots + de^{-nr}$. Show that the eventual level of the drug (after an "infinite" number of doses) is $\dfrac{d}{1-e^{-r}}$. If $r = 0.1$, find the dosage needed to maintain a drug level of 2.

44. Two bicyclists are 40 miles apart, riding toward each other at 20 mph (each). A fly starts at one bicyclist and flies toward the other bicyclist at 60 mph. When it reaches the bike, it turns around and flies back to the first bike. It continues flying back and forth until the bikes meet. Determine the distance flown on each leg of the fly's journey and find the sum of the geometric series to get the total distance flown. Verify that this is the right answer by solving the problem the easy way.

45. Suppose \$100,000 of counterfeit money is introduced into the economy. Each time the money is used, 25% of the remaining money is identified as counterfeit and removed from circulation. Determine the total amount of counterfeit money successfully used in transactions. This is an example of the **multiplier effect** in economics. Suppose that a new marking scheme on dollar bills helps raise the detection rate to 40%. Determine the reduction in the total amount of counterfeit money successfully spent.

46. In this exercise, we will find the **present value** of a plot of farmland. Assume that a crop of value \$$c$ will be planted in years 1, 2, 3 and so on, and the yearly inflation rate is r. The present value is given by

$$P = ce^{-r} + ce^{-2r} + ce^{-3r} + \cdots.$$

Find the sum of the geometric series to compute the present value.

47. Give an example where $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge but $\sum_{k=1}^{\infty}(a_k + b_k)$ converges.

48. If $\sum_{k=0}^{\infty} a_k$ converges and $\sum_{k=0}^{\infty} b_k$ diverges, is it necessarily true that $\sum_{k=0}^{\infty}(a_k + b_k)$ diverges?

49. Prove that the sum of a convergent geometric series $1 + r + r^2 + \cdots$ must be greater than $\frac{1}{2}$.

50. Prove that $\sum_{k=1}^{\infty}(a_k - a_{k-1})$ converges if and only if the sequence $\{a_n\}$ converges.

51. Prove that if the series $\sum_{k=0}^{\infty} a_k$ converges, then the series $\sum_{k=0}^{\infty} \frac{1}{a_k}$ diverges.

52. Prove that the partial sum $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ does not equal an integer for any prime $n > 1$. Is the statement true for all integers $n > 1$?

53. Suppose you repeat a game at which you have a probability p of winning each time you play. The probability that your first win comes in your nth game is $p(1-p)^{n-1}$. Compute $\sum_{n=1}^{\infty} p(1-p)^{n-1}$ and state in terms of probability why the result makes sense.

54. The **Cantor set** is one of the most famous sets in mathematics. To construct the Cantor set, start with the interval [0, 1]. Then remove the middle third, $\left(\frac{1}{3}, \frac{2}{3}\right)$. This leaves the set $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. For each of the two subintervals, remove the middle third; in this case, remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$. Continue in this way, removing the middle thirds of each remaining interval. The Cantor set is all points in [0, 1] that are *not* removed. Argue that 0, 1, $\frac{1}{3}$ and $\frac{2}{3}$ are in the Cantor set, and identify four more points in the set. It can be shown that there are an infinite number of points in the Cantor set. On the other hand, the total length of the subintervals removed is $\frac{1}{3} + 2\left(\frac{1}{9}\right) + \cdots$. Find the third term in this series, identify the series as a convergent geometric series and find the sum of the series. Given that you started with an interval of length 1, how much "length" does the Cantor set have?

EXPLORATORY EXERCISES

1. Infinite products are also of great interest to mathematicians. Numerically explore the convergence or divergence of the infinite product $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right)\cdots = \prod_{p\,=\,\text{prime}} \left(1 - \frac{1}{p^2}\right)$. Note that the product is taken over the prime numbers, not all integers. Compare your results to the number $\frac{6}{\pi^2}$.

2. In example 2.7, we showed that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$. Superimpose the graph of $f(x) = \frac{1}{x-1}$ onto Figure 8.22 and show that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln(n)$. Conclude that $\ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln(n)$. **Euler's constant** is defined by

$$\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right].$$

Look up the value of γ. (Hint: Use your CAS.) Use γ to estimate $\sum_{i=1}^{n} \frac{1}{i}$ for $n = 10{,}000$ and $n = 100{,}000$.

3. Investigate whether the sequence $a_n = \sum_{k=n}^{2n} \frac{1}{k}$ converges or diverges.

8.3 THE INTEGRAL TEST AND COMPARISON TESTS

Keep in mind that, for most series, we cannot determine whether they converge or diverge by simply looking at the sequence of partial sums. Most of the time, we will need to test a series for convergence in some indirect way. In this section, we will develop additional tests for convergence of series. The first of these is a generalization of the method we used in section 8.2 to show that the harmonic series is divergent.

For a given series $\sum_{k=1}^{\infty} a_k$, suppose that there is a function f for which

$$f(k) = a_k, \quad \text{for } k = 1, 2, \ldots,$$

where f is continuous and decreasing and $f(x) \geq 0$ for all $x \geq 1$. We consider the nth partial sum

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

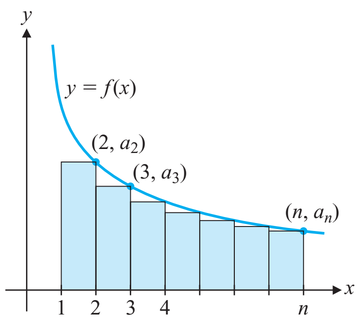


FIGURE 8.23a
$(n-1)$ rectangles, lying beneath the curve

In Figure 8.23a, we show $(n-1)$ rectangles constructed on the interval $[1, n]$, each of width 1 and with height equal to the value of the function at the right-hand endpoint of the subinterval on which it is constructed. Notice that since each rectangle lies completely beneath the curve, the sum of the areas of the $(n-1)$ rectangles shown is less than the area under the curve from $x = 1$ to $x = n$. That is,

$$0 \leq \text{ Sum of areas of } (n-1) \text{ rectangles } \leq \text{ Area under the curve } = \int_1^n f(x)\,dx. \quad (3.1)$$

Note that the area of the first rectangle is length $\times$ width $= (1)(a_2)$, the area of the second rectangle is $(1)(a_3)$ and so on. We get that the sum of the areas of the $(n-1)$ rectangles shown is

$$a_2 + a_3 + a_4 + \cdots + a_n = S_n - a_1,$$

since
$$S_n = a_1 + a_2 + \cdots + a_n.$$

Together with (3.1), this gives us

$$\begin{aligned} 0 &\leq \text{ Sum of areas of } (n-1) \text{ rectangles} \\ &= S_n - a_1 \leq \text{ Area under the curve } = \int_1^n f(x)\,dx. \end{aligned} \quad (3.2)$$

Now, suppose that the improper integral $\int_1^\infty f(x)\,dx$ converges. Then, from (3.2), we have

$$0 \leq S_n - a_1 \leq \int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx.$$

Adding a_1 to all the terms gives us

$$a_1 \leq S_n \leq a_1 + \int_1^\infty f(x)\,dx,$$

so that the sequence of partial sums $\{S_n\}_{n=1}^\infty$ is bounded. Since $\{S_n\}_{n=1}^\infty$ is also monotonic (why is that?), $\{S_n\}_{n=1}^\infty$ is convergent by Theorem 1.4 and so, the series $\sum_{k=1}^\infty a_k$ is also convergent.

In Figure 8.23b, we show $(n-1)$ rectangles constructed on the interval $[1, n]$, each of width 1, but with height equal to the value of the function at the left-hand endpoint of the subinterval on which it is constructed. In this case, the sum of the areas of the $(n-1)$ rectangles shown is greater than the area under the curve. That is,

$$\begin{aligned} 0 &\leq \text{ Area under the curve } = \int_1^n f(x)\,dx \\ &\leq \text{ Sum of areas of } (n-1) \text{ rectangles.} \end{aligned} \quad (3.3)$$

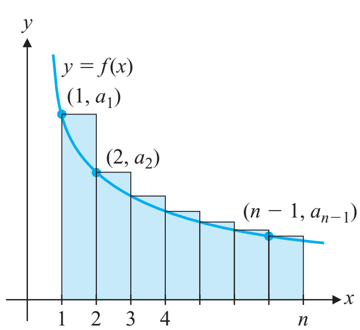


FIGURE 8.23b
$(n-1)$ rectangles, partially above the curve

Further, note that the area of the first rectangle is length $\times$ width $= (1)(a_1)$, the area of the second rectangle is $(1)(a_2)$ and so on. We get that the sum of the areas of the $(n-1)$ rectangles indicated in Figure 8.23b is

$$a_1 + a_2 + \cdots + a_{n-1} = S_{n-1}.$$

HISTORICAL NOTES

Colin Maclaurin (1698–1746) Scottish mathematician who discovered the Integral Test. Maclaurin was one of the founders of the Royal Society of Edinburgh and was a pioneer in the mathematics of actuarial studies. The Integral Test was introduced in a highly influential book that also included a new treatment of an important method for finding series of functions. Maclaurin series, as we now call them, are developed in section 8.7.

Together with (3.3), this gives us

$$\begin{aligned} 0 \le \text{Area under the curve} &= \int_1^n f(x)\,dx \\ &\le \text{Sum of areas of } (n-1) \text{ rectangles} = S_{n-1}. \end{aligned} \tag{3.4}$$

Now, suppose that the improper integral $\int_1^\infty f(x)\,dx$ diverges. Since $f(x) \ge 0$, this says that $\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty$. From (3.4), we have that

$$\int_1^n f(x)\,dx \le S_{n-1}.$$

This says that

$$\lim_{n\to\infty} S_{n-1} = \infty,$$

also. So, the sequence of partial sums $\{S_n\}_{n=1}^\infty$ diverges and hence, the series $\sum_{k=1}^\infty a_k$ diverges, too.

We summarize the results of this analysis in Theorem 3.1.

THEOREM 3.1 (Integral Test)

If $f(k) = a_k$ for all $k = 1, 2, \ldots,$ f is continuous and decreasing, and $f(x) \ge 0$ for $x \ge 1$, then $\int_1^\infty f(x)\,dx$ and $\sum_{k=1}^\infty a_k$ either *both* converge or *both* diverge.

Note that while the Integral Test might say that a given series and improper integral both converge, it does *not* say that they will converge to the same value. In fact, this is generally not the case, as we see in example 3.1.

EXAMPLE 3.1 Using the Integral Test

Investigate the convergence or divergence of the series $\sum_{k=0}^\infty \frac{1}{k^2+1}$.

Solution The graph of the first 20 partial sums shown in Figure 8.24 suggests that the series converges to some value around 2. In the accompanying table, we show some selected partial sums. Based on this, we cannot say whether the series is converging very slowly to a limit around 2.076 or whether the series is instead diverging very slowly. To determine which is the case, we must test the series further. Define $f(x) = \frac{1}{x^2+1}$. Note that f is continuous and positive everywhere and $f(k) = \frac{1}{k^2+1} = a_k$, for all $k \ge 1$. Further,

$$f'(x) = (-1)(x^2+1)^{-2}(2x) < 0,$$

for $x \in (0, \infty)$, so that f is decreasing. This says that the Integral Test applies to this series. So, we consider the improper integral

$$\begin{aligned} \int_0^\infty \frac{1}{x^2+1}\,dx &= \lim_{R\to\infty} \int_0^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \tan^{-1} x \Big|_0^R \\ &= \lim_{R\to\infty} (\tan^{-1} R - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}. \end{aligned}$$

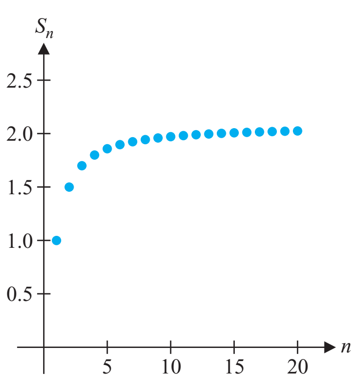


FIGURE 8.24

$S_n = \sum_{k=0}^{n-1} \frac{1}{k^2+1}$

n	$S_n = \sum_{k=0}^{n-1} \frac{1}{k^2+1}$
10	1.97189
50	2.05648
100	2.06662
200	2.07166
500	2.07467
1000	2.07567
2000	2.07617

The Integral Test says that since the improper integral converges, the series must converge, also. Now that we have established that the series is convergent, our earlier calculations give us the estimated sum 2.076. Notice that this is *not* the same as the value of the corresponding improper integral, which is $\frac{\pi}{2} \approx 1.5708$. ■

In example 3.2, we discuss an important type of series.

EXAMPLE 3.2 The p-Series

Determine for which values of p the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ (a ***p*-series**) converges.

Solution First, notice that for $p = 1$, this is the harmonic series, which diverges. For $p > 1$, define $f(x) = \frac{1}{x^p} = x^{-p}$. Notice that for $x \geq 1$, f is continuous and positive. Further,

$$f'(x) = -px^{-p-1} < 0,$$

so that f is decreasing. This says that the Integral Test applies. We now consider

$$\int_1^{\infty} x^{-p}\,dx = \lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^R$$

$$= \lim_{R\to\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1} \right) = \frac{-1}{-p+1}.$$

Since $p > 1$ implies that $-p + 1 < 0$.

In this case, the improper integral converges and so too, must the series. In the case where $p < 1$, we leave it as an exercise to show that the series diverges. ■

We summarize the result of example 3.2 as follows.

p-SERIES

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Notice that in each of examples 3.1 and 3.2, we were able to use the Integral Test to establish the convergence of a series. While you can use the partial sums of a convergent series to estimate its sum, how precise is a given estimate? First, if we estimate the sum s of the series $\sum_{k=1}^{\infty} a_k$ by the nth partial sum $S_n = \sum_{k=1}^{n} a_k$, we define the **remainder** R_n to be

$$R_n = s - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k.$$

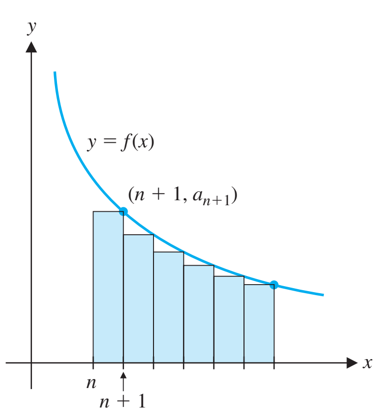


FIGURE 8.25
Estimate of the remainder

Notice that this says that the remainder R_n is the error in approximating s by S_n. For any series shown to be convergent by the Integral Test, we can estimate the size of the remainder, as follows. From Figure 8.25, observe that R_n corresponds to the sum of the areas of the indicated rectangles. Further, under the conditions of the Integral Test, this is less than the area under the curve $y = f(x)$. (Recall that this area is finite, as $\int_1^{\infty} f(x)\,dx$ converges.) That is, we have the following result.

THEOREM 3.2 (Error Estimate for the Integral Test)

Suppose that $f(k) = a_k$ for all $k = 1, 2, \ldots,$ where f is continuous and decreasing, and $f(x) \geq 0$ for all $x \geq 1$. Further, suppose that $\int_1^\infty f(x)\,dx$ converges. Then, the remainder R_n satisfies

$$0 \leq R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^\infty f(x)\,dx.$$

We can use Theorem 3.2 to estimate the error in using a partial sum to approximate the sum of a series.

EXAMPLE 3.3 Estimating the Error in a Partial Sum

Estimate the error in using the partial sum S_{100} to approximate the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

Solution First, recall that in example 3.2, we used the Integral Test to show that this series (a p-series, with $p = 3$) is convergent. From Theorem 3.2, the remainder satisfies

$$0 \leq R_{100} \leq \int_{100}^\infty \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_{100}^R \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_{100}^R$$
$$= \lim_{R\to\infty} \left(\frac{-1}{2R^2} + \frac{1}{2(100)^2}\right) = 5 \times 10^{-5}.$$

A more interesting and far more practical question related to example 3.3 is to determine the number of terms of the series necessary to obtain a given accuracy.

EXAMPLE 3.4 Finding the Number of Terms Needed for a Given Accuracy

Determine the number of terms needed to obtain an approximation to the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ correct to within 10^{-5}.

Solution Again, we already used the Integral Test to show that the series in question converges. Then, by Theorem 3.2, we have that the remainder satisfies

$$0 \leq R_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_n^R \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_n^R$$
$$= \lim_{R\to\infty} \left(\frac{-1}{2R^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}.$$

So, to ensure that the remainder is less than 10^{-5}, we require that

$$0 \leq R_n \leq \frac{1}{2n^2} \leq 10^{-5}.$$

Solving this last inequality for n yields

$$n^2 \geq \frac{10^5}{2} \quad \text{or} \quad n \geq \sqrt{\frac{10^5}{2}} = 100\sqrt{5} \approx 223.6.$$

So, taking $n \geq 224$ will guarantee the required accuracy and consequently, we have $\sum_{k=1}^{\infty} \frac{1}{k^3} \approx \sum_{k=1}^{224} \frac{1}{k^3} = 1.202047$, which is correct to within 10^{-5}, as desired.

Comparison Tests

We next present two results that allow us to compare a given series with one that is already known to be convergent or divergent, much as we did with improper integrals in section 6.6.

THEOREM 3.3 (Comparison Test)

Suppose that $0 \leq a_k \leq b_k$, for all k.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges, too.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges, too.

Intuitively, this theorem should make abundant sense: if the "larger" series converges, then the "smaller" one must also converge. Likewise, if the "smaller" series diverges, then the "larger" one must diverge, too.

PROOF

Given that $0 \leq a_k \leq b_k$ for all k, observe that the nth partial sums of the two series satisfy

$$0 \leq S_n = a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n.$$

(i) If $\sum_{k=1}^{\infty} b_k$ converges (say to B), this says that

$$0 \leq S_n \leq a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n \leq \sum_{k=1}^{\infty} b_k = B, \tag{3.5}$$

for all $n \geq 1$. From (3.5), the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums of $\sum_{k=1}^{\infty} a_k$ is bounded. Notice that $\{S_n\}_{n=1}^{\infty}$ is also increasing. (Why?) Since every bounded, monotonic sequence is convergent (see Theorem 1.4), we get that $\sum_{k=1}^{\infty} a_k$ is convergent, too.

(ii) If $\sum_{k=1}^{\infty} a_k$ is divergent, we have (since all of the terms of the series are nonnegative) that

$$\lim_{n\to\infty} (b_1 + b_2 + \cdots + b_n) \geq \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = \infty.$$

Thus, $\sum_{k=1}^{\infty} b_k$ must be divergent, also. ■

You can use the Comparison Test to test the convergence of series that look similar to series that you already know are convergent or divergent (notably, geometric series or p-series).

EXAMPLE 3.5 Using the Comparison Test for a Convergent Series

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{k^3+5k}$.

Solution The graph of the first 20 partial sums shown in Figure 8.26 suggests that the series converges to some value near 0.3. To confirm such a conjecture, we must carefully test the series. Note that for large values of k, the general term of the series looks like $\frac{1}{k^3}$, since when k is large, k^3 is much larger than $5k$. This observation is significant, since we already know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$). Further, observe that

$$\frac{1}{k^3+5k} \le \frac{1}{k^3},$$

for all $k \ge 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the Comparison Test says that $\sum_{k=1}^{\infty} \frac{1}{k^3+5k}$ converges, too. As with the Integral Test, although the Comparison Test tells us that both series converge, the two series *need not* converge to the same sum. A quick calculation of a few partial sums should convince you that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges to approximately 1.202, while $\sum_{k=1}^{\infty} \frac{1}{k^3+5k}$ converges to approximately 0.2798. (Note that this is consistent with what we saw in Figure 8.26.) ■

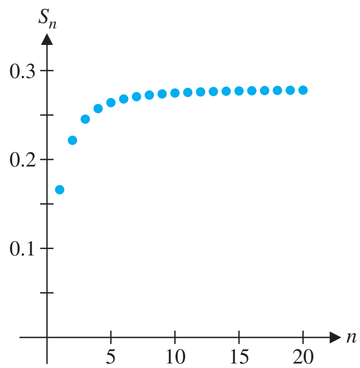


FIGURE 8.26
$S_n = \sum_{k=1}^{n} \frac{1}{k^3+5k}$

EXAMPLE 3.6 Using the Comparison Test for a Divergent Series

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}$.

Solution From the graph of the first 20 partial sums seen in Figure 8.27, it appears that the partial sums are growing very rapidly. On this basis, we would conjecture that the series diverges. Of course, to verify this, we need further testing. Notice that for k large, the general term looks like $\frac{5^k}{2^k} = \left(\frac{5}{2}\right)^k$ and we know that $\sum_{k=1}^{\infty} \left(\frac{5}{2}\right)^k$ is a divergent geometric series $\left(|r| = \frac{5}{2} > 1\right)$. Further,

$$\frac{5^k+1}{2^k-1} \ge \frac{5^k}{2^k-1} \ge \frac{5^k}{2^k} = \left(\frac{5}{2}\right)^k.$$

By the Comparison Test, $\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}$ diverges, too. ■

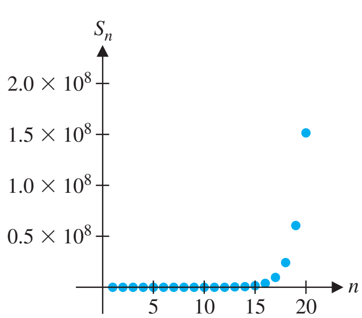


FIGURE 8.27
$S_n = \sum_{k=1}^{n} \frac{5^k+1}{2^k-1}$

There are plenty of series whose general term looks like the general term of a familiar series, but for which it is unclear how to get the inequality required for the Comparison Test to go in the right direction.

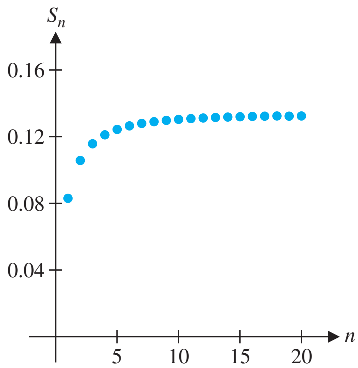


FIGURE 8.28
$S_n = \sum_{k=3}^{n+2} \frac{1}{k^3 - 5k}$

EXAMPLE 3.7 A Comparison That Does Not Work

Investigate the convergence or divergence of the series $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$.

Solution Note that this is nearly identical to example 3.5, except that there is a "$-$" sign in the denominator instead of a "$+$" sign. The graph of the first 20 partial sums seen in Figure 8.28 looks somewhat similar to the graph in Figure 8.26, except that the series appears to be converging to about 0.12. In this case, however, we have the inequality

$$\frac{1}{k^3 - 5k} \geq \frac{1}{k^3}, \quad \text{for all } k \geq 3.$$

Unfortunately, this inequality goes the *wrong way:* we know that $\sum_{k=3}^{\infty} \frac{1}{k^3}$ is a convergent p-series, but since $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$ is "larger" than this convergent series, the Comparison Test says nothing. ■

Think about what happened in example 3.7 this way: while you might observe that

$$k^2 \geq \frac{1}{k^3}, \quad \text{for all } k \geq 1$$

and you know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent, the Comparison Test says nothing about the "larger" series $\sum_{k=1}^{\infty} k^2$. In fact, we know that this last series is divergent (by the kth-term test for divergence, since $\lim_{k\to\infty} k^2 = \infty \neq 0$). To resolve this difficulty for the present problem, we will need to either make a different comparison or use the Limit Comparison Test, which follows.

NOTES

When we say $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$, we mean that the limit exists and is positive. In particular, we mean that $\lim_{k\to\infty} \frac{a_k}{b_k} \neq \infty$.

THEOREM 3.4 (Limit Comparison Test)

Suppose that $a_k, b_k > 0$ and that for some (finite) value, L, $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$. Then, either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or they *both* diverge.

PROOF

If $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$, this says that we can make $\frac{a_k}{b_k}$ as close to L as desired. So, in particular, we can make $\frac{a_k}{b_k}$ within distance $\frac{L}{2}$ of L. That is, for some number $N > 0$,

$$L - \frac{L}{2} < \frac{a_k}{b_k} < L + \frac{L}{2}, \quad \text{for all } k > N$$

or

$$\frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2}. \tag{3.6}$$

Multiplying inequality (3.6) through by b_k (recall that $b_k > 0$), we get

$$\frac{L}{2}b_k < a_k < \frac{3L}{2}b_k, \quad \text{for } k \geq N.$$

Note that this says that if $\sum_{k=1}^{\infty} a_k$ converges, then the "smaller" series $\sum_{k=1}^{\infty}\left(\frac{L}{2}b_k\right) = \frac{L}{2}\sum_{k=1}^{\infty} b_k$ must also converge, by the Comparison Test. Likewise, if $\sum_{k=1}^{\infty} a_k$ diverges, the "larger" series $\sum_{k=1}^{\infty}\left(\frac{3L}{2}b_k\right) = \frac{3L}{2}\sum_{k=1}^{\infty} b_k$ must also diverge. In the same way, if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty}\left(\frac{3L}{2}b_k\right)$ converges and so, too must the "smaller" series $\sum_{k=1}^{\infty} a_k$. Finally, if $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty}\left(\frac{L}{2}b_k\right)$ diverges and hence, the "larger" series $\sum_{k=1}^{\infty} a_k$ must diverge, also. ■

We can now use the Limit Comparison Test to test the series from example 3.7 whose convergence we have so far been unable to confirm.

EXAMPLE 3.8 Using the Limit Comparison Test

Investigate the convergence or divergence of the series $\sum_{k=3}^{\infty}\frac{1}{k^3-5k}$.

Solution Recall that we had already observed in example 3.7 that the general term $a_k = \frac{1}{k^3-5k}$ "looks like" $b_k = \frac{1}{k^3}$, for k large. We then consider the limit

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\left(a_k\frac{1}{b_k}\right) = \lim_{k\to\infty}\frac{1}{(k^3-5k)}\frac{1}{\left(\frac{1}{k^3}\right)} = \lim_{k\to\infty}\frac{1}{1-\frac{5}{k^2}} = 1 > 0.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), the Limit Comparison Test says that $\sum_{k=3}^{\infty}\frac{1}{k^3-5k}$ is also convergent, as we had originally suspected. ■

The Limit Comparison Test can be used to resolve convergence questions for a great many series. The first step in using this (like the Comparison Test) is to find another series (whose convergence or divergence is known) that "looks like" the series in question.

EXAMPLE 3.9 Using the Limit Comparison Test

Investigate the convergence or divergence of the series

$$\sum_{k=1}^{\infty}\frac{k^2-2k+7}{k^5+5k^4-3k^3+2k-1}.$$

Solution The graph of the first 20 partial sums in Figure 8.29 suggests that the series converges to a limit of about 1.61. The accompanying table of partial sums supports this conjecture.

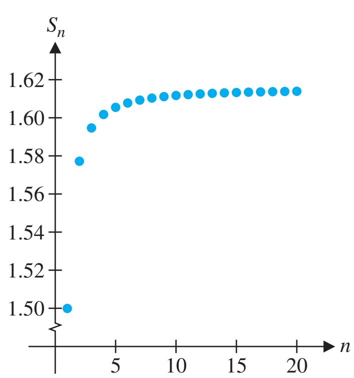


FIGURE 8.29

$$S_n = \sum_{k=1}^{n}\frac{k^2-2k+7}{k^5+5k^4-3k^3+2k-1}$$

n	$S_n = \sum_{k=1}^{n} \frac{k^2-2k+7}{k^5+5k^4-3k^3+2k-1}$
5	1.60522
10	1.61145
20	1.61365
50	1.61444
75	1.61453
100	1.61457

Notice that for k large, the general term looks like $\frac{k^2}{k^5} = \frac{1}{k^3}$ (since the terms with the largest exponents tend to dominate the expression, for large values of k). From the Limit Comparison Test, for $b_k = \frac{1}{k^3}$, we have

$$\begin{aligned}\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{k^2-2k+7}{k^5+5k^4-3k^3+2k-1} \frac{1}{\left(\frac{1}{k^3}\right)} \\ &= \lim_{k\to\infty} \frac{(k^2-2k+7)}{(k^5+5k^4-3k^3+2k-1)} \frac{k^3}{1} \\ &= \lim_{k\to\infty} \frac{(k^5-2k^4+7k^3)}{(k^5+5k^4-3k^3+2k-1)} \frac{\left(\frac{1}{k^5}\right)}{\left(\frac{1}{k^5}\right)} \\ &= \lim_{k\to\infty} \frac{1-\frac{2}{k}+\frac{7}{k^2}}{1+\frac{5}{k}-\frac{3}{k^2}+\frac{2}{k^4}-\frac{1}{k^5}} = 1 > 0.\end{aligned}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), the Limit Comparison Test says that $\sum_{k=1}^{\infty} \frac{k^2-2k+7}{k^5+5k^4-3k^3+2k-1}$ converges, also. Finally, now that we have established that the series is, in fact, convergent, we can use our table of computed partial sums to approximate the sum of the series as 1.61457. ■

BEYOND FORMULAS

Keeping track of the many convergence tests arising in the study of infinite series can be somewhat challenging. We need all of these convergence tests because there is not a single test that works for all series (although more than one test may be used for a given series). Keep in mind that each test works only for specific types of series. As a result, you must be able to distinguish one type of infinite series (such as a geometric series) from another (such as a p-series), in order to determine the right test to use.

EXERCISES 8.3

WRITING EXERCISES

1. Notice that the Comparison Test doesn't always give us information about convergence or divergence. If $a_k \le b_k$ for each k and $\sum_{k=1}^{\infty} b_k$ diverges, explain why you can't tell whether or not $\sum_{k=1}^{\infty} a_k$ diverges.

2. Explain why the Limit Comparison Test works. In particular, if $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$, explain how a_k and b_k compare and conclude that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.

3. In the Limit Comparison Test, if $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} a_k$ converges, explain why you can't tell whether or not $\sum_{k=1}^{\infty} b_k$ converges.

4. A p-series converges if $p > 1$ and diverges if $p < 1$. What happens for $p = 1$? If your friend knows that the harmonic series diverges, explain an easy way to remember the rest of the conclusion of the p-series test.

In exercises 1–34, determine convergence or divergence of the series.

1. $\sum_{k=1}^{\infty} \frac{4}{\sqrt[3]{k}}$

2. $\sum_{k=1}^{\infty} k^{-9/10}$

3. $\sum_{k=4}^{\infty} k^{-11/10}$

4. $\sum_{k=6}^{\infty} \frac{4}{\sqrt{k}}$

5. $\sum_{k=3}^{\infty} \frac{k+1}{k^2+2k+3}$

6. $\sum_{k=2}^{\infty} \frac{k^2+1}{k^3+3k+2}$

7. $\sum_{k=8}^{\infty} \frac{4}{2+4k}$

8. $\sum_{k=6}^{\infty} \frac{4}{(2+4k)^2}$

9. $\sum_{k=2}^{\infty} \frac{2}{k \ln k}$

10. $\sum_{k=2}^{\infty} \frac{3}{k(\ln k)^2}$

11. $\sum_{k=1}^{\infty} \frac{2k}{k^3+1}$

12. $\sum_{k=0}^{\infty} \frac{\sqrt{k}}{k^2+1}$

13. $\sum_{k=3}^{\infty} \frac{e^{1/k}}{k^2}$

14. $\sum_{k=4}^{\infty} \frac{\sqrt{1+1/k}}{k^2}$

15. $\sum_{k=1}^{\infty} \frac{e^{-\sqrt{k}}}{\sqrt{k}}$

16. $\sum_{k=1}^{\infty} \frac{ke^{-k^2}}{4+e^{-k}}$

17. $\sum_{k=1}^{\infty} \frac{2k^2}{k^{5/2}+2}$

18. $\sum_{k=0}^{\infty} \frac{2}{\sqrt{k^2+4}}$

19. $\sum_{k=0}^{\infty} \frac{4}{\sqrt{k^3+1}}$

20. $\sum_{k=0}^{\infty} \frac{k^2+1}{\sqrt{k^5+1}}$

21. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

22. $\sum_{k=1}^{\infty} \frac{\sin^{-1}(1/k)}{k^2}$

23. $\sum_{k=1}^{\infty} \frac{1}{\cos^2 k}$

24. $\sum_{k=1}^{\infty} \frac{e^{1/k}+1}{k^3}$

25. $\sum_{k=2}^{\infty} \frac{\ln k}{k}$

26. $\sum_{k=1}^{\infty} \frac{2+\cos k}{k}$

27. $\sum_{k=4}^{\infty} \frac{k^4+2k-1}{k^5+3k^2+1}$

28. $\sum_{k=6}^{\infty} \frac{k^3+2k+3}{k^4+2k^2+4}$

29. $\sum_{k=3}^{\infty} \frac{k+1}{k+2}$

30. $\sum_{k=2}^{\infty} \frac{k+1}{k^2+2}$

31. $\sum_{k=8}^{\infty} \frac{k+1}{k^3+2}$

32. $\sum_{k=5}^{\infty} \frac{\sqrt{k+1}}{\sqrt{k^3+2}}$

33. $\sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k}+k\sqrt{k+1}}$

34. $\sum_{k=1}^{\infty} \frac{2k+1}{(k+1)\sqrt{k}+k^2\sqrt{k+1}}$

35. In our statement of the Comparison Test, we required that $a_k \le b_k$ for all k. Explain why the conclusion would remain true if $a_k \le b_k$ for $k \ge 100$.

36. If $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ converges, prove that $\sum_{k=1}^{\infty} a_k^2$ converges.

37. Prove the following extension of the Limit Comparison Test: if $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

38. Prove the following extension of the Limit Comparison Test: if $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

39. Prove that if $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ converge, then $\sum_{k=1}^{\infty} |a_k b_k|$ converges.

40. Prove that for $a_k > 0$, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ converges. (Hint: If $x < 1$, then $x < \frac{2x}{1+x}$.)

In exercises 41–44, determine all values of p for which the series converges.

41. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$

42. $\sum_{k=0}^{\infty} \frac{1}{(a+bk)^p}$, $a > 0, b > 0$

43. $\sum_{k=2}^{\infty} \frac{\ln k}{k^p}$

44. $\sum_{k=1}^{\infty} k^{p-1} e^{kp}$

In exercises 45–50, estimate the error in using the indicated partial sum S_n to approximate the sum of the series.

45. S_{100}, $\sum_{k=1}^{\infty} \frac{1}{k^4}$

46. S_{100}, $\sum_{k=1}^{\infty} \frac{4}{k^2}$

47. S_{50}, $\sum_{k=1}^{\infty} \frac{6}{k^8}$

48. S_{80}, $\sum_{k=1}^{\infty} \frac{2}{k^2+1}$

49. S_{40}, $\sum_{k=1}^{\infty} ke^{-k^2}$

50. S_{200}, $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

In exercises 51–54, determine the number of terms needed to obtain an approximation accurate to within 10^{-6}.

51. $\sum_{k=1}^{\infty} \frac{3}{k^4}$

52. $\sum_{k=1}^{\infty} \frac{2}{k^2}$

53. $\sum_{k=1}^{\infty} ke^{-k^2}$

54. $\sum_{k=1}^{\infty} \frac{4}{k^5}$

In exercises 55 and 56, answer with "converges" or "diverges" or "can't tell." Assume that $a_k > 0$ and $b_k > 0$.

55. Assume that $\sum_{k=1}^{\infty} a_k$ converges and fill in the blanks.

(a) If $b_k \ge a_k$ for $k \ge 10$, then $\sum_{k=1}^{\infty} b_k$ ______.

(b) If $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, then $\sum_{k=1}^{\infty} b_k$ ______.

(c) If $b_k \le a_k$ for $k \ge 6$, then $\sum_{k=1}^{\infty} b_k$ ______.

(d) If $\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$, then $\sum_{k=1}^{\infty} b_k$ ______.

56. Assume that $\sum_{k=1}^{\infty} a_k$ diverges and fill in the blanks.

(a) If $b_k \ge a_k$ for $k \ge 10$, then $\sum_{k=1}^{\infty} b_k$ ______.

(b) If $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$, then $\sum_{k=1}^{\infty} b_k$ ______.

(c) If $b_k \le a_k$ for $k \ge 6$, then $\sum_{k=1}^{\infty} b_k$ ______.

(d) If $\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$, then $\sum_{k=1}^{\infty} b_k$ ______.

57. Prove that the every-other-term harmonic series $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ diverges. (Hint: Write the series as $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ and use the Limit Comparison Test.)

58. Would the every-third-term harmonic series $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots$ diverge? How about the every-fourth-term harmonic series $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots$? Make as general a statement as possible about such series.

59. **The Riemann-zeta function** is defined by $\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$ for $x > 1$. Explain why the restriction $x > 1$ is necessary. Leonhard Euler, considered to be one of the greatest and most prolific mathematicians ever, proved the remarkable result that $\zeta(x) = \prod_{p=\text{prime}} \frac{1}{\left(1 - \frac{1}{p^x}\right)}$.

60. Estimate $\zeta(2)$ numerically. Compare your result with that of exploratory exercise 1 of section 8.2.

In exercises 61–64, use your CAS or graphing calculator to numerically estimate the sum of the convergent *p*-series and identify *x* such that the sum equals $\zeta(x)$ for the Riemann-zeta function of exercise 59.

61. $\sum_{k=1}^{\infty} \frac{1}{k^4}$

62. $\sum_{k=1}^{\infty} \frac{1}{k^6}$

63. $\sum_{k=1}^{\infty} \frac{1}{k^8}$

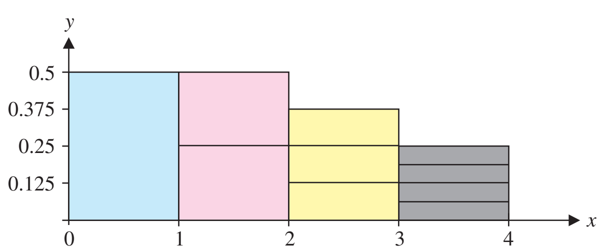
64. $\sum_{k=1}^{\infty} \frac{1}{k^{10}}$

65. Show that $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^{\ln k}}$ and $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^{k}}$ both converge.

66. Show that $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^{n}}$ diverges for any integer $n > 0$. Compare this result to exercise 65.

67. Suppose that you toss a fair coin until you get heads. How many times would you expect to toss the coin? To answer this, notice that the probability of getting heads on the first toss is $\frac{1}{2}$, getting tails then heads is $\left(\frac{1}{2}\right)^2$, getting two tails then heads is $\left(\frac{1}{2}\right)^3$ and so on. The mean number of tosses is $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$. Use the Integral Test to prove that this series converges and estimate the sum numerically.

68. A clever trick can be used to sum the series in exercise 67. The series $\sum_{k=1}^{\infty} n\left(\frac{1}{2}\right)^n$ can be visualized as the area shown in the figure. In columns of width one, we see one rectangle of height $\frac{1}{2}$, two rectangles of height $\frac{1}{4}$, three rectangles of height $\frac{1}{8}$ and so on. Start the sum by taking one rectangle from each column. The combined area of the first rectangles is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Show that this is a convergent series with sum 1. Next, take the second rectangle from each column that has at least two rectangles. The combined area of the second rectangles is $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$. Show that this is a convergent series with sum $\frac{1}{2}$. Next, take the third rectangle from each column that has at least three rectangles. The combined area from the third rectangles is $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$. Show that this is a convergent series with sum $\frac{1}{4}$. Continue this process and show that the total area of all rectangles is $1 + \frac{1}{2} + \frac{1}{4} + \cdots$. Find the sum of this convergent series.



69. This problem is sometimes called the **coupon collectors' problem.** The problem is faced by collectors of trading cards. If there are *n* different cards that make a complete set and you randomly obtain one at a time, how many cards would you expect to obtain before having a complete set? (By random, we mean that each different card has the same probability of $\frac{1}{n}$ of being the next card obtained.) In exercises 69–72, we find the answer for $n = 10$. The first step is simple; to collect one card you need to obtain one card. Now, given that you have one card, how many cards do you need to obtain to get a second (different) card? If you're lucky, the next card is it (this has probability $\frac{9}{10}$). But your next card might be a duplicate, then you get a new card (this has probability $\frac{1}{10} \cdot \frac{9}{10}$). Or you might get two duplicates and then a new card (this has probability $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{9}{10}$); and so on. The mean is $1 \cdot \frac{9}{10} + 2 \cdot \frac{1}{10} \cdot \frac{9}{10} + 3 \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{9}{10} + \cdots$ or $\sum_{k=1}^{\infty} k\left(\frac{1}{10}\right)^{k-1}\left(\frac{9}{10}\right) = \sum_{k=1}^{\infty} \frac{9k}{10^k}$. Using the same trick as in exercise 68, show that this is a convergent series with sum $\frac{10}{9}$.

70. In the situation of exercise 69, if you have two different cards out of ten, the average number of cards to get a third distinct card is $\sum_{k=1}^{\infty} \frac{8k2^{k-1}}{10^k}$; show that this is a convergent series with sum $\frac{10}{8}$.

71. Extend the results of exercises 69 and 70 to find the average number of cards you need to obtain to complete the set of ten different cards.

72. Compute the ratio of cards obtained to cards in the set in exercise 71. That is, for a set of 10 cards, on the average you need to obtain ______ times 10 cards to complete the set.

73. Generalize exercises 71 and 72 in the case of n cards in the set ($n > 2$).

74. Use the divergence of the harmonic series to state the unfortunate fact about the ratio of cards obtained to cards in the set as n increases.

EXPLORATORY EXERCISES

1. Numerically investigate the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{0.9}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{1.1}}$ and for other values of p close to 1. Can you distinguish convergent from divergent series numerically?

2. You know that $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges. This is the "smallest" p-series that diverges, in the sense that $\frac{1}{k} < \frac{1}{k^p}$ for $p < 1$. Show that $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges and $\frac{1}{k \ln k} < \frac{1}{k}$. Show that $\sum_{k=2}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$ diverges and $\frac{1}{k \ln k \ln(\ln k)} < \frac{1}{k \ln k}$. Find a series such that $\sum_{k=2}^{\infty} a_k$ diverges and $a_k < \frac{1}{k \ln k \ln(\ln k)}$. Is there a smallest divergent series?

3. In this exercise, you explore the convergence of the infinite product $P = 2^{1/4}3^{1/9}4^{1/6}\cdots$. This can be written in the form $P = \prod_{k=2}^{\infty} k^{1/k^2}$. For the partial product $P_n = \prod_{k=2}^{n} k^{1/k^2}$, use the natural logarithm to write

$$\begin{aligned} P_n &= e^{\ln P_n} = e^{\ln[2^{1/4}3^{1/9}4^{1/16}\cdots n^{1/n^2}]} = e^{Sn}, \quad \text{where} \\ S_n &= \ln[2^{1/4}3^{1/9}4^{1/16}\cdots n^{1/n^2}] \\ &= \frac{1}{4}\ln 2 + \frac{1}{9}\ln 3 + \frac{1}{16}\ln 4 + \cdots + \frac{1}{n^2}\ln n. \end{aligned}$$

By comparing to an appropriate integral and showing that the intergal converges, show that $\{S_n\}$ converges. Show that $\{P_n\}$ converges to a number between 2.33 and 2.39. Use a CAS or calculator to compute P_n for large n and see how accurate the computation is.

4. Define a function $f(x)$ in the following way for $0 \le x \le 1$. Write out the binary expansion of x. That is,

$$x = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots$$

where each a_i is either 0 or 1. Prove that this infinite series converges. Then $f(x)$ is the corresponding ternary expansion, given by

$$f(x) = \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} + \cdots$$

Prove that this series converges. There is a subtle issue here of whether the function is well defined or not. Show that $\frac{1}{2}$ can be written with $a_1 = 1$ and $a_k = 0$ for $k \ge 2$ and also with $a_1 = 0$ and $a_k = 1$ for $k \ge 2$. Show that you get different values of $f(x)$ with different representations. In such cases, we choose the representation with as few 1's as possible. Show that $f(2x) = 3f(x)$ and $f\left(x + \frac{1}{2}\right) = \frac{1}{3} + f(x)$ for $0 \le x \le \frac{1}{2}$. Use these facts to compute $\int_0^1 f(x)\,dx$. Generalize the result for any base n conversion

$$f(x) = \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots,$$

where n is an integer greater than 1.

8.4 ALTERNATING SERIES

So far, we have focused our attention on positive-term series, that is, series all of whose terms are positive. Before we consider the general case, we spend some time in this section examining *alternating series,* that is, series whose terms alternate back and forth from positive to negative.

An **alternating series** is any series of the form

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots,$$

where $a_k > 0$, for all k.

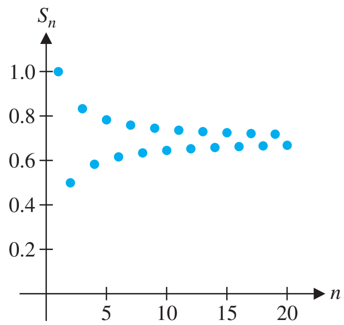


FIGURE 8.30
$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$

EXAMPLE 4.1 The Alternating Harmonic Series

Investigate the convergence or divergence of the **alternating harmonic series**

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Solution The graph of the first 20 partial sums seen in Figure 8.30 suggests that the series might converge to about 0.7. We now calculate the first few partial sums by hand. Note that

$$S_1 = 1, \qquad S_2 = 1 - \frac{1}{2} = \frac{1}{2},$$

$$S_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \qquad S_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12},$$

$$S_5 = \frac{7}{12} + \frac{1}{5} = \frac{47}{60}, \qquad S_6 = \frac{47}{60} - \frac{1}{6} = \frac{37}{60},$$

and so on. We have plotted the first 8 partial sums on the number line shown in Figure 8.31.

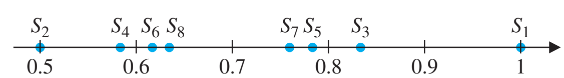


FIGURE 8.31
Partial sums of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$
1	1
2	0.5
3	0.83333
4	0.58333
5	0.78333
6	0.61667
7	0.75952
8	0.63452
9	0.74563
10	0.64563
11	0.73654
12	0.65321
13	0.73013
14	0.65871
15	0.72537
16	0.66287
17	0.7217
18	0.66614
19	0.71877
20	0.66877

Notice that the partial sums are bouncing back and forth, but seem to be zeroing in on some value. This should not be surprising, since each new term that is added or subtracted is less than the term added or subtracted to get the previous partial sum. You should notice this same zeroing-in process in the accompanying table displaying the first 20 partial sums of the series. Based on the behavior of the partial sums, it is reasonable to conjecture that the series converges to some value between 0.66877 and 0.71877. We can resolve the question of convergence definitively with Theorem 4.1. ■

THEOREM 4.1 (Alternating Series Test)

Suppose that $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \le a_k$ for all $k \ge 1$. Then, the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

Before considering the proof of Theorem 4.1, make sure that you have a clear idea what it is saying. In the case of an alternating series satisfying the hypotheses of the theorem, we start with 0 and add $a_1 > 0$ to get the first partial sum S_1. To get the next partial sum, S_2, we subtract a_2 from S_1, where $a_2 < a_1$. This says that S_2 will be between 0 and S_1. We illustrate this situation in Figure 8.32 on the following page.

Continuing in this fashion, we add a_3 to S_2 to get S_3. Since $a_3 < a_2$, we must have that $S_2 < S_3 < S_1$. Referring to Figure 8.32, notice that

$$S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1.$$

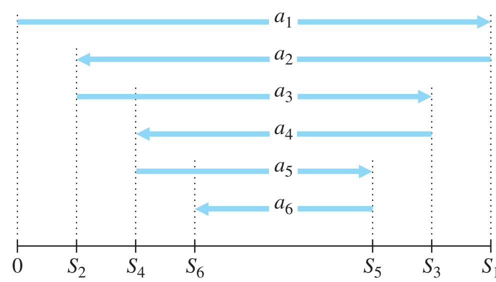


FIGURE 8.32
Convergence of the partial sums of an alternating series

In particular, this says that *all* of the odd-indexed partial sums (i.e, S_{2n+1}, for $n = 0, 1, 2, \ldots$) are larger than *all* of the even-indexed partial sums (i.e., S_{2n}, for $n = 1, 2, \ldots$). As the partial sums oscillate back and forth, they should be drawing closer and closer to some limit S, somewhere between all of the even-indexed partial sums and the odd-indexed partial sums,

$$S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1. \tag{4.1}$$

PROOF

Notice from Figure 8.32 that the even- and odd-indexed partial sums seem to behave somewhat differently. First, we consider the even-indexed partial sums. We have

$$S_2 = a_1 - a_2 > 0$$

and

$$S_4 = S_2 + (a_3 - a_4) \geq S_2,$$

since $(a_3 - a_4) \geq 0$. Likewise, for any n, we can write

$$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) \geq S_{2n-2},$$

since $(a_{2n-1} - a_{2n}) \geq 0$. This says that the sequence of even-indexed partial sums $\{S_{2n}\}_{n=1}^{\infty}$ is increasing (as we saw in Figure 8.32). Further, observe that

$$0 < S_{2n} = a_1 + (-a_2 + a_3) + (-a_4 + a_5) + \cdots + (-a_{2n-2} + a_{2n-1}) - a_{2n} \leq a_1,$$

for all n, since every term in parentheses is negative. Thus, $\{S_{2n}\}_{n=1}^{\infty}$ is both bounded (by a_1) and monotonic (increasing). By Theorem 1.4, $\{S_{2n}\}_{n=1}^{\infty}$ must be convergent to some number, say L.

Turning to the sequence of odd-indexed partial sums, notice that we have

$$S_{2n+1} = S_{2n} + a_{2n+1}.$$

From this, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L + 0 = L,$$

since $\lim_{n\to\infty} a_n = 0$. Since both the sequence of odd-indexed partial sums $\{S_{2n+1}\}_{n=0}^{\infty}$ and the sequence of even-indexed partial sums $\{S_{2n}\}_{n=1}^{\infty}$ converge to the same limit, L, we have that

$$\lim_{n\to\infty} S_n = L,$$

also. ■

EXAMPLE 4.2 Using the Alternating Series Test

Reconsider the convergence of the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

Solution Notice that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Further,
$$0 < a_{k+1} = \frac{1}{k+1} \le \frac{1}{k} = a_k, \quad \text{for all } k \ge 1.$$

By the Alternating Series Test, the series converges. (The calculations from example 4.1 give an approximate sum. An exact sum is found in exercise 43.) ■

The Alternating Series Test is straightforward, but you will sometimes need to work a bit to verify the hypotheses.

EXAMPLE 4.3 Using the Alternating Series Test

Investigate the convergence or divergence of the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k(k+3)}{k(k+1)}$.

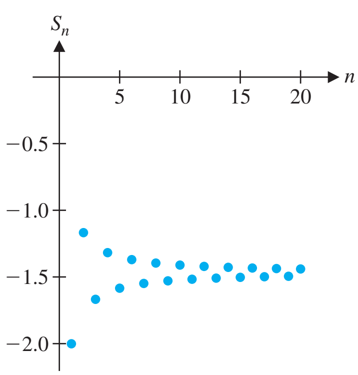


FIGURE 8.33
$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$

Solution The graph of the first 20 partial sums seen in Figure 8.33 suggests that the series converges to some value around -1.5. The following table showing some select partial sums suggests the same conclusion.

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$
50	−1.45545
100	−1.46066
200	−1.46322
300	−1.46406
400	−1.46448

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$
51	−1.47581
101	−1.47076
201	−1.46824
301	−1.46741
401	−1.46699

We can verify that the series converges by first checking that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{(k+3)}{k(k+1)} \frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\frac{1}{k} + \frac{3}{k^2}}{1 + \frac{1}{k}} = 0.$$

Next, consider the ratio of two consecutive terms:

$$\frac{a_{k+1}}{a_k} = \frac{(k+4)}{(k+1)(k+2)} \frac{k(k+1)}{(k+3)} = \frac{k^2+4k}{k^2+5k+6} < 1,$$

for all $k \ge 1$. From this, it follows that $a_{k+1} < a_k$, for all $k \ge 1$ and so, by the Alternating Series Test, the series converges. Finally, from the preceding table, we can see that the series converges to a sum between -1.46448 and -1.46699. (How can you be sure that the sum is in this interval?) ■

EXAMPLE 4.4 A Divergent Alternating Series

Determine whether the alternating series $\sum_{k=3}^{\infty} \frac{(-1)^k k}{k+2}$ converges or diverges.

Solution First, notice that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+2} = 1 \neq 0.$$

So, this alternating series is divergent, since by the kth-term test for divergence, the terms must tend to zero in order for the series to be convergent. ■

Estimating the Sum of an Alternating Series

So far, we have calculated approximate sums of series by observing that a number of successive partial sums of the series are within a given distance of one another. The underlying assumption here is that when this happens, the partial sums are also within that same distance of the sum of the series. While this is not true in general, we can say something very precise for the case of alternating series. First, note that the error in approximating the sum S by the nth partial sum S_n is $S - S_n$.

Look back at Figure 8.32 and observe that all of the even-indexed partial sums S_n of the convergent alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ lie below the sum S, while all of the odd-indexed partial sums lie above S. That is [as in (4.1)],

$$S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1.$$

This says that for n even, $$S_n \leq S \leq S_{n+1}.$$

Subtracting S_n from all terms, we get

$$0 \leq S - S_n \leq S_{n+1} - S_n = a_{n+1}.$$

Since $a_{n+1} > 0$, we have $$-a_{n+1} \leq 0 \leq S - S_n \leq a_{n+1},$$

or $$|S - S_n| \leq a_{n+1}, \quad \text{for } n \text{ even.} \tag{4.2}$$

Similarly, for n odd, we have that $$S_{n+1} \leq S \leq S_n.$$

Again subtracting S_n, we get

$$-a_{n+1} = S_{n+1} - S_n \leq S - S_n \leq 0 \leq a_{n+1}$$

or $$|S - S_n| \leq a_{n+1}, \quad \text{for } n \text{ odd.} \tag{4.3}$$

Since (4.2) and (4.3) (these are called **error bounds**) are the same, we have the same error bound whether n is even or odd. This establishes the following result.

THEOREM 4.2

Suppose that $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$ for all $k \geq 1$. Then, the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges to some number S and the error in approximating S by the nth partial sum S_n satisfies

$$|S - S_n| \leq a_{n+1}. \tag{4.4}$$

Theorem 4.2 says that the absolute value of the error in approximating S by S_n does not exceed a_{n+1} (the absolute value of the first neglected term).

EXAMPLE 4.5 Estimating the Sum of an Alternating Series

Approximate the sum of the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$ by the 40th partial sum and estimate the error in this approximation.

Solution We leave it as an exercise to show that this series is convergent. We then approximate the sum by

$$S \approx S_{40} = 0.9470326439.$$

From our error estimate (4.4), we have

$$|S - S_{40}| \leq a_{41} = \frac{1}{41^4} \approx 3.54 \times 10^{-7}.$$

This says that our approximation $S \approx 0.9470326439$ is off by no more than $\pm 3.54 \times 10^{-7}$. ■

A much more interesting question than the one asked in example 4.5 is the following. For a given convergent alternating series, how many terms must we take, in order to guarantee that our approximation is accurate to a given level? We use the same estimate of error from (4.4) to answer this question, as in example 4.6.

EXAMPLE 4.6 Finding the Number of Terms Needed for a Given Accuracy

For the convergent alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$, how many terms are needed to guarantee that S_n is within 1×10^{-10} of the actual sum S?

Solution In this case, we want to find the number of terms n for which

$$|S - S_n| \leq 1 \times 10^{-10}.$$

From (4.4), we have that $$|S - S_n| \leq a_{n+1} = \frac{1}{(n+1)^4}.$$

So, we look for n such that $$\frac{1}{(n+1)^4} \leq 1 \times 10^{-10}.$$

Solving for n, we get $$10^{10} \leq (n+1)^4,$$

so that $$\sqrt[4]{10^{10}} \leq n + 1$$

or $$n \geq \sqrt[4]{10^{10}} - 1 \approx 315.2.$$

So, if we take $n \geq 316$, we will guarantee an error of no more than 1×10^{-10}. Using the suggested number of terms, we get the approximate sum

$$S \approx S_{316} = 0.947032829447,$$

which we now know to be correct to within 1×10^{-10}. ■

BEYOND FORMULAS

When you think about infinite series, you must understand the interplay between sequences and series. Our tests for convergence involve sequences and are completely separate from the question of finding the *sum* of the series. It is important to keep reminding yourself that the sum of a convergent series is the limit of the sequence of partial sums. Often, the best we can do is approximate the sum of a series by adding together a number of terms. In this case, it becomes important to determine the accuracy of the approximation. For alternating series, this is found by examining the first neglected term. When finding an approximation with a specified accuracy, you first use the error bound in Theorem 4.2 to find how many terms you need to add. You then get an approximation with the desired accuracy by adding together that many terms.

EXERCISES 8.4

WRITING EXERCISES

1. If $a_k \geq 0$, explain in terms of partial sums why $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ is more likely to converge than $\sum_{k=1}^{\infty} a_k$.
2. Explain why in Theorem 4.2 we need the assumption that $a_{k+1} \leq a_k$. That is, what would go wrong with the proof if $a_{k+1} > a_k$?
3. The Alternating Series Test was stated for the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$. Explain the difference between $\sum_{k=1}^{\infty}(-1)^k a_k$ and $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ and explain why we could have stated the theorem for $\sum_{k=1}^{\infty}(-1)^k a_k$.
4. A common mistake is to think that if $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges. Explain why this is not true for positive-term series. This is also not true for alternating series *unless* you add one more hypothesis. State the extra hypothesis and explain why it's needed.

In exercises 1–24, determine whether the series is convergent or divergent.

1. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3}{k}$
2. $\sum_{k=1}^{\infty}(-1)^k \frac{2}{k^2}$
3. $\sum_{k=1}^{\infty}(-1)^k \frac{4}{\sqrt{k}}$
4. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{k+1}$
5. $\sum_{k=2}^{\infty}(-1)^k \frac{k}{k^2+2}$
6. $\sum_{k=7}^{\infty}(-1)^k \frac{2k-1}{k^3}$
7. $\sum_{k=5}^{\infty}(-1)^{k+1}\frac{k}{2^k}$
8. $\sum_{k=4}^{\infty}(-1)^{k+1}\frac{3^k}{k}$
9. $\sum_{k=1}^{\infty}(-1)^k \frac{4^k}{k^2}$
10. $\sum_{k=1}^{\infty}(-1)^k \frac{k+2}{4^k}$
11. $\sum_{k=1}^{\infty}\frac{2k}{k+1}$
12. $\sum_{k=1}^{\infty}\frac{4k^2}{k^2+2k+2}$
13. $\sum_{k=3}^{\infty}(-1)^k \frac{3}{\sqrt{k+1}}$
14. $\sum_{k=4}^{\infty}(-1)^k \frac{k+1}{k^3}$
15. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k!}$
16. $\sum_{k=3}^{\infty}(-1)^{k+1}\frac{k!}{3^k}$
17. $\sum_{k=2}^{\infty}(-1)^k \frac{k!}{2^k}$
18. $\sum_{k=3}^{\infty}(-1)^k \frac{4^k}{k!}$
19. $\sum_{k=5}^{\infty}(-1)^{k+1}2e^{-k}$
20. $\sum_{k=6}^{\infty}(-1)^{k+1}3e^{1/k}$
21. $\sum_{k=2}^{\infty}(-1)^k \ln k$
22. $\sum_{k=2}^{\infty}(-1)^k \frac{1}{\ln k}$
23. $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{2^k}$
24. $\sum_{k=0}^{\infty}(-1)^{k+1}2^k$

In exercises 25–32, estimate the sum of each convergent series to within 0.01.

25. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k^3}$
26. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k^3}$
27. $\sum_{k=3}^{\infty}(-1)^k \frac{k}{2^k}$
28. $\sum_{k=4}^{\infty}(-1)^k \frac{k^2}{10^k}$

29. $\sum_{k=0}^{\infty}(-1)^k \frac{3}{k!}$

30. $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{2}{k!}$

31. $\sum_{k=2}^{\infty}(-1)^{k+1} \frac{4}{k^4}$

32. $\sum_{k=3}^{\infty}(-1)^{k+1} \frac{3}{k^5}$

In exercises 33–36, determine how many terms are needed to estimate the sum of the series to within 0.0001.

33. $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{2}{k}$

34. $\sum_{k=0}^{\infty}(-1)^k \frac{2^k}{k!}$

35. $\sum_{k=0}^{\infty}(-1)^k \frac{10^k}{k!}$

36. $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k!}{k^k}$

37. In the text, we showed you one way to verify that a sequence is decreasing. As an alternative, explain why if $a_k = f(k)$ and $f'(k) < 0$, then the sequence a_k is decreasing. Use this method to prove that $a_k = \frac{k}{k^2+2}$ is decreasing.

38. Use the method of exercise 37 to prove that $a_k = \frac{k}{2^k}$ is decreasing.

39. In this exercise, you will discover why the Alternating Series Test requires that $a_{k+1} \le a_k$. If $a_k = \begin{cases} 1/k & \text{if } k \text{ is odd} \\ 1/k^2 & \text{if } k \text{ is even} \end{cases}$, argue that $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ diverges to ∞. Thus, an alternating series can diverge even if $\lim_{k\to\infty} a_k = 0$.

40. Verify that the series $\sum_{k=0}^{\infty}(-1)^k \frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ converges. It can be shown that the sum of this series is $\frac{\pi}{4}$. Given this result, we could use this series to obtain an approximation of π. How many terms would be necessary to get eight digits of π correct?

41. A person starts walking from home (at $x = 0$) toward a friend's house (at $x = 1$). Three-fourths of the way there, he changes his mind and starts walking back home. Three-fourths of the way home, he changes his mind again and starts walking back to his friend's house. If he continues this pattern of indecision, always turning around at the three-fourths mark, what will be the eventual outcome? A similar problem appeared in a national magazine and created a minor controversy due to the ambiguous wording of the problem. It is clear that the first turnaround is at $x = \frac{3}{4}$ and the second turnaround is at $\frac{3}{4} - \frac{3}{4}\left(\frac{3}{4}\right) = \frac{3}{16}$. But is the third turnaround three-fourths of the way to $x = 1$ or $x = \frac{3}{4}$? The magazine writer assumed the latter. Show that with this assumption, the person's location forms a geometric series. Find the sum of the series to find where the person ends up.

42. If the problem of exercise 41 is interpreted differently, a more interesting answer results. As before, let $x_1 = \frac{3}{4}$ and $x_2 = \frac{3}{16}$. If the next turnaround is three-fourths of the way from x_2 to 1, then $x_3 = \frac{3}{16} + \frac{3}{4}\left(1 - \frac{3}{16}\right) = \frac{3}{4} + \frac{1}{4}x_2 = \frac{51}{64}$. Three-fourths of the way back to $x = 0$ would put us at $x_4 = x_3 - \frac{3}{4}x_3 = \frac{1}{4}x_3 = \frac{51}{256}$. Show that if n is even, then $x_{n+1} = \frac{3}{4} + \frac{1}{4}x_n$ and $x_{n+2} = \frac{1}{4}x_{n+1}$. Show that the person ends up walking back and forth between two specific locations.

43. For the alternating harmonic series, show that $S_{2n} = \sum_{k=1}^{2n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}\frac{1}{n+k} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k/n}$. Identify this as a Riemann sum and show that the alternating harmonic series converges to $\ln 2$.

44. Find all values of p such that the series $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k^p}$ converges. Compare your result to the p-series of section 8.3.

45. Find a counterexample to show that the following statement is false (not always true). If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

46. Find assumptions that can be made (for example, $a_k > 0$) that make the statement in exercise 45 true.

EXPLORATORY EXERCISES

1. In this exercise, you will determine whether or not the improper integral $\int_0^1 \sin(1/x)\,dx$ converges. Argue that $\int_{1/\pi}^1 \sin(1/x)\,dx$, $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx$, $\int_{(1/3\pi)}^{1/(2\pi)} \sin(1/x)\,dx, \ldots$ exist and that (if it exists),

$$\int_0^1 \sin(1/x)\,dx = \int_{1/\pi}^1 \sin(1/x)\,dx + \int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx + \int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\,dx + \cdots.$$

Verify that the series is an alternating series and show that the hypotheses of the Alternating Series Test are met. Thus, the series and the improper integral both converge.

2. Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$, where x is a constant. Show that the series converges for $x = 1/2$; $x = -1/2$; any x such that $-1 < x \le 1$. Show that the series diverges if $x = -1$, $x < -1$ or $x > 1$. We see in exercise 5 of section 8.7 that when the series converges, it converges to $\ln(1 + x)$. Verify this numerically for $x = 1/2$ and $x = -1/2$.

8.5 ABSOLUTE CONVERGENCE AND THE RATIO TEST

Outside of the Alternating Series Test presented in section 8.4, our other tests for convergence of series (i.e., the Integral Test and the two comparison tests) apply only to series all of whose terms are *positive*. So, what do we do if we're faced with a series that has both positive and negative terms, but that is not an alternating series? For instance, look at the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^3} = \sin 1 + \frac{1}{8}\sin 2 + \frac{1}{27}\sin 3 + \frac{1}{64}\sin 4 + \cdots.$$

This has both positive and negative terms, but the terms do not alternate signs. (Calculate the first five or six terms of the series to see this for yourself.) For any such series $\sum_{k=1}^{\infty} a_k$, we can get around this problem by checking whether the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is convergent. When this happens, we say that the original series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** (or **converges absolutely**). You should note that to test the convergence of the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ (all of whose terms are positive), we have all of our earlier tests for positive-term series available to us.

EXAMPLE 5.1 Testing for Absolute Convergence

Determine whether $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$ is absolutely convergent.

Solution It is easy to show that this alternating series is convergent. (Try it!) The graph of the first 20 partial sums in Figure 8.34 suggests that the series converges to approximately 0.35. To determine absolute convergence, we need to determine whether or not the series of absolute values, $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{2^k} \right|$, is convergent. We have

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{2^k} \right| = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k,$$

which you should recognize as a convergent geometric series ($|r| = \frac{1}{2} < 1$). This says that the original series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$ converges absolutely. ■

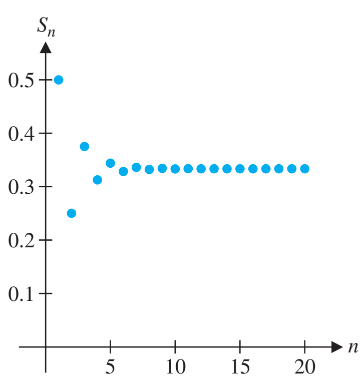


FIGURE 8.34

$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2^k}$

We'll prove shortly that every absolutely convergent series is also convergent (as in example 5.1). However, the reverse is not true; there are many series that are convergent, but not absolutely convergent. These are called **conditionally convergent** series. Can you think of an example of such a series? If so, it's probably the example that follows.

EXAMPLE 5.2 A Conditionally Convergent Series

Determine whether the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is absolutely convergent.

Solution In example 4.2, we showed that this series is convergent. To test this for absolute convergence, we consider the series of absolute values,

$$\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty}\frac{1}{k}$$

(the harmonic series), which diverges. This says that $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ converges conditionally (i.e., it converges, but does not converge absolutely). ■

THEOREM 5.1

If $\sum_{k=1}^{\infty}|a_k|$ converges, then $\sum_{k=1}^{\infty}a_k$ converges.

This result says that if a series converges absolutely, then it must also converge. Because of this, when we test series, we first test for absolute convergence. If the series converges absolutely, then we need not test any further to establish convergence.

PROOF

Notice that for any real number, x, we can say that $-|x| \le x \le |x|$. So, for any k, we have

$$-|a_k| \le a_k \le |a_k|.$$

Adding $|a_k|$ to all the terms, we get

$$0 \le a_k + |a_k| \le 2|a_k|. \tag{5.1}$$

Since $\sum_{k=1}^{\infty}a_k$ is absolutely convergent, we have that $\sum_{k=1}^{\infty}|a_k|$ and hence, also $\sum_{k=1}^{\infty}2|a_k| = 2\sum_{k=1}^{\infty}|a_k|$ is convergent. Define $b_k = a_k + |a_k|$. From (5.1),

$$0 \le b_k \le 2|a_k|$$

and so, by the Comparison Test, $\sum_{k=1}^{\infty}b_k$ is convergent. Observe that we may write

$$\begin{aligned}\sum_{k=1}^{\infty}a_k &= \sum_{k=1}^{\infty}(a_k + |a_k| - |a_k|) = \sum_{k=1}^{\infty}\underbrace{(a_k + |a_k|)}_{b_k} - \sum_{k=1}^{\infty}|a_k| \\ &= \sum_{k=1}^{\infty}b_k - \sum_{k=1}^{\infty}|a_k|.\end{aligned}$$

Since the two series on the right-hand side are convergent, it follows that $\sum_{k=1}^{\infty}a_k$ must also be convergent. ■

EXAMPLE 5.3 Testing for Absolute Convergence

Determine whether $\sum_{k=1}^{\infty}\frac{\sin k}{k^3}$ is convergent or divergent.

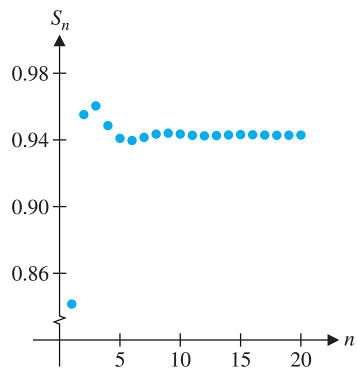


FIGURE 8.35

$S_n = \sum_{k=1}^{n} \frac{\sin k}{k^3}$

Solution Notice that while this is not a positive-term series, neither is it an alternating series. Because of this, our only choice is to test the series for absolute convergence. From the graph of the first 20 partial sums seen in Figure 8.35, it appears that the series is converging to some value around 0.94. To test for absolute convergence, we consider the series of absolute values, $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^3}\right|$. Notice that

$$\left|\frac{\sin k}{k^3}\right| = \frac{|\sin k|}{k^3} \le \frac{1}{k^3}, \tag{5.2}$$

since $|\sin k| \le 1$, for all k. Of course, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$). By the Comparison Test and (5.2), $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^3}\right|$ converges, too. Consequently, the original series $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ converges absolutely and hence, converges. ■

The Ratio Test

We next introduce a very powerful tool for testing a series for absolute convergence. This test can be applied to a wide range of series, including the extremely important case of power series that we discuss in section 8.6. As you'll see, this test is remarkably easy to use.

THEOREM 5.2 (Ratio Test)

Given $\sum_{k=1}^{\infty} a_k$, with $a_k \ne 0$ for all k, suppose that

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L.$$

Then,

(i) if $L < 1$, the series converges absolutely,
(ii) if $L > 1$ (or $L = \infty$), the series diverges and
(iii) if $L = 1$, there is no conclusion.

PROOF

(i) For $L < 1$, pick any number r with $L < r < 1$. Then, we have

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L < r.$$

For this to occur, there must be some number $N > 0$, such that for $k \ge N$,

$$\left|\frac{a_{k+1}}{a_k}\right| < r. \tag{5.3}$$

Multiplying both sides of (5.3) by $|a_k|$ gives us

$$|a_{k+1}| < r|a_k|.$$

In particular, taking $k = N$ gives us

$$|a_{N+1}| < r|a_N|$$

and taking $k = N + 1$ gives us

$$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|.$$

Likewise, $$|a_{N+3}| < r|a_{N+2}| < r^3|a_N|$$

and so on. We have $$|a_{N+k}| < r^k|a_N|, \quad \text{for } k = 1, 2, 3, \ldots .$$

Notice that $\sum_{k=1}^{\infty} |a_N|r^k = |a_N| \sum_{k=1}^{\infty} r^k$ is a convergent geometric series, since $0 < r < 1$. By the Comparison Test, it follows that $\sum_{k=1}^{\infty} |a_{N+k}| = \sum_{n=N+1}^{\infty} |a_n|$ converges, too. This says that $\sum_{n=N+1}^{\infty} a_n$ converges absolutely. Finally, since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n,$$

we also get that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) For $L > 1$, we have

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L > 1.$$

This says that there must be some number $N > 0$, such that for $k \geq N$,

$$\left| \frac{a_{k+1}}{a_k} \right| > 1. \tag{5.4}$$

Multiplying both sides of (5.4) by $|a_k|$, we get

$$|a_{k+1}| > |a_k| > 0, \quad \text{for all } k \geq N.$$

Notice that if this is the case, then

$$\lim_{k \to \infty} a_k \neq 0.$$

By the kth-term test for divergence, we now have that $\sum_{k=1}^{\infty} a_k$ diverges. ■

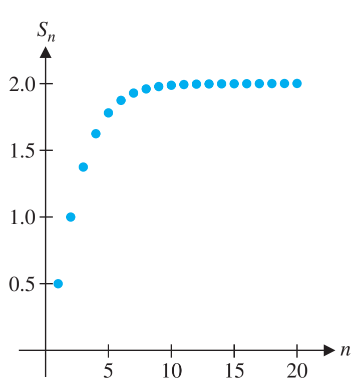


FIGURE 8.36
$S_n = \sum_{k=1}^{n} \frac{k}{2^k}$

EXAMPLE 5.4 Using the Ratio Test

Test $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2^k}$ for convergence.

Solution The graph of the first 20 partial sums of the series of absolute values, $\sum_{k=1}^{\infty} \frac{k}{2^k}$, seen in Figure 8.36, suggests that the series of absolute values converges to about 2. From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\dfrac{k+1}{2^{k+1}}}{\dfrac{k}{2^k}} = \lim_{k\to\infty}\frac{k+1}{2^{k+1}}\frac{2^k}{k} = \frac{1}{2}\lim_{k\to\infty}\frac{k+1}{k} = \frac{1}{2} < 1$$

Since $2^{k+1} = 2^k \cdot 2^1$.

and so, the series converges absolutely, as expected from Figure 8.36. ■

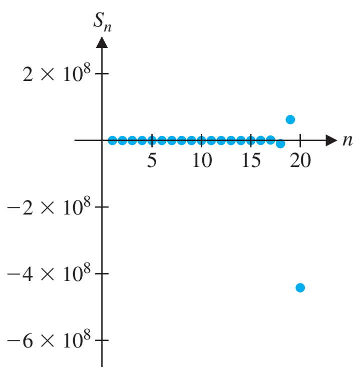


FIGURE 8.37

$S_n = \sum_{k=0}^{n-1}\frac{(-1)^k k!}{e^k}$

The Ratio Test is particularly useful when the general term of a series contains an exponential term, as in example 5.4, or a factorial, as in example 5.5.

EXAMPLE 5.5 Using the Ratio Test

Test $\sum_{k=0}^{\infty}\frac{(-1)^k k!}{e^k}$ for convergence.

Solution The graph of the first 20 partial sums of the series seen in Figure 8.37 suggests that the series diverges. We can confirm this suspicion with the Ratio Test. We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\dfrac{(k+1)!}{e^{k+1}}}{\dfrac{k!}{e^k}} = \lim_{k\to\infty}\frac{(k+1)!}{e^{k+1}}\frac{e^k}{k!}$$

$$= \lim_{k\to\infty}\frac{(k+1)k!}{e\,k!} = \frac{1}{e}\lim_{k\to\infty}\frac{k+1}{1} = \infty.$$

Since $(k+1)! = (k+1)\cdot k!$ and $e^{k+1} = e^k \cdot e^1$.

By the Ratio Test, the series diverges, as we suspected. ■

HISTORICAL NOTES

Srinivasa Ramanujan (1887–1920) Indian mathematician whose incredible discoveries about infinite series still amaze mathematicians. Largely self-taught, Ramanujan filled notebooks with conjectures about series, continued fractions and the Riemann-zeta function. Ramanujan rarely gave a proof or even justification of his results. Nevertheless, the famous English mathematician G. H. Hardy said, "They must be true because, if they weren't true, no one would have had the imagination to invent them." (See exercise 39.)

Recall that in the statement of the Ratio Test, we said that if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1,$$

then the Ratio Test yields no conclusion. By this, we mean that in such cases, the series may or may not converge and further testing is required.

EXAMPLE 5.6 A Divergent Series for Which the Ratio Test Fails

Use the Ratio Test for the harmonic series $\sum_{k=0}^{\infty}\frac{1}{k}$.

Solution We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\dfrac{1}{k+1}}{\dfrac{1}{k}} = \lim_{k\to\infty}\frac{k}{k+1} = 1.$$

In this case, the Ratio Test yields no conclusion, although we already know that the harmonic series diverges. ■

EXAMPLE 5.7 A Convergent Series for Which the Ratio Test Fails

Use the Ratio Test to test the series $\sum_{k=0}^{\infty}\frac{1}{k^2}$.

TODAY IN MATHEMATICS

Alain Connes (1947–)
A French mathematician who earned a Fields Medal in 1983 for his spectacular results in the classification of operator algebras. As a student, Connes developed a very personal understanding of mathematics. He has explained, "I first began to work in a very small place in the mathematical geography . . . . I had my own system, which was very strange because when the problems the teacher was asking fell into my system, then of course I would have an instant answer, but when they didn't—and many problems, of course, didn't fall into my system—then I would be like an idiot and I wouldn't care." As Connes' personal mathematical system expanded, he found more and more "instant answers" to important problems.

Solution Here, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1}{(k+1)^2}\frac{k^2}{1} = \lim_{k\to\infty}\frac{k^2}{k^2+2k+1} = 1.$$

So again, the Ratio Test yields no conclusion, although we already know that this is a convergent p-series. ■

Carefully examine examples 5.6 and 5.7, and you should recognize that the Ratio Test will be inconclusive for any p-series.

The Root Test

We now present one final test for convergence of series.

THEOREM 5.3 (Root Test)

Given $\sum_{k=1}^{\infty} a_k$, suppose that $\lim_{k\to\infty}\sqrt[k]{|a_k|} = L$. Then,

(i) if $L < 1$, the series converges absolutely,
(ii) if $L > 1$ (or $L = \infty$), the series diverges and
(iii) if $L = 1$, there is no conclusion.

Notice how similar the conclusion is to the conclusion of the Ratio Test. The proof is also similar to that of the Ratio Test and we leave this as an exercise.

EXAMPLE 5.8 Using the Root Test

Use the Root Test to determine the convergence or divergence of the series $\sum_{k=1}^{\infty}\left(\frac{2k+4}{5k-1}\right)^k$.

Solution In this case, we consider

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\left|\frac{2k+4}{5k-1}\right|^k} = \lim_{k\to\infty}\frac{2k+4}{5k-1} = \frac{2}{5} < 1.$$

By the Root Test, the series is absolutely convergent. ■

Summary of Convergence Tests

By this point in your study of series, it may seem as if we have thrown at you a dizzying array of different series and tests for convergence or divergence. Just how are you to keep all of these straight? The only suggestion we have is that you work through *many* problems. We provide a good assortment in the exercise set that follows this section. Some of these require the methods of this section, while others are drawn from the preceding sections (just to keep you thinking about the big picture). For the sake of convenience, we summarize our convergence tests in the table that follows.

Test	*When to Use*	*Conclusions*	*Section*
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	Converges to $\dfrac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.	8.2
***k*th-Term Test**	All series	If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.	8.2
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$, f is continuous and decreasing and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ *both* converge or *both* diverge.	8.3
***p*-series**	$\sum_{k=1}^{\infty} \dfrac{1}{k^p}$	Converges for $p > 1$; diverges for $p \leq 1$.	8.3
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.	8.3
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \dfrac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or *both* diverge.	8.3
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.	8.4
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.	8.5
Ratio Test	Any series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert \dfrac{a_{k+1}}{a_k} \right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.	8.5
Root Test	Any series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.	8.5

EXERCISES 8.5

WRITING EXERCISES

1. Suppose that two series have identical terms except that in series A all terms are positive and in series B some terms are positive and some terms are negative. Explain why series B is more likely to converge. In light of this, explain why Theorem 5.1 is true.

2. In the Ratio Test, if $\lim_{k\to\infty} \left\lvert \dfrac{a_{k+1}}{a_k} \right\rvert > 1$, which is bigger, $\lvert a_{k+1}\rvert$ or $\lvert a_k\rvert$? Explain why this implies that the series $\sum_{k=1}^{\infty} a_k$ diverges.

3. In the Ratio Test, if $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L < 1$, which is bigger, $|a_{k+1}|$ or $|a_k|$? This inequality could also hold if $L = 1$. Compare the relative sizes of $|a_{k+1}|$ and $|a_k|$ if $L = 0.8$ versus $L = 1$. Explain why $L = 0.8$ would be more likely to correspond to a convergent series than $L = 1$.

4. In many series of interest, the terms of the series involve powers of k (e.g., k^2), exponentials (e.g., 2^k) or factorials (e.g., $k!$). For which type(s) of terms is the Ratio Test likely to produce a result (i.e., a limit different from 1)? Briefly explain.

In exercises 1–38, determine whether the series is absolutely convergent, conditionally convergent or divergent.

1. $\sum_{k=0}^{\infty}(-1)^k\frac{3}{k!}$

2. $\sum_{k=0}^{\infty}(-1)^k\frac{6}{k!}$

3. $\sum_{k=0}^{\infty}(-1)^k 2^k$

4. $\sum_{k=0}^{\infty}(-1)^k\frac{2}{3^k}$

5. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{k^2+1}$

6. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2+1}{k}$

7. $\sum_{k=3}^{\infty}(-1)^k\frac{3^k}{k!}$

8. $\sum_{k=4}^{\infty}(-1)^k\frac{10^k}{k!}$

9. $\sum_{k=2}^{\infty}(-1)^{k+1}\frac{k}{2k+1}$

10. $\sum_{k=3}^{\infty}(-1)^{k+1}\frac{4}{2k+1}$

11. $\sum_{k=6}^{\infty}(-1)^k\frac{k2^k}{3^k}$

12. $\sum_{k=1}^{\infty}(-1)^k\frac{k^2 3^k}{2^k}$

13. $\sum_{k=1}^{\infty}\left(\frac{4k}{5k+1}\right)^k$

14. $\sum_{k=5}^{\infty}\left(\frac{1-3k}{4k}\right)^k$

15. $\sum_{k=1}^{\infty}\frac{-2}{k}$

16. $\sum_{k=1}^{\infty}\frac{4}{k}$

17. $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\sqrt{k}}{k+1}$

18. $\sum_{k=2}^{\infty}(-1)^{k+1}\frac{k}{k^3+1}$

19. $\sum_{k=7}^{\infty}\frac{k^2}{e^k}$

20. $\sum_{k=1}^{\infty}k^3 e^{-k}$

21. $\sum_{k=2}^{\infty}\frac{e^{3k}}{k^{3k}}$

22. $\sum_{k=4}^{\infty}\left(\frac{e^k}{k^2}\right)^k$

23. $\sum_{k=1}^{\infty}\frac{\sin k}{k^2}$

24. $\sum_{k=1}^{\infty}\frac{\cos k}{k^3}$

25. $\sum_{k=1}^{\infty}\frac{\cos k\pi}{k}$

26. $\sum_{k=1}^{\infty}\frac{\sin k\pi}{k}$

27. $\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln k}$

28. $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$

29. $\sum_{k=1}^{\infty}\frac{(-1)^k}{k\sqrt{k}}$

30. $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$

31. $\sum_{k=3}^{\infty}\frac{3}{k^k}$

32. $\sum_{k=8}^{\infty}\frac{2k}{3^k}$

33. $\sum_{k=6}^{\infty}(-1)^{k+1}\frac{k!}{4^k}$

34. $\sum_{k=4}^{\infty}(-1)^{k+1}\frac{k^2 4^k}{k!}$

35. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^{10}}{(2k)!}$

36. $\sum_{k=0}^{\infty}(-1)^k\frac{4^k}{(2k+1)!}$

37. $\sum_{k=0}^{\infty}\frac{(-2)^k(k+1)}{5^k}$

38. $\sum_{k=1}^{\infty}\frac{(-3)^k}{k^2 4^k}$

39. In the 1910s, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26{,}390k)}{(k!)^4 396^{4k}}.$$

Approximate the series with only the $k = 0$ term and show that you get 6 digits of π correct. Approximate the series using the $k = 0$ and $k = 1$ terms and show that you get 14 digits of π correct. In general, each term of this remarkable series increases the accuracy by 8 digits.

40. Prove that Ramanujan's series in exercise 39 converges.

41. To show that $\sum_{k=1}^{\infty}\frac{k!}{k^k}$ converges, use the Ratio Test and the fact that

$$\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^k=\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k=e.$$

42. Determine whether $\sum_{k=1}^{\infty}\frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$ converges or diverges.

43. Find all values of p such that $\sum_{k=1}^{\infty}\frac{p^k}{k}$ converges.

44. Find all values of p such that $\sum_{k=1}^{\infty}\frac{p^k}{k^2}$ converges.

EXPLORATORY EXERCISES

1. One reason that it is important to distinguish absolute from conditional convergence of a series is the rearrangement of series, to be explored in this exercise. Show that the series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}$ is absolutely convergent and find its sum S. Find the sum S_+ of the positive terms of the series. Find the sum S_- of the negative terms of the series. Verify that $S = S_+ + S_-$. This may seem obvious, since for the finite sums you are most familiar with, the order of addition never matters. However, you cannot separate the positive and negative terms for conditionally convergent series. For example, show that $\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}$ converges (conditionally) but that the series of positive terms and the series of negative terms both diverge. Explain in words why this will always happen for conditionally convergent series. Thus, the order of terms matters for conditionally convergent series. By exploring further, we can uncover a truly remarkable fact: for conditionally convergent series, you can reorder the terms so that the partial sums converge to *any* real number. To illustrate this, suppose we want to reorder the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ so that the partial sums converge to $\frac{\pi}{2}$. Start by pulling out positive terms $\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right)$ such that the partial sum is within 0.1 of $\frac{\pi}{2}$. Next, take the first negative term $\left(-\frac{1}{2}\right)$ and positive terms such that the partial sum is within 0.05 of $\frac{\pi}{2}$. Then take the next negative term $\left(-\frac{1}{4}\right)$ and positive terms such that the partial sum is within 0.01 of $\frac{\pi}{2}$. Argue that you could continue in this fashion to reorder the terms so that the partial sums converge to $\frac{\pi}{2}$. (Especially explain why you will never "run out of" positive terms.) Then explain why you cannot do the same with the absolutely convergent series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$.

2. In this exercise, you show that the Root Test is more general than the Ratio Test. To be precise, show that if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r \neq 1$ then $\lim_{n\to\infty} |a_n|^{1/n} = r$ by considering $\lim_{n\to\infty} \ln \left|\frac{a_{n+1}}{a_n}\right|$ and $\lim_{n\to\infty} \ln |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left|\frac{a_{k+1}}{a_k}\right|$. Interpret this result in terms of how likely the Ratio Test or Root Test is to give a definite conclusion. Show that the result is not "if and only if" by finding a sequence for which $\lim_{n\to\infty} |a_n|^{1/n} < 1$ but $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist. In spite of this, give one reason why the Ratio Test might be preferable to the Root Test.

8.6 POWER SERIES

We now expand our discussion of series to the case where the terms of the series are functions of the variable x. Pay close attention, as the primary reason for studying series is that we can use them to represent functions. This opens up numerous possibilities for us, from approximating the values of transcendental functions to calculating derivatives and integrals of such functions, to studying differential equations. As well, *defining* functions as convergent series produces an explosion of new functions available to us, including many important functions, such as the Bessel functions. We take the first few steps in this section.

As a start, consider the series

$$\sum_{k=0}^{\infty} (x-2)^k = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \cdots.$$

Notice that for each fixed x, this is a geometric series with $r = (x-2)$, which will converge whenever $|r| = |x-2| < 1$ and diverge whenever $|r| = |x-2| \geq 1$. Further, for each x with $|x-2| < 1$ (i.e., $1 < x < 3$), the series converges to

$$\frac{a}{1-r} = \frac{1}{1-(x-2)} = \frac{1}{3-x}.$$

That is, for each x in the interval $(1, 3)$, we have

$$\sum_{k=0}^{\infty} (x-2)^k = \frac{1}{3-x}.$$

For all other values of x, the series diverges. In Figure 8.38, we show a graph of $f(x) = \frac{1}{3-x}$, along with the first three partial sums P_n of this series, where

$$P_n(x) = \sum_{k=0}^{n} (x-2)^k = 1 + (x-2) + (x-2)^2 + \cdots + (x-2)^n,$$

on the interval $[1, 3]$. Notice that as n gets larger, $P_n(x)$ appears to get closer to $f(x)$, for any given x in the interval $(1, 3)$. Further, as n gets larger, $P_n(x)$ tends to stay close to $f(x)$ for a larger range of x-values.

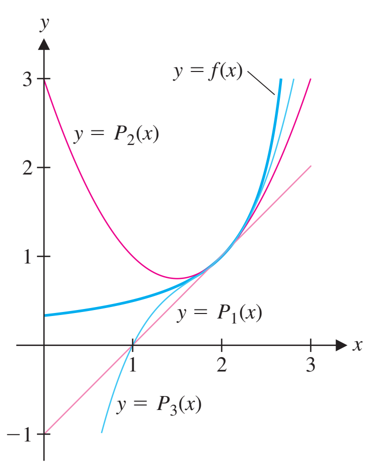


FIGURE 8.38
$y = \frac{1}{3-x}$ and the first three partial sums of $\sum_{k=0}^{\infty} (x-2)^k$

So, we have taken a series and noticed that it is equivalent to (i.e., it converges to) a *known* function on a certain interval. You might wonder why this is useful. Imagine what benefits you might find if you could take a given function (one that you don't know much about) and find an equivalent series representation. This is precisely what we do in section 8.7. For instance, we will show that for all x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{6.1}$$

As one immediate use of (6.1), suppose that you wanted to calculate $e^{1.234567}$. Using (6.1), for any given x, we can compute an approximation to e^x, simply by summing the first few terms of the equivalent power series. This is easy to do, since the partial sums of the series are simply polynomials.

In general, any series of the form

POWER SERIES

$$\sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

is called a **power series** in powers of $(x - c)$. We refer to the constants b_k, $k = 0, 1, 2, \ldots,$ as the **coefficients** of the series. The first question is: for what values of x does the series converge? Saying this another way, the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ defines a function of x. Its domain is the set of all x for which the series converges. The primary tool for investigating the convergence or divergence of a power series is the Ratio Test.

EXAMPLE 6.1 Determining Where a Power Series Converges

Determine the values of x for which the power series $\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} x^k$ converges.

Solution Using the Ratio Test, we have

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(k+1)x^{k+1}}{3^{k+2}}\frac{3^{k+1}}{kx^k}\right| \\ &= \lim_{k\to\infty}\frac{(k+1)|x|}{3k} = \frac{|x|}{3}\lim_{k\to\infty}\frac{k+1}{k} \\ &= \frac{|x|}{3} < 1,\end{aligned}$$

Since $x^{k+1} = x^k \cdot x^1$ and $3^{k+2} = 3^{k+1} \cdot 3^1$.

for $|x| < 3$ or $-3 < x < 3$. So, the series converges absolutely for $-3 < x < 3$ and diverges for $|x| > 3$ (i.e., for $x > 3$ or $x < -3$). Since the Ratio Test gives no conclusion for the endpoints $x = \pm 3$, we must test these separately.

For $x = 3$, we have the series

$$\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} x^k = \sum_{k=0}^{\infty} \frac{k}{3^{k+1}} 3^k = \sum_{k=0}^{\infty} \frac{k}{3}.$$

Since

$$\lim_{k\to\infty} \frac{k}{3} = \infty \neq 0,$$

the series diverges by the kth-term test for divergence. The series diverges when $x = -3$, for the same reason. Thus, the power series converges for all x in the interval $(-3, 3)$ and diverges for all x outside this interval. ■

Observe that example 6.1 has something in common with our introductory example. In both cases, the series have the form $\sum_{k=0}^{\infty} b_k(x-c)^k$ and there is an interval of the form $(c-r, c+r)$ on which the series converges and outside of which the series diverges. (In the case of example 6.1, notice that $c = 0$.) This interval on which a power series converges is called the **interval of convergence.** The constant r is called the **radius of convergence** (i.e., r is half the length of the interval of convergence). It turns out that there is such an interval for every power series. We have the following result.

NOTE

In part (iii) of Theorem 6.1, the series may converge at neither, one or both of the endpoints $x = c - r$ and $x = c + r$. Because the interval of convergence is centered at $x = c$, we refer to c as the **center** of the power series.

THEOREM 6.1

Given any power series, $\sum_{k=0}^{\infty} b_k(x-c)^k$, there are exactly three possibilities:

(i) The series converges absolutely for *all* $x \in (-\infty, \infty)$ and the radius of convergence is $r = \infty$;
(ii) The series converges *only* for $x = c$ (and diverges for all other values of x) and the radius of convergence is $r = 0$; or
(iii) The series converges absolutely for $x \in (c - r, c + r)$ and diverges for $x < c - r$ and for $x > c + r$, for some number r with $0 < r < \infty$.

The proof of the theorem can be found in Appendix A.

EXAMPLE 6.2 Finding the Interval and Radius of Convergence

Determine the interval and radius of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{10^k}{k!}(x-1)^k.$$

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{10^{k+1}(x-1)^{k+1}}{(k+1)!}\frac{k!}{10^k(x-1)^k}\right|$$

$$= 10|x-1| \lim_{k\to\infty}\frac{k!}{(k+1)k!}$$

Since $(x-1)^{k+1} = (x-1)^k(x-1)^1$ and $(k+1)! = (k+1)k!$.

$$= 10|x-1| \lim_{k\to\infty}\frac{1}{k+1} = 0 < 1,$$

for *all* x. This says that the series converges absolutely for all x. Thus, the interval of convergence for this series is $(-\infty, \infty)$ and the radius of convergence is $r = \infty$. ■

The interval of convergence for a power series can be a closed interval, an open interval or a half-open interval, as in example 6.3.

EXAMPLE 6.3 A Half-Open Interval of Convergence

Determine the interval and radius of convergence for the power series $\sum_{k=1}^{\infty} \frac{x^k}{k4^k}$.

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)4^{k+1}}\frac{k4^k}{x^k}\right|$$
$$= \frac{|x|}{4}\lim_{k\to\infty}\frac{k}{k+1} = \frac{|x|}{4} < 1.$$

So, we are guaranteed absolute convergence for $|x| < 4$ and divergence for $|x| > 4$. It remains only to test the endpoints of the interval: $x = \pm 4$. For $x = 4$, we have

$$\sum_{k=1}^{\infty}\frac{x^k}{k4^k} = \sum_{k=1}^{\infty}\frac{4^k}{k4^k} = \sum_{k=1}^{\infty}\frac{1}{k},$$

which you will recognize as the harmonic series, which diverges. For $x = -4$, we have

$$\sum_{k=1}^{\infty}\frac{x^k}{k4^k} = \sum_{k=1}^{\infty}\frac{(-4)^k}{k4^k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k},$$

which is the alternating harmonic series, which we know converges conditionally (see example 5.2). So, in this case, the interval of convergence is the half-open interval $[-4, 4)$ and the radius of convergence is $r = 4$. ■

Notice that (as stated in Theorem 6.1) every power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges at least for $x = c$ since for $x = c$, we have the trivial case

$$\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} a_k(c-c)^k = a_0 + \sum_{k=1}^{\infty} a_k 0^k = a_0 + 0 = a_0.$$

EXAMPLE 6.4 A Power Series That Converges at Only One Point

Determine the radius of convergence for the power series $\sum_{k=0}^{\infty} k!(x-5)^k$.

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!(x-5)^{k+1}}{k!(x-5)^k}\right|$$
$$= \lim_{k\to\infty}\frac{(k+1)k!|x-5|}{k!}$$
$$= \lim_{k\to\infty}[(k+1)|x-5|]$$
$$= \begin{cases} 0, & \text{if } x = 5 \\ \infty, & \text{if } x \neq 5 \end{cases}.$$

Thus, this power series converges only for $x = 5$ and so, its radius of convergence is $r = 0$. ■

Suppose that the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ has radius of convergence $r > 0$. Then the series converges absolutely for all x in the interval $(c-r, c+r)$ and might converge at one or both of the endpoints, $x = c - r$ and $x = c + r$. Notice that since the series converges for each $x \in (c-r, c+r)$, it defines a function f on the interval $(c-r, c+r)$,

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots.$$

It turns out that such a function is continuous and differentiable, although the proof is beyond the level of this course. In fact, we differentiate exactly the way you might expect,

Differentiating a power series

$$\boxed{\begin{aligned} f'(x) &= \frac{d}{dx}f(x) = \frac{d}{dx}[b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots] \\ &= b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots = \sum_{k=1}^{\infty} b_k k(x-c)^{k-1}, \end{aligned}}$$

where the radius of convergence of the resulting series is also r. Since we find the derivative by differentiating each term in the series, we call this **term-by-term** differentiation. Likewise, we can integrate a convergent power series term-by-term,

Integrating a power series

$$\boxed{\begin{aligned} \int f(x)\,dx &= \int \sum_{k=0}^{\infty} b_k(x-c)^k\,dx = \sum_{k=0}^{\infty} b_k \int (x-c)^k\,dx \\ &= \sum_{k=0}^{\infty} b_k \frac{(x-c)^{k+1}}{k+1} + K, \end{aligned}}$$

where the radius of convergence of the resulting series is again r and where K is a constant of integration. The proof of these two results can be found in a text on advanced calculus. It's important to recognize that these two results are *not* obvious. They are not simply an application of the rule that a derivative or integral of a sum is the sum of the derivatives or integrals, respectively, since a series is not a sum, but rather, a limit of a sum. Further, these results are true for power series, but are *not* true for series in general.

EXAMPLE 6.5 A Convergent Series Whose Series of Derivatives Diverges

Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{\sin(k^3x)}{k^2}$ and show that the series of derivatives does not converge for any x.

Solution Notice that

$$\left|\frac{\sin(k^3x)}{k^2}\right| \le \frac{1}{k^2}, \quad \text{for all } x,$$

since $|\sin(k^3x)| \le 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2 > 1$), it follows from the Comparison Test that $\sum_{k=0}^{\infty} \frac{\sin(k^3x)}{k^2}$ converges absolutely, for all x. On the other hand, the series of derivatives (found by differentiating the series term-by-term) is

$$\sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{\sin(k^3x)}{k^2}\right] = \sum_{k=1}^{\infty} \frac{k^3\cos(k^3x)}{k^2} = \sum_{k=1}^{\infty} [k\cos(k^3x)],$$

which *diverges* for all x, by the kth-term test for divergence, since the terms do not tend to zero as $k \to \infty$, for any x. ■

Keep in mind that $\sum_{k=1}^{\infty} \frac{\sin(k^3 x)}{k^2}$ is not a power series. (Why not?) The result of example 6.5 (a convergent series whose series of derivatives diverges) *cannot* occur with any power series with radius of convergence $r > 0$.

In example 6.6, we find that once we have a convergent power series representation for a given function, we can use this to obtain power series representations for any number of other functions, by substitution or by differentiating and integrating the series term-by-term.

EXAMPLE 6.6 Differentiating and Integrating a Power Series

Use the power series $\sum_{k=0}^{\infty} (-1)^k x^k$ to find power series representations of $\frac{1}{(1+x)^2}$, $\frac{1}{1+x^2}$ and $\tan^{-1} x$.

Solution Notice that $\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=0}^{\infty} (-x)^k$ is a geometric series with ratio $r = -x$. This series converges, then, whenever $|r| = |-x| = |x| < 1$, to

$$\frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}.$$

That is, for $-1 < x < 1$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k. \tag{6.2}$$

Differentiating both sides of (6.2), we get

$$\frac{-1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^k k x^{k-1}, \quad \text{for } -1 < x < 1.$$

Multiplying both sides by -1 gives us a new power series representation:

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} k x^{k-1},$$

valid for $-1 < x < 1$. Notice that we can also obtain a new power series from (6.2) by substitution. For instance, if we replace x with x^2, we get

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k (x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \tag{6.3}$$

valid for $-1 < x^2 < 1$ (which is equivalent to having $x^2 < 1$ or $-1 < x < 1$).

Integrating both sides of (6.3) gives us

$$\int \frac{1}{1+x^2}\,dx = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + c. \tag{6.4}$$

You should recognize the integral on the left-hand side of (6.4) as $\tan^{-1} x$. That is,

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} + c, \quad \text{for } -1 < x < 1. \tag{6.5}$$

Taking $x = 0$ gives us

$$\tan^{-1} 0 = \sum_{k=0}^{\infty} \frac{(-1)^k 0^{2k+1}}{2k+1} + c = c,$$

so that $c = \tan^{-1} 0 = 0$. Equation (6.5) now gives us a power series representation for $\tan^{-1} x$, namely:

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots, \quad \text{for } -1 < x < 1.$$

Notice that working as in example 6.6, we can produce power series representations of any number of functions. In section 8.7, we present a systematic method for producing power series representations for a wide range of functions.

BEYOND FORMULAS

You should think of a power series as a different form for writing functions. Just as $\frac{x}{e^x}$ can be rewritten as xe^{-x}, many functions can be rewritten as power series. In general, having another alternative for writing a function gives you one more option to consider when trying to solve a problem. Power series representations are often easier to work with than other representations. Further, power series representations have the advantage of having derivatives and integrals that are easy to compute. As a result, power series are a very useful addition to our set of problem-solving techniques.

EXERCISES 8.6

WRITING EXERCISES

1. Power series have the form $\sum_{k=0}^{\infty} a_k(x-c)^k$. Explain why the farther x is from c, the larger the terms of the series are and the less likely the series is to converge. Describe how this general trend relates to the radius of convergence.

2. Applying the Ratio Test to $\sum_{k=0}^{\infty} a_k(x-c)^k$ requires you to evaluate $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}(x-c)\right|$. For $x = c$, this limit equals 0 and the series converges. As x increases or decreases, $|x-c|$ increases. If the series has a finite radius of convergence $r > 0$, what is the value of the limit when $|x-c| = r$? Explain how the limit changes when $|x-c| < r$ and $|x-c| > r$ and how this determines the convergence or divergence of the series.

3. As shown in example 6.2, $\sum_{k=0}^{\infty} \frac{10^k}{k!}(x-1)^k$ converges for all x. If $x = 1001$, the value of $(x-1)^k = 1000^k$ gets very large very fast, as k increases. Explain why, for the series to converge, the value of $k!$ must get large faster than 1000^k. To illustrate how fast the factorial grows, compute 50!, 100! and 200! (if your calculator can).

4. In a power series representation of $\sqrt{x+1}$ about $c = 0$, explain why the radius of convergence cannot be greater than 1. (Think about the domain of $\sqrt{x+1}$.)

In exercises 1–8, find a power series representation of $f(x)$ about $c = 0$ (refer to example 6.6). Also, determine the radius and interval of convergence, and graph $f(x)$ together with the partial sums $\sum_{k=0}^{3} a_k x^k$ and $\sum_{k=0}^{6} a_k x^k$.

1. $f(x) = \frac{2}{1-x}$
2. $f(x) = \frac{3}{x-1}$
3. $f(x) = \frac{3}{1+x^2}$
4. $f(x) = \frac{2}{1-x^2}$
5. $f(x) = \frac{2x}{1-x^3}$
6. $f(x) = \frac{3x}{1+x^2}$
7. $f(x) = \frac{2}{4+x}$
8. $f(x) = \frac{3}{6-x}$

In exercises 9–14, determine the interval of convergence and the function to which the given power series converges.

9. $\sum_{k=0}^{\infty} (x+2)^k$
10. $\sum_{k=0}^{\infty} (x-3)^k$
11. $\sum_{k=0}^{\infty} (2x-1)^k$
12. $\sum_{k=0}^{\infty} (3x+1)^k$
13. $\sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^k$
14. $\sum_{k=0}^{\infty} 3\left(\frac{x}{4}\right)^k$

In exercises 15–30, determine the radius and interval of convergence.

15. $\sum_{k=0}^{\infty} \frac{2^k}{k!}(x-2)^k$

16. $\sum_{k=0}^{\infty} \frac{3^k}{k!}x^k$

17. $\sum_{k=0}^{\infty} \frac{k}{4^k}x^k$

18. $\sum_{k=0}^{\infty} \frac{k}{2^k}x^k$

19. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k3^k}(x-1)^k$

20. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k4^k}(x+2)^k$

21. $\sum_{k=0}^{\infty} k!(x+1)^k$

22. $\sum_{k=1}^{\infty} \frac{1}{k}(x-1)^k$

23. $\sum_{k=2}^{\infty} k^2(x-3)^k$

24. $\sum_{k=4}^{\infty} \frac{1}{k^2}(x+2)^k$

25. $\sum_{k=3}^{\infty} \frac{k!}{(2k)!}x^k$

26. $\sum_{k=2}^{\infty} \frac{(k!)^2}{(2k)!}x^k$

27. $\sum_{k=1}^{\infty} \frac{2^k}{k^2}(x+2)^k$

28. $\sum_{k=0}^{\infty} \frac{k^2}{k!}(x+1)^k$

29. $\sum_{k=1}^{\infty} \frac{4^k}{\sqrt{k}}x^k$

30. $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{\sqrt{k}}$

In exercises 31–36, find a power series representation and radius of convergence by integrating or differentiating one of the series from exercises 1–8.

31. $f(x) = 3\tan^{-1} x$

32. $f(x) = 2\ln(1-x)$

33. $f(x) = \frac{2x}{(1-x^2)^2}$

34. $f(x) = \frac{3}{(x-1)^2}$

35. $f(x) = \ln(1+x^2)$

36. $f(x) = \ln(4+x)$

In exercises 37–40, find the interval of convergence of the (nonpower) series and the corresponding series of derivatives.

37. $\sum_{k=1}^{\infty} \frac{\cos(k^3x)}{k^2}$

38. $\sum_{k=1}^{\infty} \frac{\cos(x/k)}{k}$

39. $\sum_{k=0}^{\infty} e^{kx}$

40. $\sum_{k=0}^{\infty} e^{-2kx}$

41. For any constants a and $b > 0$, determine the interval and radius of convergence of $\sum_{k=0}^{\infty} \frac{(x-a)^k}{b^k}$.

42. Prove that if $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, then $\sum_{k=0}^{\infty} a_k x^{2k}$ has radius of convergence $\sqrt{r}$.

43. If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty} a_k (x-c)^k$ for any constant c.

44. If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty} a_k \left(\frac{x}{b}\right)^k$ for any constant $b \neq 0$.

45. Show that $f(x) = \frac{x+1}{(1-x)^2} = \frac{\frac{2x}{1-x}+1}{1-x}$ has the power series representation $f(x) = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$. Find the radius of convergence. Set $x = \frac{1}{1000}$ and discuss the interesting decimal representation of $\frac{1{,}001{,}000}{998{,}001}$.

46. Use long division to show that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$.

47. Even great mathematicians can make mistakes. Leonhard Euler started with the equation $\frac{x}{x-1} + \frac{x}{1-x} = 0$, rewrote it as $\frac{1}{1-1/x} + \frac{x}{1-x} = 0$, found power series representations for each function and concluded that $\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + \cdots = 0$. Substitute $x = 1$ to show that the conclusion is false, then find the mistake in Euler's derivation.

48. If your CAS or calculator has a command named "Taylor," use it to verify your answers to exercises 31–36.

49. An **electric dipole** consists of a charge q at $x = 1$ and a charge $-q$ at $x = -1$. The electric field at any $x > 1$ is given by $E(x) = \frac{kq}{(x-1)^2} - \frac{kq}{(x+1)^2}$, for some constant k. Find a power series representation for $E(x)$.

50. Show that a power series representation of $f(x) = \ln(1+x^2)$ is given by $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{k+1}$. For the partial sums $P_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+2}}{k+1}$, compute $|f(0.9) - P_n(0.9)|$ for each of $n = 2, 4, 6$. Discuss the pattern. Then compute $|f(1.1) - P_n(1.1)|$ for each of $n = 2, 4, 6$. Discuss the pattern. Discuss the relevance of the radius of convergence to these calculations.

EXPLORATORY EXERCISES

1. Note that the radius of convergence in each of exercises 1–5 is 1. Given that the functions in exercises 1, 2, 4 and 5 are undefined at $x = 1$, explain why the radius of convergence can't be larger than 1. The restricted radius in exercise 3 can be understood using complex numbers. Show that $1 + x^2 = 0$ for $x = \pm i$, where $i = \sqrt{-1}$. In general, a complex number $a + bi$ is associated with the point (a, b). Find the "distance" between the complex numbers 0 and i by finding the distance between the associated points (0, 0) and (0, 1). Discuss how this compares to the radius of convergence. Then use the ideas in this exercise to quickly conjecture the radius of convergence of power series with center $c = 0$ for the functions $f(x) = \dfrac{4}{1+4x}$, $f(x) = \dfrac{2}{4+x}$ and $f(x) = \dfrac{2}{4+x^2}$.

2. For each series $f(x)$, compare the intervals of convergence of $f(x)$ and $\int f(x)\,dx$, where the antiderivative is taken term-by-term. (a) $f(x) = \sum_{k=0}^{\infty}(-1)^k x^k$; (b) $f(x) = \sum_{k=0}^{\infty}\sqrt{k}x^k$; (c) $f(x) = \sum_{k=0}^{\infty}\frac{1}{k}x^k$. As stated in the text, the radius of convergence remains the same after integration (or differentiation). Based on the examples in this exercise, does integration make it more or less likely that the series will converge at the endpoints? Conversely, will differentiation make it more or less likely that the series will converge at the endpoints?

3. Let $\{f_k(x)\}$ be a sequence of functions defined on a set E. The **Weierstrass M-test** states that if there exist constants M_k such that $|f_k(x)| \le M_k$ for each x and $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{k=1}^{\infty} f_k(x)$ converges (uniformly) for each x in E. Prove that $\sum_{k=1}^{\infty}\frac{1}{k^2+x^2}$ and $\sum_{k=1}^{\infty} x^2 e^{-kx}$ converge (uniformly) for all x. "Uniformly" in this exercise refers to the rate at which the infinite series converges to its sum. A precise definition can be found in an advanced calculus book. We explore the main idea of the definition in this exercise. Explain why you would expect the convergence of the series $\sum_{k=1}^{\infty}\frac{1}{k^2+x^2}$ to be slowest at $x = 0$. Now, numerically explore the following question. Defining $f(x) = \sum_{k=1}^{\infty}\frac{1}{k^2+x^2}$ and $S_n(x) = \sum_{k=1}^{\infty}\frac{1}{k^2+x^2}$, is there an integer N such that if $n > N$ then $|f(x) - S_n(x)| < 0.01$ for all x?

8.7 TAYLOR SERIES

Representation of Functions as Power-Series

In this section, we develop a compelling reason for considering series. They are not merely a mathematical curiosity, but rather, are an essential means for exploring and computing with transcendental functions (e.g., $\sin x$, $\cos x$, $\ln x$, e^x, etc.).

Suppose that the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ has radius of convergence $r > 0$. As we've observed, this means that the series converges absolutely to some function f on the interval $(c - r, c + r)$. We have

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4 + \cdots,$$

for each $x \in (c - r, c + r)$. Differentiating term-by-term, we get that

$$f'(x) = \sum_{k=0}^{\infty} b_k k(x-c)^{k-1} = b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + 4b_4(x-c)^3 + \cdots,$$

again, for each $x \in (c - r, c + r)$. Likewise, we get

$$f''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(x-c)^{k-2} = 2b_2 + 3\cdot 2b_3(x-c) + 4\cdot 3b_4(x-c)^2 + \cdots$$

and $$f'''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(k-2)(x-c)^{k-3} = 3\cdot 2b_3 + 4\cdot 3\cdot 2b_4(x-c) + \cdots$$

and so on (all valid for $c - r < x < c + r$). Notice that if we substitute $x = c$ in each of the above derivatives, all the terms of the series drop out, except one. We get

$$f(c) = b_0,$$
$$f'(c) = b_1,$$
$$f''(c) = 2b_2,$$
$$f'''(c) = 3!\,b_3$$

and so on. Observe that in general, we have

$$f^{(k)}(c) = k!\, b_k. \tag{7.1}$$

Solving (7.1) for b_k, we have

$$b_k = \frac{f^{(k)}(c)}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$

To summarize, we found that if $\sum_{k=0}^{\infty} b_k(x-c)^k$ is a convergent power series with radius of convergence $r > 0$, then the series converges to some function f that we can write as

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k, \quad \text{for } x \in (c-r, c+r).$$

Now, think about this problem from another angle. Instead of starting with a series, suppose that you start with an infinitely differentiable function, f (i.e., f can be differentiated infinitely often). Then, we can construct the series

Taylor Series Expansion of $f(x)$ about $x = c$

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,}$$

called a **Taylor series** expansion for f. (See the historical note on Brook Taylor in section 6.2.) There are two important questions we need to answer.

- Does a series constructed in this way converge? If so, what is its radius of convergence?
- If the series converges, it converges to a function. Does it converge to f?

We can answer the first of these questions on a case-by-case basis, usually by applying the Ratio Test. The second question will require further insight.

EXAMPLE 7.1 Constructing a Taylor Series Expansion

Construct the Taylor series expansion for $f(x) = e^x$, about $x = 0$ (i.e., take $c = 0$).

Solution Here, we have the extremely simple case where

$$f'(x) = e^x,\ f''(x) = e^x \text{ and so on, } \ f^{(k)}(x) = e^x, \quad \text{for } k = 0, 1, 2, \ldots.$$

This gives us the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{e^0}{k!}x^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{|x|^{k+1}}{(k+1)!}\frac{k!}{|x|^k} = |x| \lim_{k\to\infty} \frac{k!}{(k+1)k!}$$
$$= |x| \lim_{k\to\infty} \frac{1}{k+1} = |x|(0) = 0 < 1, \quad \text{for all } x.$$

So, the Taylor series $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ converges absolutely for all real numbers x. At this point, though, we do not know the function to which the series converges. (Could it be e^x?) ■

REMARK 7.1

The special case of a Taylor series expansion about $x = 0$ is often called a **Maclaurin series.** (See the historical note about Colin Maclaurin in section 8.3.) That is, the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ is the Maclaurin series expansion for f.

Before we present any further examples of Taylor series, let's see if we can determine the function to which a given Taylor series converges. First, notice that the partial sums of a Taylor series (like those for any power series) are simply polynomials. We define

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n. \end{aligned}$$

Observe that $P_n(x)$ is a polynomial of degree n, as $\frac{f^{(k)}(c)}{k!}$ is a constant for each k. We refer to P_n as the **Taylor polynomial of degree *n*** for f expanded about $x = c$.

EXAMPLE 7.2 Constructing and Graphing Taylor Polynomials

For $f(x) = e^x$, find the Taylor polynomial of degree n expanded about $x = 0$.

Solution As in example 7.1, we have that $f^{(k)}(x) = e^x$, for all k. So, we have that the nth-degree Taylor polynomial is

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{e^0}{k!}x^k \\ &= \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}. \end{aligned}$$

Since we established in example 7.1 that the Taylor series for $f(x) = e^x$ about $x = 0$ converges for all x, this says that the sequence of partial sums (i.e., the sequence of Taylor polynomials) converges for all x. In an effort to determine the function to which the Taylor polynomials are converging, we have plotted $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$, together with the graph of $f(x) = e^x$ in Figures 8.39a–d, respectively.

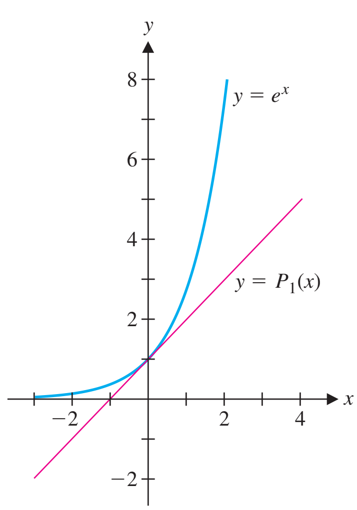


FIGURE 8.39a
$y = e^x$ and $y = P_1(x)$

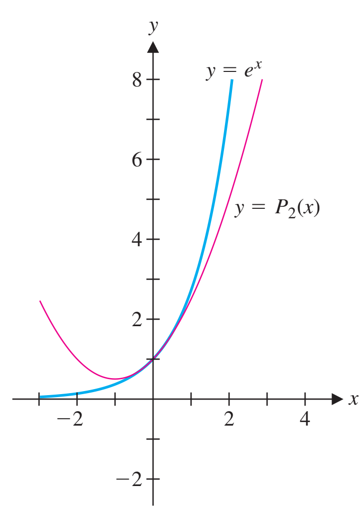


FIGURE 8.39b
$y = e^x$ and $y = P_2(x)$

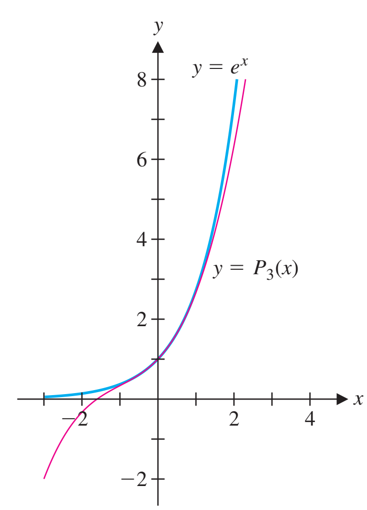


FIGURE 8.39c
$y = e^x$ and $y = P_3(x)$

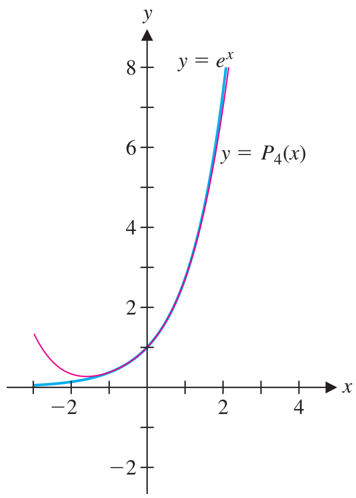


FIGURE 8.39d
$y = e^x$ and $y = P_4(x)$

Notice that as n gets larger, the graphs of $P_n(x)$ appear (at least on the interval displayed) to be approaching the graph of $f(x) = e^x$. Since we know that the Taylor series converges and the graphical evidence suggests that the partial sums of the series are approaching $f(x) = e^x$, it is reasonable to conjecture that the series converges to e^x.

This is, in fact, exactly what is happening, as we can prove using Theorems 7.1 and 7.2. ■

THEOREM 7.1 (Taylor's Theorem)

Suppose that f has $(n+1)$ derivatives on the interval $(c-r, c+r)$, for some $r > 0$. Then, for $x \in (c-r, c+r)$, $f(x) \approx P_n(x)$ and the error in using $P_n(x)$ to approximate $f(x)$ is

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}, \tag{7.2}$$

for some number z between x and c.

REMARK 7.2

Observe that for $n = 0$, Taylor's Theorem simplifies to a very familiar result. We have

$$\begin{aligned} R_0(x) &= f(x) - P_0(x) \\ &= \frac{f'(z)}{(0+1)!}(x-c)^{0+1}. \end{aligned}$$

Since $P_0(x) = f(c)$, we have simply

$$f(x) - f(c) = f'(z)(x-c).$$

Dividing by $(x-c)$, gives us

$$\frac{f(x) - f(c)}{x-c} = f'(z),$$

which is the conclusion of the Mean Value Theorem. In this way, observe that Taylor's Theorem is a generalization of the Mean Value Theorem.

The error term $R_n(x)$ in (7.2) is often called the **remainder term.** Note that this term looks very much like the first neglected term of the Taylor series, except that $f^{(n+1)}$ is evaluated at some (unknown) number z between x and c, instead of at c. This remainder term serves two purposes: it enables us to obtain an estimate of the error in using a Taylor polynomial to approximate a given function and as we'll see in Theorem 7.2, it gives us the means to prove that a Taylor series for a given function f converges to f.

The proof of Taylor's Theorem is somewhat technical and so we leave it for the end of the section.

Note: If we could show that

$$\lim_{n\to\infty} R_n(x) = 0, \quad \text{for } all \ x \text{ in } (c-r, c+r),$$

then we would have that

$$0 = \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} [f(x) - P_n(x)] = f(x) - \lim_{n\to\infty} P_n(x)$$

or

$$\lim_{n\to\infty} P_n(x) = f(x), \text{ for } all \ x \in (c-r, c+r).$$

That is, the sequence of partial sums of the Taylor series (i.e., the sequence of Taylor polynomials) converges to $f(x)$ for each $x \in (c-r, c+r)$. We summarize this in Theorem 7.2.

THEOREM 7.2

Suppose that f has derivatives of all orders in the interval $(c-r, c+r)$, for some $r > 0$ and $\lim_{n\to\infty} R_n(x) = 0$, for all x in $(c-r, c+r)$. Then, the Taylor series for f expanded about $x = c$ converges to $f(x)$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,$$

for all x in $(c-r, c+r)$.

We now return to the Taylor series expansion of $f(x) = e^x$ about $x = 0$, constructed in example 7.1 and investigated further in example 7.2 and prove that it converges to e^x, as we had suspected.

EXAMPLE 7.3 Proving That a Taylor Series Converges to the Desired Function

Show that the Taylor series for $f(x) = e^x$ expanded about $x = 0$ converges to e^x.

Solution We already found the indicated Taylor series, $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ in example 7.1. Here, we have $f^{(k)}(x) = e^x$, for all $k = 0, 1, 2, \ldots$. This gives us the remainder term

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1} = \frac{e^z}{(n+1)!}x^{n+1}, \tag{7.3}$$

where z is somewhere between x and 0 (and depends also on the value of n). We first find a bound on the size of e^z. Notice that if $x > 0$, then $0 < z < x$ and so,

$$e^z < e^x.$$

If $x \le 0$, then $x \le z \le 0$, so that

$$e^z \le e^0 = 1.$$

We define M to be the larger of these two bounds on e^z. That is, we let

$$M = \max\{e^x, 1\}.$$

Then, for any x and any n, we have

$$e^z \le M.$$

Together with (7.3), this gives us the error estimate

$$|R_n(x)| = \frac{e^z}{(n+1)!}|x|^{n+1} \le M\frac{|x|^{n+1}}{(n+1)!}. \tag{7.4}$$

To prove that the Taylor series converges to e^x, we want to use (7.4) to show that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$, for all x. However, for any given x, we cannot compute $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$ directly. Instead, we use the following indirect approach. We test the *series* $\displaystyle\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ using the Ratio Test. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+2}}{(n+2)!}\,\frac{(n+1)!}{|x|^{n+1}} = |x|\lim_{n\to\infty}\frac{1}{n+2} = 0 < 1,$$

for all x. This says that the series $\displaystyle\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ converges absolutely for all x. By the kth-term test for divergence, it follows that the general term must tend to 0 as $n \to \infty$, for all x. That is,

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

and so, from (7.4), $\lim_{n\to\infty} R_n(x) = 0$, for all x. From Theorem 7.2, we now conclude that the Taylor series converges to e^x for all x. That is,

$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.} \tag{7.5}$$

When constructing a Taylor series expansion, it is essential to *accurately* calculate enough derivatives for you to recognize the general form of the nth derivative. So, take your time and **BE CAREFUL!** Once this is done, you need only show that $R_n(x) \to 0$, as $n \to \infty$, for all x, to ensure that the series converges to the function you are expanding.

One of the reasons for calculating Taylor series is that we can use their partial sums to compute approximate values of a function.

M	$\sum_{k=0}^{M} \frac{1}{k!}$
5	2.716666667
10	2.718281801
15	2.718281828
20	2.718281828

EXAMPLE 7.4 Using a Taylor Series to Obtain an Approximation of e

Use the Taylor series for e^x in (7.5) to obtain an approximation to the number e.

Solution We have

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!}1^k = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

We list some partial sums of this series in the accompanying table. From this we get the very accurate approximation

$$e \approx 2.718281828.$$

EXAMPLE 7.5 A Taylor Series Expansion of sin *x*

Find the Taylor series for $f(x) = \sin x$, expanded about $x = \frac{\pi}{2}$ and prove that the series converges to $\sin x$ for all x.

Solution In this case, the Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x - \frac{\pi}{2}\right)^k.$$

First, we compute some derivatives and their value at $x = \frac{\pi}{2}$. We have

$$\begin{aligned}
f(x) &= \sin x & f\left(\tfrac{\pi}{2}\right) &= 1,\\
f'(x) &= \cos x & f'\left(\tfrac{\pi}{2}\right) &= 0,\\
f''(x) &= -\sin x & f''\left(\tfrac{\pi}{2}\right) &= -1,\\
f'''(x) &= -\cos x & f'''\left(\tfrac{\pi}{2}\right) &= 0,\\
f^{(4)}(x) &= \sin x & f^{(4)}\left(\tfrac{\pi}{2}\right) &= 1
\end{aligned}$$

and so on. Recognizing that every other term is zero and every other term is ± 1, we see that the Taylor series is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{2}\right)}{k!}\left(x - \frac{\pi}{2}\right)^k &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k}.
\end{aligned}$$

In order to test this series for convergence, we consider the remainder term

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \left(x - \frac{\pi}{2}\right)^{n+1} \right|, \tag{7.6}$$

for some z between x and $\frac{\pi}{2}$. From our derivative calculation, note that

$$f^{(n+1)}(z) = \begin{cases} \pm\cos z, & \text{if } n \text{ is even} \\ \pm\sin z, & \text{if } n \text{ is odd} \end{cases}.$$

From this, it follows that
$$|f^{(n+1)}(z)| \le 1,$$
for every n. (Notice that this is true whether n is even or odd.) From (7.6), we now have

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| \left| x - \frac{\pi}{2} \right|^{n+1} \le \frac{1}{(n+1)!} \left| x - \frac{\pi}{2} \right|^{n+1} \to 0,$$

as $n \to \infty$, for every x, as in example 7.3. This says that

$$\boxed{\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots,}$$

for all x. In Figures 8.40a–d, we show graphs of $f(x) = \sin x$ together with the Taylor polynomials $P_2(x)$, $P_4(x)$, $P_6(x)$ and $P_8(x)$ (the first few partial sums of the series). Notice that the higher the degree of the Taylor polynomial is, the larger the interval is over which the polynomial provides a close approximation to $f(x) = \sin x$.

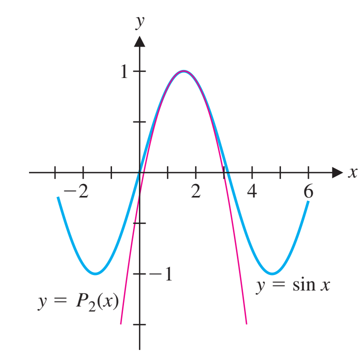


FIGURE 8.40a
$y = \sin x$ and $y = P_2(x)$

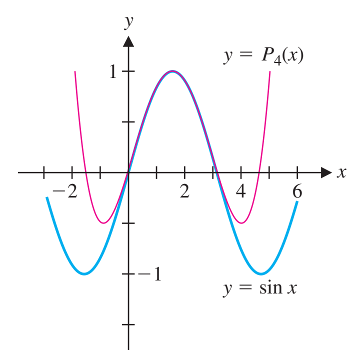


FIGURE 8.40b
$y = \sin x$ and $y = P_4(x)$

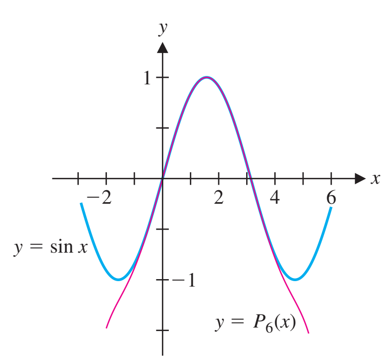


FIGURE 8.40c
$y = \sin x$ and $y = P_6(x)$

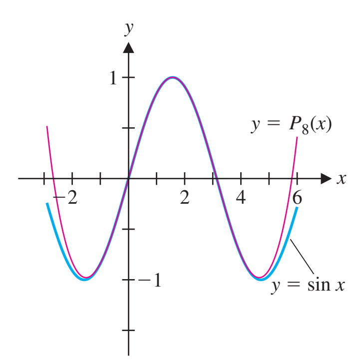


FIGURE 8.40d
$y = \sin x$ and $y = P_8(x)$

In example 7.6, we illustrate how to use Taylor's Theorem to estimate the error in using a Taylor polynomial to approximate the value of a function.

EXAMPLE 7.6 Estimating the Error in a Taylor Polynomial Approximation

Expand $f(x) = \ln x$ in a Taylor series about a convenient point and use a Taylor polynomial of degree 4 to approximate the value of $\ln(1.1)$. Then, estimate the error in this approximation.

Solution First, note that since $\ln 1$ is known exactly and 1 is close to 1.1 (why would this matter?), we expand $f(x) = \ln x$ in a Taylor series about $x = 1$. We compute an adequate number of derivatives so that the pattern becomes clear. We have

$$\begin{aligned} f(x) &= \ln x & f(1) &= 0 \\ f'(x) &= x^{-1} & f'(1) &= 1 \\ f''(x) &= -x^{-2} & f''(1) &= -1 \\ f'''(x) &= 2x^{-3} & f'''(1) &= 2 \\ f^{(4)}(x) &= -3 \cdot 2x^{-4} & f^{(4)}(1) &= -3! \\ f^{(5)}(x) &= 4!\,x^{-5} & f^{(5)}(1) &= 4! \\ &\vdots & &\vdots \\ f^{(k)}(x) &= (-1)^{k+1}(k-1)!\,x^{-k} & f^{(k)}(1) &= (-1)^{k+1}(k-1)! \end{aligned}$$

We get the Taylor series

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots + (-1)^{k+1}\frac{(k-1)!}{k!}(x-1)^k + \cdots \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k. \end{aligned}$$

We leave it as an exercise to show that the series converges to $f(x) = \ln x$, for $0 < x < 2$. The Taylor polynomial $P_4(x)$ is then

$$\begin{aligned} P_4(x) &= \sum_{k=0}^{4} \frac{(-1)^{k+1}}{k}(x-1)^k \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4. \end{aligned}$$

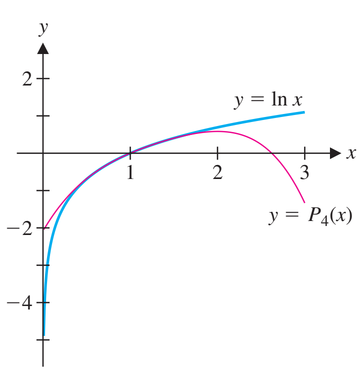


FIGURE 8.41
$y = \ln x$ and $y = P_4(x)$

We show a graph of $y = \ln x$ and $y = P_4(x)$ in Figure 8.41. Taking $x = 1.1$ gives us the approximation

$$\ln(1.1) \approx P_4(1.1) = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \approx 0.095308333.$$

We can use the remainder term to estimate the error in this approximation. We have

$$\begin{aligned} |Error| &= |\ln(1.1) - P_4(1.1)| = |R_4(1.1)| \\ &= \left| \frac{f^{(4+1)}(z)}{(4+1)!}(1.1-1)^{4+1} \right| = \frac{4!|z|^{-5}}{5!}(0.1)^5, \end{aligned}$$

where z is between 1 and 1.1. This gives us the following bound on the error:

$$|Error| = \frac{(0.1)^5}{5z^5} < \frac{(0.1)^5}{5(1^5)} = 0.000002,$$

since $1 < z < 1.1$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. This says that the approximation $\ln(1.1) \approx 0.095308333$ is off by no more than ± 0.000002. ■

A more significant problem related to example 7.6 is to determine how many terms of the Taylor series are needed in order to guarantee a given accuracy. We use the remainder term to accomplish this in example 7.7.

EXAMPLE 7.7 Finding the Number of Terms Needed for a Given Accuracy

Find the number of terms in the Taylor series for $f(x) = \ln x$ expanded about $x = 1$ that will guarantee an accuracy of at least 1×10^{-10} in the approximation of (a) $\ln(1.1)$ and (b) $\ln(1.5)$.

Solution (a) From our calculations in example 7.6 and (7.2), we have that for some number z between 1 and 1.1,

$$|R_n(1.1)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.1-1)^{n+1}\right|$$
$$= \frac{n!|z|^{-n-1}}{(n+1)!}(0.1)^{n+1} = \frac{(0.1)^{n+1}}{(n+1)z^{n+1}} < \frac{(0.1)^{n+1}}{n+1}.$$

Further, since we want the error to be less than 1×10^{-10}, we require that

$$|R_n(1.1)| < \frac{(0.1)^{n+1}}{n+1} < 1 \times 10^{-10}.$$

You can solve this inequality for n by trial and error, to find that $n = 9$ will guarantee the required accuracy. Notice that larger values of n will also guarantee this accuracy, since $\frac{(0.1)^{n+1}}{n+1}$ is a decreasing function of n. We then have the approximation

$$\ln(1.1) \approx P_9(1.1) = \sum_{k=0}^{9} \frac{(-1)^{k+1}}{k}(1.1-1)^k$$
$$= (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 + \frac{1}{5}(0.1)^5$$
$$- \frac{1}{6}(0.1)^6 + \frac{1}{7}(0.1)^7 - \frac{1}{8}(0.1)^8 + \frac{1}{9}(0.1)^9$$
$$\approx 0.095310179813,$$

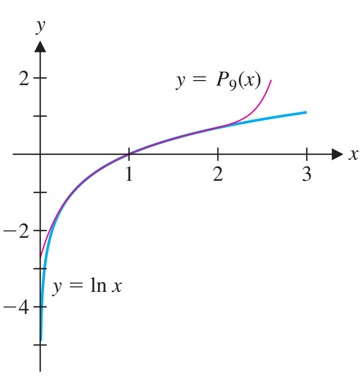


FIGURE 8.42
$y = \ln x$ and $y = P_9(x)$

which from our error estimate we know is correct to within 1×10^{-10}. We show a graph of $y = \ln x$ and $y = P_9(x)$ in Figure 8.42. In comparing Figure 8.42 with Figure 8.41, observe that while $P_9(x)$ provides an improved approximation to $P_4(x)$ over the interval of convergence (0, 2), it does not provide a better approximation outside of this interval.

(b) Similarly, notice that for some number z between 1 and 1.5,

$$|R_n(1.5)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.5-1)^{n+1}\right| = \frac{n!|z|^{-n-1}}{(n+1)!}(0.5)^{n+1}$$
$$= \frac{(0.5)^{n+1}}{(n+1)z^{n+1}} < \frac{(0.5)^{n+1}}{n+1},$$

since $1 < z < 1.5$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. So, here we require that

$$|R_n(1.5)| < \frac{(0.5)^{n+1}}{n+1} < 1 \times 10^{-10}.$$

Solving this by trial and error shows that $n = 28$ will guarantee the required accuracy. Observe that this says that to obtain the same accuracy, many more terms are needed to approximate $f(1.5)$ than for $f(1.1)$. This further illustrates the general principle that the farther away x is from the point about which we expand, the slower the convergence of the Taylor series will be. ■

For your convenience, we have compiled a list of common Taylor series in the following table.

Taylor Series	*Interval of Convergence*	*Where to Find*
$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$	$(-\infty, \infty)$	examples 7.1 and 7.3
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$	$(-\infty, \infty)$	exercise 2
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$	$(-\infty, \infty)$	exercise 1
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots$	$(-\infty, \infty)$	example 7.5
$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k \quad = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$	$(0, 2]$	examples 7.6, 7.7
$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \quad = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$	$(-1, 1)$	example 6.6

Notice that once you have found a Taylor series expansion for a given function, you can find any number of other Taylor series simply by making a substitution.

EXAMPLE 7.8 Finding New Taylor Series from Old Ones

Find Taylor series in powers of x for e^{2x}, e^{x^2} and e^{-2x}.

Solution Rather than compute the Taylor series for these functions from scratch, recall that we had established in example 7.3 that

$$e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots, \tag{7.7}$$

for all $t \in (-\infty, \infty)$. We use the variable t here instead of x, so that we can more easily make substitutions. Taking $t = 2x$ in (7.7), we get the new Taylor series:

$$e^{2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(2x)^k = \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \cdots.$$

Similarly, letting $t = x^2$ in (7.7), we get the Taylor series

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^{2k} = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots.$$

Finally, taking $t = -2x$ in (7.7), we get

$$e^{-2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(-2x)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}2^k x^k = 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \cdots.$$

Notice that all of these last three series converge for all $x \in (-\infty, \infty)$. (Why is that?) ■

Proof of Taylor's Theorem

Recall that we had observed that the Mean Value Theorem was a special case of Taylor's Theorem. As it turns out, the proof of Taylor's Theorem parallels that of the Mean Value Theorem. Both make use of Rolle's Theorem: If g is continuous on the interval $[a, b]$ and differentiable on (a, b) and $g(a) = g(b)$, then there is a number $z \in (a, b)$ for which $g'(z) = 0$. As with the proof of the Mean Value Theorem, for a *fixed* $x \in (c - r, c + r)$, we define the function

$$\begin{aligned} g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2!}f''(t)(x-t)^2 - \frac{1}{3!}f'''(t)(x-t)^3 \\ - \cdots - \frac{1}{n!}f^{(n)}(t)(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-c)^{n+1}}, \end{aligned}$$

where $R_n(x)$ is the remainder term, $R_n(x) = f(x) - P_n(x)$. If we take $t = x$, notice that

$$g(x) = f(x) - f(x) - 0 - 0 - \cdots - 0 = 0$$

and if we take $t = c$, we get

$$\begin{aligned} g(c) &= f(x) - f(c) - f'(c)(x-c) - \frac{1}{2!}f''(c)(x-c)^2 - \frac{1}{3!}f'''(c)(x-c)^3 \\ &\quad - \cdots - \frac{1}{n!}f^{(n)}(c)(x-c)^n - R_n(x)\frac{(x-c)^{n+1}}{(x-c)^{n+1}} \\ &= f(x) - P_n(x) - R_n(x) = R_n(x) - R_n(x) = 0. \end{aligned}$$

By Rolle's Theorem, there must be some number z between x and c for which $g'(z) = 0$. Differentiating our expression for $g(t)$ (with respect to t), we get (beware of all the product rules!)

$$\begin{aligned} g'(t) &= 0 - f'(t) - f'(t)(-1) - f''(t)(x-t) - \frac{1}{2}f''(t)(2)(x-t)(-1) \\ &\quad - \frac{1}{2}f'''(t)(x-t)^2 - \cdots - \frac{1}{n!}f^{(n)}(t)(n)(x-t)^{n-1}(-1) \\ &\quad - \frac{1}{n!}f^{(n+1)}(t)(x-t)^n - R_n(x)\frac{(n+1)(x-t)^n(-1)}{(x-c)^{n+1}} \\ &= -\frac{1}{n!}f^{(n+1)}(t)(x-t)^n + R_n(x)\frac{(n+1)(x-t)^n}{(x-c)^{n+1}}, \end{aligned}$$

after most of the terms cancel. So, taking $t = z$, we have that

$$0 = g'(z) = -\frac{1}{n!}f^{(n+1)}(z)(x-z)^n + R_n(x)\frac{(n+1)(x-z)^n}{(x-c)^{n+1}}.$$

Solving this for the term containing $R_n(x)$, we get

$$R_n(x)\frac{(n+1)(x-z)^n}{(x-c)^{n+1}} = \frac{1}{n!}f^{(n+1)}(z)(x-z)^n$$

and finally,

$$\begin{aligned} R_n(x) &= \frac{1}{n!}f^{(n+1)}(z)(x-z)^n\frac{(x-c)^{n+1}}{(n+1)(x-z)^n} \\ &= \frac{f^{(n+1)}(z)}{(n+1)n!}(x-c)^{n+1} \\ &= \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}, \end{aligned}$$

as we had claimed.

BEYOND FORMULAS

You should think of this section as giving a general procedure for finding power series representations, where section 8.6 solves that problem only for special cases. Further, we can utilize the idea that if a function is the sum of a convergent power series, then we can approximate the function with a partial sum of the series. Taylor's Theorem provides us with an estimate of the error in a given approximation and tells us that (in general) the approximation is improved by taking more terms.

EXERCISES 8.7

WRITING EXERCISES

1. Describe how the Taylor polynomial with $n = 1$ compares to the linear approximation. (See section 3.1.) Give an analogous interpretation of the Taylor polynomial with $n = 2$. That is, how do various graphical properties (position, slope, concavity) of the Taylor polynomial compare with those of the function $f(x)$ at $x = c$?

2. Briefly discuss how a computer might use Taylor polynomials to compute sin(1.2). In particular, how would the computer know how many terms to compute? How would the number of terms necessary to compute sin(1.2) compare to the number needed to compute sin(100)? Describe a trick that would make it much easier for the computer to compute sin(100). (Hint: The sine function is periodic.)

3. Taylor polynomials are built up from a knowledge of $f(c)$, $f'(c)$, $f''(c)$ and so on. Explain in graphical terms why information at one point (e.g., position, slope, concavity, etc.) can be used to construct the graph of the function on the entire interval of convergence.

4. If $f(c)$ is the position of an object at time $t = c$, then $f'(c)$ is the object's velocity and $f''(c)$ is the object's acceleration at time c. Explain in physical terms how knowledge of these values at one time (plus $f'''(c)$, etc.) can be used to predict the position of the object on the interval of convergence.

5. Our table of common Taylor series lists two different series for $\sin x$. Explain how the same function could have two different Taylor series representations. For a given problem (e.g., approximate sin 2), explain how you would choose which Taylor series to use.

6. Explain why the Taylor series with center $c = 0$ of $f(x) = x^2 - 1$ is simply $x^2 - 1$.

In exercises 1–8, find the Maclaurin series (i.e., Taylor series about $c = 0$) and its interval of convergence.

1. $f(x) = \cos x$
2. $f(x) = \sin x$
3. $f(x) = e^{2x}$
4. $f(x) = \cos 2x$
5. $f(x) = \ln(1 + x)$
6. $f(x) = e^{-x}$
7. $f(x) = 1/(1 + x)^2$
8. $f(x) = 1/(1 - x)$

In exercises 9–14, find the Taylor series about the indicated center and determine the interval of convergence.

9. $f(x) = e^{x-1}, c = 1$

10. $f(x) = \cos x, c = -\pi/2$

11. $f(x) = \ln x, c = e$

12. $f(x) = e^x, c = 2$

13. $f(x) = 1/x, c = 1$

14. $f(x) = 1/x, c = -1$

In exercises 15–20, graph $f(x)$ and the Taylor polynomials for the indicated center c and degree n.

15. $f(x) = \sqrt{x}, c = 1, n = 3; n = 6$

16. $f(x) = \dfrac{1}{1+x}, c = 0, n = 4; n = 8$

17. $f(x) = e^x, c = 2, n = 3; n = 6$

18. $f(x) = \cos x, c = \pi/2, n = 4; n = 8$

19. $f(x) = \sin^{-1} x, c = 0, n = 3; n = 5$

20. $f(x) = \ln x, c = 1, n = 4; n = 8$

In exercises 21–24, prove that the Taylor series converges to $f(x)$ by showing that $R_n(x) \to 0$ as $n \to \infty$.

21. $\sin x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

22. $\cos x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

23. $\ln x = \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}, 1 \le x \le 2$

24. $e^{-x} = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

In exercises 25–28, (a) use a Taylor polynomial of degree 4 to approximate the given number, (b) estimate the error in the approximation and (c) estimate the number of terms needed in a Taylor polynomial to guarantee an accuracy of 10^{-10}.

25. $\ln(1.05)$

26. $\ln(0.9)$

27. $\sqrt{1.1}$

28. $\sqrt{1.2}$

In exercises 29–32, use a Taylor series to verify the given formula.

29. $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!} = e^2$

30. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} = 0$

31. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$

32. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$

In exercises 33–38, use a known Taylor series to find the Taylor series about $c = 0$ for the given function and find its radius of convergence.

33. $f(x) = e^{-3x}$

34. $f(x) = \dfrac{e^x - 1}{x}$

35. $f(x) = xe^{-x^2}$

36. $f(x) = \sin x^2$

37. $f(x) = x \sin 2x$

38. $f(x) = \cos x^3$

39. You may have wondered why it is necessary to show that $\lim_{n\to\infty} R_n(x) = 0$ to conclude that a Taylor series converges to $f(x)$. Consider $f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$. Show that $f'(0) = f''(0) = 0$. (Hint: Use the fact that $\displaystyle\lim_{h\to 0} \frac{e^{-1/h^2}}{h^n} = 0$ for any positive integer n.) It turns out that $f^{(n)}(0) = 0$ for all n. Thus, the Taylor series of $f(x)$ about $c = 0$ equals 0, a convergent "series" that does not converge to $f(x)$.

40. In many applications, the **error function** $\text{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-u^2}\, du$ is important. Compute and graph the fourth-order Taylor polynomial for erf(x) about $c = 0$.

41. Find the Taylor series of $f(x) = |x|$ with center $c = 1$. Argue that the radius of convergence is ∞. However, show that the Taylor series of $f(x)$ does not converge to $f(x)$ for all x.

42. Find the Maclaurin series of $f(x) = \sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}$ for some nonzero constant a.

43. Prove that if f and g are functions such that $f''(x)$ and $g''(x)$ exist for all x and $\displaystyle\lim_{x\to a} \frac{f(x) - g(x)}{(x-a)^2} = 0$, then $f(a) = g(a)$, $f'(a) = g'(a)$ and $f''(a) = g''(a)$. What does this imply about the Taylor series for $f(x)$ and $g(x)$?

44. Generalize exercise 43 by proving that if f and g are functions such that for some positive integer n, $f^{(n)}(x)$ and $g^{(n)}(x)$ exist for all x and $\displaystyle\lim_{x\to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$, then $f^{(k)}(a) = g^{(k)}(a)$ for $0 \le k \le n$.

45. We have seen that $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} + \cdots$. Determine how many terms are needed to approximate sin 1 to within 10^{-5}. Show that $\sin 1 = \int_0^1 \cos x\, dx$. Determine how many points are needed for Simpson's Rule to approximate this integral to within 10^{-5}. Compare the efficiency of Maclaurin series and Simpson's Rule for this problem.

46. As in exercise 45, compare the efficiency of Maclaurin series and Simpson's Rule in estimating e to within 10^{-5}.

47. Find the first five terms in the Taylor series about $c = 0$ for $f(x) = e^x \sin x$ and compare to the product of the Taylor polynomials about $c = 0$ of e^x and $\sin x$.

48. Find the first five terms in the Taylor series about $c = 0$ for $f(x) = \tan x$ and compare to the quotient of the Taylor polynomials about $c = 0$ of $\sin x$ and $\cos x$.

49. Find the first four nonzero terms in the Maclaurin series of $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$ and compare to the Maclaurin series for $\sin x$.

50. Find the Taylor series of $f(x) = x \ln x$ about $c = 1$. Compare to the Taylor series for $\ln x$ about $c = 1$.

51. Suppose that a plane is at location $f(0) = 10$ miles with velocity $f'(0) = 10$ miles/min, acceleration $f''(0) = 2$ miles/min^2

and $f'''(0) = -1$ miles/min^3. Predict the location of the plane at time $t = 2$ min.

52. Suppose that an astronaut is at (0, 0) and the moon is represented by a circle of radius 1 centered at (10, 5). The astronaut's capsule follows a path $y = f(x)$ with current position $f(0) = 0$, slope $f'(0) = 1/5$, concavity $f''(0) = -1/10$, $f'''(0) = 1/25$, $f^{(4)}(0) = 1/25$ and $f^{(5)}(0) = -1/50$. Graph a Taylor polynomial approximation of $f(x)$. Based on your current information, do you advise the astronaut to change paths? How confident are you in the accuracy of your approximation?

53. Find the Taylor series for e^x about a general center c.

54. Find the Taylor series for $\sqrt{x}$ about a general center $c = a^2$.

Exercises 55–58 involve the binomial expansion.

55. Show that the Maclaurin series for $(1+x)^r$ is $1 + \sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k$, for any constant r.

56. Simplify the series in exercise 55 for $r = 2$; $r = 3$; r is a positive integer.

57. Use the result of exercise 55 to write out the Maclaurin series for $f(x) = \sqrt{1+x}$.

58. Use the result of exercise 55 to write out the Maclaurin series for $f(x) = (1+x)^{3/2}$.

59. Find the Maclaurin series of $f(x) = \cosh x$ and $f(x) = \sinh x$. Compare to the Maclaurin series of $\cos x$ and $\sin x$.

60. Use the Maclaurin series for $\tan x$ and the result of exercise 59 to conjecture the Maclaurin series for $\tanh x$.

EXPLORATORY EXERCISES

1. Almost all of our series results apply to series of complex numbers. Defining $i = \sqrt{-1}$, show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and so on. Replacing x with ix in the Maclaurin series for e^x, separate terms containing i from those that don't contain i (after the simplifications indicated above) and derive **Euler's formula:** $e^{ix} = \cos x + i \sin x$.

2. Using the technique of exercise 1, show that $\cos(ix) = \cosh x$ and $\sin(ix) = i \sinh x$. That is, the trig functions and their hyperbolic counterparts are closely related as functions of complex variables.

3. The method used in examples 7.3, 7.5, 7.6 and 7.7 does not require us to actually find $R_n(x)$, but to approximate it with a worst-case bound. Often this approximation is fairly close to $R_n(x)$, but this is not always true. As an extreme example of this, show that the bound on $R_n(x)$ for $f(x) = \ln x$ about $c = 1$ (see exercise 23) increases without bound for $0 < x < \frac{1}{2}$. Explain why this does not necessarily mean that the actual error increases without bound. In fact, $R_n(x) \to 0$ for $0 < x < \frac{1}{2}$ but we must show this using some other method. Use integration of an appropriate power series to show that $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$ converges to $\ln x$ for $0 < x < \frac{1}{2}$.

4. Verify numerically that if a_1 is close to π, the sequence $a_{n+1} = a_n + \sin a_n$ converges to π. (In other words, if a_n is an approximation of π, then $a_n + \sin a_n$ is a better approximation.) To prove this, find the Taylor series for $\sin x$ about $c = \pi$. Use this to show that if $\pi < a_n < 2\pi$, then $\pi < a_{n+1} < a_n$. Similarly, show that if $0 < a_n < \pi$, then $a_n < a_{n+1} < \pi$.

8.8 APPLICATIONS OF TAYLOR SERIES

In section 8.7, we developed the concept of a Taylor series expansion and gave many illustrations of how to compute these. In this section, we expand on our earlier presentation, by giving a few examples of how Taylor series are used to approximate the values of transcendental functions, evaluate limits and integrals and define important new functions. These represent but a small sampling of the important applications of Taylor series.

First, consider how calculators and computers might calculate values of transcendental functions, such as sin(1.234567). We illustrate this in example 8.1.

EXAMPLE 8.1 Using Taylor Polynomials to Approximate a Sine Value

Use a Taylor series to approximate sin(1.234567) accurate to within 10^{-11}.

Solution In section 8.7, we left it as an exercise to show that the Taylor series expansion for $f(x) = \sin x$ about $x = 0$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Notice that if we take $x = 1.234567$, the series representation of $\sin 1.234567$ is

$$\sin 1.234567 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (1.234567)^{2k+1},$$

which is an alternating series. We can use a partial sum of this series to approximate the desired value, but how many terms will we need for the desired accuracy? Recall that for alternating series, the error in a partial sum is bounded by the absolute value of the first neglected term. (Note that you could also use the remainder term from Taylor's Theorem to bound the error.) To ensure that the error is less than 10^{-11}, we must find an integer k such that $\frac{1.234567^{2k+1}}{(2k+1)!} < 10^{-11}$. By trial and error, we find that

$$\frac{1.234567^{17}}{17!} \approx 1.010836 \times 10^{-13} < 10^{-11},$$

so that $k = 8$ will do. This says that the first neglected term corresponds to $k = 8$ and so, we compute the partial sum

$$\begin{aligned} \sin 1.234567 &\approx \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!} (1.234567)^{2k+1} \\ &= 1.234567 - \frac{1.234567^3}{3!} + \frac{1.234567^5}{5!} - \frac{1.234567^7}{7!} + \cdots - \frac{1.234567^{15}}{15!} \\ &\approx 0.94400543137. \end{aligned}$$

Check your calculator or computer to verify that this matches your calculator's estimate.

In example 8.1, while we produced an approximation with the desired accuracy, we did not do this in the most efficient fashion, as we simply grabbed the most handy Taylor series expansion of $f(x) = \sin x$. You should try to resist the impulse to automatically use the Taylor series expansion about $x = 0$ (i.e., the Maclaurin series), rather than making a more efficient choice. We illustrate this in example 8.2.

EXAMPLE 8.2 Choosing a More Appropriate Taylor Series Expansion

Repeat example 8.1, but this time, make a more appropriate choice of the Taylor series.

Solution Recall that Taylor series converge much faster close to the point about which you expand, than they do far away. Given this and the fact that we know the exact value of $\sin x$ at only a few points, you should quickly recognize that a series expanded about $x = \frac{\pi}{2} \approx 1.57$ is a better choice for computing $\sin 1.234567$ than one expanded about $x = 0$. (Another reasonable choice is the Taylor series expansion about $x = \frac{\pi}{3}$.) In example 7.5, recall that we had found that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Taking $x = 1.234567$ gives us

$$\begin{aligned}\sin 1.234567 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k} \\ &= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 - \cdots,\end{aligned}$$

which is again an alternating series. Using the remainder term from Taylor's Theorem to bound the error, we have that

$$\begin{aligned}|R_n(1.234567)| &= \left|\frac{f^{(2n+2)}(z)}{(2n+2)!}\right| \left|1.234567 - \frac{\pi}{2}\right|^{2n+2} \\ &\le \frac{\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}}{(2n+2)!}.\end{aligned}$$

(Note that we get the same error bound if we use the error bound for an alternating series.) By trial and error, you can find that

$$\frac{\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}}{(2n+2)!} < 10^{-11}$$

for $n = 4$, so that an approximation with the required degree of accuracy is

$$\begin{aligned}\sin 1.234567 &\approx \sum_{k=0}^{4} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k} \\ &= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 \\ &\quad - \frac{1}{6!}\left(1.234567 - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(1.234567 - \frac{\pi}{2}\right)^8 \\ &\approx 0.94400543137.\end{aligned}$$

Compare this result to example 8.1, where we needed many more terms of the Taylor series to obtain the same degree of accuracy. ■

We can also use Taylor series to quickly conjecture the value of difficult limits. Be careful, though: the theory of when these conjectures are guaranteed to be correct is beyond the level of this text. However, we can certainly obtain helpful hints about certain limits.

EXAMPLE 8.3 Using Taylor Polynomials to Conjecture the Value of a Limit

Use Taylor series to conjecture $\lim_{x \to 0} \dfrac{\sin x^3 - x^3}{x^9}$.

Solution Again recall that the Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Substituting x^3 for x gives us

$$\sin x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^3)^{2k+1} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

This gives us

$$\frac{\sin x^3 - x^3}{x^9} = \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9} = -\frac{1}{3!} + \frac{x^6}{5!} + \cdots$$

and so, we conjecture that

$$\lim_{x \to 0} \frac{\sin x^3 - x^3}{x^9} = -\frac{1}{3!} = -\frac{1}{6}.$$

You can verify that this limit is correct using l'Hôpital's Rule (three times, simplifying each time). ■

Since Taylor polynomials are used to approximate functions on a given interval and since polynomials are easy to integrate, we can use a Taylor polynomial to obtain an approximation of a definite integral. It turns out that such an approximation is often better than that obtained from the numerical methods developed in section 4.7. We illustrate this in example 8.4.

EXAMPLE 8.4 Using Taylor Series to Approximate a Definite Integral

Use a Taylor polynomial with $n = 8$ to approximate $\int_{-1}^{1} \cos(x^2)\,dx$.

Solution Since we do not know an antiderivative of $\cos(x^2)$, we must rely on a numerical approximation of the integral. Since we are integrating on the interval $(-1, 1)$, a Maclaurin series expansion (i.e., a Taylor series expansion about $x = 0$) is a good choice. We have

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

which converges on all of $(-\infty, \infty)$. Replacing x by x^2 gives us

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots,$$

so that

$$\cos(x^2) \approx 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8.$$

This leads us to the approximation

$$\begin{aligned}\int_{-1}^{1} \cos(x^2)\,dx &\approx \int_{-1}^{1} \left(1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8\right) dx \\ &= \left(x - \frac{x^5}{10} + \frac{x^9}{216}\right)\Bigg|_{x=-1}^{x=1} \\ &= \frac{977}{540} \approx 1.809259.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1} \cos(x^2)\,dx \approx 1.809048$, so our approximation appears to be very accurate. ■

You might reasonably argue that we don't need Taylor series to obtain approximations like those in example 8.4, as you could always use other, simpler numerical methods like Simpson's Rule to do the job. That's often true, but just try to use Simpson's Rule on the integral in example 8.5.

EXAMPLE 8.5 Using Taylor Series to Approximate the Value of an Integral

Use a Taylor polynomial with $n = 5$ to approximate $\int_{-1}^{1} \frac{\sin x}{x}\,dx$.

Solution Note that you do not know an antiderivative of $\frac{\sin x}{x}$. Further, while the integrand is discontinuous at $x = 0$, this does *not* need to be treated as an improper integral, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$. (This says that the integrand has a removable discontinuity at $x = 0$.) From the first few terms of the Maclaurin series for $f(x) = \sin x$, we have the Taylor polynomial approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

so that

$$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!}.$$

Consequently,

$$\begin{aligned}\int_{-1}^{1} \frac{\sin x}{x}\,dx &\approx \int_{-1}^{1}\left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)dx \\ &= \left(x - \frac{x^3}{18} + \frac{x^5}{600}\right)\Bigg|_{x=-1}^{x=1} \\ &= \left(1 - \frac{1}{18} + \frac{1}{600}\right) - \left(-1 + \frac{1}{18} - \frac{1}{600}\right) \\ &= \frac{1703}{900} \approx 1.89222.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1} \frac{\sin x}{x}\,dx \approx 1.89216$, so our approximation is quite good. On the other hand, if you try to apply Simpson's Rule or Trapezoidal Rule, the algorithm will not work, as they will attempt to evaluate $\frac{\sin x}{x}$ at $x = 0$. ■

While you have now calculated Taylor series expansions of many familiar functions, many other functions are actually *defined* by a power series. These include many functions in the very important class of **special functions** that frequently arise in physics and engineering applications. One important family of special functions are the Bessel functions, which arise in the study of fluid mechanics, acoustics, wave propagation and other areas of applied mathematics. The **Bessel function of order *p*** is defined by the power series

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k+p} k!(k+p)!}, \tag{8.1}$$

for nonnegative integers p. Bessel functions arise in the solution of the differential equation $x^2 y'' + xy' + (x^2 - p^2)y = 0$. In examples 8.6 and 8.7, we explore several interesting properties of Bessel functions.

EXAMPLE 8.6 The Radius of Convergence of a Bessel Function

Find the radius of convergence for the series defining the Bessel function $J_0(x)$.

Solution From equation (8.1) with $p = 0$, we have $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$. The Ratio Test gives us

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}}{2^{2k+2}[(k+1)!]^2}\,\frac{2^{2k}(k!)^2}{x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{4(k+1)^2}\right| = 0 < 1,$$

for all x. The series then converges absolutely for all x and so, the radius of convergence is ∞. ■

In example 8.7, we explore an interesting relationship between the zeros of two Bessel functions.

EXAMPLE 8.7 The Zeros of Bessel Functions

Verify graphically that on the interval [0, 10], the zeros of $J_0(x)$ and $J_1(x)$ alternate.

Solution Unless you have a CAS with these Bessel functions available as built-in functions, you will need to graph partial sums of the defining series:

$$J_0(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \quad \text{and} \quad J_1(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}.$$

Before graphing these, you must first determine how large n should be in order to produce a reasonable graph. Notice that for each fixed $x > 0$, both of the defining series are alternating series. Consequently, the error in using a partial sum to approximate the function is bounded by the first neglected term. That is,

$$\left|J_0(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2n+2}}{2^{2n+2}[(n+1)!]^2}$$

and

$$\left|J_1(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2n+3}}{2^{2n+3}(n+1)!(n+2)!},$$

with the maximum error in each occurring at $x = 10$. Notice that for $n = 12$, we have that

$$\left|J_0(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2(12)+2}}{2^{2(12)+2}[(12+1)!]^2} \le \frac{10^{26}}{2^{26}(13!)^2} < 0.04$$

and

$$\left|J_1(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2(12)+3}}{2^{2(12)+3}(12+1)!(12+2)!} \le \frac{10^{27}}{2^{27}(13!)(14!)} < 0.04.$$

So, in either case, using a partial sum with $n = 12$ results in an approximation that is within 0.04 of the correct value for each x in the interval [0, 10]. This is plenty of accuracy for our present purposes. Figure 8.43 shows graphs of partial sums with $n = 12$ for $J_0(x)$ and $J_1(x)$.

Notice that $J_1(0) = 0$ and in the figure, you can clearly see that $J_0(x) = 0$ at about $x = 2.4$, $J_1(x) = 0$ at about $x = 3.9$, $J_0(x) = 0$ at about $x = 5.6$, $J_1(x) = 0$ at about $x = 7.0$ and $J_0(x) = 0$ at about $x = 8.8$. From this, it is now apparent that the zeros of $J_0(x)$ and $J_1(x)$ do indeed alternate on the interval [0, 10]. ■

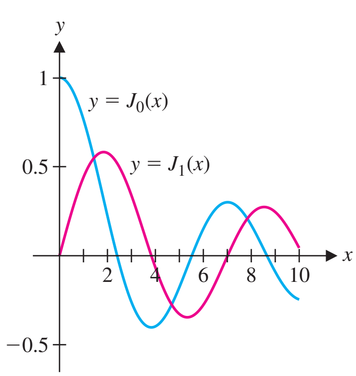


FIGURE 8.43
$y = J_0(x)$ and $y = J_1(x)$

It turns out that the result of example 8.7 generalizes to any interval of positive numbers and any two Bessel functions of consecutive order. That is, between consecutive zeros of

$J_p(x)$ is a zero of $J_{p+1}(x)$ and between consecutive zeros of $J_{p+1}(x)$ is a zero of $J_p(x)$. We explore this further in the exercises.

The Binomial Series

You are already familiar with the Binomial Theorem, which states that for any positive integer n,

$$(a+b)^n = a^n + na^{n-1}b + n\,(n-1)\,a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

We often write this as
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where we use the shorthand notation $\binom{n}{k}$ to denote the binomial coefficient, defined by

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = \frac{n(n-1)}{2} \quad \text{and}$$
$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{for } k \geq 3.$$

For the case where $a = 1$ and $b = x$, the Binomial Theorem simplifies to

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Newton discovered that this result could be extended to include values of n other than positive integers. What resulted is a special type of power series known as the *binomial series,* which has important applications in statistics and physics. We begin by deriving the Maclaurin series for $f(x) = (1+x)^n$, for some constant $n \neq 0$. Computing derivatives and evaluating these at $x = 0$, we have

$$\begin{aligned}
f(x) &= (1+x)^n & f(0) &= 1\\
f'(x) &= n(1+x)^{n-1} & f'(0) &= n\\
f''(x) &= n(n-1)(1+x)^{n-2} & f''(0) &= n(n-1)\\
&\ \vdots & &\ \vdots\\
f^{(k)}(x) &= n(n-1)\cdots(n-k+1) & f^{(k)}(0) &= n(n-1)\cdots(n-k+1).
\end{aligned}$$

We call the resulting Maclaurin series the **binomial series,** given by

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= 1 + nx + n(n-1)\frac{x^2}{2!} + \cdots + n(n-1)\cdots(n-k+1)\frac{x^k}{k!} + \cdots\\
&= \sum_{k=0}^{\infty} \binom{n}{k} x^k.
\end{aligned}$$

From the Ratio Test, we have

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{n(n-1)\cdots(n-k+1)(n-k)x^{k+1}}{(k+1)!}\,\frac{k!}{n(n-1)\cdots(n-k+1)x^k}\right|\\
&= |x|\lim_{k\to\infty}\frac{|n-k|}{k+1} = |x|,
\end{aligned}$$

so that the binomial series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. By showing that the remainder term $R_k(x)$ tends to zero as $k \to \infty$, we can confirm that the binomial series converges to $(1+x)^n$ for $|x| < 1$. We state this formally in Theorem 8.1.

THEOREM 8.1 (Binomial Series)

For any real number r, $(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$, for $-1 < x < 1$.

As seen in the exercises, for some values of the exponent r, the binomial series also converges at one or both of the endpoints $x = \pm 1$.

EXAMPLE 8.8 Using the Binomial Series

Using the binomial series, find a Maclaurin series for $f(x) = \sqrt{1+x}$ and use it to approximate $\sqrt{17}$ accurate to within 0.000001.

Solution From the binomial series with $r = \frac{1}{2}$, we have

$$\begin{aligned}\sqrt{1+x} = (1+x)^{1/2} &= \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots,\end{aligned}$$

for $-1 < x < 1$. To use this to approximate $\sqrt{17}$, we first rewrite it in a form involving $\sqrt{1+x}$, for $-1 < x < 1$. Observe that we can do this by writing

$$\sqrt{17} = \sqrt{16 \cdot \frac{17}{16}} = 4\sqrt{\frac{17}{16}} = 4\sqrt{1 + \frac{1}{16}}.$$

Since $x = \frac{1}{16}$ is in the interval of convergence, $-1 < x < 1$, the binomial series gives us

$$\sqrt{17} = 4\sqrt{1 + \frac{1}{16}} = 4\left[1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3 - \frac{5}{128}\left(\frac{1}{16}\right)^4 + \cdots\right].$$

Since this is an alternating series, the error in using the first n terms to approximate the sum is bounded by the first neglected term. So, if we use only the first three terms of the series, the error is bounded by $\frac{1}{16}\left(\frac{1}{16}\right)^3 \approx 0.000015 > 0.000001$. Similarly, if we use the first four terms of the series to approximate the sum, the error is bounded by $\frac{5}{128}\left(\frac{1}{16}\right)^4 \approx 0.0000006 < 0.000001$, as desired. So, we can achieve the desired accuracy by summing the first four terms of the series:

$$\sqrt{17} \approx 4\left[1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3\right] \approx 4.1231079,$$

where this approximation is accurate to within the desired accuracy. ■

EXERCISES 8.8

WRITING EXERCISES

1. In example 8.2, we showed that an expansion about $x = \frac{\pi}{2}$ is more accurate for approximating sin(1.234567) than an expansion about $x = 0$ with the same number of terms. Explain why an expansion about $x = 1.2$ would be even more efficient, but is not practical.

2. Assuming that you don't need to rederive the Maclaurin series for cos x, compare the amount of work done in example 8.4 to the work needed to compute a Simpson's Rule approximation with $n = 16$.

3. In equation (8.1), we defined the Bessel functions as series. This may seem like a convoluted way of defining a function, but compare the levels of difficulty doing the following with a Bessel function versus $\sin x$: computing $f(0)$, computing $f(1.2)$, evaluating $f(2x)$, computing $f'(x)$, computing $\int f(x)\,dx$ and computing $\int_0^1 f(x)\,dx$.

4. Discuss how you might estimate the error in the approximation of example 8.4.

In exercises 1–6, use an appropriate Taylor series to approximate the given value, accurate to within 10^{-11}.

1. $\sin 1.61$
2. $\sin 6.32$
3. $\cos 0.34$
4. $\cos 3.04$
5. $e^{-0.2}$
6. $e^{0.4}$

In exercises 7–12, use a known Taylor series to conjecture the value of the limit.

7. $\lim_{x\to 0} \dfrac{\cos x^2 - 1}{x^4}$
8. $\lim_{x\to 0} \dfrac{\sin x^2 - x^2}{x^6}$
9. $\lim_{x\to 1} \dfrac{\ln x - (x-1)}{(x-1)^2}$
10. $\lim_{x\to 0} \dfrac{\tan^{-1} x - x}{x^3}$
11. $\lim_{x\to 0} \dfrac{e^x - 1}{x}$
12. $\lim_{x\to 0} \dfrac{e^{-2x} - 1}{x}$

In exercises 13–18, use a known Taylor polynomial with n nonzero terms to estimate the value of the integral.

13. $\displaystyle\int_{-1}^{1} \frac{\sin x}{x}\,dx,\ n = 3$
14. $\displaystyle\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2\,dx,\ n = 4$
15. $\displaystyle\int_{-1}^{1} e^{-x^2}\,dx,\ n = 5$
16. $\displaystyle\int_{0}^{1} \tan^{-1} x\,dx,\ n = 5$
17. $\displaystyle\int_{1}^{2} \ln x\,dx,\ n = 5$
18. $\displaystyle\int_{0}^{1} e^{\sqrt{x}}\,dx,\ n = 4$

19. Find the radius of convergence of $J_1(x)$.

20. Find the radius of convergence of $J_2(x)$.

21. Find the number of terms needed to approximate $J_2(x)$ within 0.04 for x in the interval [0, 10].

22. Show graphically that the zeros of $J_1(x)$ and $J_2(x)$ alternate on the interval (0, 10].

23. Einstein's theory of relativity states that the mass of an object traveling at velocity v is $m(v) = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the rest mass of the object and c is the speed of light. Show that $m \approx m_0 + \left(\dfrac{m_0}{2c^2}\right)v^2$. Use this approximation to estimate how large v would need to be to increase the mass by 10%.

24. Find the fourth-degree Taylor polynomial expanded about $v = 0$, for $m(v)$ in exercise 23.

25. The weight (force due to gravity) of an object of mass m and altitude x miles above the surface of the earth is $w(x) = \dfrac{mgR^2}{(R+x)^2}$, where R is the radius of the earth and g is the acceleration due to gravity. Show that $w(x) \approx mg(1 - 2x/R)$. Estimate how large x would need to be to reduce the weight by 10%.

26. Find the second-degree Taylor polynomial for $w(x)$ in exercise 25. Use it to estimate how large x needs to be to reduce the weight by 10%.

27. Based on your answers to exercises 25 and 26, is weight significantly different at a high-altitude location (e.g., 7500 ft) compared to sea level?

28. The radius of the earth is up to 300 miles larger at the equator than it is at the poles. Which would have a larger effect on weight, altitude or latitude?

In exercises 29–32, use the Maclaurin series expansion $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \cdots$.

29. The tangential component of the space shuttle's velocity during reentry is approximately $v(t) = v_c \tanh\left(\dfrac{g}{v_c}t + \tanh^{-1}\dfrac{v_0}{v_c}\right)$, where v_0 is the velocity at time 0 and v_c is the terminal velocity (see Long and Weiss, *The American Mathematical Monthly,* February 1999). If $\tanh^{-1}\dfrac{v_0}{v_c} = \dfrac{1}{2}$, show that $v(t) \approx gt + \dfrac{1}{2}v_c$. Is this estimate of $v(t)$ too large or too small?

30. Show that in exercise 29, $v(t) \to v_c$ as $t \to \infty$. Use the approximation in exercise 29 to estimate the time needed to reach 90% of the terminal velocity.

31. The downward velocity of a sky diver of mass m is $v(t) = \sqrt{40mg}\ \tanh\left(\sqrt{\dfrac{g}{40m}}t\right)$. Show that $v(t) \approx gt - \dfrac{g^2}{120m}t^3$.

32. The velocity of a water wave of length L in water of depth h satisfies the equation $v^2 = \dfrac{gL}{2\pi}\tanh\dfrac{2\pi h}{L}$. Show that $v \approx \sqrt{gh}$.

In exercises 33–36, use the Binomial Theorem to find the first five terms of the Maclaurin series.

33. $f(x) = \dfrac{1}{\sqrt{1-x}}$
34. $f(x) = \sqrt[3]{1+2x}$
35. $f(x) = \dfrac{6}{\sqrt[3]{1+3x}}$
36. $f(x) = (1+x^2)^{4/5}$

In exercises 37 and 38, use the Binomial Theorem to approximate the value to within 10^{-6}.

37. (a) $\sqrt{26}$ (b) $\sqrt{24}$
38. (a) $\dfrac{2}{\sqrt[3]{9}}$ (b) $\sqrt[4]{17}$

39. Apply the Binomial Theorem to $(x+4)^3$ and $(1-2x)^4$. Determine the number of nonzero terms in the binomial expansion for any positive integer n.

40. If n and k are positive integers with $n > k$, show that $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

41. Use exercise 33 to find the Maclaurin series for $\dfrac{1}{\sqrt{1-x^2}}$ and use it to find the Maclaurin series for $\sin^{-1} x$.

42. Use the Binomial Theorem to find the Maclaurin series for $(1+2x)^{4/3}$ and compare this series to that of exercise 34.

43. Use a Taylor polynomial to estimate $\int_0^{\pi} \frac{\sin x}{x}\,dx$ accurate to within 0.00001. (This value will be used in the next section.)

44. Use a Taylor polynomial to conjecture the value of $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$ and then confirm your conjecture using l'Hôpital's Rule.

45. The energy density of electromagnetic radiation at wavelength λ from a black body at temperature T (degrees Kelvin) is given by **Planck's law** of black body radiation: $f(\lambda) = \dfrac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)}$, where h is Planck's constant, c is the speed of light and k is Boltzmann's constant. To find the wavelength of peak emission, maximize $f(\lambda)$ by minimizing $g(\lambda) = \lambda^5(e^{hc/\lambda kT} - 1)$. Use a Taylor polynomial for e^x with $n = 7$ to expand the expression in parentheses and find the critical number of the resulting function. (Hint: Use $\frac{hc}{k} \approx 0.014$.) Compare this to **Wien's law:** $\lambda_{max} = \dfrac{0.002898}{T}$. Wien's law is accurate for small λ. Discuss the flaw in our use of Maclaurin series.

46. Use a Taylor polynomial for e^x to expand the denominator in Planck's law of exercise 45 and show that $f(\lambda) \approx \dfrac{8\pi kT}{\lambda^4}$. State whether this approximation is better for small or large wavelengths λ. This is known in physics as the **Rayleigh-Jeans law.**

47. The power of a reflecting telescope is proportional to the surface area S of the parabolic reflector, where $S = \dfrac{8\pi}{3}c^2\left[\left(\dfrac{d^2}{16c^2}+1\right)^{3/2} - 1\right]$. Here, d is the diameter of the parabolic reflector, which has depth k with $c = \dfrac{d^2}{4k}$. Expand the term $\left(\dfrac{d^2}{16c^2}+1\right)^{3/2}$ and show that if $\dfrac{d^2}{16c^2}$ is small, then $S \approx \dfrac{\pi d^2}{4}$.

48. A disk of radius a has a charge of constant density σ. Point P lies at a distance r directly above the disk. The **electrical potential** at point P is given by $V = 2\pi\sigma(\sqrt{r^2+a^2} - r)$. Show that for large r, $V \approx \dfrac{\pi a^2 \sigma}{r}$.

EXPLORATORY EXERCISES

1. The Bessel functions and **Legendre polynomials** are examples of the so-called special functions. For nonnegative integers n, the Legendre polynomials are defined by

$$P_n(x) = 2^{-n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{(n-k)!k!(n-2k)!}x^{n-2k}.$$

Here, $[n/2]$ is the greatest integer less than or equal to $n/2$ (for example, $[1/2] = 0$ and $[2/2] = 1$). Show that $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. Show that for these three functions,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \quad \text{for } m \neq n.$$

This fact, which is true for all Legendre polynomials, is called the **orthogonality condition.** Orthogonal functions are commonly used to provide simple representations of complicated functions.

2. Use the Ratio Test to show that the radius of convergence of $\sum_{k=0}^{\infty}\binom{n}{k}x^k$ is 1. (a) If $n \le -1$, show that the interval of convergence is $(-1, 1)$. (b) If $n > 0$ and n is not an integer, show that the interval of convergence is $[-1, 1]$. (c) If $-1 < n < 0$, show that the interval of convergence is $(-1, 1]$.

3. Suppose that p is an approximation of π with $|p - \pi| < 0.001$. Explain why p has two digits of accuracy and has a decimal expansion that starts $p = 3.14\ldots$. Use Taylor's Theorem to show that $p + \sin p$ has six digits of accuracy. In general, if p has n digits of accuracy, show that $p + \sin p$ has $3n$ digits of accuracy. Compare this to the accuracy of $p - \tan p$.

8.9 FOURIER SERIES

Many phenomena we encounter in the world around us are periodic in nature. That is, they repeat themselves over and over again. For instance, light, sound, radio waves and x-rays are all periodic. For such phenomena, Taylor polynomial approximations have shortcomings. As x gets farther away from c (the point about which you expanded), the difference between the function and a given Taylor polynomial grows. Such behavior is illustrated in Figure 8.44 for the case of $f(x) = \sin x$ expanded about $x = \frac{\pi}{2}$.

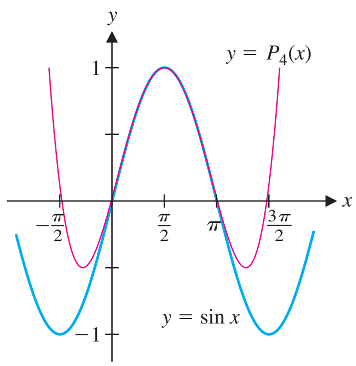


FIGURE 8.44
$y = \sin x$ and $y = P_4(x)$

Because Taylor polynomials provide an accurate approximation only in the vicinity of c, we say that they are accurate *locally*. In general, no matter how large you make n, the approximation is still only valid locally. In many situations, notably in communications, we need to find an approximation to a given periodic function that is valid *globally* (i.e., for all x). For this reason, we construct a different type of series expansion for periodic functions, one where each of the terms in the expansion is periodic.

Recall that we say that a function f is **periodic** of **period** $T > 0$ if $f(x+T) = f(x)$, for all x in the domain of f. Can you think of any periodic functions? Surely, $\sin x$ and $\cos x$ come to mind. These are both periodic of period 2π. Further, $\sin(2x)$, $\cos(2x)$, $\sin(3x)$, $\cos(3x)$ and so on are all periodic of period 2π. In fact,

$$\sin(kx) \text{ and } \cos(kx), \quad \text{for } k = 1, 2, 3, \ldots$$

are each periodic of period 2π, as follows. For any integer k, let $f(x) = \sin(kx)$. We then have

$$f(x+2\pi) = \sin[k(x+2\pi)] = \sin(kx + 2k\pi) = \sin(kx) = f(x).$$

Likewise, you can show that $\cos(kx)$ has period 2π.

So, if you wanted to expand a periodic function of period 2π in a series, you might consider a series each of whose terms has period 2π, for instance,

FOURIER SERIES

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)],$$

called a **Fourier series.** Notice that if the series converges, it will converge to a periodic function whose period is 2π, since every term in the series has period 2π. The coefficients of the series, $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, are called the **Fourier coefficients.** You may have noticed the unusual way in which we wrote the leading term of the series $\left(\frac{a_0}{2}\right)$. We did this in order to simplify the formulas for computing these coefficients, as we'll see later.

There are a number of important questions we must address.

- What functions can be expanded in a Fourier series?
- How do we compute the Fourier coefficients?
- Does the Fourier series converge? If so, to what function does the series converge?

We begin our investigation much as we did with power series. Suppose that a given Fourier series converges on the interval $[-\pi, \pi]$. It then represents a function f on that interval,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)], \tag{9.1}$$

where f must be periodic outside of $[-\pi, \pi]$. Although some of the details of the proof are beyond the level of this course, we want to give you some idea of how the Fourier coefficients are computed. If we integrate both sides of equation (9.1) with respect to x on the interval $[-\pi, \pi]$, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]\,dx \\ &= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{k=1}^{\infty} \left[a_k \int_{-\pi}^{\pi} \cos(kx)\,dx + b_k \int_{-\pi}^{\pi} \sin(kx)\,dx\right], \end{aligned} \tag{9.2}$$

HISTORICAL NOTES

Jean Baptiste Joseph Fourier (1768–1830) French mathematician who invented Fourier series. Fourier was heavily involved in French politics, becoming a member of the Revolutionary Committee, serving as scientific advisor to Napoleon and establishing educational facilities in Egypt. Fourier held numerous offices, including secretary of the Cairo Institute and Prefect of Grenoble. Fourier introduced his trigonometric series as an essential technique for developing his highly original and revolutionary theory of heat.

assuming we can interchange the order of integration and summation. In general, the order may *not* be interchanged (this is beyond the level of this course), but for many Fourier series, doing so is permissible. Observe that for every $k = 1, 2, 3, \ldots$, we have

$$\int_{-\pi}^{\pi} \cos(kx)\,dx = \frac{1}{k}\sin(kx)\Big|_{-\pi}^{\pi} = \frac{1}{k}[\sin(k\pi) - \sin(-k\pi)] = 0$$

and
$$\int_{-\pi}^{\pi} \sin(kx)\,dx = -\frac{1}{k}\cos(kx)\Big|_{-\pi}^{\pi} = -\frac{1}{k}[\cos(k\pi) - \cos(-k\pi)] = 0.$$

This reduces equation (9.2) to simply

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = a_0\,\pi.$$

Solving this for a_0, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{9.3}$$

Similarly, if we multiply both sides of equation (9.1) by $\cos(nx)$ (where n is an integer, $n \geq 1$) and then integrate with respect to x on the interval $[-\pi, \pi]$, we get

$$\begin{aligned}
\int_{-\pi}^{\pi} & f(x)\cos(nx)\,dx \\
&= \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(nx)\,dx \\
&\qquad + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} [a_k\cos(kx)\cos(nx) + b_k\sin(kx)\cos(nx)]\,dx \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi} \cos(nx)\,dx \\
&\qquad + \sum_{k=1}^{\infty}\left[a_k\int_{-\pi}^{\pi}\cos(kx)\cos(nx)\,dx + b_k\int_{-\pi}^{\pi}\sin(kx)\cos(nx)\,dx\right],
\end{aligned} \tag{9.4}$$

again assuming we can interchange the order of integration and summation. Next, recall that

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0, \quad \text{for all } n = 1, 2, \ldots.$$

It's a straightforward, yet lengthy exercise to show that

$$\int_{-\pi}^{\pi} \sin(kx)\cos(nx)\,dx = 0, \quad \text{for all } n = 1, 2, \ldots \text{ and for all } k = 1, 2, \ldots$$

and that
$$\int_{-\pi}^{\pi} \cos(kx)\cos(nx)\,dx = \begin{cases} 0, & \text{if } n \neq k \\ \pi, & \text{if } n = k \end{cases}.$$

Notice that this says that every term in the series in equation (9.4) except one (the term corresponding to $k = n$) is zero and equation (9.4) reduces to simply

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = a_n\pi.$$

This gives us (after substituting k for n)

Fourier coefficients

$$\boxed{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \quad \text{for } k = 1, 2, 3, \ldots.} \tag{9.5}$$

Similarly, multiplying both sides of equation (9.1) by $\sin(nx)$ and integrating from $-\pi$ to π gives us

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx, \quad \text{for } k = 1, 2, 3, \ldots. \tag{9.6}$$

Equations (9.3), (9.5) and (9.6) are called the **Euler-Fourier formulas.** Notice that equation (9.3) is the same as (9.5) with $k = 0$. (This was the reason we chose the leading term of the series to be $\frac{a_0}{2}$, instead of simply a_0.)

Let's summarize what we've done so far. We have observed that if a Fourier series converges on some interval, then it converges to a function f where the Fourier coefficients satisfy the Euler-Fourier formulas (9.3), (9.5) and (9.6).

Just as we did with power series, given any integrable function f, we can compute the coefficients in (9.3), (9.5) and (9.6) and write down a Fourier series. But, will the series converge and if it does, to what function will it converge? We'll answer these questions shortly. For the moment, let's simply compute the terms of a Fourier series and see what we can observe.

TODAY IN MATHEMATICS

Ingrid Daubechies (1954–)
A Belgian physicist and mathematician who pioneered the use of wavelets, which extend the ideas of Fourier series. In a talk on the relationship between algorithms and analysis, she explained that her wavelet research was of a type "stimulated by the requirements of engineering design rather than natural science problems, but equally interesting and possibly far-reaching." To meet the needs of an efficient image compression algorithm, she created the first continuous wavelet corresponding to a fast algorithm. The Daubechies wavelets are now the most commonly used wavelets in applications and were instrumental in the explosion of wavelet applications in areas as diverse as FBI fingerprinting, magnetic resonance imaging (MRI) and digital storage formats such as JPEG-2000.

EXAMPLE 9.1 Finding a Fourier Series Expansion

Find the Fourier series corresponding to the **square-wave** function

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \le 0 \\ 1, & \text{if } 0 < x \le \pi \end{cases},$$

where f is assumed to be periodic outside of the interval $[-\pi, \pi]$ (see the graph in Figure 8.45).

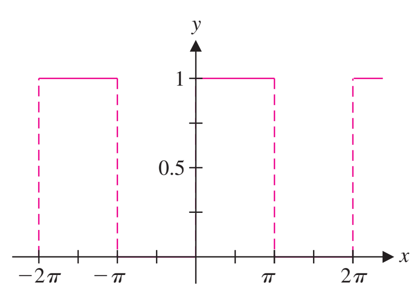


FIGURE 8.45
Square-wave function

Solution Even though a_0 satisfies the same formula as a_k for $k \ge 1$, we must always compute a_0 separately from the rest of the a_k's. From equation (9.3), we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{\pi}{\pi} = 1.$$

From (9.5), we also have that for $k \ge 1$,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(0)\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\cos(kx)\,dx \\ &= \frac{1}{\pi k}\sin(kx)\Big|_0^{\pi} = \frac{1}{\pi k}[\sin(k\pi) - \sin(0)] = 0. \end{aligned}$$

Finally, from (9.6), we have

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(0)\sin(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin(kx)\,dx \\ &= -\frac{1}{\pi k}\cos(kx)\Big|_0^{\pi} = -\frac{1}{\pi k}[\cos(k\pi) - \cos(0)] = -\frac{1}{\pi k}[(-1)^k - 1] \\ &= \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{2}{\pi k}, & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

Notice that we can write the even- and odd-indexed coefficients separately as $b_{2k} = 0$, for $k = 1, 2, \ldots$ and $b_{2k-1} = \dfrac{2}{(2k-1)\pi}$, for $k = 1, 2, \ldots$. We then have the Fourier series

$$\begin{aligned} \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)] &= \frac{1}{2} + \sum_{k=1}^{\infty} b_k\sin(kx) = \frac{1}{2} + \sum_{k=1}^{\infty} b_{2k-1}\sin[(2k-1)x] \\ &= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin[(2k-1)x] \\ &= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x) + \cdots. \end{aligned}$$

None of our existing convergence tests are appropriate for this series. Instead, we consider the graphs of the first few partial sums of the series defined by

$$F_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\frac{2}{(2k-1)\pi}\sin[(2k-1)x].$$

In Figures 8.46a–d, we graph a number of these partial sums.

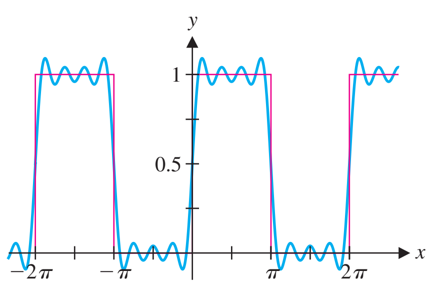


FIGURE 8.46a
$y = F_4(x)$ and $y = f(x)$

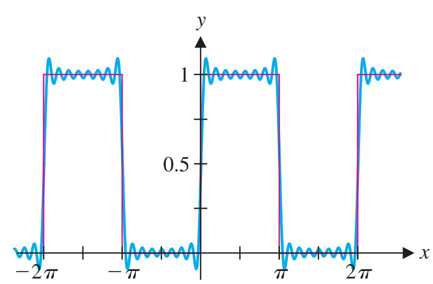


FIGURE 8.46b
$y = F_8(x)$ and $y = f(x)$

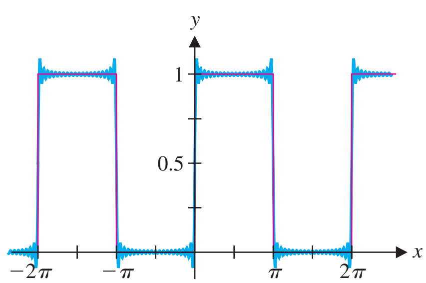


FIGURE 8.46c
$y = F_{20}(x)$ and $y = f(x)$

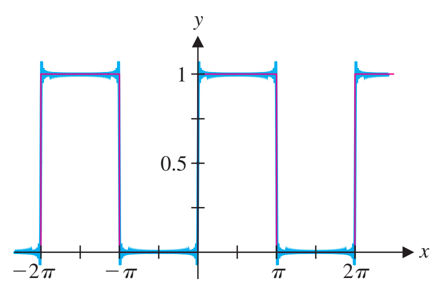


FIGURE 8.46d
$y = F_{50}(x)$ and $y = f(x)$

Notice that as n gets larger and larger, the graph of $F_n(x)$ appears to be approaching the graph of the square-wave function $f(x)$ shown in red and seen in Figure 8.45. Based on this, we might conjecture that the Fourier series converges to the function $f(x)$. As it turns out, this is not quite correct. We'll soon see that the series converges to $f(x)$ everywhere, *except* at points of discontinuity. ■

Next, we give an example of constructing a Fourier series for another common waveform.

EXAMPLE 9.2 A Fourier Series Expansion for the Triangular-Wave Function

Find the Fourier series expansion of $f(x) = |x|$, for $-\pi \le x \le \pi$, where f is assumed to be periodic, of period 2π, outside of the interval $[-\pi, \pi]$.

Solution In this case, f is the **triangular-wave** function graphed in Figure 8.47. From the Euler-Fourier formulas, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx$$

$$= -\frac{1}{\pi}\frac{x^2}{2}\bigg|_{-\pi}^{0} + \frac{1}{\pi}\frac{x^2}{2}\bigg|_{0}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

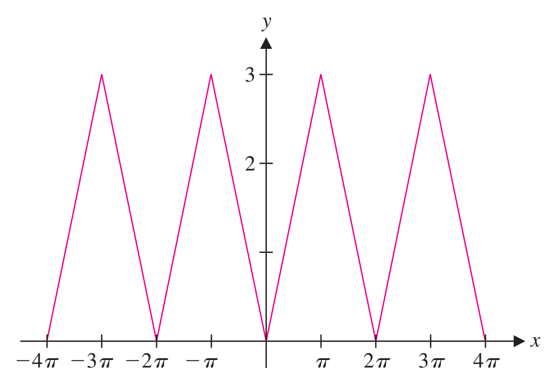


FIGURE 8.47
Triangular-wave function

Similarly, for each $k \ge 1$, we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(kx)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos(kx)\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)\, dx.$$

Both integrals require the same integration by parts. We let

$$u = x \qquad dv = \cos(kx)\, dx$$

$$du = dx \qquad v = \frac{1}{k}\sin(kx)$$

so that

$$\begin{aligned}
a_k &= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos(kx)\,dx + \frac{1}{\pi}\int_0^{\pi} x\cos(kx)\,dx \\
&= -\frac{1}{\pi}\left[\frac{x}{k}\sin(kx)\right]_{-\pi}^{0} + \frac{1}{\pi k}\int_{-\pi}^{0}\sin(kx)\,dx + \frac{1}{\pi}\left[\frac{x}{k}\sin(kx)\right]_0^{\pi} - \frac{1}{\pi k}\int_0^{\pi}\sin(kx)\,dx \\
&= -\frac{1}{\pi}\left[0+\frac{\pi}{k}\sin(-\pi k)\right] - \frac{1}{\pi k^2}\cos(kx)\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\pi}{k}\sin(\pi k)-0\right] + \frac{1}{\pi k^2}\cos(kx)\Big|_0^{\pi} \\
&= 0 - \frac{1}{\pi k^2}[\cos 0 - \cos(-k\pi)] + 0 + \frac{1}{\pi k^2}[\cos(k\pi) - \cos 0] \\
&= \frac{2}{\pi k^2}[\cos(k\pi) - 1] = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{-4}{\pi k^2}, & \text{if } k \text{ is odd} \end{cases}.
\end{aligned}$$

Since $\sin \pi k = 0$ and $\sin(-\pi k) = 0$.

Since $\cos(k\pi) = 1$ when k is even, and $\cos(k\pi) = -1$ when k is odd.

Writing the even- and odd-indexed coefficients separately, we have $a_{2k} = 0$, for $k = 1, 2, \ldots$ and $a_{2k-1} = \dfrac{-4}{\pi(2k-1)^2}$, for $k = 1, 2, \ldots$. We leave it as an exercise to show that

$$b_k = 0, \quad \text{for all } k.$$

This gives us the Fourier series

$$\begin{aligned}
\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)] &= \frac{\pi}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) = \frac{\pi}{2} + \sum_{k=1}^{\infty} a_{2k-1}\cos[(2k-1)x] \\
&= \frac{\pi}{2} - \sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)^2}\cos[(2k-1)x] \\
&= \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x) - \cdots.
\end{aligned}$$

You can show that this series converges absolutely for all x, by using the Comparison Test, since

$$|a_k| = \left|\frac{4}{\pi(2k-1)^2}\cos(2k-1)x\right| \le \frac{4}{\pi(2k-1)^2}$$

and the series $\sum_{k=1}^{\infty}\dfrac{4}{\pi(2k-1)^2}$ converges. (Hint: Compare this last series to the convergent p-series $\sum_{k=1}^{\infty}\dfrac{1}{k^2}$, using the Limit Comparison Test.) To get an idea of the function to which the series converges, we plot several of the partial sums of the series,

$$F_n(x) = \frac{\pi}{2} - \sum_{k=1}^{n}\frac{4}{\pi(2k-1)^2}\cos[(2k-1)x].$$

See if you can conjecture the sum of the series by looking at Figures 8.48a–d. Notice how quickly the partial sums of the series appear to converge to the triangular-wave function f (shown in red; also see Figure 8.47). We'll see later that the Fourier series converges to $f(x)$ for all x. There's something further to note here: the accuracy of the

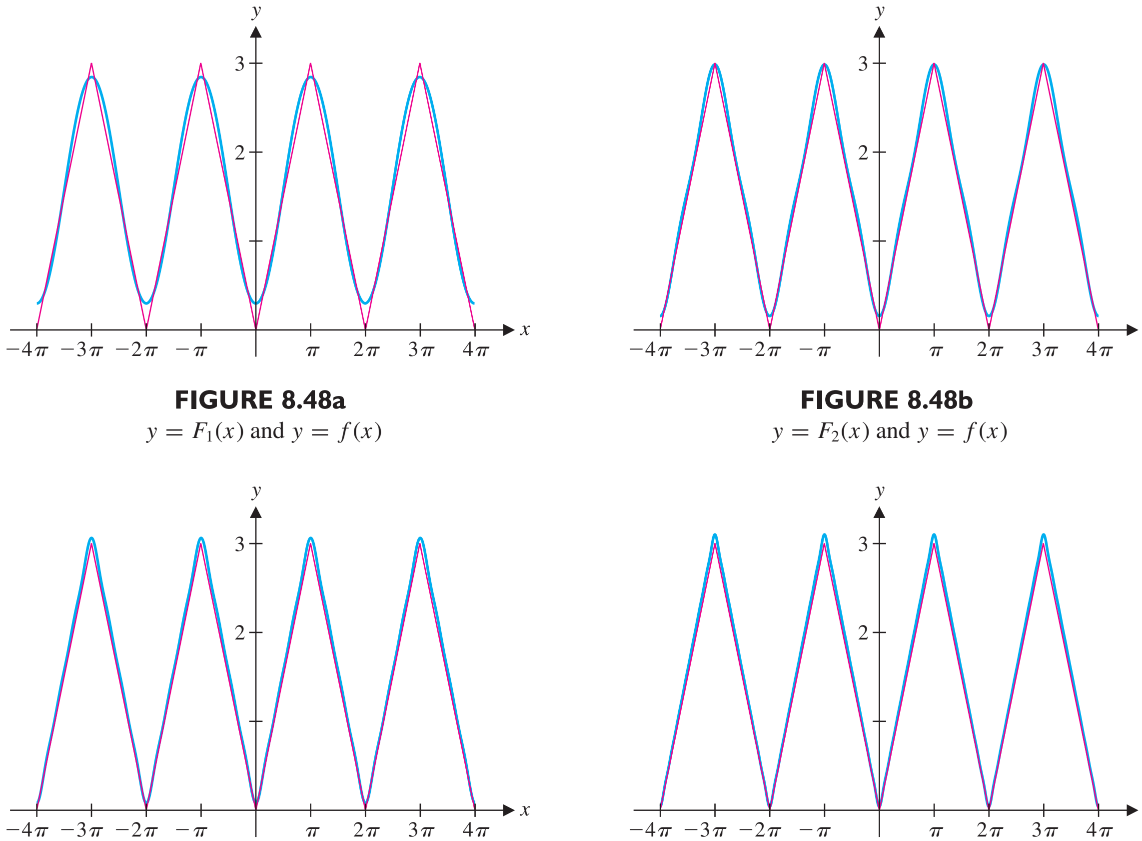


FIGURE 8.48a
$y = F_1(x)$ and $y = f(x)$

FIGURE 8.48b
$y = F_2(x)$ and $y = f(x)$

FIGURE 8.48c
$y = F_4(x)$ and $y = f(x)$

FIGURE 8.48d
$y = F_8(x)$ and $y = f(x)$

approximation is fairly uniform. That is, the difference between a given partial sum and $f(x)$ is roughly the same for each x. Take care to distinguish this behavior from that of Taylor polynomial approximations, where the farther you get away from the point about which you've expanded, the worse the approximation tends to get. ■

Functions of Period Other Than 2π

Now, suppose you have a function f that is periodic of period T, but $T \neq 2\pi$. In this case, we want to expand f in a series of simple functions of period T. First, define $l = \frac{T}{2}$ and notice that

$$\cos\left(\frac{k\pi x}{l}\right) \quad \text{and} \quad \sin\left(\frac{k\pi x}{l}\right)$$

are periodic of period $T = 2l$, for each $k = 1, 2, \ldots$. The Fourier series expansion of f of period $2l$ is then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right)\right].$$

We leave it as an exercise to show that the Fourier coefficients in this case are given by the Euler-Fourier formulas:

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi x}{l}\right)dx, \text{ for } k = 0, 1, 2, \ldots \tag{9.7}$$

and

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{k\pi x}{l}\right)dx, \text{ for } k = 1, 2, 3, \ldots. \tag{9.8}$$

Notice that (9.3), (9.5) and (9.6) are equivalent to (9.7) and (9.8) with $l = \pi$.

EXAMPLE 9.3 A Fourier Series Expansion for a Square-Wave Function

Find a Fourier series expansion for the function

$$f(x) = \begin{cases} -2, & \text{if } -1 < x \le 0 \\ 2, & \text{if } 0 < x \le 1 \end{cases},$$

where f is defined so that it is periodic of period 2 outside of the interval $[-1, 1]$.

Solution The graph of f is the square wave seen in Figure 8.49. From the Euler-Fourier formulas (9.7) and (9.8) with $l = 1$, we have

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(-2)\,dx + \int_{0}^{1} 2\,dx = 0.$$

Likewise, we get

$$a_k = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{k\pi x}{1}\right)dx = 0, \quad \text{for } k = 1, 2, 3, \ldots.$$

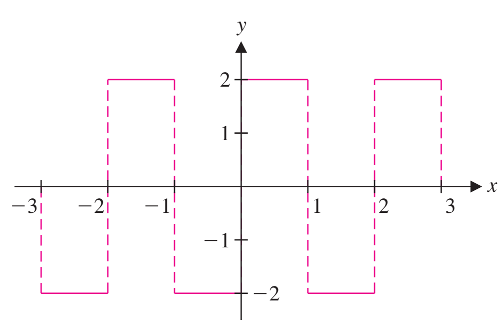


FIGURE 8.49
Square wave

Finally, we have

$$\begin{aligned} b_k &= \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{k\pi x}{1}\right)dx = \int_{-1}^{0}(-2)\sin(k\pi x)\,dx + \int_{0}^{1} 2\sin(k\pi x)\,dx \\ &= \frac{2}{k\pi}\cos(k\pi x)\Big|_{-1}^{0} - \frac{2}{k\pi}\cos(k\pi x)\Big|_{0}^{1} = \frac{4}{k\pi}[\cos 0 - \cos(k\pi)] \\ &= \frac{4}{k\pi}[1 - \cos(k\pi)] = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{8}{k\pi}, & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

Since $\cos(k\pi) = 1$ when k is even, and $\cos(k\pi) = -1$ when k is odd.

This gives us the Fourier series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}[a_k\cos(k\pi x)+b_k\sin(k\pi x)]=\sum_{k=1}^{\infty}b_k\sin(k\pi x)=\sum_{k=1}^{\infty}b_{2k-1}\sin[(2k-1)\pi x]$$

$$=\sum_{k=1}^{\infty}\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x].$$

Since $b_{2k}=0$ and $b_{2k-1}=\dfrac{8}{(2k-1)\pi}$.

Although we as yet have no tools for determining the convergence or divergence of this series, we graph a few of the partial sums of the series,

$$F_n(x)=\sum_{k=1}^{n}\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x]$$

in Figures 8.50a–d. From the graphs, it appears that the series is converging to the square-wave function f, except at the points of discontinuity, $x=0,\pm1,\pm2,\pm3,\ldots$. At those points, the series appears to converge to 0. You can easily verify this by observing that the terms of the series are

$$\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x]=0,\quad \text{for integer values of } x.$$

Since each term in the series is zero, the series converges to 0 at all integer values of x. You might think of this as follows: at the points where f is discontinuous, the series converges to the average of the two function values on either side of the discontinuity. As we will see, this is typical of the convergence of Fourier series.

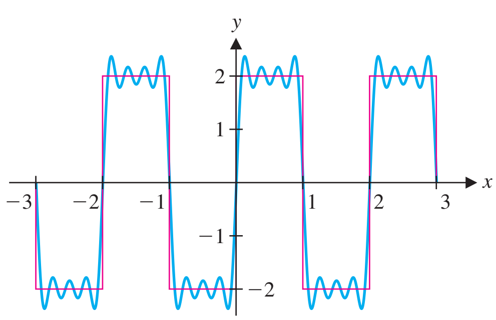


FIGURE 8.50a
$y=F_4(x)$ and $y=f(x)$

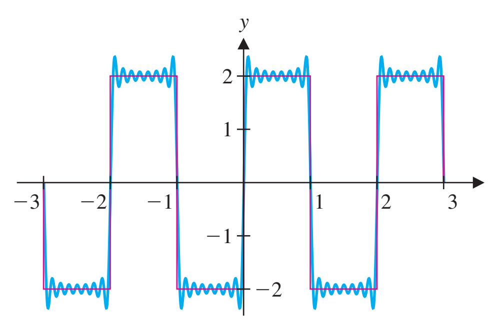


FIGURE 8.50b
$y=F_8(x)$ and $y=f(x)$

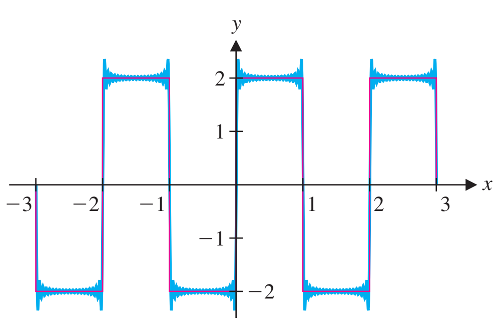


FIGURE 8.50c
$y=F_{20}(x)$ and $y=f(x)$

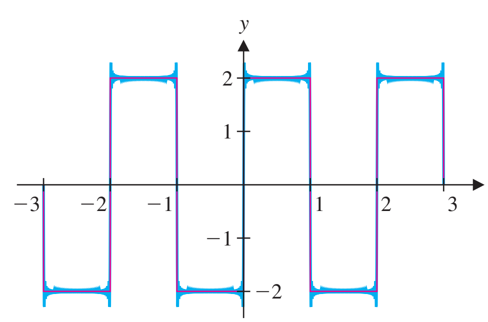


FIGURE 8.50d
$y=F_{50}(x)$ and $y=f(x)$

We now state the following major result on the convergence of Fourier series.

THEOREM 9.1 (Fourier Convergence Theorem)

Suppose that f is periodic of period $2l$ and that f and f' are continuous on the interval $[-l, l]$, except for at most a finite number of jump discontinuities. Then, f has a convergent Fourier series expansion. Further, the series converges to $f(x)$, when f is continuous at x and to

$$\frac{1}{2}\left[\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t)\right]$$

at any points x where f is discontinuous.

The proof of the theorem is beyond the level of this text and can be found in texts on advanced calculus or Fourier analysis.

REMARK 9.1

The Fourier Convergence Theorem says that a Fourier series may converge to a discontinuous function, even though every term in the series is continuous (and differentiable) for all x.

EXAMPLE 9.4 Proving Convergence of a Fourier Series

Use the Fourier Convergence Theorem to prove that the Fourier series expansion of period 2π,

$$\frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi} \cos[(2k-1)x],$$

derived in example 9.2, for $f(x) = |x|$, for $-\pi \le x \le \pi$ and periodic outside of $[-\pi, \pi]$, converges to $f(x)$ everywhere.

Solution First, note that f is continuous everywhere (see Figure 8.47). We also have that since

$$f(x) = |x| = \begin{cases} -x, & \text{if } -\pi \le x < 0 \\ x, & \text{if } 0 \le x < \pi \end{cases}$$

and is periodic outside $[-\pi, \pi]$, then

$$f'(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}.$$

So, f' is also continuous on $[-\pi, \pi]$, except for jump discontinuities at $x = 0$ and $x = \pm\pi$. From the Fourier Convergence Theorem, we now have that the Fourier series converges to $f(x)$ everywhere (since f is continuous everywhere). Because of this, we write

$$f(x) = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi} \cos(2k-1)x,$$

for all x. ■

As you can see from the Fourier Convergence Theorem, Fourier series do not always converge to the function you are expanding.

EXAMPLE 9.5 Investigating Convergence of a Fourier Series

Use the Fourier Convergence Theorem to investigate the convergence of the Fourier series

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin[(2k-1)\pi x],$$

derived as an expansion of the square-wave function

$$f(x) = \begin{cases} -2, & \text{if } -1 < x \le 0 \\ 2, & \text{if } 0 < x \le 1 \end{cases},$$

where f is taken to be periodic outside of $[-1, 1]$ (see example 9.3).

Solution First, note that f is continuous, except for jump discontinuities at $x = 0, \pm 1, \pm 2, \ldots$. Further,

$$f'(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 0, & \text{if } 0 < x < 1 \end{cases}$$

and is periodic outside of $[-1, 1]$. Thus, f' is also continuous everywhere, except at integer values of x where f' is undefined. From the Fourier Convergence Theorem, the Fourier series will converge to $f(x)$ everywhere, except at the discontinuities, $x = 0, \pm 1, \pm 2, \ldots$, where the series converges to the average of the one-sided limits, that is, 0. A graph of the function to which the series converges is shown in Figure 8.51. Since the series does not converge to f everywhere, we cannot say that the function and the series are *equal*. In this case, we usually write

$$f(x) \sim \sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin[(2k-1)\pi x]$$

to indicate that the series *corresponds* to f (but is not necessarily equal to f). In the case of Fourier series, this says that the series converges to $f(x)$ at every x where f is continuous and to the average of the one-sided limits at any jump discontinuities. Notice that this is the behavior seen in the graphs of the partial sums of the series seen in Figures 8.50a–d.

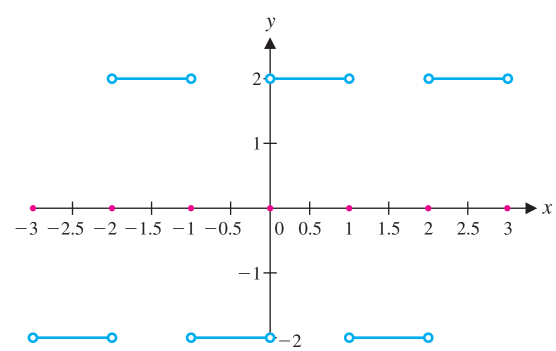


FIGURE 8.51

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin[(2k-1)\pi x]$$

Fourier Series and Music Synthesizers

Fourier series are widely used in engineering, physics, chemistry and so on. We give you a sense of how they are used with the following brief discussion of music synthesizers and a variety of exercises.

Suppose that you had a music machine that could generate pure tones at various pitches and volumes. What types of sounds could you synthesize by combining several pure tones together? To answer this question, we first translate the problem into mathematics. A pure

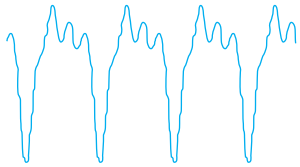


FIGURE 8.52
Saxophone waveform

tone can be modeled by $A \sin \omega t$, where the amplitude A determines the volume and the frequency ω determines the pitch. For example, to mimic a saxophone, you must match the characteristic waveform of a saxophone (see Figure 8.52). The shape of the waveform affects the **timbre** of the tone, a quality most humans readily discern. (A saxophone *sounds* different than a trumpet, doesn't it?)

Consider the following music synthesizer problem. Given a waveform such as the one shown in Figure 8.52, can you add together several pure tones of the form $A \sin \omega t$ to approximate the waveform? Note that if the pure tones are of the form $b_1 \sin t$, $b_2 \sin 2t$, $b_3 \sin 3t$ and so on, this is essentially a Fourier series problem. That is, we want to approximate a given wave function $f(t)$ by a sum of these pure tones, as follows:

$$f(t) \approx b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots + b_n \sin nt.$$

FIGURE 8.53
A graphic equalizer

Although the cosine terms are all missing, notice that this is the partial sum of a Fourier series. (Such series are called **Fourier sine series** and are explored in the exercises.) For music synthesizers, the Fourier coefficients are simply the amplitudes of the various harmonics in a given waveform. In this context, you can think of the bass and treble knobs on a stereo as manipulating the amplitudes of different terms in a Fourier series. Cranking up the bass emphasizes low-frequency terms (i.e., increases the coefficients of the first few terms of the Fourier series), while turning up the treble emphasizes the high-frequency terms. An equalizer (see Figure 8.53) gives you more direct control of individual frequencies.

In general, the idea of analyzing a wave phenomenon by breaking the wave down into its component frequencies is essential to much of modern science and engineering. This type of *spectral analysis* is used in numerous scientific disciplines.

BEYOND FORMULAS

Fourier series provide an alternative to power series for representing functions. Which representation is more useful depends on the specifics of the problem you are working on. Fourier series and its extensions (including wavelets) are used to represent wave phenomena such as sight and sound. In our digital age, such applications are everywhere.

EXERCISES 8.9

WRITING EXERCISES

1. Explain why the Fourier series of $f(x) = 1 + 3\cos x - \sin 2x$ on the interval $[-\pi, \pi]$ is simply $1 + 3\cos x - \sin 2x$. (Hint: Explain what the goal of a Fourier series representation is and note that in this case no work needs to be done.) Would this change if the interval were $[-1, 1]$ instead?

2. Polynomials are built up from the basic operations of arithmetic. We often use Taylor series to rewrite an awkward function (e.g., $\sin x$) into arithmetic form. Many natural phenomena are waves, which are well modeled by sines and cosines. Discuss the extent to which the following statement is true: Fourier series allow us to rewrite algebraic functions (e.g., x^2) into a natural (wave) form.

3. Theorem 9.1 states that a Fourier series may converge to a function with jump discontinuities. In examples 9.1 and 9.3, identify the locations of the jump discontinuities and the values to which the Fourier series converges at these points. In what way are these values reasonable compromises?

4. Carefully examine Figures 8.46 and 8.50. For which x's does the Fourier series seem to converge rapidly? Slowly? Note that for every n, the partial sum $F_n(x)$ passes *exactly* through the limiting point for jump discontinuities. Describe the behavior of the partial sums *near* the jump discontinuities. This overshoot/undershoot behavior is referred to as the **Gibbs phenomenon.**

In exercises 1–8, find the Fourier series of the function on the interval $[-\pi, \pi]$. Graph the function and the partial sums $F_4(x)$ and $F_8(x)$ on the interval $[-2\pi, 2\pi]$.

1. $f(x) = x$

2. $f(x) = x^2$

3. $f(x) = 2|x|$

4. $f(x) = 3x$

5. $f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0 \\ -1, & \text{if } 0 < x < \pi \end{cases}$

6. $f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0 \\ 0, & \text{if } 0 < x < \pi \end{cases}$

7. $f(x) = 3\sin 2x$

8. $f(x) = 2\sin 3x$

In exercises 9–14, find the Fourier series of the function on the given interval.

9. $f(x) = -x, [-1, 1]$

10. $f(x) = |x|, [-1, 1]$

11. $f(x) = x^2, [-1, 1]$

12. $f(x) = 3x, [-2, 2]$

13. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1 \end{cases}$

14. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 1 - x, & \text{if } 0 < x < 1 \end{cases}$

In exercises 15–20, do not compute the Fourier series, but graph the function to which the Fourier series converges, showing at least three full periods.

15. $f(x) = x, [-2, 2]$

16. $f(x) = x^2, [-3, 3]$

17. $f(x) = \begin{cases} -x, & \text{if } -1 < x < 0 \\ 0, & \text{if } 0 < x < 1 \end{cases}$

18. $f(x) = \begin{cases} 1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$

19. $f(x) = \begin{cases} -1, & \text{if } -2 < x < -1 \\ 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } 1 < x < 2 \end{cases}$

20. $f(x) = \begin{cases} 2, & \text{if } -2 < x < -1 \\ -2, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$

21. Substitute $x = 1$ into the Fourier series formula of exercise 11 to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

22. Use the Fourier series of example 9.1 to prove that $\sum_{k=1}^{\infty} \frac{\sin(2k-1)}{2k-1} = \frac{\pi}{4}$.

23. Use the Fourier series of example 9.2 to prove that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

24. Combine the results of exercises 21 and 23 to find $\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$.

In exercises 25–28, use the Fourier Convergence Theorem to investigate the convergence of the Fourier series in the given exercise.

25. exercise 5

26. exercise 7

27. exercise 9

28. exercise 17

29. You have undoubtedly noticed that many Fourier series consist of only cosine or only sine terms. This can be easily understood in terms of even and odd functions. A function f is **even** if $f(-x) = f(x)$ for all x. A function is **odd** if $f(-x) = -f(x)$ for all x. Show that $\cos x$ is even, $\sin x$ is odd and $\cos x + \sin x$ is neither.

30. If f is even, show that $g(x) = f(x)\cos x$ is even and $h(x) = f(x)\sin x$ is odd.

31. If f is odd, show that $g(x) = f(x)\cos x$ is odd and $h(x) = f(x)\sin x$ is even. If f and g are even, what can you say about fg?

32. If f is even and g is odd, what can you say about fg? If f and g are odd, what can you say about fg?

33. Prove the general Euler-Fourier formulas (9.7) and (9.8).

34. If g is an odd function (see exercise 29), show that $\int_{-l}^{l} g(x)\,dx = 0$ for any (positive) constant l. (Hint: Compare $\int_{-l}^{0} g(x)\,dx$ and $\int_{0}^{l} g(x)\,dx$. You will need to make the change of variable $t = -x$ in one of the integrals.) Using the results of exercise 30, show that if f is even, then $b_k = 0$ for all k and the Fourier series of $f(x)$ consists only of a constant and cosine terms. If f is odd, show that $a_k = 0$ for all k and the Fourier series of $f(x)$ consists only of sine terms.

In exercises 35–38, use the even/odd properties of $f(x)$ to predict (don't compute) whether the Fourier series will contain only cosine terms, only sine terms or both.

35. $f(x) = x^3$

36. $f(x) = x^4$

37. $f(x) = e^x$

38. $f(x) = |x|$

39. The function $f(x) = \begin{cases} -1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$ is neither even nor odd, but can be written as $f(x) = g(x) + 1$ where $g(x) = \begin{cases} -2, & \text{if } -2 < x < 0 \\ 2, & \text{if } 0 < x < 2 \end{cases}$. Explain why the Fourier series of $f(x)$ will contain sine terms and the constant 1, but no cosine terms.

40. Suppose that you want to find the Fourier series of $f(x) = x + x^2$. Explain why to find b_k you would need only to integrate $x \sin\left(\frac{k\pi x}{l}\right)$ and to find a_k you would need only to integrate $x^2 \cos\left(\frac{k\pi x}{l}\right)$.

Exercises 41–44 are adapted from the owner's manual of a high-end music synthesizer.

41. A fundamental choice to be made when generating a new tone on a music synthesizer is the waveform. The options are sawtooth, square and pulse. You worked with the sawtooth wave in exercise 9. Graph the limiting function for the function in exercise 9 on the interval $[-4, 4]$. Explain why "sawtooth" is a good name. A square wave is shown in Figure 8.49. A pulse wave of period 2 with width $1/n$ is generated by $f(x) = \begin{cases} -2, & \text{if } 1/n < |x| < 1 \\ 2, & \text{if } |x| \le 1/n \end{cases}$. Graph pulse waves of width 1/3 and 1/4 on the interval $[-4, 4]$.

42. The **harmonic content** of a wave equals the ratio of integral harmonic waves to the fundamental wave. To understand what this means, write the Fourier series of exercise 9 as $\frac{2}{\pi}(-\sin \pi x + \frac{1}{2}\sin 2\pi x - \frac{1}{3}\sin 3\pi x + \frac{1}{4}\sin 4\pi x - \cdots)$. The harmonic content of the sawtooth wave is $\frac{1}{n}$. Explain how this relates to the relative sizes of the Fourier coefficients. The harmonic content of the square wave is $\frac{1}{n}$ with even-numbered harmonics missing. Compare this description to the Fourier series of example 9.3. The harmonic content of the pulse wave of width $\frac{1}{3}$ is $\frac{1}{n}$ with every third harmonic missing. Without computing the Fourier coefficients, write out the general form of the Fourier series of $f(x) = \begin{cases} -2, & \text{if } 1/3 < |x| < 1 \\ 2, & \text{if } |x| \le 1/3 \end{cases}$.

43. The cutoff frequency setting on a music synthesizer has a dramatic effect on the timbre of the tone produced. In terms of harmonic content (see exercise 42), when the cutoff frequency is set at $n > 0$, all harmonics beyond the nth harmonic are set equal to 0. In Fourier series terms, explain how this corresponds to the partial sum $F_n(x)$. For the sawtooth and square waves, graph the waveforms with the cutoff frequency set at 4. Compare these to the waveforms with the cutoff frequency set at 2. As the setting is lowered, you hear more of a "pure" tone. Briefly explain why.

44. The resonance setting on a music synthesizer also changes timbre significantly. Set at 1, you get the basic waveform (e.g., sawtooth or square). Set at 2, the harmonic content of the first four harmonics are divided by 2, the fifth harmonic is multiplied by $\frac{3}{4}$, the sixth harmonic is left the same, the seventh harmonic is divided by 2 and the remaining harmonics are set to 0. Graph the sawtooth and square waves with resonance set to 2. Which one is starting to resemble the saxophone waveform of Figure 8.52?

45. Piano tuning is relatively simple, due to the phenomenon studied in this exercise. Compare the graphs of $\sin 8t + \sin 8.2t$ and $2\sin 8t$. Note especially that the amplitude of $\sin 8t + \sin 8.2t$ appears to slowly rise and fall. In the trigonometric identity $\sin 8t + \sin 8.2t = [2\cos(0.2t)]\sin(8.1t)$, think of $2\cos(0.2t)$ as the amplitude of $\sin(8.1t)$ and explain why the amplitude varies slowly. Piano tuners often start by striking a tuning fork of a certain pitch (e.g., $\sin 8t$) and then striking the corresponding piano note. If the piano is slightly out-of-tune (e.g., $\sin 8.2t$), the tuning fork plus piano produces a combined tone that noticeably increases and decreases in volume. Use your graph to explain why this occurs.

46. The function $\sin 8\pi t$ represents a 4-Hz signal (1 Hz equals 1 cycle per second) if t is measured in seconds. If you received this signal, your task might be to take your measurements of the signal and try to reconstruct the function. For example, if you measured three samples per second, you would have the data $f(0) = 0$, $f(1/3) = \sqrt{3}/2$, $f(2/3) = -\sqrt{3}/2$ and $f(1) = 0$. Knowing the signal is of the form $A \sin Bt$, you would use the data to try to solve for A and B. In this case, you don't have enough information to guarantee getting the right values for A and B. Prove this by finding several values of A and B with $B \ne 8\pi$ that match the data. A famous result of H. Nyquist from 1928 states that to reconstruct a signal of frequency f you need at least $2f$ samples.

47. The energy of a signal $f(x)$ on the interval $[-\pi, \pi]$ is defined by $E = \dfrac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx$. If $f(x)$ has a Fourier series $f(x) = \dfrac{a_0}{2} + \sum_{k=0}^{\infty}[a_k \cos kx + b_k \sin kx]$, show that $E = A_0^2 + A_1^2 + A_2^2 + \cdots$, where $A_k = \sqrt{a_k^2 + b_k^2}$. The sequence $\{A_k\}$ is called the **energy spectrum** of $f(x)$.

48. Carefully examine the graphs in Figure 8.46. There is a Gibbs phenomenon at $x = 0$. Does it appear that the size of the Gibbs overshoot changes as the number of terms increases? We examine that here. For the partial sum $F_n(x)$ as defined in example 9.1, it can be shown that the absolute maximum occurs at $\dfrac{\pi}{2n}$. Evaluate $F_n\left(\dfrac{\pi}{2n}\right)$ for $n = 4$, $n = 6$ and $n = 8$. Show that for large n, the size of the bump is $\left|F_n\left(\dfrac{\pi}{2n}\right) - f\left(\dfrac{\pi}{2n}\right)\right| \approx 0.09$. Gibbs showed that, in general, the size of the bump at a jump discontinuity is about 0.09 times the size of the jump.

49. Some fixes have been devised to reduce the Gibbs phenomenon. Define the **σ-factors** by $\sigma_k = \dfrac{\sin\left(\dfrac{k\pi}{n}\right)}{\dfrac{k\pi}{n}}$ for $k = 1, 2, \ldots, n$ and consider the modified Fourier sum $\dfrac{a_0}{2} + \sum_{k=0}^{n}[a_k\sigma_k \cos kx + b_k\sigma_k \sin kx]$. For example 9.1, plot the modified sums for $n = 4$ and $n = 8$ and compare to Figure 8.46: $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$, F_{2n-1} has critical point at $\pi/2n$ and $\lim_{n\to\infty} F_{2n-1}\left(\frac{\pi}{2n}\right) = \frac{2}{\pi}\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx 1.18$.

EXPLORATORY EXERCISES

1. Suppose that you wanted to approximate a waveform with sine functions (no cosines), as in the music synthesizer problem. Such a **Fourier sine series** will be derived in this exercise. You essentially use Fourier series with a trick to guarantee sine terms only. Start with your waveform as a function defined on the interval $[0, l]$, for some length l. Then define a function $g(x)$ that equals $f(x)$ on $[0, l]$ *and* that is an odd function. Show that $g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le l \\ -f(-x) & \text{if } -l < x < 0 \end{cases}$ works. Explain why the Fourier series expansion of $g(x)$ on $[-l, l]$ would contain sine terms only. This series is the sine series expansion of $f(x)$. Show the following helpful shortcut: the sine series coefficients are

$$b_k = \frac{1}{l}\int_{-l}^{l} g(x)\sin\left(\frac{k\pi}{l}\right)dx = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{k\pi}{l}\right)dx.$$

Then compute the sine series expansion of $f(x) = x^2$ on $[0, 1]$ and graph the limit function on $[-3, 3]$. Analogous to the above, develop a **Fourier cosine series** and find the cosine series expansion of $f(x) = x$ on $[0, 1]$.

2. Fourier series are a part of the field of **Fourier analysis,** which is central to many engineering applications. Fourier analysis includes the Fourier transforms (and the FFT or Fast Fourier Transform) and inverse Fourier transforms, to which you will get a brief introduction in this exercise. Given measurements of a signal (waveform), the goal is to construct the Fourier series of a function. To start with a simple version of the problem, suppose the signal has the form $f(x) = \frac{a_0}{2} + a_1\cos\pi x + a_2\cos 2\pi x + b_1\sin\pi x + b_2\sin 2\pi x$ and you have the measurements $f(-1) = 0$, $f(-\frac{1}{2}) = 1$, $f(0) = 2$, $f(\frac{1}{2}) = 1$ and $f(1) = 0$. Substituting into the general equation for $f(x)$, show that

$$f(-1) = \frac{a_0}{2} - a_1 + a_2 = 0$$

$$f\left(-\frac{1}{2}\right) = \frac{a_0}{2} - a_2 - b_1 = 1$$

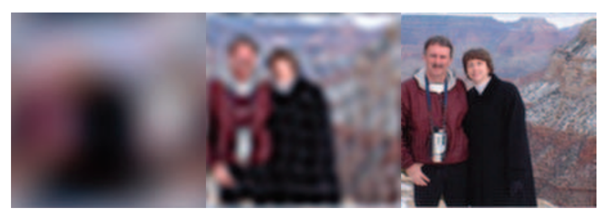
$$f(0) = \frac{a_0}{2} + a_1 + a_2 = 2$$

$$f\left(\frac{1}{2}\right) = \frac{a_0}{2} - a_2 + b_1 = 1$$

$$\text{and} \quad f(1) = \frac{a_0}{2} - a_1 + a_2 = 0.$$

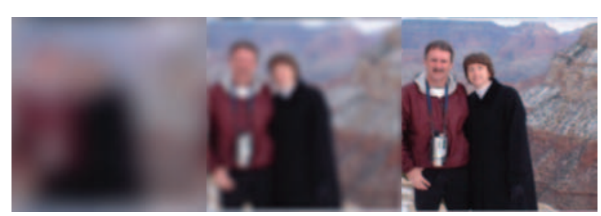
Note that the first and last equations are identical and that b_2 never appears in an equation. Thus, you have four equations and four unknowns. Solve the equations. (Hint: Start by comparing the second and fourth equations, then the third and fifth equations.) You should conclude that $f(x) = 1 + \cos\pi x + b_2\sin\pi x$, with no information about b_2. To determine b_2, we would need another function value. In general, the number of measurements determines how many terms you can find in the Fourier series. (See exercise 46.) Fortunately, there is an easier way of determining the Fourier coefficients. Recall that $a_k = \int_{-1}^{1} f(x)\cos k\pi x\,dx$ and $b_k = \int_{-1}^{1} f(x)\sin k\pi x\,dx$. You can estimate the integral using function values at $x = -1/2, x = 0, x = 1/2$ and $x = 1$. Find a version of a Riemann sum approximation that gives $a_0 = 2, a_1 = 1, a_2 = 0$ and $b_1 = 0$. What value is given for b_2? Use this Riemann sum rule to find the appropriate coefficients for the data $f(-\frac{3}{4}) = \frac{3}{4}$, $f(-\frac{1}{2}) = \frac{1}{2}$, $f(-\frac{1}{4}) = \frac{1}{4}$, $f(0) = 0$, $f(\frac{1}{4}) = -\frac{1}{4}$, $f(\frac{1}{2}) = -\frac{1}{2}$, $f(\frac{3}{4}) = -\frac{3}{4}$ and $f(1) = -1$. Compare to the Fourier series of exercise 9.

3. Fourier series have been used extensively in processing digital information, including digital photographs as well as music synthesis. A digital photograph stored in "bitmap" format can be thought of as three functions $f_R(x, y)$, $f_G(x, y)$ and $f_B(x, y)$. For example, $f_R(x, y)$ could be the amount of red content in the pixel that contains the point (x, y). Briefly explain what $f_G(x, y)$ and $f_B(x, y)$ would represent and how the three functions could be combined to create a color picture. A sine series for a function $f(x)$ on the interval $[0, L]$ is $\sum_{k=1}^{\infty} b_k\sin\left(\frac{k\pi x}{L}\right)$ where $b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx$. Describe what a sine series for a function $f(x, y)$ with $0 \le x \le L$ and $0 \le y \le M$ would look like. If possible, take your favorite photograph in bitmap format and write a program to find Fourier approximations. The accompanying images were created in this way. The first three images show Fourier approximations with 2, 10 and 50 terms, respectively. Notice that while the 50-term approximation is fairly sharp, there are some ripples (or "ghosts") outlining the two people; the ripples are more obvious in the 10-term image. Briefly explain how these ripples relate to the Gibbs phenomenon.



2 terms 10 terms 50 terms

In exercise 49, a σ-correction is introduced that reduces the Gibbs phenomenon. The next three images show the same picture using σ-corrected Fourier approximations with 2, 10 and 50 terms, respectively. Describe how the correction of the Gibbs phenomenon shows up in the images. Based on these images, how does the rate of convergence for Fourier series compare to σ-corrected Fourier series?



2 terms 10 terms 50 terms

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Sequence	Limit of sequence	Squeeze Theorem
Infinite series	Partial sum	Series converges
Series diverges	Geometric series	kth-term test for divergence
Harmonic series	Integral Test	p-Series
Comparison Test	Limit Comparison Test	Alternating Series Test
Conditional convergence	Absolute convergence	Alternating harmonic series
Ratio Test	Root Test	Power series
Radius of convergence	Taylor series	Taylor polynomial
Taylor's Theorem	Fourier series	

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. An increasing sequence diverges to infinity.
2. As n increases, $n!$ increases faster than 10^n.
3. If the sequence a_n diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.
4. If a_k decreases to 0 as $k \to \infty$, then $\sum_{k=1}^{\infty} a_k$ diverges.
5. If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{k=1}^{\infty} a_k$ converges for $a_k = f(k)$.
6. If the Comparison Test can be used to determine the convergence or divergence of a series, then the Limit Comparison Test can also determine the convergence or divergence of the series.
7. Using the Alternating Series Test, if $\lim_{k\to\infty} a_k \neq 0$, then you can conclude that $\sum_{k=1}^{\infty} a_k$ diverges.
8. The difference between a partial sum of a convergent series and its sum is less than the first neglected term in the series.
9. If a series is conditionally convergent, then the Ratio Test will be inconclusive.
10. A series with all negative terms cannot be conditionally convergent.
11. If $\sum_{k=1}^{\infty} |a_k|$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.
12. A series may be integrated term-by-term and the interval of convergence will remain the same.
13. A Taylor series of a function f is simply a power series representation of f.
14. The more terms in a Taylor polynomial, the better the approximation.
15. The Fourier series of x^2 converges to x^2 for all x.

In exercises 1–8, determine whether the sequence converges or diverges. If it converges, give the limit.

1. $a_n = \dfrac{4}{3+n}$
2. $a_n = \dfrac{3n}{1+n}$
3. $a_n = (-1)^n \dfrac{n}{n^2+4}$
4. $a_n = (-1)^n \dfrac{n}{n+4}$
5. $a_n = \dfrac{4^n}{n!}$
6. $a_n = \dfrac{n!}{n^n}$
7. $a_n = \cos \pi n$
8. $a_n = \dfrac{\cos n\pi}{n}$

In exercises 9–18, answer with "converges", "diverges" or "can't tell."

9. If $\lim_{k\to\infty} a_k = 1$, then $\sum_{k=1}^{\infty} a_k$ ______.
10. If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.
11. If $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = 1$, then $\sum_{k=1}^{\infty} a_k$ ______.
12. If $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.
13. If $\lim_{k\to\infty} a_k = \dfrac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.
14. If $\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \dfrac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.
15. If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \dfrac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.
16. If $\lim_{k\to\infty} k^2 a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.

Review Exercises

17. If $p > 1$, then $\sum_{k=1}^{\infty} \frac{8}{k^p}$ ______.

18. If $r > 1$, then $\sum_{k=1}^{\infty} ar^k$ ______.

In exercises 19–22, find the sum of the convergent series.

19. $\sum_{k=0}^{\infty} 4\left(\frac{1}{2}\right)^k$

20. $\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$

21. $\sum_{k=0}^{\infty} 4^{-k}$

22. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{4^k}$

In exercises 23 and 24, estimate the sum of the series to within 0.01.

23. $\sum_{k=0}^{\infty} (-1)^k \frac{k}{k^4+1}$

24. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{3}{k!}$

In exercises 25–44, determine whether the series converges or diverges.

25. $\sum_{k=0}^{\infty} \frac{2k}{k+3}$

26. $\sum_{k=0}^{\infty} (-1)^k \frac{2k}{k+3}$

27. $\sum_{k=0}^{\infty} (-1)^k \frac{4}{\sqrt{k+1}}$

28. $\sum_{k=0}^{\infty} \frac{4}{\sqrt{k+1}}$

29. $\sum_{k=1}^{\infty} 3k^{-7/8}$

30. $\sum_{k=1}^{\infty} 2k^{-8/7}$

31. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$

32. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^3+1}}$

33. $\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$

34. $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k}$

35. $\sum_{k=1}^{\infty} (-1)^k \ln\left(1+\frac{1}{k}\right)$

36. $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}}$

37. $\sum_{k=1}^{\infty} \frac{2}{(k+3)^2}$

38. $\sum_{k=2}^{\infty} \frac{4}{k\ln k}$

39. $\sum_{k=1}^{\infty} \frac{k!}{3^k}$

40. $\sum_{k=1}^{\infty} \frac{k}{3^k}$

41. $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$

42. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{\ln k+1}}$

43. $\sum_{k=1}^{\infty} \frac{4^k}{(k!)^2}$

44. $\sum_{k=1}^{\infty} \frac{k^2+4}{k^3+3k+1}$

In exercises 45–48, determine whether the series converges absolutely, converges conditionally or diverges.

45. $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2+1}$

46. $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k+1}$

47. $\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}}$

48. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{\ln k+1}$

In exercises 49 and 50, find all values of p for which the series converges.

49. $\sum_{k=1}^{\infty} \frac{2}{(3+k)^p}$

50. $\sum_{k=1}^{\infty} e^{kp}$

In exercises 51 and 52, determine the number of terms necessary to estimate the sum of the series to within 10^{-6}.

51. $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k^2}$

52. $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$

In exercises 53–56, find a power series representation for the function. Find the radius of convergence.

53. $\frac{1}{4+x}$

54. $\frac{2}{6-x}$

55. $\frac{3}{3+x^2}$

56. $\frac{2}{1+4x^2}$

In exercises 57 and 58, use the series from exercises 53 and 54 to find a power series and its radius of convergence.

57. $\ln(4+x)$

58. $\ln(6-x)$

In exercises 59–66, find the interval of convergence.

59. $\sum_{k=0}^{\infty} (-1)^k 2x^k$

60. $\sum_{k=0}^{\infty} (-1)^k (2x)^k$

61. $\sum_{k=1}^{\infty} (-1)^k \frac{2}{k} x^k$

62. $\sum_{k=1}^{\infty} \frac{-3}{\sqrt{k}} \left(\frac{x}{2}\right)^k$

63. $\sum_{k=0}^{\infty} \frac{4}{k!} (x-2)^k$

64. $\sum_{k=0}^{\infty} k^2 (x+3)^k$

65. $\sum_{k=0}^{\infty} 3^k (x-2)^k$

66. $\sum_{k=0}^{\infty} \frac{k}{4^k} (x+1)^k$

In exercises 67 and 68, derive the Taylor series of $f(x)$ about the center $x = c$.

67. $f(x) = \sin x$, $c = 0$

68. $f(x) = \frac{1}{x}$, $c = 1$

In exercises 69 and 70, find the Taylor polynomial $P_4(x)$. Graph $f(x)$ and $P_4(x)$.

69. $f(x) = \ln x$, $c = 1$

70. $f(x) = \frac{1}{\sqrt{x}}$, $c = 1$

Review Exercises

In exercises 71 and 72, use the Taylor polynomials from exercises 69 and 70 to estimate the given values. Determine the order of the Taylor polynomial needed to estimate the value to within 10^{-8}.

71. $\ln 1.2$ **72.** $\dfrac{1}{\sqrt{1.1}}$

In exercises 73 and 74, use a known Taylor series to find a Taylor series of the function and find its radius of convergence.

73. e^{-3x^2} **74.** $\sin 4x$

In exercises 75 and 76, use the first five nonzero terms of a known Taylor series to estimate the value of the integral.

75. $\displaystyle\int_0^1 \tan^{-1} x\,dx$ **76.** $\displaystyle\int_0^2 e^{-3x^2}\,dx$

In exercises 77 and 78, derive the Fourier series of the function.

77. $f(x) = x,\ -2 \le x \le 2$

78. $f(x) = \begin{cases} 0 & \text{if } -\pi < x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$

In exercises 79–82, graph at least three periods of the function to which the Fourier series converges.

79. $f(x) = x^2,\ -1 \le x \le 1$

80. $f(x) = 2x,\ -2 \le x \le 2$

81. $f(x) = \begin{cases} -1 & \text{if } -1 < x \le 0 \\ 1 & \text{if } 0 < x \le 1 \end{cases}$

82. $f(x) = \begin{cases} 0 & \text{if } -2 < x \le 0 \\ x & \text{if } 0 < x \le 2 \end{cases}$

83. Suppose you and your friend take turns tossing a coin. The first one to get a head wins. Obviously, the person who goes first has an advantage, but how much of an advantage is it? If you go first, the probability that you win on your first toss is $\frac{1}{2}$, the probability that you win on your second toss is $\frac{1}{8}$, the probability that you win on your third toss is $\frac{1}{32}$ and so on. Sum a geometric series to find the probability that you win.

84. In a game similar to that of exercise 83, the first one to roll a 4 on a six-sided die wins. Is this game more fair than the previous game? The probabilities of winning on the first, second and third roll are $\frac{1}{6}$, $\frac{25}{216}$ and $\frac{625}{7776}$, respectively. Sum a geometric series to find the probability that you win.

85. Recall the Fibonacci sequence defined by $a_0 = 1$, $a_1 = 1, a_2 = 2$ and $a_{n+1} = a_n + a_{n-1}$. Prove the following fact: $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$. (This number, known to the ancient Greeks, is called the **golden ratio.**) (Hint: Start with $a_{n+1} = a_n + a_{n-1}$ and divide by a_n. If $r = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, argue that $\displaystyle\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = \frac{1}{r}$ and then solve the equation $r = 1 + \frac{1}{r}$.)

86. The Fibonacci sequence can be visualized with the following construction. Start with two side-by-side squares of side 1 (Figure A). Above them, draw a square (Figure B), which will have side 2. To the left of that, draw a square (Figure C), which will have side 3. Continue to spiral around, drawing squares that have sides given by the Fibonacci sequence. For each bounding rectangle in Figures A–C, compute the ratio of the sides of the rectangle. (Hint: Start with $\frac{2}{1}$ and then $\frac{3}{2}$.) Find the limit of the ratios as the construction process continues. The Greeks proclaimed this to be the most "pleasing" of all rectangles, building the Parthenon and other important buildings with these proportions. (See *The Divine Proportion* by H. E. Huntley.)

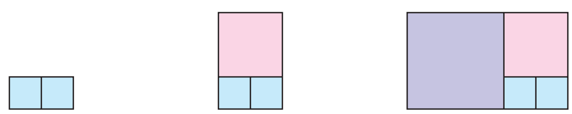


FIGURE A **FIGURE B** **FIGURE C**

87. Another type of sequence studied by mathematicians is the **continued fraction.** Numerically explore the sequence $1 + \dfrac{1}{1}, 1 + \dfrac{1}{1+\frac{1}{1}}, 1 + \dfrac{1}{1+\frac{1}{1+\frac{1}{1}}}$ and so on. This is yet another occurrence of the golden ratio. Viscount Brouncker, a seventeenth-century English mathematician, showed that the sequence $1 + \dfrac{1^2}{2}, 1 + \dfrac{1^2}{2+\frac{3^2}{2}}, 1 + \dfrac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2}}}$ and so on, converges to $\dfrac{4}{\pi}$. (See *A History of Pi* by Petr Beckmann.) Explore this sequence numerically.

88. For the power series $\dfrac{1}{1-x-x^2} = c_1 + c_2x + c_3x^2 + \cdots$, show that the constants c_i are the Fibonacci numbers. Substitute $x = \dfrac{1}{1000}$ to find the interesting decimal representation for $\dfrac{1{,}000{,}000}{998{,}999}$.

EXPLORATORY EXERCISES

1. The challenge here is to determine $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$ as completely as possible. Start by finding the interval of convergence. Find the sum for the special cases (a) $x = 0$ and (b) $x = 1$. For $0 < x < 1$, do the following: (c) Rewrite the series using the

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partial fractions expansion of $\frac{1}{k(k+1)}$. (d) Because the series converges absolutely, it is legal to rearrange terms. Do so and rewrite the series as $x + \frac{x-1}{x}\left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\right]$. (e) Identify the series in brackets as $\int \left(\sum_{k=1}^{\infty} x^k\right) dx$, evaluate the series and then integrate term-by-term. (f) Replace the term in brackets in part (d) with its value obtained in part (e). (g) The next case is for $-1 < x < 0$. Use the technique in parts (c)–(f) to find the sum. (h) Evaluate the sum at $x = -1$ using the fact that the alternating harmonic series sums to ln 2. (Used by permission of the Virginia Tech Mathematics Contest. Solution suggested by Gregory Minton.)

2. You have used Fourier series to show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Here, you will use a version of **Vièta's formula** to give an alternative derivation. Start by using a Maclaurin series for $\sin x$ to derive a series for $f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$. Then find the zeros of $f(x)$. Vièta's formula states that the sum of the reciprocals of the zeros of $f(x)$ equals the negative of the coefficient of the linear term in the Maclaurin series of $f(x)$ divided by the constant term. Take this equation and multiply by π^2 to get the desired formula. Use the same method with a different function to show that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.