First-Order Differential Equations

CHAPTER 7

The discovery of a well-preserved fossil can provide paleontologists with priceless clues about the early history of life on Earth. In 1993, an amateur fossil hunter named Ruben Carolini found the bones of a massive dinosaur in southern Argentina. The new species of dinosaur, named *Giganotosaurus,* replaced *Tyrannosaurus Rex* as the largest known carnivore. Measuring up to 45 feet in length and standing 12 feet high at the hip bone, *Giganotosaurus* is estimated to have weighed in at about 8 tons.

To place finds like *Giganotosaurus* in their correct historical perspective, paleontologists employ several techniques for dating fossils. The most well known of these is radiocarbon dating using carbon-14, an unstable isotope of carbon formed by collisions of cosmic rays with nitrogen atoms in the upper atmosphere. In living plants and animals, the ratio of the amount of carbon-14 to the total amount of carbon is constant. When a plant or animal dies, it stops taking in carbon-14 and the existing carbon-14 begins to decay, at a constant (though nearly imperceptible) percentage rate. An accurate measurement of the proportion of carbon-14 remaining can then be converted into an estimate of the time of death. Carbon-14 is suitable for this kind of dating because its decay rate is so very slow. Estimates from carbon-14 dating are considered to be reliable for fossils dating back tens of thousands of years.

Dating using other radioisotopes with slower decay rates than that of carbon-14 works on the same basic principle, but cannot be applied directly to the fossilized organism. Instead, these isotopes can be used to accurately date very old rock or sediment that surrounds the fossils. Using such techniques, paleontologists believe that *Giganotosaurus* lived about 100 million years ago. This is critical information to scientists studying life near the end of the Mesozoic era. For example, based on this method of dating, it is apparent that *Giganotosaurus* did not live at the same time and therefore did not compete with the smaller but stronger *Tyrannosaurus Rex*.

The mathematics underlying carbon-14 and other radioisotope dating techniques is developed in this chapter. Amazingly, the same mathematics can be applied to computing the balance in your bank account and to estimating the population of a bacterial colony. The study of differential equations is full of surprising connections like this. An understanding of differential equations provides you with essential tools to analyze many important phenomena beginning with basic physical principles. In this chapter, we introduce the basic theory and a few common applications of some elementary differential equations. In Chapter 15, we return to the topic of differential equations and present additional examples. However, a more thorough examination of this vast field will need to wait for a course focused on this topic.

7.1 MODELING WITH DIFFERENTIAL EQUATIONS

Growth and Decay Problems

Time (hours)	*Number of Bacteria (millions per ml)*
0	1.2
0.5	2.5
1	5.1
1.5	11.0
2	23.0
2.5	45.0
3	91.0
3.5	180.0
4	350.0

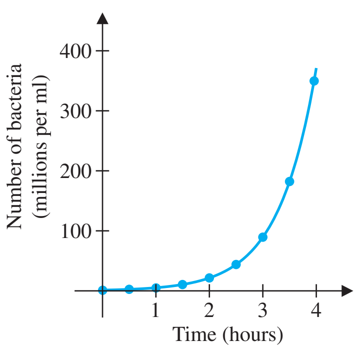


FIGURE 7.1
Growth of bacteria

In this age, we are all keenly aware of how infection by microorganisms such as *Escherichia coli* (*E. coli*) causes disease. Many organisms (such as *E. coli*) produce a toxin that can cause sickness or even death. Some bacteria can reproduce in our bodies at a surprisingly fast rate, overwhelming our bodies' natural defenses with the sheer volume of toxin they are producing. The table shown in the margin indicates the number of *E. coli* bacteria (in millions of bacteria per ml) in a laboratory culture measured at half-hour intervals during the course of an experiment. We have plotted the number of bacteria per milliliter versus time in Figure 7.1. What would you say the graph most resembles? If you said, "an exponential," you guessed right. Careful analysis of experimental data has shown that many populations grow at a rate proportional to their current level. This is quite easily observed in bacterial cultures, where the bacteria reproduce by binary fission (i.e., each cell reproduces by dividing into two cells). In this case, the rate at which the bacterial culture grows is directly proportional to the current population (until such time as resources become scarce or overcrowding becomes a limiting factor). If we let $y(t)$ represent the number of bacteria in a culture at time t, then the rate of change of the population with respect to time is $y'(t)$. Thus, since $y'(t)$ is proportional to $y(t)$, we have

$$y'(t) = ky(t), \tag{1.1}$$

for some constant of proportionality k (the **growth constant**). Since equation (1.1) involves the derivative of an unknown function, we call it a **differential equation.** Our aim is to *solve* the differential equation, that is, find the *function* $y(t)$. Assuming that $y(t) > 0$ (this is a reasonable assumption, since $y(t)$ represents a population), we have

$$\frac{y'(t)}{y(t)} = k. \tag{1.2}$$

Integrating both sides of equation (1.2) with respect to t, we obtain

$$\int \frac{y'(t)}{y(t)}\,dt = \int k\,dt. \tag{1.3}$$

Substituting $y = y(t)$ in the integral on the left-hand side, we have $dy = y'(t)\,dt$ and so, (1.3) becomes

$$\int \frac{1}{y}\,dy = \int k\,dt.$$

Evaluating these integrals, we obtain

$$\ln|y| + c_1 = kt + c_2,$$

where c_1 and c_2 are constants of integration. Subtracting c_1 from both sides yields

$$\ln|y| = kt + (c_2 - c_1) = kt + c,$$

for some constant c. Since $y(t) > 0$, we have

$$\ln y(t) = kt + c$$

and taking exponentials of both sides, we get

$$y(t) = e^{\ln y(t)} = e^{kt+c} = e^{kt}e^{c}.$$

Since c is an arbitrary constant, we write $A = e^c$ and get

$$y(t) = Ae^{kt}. \tag{1.4}$$

We refer to (1.4) as the **general solution** of the differential equation (1.1). For $k > 0$, equation (1.4) is called an **exponential growth law** and for $k < 0$, it is an **exponential decay law.** (Think about the distinction.)

In example 1.1, we examine how an exponential growth law predicts the number of cells in a bacterial culture.

EXAMPLE 1.1 Exponential Growth of a Bacterial Colony

A freshly inoculated bacterial culture of *Streptococcus A* (a common group of microorganisms that cause strep throat) contains 100 cells. When the culture is checked 60 minutes later, it is determined that there are 450 cells present. Assuming exponential growth, determine the number of cells present at any time t (measured in minutes) and find the doubling time.

Solution Exponential growth means that

$$y'(t) = ky(t)$$

and hence, from (1.4),

$$y(t) = Ae^{kt}, \tag{1.5}$$

where A and k are constants to be determined. If we set the starting time as $t = 0$, we have

$$y(0) = 100. \tag{1.6}$$

Equation (1.6) is called an **initial condition.** Setting $t = 0$ in (1.5), we now have

$$100 = y(0) = Ae^0 = A$$

and hence,

$$y(t) = 100\,e^{kt}.$$

We can use the second observation to determine the value of the growth constant k. We have

$$450 = y(60) = 100\,e^{60k}.$$

Dividing both sides by 100 and taking the natural logarithm of both sides, we have

$$\ln 4.5 = \ln e^{60k} = 60k,$$

so that

$$k = \frac{\ln 4.5}{60} \approx 0.02507.$$

We now have a formula representing the number of cells present at any time t:

$$y(t) = 100\,e^{kt} = 100\exp\left(\frac{\ln 4.5}{60}t\right).$$

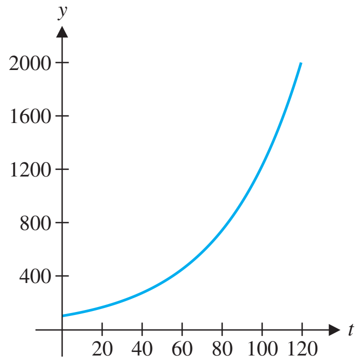


FIGURE 7.2
$y = 100\,e^{\left(\frac{\ln 4.5}{60}t\right)}$

See Figure 7.2 for a graph of the projected bacterial growth over the first 120 minutes. One further question of interest to microbiologists is the **doubling time,** that is, the time it takes for the number of cells to double. We can find this by solving for the time t for which $y(t) = 2y(0) = 200$. We have

$$200 = y(t) = 100 \exp\left(\frac{\ln 4.5}{60}t\right).$$

Dividing both sides by 100 and taking logarithms, we obtain

$$\ln 2 = \frac{\ln 4.5}{60}t,$$

so that

$$t = \frac{60 \ln 2}{\ln 4.5} \approx 27.65.$$

So, the doubling time for this culture of *Streptococcus A* is about 28 minutes. The doubling time for a bacterium depends on the specific strain of bacteria, as well as the quality and quantity of the food supply, the temperature and other environmental factors. However, it is not dependent on the initial population. Here, you can easily check that the population reaches 400 at time

$$t = \frac{120 \ln 2}{\ln 4.5} \approx 55.3$$

(exactly double the time it took to reach 200).

That is, the initial population of 100 doubles to 200 in approximately 28 minutes and it doubles again (to 400) in another 28 minutes and so on. ■

Numerous physical phenomena satisfy exponential growth or decay laws. For instance, experiments have shown that the rate at which a radioactive element decays is directly proportional to the amount present. (Recall that radioactive elements are chemically unstable elements that gradually decay into other, more stable elements.) Let $y(t)$ be the amount (mass) of a radioactive element present at time t. Then, we have that the rate of change (rate of decay) of $y(t)$ satisfies

$$y'(t) = ky(t). \tag{1.7}$$

Note that (1.7) is precisely the same differential equation as (1.1), encountered in example 1.1 for the growth of bacteria and hence, from (1.4), we have that

$$y(t) = Ae^{kt},$$

for some constants A and k (here, the **decay constant**) to be determined.

It is common to discuss the decay rate of a radioactive element in terms of its **half-life,** the time required for half of the initial quantity to decay into other elements. For instance, scientists have calculated that the half-life of carbon-14 (^{14}C) is approximately 5730 years. That is, if you have 2 grams of ^{14}C today and you come back in 5730 years, you will have approximately 1 gram of ^{14}C remaining. It is this long half-life and the fact that living creatures continually take in ^{14}C that make ^{14}C measurements useful for radiocarbon dating. (See the exercise set for more on this important application.)

EXAMPLE 1.2 Radioactive Decay

If you have 50 grams of ^{14}C today, how much will be left in 100 years?

Solution Let $y(t)$ be the mass (in grams) of ^{14}C present at time t. Then, we have

$$y'(t) = ky(t)$$

and as we have already seen, $y(t) = Ae^{kt}$.

The initial condition is $y(0) = 50$, so that

$$50 = y(0) = Ae^{0} = A$$

and $$y(t) = 50\,e^{kt}.$$

To find the decay constant k, we use the half-life:

$$25 = y(5730) = 50\,e^{5730k}.$$

Dividing both sides by 50 and taking logarithms gives us

$$\ln\frac{1}{2} = \ln\ e^{5730k} = 5730k,$$

so that $$k = \frac{\ln\frac{1}{2}}{5730} \approx -1.20968 \times 10^{-4}.$$

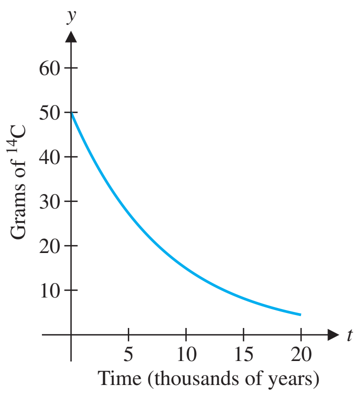


FIGURE 7.3
Decay of ^{14}C

A graph of the mass of ^{14}C as a function of time is seen in Figure 7.3. Notice the extremely large time scale shown. This should give you an idea of the incredibly slow rate of decay of ^{14}C. Finally, notice that if we start with 50 grams, then the amount left after 100 years is

$$y(100) = 50e^{100k} \approx 49.3988 \text{ grams.}$$

A mathematically similar physical principle is **Newton's Law of Cooling.** If you introduce a hot object into cool surroundings (or equivalently, a cold object into warm surroundings), the rate at which the object cools (or warms) is not proportional to its temperature, but rather, to the difference in temperature between the object and its surroundings. Symbolically, if we let $y(t)$ be the temperature of the object at time t and let T_a be the temperature of the surroundings (the **ambient temperature,** which we assume to be constant), we have the differential equation

$$y'(t) = k[y(t) - T_a]. \tag{1.8}$$

Notice that (1.8) is not the same as the differential equation describing exponential growth or decay. (Compare these; what's the difference?) Even so, we can approach finding a solution in the same way. In the case of cooling, we assume that

$$T_a < y(t).$$

(Why is it fair to assume this?) If we divide both sides of equation (1.8) by $y(t) - T_a$ and then integrate both sides, we obtain

$$\int \frac{y'(t)}{y(t) - T_a}\,dt = \int k\,dt = kt + c_1. \tag{1.9}$$

Notice that we can evaluate the integral on the left-hand side by making the substitution $u = y(t) - T_a$, so that $du = y'(t)\,dt$. Thus, we have

$$\int \frac{y'(t)}{y(t) - T_a}\,dt = \int \frac{1}{u}\,du = \ln|u| + c_2 = \ln|y(t) - T_a| + c_2$$
$$= \ln[y(t) - T_a] + c_2,$$

since $y(t) - T_a > 0$. From (1.9), we now have

$$\ln[y(t) - T_a] + c_2 = kt + c_1 \quad \text{or} \quad \ln[y(t) - T_a] = kt + c,$$

where we have combined the two constants of integration. Taking exponentials of both sides, we obtain

$$y(t) - T_a = e^{kt+c} = e^{kt}e^c.$$

Finally, for convenience, we write $A = e^c$, to obtain

$$y(t) = Ae^{kt} + T_a,$$

where A and k are constants to be determined.

We illustrate Newton's Law of Cooling in example 1.3.

EXAMPLE 1.3 Newton's Law of Cooling for a Cup of Coffee

A cup of fast-food coffee is 180°F when freshly poured. After 2 minutes in a room at 70°F, the coffee has cooled to 165°F. Find the temperature at any time t and find the time at which the coffee has cooled to 120°F.

Solution Letting $y(t)$ be the temperature of the coffee at time t, we have

$$y'(t) = k[y(t) - 70].$$

Proceeding as above, we obtain

$$y(t) = Ae^{kt} + 70.$$

Observe that the initial condition here is the initial temperature, $y(0) = 180$. This gives us

$$180 = y(0) = Ae^0 + 70 = A + 70,$$

so that $A = 110$ and

$$y(t) = 110\, e^{kt} + 70.$$

We can now use the second measured temperature to solve for the constant k. We have

$$165 = y(2) = 110\, e^{2k} + 70.$$

Subtracting 70 from both sides and dividing by 110, we have

$$e^{2k} = \frac{165 - 70}{110} = \frac{95}{110}.$$

Taking logarithms of both sides yields

$$2k = \ln\left(\frac{95}{110}\right)$$

and hence,

$$k = \frac{1}{2}\ln\left(\frac{95}{110}\right) \approx -0.0733017.$$

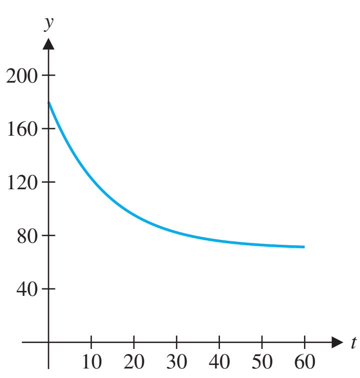


FIGURE 7.4
Temperature of coffee

A graph of the projected temperature against time is shown in Figure 7.4. From Figure 7.4, you might observe that the temperature appears to have fallen to 120°F after about 10 minutes. We can solve this symbolically by finding the time t for which

$$120 = y(t) = 110\, e^{kt} + 70.$$

It is not hard to solve this to obtain

$$t = \frac{1}{k}\ln\frac{5}{11} \approx 10.76 \text{ minutes}.$$

The details are left as an exercise. ■

Compound Interest

If a bank agrees to pay you 8% (annual) interest on your investment of \$10,000, then at the end of a year, you will have

$$\$10{,}000 + (0.08)\$10{,}000 = \$10{,}000(1 + 0.08) = \$10{,}800.$$

On the other hand, if the bank agrees to pay you interest twice a year at the same 8% annual rate, you receive $\frac{8}{2}$% interest twice each year. At the end of the year, you will have

$$\begin{aligned}\$10{,}000\left(1+\frac{0.08}{2}\right)\left(1+\frac{0.08}{2}\right) &= \$10{,}000\left(1+\frac{0.08}{2}\right)^2 \\ &= \$10{,}816.\end{aligned}$$

Continuing in this fashion, notice that paying (compounding) interest monthly would pay $\frac{8}{12}$% each month (period), resulting in a balance of

$$\$10{,}000\left(1+\frac{0.08}{12}\right)^{12} \approx \$10{,}830.00.$$

Further, if interest is compounded daily, you would end up with

$$\$10{,}000\left(1+\frac{0.08}{365}\right)^{365} \approx \$10{,}832.78.$$

It should be evident that the more often interest is compounded, the greater the interest will be. A reasonable question to ask is whether there is a limit to how much interest can accrue on a given investment at a given interest rate. If n is the number of times per year that interest is compounded, we wish to calculate the annual percentage yield (APY) under **continuous compounding,**

$$\text{APY} = \lim_{n\to\infty}\left(1+\frac{0.08}{n}\right)^n - 1.$$

To determine this limit, you must recall (see Chapter 0) that

$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$

Notice that if we make the change of variable $n = 0.08m$, then we have

$$\begin{aligned}\text{APY} &= \lim_{m\to\infty}\left(1+\frac{0.08}{0.08m}\right)^{0.08m} - 1 \\ &= \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^{0.08} - 1 \\ &= e^{0.08} - 1 \approx 0.083287.\end{aligned}$$

Under continuous compounding, you would thus earn approximately 8.3% or

$$\$10{,}000(e^{0.08} - 1) \approx \$832.87$$

in interest, leaving your investment with a total value of \$10,832.87. More generally, suppose that you invest $\$P$ at an annual interest rate r, compounded n times per year. Then the value of your investment after t years is

$$\$P\left(1+\frac{r}{n}\right)^{nt}.$$

Under continuous compounding (i.e., taking the limit as $n \to \infty$), this becomes

$$\$Pe^{rt}. \tag{1.10}$$

Alternatively, if $y(t)$ is the value of your investment after t years, with continuous compounding, the rate of change of $y(t)$ is proportional to $y(t)$. That is,

$$y'(t) = ry(t),$$

where r is the annual interest rate. From (1.4), we have

$$y(t) = Ae^{rt}.$$

For an initial investment of $\$P$, we have

$$\$P = y(0) = Ae^0 = A,$$

so that

$$y(t) = \$Pe^{rt},$$

which is the same as (1.10).

EXAMPLE 1.4 Comparing Forms of Compounding Interest

If you invest \$7000 at an annual interest rate of 5.75%, compare the value of your investment after 5 years under various forms of compounding.

Solution With annual compounding, the value is

$$\$7000\left(1 + \frac{0.0575}{1}\right)^5 \approx \$9257.63.$$

With monthly compounding, this becomes

$$\$7000\left(1 + \frac{0.0575}{12}\right)^{12(5)} \approx \$9325.23.$$

With daily compounding, this yields

$$\$7000\left(1 + \frac{0.0575}{365}\right)^{365(5)} \approx \$9331.42.$$

Finally, with continuous compounding, the value is

$$\$7000\, e^{0.0575(5)} \approx \$9331.63. \blacksquare$$

The mathematics used to describe the compounding of interest also applies to accounts that are decreasing in value.

EXAMPLE 1.5 Depreciation of Assets

(a) Suppose that the value of a \$10,000 asset decreases continuously at a constant rate of 24% per year. Find its worth after 10 years; after 20 years. (b) Compare these values to a \$10,000 asset that is depreciated to no value in 20 years using linear depreciation.

Solution The value $v(t)$ of any quantity that is changing at a constant rate r satisfies $v' = rv$. Here, $r = -0.24$, so that

$$v(t) = Ae^{-0.24t}.$$

Since the value of the asset is initially 10,000, we have

$$10{,}000 = v(0) = Ae^0 = A.$$

We now have
$$v(t) = 10{,}000\,e^{-0.24t}.$$
At time $t = 10$, the value of the asset is then
$$\$10{,}000\,e^{-0.24(10)} \approx \$907.18$$
and at time $t = 20$, the value has decreased to
$$\$10{,}000\,e^{-0.24(20)} \approx \$82.30.$$
For part (b), linear depreciation means we use a linear function $v(t) = mt + b$ for the value of the asset. We start with $v(0) = 10{,}000$ and end at $v(20) = 0$. From $v(0) = 10{,}000$ we get $b = 10{,}000$ and using the points (0,10,000) and (20, 0), we compute the slope $m = \dfrac{10{,}000}{-20} = -500$. We then have
$$v(t) = -500t + 10{,}000.$$

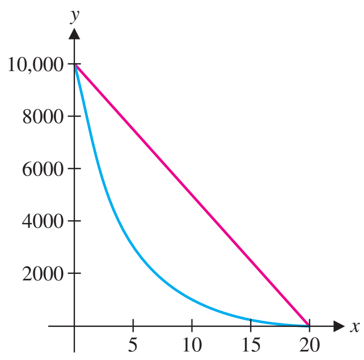


FIGURE 7.5
Linear versus exponential depreciation

At time $t = 10$, $v(10) = \$5000$. Notice that this is considerably more than the approximately \$900 that exponential depreciation gave us. By time $t = 20$, however, the linear depreciation value of \$0 is less than the exponential depreciation value of \$82.30. The graphs in Figure 7.5 illustrate these comparisons. ■

BEYOND FORMULAS

With a basic understanding of differential equations, you can model a diverse collection of physical phenomena arising in the sciences and engineering. Understanding the physical assumptions that go into the model allows you to interpret the physical meaning of the solution. Moreover, part of the power of mathematics lies in its generality. In this case, the same differential equation may model a collection of vastly different phenomena. In this sense, knowing a little bit of mathematics goes a long way. Having solved for doubling time in example 1.1, if you are told that the value of an investment or the size of a tumor is modeled by the same equation, what can you conclude?

EXERCISES 7.1

WRITING EXERCISES

1. A linear function is defined by constant slope. If a population showed a constant numerical increase year by year, explain why the population could be represented by a linear function. If the population showed a constant *percentage* increase instead, explain why the population could be represented by an exponential function.
2. If a population has a constant birthrate and a constant death rate (smaller than the birthrate), describe what the population would look like over time. In the United States, is the death rate increasing, decreasing or staying the same? Given this, why is there concern about reducing the birthrate?
3. Explain, in monetary terms, why for a given interest rate the more times the interest is compounded the more money is in the account at the end of a year.
4. In the growth and decay examples, the constant A turned out to be equal to the initial value. In the cooling examples, the constant A did not equal the initial value. Explain why the cooling example worked differently.

In exercises 1–8, find the solution of the given differential equation satisfying the indicated initial condition.

1. $y' = 4y$, $y(0) = 2$
2. $y' = 3y$, $y(0) = -2$
3. $y' = -3y$, $y(0) = 5$
4. $y' = -2y$, $y(0) = -6$
5. $y' = 2y$, $y(1) = 2$
6. $y' = -y$, $y(1) = 2$
7. $y' = -3$, $y(0) = 3$
8. $y' = -2$, $y(0) = -8$

Exercises 9–16 involve exponential growth.

9. Suppose a bacterial culture doubles in population every 4 hours. If the population is initially 100, find an equation for the population at any time. Determine when the population will reach 6000.

10. Suppose a bacterial culture triples in population every 5 hours. If the population is initially 200, find an equation for the population at any time. Determine when the population will reach 20,000.

11. Suppose a bacterial culture initially has 400 cells. After 1 hour, the population has increased to 800. Find an equation for the population at any time. What will the population be after 10 hours?

12. Suppose a bacterial culture initially has 100 cells. After 2 hours, the population has increased to 400. Find an equation for the population at any time. What will the population be after 8 hours?

13. A bacterial culture grows exponentially with growth constant 0.44 hour^{-1}. Find its doubling time.

14. A bacterial culture grows exponentially with growth constant 0.12 hour^{-1}. Find its doubling time.

15. Suppose that a population of *E. coli* doubles every 20 minutes. A treatment of the infection removes 90% of the *E. coli* present and is timed to accomplish the following. The population starts at size 10^8, grows for T minutes, the treatment is applied and the population returns to size 10^8. Find the time T.

16. Research by Meadows, Meadows, Randers and Behrens indicates that the earth has 3.2×10^9 acres of arable land available. The world population of 1950 required 10^9 acres to sustain it, and the population of 1980 required 2×10^9 acres. If the required acreage grows at a constant percentage rate, in what year will the population reach the maximum sustainable size?

17. Suppose some quantity is increasing exponentially (e.g., the number of cells in a bacterial culture) with growth rate r. Show that the doubling time is $\frac{\ln 2}{r}$.

18. Suppose some quantity is decaying exponentially with decay constant r. Show that the half-life is $-\frac{\ln 2}{r}$. What is the difference between the half-life here and the doubling time in exercise 17?

Exercises 19–26 involve exponential decay.

19. The radioactive element iodine-131 has a decay constant of $-1.3863\ \text{day}^{-1}$. Find its half-life.

20. The radioactive element cesium-137 has a decay constant of $-0.023\ \text{year}^{-1}$. Find its half-life.

21. The half-life of morphine in the human bloodstream is 3 hours. If initially there is 0.4 mg of morphine in the bloodstream, find an equation for the amount in the bloodstream at any time. When does the amount drop below 0.01 mg?

22. Given a half-life of 3 hours, by what percentage will the amount of morphine in the bloodstream have decreased in 1 day? (See exercise 21.)

23. Strontium-90 is a dangerous radioactive isotope. Because of its similarity to calcium, it is easily absorbed into human bones. The half-life of strontium-90 is 28 years. If a certain amount is absorbed into the bones due to exposure to a nuclear explosion, what percentage will remain after 50 years?

24. The half-life of uranium ^{235}U is approximately 0.7×10^9 years. If 50 grams are buried at a nuclear waste site, how much will remain after 100 years?

25. Scientists dating a fossil estimate that 20% of the original amount of carbon-14 is present. Recalling that the half-life is 5730 years, approximately how old is the fossil?

26. If a fossil is 1 million years old, what percentage of its original carbon-14 should remain?

Exercises 27–34 involve Newton's Law of Cooling.

27. A bowl of porridge at 200°F (too hot) is placed in a 70°F room. One minute later the porridge has cooled to 180°F. When will the temperature be 120°F (just right)?

28. A smaller bowl of porridge served at 200°F cools to 160°F in 1 minute. What temperature (too cold) will this porridge be when the bowl of exercise 27 has reached 120°F (just right)?

29. A cold drink is poured out at 50°F. After 2 minutes of sitting in a 70°F room, its temperature has risen to 56°F. Find the drink's temperature at any time t.

30. For the cold drink in exercise 29, what will the temperature be after 10 minutes? When will the drink have warmed to 66°F?

31. At 10:07 P.M. you find a secret agent murdered. Next to him is a martini that got shaken before the secret agent could stir it. Room temperature is 70°F. The martini warms from 60°F to 61°F in the 2 minutes from 10:07 P.M. to 10:09 P.M. If the secret agent's martinis are always served at 40°F, what was the time of death?

32. Twenty minutes after being served a cup of fast-food coffee, it is still too hot to drink at 160°F. Two minutes later, the temperature has dropped to 158°F. Your friend, whose coffee is also too hot to drink, speculates that since the temperature is dropping an average of 1 degree per minute, it was served at 180°F. Explain what is wrong with this logic. Was the actual serving temperature hotter or cooler than 180°F?

33. Find the actual serving temperature in exercise 32 if room temperature is 68°F.

34. For the cup of coffee in example 1.3, suppose that the goal is to have the coffee cool to 120°F in 5 minutes. At what temperature should the coffee be served?

Exercises 35–38 involve compound interest.

35. If you invest \$1000 at an annual interest rate of 8%, compare the value of the investment after 1 year under the following forms of compounding: annual, monthly, daily, continuous.

36. Repeat exercise 35 for the value of the investment after 5 years.

37. Person A invests \$10,000 in 1990 and person B invests \$20,000 in 2000. If both receive 12% interest (compounded continuously), what are the values of the investments in 2010?

38. Repeat exercise 37 for an interest rate of 4%. Then determine the interest rate such that person A ends up exactly even with person B. (Hint: You want person A to have \$20,000 in 2000.)

39. One of the authors bought a set of basketball trading cards in 1985 for \$34. In 1995, the "book price" for this set was \$9800. Assuming a constant percentage return on this investment, find an equation for the worth of the set at time t years (where $t = 0$ corresponds to 1985). At this rate of return, what would the set have been worth in 2005?

40. The author also bought a set of baseball cards in 1985, costing \$22. In 1995, this set was worth \$32. At this rate of return, what would the set have been worth in 2005?

41. In 1975, income between \$16,000 and \$20,000 was taxed at 28%. In 1988, income between \$16,000 and \$20,000 was taxed at 15%. This makes it seem as if taxes went down considerably between 1975 and 1988. Taking inflation into account, briefly explain why this is not a valid comparison.

42. To make the comparison in exercise 41 a little fairer, note that income above \$30,000 was taxed at 28% in 1988 and assume that inflation averaged 5.5% between 1975 and 1988. Adjust \$16,000 for inflation by computing its value after increasing continuously at 5.5% for 13 years. Based on this calculation, how do the tax rates compare?

43. Suppose the income tax structure is as follows: the first \$30,000 is taxed at 15%, the remainder is taxed at 28%. Compute the tax T_1 on an income of \$40,000. Now, suppose that inflation is 5% and you receive a cost of living (5%) raise to \$42,000. Compute the tax T_2 on this income. To compare the taxes, you should adjust the tax T_1 for inflation (add 5%).

44. In exercise 43, the tax code stayed the same, but (adjusted for inflation) the tax owed did not stay the same. Briefly explain why this happened. What could be done to make the tax owed remain constant?

45. Suppose that the value of a \$40,000 asset decreases at a constant percentage rate of 10%. Find its worth after (a) 10 years and (b) 20 years. Compare these values to a \$40,000 asset that is depreciated to no value in 20 years using linear depreciation.

46. Suppose that the value of a \$400,000 asset decreases at a constant percentage rate of 40%. Find its worth after (a) 5 years and (b) 10 years. Compare these values to a \$40,000 asset that is depreciated to no value in 10 years using linear depreciation.

47. One of the mysteries in population biology is how populations regulate themselves. The most famous myth involves lemmings diving off of cliffs at times of overpopulation. It *is* true that lemming populations rise and fall dramatically, for whatever reason (not including suicide). Animal ecologists draw graphs to visualize the rises and falls of animal populations. Instead of graphing population versus time, ecologists graph the logarithm of population versus time. To understand why, note that a population drop from 1000 to 500 would represent the same percentage decrease as a drop from 10 to 5. Show that the slopes of the drops are different, so that these drops would appear to be different on a population/time graph. However, show that the slopes of the drops in the logarithms (e.g., ln 1000 to ln 500) are the same. In general, if a population were changing at a constant percentage rate, what would the graph of population versus time look like? What would the graph of the logarithm of population versus time look like?

48. It has been conjectured that half the people who have ever lived are still alive today. To see whether this is plausible, assume that humans have maintained a constant birthrate b and death rate d. Show that the statement is true if and only if $b \geq 2d$.

49. Using the bacterial population data at the beginning of this section, define x to be time and y to be the natural logarithm of the population. Plot the data points (x, y) and comment on how close the data are to being linear. Take two representative points and find an equation of the line through the two points. Then find the population function $p(x) = e^{y(x)}$.

50. If your calculator or CAS does exponential regression, compare the regression equation to your model from exercise 49.

51. As in exercise 49, find an exponential model for the population data (0, 10), (1, 15), (2, 22), (3, 33) and (4, 49).

52. As in exercise 49, find an exponential model for the population data (0, 20), (1, 16), (2, 13), (3, 11) and (4, 9).

53. Use the method of exercise 49 to fit an exponential model to the following data representing percentage of the U.S. population classified as living on rural farms (data from the U.S. Census Bureau).

Year	1960	1970	1980	1990
% Pop. Farm	7.5	5.2	2.5	1.6

54. Use the method of exercise 49 to fit an exponential model to the following data representing percentage of the U.S. population classified as living in urban areas (data from the U.S. Census Bureau).

Year	1960	1970	1980	1990
% Pop. Urban	69.9	73.5	73.7	75.2

55. An Internet site reports that the antidepressant drug amitriptyline has a half-life in humans of 31–46 hours. For a dosage of 150 mg, compare the amounts left in the bloodstream after one day for a person for whom the half-life is 31 hours versus a person for whom the half-life is 46 hours. Is this a large difference?

56. It is reported that Prozac® has a half-life of 2 to 3 days but may be found in your system for several weeks after you stop taking it. What percentage of the original dosage would remain after 2 weeks if the half-life is 2 days? How much would remain if the half-life is 3 days?

57. The antibiotic ertapenem has a half-life of 4 hours in the human bloodstream. The dosage is 1 gm per day. Find and graph the amount in the bloodstream t hours after taking it ($0 \le t \le 24$).

58. Compare your answer to exercise 57 with a similar drug that is taken with a dosage of 1 gm four times a day and has a half-life of 1 hour. (Note that you will have to do four separate calculations here.)

59. A bank offers to sell a bank note that will reach a maturity value of \$10,000 in 10 years. How much should you pay for it now if you wish to receive an 8% return on your investment? (Note: This is called the **present value** of the bank note.) Show that in general, the present value of an item worth $\$P$ in t years with constant interest rate r is given by $\$Pe^{-rt}$.

60. Suppose that the value of a piece of land t years from now is $\$40{,}000e^{2\sqrt{t}}$. Given 6% annual inflation, find t that maximizes the present value of your investment: $\$40{,}000e^{2\sqrt{t}-0.06t}$.

61. Suppose that a business has an income stream of $\$P(t)$. The present value at interest rate r of this income for the next T years is $\int_0^T P(t)e^{-rt}\,dt$. Compare the present values at 5% for three people with the following salaries for 3 years: A: $P(t) = 60{,}000$; B: $P(t) = 60{,}000 + 3000t$; and C: $P(t) = 60{,}000e^{0.05t}$.

62. The **future value** of an income stream after T years at interest rate r is given by $\int_0^T P(t)e^{r(T-t)}\,dt$. Calculate the future value for cases A, B and C in exercise 61. Briefly describe the difference between what present value and future value measure.

63. If you win a "million dollar" lottery, would you be better off getting your money in four annual installments of \$280,000 or in one lump sum of \$1 million? Assume 8% interest and payments made at the beginning of the year.

64. The "Rule of 72" is used by many investors to quickly estimate how fast an investment will double in value. For example, at 8% the rule suggests that the doubling time will be about $\frac{72}{8} = 9$ years. Calculate the actual doubling time. Explain why a "Rule of 69" would be more accurate. Give at least one reason why the number 72 is used instead.

EXPLORATORY EXERCISES

1. In the text, we briefly discussed the use of the radioactive isotope carbon-14 to date fossils. We elaborate on that here. The amount of carbon-14 in the atmosphere is largely determined by cosmic ray bombardment, with nuclear testing also playing a role. Living organisms maintain a constant level of carbon-14 through exchanges with the environment. At death, the organism no longer takes in carbon-14, so the carbon-14 level decreases with the 5730-year half-life. Scientists can measure the rate of disintegration of carbon-14. (You might visualize a Geiger counter.) If $y(t)$ is the amount of carbon-14 remaining at time t, the rate of change is $y'(t) = ky(t)$. The main assumption is that the rate of disintegration at the time of death is the same as it is now for living organisms. Mathematically, this rate corresponds to $ky(0)$. The ratio of disintegration rates is $y(t)/y(0)$. Given this ratio, describe how to determine the time t. In particular, suppose that $ky(t) = -2.4$ (disintegrations per minute) and $ky(0) = -6.7$ (disintegrations per minute). Solve for t. Different inaccuracies can creep into this process. First, suppose the assumption of constant carbon-14 levels is wrong. If $ky(0)$ is decreased by 5%, by what percentage does the estimate of the time t change? There may also be inaccuracies in the measurement of the current disintegration rate. If $ky(t)$ is decreased by 5%, by what percentage does the estimate of the time change? Roughly, how do errors in the measurements translate to errors in the estimate of the time?

2. The carbon-14 method of dating fossils is discussed in exercise 1. Here, we discuss the **potassium-argon dating** method, used to date old volcanic rock. The background theory is that radioactive potassium-40 decays very slowly, with a half-life of about 1.3 billion years. Approximately 11% of the potassium-40 that decays produces argon-40. Argon escapes from molten lava but is trapped in cool rock, so the amount of argon can be used to measure how much time has passed since the lava cooled. First, show that the amount $K(t)$ of potassium-40 and the amount $A(t)$ of argon-40 satisfy the equations

$$\frac{dK}{dt} = -0.0000000005305K(t) \quad \text{and}$$

$$\frac{dA}{dt} = 0.0000000000585K(t).$$

A change of variables improves the look of these numbers. If $s = \frac{t}{10^{10}}$, show that

$$\frac{dK}{ds} = -5.305K(s) \quad \text{and} \quad \frac{dA}{ds} = 0.585K(s).$$

Choose units of measurement so that $K(0) = 1$. If we measure the current ratio of argon-40 to potassium-40 in a rock, this is $A(0)$. If the rock cooled T years ago, then $A(-T) = 0$, since argon-40 does not remain in molten lava. For a ratio of 0.00012, find the age of the rock. Comment on why this method is used to date very old rocks.

3. Three confused hunting dogs start at the vertices of an equilateral triangle of side 1. Each dog runs with a constant speed aimed directly at the dog that is positioned clockwise from it. The chase stops when the dogs meet in the middle (having grabbed each other by their tails). How far does each dog run? [Hints: Represent the position of each dog in polar coordinates (r, θ) with the center of the triangle at the origin. By symmetry, each dog has the same r-value, and if one dog has angle θ, then it is aimed at the dog with angle $\theta - \frac{2\pi}{3}$. Set up a differential equation for the motion of one dog and show that there is a solution if $r'(\theta) = \sqrt{3}r$. Use the arc length formula $L = \int_{\theta_1}^{\theta_2} \sqrt{[r'(\theta)]^2 + [r(\theta)]^2}\, d\theta$.]

4. To generalize exercise 3, suppose that there are n dogs starting at the vertices of a regular n-gon of side s. If α is the interior angle from the center of the n-gon to adjacent vertices, show that the distance run by each dog equals $\frac{s}{1 - \cos \alpha}$. What happens to the distance as n increases without bound? Explain this in terms of the paths of the dogs.

7.2 SEPARABLE DIFFERENTIAL EQUATIONS

In section 7.1, we solved two different differential equations:

$$y'(t) = ky(t) \quad \text{and} \quad y'(t) = k[y(t) - T_a].$$

These are both examples of *separable* differential equations. We will examine this type of equation at some length in this section. First, we consider the more general **first-order ordinary differential equation**

$$y' = f(x, y). \tag{2.1}$$

Here, the derivative y' of some unknown function $y(x)$ is given as a function f of both x and y. Our objective is to find some function $y(x)$ (a **solution**) that satisfies equation (2.1). The equation is **first-order,** since it involves only the first derivative of the unknown function. We will consider the case where the x's and y's can be separated. We call equation (2.1) **separable** if we can separate the variables, i.e., if we can rewrite it in the form

$$g(y)y' = h(x),$$

where all of the x's are on one side of the equation and all of the y's are on the other side.

NOTE

Do not be distracted by the letter used for the independent variable. We frequently use the independent variable x, as in equation (2.1). Whenever the independent variable represents time, we use t as the independent variable, in order to reinforce this connection, as we did in example 1.2. There, the equation describing radioactive decay was given as

$$y'(t) = ky(t).$$

EXAMPLE 2.1 A Separable Differential Equation

Determine whether the differential equation

$$y' = xy^2 - 2xy$$

is separable.

Solution Notice that this equation is separable, since we can rewrite it as

$$y' = x(y^2 - 2y)$$

and then divide by $(y^2 - 2y)$ (assuming this is not zero), to obtain

$$\frac{1}{y^2 - 2y} y' = x.$$

EXAMPLE 2.2 An Equation That Is Not Separable

The equation
$$y' = xy^2 - 2x^2y$$
is not separable, as there is no way to separate the x's and the y's. (Try this for yourself!) ■

Essentially, the x's and y's must be separated by multiplication or division in order for a differential equation to be separable. Notice that in example 2.2, you can factor to get $y' = xy(y - 2x)$, but the subtraction $y - 2x$ keeps this equation from being separable.

Separable differential equations are of interest because there is a very simple means of solving them. Notice that if we integrate both sides of
$$g(y)y'(x) = h(x)$$
with respect to x, we get
$$\int g(y)y'(x)\,dx = \int h(x)\,dx. \tag{2.2}$$
Since $dy = y'(x)\,dx$, the integral on the left-hand side of (2.2) becomes
$$\int g(y)\,\underbrace{y'(x)\,dx}_{dy} = \int g(y)\,dy.$$
Consequently, from (2.2), we have
$$\int g(y)\,dy = \int h(x)\,dx.$$
So, provided we can evaluate both of these integrals, we have an equation relating x and y, which no longer involves y'.

EXAMPLE 2.3 Solving a Separable Equation

Solve the differential equation
$$y' = \frac{x^2 + 7x + 3}{y^2}.$$

Solution Separating variables, observe that we have
$$y^2y' = x^2 + 7x + 3.$$
Integrating both sides with respect to x, we obtain
$$\int y^2y'(x)\,dx = \int (x^2 + 7x + 3)\,dx$$
or
$$\int y^2dy = \int (x^2 + 7x + 3)\,dx.$$
Performing the indicated integrations yields
$$\frac{y^3}{3} = \frac{x^3}{3} + 7\frac{x^2}{2} + 3x + c,$$
where we have combined the two constants of integration into one on the right-hand side.

Solving for y, we get
$$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 3c}.$$

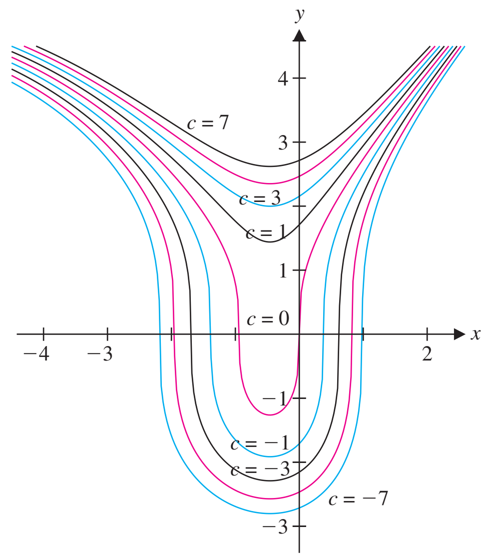


FIGURE 7.6
A family of solutions

Notice that for each value of c, we get a different solution of the differential equation. This is called a **family of solutions** (or the **general solution**) of the differential equation. In Figure 7.6, we have plotted a number of the members of this family of solutions. ■

In general, the solution of a first-order separable equation will include an arbitrary constant (the constant of integration). To select just one of these solution curves, we specify a single point through which the solution curve must pass, say (x_0, y_0). That is, we require that

$$y(x_0) = y_0.$$

This is called an **initial condition** (since this condition often specifies the initial state of a physical system). Such a differential equation together with an initial condition is referred to as an **initial value problem** (IVP).

EXAMPLE 2.4 Solving an Initial Value Problem

Solve the IVP
$$y' = \frac{x^2 + 7x + 3}{y^2}, \quad y(0) = 3.$$

Solution In example 2.3, we found that the general solution of the differential equation is

$$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 3c}.$$

From the initial condition, we now have

$$3 = y(0) = \sqrt[3]{0 + 3c} = \sqrt[3]{3c}$$

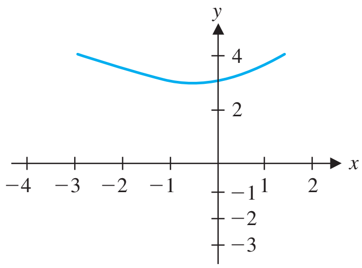


FIGURE 7.7
$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 27}$

and hence, $c = 9$. The solution of the IVP is then

$$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 27}.$$

We show a graph of this solution in Figure 7.7. Notice that this graph would fit above the curves shown in Figure 7.6. We'll explore the effects of other initial conditions in the exercises. ■

We are not always as fortunate as we were in example 2.4. There, we were able to obtain an explicit representation of the solution (i.e., we found a formula for y in terms of x). Most often, we must settle for an *implicit* representation of the solution, that is, an equation relating x and y that cannot be solved for y in terms of x alone.

EXAMPLE 2.5 An Initial Value Problem That Has Only an Implicit Solution

Find the solution of the IVP

$$y' = \frac{9x^2 - \sin x}{\cos y + 5e^y}, \qquad y(0) = \pi.$$

Solution First, note that the differential equation is separable, since we can rewrite it as

$$(\cos y + 5e^y)y'(x) = 9x^2 - \sin x.$$

Integrating both sides of this equation with respect to x, we find

$$\int (\cos y + 5e^y)y'(x)\,dx = \int (9x^2 - \sin x)\,dx$$

or

$$\int (\cos y + 5e^y)\,dy = \int (9x^2 - \sin x)\,dx.$$

Evaluating the integrals, we obtain

$$\sin y + 5e^y = 3x^3 + \cos x + c. \qquad (2.3)$$

Notice that there is no way to solve this equation explicitly for y in terms of x. However, you can still picture the graphs of some members of this family of solutions by using the implicit plot mode on your graphing utility. Several of these are plotted in Figure 7.8a. Even though we have not solved for y explicitly in terms of x, we can still use the initial condition. Substituting $x = 0$ and $y = \pi$ into equation (2.3), we have

$$\sin \pi + 5e^\pi = 0 + \cos 0 + c$$

or

$$5e^\pi - 1 = c.$$

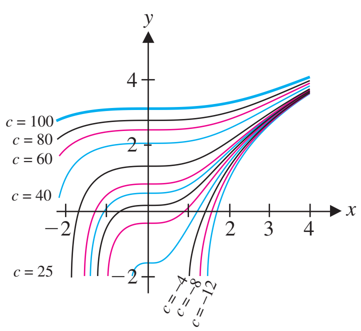


FIGURE 7.8a
A family of solutions

This leaves us with

$$\sin y + 5e^y = 3x^3 + \cos x + 5e^\pi - 1$$

as an implicit representation of the solution of the IVP. Although we cannot solve for y in terms of x alone, given any particular value for x, we can use Newton's method (or some other numerical method) to approximate the value of the corresponding y. This is essentially what your CAS does (with many, many points) when you use it to plot a graph in implicit mode. We plot the solution of the IVP in Figure 7.8b. ■

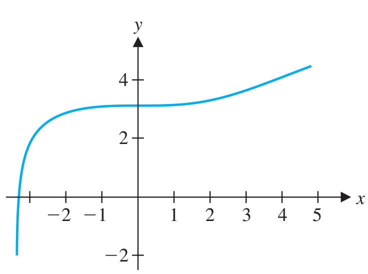


FIGURE 7.8b
The solution of the IVP

Logistic Growth

In section 7.1, we introduced the differential equation

$$y' = ky$$

as a model of bacterial population growth, valid for populations growing with unlimited resources and with unlimited room for growth. Of course, *all* populations have factors that eventually limit their growth. Thus, this particular model generally provides useful information only for relatively short periods of time.

An alternative model of population growth assumes that there is a maximum sustainable population, M (called the **carrying capacity**), determined by the available resources. Further, as the population size approaches M (as available resources become more scarce), the population growth will slow. To reflect this, we assume that the rate of population growth is jointly proportional to the present population level and the difference between the current level and the maximum, M. That is, if $y(t)$ is the population at time t, we assume that

$$y'(t) = ky(M - y).$$

This differential equation is referred to as the **logistic equation.**

HISTORICAL NOTES

Pierre Verhulst (1804–1849) A Belgian mathematician who proposed the logistic model for population growth. Verhulst was a professor of mathematics in Brussels and did research on number theory and social statistics. His most important contribution was the logistic equation (also called the Verhulst equation) giving the first realistic model of a population with limited resources. It is worth noting that Verhulst's estimate of Belgium's equilibrium population closely matches the current Belgian population.

Two special solutions of this differential equation are apparent. The constant functions $y = 0$ and $y = M$ are both solutions of this differential equation. These are called **equilibrium solutions** since, under the assumption of logistic growth, once a population hits one of these levels, it remains there for all time. If $y \neq 0$ and $y \neq M$, we can solve the differential equation, since it is separable, as

$$\frac{1}{y(M - y)} y'(t) = k. \tag{2.4}$$

Integrating both sides with respect to t, we obtain

$$\int \frac{1}{y(M - y)} y'(t)\, dt = \int k\, dt$$

or

$$\int \frac{1}{y(M - y)}\, dy = \int k\, dt. \tag{2.5}$$

We use partial fractions to write

$$\frac{1}{y(M - y)} = \frac{1}{My} + \frac{1}{M(M - y)}.$$

Observe that from (2.5) we now have

$$\int \left[\frac{1}{My} + \frac{1}{M(M - y)}\right] dy = \int k\, dt.$$

Carrying out the integrations gives us

$$\frac{1}{M} \ln |y| - \frac{1}{M} \ln |M - y| = kt + c.$$

Multiplying both sides by M and using the fact that $0 < y < M$, we have

$$\ln y - \ln(M - y) = kMt + Mc.$$

Taking exponentials of both sides, we find

$$\exp[\ln y - \ln(M - y)] = e^{kMt+Mc} = e^{kMt}e^{Mc}.$$

Next, using rules of exponentials and logarithms and replacing the constant term e^{Mc} by a new constant A, we obtain

$$\frac{y}{M-y} = Ae^{kMt}.$$

To solve this for y, we first multiply both sides by $(M - y)$ to obtain

$$\begin{aligned} y &= Ae^{kMt}(M-y) \\ &= AMe^{kMt} - Ae^{kMt}y. \end{aligned}$$

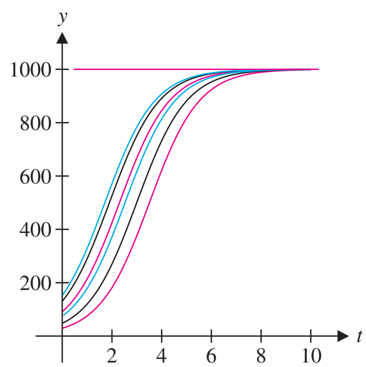


FIGURE 7.9
Several solution curves

Combining the two y terms, we find

$$y(1 + Ae^{kMt}) = AMe^{kMt},$$

which gives us the explicit solution of the logistic equation,

$$y = \frac{AMe^{kMt}}{1 + Ae^{kMt}}. \tag{2.6}$$

In Figure 7.9, we plot a number of these solution curves for various values of A (for the case where $M = 1000$ and $k = 0.001$), along with the equilibrium solution $y = 1000$. You can see from Figure 7.9 that logistic growth consists of nearly exponential growth initially, followed by the graph becoming concave down and then asymptotically approaching the maximum, M.

EXAMPLE 2.6 Solving a Logistic Growth Problem

Given a maximum sustainable population of $M = 1000$ (this could be measured in millions or tons, etc.) and growth rate $k = 0.007$, find an expression for the population at any time t, given an initial population of $y(0) = 350$ and assuming logistic growth.

Solution From the solution (2.6) of the logistic equation, we have $kM = 7$ and

$$y = \frac{1000Ae^{7t}}{1 + Ae^{7t}}.$$

From the initial condition, we have

$$350 = y(0) = \frac{1000A}{1 + A}.$$

Solving for A, we obtain $A = \frac{35}{65}$, which gives us the solution of the IVP

$$y = \frac{35{,}000e^{7t}}{65 + 35e^{7t}}.$$

This solution is plotted in Figure 7.10. ■

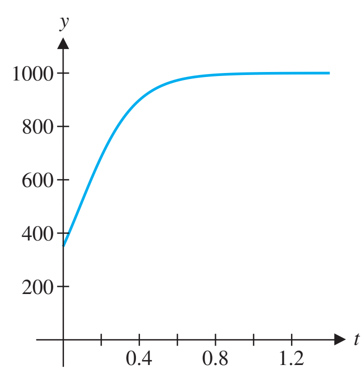


FIGURE 7.10
$y = \dfrac{35{,}000e^{7t}}{65 + 35e^{7t}}$

Note that, in practice, the values of M and k are not known and must be estimated from a careful study of the particular population. We explore these issues further in the exercises.

In our final example, we consider an investment plan. Before working the problems, try to make an educated guess of the answer in advance. For investments over long periods of time, most people are surprised at how rapidly the money accumulates.

EXAMPLE 2.7 Investment Strategies for Making a Million

Money is invested at 8% interest compounded continuously. If deposits are made continuously at the rate of \$2000 per year, find the size of the initial investment needed to reach \$1 million in 20 years.

Solution Here, interest is earned at the rate of 8% and additional deposits are assumed to be made on a continuous basis. If the deposit rate is $\$d$ per year, the amount $A(t)$ in the account after t years satisfies the differential equation

$$\frac{dA}{dt} = 0.08A + d.$$

This equation is separable and can be solved by dividing both sides by $0.08A + d$ and integrating. We have

$$\int \frac{1}{0.08A + d}\,dA = \int 1\,dt,$$

so that

$$\frac{1}{0.08}\ln|0.08A + d| = t + c.$$

Using $d = 2000$, we have

$$12.5\ln|0.08A + 2000| = t + c.$$

Setting $t = 0$ and taking $A(0) = x$ gives us the constant of integration:

$$12.5\ln|0.08x + 2000| = c,$$

so that

$$12.5\ln|0.08A + 2000| = t + 12.5\ln|0.08x + 2000|. \tag{2.7}$$

To find the value of x such that $A(20) = 1{,}000{,}000$, we substitute $t = 20$ and $A = 1{,}000{,}000$ into equation (2.7) to obtain

$$12.5\ln|0.08(1{,}000{,}000) + 2000| = 20 + 12.5\ln|0.08x + 2000|$$

or

$$12.5\ln|82{,}000| = 20 + 12.5\ln|0.08x + 2000|.$$

We can solve this for x, by subtracting 20 from both sides and then dividing by 12.5, to obtain

$$\frac{12.5\ln 82{,}000 - 20}{12.5} = \ln|0.08x + 2000|.$$

Taking the exponential of both sides, we now have

$$e^{(12.5\ln 82{,}000 - 20)/12.5} = 0.08x + 2000.$$

Solving this for x yields

$$x = \frac{e^{\ln 82{,}000 - 1.6} - 2000}{0.08} \approx 181{,}943.93.$$

So, the initial investment needs to be \$181,943.93 (slightly less than \$200,000) in order to be worth \$1 million at the end of 20 years. ■

To be fair, the numbers in example 2.7 (like most investment numbers) must be interpreted carefully. Of course, 20 years from now, \$1 million likely won't buy as much as \$1 million does today. For instance, the value of a million dollars adjusted for 8%

annual inflation would be $\$1{,}000{,}000e^{-0.08(20)} \approx \$201{,}896$, which is not much larger than the $181,943 initial investment required. However, if inflation is only 4%, then the value of a million dollars (in current dollars) is $449,328. The lesson here is the obvious one: Be sure to invest money at an interest rate that exceeds the rate of inflation.

EXERCISES 7.2

WRITING EXERCISES

1. Discuss the differences between solving algebraic equations (e.g., $x^2 - 1 = 0$) and solving differential equations. Especially note what type of mathematical object you are solving for.

2. A differential equation is not separable if it can't be written in the form $g(y)y' = h(x)$. If you have an equation that you can't write in this form, how do you know whether it's really impossible or you just haven't figured it out yet? Discuss some general forms (e.g., $x + y$ and xy) that give you clues as to whether the equation is likely to separate or not.

3. The solution curves in Figures 7.6, 7.8a and 7.9 do not appear to cross. In fact, they never intersect. If solution curves crossed at the point (x_1, y_1), then there would be two solutions satisfying the initial condition $y(x_1) = y_1$. Explain why this does not happen. In terms of the logistic equation as a model of population growth, explain why it is important to know that this does not happen.

4. The logistic equation includes a term in the differential equation that slows population growth as the population increases. Discuss some of the reasons why this occurs in real populations (human, animal and plant).

In exercises 1–8, determine whether the differential equation is separable.

1. $y' = (3x + 1)\cos y$
2. $y' = 2x(\cos y - 1)$
3. $y' = (3x + y)\cos y$
4. $y' = 2x(y - x)$
5. $y' = x^2y + y\cos x$
6. $y' = 2x\cos y - xy^3$
7. $y' = x^2y - x\cos y$
8. $y' = x^3 - 2x + 1$

In exercises 9–22, the differential equation is separable. Find the general solution, in an explicit form if possible. Sketch several members of the family of solutions.

9. $y' = (x^2 + 1)y$
10. $y' = 2x(y - 1)$
11. $y' = 2x^2y^2$
12. $y' = 2(y^2 + 1)$
13. $y' = \dfrac{6x^2}{y(1 + x^3)}$
14. $y' = \dfrac{3x}{y + 1}$
15. $y' = \dfrac{2xe^y}{ye^x}$
16. $y' = \dfrac{\sqrt{1 - y^2}}{x\ln x}$
17. $y' = y^2 - y$
18. $y' = x\cos^2 y$
19. $y' = \dfrac{xy}{1 + x^2}$
20. $y' = \dfrac{2}{xy + y}$
21. $y' = \dfrac{\cos^2 y}{4x - 3}$
22. $y' = \dfrac{(y^2 + 1)\ln x}{4y}$

In exercises 23–30, solve the IVP, explicitly if possible.

23. $y' = 3(x + 1)^2y,\ y(0) = 1$
24. $y' = \dfrac{x - 1}{y^2},\ y(0) = 2$
25. $y' = \dfrac{4x^2}{y},\ y(0) = 2$
26. $y' = \dfrac{x - 1}{y},\ y(0) = -2$
27. $y' = \dfrac{4y}{x + 3},\ y(-2) = 1$
28. $y' = \dfrac{3x}{4y + 1},\ y(1) = 4$
29. $y' = \dfrac{4x}{\cos y},\ y(0) = 0$
30. $y' = \dfrac{\tan y}{x},\ y(1) = \dfrac{\pi}{2}$

In exercises 31–36, use equation (2.6) to help solve the IVP.

31. $y' = 3y(2 - y),\ y(0) = 1$
32. $y' = y(3 - y),\ y(0) = 2$
33. $y' = 2y(5 - y),\ y(0) = 4$
34. $y' = y(2 - y),\ y(0) = 1$
35. $y' = y(1 - y),\ y(0) = \frac{3}{4}$
36. $y' = y(3 - y),\ y(0) = 0$

37. The logistic equation is sometimes written in the form $y'(t) = ry(t)(1 - y(t)/M)$. Show that this is equivalent to equation (2.4) with $r/M = k$. Biologists have measured the values of the carrying capacity M and growth rate r for a variety of fish. Just for the halibut, approximate values are $r = 0.71$ year^{-1} and $M = 8 \times 10^7$ kg. If the initial biomass of halibut is $y(0) = 2 \times 10^7$ kg, find an equation for the biomass of halibut at any time. Sketch a graph of the biomass as a function of time.

38. Estimate how long it will take for the biomass of the halibut in exercise 37 to get within 10% of the carrying capacity.

39. Find the solution of equation (2.4) if $y(t) > M$.

40. Use the solution found in exercise 39 and the parameters for the halibut in exercise 37 to answer the following question. If the halibut biomass explodes to 3×10^8 kg, how long will it take for the population to drop back to within 10% of the carrying capacity?

41. In example 2.4, find and graph the solution passing through (0, 0).

42. In exercise 41, notice that the initial value problem is $y' = \dfrac{x^2+7x+3}{y^2}$ with $y(0) = 0$. If you substitute $y = 0$ into the differential equation, what is $y'(0)$? Verify that your answer in exercise 41 has this property. Describe what is happening graphically at $x = 0$. (See Figure 7.6.)

43. For the differential equation $y' = \dfrac{x^2+7x+3}{y^2}$ used in exercises 41 and 42, notice that $y'(x)$ does not exist at any x for which $y(x) = 0$. Given the solution of example 2.4, this occurs if $x^3 + \frac{21}{2}x^2 + 9x + 3c = 0$. Find the values c_1 and c_2 such that this equation has three real solutions if and only if $c_1 < c < c_2$.

44. Graph the solution of $y' = \dfrac{x^2+7x+3}{y^2}$ with $c = c_2$. (See exercise 43.)

45. For $c = c_2$ in exercise 43, argue that the solution to $y' = \dfrac{x^2+7x+3}{y^2}$ with $y(0) = \sqrt[3]{3c_2}$ has two points with vertical tangent lines.

46. Estimate the locations of the three points with vertical tangent lines in exercise 41.

Exercises 47–56 relate to money investments.

47. If continuous deposits are made into an account at the rate of \$2000 per year and interest is earned at 6% compounded continuously, find the size of the initial investment needed to reach \$1 million in twenty years. Comparing your answer to that of example 2.7, how much difference does interest rate make?

48. If \$10,000 is invested initially at 6% interest compounded continuously, find the (yearly) continuous deposit rate needed to reach \$1 million in twenty years. Comparing your answer to that of exercise 47, how much difference does an initial deposit make?

49. A house mortgage is a loan that is to be paid over a fixed period of time. Suppose \$150,000 is borrowed at 8% interest. If the monthly payment is \$$P$, then explain why the equation $A'(t) = 0.08A(t) - 12P$, $A(0) = 150{,}000$ is a model of the amount owed after t years. For a 30-year mortgage, the payment P is set so that $A(30) = 0$. Find P. Then, compute the total amount paid and the amount of interest paid.

50. Rework exercise 49 with a 7.5% loan. Does the half-percent decrease in interest rate make a difference?

51. Rework exercise 49 with a 15-year mortgage. Compare the monthly payments and total amount paid.

52. Rework exercise 49 with a loan of \$125,000. How much difference does it make to add an additional \$25,000 down payment?

53. A person contributes \$10,000 per year to a retirement fund continuously for 10 years until age 40 but makes no initial payment and no further payments. At 8% interest, what is the value of the fund at age 65?

54. A person contributes \$20,000 per year to a retirement fund from age 40 to age 65 but makes no initial payment. At 8% interest, what is the value of the fund at age 65? Compare with your answer to exercise 53.

55. Find the interest rate r at which the investors of exercises 53 and 54 have equal retirement funds.

56. An endowment is seeded with \$1,000,000 invested with interest compounded continuously at 10%. Determine the amount that can be withdrawn (continuously) annually so that the endowment lasts thirty years.

Exercises 57–60 relate to reversible bimolecular chemical reactions, where molecules A and B combine to form two other molecules C and D and vice versa. If $x(t)$ and $y(t)$ are the concentrations of C and D, respectively and the initial concentrations of A, B, C and D are a, b, c and d, respectively, then the reaction is modeled by

$$\boldsymbol{x'(t) = k_1(a+c-x)(b+c-x) - k_{-1}x(d-c+x)}$$

for rate constants k_1 and k_{-1}.

57. If $k_1 = 1$, $k_{-1} = 0.625$, $a + c = 0.4$, $b + c = 0.6$, $c = d$ and $x(0) = 0.2$, find the concentration $x(t)$. Graph $x(t)$ and find the eventual concentration level.

58. Repeat exercise 57 with (a) $x(0) = 0.3$ and (b) $x(0) = 0.6$. Briefly explain what is physically impossible about the initial condition in part (b).

59. For the bimolecular reaction with $k_1 = 0.6$, $k_{-1} = 0.4$, $a + c = 0.5$, $b + c = 0.6$ and $c = d$, write the differential equation for the concentration of C. For $x(0) = 0.2$, solve for the concentration at any time and graph the solution.

60. For the bimolecular reaction with $k_1 = 1.0$, $k_{-1} = 0.4$, $a + c = 0.6$, $b + c = 0.4$ and $d - c = 0.1$, write the differential equation for the concentration of C. For $x(0) = 0.2$, solve for the concentration at any time and graph the solution.

61. In a **second-order chemical reaction,** one molecule each of substances A and B combine to produce one molecule of substance X. If a and b are the initial concentrations of A and B, respectively, the concentration x of the substance X satisfies the differential equation $x' = r(a - x)(b - x)$ for some positive rate constant r. (a) If $r = 0.4$, $a = 6$, $b = 8$ and $x(0) = 0$, find $x(t)$ and $\lim_{t\to\infty} x(t)$. Explain this answer in terms of the chemical process. (b) Repeat part (a) with $r = 0.6$. Graph the solutions and discuss differences and similarities.

62. In a second-order chemical reaction, if there is initially 10 g of substance A available and 12 g of substance B available, then the amount $x(t)$ of substance X formed by time t satisfies the IVP $x'(t) = r(10 - x)(12 - x)$, $x(0) = 0$. Explain why, physically, it makes sense that $0 \le x < 10$. Solve the IVP and indicate where you need this assumption.

Exercises 63–68 relate to logistic growth with harvesting. Suppose that a population in isolation satisfies the logistic equation $y'(t) = ky(M - y)$. If the population is harvested (for example, by fishing) at the rate R, then the population model becomes $y'(t) = ky(M - y) - R$.

63. Suppose that a species of fish has population in hundreds of thousands that follows the logistic model with $k = 0.025$ and $M = 8$. Determine the long-term effect on population if the initial population is 800,000 $[y(0) = 8]$ and fishing removes fish at the rate of 20,000 per year.

64. Repeat exercise 63 if fish are removed at the rate of 60,000 per year.

65. For the fishing model $P'(t) = 0.025P(t)[8 - P(t)] - R$ (see exercise 63), the population is constant if $P'(t) = P^2 - 8P + 40R = 0$. The solutions are called equilibrium points. Compare the equilibrium points for exercises 63 and 64.

66. Determine the critical fishing level R_c such that there are two equilibrium points if and only if $R < R_c$.

67. Solve the population model

$$\begin{aligned} P'(t) &= 0.05P(t)[8 - P(t)] - 0.6 \\ &= 0.4P(t)[1 - P(t)/8] - 0.6 \end{aligned}$$

with $P(0) > 2$ and determine the limiting amount $\lim_{t\to\infty} P(t)$. What happens if $P(0) < 2$?

68. The constant 0.4 in exercise 67 represents the natural growth rate of the species. Comparing answers to exercises 63 and 67, discuss how this constant affects the population size.

69. The resale value $r(t)$ of a machine decreases at a rate proportional to the difference between the current price and the scrap value S. Write a differential equation for r. If the machine sells new for \$14,000, is worth \$8000 in 4 years and has a scrap value of \$1000, find an equation for the resale value at any time.

70. A granary is filled with 6000 kg of grain. The grain is shipped out at a constant rate of 1000 kg per month. Storage costs equal 2 cents per kg per month. Let $S(t)$ be the total storage charge for t months. Write a differential equation for S with $0 \le 1 \le 6$. Solve the initial value problem for $S(t)$. What is the total storage bill for six months?

71. The population models $P'(t) = kP(t)$ and $P'(t) = k[P(t)]^{1.1}$ look very similar. The first is called exponential growth and is studied in detail in section 7.1. The second is sometimes called a **doomsday model.** Solve the general doomsday equation. Assuming that $P(0)$ and k are positive, find the time at which the population becomes infinite.

72. Suppose that the thrust of a boat's propeller produces a constant acceleration, but that friction with water produces a deceleration that is proportional to the square of the speed of the boat. Write a differential equation for the speed v of the boat. Find equilibrium points and use a slope diagram to determine the eventual speed of the boat.

73. For the logistic equation $y'(t) = ky(M - y)$, show that a graph of $\frac{1}{y}y'$ as a function of y produces a linear graph. Given the slope m and intercept b of this line, explain how to compute the model parameters k and M.

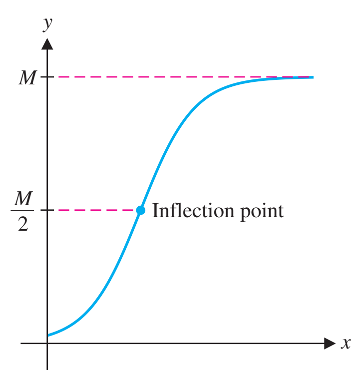
74. You suspect that a fish population follows a logistic equation. Use the following data to estimate k and M, as in exercise 73. Predict the eventual population of the fish.

t	2	3	4	5
y	1197	1291	1380	1462

75. The downward velocity of a falling object is modeled by the differential equation $\frac{dv}{dt} = 32 - 0.4v^2$. If $v(0) = 0$ ft/s, the velocity will increase to a **terminal velocity.** The terminal velocity is an equilibrium solution where the upward air drag exactly cancels the downward gravitational force. Find the terminal velocity.

76. Suppose that f is a function such that $f(x) \ge 0$ and $f'(x) < 0$ for $x > 0$. Show that the area of the triangle with sides $x = 0$, $y = 0$ and the tangent line to $y = f(x)$ at $x = a > 0$ is $A(a) = -\frac{1}{2}\{a^2 f'(a) - 2af(a) + [f(a)]^2/f'(a)\}$. To find a curve such that this area is the same for any choice of $a > 0$, solve the equation $\frac{dA}{da} = 0$.

77. It is an interesting fact that the inflection point in the solution of a logistic equation (see figure) occurs at $y = \frac{1}{2}M$. To verify this, you do not want to compute two derivatives of equation (2.6) and solve $y'' = 0$. This would be quite ugly and would give you the solution in terms of t, instead of y. Instead, a more abstract approach works well. Start with the differential equation $y' = ky(M - y)$ and take derivatives of both sides.



(Hint: Use the product and chain rules on the right-hand side.) You should find that $y'' = ky'(M - 2y)$. Then, $y'' = 0$ if and only if $y' = 0$ or $y = \frac{1}{2}M$. Rule out $y' = 0$ by describing how the solution behaves at the equilibrium values.

78. The differential equation $y' = -ay\ln(y/b)$ (for positive constants a and b) arises in the study of the growth of some animal tumors. Solve the differential equation and sketch several members of the family of solutions. What adjective (e.g., rapid, moderate, slow) would you use to characterize this type of growth?

79. Use the technique of exercise 77 to show that solutions of $y' = -ay\ln(y/b)$ for positive constants a and b have at most one inflection point, which occurs at $y = b/a$.

EXPLORATORY EXERCISES

1. Look up the census figures for the U.S. population starting in 1790. (You can find this information in any library, in virtually any almanac or encyclopedia.) Plot the data on a graph of population versus time. Does this look like the solution of a logistic equation? Briefly explain. If you wanted to model these data with a logistic function, you would need to estimate values for k and M. As shown in exercise 77, M equals twice the height of the inflection point. Explain why (for the logistic curve) the inflection point represents the point of maximum slope. Estimate this for the population data. To estimate r, note that for small populations, the logistic equation $y' = ry(1 - y/k) \approx ry$. Then r equals the rate of exponential increase. Show that for the first 50 years, the U.S. population growth was approximately exponential and find the percentage increase as an estimate of r. With these values of r and M and the initial population in 1790, find a function describing the population. Test this model by comparing actual populations to predicted populations for several years.

2. An object traveling through the air is acted on by gravity (acting vertically), air resistance (acting in the direction opposite velocity) and other forces (such as a motor). An equation for the horizontal motion of a jet plane is $v' = c - f(v)/m$, where c is the thrust of the motor and $f(v)$ is the air resistance force. For some ranges of velocity, the air resistance actually *drops* substantially for higher velocities as the air around the object becomes turbulent. For example, suppose that $v' = 32{,}000 - f(v)$, where $f(v) = \begin{cases} 0.8v^2 & \text{if } 0 \le v \le 100 \\ 0.2v^2 & \text{if } 100 < v \end{cases}$. To solve the initial value problem $v' = 32{,}000 - f(v)$, $v(0) = 0$, start with the initial value problem $v' = 32{,}000 - 0.8v^2$, $v(0) = 0$. Solve this IVP $\left[\text{Hint: } \dfrac{1}{40{,}000 - v^2} = \dfrac{1}{400}\left(\dfrac{1}{200 + v} + \dfrac{1}{200 - v}\right)\right]$ and determine the time t such that $v(t) = 100$. From this time forward, the equation becomes $v' = 32{,}000 - 0.2v^2$. Solve the IVP $v' = 32{,}000 - 0.2v^2$, $v(0) = 100$. Put this solution together with the previous solution to piece together a solution valid for all time.

3. Solve the initial value problems $\frac{dy}{dt} = 2(1 - y)(2 - y)(3 - y)$ with (a) $y(0) = 0$, (b) $y(0) = 1.5$, (c) $y(0) = 2.5$ and (d) $y(0) = 4$. State as completely as possible how the limit $\lim_{t\to\infty} y(t)$ depends on $y(0)$.

7.3 DIRECTION FIELDS AND EULER'S METHOD

In section 7.2, we saw how to solve some simple first-order differential equations, namely, those that are separable. While there are numerous other special cases of differential equations whose solutions are known (you will encounter many of these in any beginning course in differential equations), the vast majority cannot be solved exactly. For instance, the equation

$$y' = x^2 + y^2 + 1$$

is not separable and cannot be solved using our current techniques. Nevertheless, *some* information about the solution(s) can be determined. In particular, since $y' = x^2 + y^2 + 1 > 0$, we can conclude that every solution is an increasing function. This type of information is called *qualitative,* since it tells us about some quality of the solution without providing any specific quantitative information.

In this section, we examine first-order differential equations in a more general setting. We consider any first-order equation of the form

$$y' = f(x, y). \tag{3.1}$$

While we cannot solve all such equations, it turns out that there are many numerical methods available for approximating the solution of such problems. In this section, we will study one such method, called Euler's method.

HISTORICAL NOTES

Leonhard Euler (1707–1783) A Swiss mathematician regarded as the most prolific mathematician of all time. Euler's complete works fill over 100 large volumes, with much of his work being completed in the last 17 years of his life after losing his eyesight. Euler made important and lasting contributions in numerous research fields, including calculus, number theory, calculus of variations, complex variables, graph theory and differential geometry. Mathematics author George Simmons calls Euler, "the Shakespeare of mathematics—universal, richly detailed and inexhaustible."

We begin by observing that any solution of equation (3.1) is a function $y = y(x)$ whose slope at any particular point (x, y) is given by $f(x, y)$. To get an idea of what a solution curve looks like, we draw a short line segment through each of a sequence of points (x, y), with slope $f(x, y)$, respectively. This collection of line segments is called the **direction field** or **slope field** of the differential equation. Notice that if a particular solution curve passes through a given point (x, y), then its slope at that point is $f(x, y)$. Thus, the direction field gives an indication of the behavior of the family of solutions of a differential equation.

EXAMPLE 3.1 Constructing a Direction Field

Construct the direction field for

$$y' = \frac{1}{2}y. \tag{3.2}$$

Solution All that needs to be done is to plot a number of points and then through each point (x, y), draw a short line segment with slope $f(x, y)$. For example, at the point $(0, 1)$, draw a short line segment with slope

$$y'(0) = f(0, 1) = \tfrac{1}{2}(1) = \tfrac{1}{2}.$$

Draw similar segments at 25 to 30 points. This is a bit tedious to do by hand, but a good graphing utility can do this for you with minimal effort. See Figure 7.11a for the direction field for equation (3.2). Notice that equation (3.2) is separable. We leave it as an exercise to produce the general solution

$$y = Ae^{\frac{1}{2}x}.$$

We plot a number of the curves in this family of solutions in Figure 7.11b using the same graphing window we used for Figure 7.11a. Notice that if you connected some of the line segments in Figure 7.11a, you would obtain a close approximation to the exponential curves depicted in Figure 7.11b. This is significant because the direction field was constructed using only elementary algebra, *without* first solving the differential equation. That is, by constructing the direction field, we obtain a reasonably good picture of how the solution curves behave. This is qualitative information about the solution: we get a graphical idea of how solutions behave, but no details, such as the value of a solution at a specific point. We'll see later in this section that we can obtain approximate values of the solution of an IVP numerically.

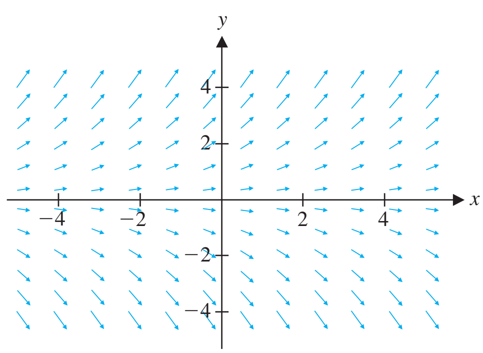


FIGURE 7.11a
Direction field for $y' = \frac{1}{2}y$

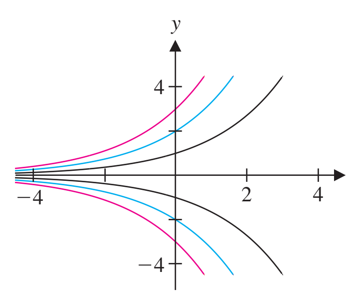


FIGURE 7.11b
Several solutions of $y' = \frac{1}{2}y$

As we have already seen, differential equations are used to describe a wide variety of phenomena in science and engineering. Among many other applications, differential equations are used to find flow lines or equipotential lines for electromagnetic fields. In such cases, it is very helpful to visualize solutions graphically, so as to gain an intuitive understanding of the behavior of such solutions and the physical phenomena they are modeling.

EXAMPLE 3.2 Using a Direction Field to Visualize the Behavior of Solutions

Construct the direction field for

$$y' = x + e^{-y}.$$

Solution There's really no trick to this; just draw a number of line segments with the correct slope. Again, we let our CAS do this for us and obtained the direction field in Figure 7.12a. Unlike example 3.1, you do not know how to solve this differential equation exactly. Even so, you should be able to clearly see from the direction field how solutions behave. For example, solutions that start out in the second quadrant initially decrease very rapidly, may dip into the third quadrant and then get pulled into the first quadrant and increase quite rapidly toward infinity. This is quite a bit of information to have determined using little more than elementary algebra. In Figure 7.12b, we have plotted the solution of the differential equation that also satisfies the initial condition $y(-4) = 2$. We'll see how to generate such an approximate solution later in this section. Note how well this corresponds with what you get by connecting a few of the line segments in Figure 7.12a.

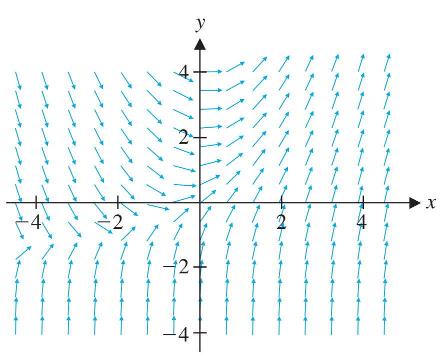


FIGURE 7.12a
Direction field for $y' = x + e^{-y}$

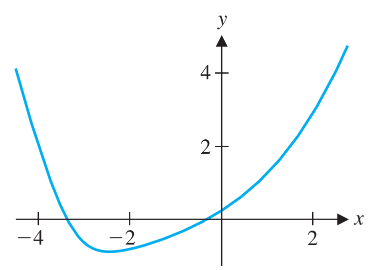


FIGURE 7.12b
Solution of $y' = x + e^{-y}$ passing through $(-4, 2)$

You have already seen (in sections 7.1 and 7.2) how differential equation models can provide important information about how populations change over time. A model that includes a **critical threshold** is

$$P'(t) = -2[1 - P(t)][2 - P(t)]P(t),$$

where $P(t)$ represents the size of a population at time t.

A simple context in which to understand a critical threshold is with the problem of the sudden infestations of pests. For instance, suppose that you have some method for removing ants from your home. As long as the reproductive rate of the ants is lower than your removal rate, you will keep the ant population under control. However, as soon as your removal rate becomes less than the ant reproductive rate (i.e., the removal rate crosses a critical threshold), you won't be able to keep up with the extra ants and you will suddenly be faced with a big ant problem. We see this type of behavior in example 3.3.

EXAMPLE 3.3 Population Growth with a Critical Threshold

Draw the direction field for

$$P'(t) = -2[1 - P(t)][2 - P(t)]P(t)$$

and discuss the eventual size of the population.

Solution The direction field is particularly easy to sketch here since the right-hand side depends on P, but not on t. If $P(t) = 0$, then $P'(t) = 0$, also, so that the direction field is horizontal. The same is true for $P(t) = 1$ and $P(t) = 2$. If $0 < P(t) < 1$, then $P'(t) < 0$ and the solution decreases. If $1 < P(t) < 2$, then $P'(t) > 0$ and the solution increases. Finally, if $P(t) > 2$, then $P'(t) < 0$ and the solution decreases. Putting all of these pieces together, we get the direction field seen in Figure 7.13. The constant solutions $P(t) = 0$, $P(t) = 1$ and $P(t) = 2$ are called **equilibrium solutions.** $P(t) = 1$ is called an **unstable equilibrium,** since populations that start near 1 don't remain close to 1. $P(t) = 0$ and $P(t) = 2$ are called **stable equilibria,** since populations either rise to 2 or drop to 0 (extinction), depending on which side of the critical threshold $P(t) = 1$ they are on. (Look again at Figure 7.13.)

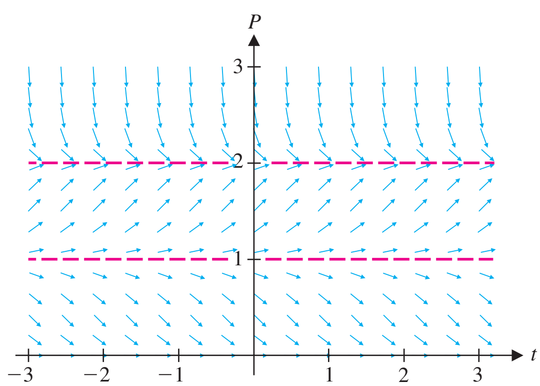


FIGURE 7.13
Direction field for $P'(t) = -2[1 - P(t)][2 - P(t)]P(t)$

In cases where you are interested in finding a particular solution, the numerous arrows of a direction field can be distracting. Euler's method, developed below, enables you to approximate a single solution curve. The method is quite simple, based almost entirely on the idea of a direction field. However, Euler's method does not provide particularly accurate approximations. More accurate methods will be explored in the exercises.

Consider the IVP

$$y' = f(x, y), \quad y(x_0) = y_0.$$

We must emphasize once again that, assuming there is a solution $y = y(x)$, the differential equation tells us that the slope of the tangent line to the solution curve at any point (x, y) is

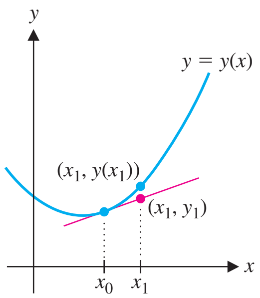


FIGURE 7.14
Tangent line approximation

given by $f(x, y)$. Remember that the tangent line to a curve stays close to that curve near the point of tangency. Notice that we already know one point on the graph of $y = y(x)$, namely, the initial point (x_0, y_0). Referring to Figure 7.14, if we would like to approximate the value of the solution at $x = x_1$ [i.e., $y(x_1)$] and if x_1 is not too far from x_0, then we could follow the tangent line at (x_0, y_0) to the point corresponding to $x = x_1$ and use the y-value at that point (call it y_1) as an approximation to $y(x_1)$. This is virtually the same thinking we employed when we devised Newton's method and differential (tangent line) approximations. The equation of the tangent line at $x = x_0$ is

$$y = y_0 + y'(x_0)(x - x_0).$$

Thus, an approximation to the value of the solution at $x = x_1$ is the y-coordinate of the point on the tangent line corresponding to $x = x_1$, that is,

$$y(x_1) \approx y_1 = y_0 + y'(x_0)(x_1 - x_0). \tag{3.3}$$

You have only to glance at Figure 7.14 to realize that this approximation is valid only when x_1 is close to x_0. In solving an IVP, though, we are usually interested in finding the value of the solution on an interval $[a, b]$ of the x-axis. With Euler's method, we settle for finding an approximate solution at a sequence of points in the interval $[a, b]$. First, we partition the interval $[a, b]$ into n equal-sized pieces (a *regular partition;* where did you see this notion before?):

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where

$$x_{i+1} - x_i = h,$$

for all $i = 0, 1, \ldots, n - 1$. We call h the **step size.** From the tangent line approximation (3.3), we already have

$$\begin{aligned} y(x_1) \approx y_1 &= y_0 + y'(x_0)(x_1 - x_0) \\ &= y_0 + hf(x_0, y_0), \end{aligned}$$

where we have replaced $(x_1 - x_0)$ by the step size, h and used the differential equation to write $y'(x_0) = f(x_0, y_0)$. To approximate the value of $y(x_2)$, we could use the tangent line at the point $(x_1, y(x_1))$ to produce a tangent line approximation, but we don't know the y-coordinate of the point of tangency, $y(x_1)$. We do, however, have an approximation for this, produced in the preceding step. So, we make the further approximation

$$\begin{aligned} y(x_2) &\approx y(x_1) + y'(x_1)(x_2 - x_1) \\ &= y(x_1) + hf(x_1, y(x_1)), \end{aligned}$$

where we have used the differential equation to replace $y'(x_1)$ by $f(x_1, y(x_1))$ and used the fact that $x_2 - x_1 = h$. Finally, we approximate $y(x_1)$ by the approximation obtained in the previous step, y_1, to obtain

$$\begin{aligned} y(x_2) &\approx y(x_1) + hf(x_1, y(x_1)) \\ &\approx y_1 + hf(x_1, y_1) = y_2. \end{aligned}$$

Continuing in this way, we obtain the sequence of approximate values

EULER'S METHOD

$$y(x_{i+1}) \approx y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for } i = 0, 1, 2 \ldots. \tag{3.4}$$

This tangent line method of approximation is called **Euler's method.**

EXAMPLE 3.4 Using Euler's Method

Use Euler's method to approximate the solution of the IVP

$$y' = y, \quad y(0) = 1.$$

Solution You can probably solve this equation by inspection, but if not, notice that it's separable and that the solution of the IVP is $y = y(x) = e^x$. We will use this exact solution to evaluate the performance of Euler's method. From (3.4) with $f(x, y) = y$ and taking $h = 1$, we have

$$\begin{aligned} y(x_1) \approx y_1 &= y_0 + hf(x_0, y_0) \\ &= y_0 + hy_0 = 1 + 1(1) = 2. \end{aligned}$$

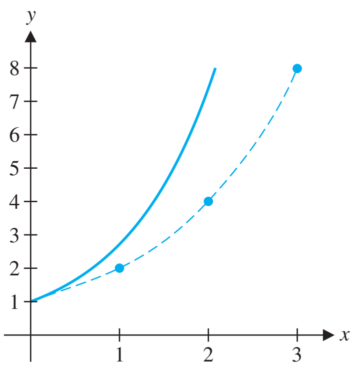


FIGURE 7.15
Exact solution versus the approximate solution (dashed line)

Likewise, for further approximations, we have

$$\begin{aligned} y(x_2) \approx y_2 &= y_1 + hf(x_1, y_1) \\ &= y_1 + hy_1 = 2 + 1(2) = 4, \\ y(x_3) \approx y_3 &= y_2 + hf(x_2, y_2) \\ &= y_2 + hy_2 \\ &= 4 + 1(4) = 8 \end{aligned}$$

and so on. In this way, we construct a sequence of approximate values of the solution function. In Figure 7.15, we have plotted the exact solution (solid line) against the approximate solution obtained from Euler's method (dashed line). Notice how the error grows as x gets farther and farther from the initial point. This is characteristic of Euler's method (and other similar methods). This growth in error becomes even more apparent if we look at a table of values of the approximate and exact solutions together. We display these in the table that follows, where we have used $h = 0.1$ (values are displayed to seven digits).

x	*Euler*	*Exact*	*Error = Exact − Euler*
0.1	1.1	1.1051709	0.0051709
0.2	1.21	1.2214028	0.0114028
0.3	1.331	1.3498588	0.0188588
0.4	1.4641	1.4918247	0.0277247
0.5	1.61051	1.6487213	0.0382113
0.6	1.771561	1.8221188	0.0505578
0.7	1.9487171	2.0137527	0.0650356
0.8	2.1435888	2.2255409	0.0819521
0.9	2.3579477	2.4596031	0.1016554
1.0	2.5937425	2.7182818	0.1245393

As you might expect from our development of Euler's method, the smaller we make h, the more accurate the approximation at a given point tends to be. As well, the smaller the value of h, the more steps it takes to reach a given value of x. In the following table, we display the Euler's method approximation, the error and the number of steps needed to reach $x = 1.0$. Here, the exact value of the solution is $y = e^1 \approx 2.718281828459$.

h	*Euler*	*Error = Exact − Euler*	*Number of Steps*
1.0	2	0.7182818	1
0.5	2.25	0.4682818	2
0.25	2.4414063	0.2768756	4
0.125	2.5657845	0.1524973	8
0.0625	2.6379285	0.0803533	16
0.03125	2.6769901	0.0412917	32
0.015625	2.697345	0.0209369	64
0.0078125	2.707739	0.0105428	128
0.00390625	2.7129916	0.0052902	256

From the table, observe that each time the step size h is cut in half, the error is also cut roughly in half. This increased accuracy though, comes at the cost of the additional steps of Euler's methods required to reach a given point (doubled each time h is halved). ■

The point of having a numerical method, of course, is to find meaningful approximations to the solution of problems that we do not know how to solve exactly. Example 3.5 is of this type.

TODAY IN MATHEMATICS

Kay McNulty (1921–)
An Irish mathematician who became one of the first computer software designers. In World War II, before computers, she approximated solutions of projectile differential equations. McNulty says, "We did have desk calculators at that time, mechanical and driven with electric motors, that could do simple arithmetic. You'd do a multiplication and when the answer appeared, you had to write it down to reenter it into the machine to do the next calculation. We were preparing a firing table for each gun, with maybe 1800 simple trajectories. To hand-compute just one of these trajectories took 30 or 40 hours of sitting at a desk with paper and a calculator. . . . Actually, my title working for the ballistics project was 'computer'. . . . ENIAC made me, one of the first 'computers,' obsolete."

EXAMPLE 3.5 Finding an Approximate Solution

Find an approximate solution for the IVP

$$y' = x^2 + y^2,\ y(-1) = -\frac{1}{2}.$$

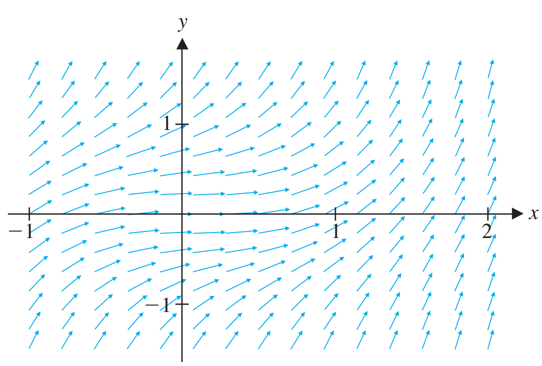


FIGURE 7.16
Direction field for $y' = x^2 + y^2$

Solution First let's take a look at the direction field, so that we can see how solutions to this differential equation should behave (see Figure 7.16). Using Euler's method with $h = 0.1$, we get

$$\begin{aligned} y(x_1) \approx y_1 &= y_0 + hf(x_0, y_0) \\ &= y_0 + h\left(x_0^2 + y_0^2\right) \\ &= -\frac{1}{2} + 0.1\left[(-1)^2 + \left(-\frac{1}{2}\right)^2\right] = -0.375 \end{aligned}$$

and
$$\begin{aligned} y(x_2) \approx y_2 &= y_1 + hf(x_1, y_1) \\ &= y_1 + h\left(x_1^2 + y_1^2\right) \\ &= -0.375 + 0.1[(-0.9)^2 + (-0.375)^2] = -0.2799375 \end{aligned}$$

and so on. Continuing in this way, we generate the table of values that follows.

x	*Euler*	*x*	*Euler*	*x*	*Euler*
−0.9	−0.375	0.1	−0.0575822	1.1	0.3369751
−0.8	−0.2799375	0.2	−0.0562506	1.2	0.4693303
−0.7	−0.208101	0.3	−0.0519342	1.3	0.6353574
−0.6	−0.1547704	0.4	−0.0426645	1.4	0.8447253
−0.5	−0.116375	0.5	−0.0264825	1.5	1.1120813
−0.4	−0.0900207	0.6	−0.0014123	1.6	1.4607538
−0.3	−0.0732103	0.7	0.0345879	1.7	1.9301340
−0.2	−0.0636743	0.8	0.0837075	1.8	2.5916757
−0.1	−0.0592689	0.9	0.1484082	1.9	3.587354
0.0	−0.0579176	1.0	0.2316107	2.0	5.235265

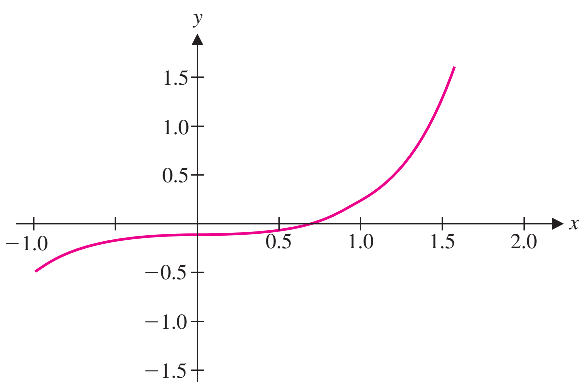


FIGURE 7.17a
Approximate solution of $y' = x^2 + y^2$, passing through $(-1, -\frac{1}{2})$

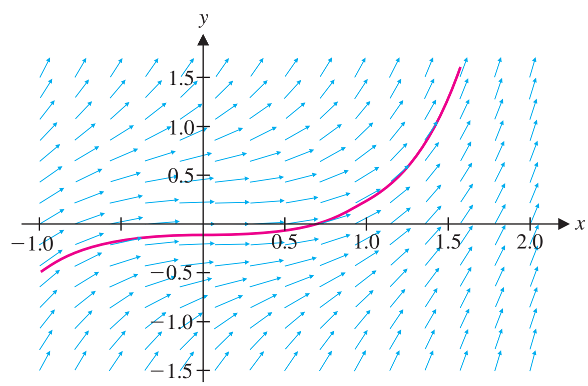


FIGURE 7.17b
Approximate solution superimposed on the direction field

In Figure 7.17a, we display a smooth curve connecting the data points in the preceding table. Take particular note of how well this corresponds with the direction field in Figure 7.16. To make this correspondence more apparent, we show a graph of the approximate solution superimposed on the direction field in Figure 7.17b. Since this corresponds so well with the behavior you expect from the direction field, you should expect that there are no gross errors in this approximate solution. (Certainly, there is always some level of round-off and other numerical errors.) ■

We can expand on the concept of equilibrium solution, which we introduced briefly in example 3.3. More generally, we say that the constant function $y = c$ is an **equilibrium solution** of the differential equation $y' = f(t, y)$ if $f(t, c) = 0$ for all t. In simple terms, this says that $y = c$ is an equilibrium solution of the differential equation $y' = f(t, y)$ if the substitution $y = c$ reduces the equation to simply $y' = 0$. Observe that this, in turn, says that $y(t) = c$ is a (constant) solution of the differential equation. In example 3.6, notice that finding equilibrium solutions requires only basic algebra.

EXAMPLE 3.6 Finding Equilibrium Solutions

Find all equilibrium solutions of (a) $y'(t) = k[y(t) - 70]$ and (b) $y'(t) = 2y(t)[4 - y(t)]$.

Solution An equilibrium solution is a constant solution that reduces the equation to $y'(t) = 0$. For part (a), this gives us

$$0 = y'(t) = k[y(t) - 70] \quad \text{or} \quad 0 = y(t) - 70.$$

The only equilibrium solution is then $y = 70$. For part (b), we want

$$0 = 2y(t)[4 - y(t)] \quad \text{or} \quad 0 = y(t)[4 - y(t)].$$

So, in this case, there are two equilibrium solutions: $y = 0$ and $y = 4$. ■

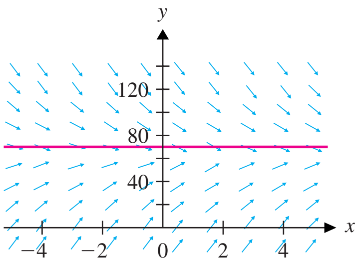


FIGURE 7.18a
Direction field

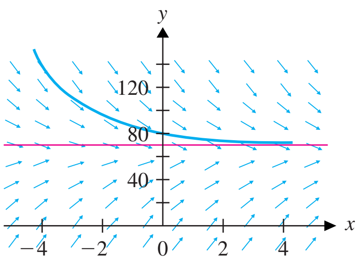


FIGURE 7.18b
Solution curve starting above $y = 70$

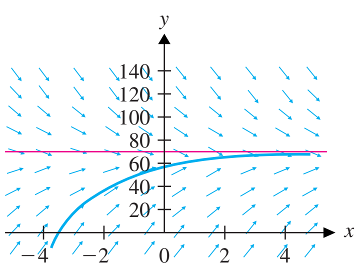


FIGURE 7.18c
Solution curve starting below $y = 70$

There is some special significance to an equilibrium solution, which we describe from a sketch of the direction field. Start with the differential equation $y'(t) = k[y(t) - 70]$ for some negative constant k. Notice that if $y(t) > 70$, then $y'(t) = k[y(t) - 70] < 0$ (since k is negative). Of course, $y'(t) < 0$ means that the solution is decreasing. Similarly, when $y(t) < 70$, we have that $y'(t) = k[y(t) - 70] > 0$, so that the solution is increasing. Observe that the direction field sketched in Figure 7.18a suggests that $y(t) \to 70$ as $t \to \infty$, since all arrows point toward the line $y = 70$. More precisely, if a solution curve lies slightly above the line $y = 70$, notice that the solution decreases, toward $y = 70$, as indicated in Figure 7.18b. Similarly, if the solution curve lies slightly below $y = 70$, then the solution increases toward $y = 70$, as shown in Figure 7.18c. You should observe that we obtained this information without solving the differential equation.

We say that an equilibrium solution is **stable** if solutions close to the equilibrium solution tend to approach that solution as $t \to \infty$. Observe that this is the behavior indicated in Figures 7.18a to 7.18c, so that the solution $y = 70$ is stable. Alternatively, an equilibrium solution is **unstable** if solutions close to the equilibrium solution tend to get further away from that solution as $t \to \infty$.

In example 3.6, part (b), we found that $y'(t) = 2y(t)[4 - y(t)]$ has the two equilibrium solutions $y = 0$ and $y = 4$. We now use a direction field to determine whether these solutions are stable or unstable.

EXAMPLE 3.7 Determining the Stability of Equilibrium Solutions

Draw a direction field for $y'(t) = 2y(t)[4 - y(t)]$ and determine the stability of all equilibrium solutions.

Solution We previously determined that the equilibrium solutions are $y = 0$ and $y = 4$.

We add the horizontal lines $y = 0$ and $y = 4$ to the direction field, as shown in Figure 7.19.

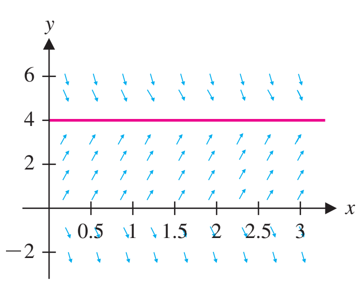


FIGURE 7.19
Direction field for $y' = 2y(4 - y)$

Observe that the behavior is distinctly different in each of three regions in this diagram: $y > 4$, $0 < y < 4$ and $y < 0$. We analyze each separately. First, observe that if $y(t) > 4$, then $y'(t) = 2y(t)[4 - y(t)] < 0$ (since $2y$ is positive, but $4 - y$ is negative). Next, if $0 < y(t) < 4$, then $y'(t) = 2y(t)[4 - y(t)] > 0$ (since $2y$ and $4-y$ are both positive in this case). Finally, if $y(t) < 0$, then $y'(t) = 2y(t)[4 - y(t)] < 0$. In Figure 7.19, the arrows on either side of the line $y = 4$ all point toward $y = 4$. This indicates that $y = 4$ is stable. By contrast, the arrows on either side of $y = 0$ point away from $y = 0$, indicating that $y = 0$ is unstable. ■

Notice that the direction field in example 3.7 gives strong evidence that if $y(0) > 0$, then the limiting value is $\lim_{t\to\infty} y(t) = 4$. [Think about why the condition $y(0) > 0$ is needed here.]

BEYOND FORMULAS

Numerical approximations of solutions of differential equations are basic tools of the trade for modern engineers and scientists. Euler's method, presented in this section, is one of the least accurate methods in use today, but its simplicity makes it useful in a variety of applications. Most differential equations cannot be solved exactly. For such cases, we need reliable numerical methods to obtain approximate values of the solution. What other types of calculations have you seen that typically must be approximated?

EXERCISES 7.3

WRITING EXERCISES

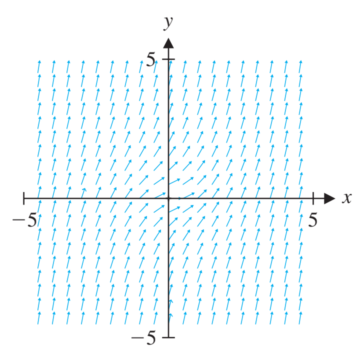
1. For Euler's method, explain why using a smaller step size should produce a better approximation.
2. Look back at the direction field in Figure 7.16 and the Euler's method solution in Figure 7.17a. Describe how the direction field gives you a more accurate sense of the exact solution. Given this, explain why Euler's method is important. (Hint: How would you get a table of approximate values of the solution from a direction field?)
3. In the situation of example 3.3, if you only needed to know the stability of an equilibrium solution, explain why a qualitative method is preferred over trying to solve the differential equation. Describe one situation in which you would need to solve the equation.
4. Imagine superimposing solution curves over Figure 7.12a. Explain why the Euler's method approximation takes you from one solution curve to a nearby one. Use one of the examples in this section to describe how such a small error could lead to very large errors in approximations for large values of x.

In exercises 1–6, construct four of the direction field arrows by hand and use your CAS or calculator to do the rest. Describe the general pattern of solutions.

1. $y' = x + 4y$
2. $y' = \sqrt{x^2 + y^2}$
3. $y' = 2y - y^2$
4. $y' = y^3 - 1$
5. $y' = 2xy - y^2$
6. $y' = y^3 - x$

In exercises 7–12, match each differential equation to the correct direction field.

7. $y' = 2 - xy$
8. $y' = 1 + 2xy$
9. $y' = x\cos 3y$
10. $y' = y\cos 3x$
11. $y' = \sqrt{x^2 + y^2}$
12. $y' = \ln(x^2 + y^2)$



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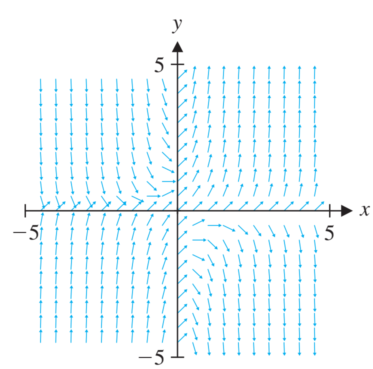

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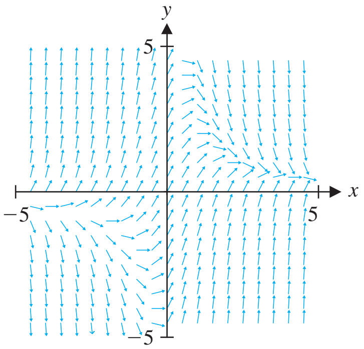

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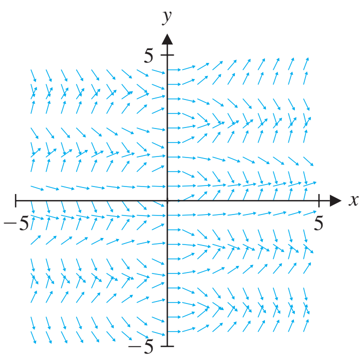

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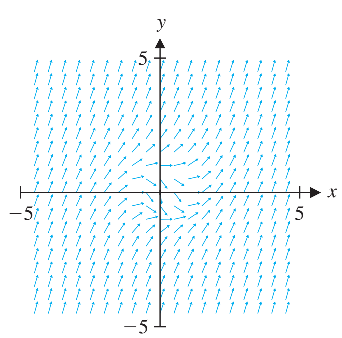

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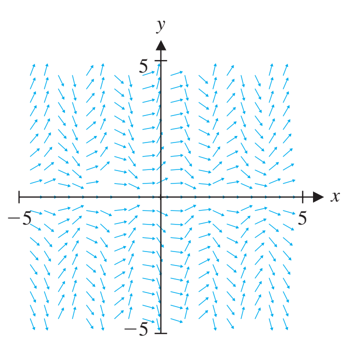


FIELD F

In exercises 13–20, use Euler's method with $h = 0.1$ and $h = 0.05$ to approximate $y(1)$ and $y(2)$. Show the first two steps by hand.

13. $y' = 2xy$, $y(0) = 1$

14. $y' = x/y$, $y(0) = 2$

15. $y' = 4y - y^2$, $y(0) = 1$

16. $y' = x/y^2$, $y(0) = 2$

17. $y' = 1 - y + e^{-x}$, $y(0) = 3$

18. $y' = \sin y - x^2$, $y(0) = 1$

19. $y' = \sqrt{x + y}$, $y(0) = 1$

20. $y' = \sqrt{x^2 + y^2}$, $y(0) = 4$

21. Find the exact solutions in exercises 13 and 14, and compare $y(1)$ and $y(2)$ to the Euler's method approximations.

22. Find the exact solutions in exercises 15 and 16, and compare $y(1)$ and $y(2)$ to the Euler's method approximations.

23. Sketch the direction fields for exercises 17 and 18, highlight the curve corresponding to the given initial condition and compare the Euler's method approximations to the location of the curve at $x = 1$ and $x = 2$.

24. Sketch the direction fields for exercises 19 and 20, highlight the curve corresponding to the given initial condition and compare the Euler's method approximations to the location of the curve at $x = 1$ and $x = 2$.

In exercises 25–30, find the equilibrium solutions and determine which are stable and which are unstable.

25. $y' = 2y - y^2$

26. $y' = y^3 - 1$

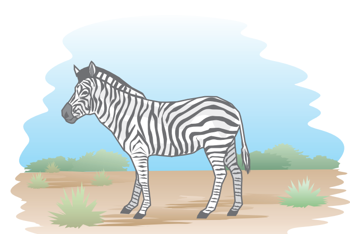
27. $y' = y^2 - y^4$

28. $y' = e^{-y} - 1$

29. $y' = (1 - y)\sqrt{1 + y^2}$

30. $y' = \sqrt{1 - y^2}$

31. Zebra stripes and patterns on butterfly wings are thought to be the result of gene-activated chemical processes. Suppose $g(t)$ is the amount of gene that is activated at time t. The differential equation $g' = -g + \dfrac{3g^2}{1 + g^2}$ has been used to model the process. Show that there are three equilibrium solutions: 0 and two positive solutions a and b, with $a < b$. Show that $g' > 0$ if $a < g < b$ and $g' < 0$ if $0 < g < a$ or $g > b$. Explain why $\lim_{t \to \infty} g(t)$ could be 0 or b, depending on the initial amount of activated gene. Suppose that a patch of zebra skin extends from $x = 0$ to $x = 4\pi$ with an initial activated-gene distribution $g(0) = \frac{3}{2} + \frac{3}{2}\sin x$ at location x. If black corresponds to an eventual activated-gene level of 0 and white corresponds to an eventual activated-gene level of b, show what the zebra stripes will look like.



32. Many species of trees are plagued by sudden infestations of worms. Let $x(t)$ be the population of a species of worm on a particular tree. For some species, a model for population change is $x' = 0.1x(1 - x/k) - x^2/(1 + x^2)$ for some positive constant k. If $k = 10$, show that there is only one positive equilibrium solution. If $k = 50$, show that there are three positive equilibrium solutions. Sketch the direction field for $k = 50$. Explain why the middle equilibrium value is called a *threshold*. An outbreak of worms corresponds to crossing the threshold for a large value of k (k is determined by the resources available to the worms).

33. Apply Euler's method with $h = 0.1$ to the initial value problem $y' = y^2 - 1$, $y(0) = 3$ and estimate $y(0.5)$. Repeat with $h = 0.05$ and $h = 0.01$. In general, Euler's method is more accurate with smaller h-values. Conjecture how the exact solution behaves in this example. (This is explored further in exercises 34–36.)

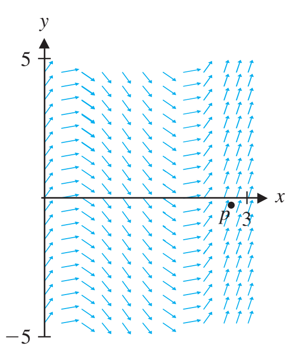
34. Show that $f(x) = \dfrac{2 + e^{2x}}{2 - e^{2x}}$ is a solution of the initial value problem in exercise 33. Compute $f(0.1)$, $f(0.2)$, $f(0.3)$, $f(0.4)$ and $f(0.5)$, and compare to the approximations in exercise 33.

35. Graph the solution of $y' = y^2 - 1$, $y(0) = 3$, given in exercise 34. Find an equation of the vertical asymptote. Explain why Euler's method would be "unaware" of the existence of this asymptote and would therefore provide very unreliable approximations.

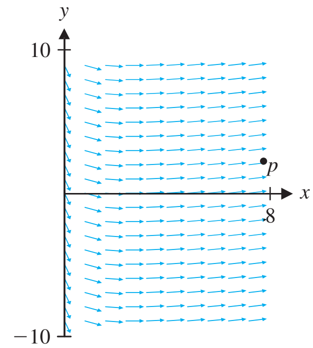
36. In exercises 33–35, suppose that x represents time (in hours) and y represents the force (in newtons) exerted on an arm of a robot. Explain what happens to the arm. Given this, explain why the negative function values in exercise 34 are irrelevant and, in some sense, the Euler's method approximations in exercise 33 give useful information.

In exercises 37–40, use the direction field to sketch solution curves and estimate the initial value $y(0)$ such that the solution curve would pass through the given point P. In exercises 37 and 38, solve the equation and determine how accurate your estimate is. In exercises 39 and 40, use a CAS if available to determine how accurate your estimate is.

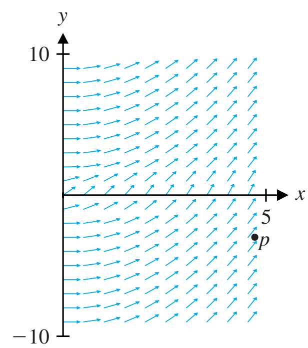
37. $y' = x^2 - 4x + 2$, $P(3, 0)$



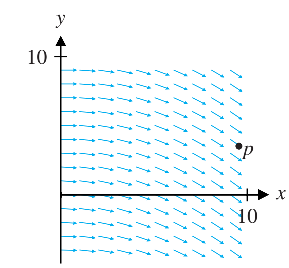
38. $y' = \dfrac{x - 3}{4x + 1}$, $P(8, 1)$



39. $y' = 0.2x + e^{-y^2}$, $P(5, -3)$



40. $y' = -0.1x - 0.1e^{-y^2/20}$, $P(10, 4)$



EXPLORATORY EXERCISES

1. There are several ways of deriving the Euler's method formula. One benefit of having an alternative derivation is that it may suggest an improvement of the method. Here, we use an alternative form of Euler's method to derive a method known as the **Improved Euler's method.** Start with the differential equation $y'(x) = f(x, y(x))$ and integrate both sides from $x = x_n$ to $x = x_{n+1}$. Show that $y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$. Given $y(x_n)$, then, you can estimate $y(x_{n+1})$ by estimating the integral $\int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$. One such estimate is a Riemann sum using left-endpoint evaluation, given by $f(x_n, y(x_n))\Delta x$. Show that with this estimate you get Euler's method. There are numerous ways of getting better estimates of the integral. One is to use the Trapezoidal Rule, $\displaystyle\int_{x_n}^{x_{n+1}} f(x, y(x))\,dx \approx \frac{f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))}{2}\Delta x.$ The drawback with this estimate is that you know $y(x_n)$ but you do *not* know $y(x_{n+1})$. Briefly explain why this statement is correct. The way out is to use Euler's method; you do not know $y(x_{n+1})$ but you can approximate it by $y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n))$. Put all of this together to get the Improved Euler's method:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))].$$

Use the Improved Euler's method for the IVP $y' = y$, $y(0) = 1$ with $h = 0.1$ to compute y_1, y_2 and y_3. Compare to the exact values and the Euler's method approximations given in example 3.4.

2. As in exercise 1, derive a numerical approximation method based on (a) the Midpoint rule and (b) Simpson's rule. Compare your results to those obtained in example 3.4 and exercise 1.

3. In sections 7.1 and 7.2, you explored some differential equation models of population growth. An obvious flaw in those models was the consideration of a single species in isolation. In this exercise, you will investigate a **predator-prey** model. In this case, there are two species, X and Y, with populations $x(t)$ and $y(t)$, respectively. The general form of the model is

$$\begin{aligned} x'(t) &= ax(t) - bx(t)y(t) \\ y'(t) &= bx(t)y(t) - cy(t) \end{aligned}$$

for positive constants a, b and c. First, look carefully at the equations. The term $bx(t)y(t)$ is included to represent the effects of encounters between the species. This effect is negative on species X and positive on species Y. If $b = 0$, the species don't interact at all. In this case, show that species Y dies out (with death rate c) and species X thrives (with growth rate a). Given all of this, explain why X must be the prey and Y the predator. Next, you should find the equilibrium point for coexistence. That is, find positive values $\bar{x}$ and $\bar{y}$ such that both $x'(t) = 0$ and $y'(t) = 0$. For this problem, think of X as an insect that damages farmers' crops and Y as a natural predator (e.g., a bat). A farmer might decide to use a pesticide to reduce the damage caused by the X's. Briefly explain why the effects of the pesticide might be to decrease the value of a and increase the value of c. Now, determine how these changes affect the equilibrium values. Show that the pest population X actually increases and the predator population Y decreases. Explain, in terms of the interaction between predator and prey, how this could happen. The moral is that the long-range effects of pesticides can be the exact opposite of the short-range (and desired) effects.

7.4 SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

In this section, we consider systems of two or more first-order differential equations. Recall that the exponential growth described in example 1.1 and the logistic growth shown in example 2.6 both model the population of a single organism in isolation. A more realistic model would account for interactions between two species, where interactions significantly impact both populations. For instance, the population of rabbits in a given area is negatively affected by the presence of various predators (such as foxes), while the population of predators will grow in response to an abundant supply of prey and decrease when the prey are less plentiful. We begin with the analysis of such a **predator-prey** model, where a species of predators depends on a species of prey for food.

Predator-Prey Systems

Suppose that the population of the prey (in hundreds of animals) is given by $x(t)$. This species thrives in its environment, except for interactions with a predator, with population $y(t)$ (in hundreds of animals). Without any predators, we assume that $x(t)$ satisfies the logistic equation $x'(t) = bx(t) - c[x(t)]^2$, for positive constants b and c. The negative effect of the predator should be proportional to the number of interactions between the species, which we assume to be proportional to $x(t)y(t)$, since the number of interactions increases as $x(t)$ or $y(t)$ increases. This leads us to the model

$$x'(t) = bx(t) - c[x(t)]^2 - k_1\, x(t)y(t),$$

for some positive constant k_1. On the other hand, the population of predators depends on interactions between the two species to survive. We assume that without any available prey, the population $y(t)$ decays exponentially. Interactions between predator and prey have a positive influence on the population of predators. This gives us the model

$$y'(t) = -d\, y(t) + k_2\, x(t)y(t),$$

for positive constants d and k_2.

Putting these equations together, we have a **system** of first-order differential equations:

PREDATOR-PREY EQUATIONS

$$\begin{aligned} x'(t) &= b\,x(t) - c[x(t)]^2 - k_1\, x(t)y(t) \\ y'(t) &= -d\, y(t) + k_2\, x(t)y(t). \end{aligned}$$

A **solution** of this system is a pair of functions, $x(t)$ and $y(t)$, that satisfy both of the equations. We refer to this system as **coupled,** since we must solve the equations together, as each of $x'(t)$ and $y'(t)$ depends on both $x(t)$ and $y(t)$. Although solving such a system is beyond the level of this course, we can learn something about the solutions using graphical methods. The analysis of this system proceeds similarly to examples 3.6 and 3.7. As before, it is helpful to first find equilibrium solutions (solutions for which both $x'(t) = 0$ and $y'(t) = 0$).

EXAMPLE 4.1 Finding Equilibrium Solutions of a System of Equations

Find and interpret all equilibrium solutions of the predator-prey model

$$\begin{cases} x'(t) = 0.2x(t) - 0.1[x(t)]^2 - 0.4x(t)y(t) \\ y'(t) = -0.1y(t) + 0.1x(t)y(t), \end{cases}$$

where x and y represent the populations (in hundreds of animals) of a prey and a predator, respectively.

Solution If (x, y) is an equilibrium solution, then the constant functions $x(t) = x$ and $y(t) = y$ satisfy the system of equations with $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have

$$\begin{aligned} 0 &= 0.2x - 0.1x^2 - 0.4xy \\ 0 &= -0.1y + 0.1xy \end{aligned}$$

This is now a system of two (nonlinear) equations for the two unknowns x and y. There is no general method for solving systems of nonlinear algebraic equations exactly. In this case, you should solve the simpler equation carefully and then substitute solutions back into the more complicated equation. Notice that both equations factor, to give

$$\begin{aligned} 0 &= 0.1x(2 - x - 4y) \\ 0 &= 0.1y(-1 + x). \end{aligned}$$

The second equation has solutions $y = 0$ and $x = 1$. We now substitute these solutions one at a time into the first equation.

Taking $y = 0$, the first equation becomes $0 = 0.1x(2 - x)$, which has the solutions $x = 0$ and $x = 2$. This says that $(0, 0)$ and $(2, 0)$ are equilibrium solutions of the system.

Note that the equilibrium point (0, 0) corresponds to the case where there are no predators or prey, while (2, 0) corresponds to the case where there are prey but no predators.

Taking $x = 1$, the first equation becomes $0 = 0.2 - 0.1 - 0.4y$, which has the solution $y = \frac{0.1}{0.4} = 0.25$. A third equilibrium solution is then (1, 0.25), corresponding to having both populations constant, with four times as many prey as predators.

Since we have now considered both solutions from the second equation, we have found all equilibrium solutions of the system: (0, 0), (2, 0) and (1, 0.25). ■

Next, we analyze the stability of each equilibrium solution. We can infer from this which solution corresponds to the natural balance of the populations. More advanced techniques for determining stability can be found in most differential equations texts. For simplicity, we use a graphical technique involving a plot called the *phase portrait* to determine the stability. For the system in example 4.1, we can eliminate the time variable, by observing that by the chain rule,

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-0.1y + 0.1xy}{0.2x - 0.1x^2 - 0.4xy}.$$

Observe that this is simply a first-order differential equation for y as a function of x. In this case, we refer to the xy-plane as the **phase plane** for the original system. A **phase portrait** is a sketch of a number of solution curves of the differential equation in the xy-plane. We illustrate this in example 4.2.

EXAMPLE 4.2 Using a Direction Field to Sketch a Phase Portrait

Sketch a direction field of $\frac{dy}{dx} = \frac{-0.1y + 0.1xy}{0.2x - 0.1x^2 - 0.4xy}$, and use the resulting phase portrait to determine the stability of the three equilibrium points (0, 0), (2, 0) and (1, 0.25).

Solution The direction field generated by our CAS (see Figure 7.20) is not especially helpful, largely because it does not show sufficient detail near the equilibrium points.

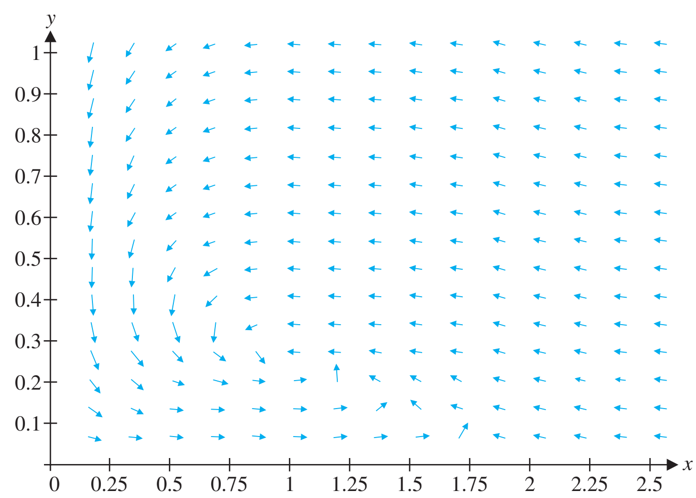


FIGURE 7.20
Direction field

To more clearly see the behavior of solutions near each equilibrium solution, we zoom in on each equilibrium point in turn and plot a number of solution curves, as shown in Figures 7.21a, 7.21b and 7.21c.

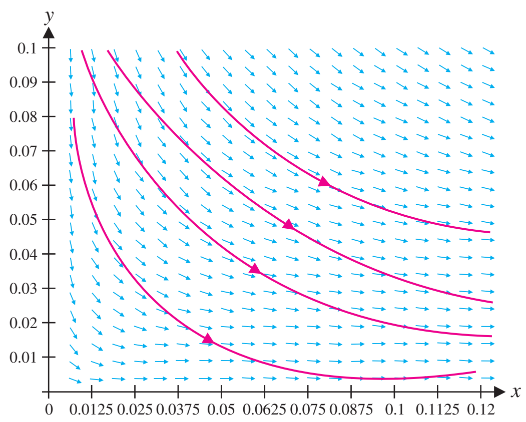


FIGURE 7.21a
Phase portrait near (0, 0)

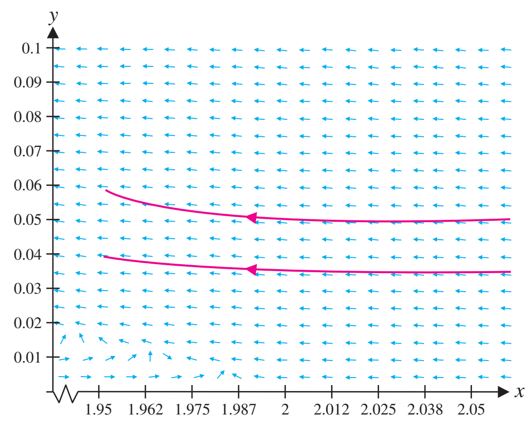


FIGURE 7.21b
Phase portrait near (2, 0)

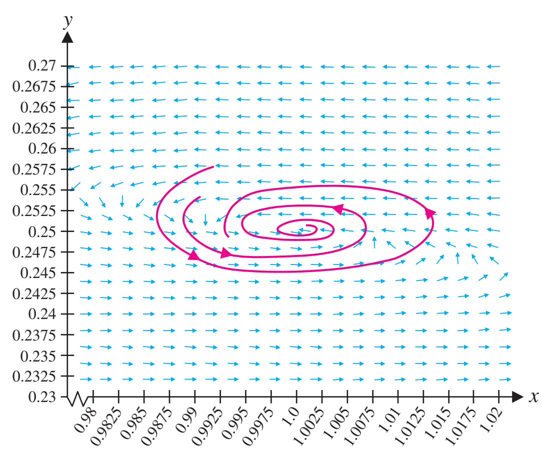


FIGURE 7.21c
Phase portrait near (1, 0.25)

Since the arrows in Figure 7.21a point away from (0, 0), we refer to (0, 0) as an **unstable equilibrium.** That is, solutions that start out close to (0, 0) move *away* from that point as $t \to \infty$. Similarly, most of the arrows in Figure 7.21b point away from (2, 0) and so, we conclude that (2, 0) is also unstable. Finally, in Figure 7.21c, the arrows spiral in toward the point (1, 0.25), indicating that solutions that start out near (1, 0.25) tend toward that point as $t \to \infty$, making this is a **stable equilibrium.** From this, we conclude that the naturally balanced state is for the two species to coexist, with 4 times as many prey as predators. ■

We next consider a two-species system where the species compete for the same resources and/or space. General equations describing this case (where species X has population $x(t)$ and species Y has population $y(t)$) are

$$x'(t) = b_1 x(t) - c_1[x(t)]^2 - k_1 x(t)y(t)$$
$$y'(t) = b_2 y(t) - c_2[y(t)]^2 - k_2 x(t)y(t),$$

for positive constants b_1, b_2, c_1, c_2, k_1 and k_2. Notice how this differs from the predator-prey case. As before, species X grows logistically in the absence of species Y. However, here, species Y also grows logistically in the absence of species X. Further, the interaction terms $k_1 x(t)y(t)$ and $k_2 x(t)y(t)$ are now negative for both species. So, in this case, both species survive nicely on their own but are hurt by the presence of the other. As with predator-prey systems, our focus here is on finding equilibrium solutions.

EXAMPLE 4.3 Finding Equilibrium Solutions of a System of Equations

Find and interpret all equilibrium solutions of the competing species model

$$\begin{cases} x'(t) = 0.4x(t) - 0.1[x(t)]^2 - 0.4x(t)y(t) \\ y'(t) = 0.3y(t) - 0.2[y(t)]^2 - 0.1x(t)y(t). \end{cases}$$

Solution If (x, y) is an equilibrium solution, then the constant functions $x(t) = x$ and $y(t) = y$ satisfy the system of equations with $x'(t) = 0$ and $y'(t) = 0$. Substituting into the equations, we have

$$0 = 0.4x - 0.1x^2 - 0.4xy$$
$$0 = 0.3y - 0.2y^2 - 0.1xy.$$

Notice that both equations factor, to give

$$0 = 0.1x(4 - x - 4y)$$
$$0 = 0.1y(3 - 2y - x).$$

The equations are equally complicated, so we work with both equations simultaneously.

From the top equation, either $x = 0$ or $x + 4y = 4$. From the bottom equation, either $y = 0$ or $x + 2y = 3$. Summarizing, we have

$$x = 0 \quad \text{or} \quad x + 4y = 4$$

and

$$y = 0 \quad \text{or} \quad x + 2y = 3.$$

Taking $x = 0$ from the top line and $y = 0$ from the bottom gives us the equilibrium solution (0, 0). Taking $x = 0$ from the top and substituting into $x + 2y = 3$ on the bottom, we get $y = \frac{3}{2}$ so that $(0, \frac{3}{2})$ is a second equilibrium solution. Note that (0, 0) corresponds to the case where neither species exists and $(0, \frac{3}{2})$ corresponds to the case where species Y exists but species X does not.

Substituting $y = 0$ from the bottom line into $x + 4y = 4$, we get $x = 4$ so that (4, 0) is an equilibrium solution, corresponding to the case where species X exists but species Y does not. The fourth and last possibility has $x + 4y = 4$ and $x + 2y = 3$. Subtracting the equations gives $2y = 1$ or $y = \frac{1}{2}$. With $y = \frac{1}{2}$, $x + 2y = 3$ gives us $x = 2$. The final equilibrium solution is then $(2, \frac{1}{2})$. In this case, both species exist, with 4 times as many of species X.

Since we have now considered all combinations that make *both* equations true, we have found all equilibrium solutions of the system: (0, 0), $(0, \frac{3}{2})$, (4, 0) and $(2, \frac{1}{2})$. ■

We explore predator-prey systems and models for competing species further in the exercises.

Systems of first-order differential equations also arise when we rewrite a single higher-order differential equation as a system of first-order equations. There are several reasons for doing this, notably so that the theory and numerical approximation schemes for first-order equations (such as Euler's method from section 7.3) can be applied.

A falling object is acted on by two primary forces, gravity (pulling down) and air drag (pushing in the direction opposite the motion). In section 5.5, we solved a number of problems by ignoring air drag and assuming that the force due to gravity (i.e., the weight) is constant. While these assumptions lead to solvable equations, in many important applications, neither assumption is valid. Air drag is frequently described as proportional to the square of velocity. Further, the weight of an object of constant mass is not constant, but rather, depends on its distance from the center of the earth. In this case, if we take y as the height of the object above the surface of the earth, then the velocity is y' and the acceleration is y''. The air drag is then $c(y')^2$, for some positive constant c (the **drag coefficient**) and the weight is $-\frac{mgR^2}{(R+y)^2}$, where R is the radius of the earth. Newton's second law $F = ma$ then gives us

$$-\frac{mgR^2}{(R+y)^2} + c(y')^2 = my''.$$

Since this equation involves y, y' and y'', we refer to this as a **second-order differential equation.** In example 4.4, we see how to write this as a system of first-order equations.

EXAMPLE 4.4 Writing a Second-Order Equation as a System of First Order Equations

Write the equation $y'' = 0.1(y')^2 - \frac{1600}{(40+y)^2}$ as a system of first-order equations. Then, find all equilibrium points and interpret the result.

Solution The idea is to define new functions u and v where $u = y$ and $v = y'$. We then have $u' = y' = v$ and

$$v' = y'' = 0.1(y')^2 - \frac{1600}{(40+y)^2} = 0.1v^2 - \frac{1600}{(40+u)^2}.$$

Summarizing, we have the system of first-order equations

$$\begin{aligned} u' &= v \\ v' &= 0.1v^2 - \frac{1600}{(40+u)^2}. \end{aligned}$$

The equilibrium points are then solutions of

$$\begin{aligned} 0 &= v \\ 0 &= 0.1v^2 - \frac{1600}{(40+u)^2}. \end{aligned}$$

With $v = 0$, observe that the second equation has no solution, so that there are no equilibrium points. For a falling object, the position (u) is not constant and so, there are no equilibrium solutions. ■

Some graphing calculators will graph solutions of differential equations, but the equations must be written as a single first-order equation or a system of first-order equations. With the technique shown in example 4.4, you can use such a calculator to graph solutions of higher-order equations.

EXERCISES 7.4

WRITING EXERCISES

1. Explain why in the general predator-prey model, the interaction term k_1xy is subtracted in the prey equation and k_2xy is added in the predator equation.

2. In general, explain why you would expect the constants for the interaction terms in the predator-prey model to satisfy $k_1 > k_2$.

3. In example 4.1, the second equilibrium equation is solved to get $x = 1$ and $y = 0$. Explain why this does not mean that $(1, 0)$ is an equilibrium point.

4. If the populations in a predator-prey or other system approach constant values, explain why the values must come from an equilibrium point.

In exercises 1–6, find and interpret all equilibrium points for the predator-prey model.

1. $\begin{cases} x' = 0.2x - 0.2x^2 - 0.4xy \\ y' = -0.1y + 0.2xy \end{cases}$

2. $\begin{cases} x' = 0.4x - 0.1x^2 - 0.2xy \\ y' = -0.2y + 0.1xy \end{cases}$

3. $\begin{cases} x' = 0.3x - 0.1x^2 - 0.2xy \\ y' = -0.2y + 0.1xy \end{cases}$

4. $\begin{cases} x' = 0.1x - 0.1x^2 - 0.4xy \\ y' = -0.1y + 0.2xy \end{cases}$

5. $\begin{cases} x' = 0.2x - 0.1x^2 - 0.4xy \\ y' = -0.3y + 0.1xy \end{cases}$

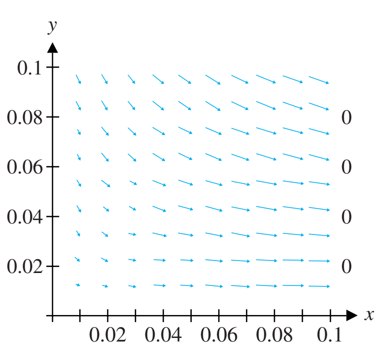
6. $\begin{cases} x' = 0.2x - 0.1x^2 - 0.4xy \\ y' = -0.2y + 0.1xy \end{cases}$

In exercises 7–10, use direction fields to determine the stability of each equilibrium point found in the given exercise.

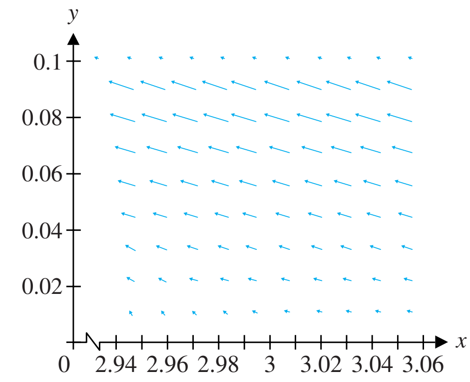
7. exercise 1 **8.** exercise 2 **9.** exercise 5 **10.** exercise 6

In exercises 11–16, use the direction fields to determine the stability of each point.

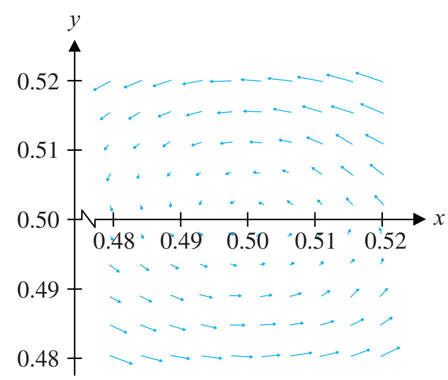
11. The point (0, 0)



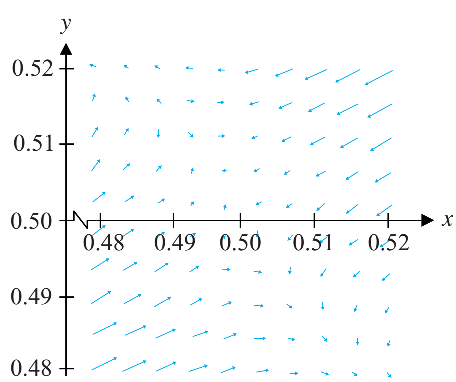
12. The point (3, 0)



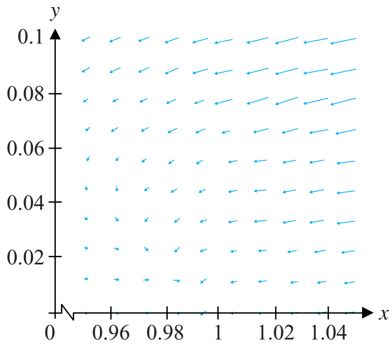
13. The point (0.5, 0.5)



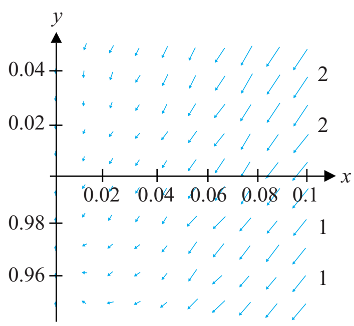
14. The point (0.5, 0.5)



15. The point (1, 0)



16. The point (0, 2)



In exercises 17–22, find and interpret all equilibrium points for the competing species model. (Hint: There are four equilibrium points in exercise 17.)

17. $\begin{cases} x' = 0.3x - 0.2x^2 - 0.1xy \\ y' = 0.2y - 0.1y^2 - 0.1xy \end{cases}$ **18.** $\begin{cases} x' = 0.4x - 0.1x^2 - 0.2xy \\ y' = 0.5y - 0.4y^2 - 0.1xy \end{cases}$

19. $\begin{cases} x' = 0.3x - 0.2x^2 - 0.2xy \\ y' = 0.2y - 0.1y^2 - 0.2xy \end{cases}$ **20.** $\begin{cases} x' = 0.4x - 0.3x^2 - 0.1xy \\ y' = 0.3y - 0.2y^2 - 0.1xy \end{cases}$

21. $\begin{cases} x' = 0.2x - 0.2x^2 - 0.1xy \\ y' = 0.1y - 0.1y^2 - 0.2xy \end{cases}$ **22.** $\begin{cases} x' = 0.1x - 0.2x^2 - 0.1xy \\ y' = 0.3y - 0.2y^2 - 0.1xy \end{cases}$

23. The population models in exercises 17–22 are *competing species models*. Suppose that $x(t)$ and $y(t)$ are the populations of two species of animals that compete for the same plant food. Explain why the interaction terms for both species are negative.

24. The **competitive exclusion principle** in biology states that two species with the same niche cannot coexist in the same ecology. Explain why the existence in exercise 17 of an equilibrium point with both species existing does not necessarily contradict this principle.

25. Use direction fields to determine the stability of each equilibrium point in exercise 17. Do your results contradict or affirm the competitive exclusion principle?

26. Use direction fields to determine the stability of each equilibrium point in exercise 18. Do your results contradict or affirm the competitive exclusion principle?

27. If you have a CAS that can solve systems of equations, sketch solutions of the system of exercise 1 with the initial conditions (a) $x = 1, y = 1$; (b) $x = 0.2, y = 0.4$; (c) $x = 1, y = 0$.

28. If you have a CAS that can solve systems of equations, sketch solutions of the system of exercise 17 with the initial conditions (a) $x = 0.5, y = 0.5$; (b) $x = 0.2, y = 0.4$; (c) $x = 0, y = 0.5$.

In exercises 29–32, write the second-order equation as a system of first-order equations.

29. $y'' + 2xy' + 4y = 4x^2$ **30.** $y'' - 3y' + 3\sqrt{xy} = 4$

31. $y'' - (\cos x)y' + xy^2 = 2x$ **32.** $xy'' + 3(y')^2 = y + 2x$

33. To write a third-order equation as a system of equations, define $u_1 = y$, $u_2 = y'$ and $u_3 = y''$ and compute derivatives as in example 4.4. Write $y''' + 2xy'' - 4y' + 2y = x^2$ as a system of first-order equations.

34. As in exercise 33, write $y''' - 2x^2y' + y^2 = 2$ as a system of first-order equations.

35. As in exercise 33, write $y^{(4)} - 2y''' + xy' = 2 - e^x$ as a system of first-order equations.

36. As in exercise 33, write $y^{(4)} - 2y''y' + (\cos x)y^2 = 0$ as a system of first-order equations.

37. Euler's method applied to the system of equations $x' = f(x, y), x(0) = x_0, y' = g(x, y), y(0) = y_0$ is given by

$$x_{n+1} = x_n + hf(x_n, y_n), \qquad y_{n+1} = y_n + hg(x_n, y_n).$$

Use Euler's method with $h = 0.1$ to estimate the solution at $t = 1$ for exercise 3 with $x(0) = y(0) = 0.2$.

38. Use Euler's method with $h = 0.1$ to estimate the solution at $t = 1$ for exercise 17 with $x(0) = y(0) = 0.2$.

In exercises 39–42, find all equilibrium points.

39. $\begin{cases} x' = (x^2 - 4)(y^2 - 9) \\ y' = x^2 - 2xy \end{cases}$ **40.** $\begin{cases} x' = (x - y)(1 - x - y) \\ y' = 2x - xy \end{cases}$

41. $\begin{cases} x' = (2 + x)(y - x) \\ y' = (4 - x)(x + y) \end{cases}$ **42.** $\begin{cases} x' = -x + y \\ y' = y + x^2 \end{cases}$

43. For the predator-prey model $\begin{cases} x' = 0.4x - 0.1x^2 - 0.2xy \\ y' = -0.5y + 0.1xy \end{cases}$ show that the species cannot coexist. If the death rate 0.5 of species Y could be reduced, determine how much it would have to decrease before the species can coexist.

44. In exercise 43, if the death rate of species Y stays constant but the birthrate 0.4 of species X can be increased, determine how much it would have to increase before the species can coexist. Briefly explain why an increase in the birthrate of species X could help species Y survive.

45. For the general predator-prey model $\begin{cases} x' = bx - cx^2 - k_1xy \\ y' = -dy + k_2xy \end{cases}$ show that the species can coexist if and only if $bk_2 > cd$.

46. In the predator-prey model of exercise 45, the prey could be a pest insect that attacks a farmer's crop and the predator, a natural predator (e.g., a bat) of the pest. Assume that $c = 0$ and the coexistence equilibrium point is stable. The effect of a pesticide would be to reduce the birthrate b of the pest. It could also potentially increase the death rate d of the predator. If this happens, state the effect on the coexistence equilibrium point. Is this the desired effect of the pesticide?

EXPLORATORY EXERCISES

1. The equations of motion for a golf ball with position (x, y, z) measured in feet and spin defined by parameters a, b and c are given by

$$x'' = -0.0014x'\sqrt{(x')^2 + (y')^2 + (z')^2} + 0.001(bz' - cy')$$
$$y'' = -0.0014y'\sqrt{(x')^2 + (y')^2 + (z')^2} + 0.001(cx' - az')$$
$$z'' = -32 - 0.0014z'\sqrt{(x')^2 + (y')^2 + (z')^2} + 0.001(ay' - bx')$$

Write this as a system of six first-order equations and use a CAS to generate graphical solutions to answer the following questions. Think of x as being measured left to right, y measured downrange and z measuring height above the ground. (a) The ball starts at (0, 0) and has an initial velocity with $x'(0) = 0$ and $y'(0)$ and $z'(0)$ being the components of a speed of 260 ft/s launched at 18° above the horizontal. The spin is backspin at 2200 rpm, so that $a = 220$ and $b = c = 0$. Estimate the maximum height of the ball and the horizontal range. (b) Repeat part (a), except assume there is now some sidespin so that $a = 210$ and $b = c = -46$. Estimate the maximum height of the ball, the horizontal range and the amount of left/right distance that the ball hooks or slices.

2. In this exercise, we expand the predator-prey model of example 4.1 to a model of a small ecology with one predator and two prey. To start, let x and y be the populations of the prey species and z the predator population. Consider the model

$$x'(t) = b_1x(t) - k_1x(t)z(t)$$
$$y'(t) = b_2y(t) - k_2y(t)z(t)$$
$$z'(t) = -dz(t) + k_3x(t)z(t) + k_4y(t)z(t)$$

for positive constants b_1, b_2, d and $k_1, \dots, k_4$. Notice that in the absence of the predator, each prey population grows exponentially. Assuming that the predator population is reasonably large and stable, explain why it might be an acceptable simplification to leave out the x^2 and y^2 terms for logistic growth. According to this model, are there significant interactions between the x and y populations? Find all equilibrium points and determine the conditions under which all three species could coexist. Repeat this with the logistic terms restored.

$$x'(t) = b_1x(t) - c_1[x(t)]^2 - k_1x(t)z(t)$$
$$y'(t) = b_2y(t) - c_2[y(t)]^2 - k_2y(t)z(t)$$
$$z'(t) = -dz(t) + k_3x(t)z(t) + k_4y(t)z(t)$$

Does it make any difference which model is used?

3. For the general competing species model

$$x'(t) = b_1x(t) - c_1[x(t)]^2 - k_1x(t)y(t)$$
$$y'(t) = b_2y(t) - c_2[y(t)]^2 - k_2x(t)y(t)$$

show that the species cannot coexist under either of the following conditions:

(a) $\frac{b_1}{k_1} > \frac{b_2}{c_2}$ and $\frac{b_1}{c_1} > \frac{b_2}{k_2}$ or (b) $\frac{b_2}{k_2} > \frac{b_1}{c_1}$ and $\frac{b_2}{c_2} > \frac{b_1}{k_1}$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Differential equation	Doubling time	Half-life
Newton's Law of Cooling	Equilibrium solution	Stable
Separable equation	Logistic growth	Euler's method
Direction field		System of equations
		Phase portrait
		Predator-prey systems

Review Exercises

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. For exponential growth and decay, the rate of change is constant.
2. For logistic growth, the rate of change is proportional to the amount present.
3. Any separable equation can be solved for y as a function of x.
4. The direction field of a differential equation is tangent to the solution.
5. The smaller h is, the more accurate Euler's method is.
6. An equilibrium point of a system of two equations and unknown functions x and y is any value of x such that $x' = 0$ or $y' = 0$.
7. A phase portrait shows several solutions on the same graph.

In exercises 1–6, solve the IVP.

1. $y' = 2y,\ y(0) = 3$
2. $y' = -3y,\ y(0) = 2$
3. $y' = \dfrac{2x}{y},\ y(0) = 2$
4. $y' = -3xy^2,\ y(0) = 4$
5. $y' = \sqrt{xy},\ y(1) = 4$
6. $y' = x + y^2x,\ y(0) = 1$
7. A bacterial culture has an initial population 10^4 and doubles every 2 hours. Find an equation for the population at any time t and determine when the population reaches 10^6.
8. An organism has population 100 at time $t = 0$ and population 140 at time $t = 2$. Find an equation for the population at any time and determine the population at time $t = 6$.
9. The half-life of nicotine in the human bloodstream is 2 hours. If there is initially 2 mg of nicotine present, find an equation for the amount at any time t and determine when the nicotine level reaches 0.1 mg.
10. If the half-life of a radioactive material is 3 hours, what percentage of the material will be left after 9 hours? 11 hours?
11. If you invest \$2000 at 8% compounded continuously, how long will it take the investment to double?
12. If you invest \$4000 at 6% compounded continuously, how much will the investment be worth in 10 years?
13. A cup of coffee is served at 180°F in a room with temperature 68°F. After 1 minute, the temperature has dropped to 176°F. Find an equation for the temperature at any time and determine when the temperature will reach 120°F.
14. A cold drink is served at 46°F in a room with temperature 70°F. After 4 minutes, the temperature has increased to 48°F. Find an equation for the temperature at any time and determine when the temperature will reach 58°F.

In exercises 15–18, solve each separable equation, explicitly if possible.

15. $y' = 2x^3y$
16. $y' = \dfrac{y}{\sqrt{1-x^2}}$
17. $y' = \dfrac{4}{(y^2+y)(1+x^2)}$
18. $y' = e^{x+y}$

In exercises 19–22, find all equilibrium solutions and determine which are stable and which are unstable.

19. $y' = 3y(2-y)$
20. $y' = y(1-y^2)$
21. $y' = -y\sqrt{1+y^2}$
22. $y' = y + \dfrac{2y}{1-y}$

In exercises 23–26, sketch the direction field.

23. $y' = -x(4-y)$
24. $y' = 4x - y^2$
25. $y' = 2xy - y^2$
26. $y' = 4x - y$
27. Suppose that the concentration x of chemical in a bimolecular reaction satisfies the differential equation $x'(t) = (0.3 - x)(0.4 - x) - 0.25x^2$. For (a) $x(0) = 0.1$ and (b) $x(0) = 0.4$, find the concentration at any time. Graph the solutions. Explain what is physically impossible about problem (b).
28. For exercise 27, find equilibrium solutions and use a slope diagram to determine the stability of each equilibrium.
29. In the second-order chemical reaction $x' = r(a-x)(b-x)$, suppose that A and B are the same (thus, $a = b$). Identify the values of x that are possible. Draw the direction field and determine the limiting amount $\lim_{t\to\infty} x(t)$. Verify your answer by solving for x. Interpret the physical significance of a in this case.
30. In an **autocatalytic reaction,** a substance reacts with itself. Explain why the concentration would satisfy the differential equation $x' = rx(1-x)$. Identify the values of x that are possible. Draw the direction field and determine the limiting amount $\lim_{t\to\infty} x(t)$. Verify your answer by solving for x.
31. Suppose that \$100,000 is invested initially and continuous deposits are made at the rate of \$20,000 per year. Interest is compounded continuously at 10%. How much time will it take for the account to reach \$1 million?

Review Exercises

32. Rework exercise 31 with the $20,000 payment made at the end of each year instead of continuously.

In exercises 33–36, identify the system of equations as a predator-prey model or a competing species model. Find and interpret all equilibrium points.

33. $\begin{cases} x' = 0.1x - 0.1x^2 - 0.2xy \\ y' = -0.1y + 0.1xy \end{cases}$

34. $\begin{cases} x' = 0.2x - 0.1x^2 - 0.2xy \\ y' = 0.1y - 0.1y^2 - 0.1xy \end{cases}$

35. $\begin{cases} x' = 0.5x - 0.1x^2 - 0.2xy \\ y' = 0.4y - 0.1y^2 - 0.2xy \end{cases}$

36. $\begin{cases} x' = 0.4x - 0.1x^2 - 0.2xy \\ y' = -0.2y + 0.1xy \end{cases}$

37. Use direction fields to determine the stability of each equilibrium point in exercise 33.

38. Use direction fields to determine the stability of each equilibrium point in exercise 35.

39. Write the second-order equation $y'' - 4x^2y' + 2y = 4xy - 1$ as a system of first-order equations.

40. If you have a CAS that can solve systems of equations, sketch solutions of the system of exercise 33 with the initial conditions (a) $x = 0.4$, $y = 0.1$; (b) $x = 0.1$, $y = 0.4$.

EXPLORATORY EXERCISES

1. In this exercise, we compare two models of the vertical velocity of a falling object. Forces acting on the object are gravity and air drag. From experience, you know that the faster an object moves, the more air drag there is. But, is the drag force proportional to velocity v or the square of velocity v^2? It turns out that the answer depends on the shape and speed of the object. The goal of this exercise is to explore how much difference it makes which model is used. Define the following models for a falling object with $v \le 0$ (units of meters and seconds):

$$\text{Model 1:} \quad \frac{dv}{dt} = -9.8 + 0.7v$$

$$\text{Model 2:} \quad \frac{dv}{dt} = -9.8 + 0.05v^2.$$

Solve each equation with the initial condition $v(0) = 0$. Graph the two solutions on the same axes and discuss similarities and differences. Show that in both cases the limiting velocity is $\lim_{t\to\infty} v(t) = 14$ m/s. In each case, determine the time required to reach 4 m/s and the time required to reach 13 m/s. Summarizing, discuss how much difference it makes which model you use.

2. In this exercise, we compare the two drag models for objects moving horizontally. Since gravity does not affect horizontal motion, if $v \ge 0$, the models are

$$\text{Model 1:} \quad \frac{dv}{dt} = -c_1 v$$

$$\text{Model 2:} \quad \frac{dv}{dt} = -c_2 v^2,$$

for positive constants c_1 and c_2. Explain why the negative signs are needed. If $v \le 0$, how would the equations change? For a pitched baseball, physicists find that the second model is more accurate. The drag coefficient is approximately $c_2 = 0.0025$ if v is measured in ft/s. For comparison purposes, find the value of c_1 such that $c_1 v = c_2 v^2$ for $v = 132$ ft/s (this is a 90-mph pitch). Then solve each equation with initial condition $v(0) = 132$. Find the time it takes the ball to reach home plate 60 feet away. Find the velocity of the ball when it reaches home plate. How much difference is there in the two models? For a tennis serve, use the second model with $c_2 = 0.003$ to estimate how much a 140-mph serve has slowed by the time it reaches the service line 60 feet away. Both baseball and tennis use radar guns to measure speeds. Based on your calculations, does it make much of a difference at which point the speed of a ball is measured?