Vectors and the Geometry of Space

CHAPTER 10

The Bristol Motor Speedway is one of the most popular racetracks on the NASCAR circuit. Races there are considered the most intense of the year, thanks to a combination of high speed, tight quarters and high-energy crowds. With 43 drivers going over 120 miles per hour on a half-mile oval for two and a half hours with 160,000 people cheering them on, the experience is both thrilling and exhausting for all involved.

Auto racing at all levels is a highly technological competition. A Formula One race car seems to have more in common with an airplane than with a standard car. At speeds of 100–200 mph, air resistance forces on the car can reach hurricane levels. The racing engineer's task is to design the car aerodynamically so that these forces help keep the car on the road. The large wings on the back of cars create a downward force that improves the traction of the car. So-called "ground effect" downforces have an even greater impact. Here, the entire underside of the car is shaped like an upside-down airplane wing and air drawn underneath the car generates a tremendous downward force. Such designs are so successful that the downward forces exceed three times the weight of the car. This means that theoretically the car could race upside down!

Stock car racers have additional challenges, since intricate rules severely limit the extent to which the cars can be modified. So, how do stock cars safely speed around the Bristol Motor Speedway? The track is an oval only 0.533 mile in length and racers regularly exceed 120 miles per hour, completing a lap in just over 15 seconds. These speeds would be unsafe if the track were not specially designed for high-speed racing. In particular, the Bristol track is steeply banked, with a 16-degree bank on straightaways and a spectacular 36-degree bank in the corners.

As you will see in the exercises in section 10.3, the banking of a road changes the role of gravity. In effect, part of the weight of the car is diverted into a force that helps the car make its turn safely. In this chapter, we introduce vectors and develop the calculations needed to resolve vectors into components. This is a fundamental tool for engineers designing race cars and racetracks.

This chapter represents a crossroads from the primarily two-dimensional world of first-year calculus to the three-dimensional world of many important scientific and engineering problems. The rest of the calculus we develop in this book builds directly on the basic ideas developed here.

10.1 VECTORS IN THE PLANE

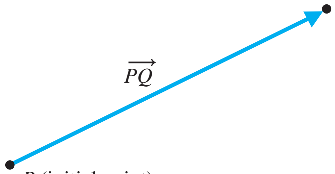


FIGURE 10.1
Directed line segment

We have already considered very simple models of velocity and acceleration in one dimension, but these are *not* generally one-dimensional quantities. In particular, to describe the velocity of a moving object, we must specify both its speed *and* the direction in which it's moving. In fact, velocity, acceleration and force each have both a *size* (e.g., speed) and a *direction*. We represent such a quantity graphically as a directed line segment, that is, a line segment with a specific direction (i.e., an arrow).

We denote the directed line segment extending from the point P (the **initial point**) to the point Q (the **terminal point**) as $\overrightarrow{PQ}$ (see Figure 10.1). We refer to the length of $\overrightarrow{PQ}$ as its **magnitude,** denoted $\|\overrightarrow{PQ}\|$. Mathematically, we consider all directed line segments with the same magnitude and direction to be equivalent, regardless of the location of their initial point and we use the term **vector** to describe any quantity that has both a magnitude and a direction. We should emphasize that the location of the initial point is not relevant; only the magnitude and direction matter. In other words, if $\overrightarrow{PQ}$ is the directed line segment from the initial point P to the terminal point Q, then the corresponding vector $\mathbf{v}$ represents $\overrightarrow{PQ}$ as well as every other directed line segment having the same magnitude and direction as $\overrightarrow{PQ}$. In Figure 10.2, we indicate three vectors that are all considered to be equivalent, even though their initial points are different. In this case, we write

$$\mathbf{a} = \mathbf{b} = \mathbf{c}.$$

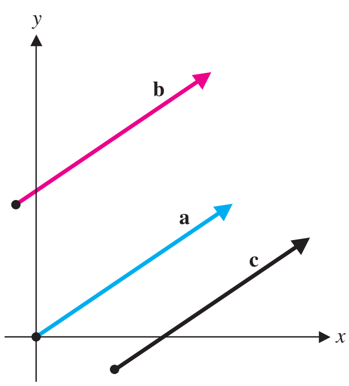


FIGURE 10.2
Equivalent vectors

When considering vectors, it is often helpful to think of them as representing some specific physical quantity. For instance, when you see the vector $\overrightarrow{PQ}$, you might imagine moving an object from the initial point P to the terminal point Q. In this case, the magnitude of the vector would represent the distance the object is moved and the direction of the vector would point from the starting position to the final position.

In this text, we usually denote vectors by boldface characters such as **a**, **b** and **c**, as seen in Figure 10.2. Since you will not be able to write in boldface, you should use the arrow notation (e.g., $\overrightarrow{a}$). When discussing vectors, we refer to real numbers as **scalars.** It is *very important* that you begin now to carefully distinguish between vector and scalar quantities. This will save you immense frustration both now and as you progress through the remainder of this text.

Look carefully at the three vectors shown in Figure 10.3a. If you think of the vector $\overrightarrow{AB}$ as representing the displacement of a particle from the point A to the point B, notice that the end result of displacing the particle from A to B (corresponding to the vector $\overrightarrow{AB}$), followed by displacing the particle from B to C (corresponding to the vector $\overrightarrow{BC}$) is the same as displacing the particle directly from A to C, which corresponds to the vector $\overrightarrow{AC}$

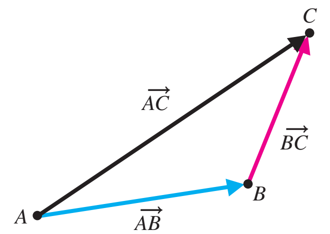


FIGURE 10.3a
Resultant vector

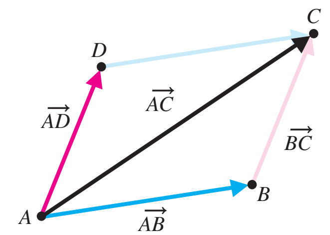


FIGURE 10.3b
Sum of two vectors

(called the **resultant vector**). We call $\overrightarrow{AC}$ the **sum** of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

Given two vectors that we want to add, we locate their initial points at the same point, translate the initial point of one to the terminal point of the other and complete the parallelogram, as indicated in Figure 10.3b. The vector lying along the diagonal, with initial point at A and terminal point at C is the sum

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}.$$

A second basic arithmetic operation for vectors is **scalar multiplication.** If we multiply a vector **u** by a scalar (a real number) $c > 0$, the resulting vector will have the same direction as **u**, but will have magnitude $c\|\mathbf{u}\|$. On the other hand, multiplying a vector **u** by a scalar $c < 0$ will result in a vector with opposite direction from **u** and magnitude $|c|\|\mathbf{u}\|$ (see Figure 10.4).

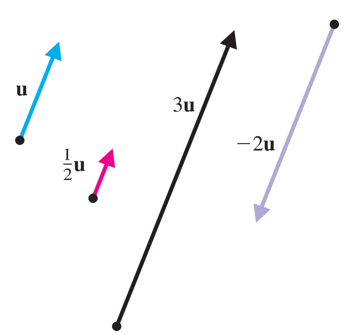


FIGURE 10.4
Scalar multiplication

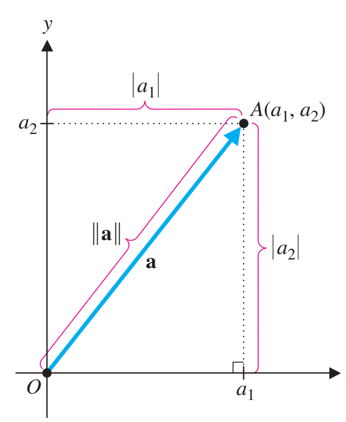


FIGURE 10.5
Position vector $\mathbf{a} = \langle a_1, a_2 \rangle$

Since the location of the initial point is irrelevant, we typically draw vectors with their initial point located at the origin. Such a vector is called a **position vector.** Notice that the terminal point of a position vector will completely determine the vector, so that specifying the terminal point will also specify the vector. For the position vector **a** with initial point at the origin and terminal point at the point $A(a_1, a_2)$ (see Figure 10.5), we denote the vector by

$$\mathbf{a} = \overrightarrow{OA} = \langle a_1, a_2 \rangle.$$

We call a_1 and a_2 the **components** of the vector **a**; a_1 is the **first component** and a_2 is the **second component.** Be careful to distinguish between the *point* (a_1, a_2) and the position vector $\langle a_1, a_2 \rangle$. Note from Figure 10.5 that the magnitude of the position vector **a** follows

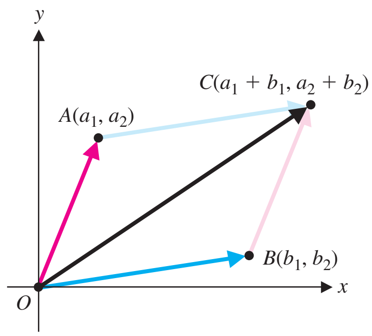


FIGURE 10.6
Adding position vectors

directly from the Pythagorean Theorem. We have

$$\boxed{\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}.} \qquad \text{Magnitude of a vector} \tag{1.1}$$

Notice that it follows from the definition that for two position vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, $\mathbf{a} = \mathbf{b}$ if and only if their terminal points are the same, that is if $a_1 = b_1$ and $a_2 = b_2$. In other words, two position vectors are equal only when their corresponding components are equal.

To add two position vectors, $\overrightarrow{OA} = \langle a_1, a_2 \rangle$ and $\overrightarrow{OB} = \langle b_1, b_2 \rangle$, we draw the position vectors in Figure 10.6 and complete the parallelogram, as before. From Figure 10.6, we have

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}.$$

Writing down the position vectors in their component form, we take this as our definition of vector addition:

$$\boxed{\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle.} \qquad \text{Vector addition} \tag{1.2}$$

So, to add two vectors, we simply add the corresponding components. For this reason, we say that addition of vectors is done **componentwise.** Similarly, we *define* subtraction of vectors componentwise, so that

$$\boxed{\langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle.} \qquad \text{Vector subtraction} \tag{1.3}$$

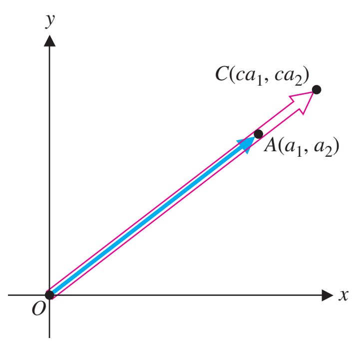


FIGURE 10.7a
Scalar multiplication ($c > 1$)

We give a geometric interpretation of subtraction later in this section.

Recall that if we multiply a vector $\mathbf{a}$ by a scalar c, the result is a vector in the same direction as $\mathbf{a}$ (for $c > 0$) or the opposite direction as $\mathbf{a}$ (for $c < 0$), in each case with magnitude $|c|\|\mathbf{a}\|$. We indicate the case of a position vector $\mathbf{a} = \langle a_1, a_2 \rangle$ and scalar multiple $c > 1$ in Figure 10.7a and for $0 < c < 1$ in Figure 10.7b. The situation for $c < 0$ is illustrated in Figures 10.7c and 10.7d.

For the case where $c > 0$, notice that a vector in the same direction as $\mathbf{a}$, but with magnitude $|c|\|\mathbf{a}\|$, is the position vector $\langle ca_1, ca_2 \rangle$, since

$$\begin{aligned} \|\langle ca_1, ca_2 \rangle\| &= \sqrt{(ca_1)^2 + (ca_2)^2} = \sqrt{c^2 a_1^2 + c^2 a_2^2} \\ &= |c|\sqrt{a_1^2 + a_2^2} = |c|\|\mathbf{a}\|. \end{aligned}$$

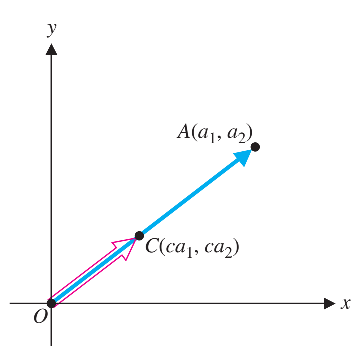


FIGURE 10.7b
Scalar multiplication ($0 < c < 1$)

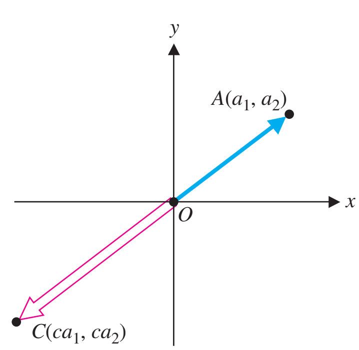


FIGURE 10.7c
Scalar multiplication ($c < -1$)

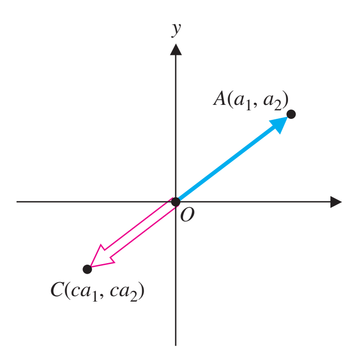


FIGURE 10.7d
Scalar multiplication ($-1 < c < 0$)

Similarly, if $c < 0$, you can show that $\langle ca_1, ca_2\rangle$ is a vector in the **opposite** direction from **a**, with magnitude $|c|\|\mathbf{a}\|$. For this reason, we define scalar multiplication of position vectors by

Scalar multiplication

$$\boxed{c\langle a_1, a_2\rangle = \langle ca_1, ca_2\rangle,} \tag{1.4}$$

for any scalar c. Further, notice that this says that

$$\boxed{\|c\mathbf{a}\| = |c|\|\mathbf{a}\|.} \tag{1.5}$$

EXAMPLE 1.1 Vector Arithmetic

For vectors $\mathbf{a} = \langle 2, 1\rangle$ and $\mathbf{b} = \langle 3, -2\rangle$, compute (a) $\mathbf{a} + \mathbf{b}$, (b) $2\mathbf{a}$, (c) $2\mathbf{a} + 3\mathbf{b}$, (d) $2\mathbf{a} - 3\mathbf{b}$ and (e) $\|2\mathbf{a} - 3\mathbf{b}\|$.

Solution (a) From (1.2), we have

$$\mathbf{a} + \mathbf{b} = \langle 2, 1\rangle + \langle 3, -2\rangle = \langle 2 + 3, 1 - 2\rangle = \langle 5, -1\rangle.$$

(b) From (1.4), we have

$$2\mathbf{a} = 2\langle 2, 1\rangle = \langle 2 \cdot 2, 2 \cdot 1\rangle = \langle 4, 2\rangle.$$

(c) From (1.2) and (1.4), we have

$$2\mathbf{a} + 3\mathbf{b} = 2\langle 2, 1\rangle + 3\langle 3, -2\rangle = \langle 4, 2\rangle + \langle 9, -6\rangle = \langle 13, -4\rangle.$$

(d) From (1.3) and (1.4), we have

$$2\mathbf{a} - 3\mathbf{b} = 2\langle 2, 1\rangle - 3\langle 3, -2\rangle = \langle 4, 2\rangle - \langle 9, -6\rangle = \langle -5, 8\rangle.$$

(e) Finally, from (1.1), we have

$$\|2\mathbf{a} - 3\mathbf{b}\| = \|\langle -5, 8\rangle\| = \sqrt{25 + 64} = \sqrt{89}. \blacksquare$$

Observe that if we multiply any vector (with any direction) by the scalar $c = 0$, we get a vector with zero length, the **zero vector:**

$$\mathbf{0} = \langle 0, 0\rangle.$$

Further, notice that this is the *only* vector with zero length. (Why is that?) The zero vector also has no particular direction. Finally, we define the **additive inverse** $-\mathbf{a}$ of a vector **a** in the expected way:

$$-\mathbf{a} = -\langle a_1, a_2\rangle = (-1)\langle a_1, a_2\rangle = \langle -a_1, -a_2\rangle.$$

Notice that this says that the vector $-\mathbf{a}$ is a vector with the **opposite** direction as **a** and since

$$\|-\mathbf{a}\| = \|(-1)\langle a_1, a_2\rangle\| = |-1|\|\mathbf{a}\| = \|\mathbf{a}\|,$$

$-\mathbf{a}$ has the same length as **a**.

DEFINITION 1.1

Two vectors having the same or opposite direction are called **parallel.**

It then follows that two (nonzero) position vectors **a** and **b** are parallel if and only if $\mathbf{b} = c\mathbf{a}$, for some scalar c. In this event, we say that **b** is a *scalar multiple* of **a**.

EXAMPLE 1.2 Determining When Two Vectors Are Parallel

Determine whether the given pair of vectors is parallel: (a) $\mathbf{a} = \langle 2, 3\rangle$ and $\mathbf{b} = \langle 4, 5\rangle$, (b) $\mathbf{a} = \langle 2, 3\rangle$ and $\mathbf{b} = \langle -4, -6\rangle$.

Solution (a) Notice that from (1.4), we have that if $\mathbf{b} = c\mathbf{a}$, then

$$\langle 4, 5\rangle = c\langle 2, 3\rangle = \langle 2c, 3c\rangle.$$

For this to hold, the corresponding components of the two vectors must be equal. That is, $4 = 2c$ (so that $c = 2$) *and* $5 = 3c$ (so that $c = 5/3$). This is a contradiction and so, **a** and **b** are not parallel.

(b) Again, from (1.4), we have

$$\langle -4, -6\rangle = c\langle 2, 3\rangle = \langle 2c, 3c\rangle.$$

In this case, we have $-4 = 2c$ (so that $c = -2$) and $-6 = 3c$ (which again leads us to $c = -2$). This says that $-2\mathbf{a} = \langle -4, -6\rangle = \mathbf{b}$ and so, $\langle 2, 3\rangle$ and $\langle -4, 6\rangle$ are parallel. ■

We denote the set of all position vectors in two-dimensional space by

$$V_2 = \{\langle x, y\rangle | x, y \in \mathbb{R}\}.$$

You can easily show that the rules of algebra given in Theorem 1.1 hold for vectors in V_2.

THEOREM 1.1

For any vectors **a**, **b** and **c** in V_2, and any scalars d and e in $\mathbb{R}$, the following hold:

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
(ii) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
(iii) $\mathbf{a} + \mathbf{0} = \mathbf{a}$ (zero vector)
(iv) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (additive inverse)
(v) $d(\mathbf{a} + \mathbf{b}) = d\mathbf{a} + d\mathbf{b}$ (distributive law)
(vi) $(d + e)\,\mathbf{a} = d\mathbf{a} + e\mathbf{a}$ (distributive law)
(vii) $(1)\,\mathbf{a} = \mathbf{a}$ (multiplication by 1) and
(viii) $(0)\,\mathbf{a} = \mathbf{0}$ (multiplication by 0).

PROOF

We prove the first of these and leave the rest as exercises. By definition,

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \langle a_1, a_2\rangle + \langle b_1, b_2\rangle = \langle a_1 + b_1, a_2 + b_2\rangle \\ &= \langle b_1 + a_1, b_2 + a_2\rangle = \mathbf{b} + \mathbf{a}. \ ■\end{aligned}$$

Since addition of real numbers is commutative.

Notice that using the commutativity and associativity of vector addition, we have

$$\mathbf{b} + (\mathbf{a} - \mathbf{b}) = (\mathbf{a} - \mathbf{b}) + \mathbf{b} = \mathbf{a} + (-\mathbf{b} + \mathbf{b}) = \mathbf{a} + \mathbf{0} = \mathbf{a}.$$

From our graphical interpretation of vector addition, we get Figure 10.8. Notice that this now gives us a geometric interpretation of vector subtraction.

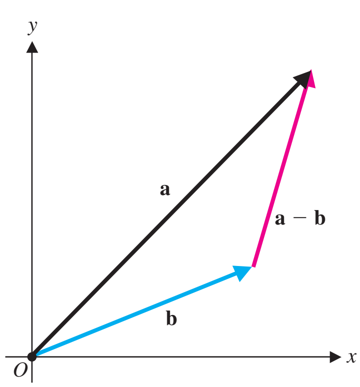


FIGURE 10.8
$\mathbf{b} + (\mathbf{a} - \mathbf{b}) = \mathbf{a}$

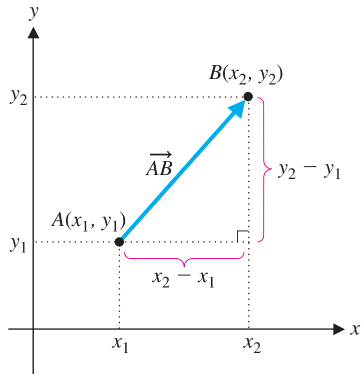


FIGURE 10.9
Vector from A to B

For any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, observe from Figure 10.9 that the vector $\overrightarrow{AB}$ corresponds to the position vector $\langle x_2 - x_1, y_2 - y_1 \rangle$.

EXAMPLE 1.3 Finding a Position Vector

Find the vector with (a) initial point at $A(2, 3)$ and terminal point at $B(3, -1)$ and (b) initial point at B and terminal point at A.

Solution (a) We show this graphically in Figure 10.10a. Notice that

$$\overrightarrow{AB} = \langle 3 - 2, -1 - 3 \rangle = \langle 1, -4 \rangle.$$

(b) Similarly, the vector with initial point at $B(3, -1)$ and terminal point at $A(2, 3)$ is given by

$$\overrightarrow{BA} = \langle 2 - 3, 3 - (-1) \rangle = \langle 2 - 3, 3 + 1 \rangle = \langle -1, 4 \rangle.$$

We indicate this graphically in Figure 10.10b. ■

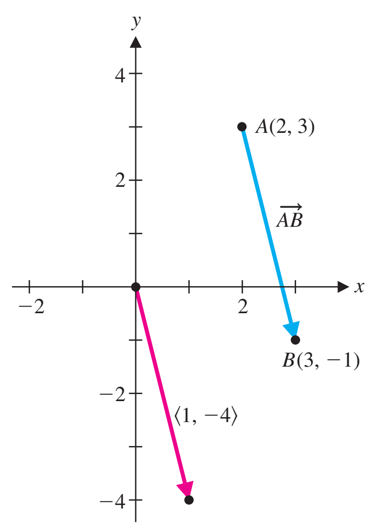


FIGURE 10.10a
$\overrightarrow{AB} = \langle 1, -4 \rangle$

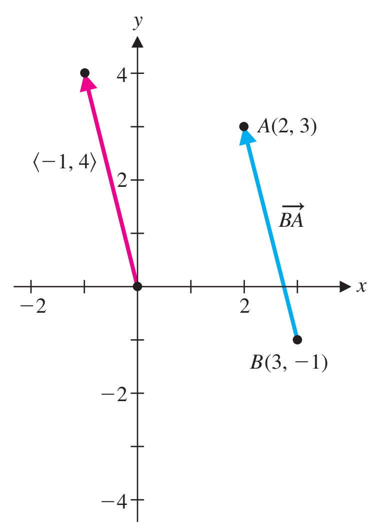


FIGURE 10.10b
$\overrightarrow{BA} = \langle -1, 4 \rangle$

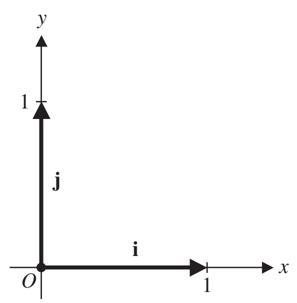


FIGURE 10.11
Standard basis

We often find it convenient to write vectors in terms of some standard vectors. We define the **standard basis vectors i** and **j** by

$$\boxed{\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle}$$

(see Figure 10.11). Notice that $\|\mathbf{i}\| = \|\mathbf{j}\| = 1$. Any vector $\mathbf{a}$ with $\|\mathbf{a}\| = 1$ is called a **unit vector.** So, $\mathbf{i}$ and $\mathbf{j}$ are unit vectors.

Finally, we say that $\mathbf{i}$ and $\mathbf{j}$ form a **basis** for V_2, since we can write any vector $\mathbf{a} \in V_2$ uniquely in terms of $\mathbf{i}$ and $\mathbf{j}$, as follows:

$$\boxed{\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}.}$$

We call a_1 and a_2 the **horizontal** and **vertical components** of $\mathbf{a}$, respectively.

For any nonzero vector, we can always find a unit vector with the same direction, as in Theorem 1.2.

THEOREM 1.2 (Unit Vector)

For any nonzero position vector $\mathbf{a} = \langle a_1, a_2 \rangle$, a unit vector having the same direction as $\mathbf{a}$ is given by

$$\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a}.$$

The process of dividing a nonzero vector by its magnitude is sometimes called **normalization.** (A vector's magnitude is sometimes called its **norm.**) As we'll see, some problems are simplified by using normalized vectors.

PROOF

First, notice that since $\mathbf{a} \neq \mathbf{0}$, $\|\mathbf{a}\| > 0$ and so, $\mathbf{u}$ is a *positive* scalar multiple of $\mathbf{a}$. This says that $\mathbf{u}$ and $\mathbf{a}$ have the same direction. To see that $\mathbf{u}$ is a unit vector, notice that since $\dfrac{1}{\|\mathbf{a}\|}$ is a positive scalar, we have from (1.5) that

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{a}\|}\mathbf{a} \right\| = \frac{1}{\|\mathbf{a}\|}\|\mathbf{a}\| = 1. \quad \blacksquare$$

EXAMPLE 1.4 Finding a Unit Vector

Find a unit vector in the same direction as $\mathbf{a} = \langle 3, -4 \rangle$.

Solution First, note that

$$\|\mathbf{a}\| = \|\langle 3, -4 \rangle\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5.$$

A unit vector in the same direction as $\mathbf{a}$ is then

$$\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a} = \frac{1}{5}\langle 3, -4 \rangle = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle. \quad \blacksquare$$

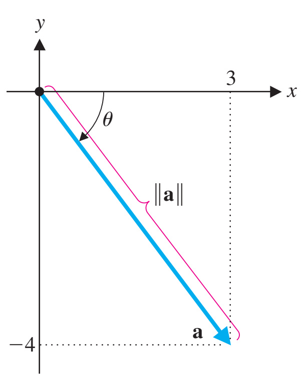


FIGURE 10.12
Polar form of a vector

It is often convenient to write a vector explicitly in terms of its magnitude and direction. For instance, in example 1.4, we found that the magnitude of $\mathbf{a} = \langle 3, -4 \rangle$ is $\|\mathbf{a}\| = 5$, while its direction is indicated by the unit vector $\langle \frac{3}{5}, -\frac{4}{5} \rangle$. Notice that we can now write $\mathbf{a} = 5\langle \frac{3}{5}, -\frac{4}{5} \rangle$. Graphically, we can represent $\mathbf{a}$ as a position vector (see Figure 10.12). Notice also that if θ is the angle between the positive x-axis and $\mathbf{a}$, then

$$\mathbf{a} = 5\langle \cos\theta, \sin\theta \rangle,$$

where $\theta = \tan^{-1}\left(-\frac{4}{3}\right) \approx -0.93$. This representation is called the **polar form** of the vector $\mathbf{a}$. Note that this corresponds to writing the rectangular point $(3, -4)$ as the polar point (r, θ), where $r = \|\mathbf{a}\|$.

We close this section with two applications of vector arithmetic. Whenever two or more forces are acting on an object, the net force acting on the object (often referred to as the **resultant force**) is simply the sum of all of the force vectors. That is, the net effect of two or more forces acting on an object is the same as a single force (given by the sum) applied to the object.

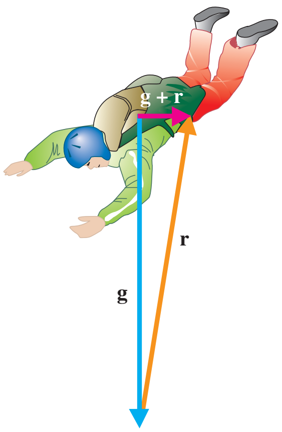


FIGURE 10.13
Forces on a sky diver

EXAMPLE 1.5 Finding the Net Force Acting on a Sky Diver

At a certain point during a jump, there are two principal forces acting on a sky diver: gravity exerting a force of 180 pounds straight down and air resistance exerting a force of 180 pounds up and 30 pounds to the right. What is the net force acting on the sky diver?

Solution We write the gravity force vector as $\mathbf{g} = \langle 0, -180 \rangle$ and the air resistance force vector as $\mathbf{r} = \langle 30, 180 \rangle$. The net force on the sky diver is the sum of the two forces, $\mathbf{g} + \mathbf{r} = \langle 30, 0 \rangle$. We illustrate the forces in Figure 10.13. Notice that at this point, the vertical forces are balanced, producing a "free-fall" vertically, so that the sky diver is neither accelerating nor decelerating vertically. The net force is purely horizontal, combating the horizontal motion of the sky diver after jumping from the plane. ■

When flying an airplane, it's important to consider the velocity of the air in which you are flying. Observe that the effect of the velocity of the air can be quite significant. Think about it this way: if a plane flies at 200 mph (its airspeed) and the air in which the plane is moving is itself moving at 35 mph in the same direction (i.e., there is a 35 mph tailwind), then the effective speed of the plane is 235 mph. Conversely, if the same 35 mph wind is moving in exactly the opposite direction (i.e., there is a 35 mph headwind), then the plane's effective speed is only 165 mph. If the wind is blowing in a direction that's not parallel to the plane's direction of travel, we need to add the velocity vectors corresponding to the plane's airspeed and the wind to get the effective velocity. We illustrate this in example 1.6.

EXAMPLE 1.6 Steering an Aircraft in a Headwind and a Crosswind

An airplane has an airspeed of 400 mph. Suppose that the wind velocity is given by the vector $\mathbf{w} = \langle 20, 30 \rangle$. In what direction should the airplane head in order to fly due west (i.e., in the direction of the unit vector $-\mathbf{i} = \langle -1, 0 \rangle$)?

Solution We illustrate the velocity vectors for the airplane and the wind in Figure 10.14. We let the airplane's velocity vector be $\mathbf{v} = \langle x, y \rangle$. The effective velocity of the plane is then $\mathbf{v} + \mathbf{w}$, which we set equal to $\langle c, 0 \rangle$, for some negative constant c. Since

$$\mathbf{v} + \mathbf{w} = \langle x + 20, y + 30 \rangle = \langle c, 0 \rangle,$$

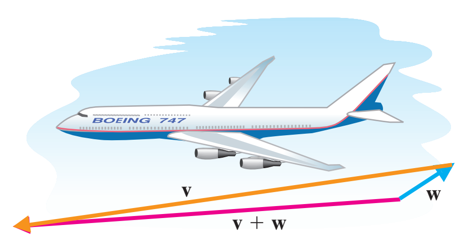


FIGURE 10.14
Forces on an airplane

we must have $x + 20 = c$ and $y + 30 = 0$, so that $y = -30$. Further, since the plane's airspeed is 400 mph, we must have $400 = \|\mathbf{v}\| = \sqrt{x^2 + y^2} = \sqrt{x^2 + 900}$. Squaring this gives us $x^2 + 900 = 160{,}000$, so that $x = -\sqrt{159{,}100}$. (We take the negative square root so that the plane heads westward.) Consequently, the plane should head in the direction of $\mathbf{v} = \langle -\sqrt{159{,}100}, -30 \rangle$, which points left and down, or southwest, at an angle of $\tan^{-1}(30/\sqrt{159{,}100}) \approx 4°$ below due west. ■

BEYOND FORMULAS

It is important that you understand vectors in both symbolic and graphical terms. Much of the notation introduced in this section is used to simplify calculations. However, the visualization of vectors as directed line segments is often the key to identifying which calculation is appropriate. For example, notice in example 1.6 that Figure 10.14 leads directly to an equation. The equation is more easily solved than the corresponding trigonometric problem implied by Figure 10.14. What are some of the ways in which symbolic and graphical representations reinforce each other in one-variable calculus?

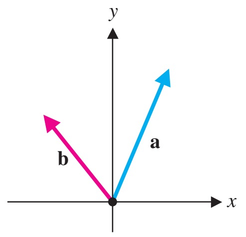
EXERCISES 10.1

WRITING EXERCISES

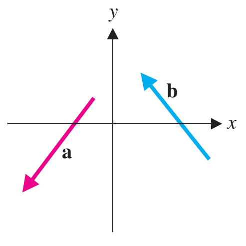
1. Discuss whether each of the following is a vector or a scalar quantity: force, area, weight, height, temperature, wind velocity.
2. Some athletes are blessed with "good acceleration." In calculus, we define acceleration as the rate of change of velocity. Keeping in mind that the velocity vector has magnitude (i.e., speed) and direction, discuss why the ability to accelerate rapidly is beneficial.
3. The location of the initial point of a vector is irrelevant. Using the example of a velocity vector, explain why we want to focus on the magnitude of the vector and its direction, but not on the initial point.
4. Describe the changes that occur when a vector is multiplied by a scalar $c \neq 0$. In your discussion, consider both positive and negative scalars, discuss changes both in the components of the vector and in its graphical representation, and consider the specific case of a velocity vector.

In exercises 1 and 2, sketch the vectors 2a, −3b, a + b and 2a − 3b.

1.



2.



In exercises 3–6, compute, a + b, a − 2b, 3a and ‖5b − 2a‖.

3. $\mathbf{a} = \langle 2, 4\rangle, \mathbf{b} = \langle 3, -1\rangle$
4. $\mathbf{a} = \langle 3, -2\rangle, \mathbf{b} = \langle 2, 0\rangle$
5. $\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = 3\mathbf{i} - \mathbf{j}$
6. $\mathbf{a} = -2\mathbf{i} + \mathbf{j}, \mathbf{b} = 3\mathbf{i}$
7. For exercises 3 and 4, illustrate the sum $\mathbf{a} + \mathbf{b}$ graphically.
8. For exercises 5 and 6, illustrate the difference $\mathbf{a} - \mathbf{b}$ graphically.

In exercises 9–14, determine whether the vectors a and b are parallel.

9. $\mathbf{a} = \langle 2, 1\rangle, \mathbf{b} = \langle -4, -2\rangle$
10. $\mathbf{a} = \langle 1, -2\rangle, \mathbf{b} = \langle 2, 1\rangle$
11. $\mathbf{a} = \langle -2, 3\rangle, \mathbf{b} = \langle 4, 6\rangle$
12. $\mathbf{a} = \langle 1, -2\rangle, \mathbf{b} = \langle -4, 8\rangle$
13. $\mathbf{a} = \mathbf{i} + 2\mathbf{j}, \mathbf{b} = 3\mathbf{i} + 6\mathbf{j}$
14. $\mathbf{a} = -2\mathbf{i} + \mathbf{j}, \mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$

In exercises 15–18, find the vector with initial point *A* and terminal point *B*.

15. $A = (2, 3), B = (5, 4)$
16. $A = (4, 3), B = (1, 0)$
17. $A = (-1, 2), B = (1, -1)$
18. $A = (1, 1), B = (-2, 4)$

In exercises 19–24, (a) find a unit vector in the same direction as the given vector and (b) write the given vector in polar form.

19. $\langle 4, -3\rangle$
20. $\langle 3, 6\rangle$
21. $2\mathbf{i} - 4\mathbf{j}$
22. $4\mathbf{i}$
23. from $(2, 1)$ to $(5, 2)$
24. from $(5, -1)$ to $(2, 3)$

In exercises 25–30, find a vector with the given magnitude in the same direction as the given vector.

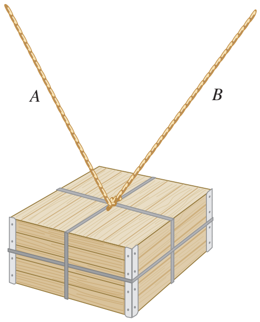
25. magnitude 3, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$
26. magnitude 4, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$
27. magnitude 29, $\mathbf{v} = \langle 2, 5\rangle$
28. magnitude 10, $\mathbf{v} = \langle 3, 1\rangle$
29. magnitude 4, $\mathbf{v} = \langle 3, 0\rangle$
30. magnitude 5, $\mathbf{v} = \langle 0, -2\rangle$
31. Suppose that there are two forces acting on a sky diver: gravity at 150 pounds down and air resistance at 140 pounds up and 20 pounds to the right. What is the net force acting on the sky diver?

32. Suppose that there are two forces acting on a sky diver: gravity at 200 pounds down and air resistance at 180 pounds up and 40 pounds to the right. What is the net force acting on the sky diver?

33. Suppose that there are two forces acting on a sky diver: gravity at 200 pounds down and air resistance. If the net force is 10 pounds down and 30 pounds to the right, what is the force of air resistance acting on the sky diver?

34. Suppose that there are two forces acting on a sky diver: gravity at 180 pounds down and air resistance. If the net force is 20 pounds down and 20 pounds to the left, what is the force of air resistance acting on the sky diver?

35. In the accompanying figure, two ropes are attached to a large crate. Suppose that rope A exerts a force of $\langle -164, 115\rangle$ pounds on the crate and rope B exerts a force of $\langle 177, 177\rangle$ pounds on the crate. If the crate weighs 275 pounds, what is the net force acting on the crate? Based on your answer, which way will the crate move?



36. Repeat exercise 35 with forces of $\langle -131, 92\rangle$ pounds from rope A and $\langle 92, 92\rangle$ from rope B.

37. The thrust of an airplane's engines produces a speed of 300 mph in still air. The wind velocity is given by $\langle 30, -20\rangle$. In what direction should the airplane head to fly due west?

38. The thrust of an airplane's engines produces a speed of 600 mph in still air. The wind velocity is given by $\langle -30, 60\rangle$. In what direction should the airplane head to fly due west?

39. The thrust of an airplane's engines produces a speed of 400 mph in still air. The wind velocity is given by $\langle -20, 30\rangle$. In what direction should the airplane head to fly due north?

40. The thrust of an airplane's engines produces a speed of 300 mph in still air. The wind velocity is given by $\langle 50, 0\rangle$. In what direction should the airplane head to fly due north?

41. A paperboy is riding at 10 ft/s on a bicycle and tosses a paper over his left shoulder at 50 ft/s. If the porch is 50 ft off the road, how far up the street should the paperboy release the paper to hit the porch?

42. A papergirl is riding at 12 ft/s on a bicycle and tosses a paper over her left shoulder at 48 ft/s. If the porch is 40 ft off the road, how far up the street should the papergirl release the paper to hit the porch?

43. The water from a fire hose exerts a force of 200 pounds on the person holding the hose. The nozzle of the hose weighs 20 pounds. What force is required to hold the hose horizontal? At what angle to the horizontal is this force applied?

44. Repeat exercise 43 for holding the hose at a 45° angle to the horizontal.

45. A person is paddling a kayak in a river with a current of 1 ft/s. The kayaker is aimed at the far shore, perpendicular to the current. The kayak's speed in still water would be 4 ft/s. Find the kayak's actual speed and the angle between the kayak's direction and the far shore.

46. For the kayak in exercise 45, find the direction that the kayaker would need to paddle to go straight across the river. How does this angle compare to the angle found in exercise 45?

47. If vector $\mathbf{a}$ has magnitude $\|\mathbf{a}\| = 3$ and vector $\mathbf{b}$ has magnitude $\|\mathbf{b}\| = 4$, what is the largest possible magnitude for the vector $\mathbf{a}+\mathbf{b}$? What is the smallest possible magnitude for the vector $\mathbf{a}+\mathbf{b}$? What will be the magnitude of $\mathbf{a}+\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular?

48. Use vectors to show that the points $(1, 2)$, $(3, 1)$, $(4, 3)$ and $(2, 4)$ form a parallelogram.

49. Prove the associativity property of Theorem 1.1.

50. Prove the distributive laws of Theorem 1.1.

51. For vectors $\mathbf{a} = \langle 2, 3\rangle$ and $\mathbf{b} = \langle 1, 4\rangle$, compare $\|\mathbf{a}+\mathbf{b}\|$ and $\|\mathbf{a}\| + \|\mathbf{b}\|$. Repeat this comparison for two other choices of $\mathbf{a}$ and $\mathbf{b}$. Use the sketch in Figure 10.6 to explain why $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$ for any vectors $\mathbf{a}$ and $\mathbf{b}$.

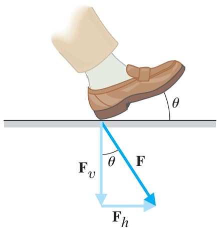
52. To prove that $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$ for $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$, start by showing that $2a_1a_2b_1b_2 \le a_1^2b_2^2 + a_2^2b_1^2$. [Hint: Compute $(a_1b_2 - a_2b_1)^2$.] Then, show that $a_1b_1 + a_2b_2 \le \sqrt{a_1^2+a_2^2}\sqrt{b_1^2+b_2^2}$. (Hint: Square both sides and use the previous result.) Finally, compute $\|\mathbf{a}+\mathbf{b}\|^2 - (\|\mathbf{a}\| + \|\mathbf{b}\|)^2$ and use the previous inequality to show that this is less than or equal to 0.

53. In exercises 51 and 52, you explored the inequality $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$. Use the geometric interpretation of Figure 10.6 to conjecture the circumstances under which $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$. Similarly, use a geometric interpretation to determine circumstances under which $\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$. In general, what is the relationship between $\|\mathbf{a}+\mathbf{b}\|^2$ and $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$ (i.e., which is larger)?

EXPLORATORY EXERCISES

1. The figure shows a foot striking the ground, exerting a force of $\mathbf{F}$ pounds at an angle of θ from the vertical. The force is resolved into vertical and horizontal components $\mathbf{F}_v$ and $\mathbf{F}_h$, respectively. The friction force between floor and foot is $\mathbf{F}_f$, where $\|\mathbf{F}_f\| = \mu\|\mathbf{F}_v\|$ for a positive constant μ known as the **coefficient of friction.** Explain why the foot will slip if $\|\mathbf{F}_h\| > \|\mathbf{F}_f\|$ and show that this happens if and only if

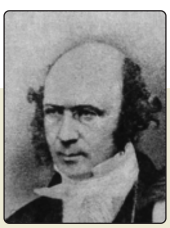
$\tan\theta > \mu$. Compare the angles θ at which slipping occurs for coefficients $\mu = 0.6$, $\mu = 0.4$ and $\mu = 0.2$.



2. The vectors **i** and **j** are not the only basis vectors that can be used. In fact, any two nonzero and nonparallel vectors can be used as basis vectors for two-dimensional space. To see this, define $\mathbf{a} = \langle 1, 1\rangle$ and $\mathbf{b} = \langle 1, -1\rangle$. To write the vector $\langle 5, 1\rangle$ in terms of these vectors, we want constants c_1 and c_2 such that $\langle 5, 1\rangle = c_1\mathbf{a} + c_2\mathbf{b}$. Show that this requires that $c_1 + c_2 = 5$ and $c_1 - c_2 = 1$, and then solve for c_1 and c_2. Show that any vector $\langle x, y\rangle$ can be represented uniquely in terms of **a** and **b**. Determine as completely as possible the set of all vectors **v** such that **a** and **v** form a basis.

10.2 VECTORS IN SPACE

We now extend several ideas from the two-dimensional Euclidean space, $\mathbb{R}^2$ to the three-dimensional Euclidean space, $\mathbb{R}^3$. We specify each point in three dimensions by an ordered triple (a, b, c), where the coordinates a, b and c represent the (signed) distance from the origin along each of three coordinates axes (x, y and z), as indicated in Figure 10.15a. This orientation of the axes is an example of a **right-handed** coordinate system. That is, if you align the fingers of your right hand along the positive x-axis and then curl them toward the positive y-axis, your thumb will point in the direction of the positive z-axis (see Figure 10.15b).



HISTORICAL NOTES

William Rowan Hamilton (1805–1865)
Irish mathematician who first defined and developed the theory of vectors. Hamilton was an outstanding student who was appointed Professor of Astronomy at Trinity College while still an undergraduate. After publishing several papers in the field of optics, Hamilton developed an innovative and highly influential approach to dynamics. He then became obsessed with the development of his theory of "quaternions" in which he also defined vectors. Hamilton thought that quaternions would revolutionize mathematical physics, but vectors have proved to be his most important contribution to mathematics.

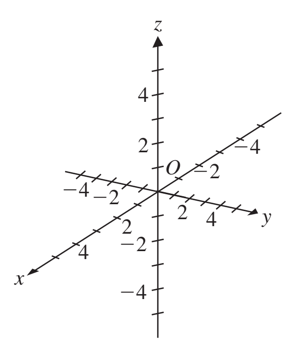


FIGURE 10.15a
Coordinate axes in $\mathbb{R}^3$

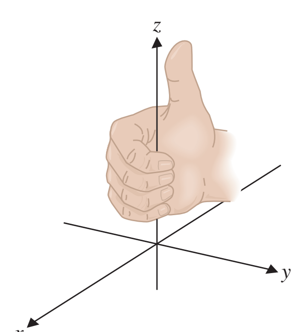


FIGURE 10.15b
Right-handed system

To locate the point $(a, b, c) \in \mathbb{R}^3$, where a, b and c are all positive, first move along the x-axis a distance of a units from the origin. This will put you at the point $(a, 0, 0)$. Continuing from this point, move parallel to the y-axis a distance of b units from $(a, 0, 0)$. This leaves you at the point $(a, b, 0)$. Finally, continuing from this point, move c units parallel to the z-axis. This is the location of the point (a, b, c) (see Figure 10.16).

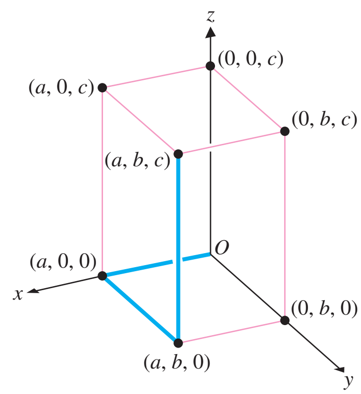


FIGURE 10.16
Locating the point (a, b, c)

EXAMPLE 2.1 Plotting Points in Three Dimensions

Plot the points (1, 2, 3), (3, −2, 4) and (−1, 3, −2).

Solution Working as indicated above, we see the points plotted in Figures 10.17a, 10.17b and 10.17c, respectively.

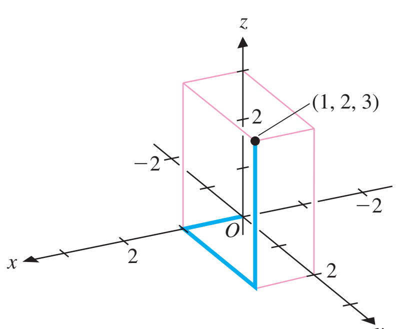


FIGURE 10.17a
The point (1, 2, 3)

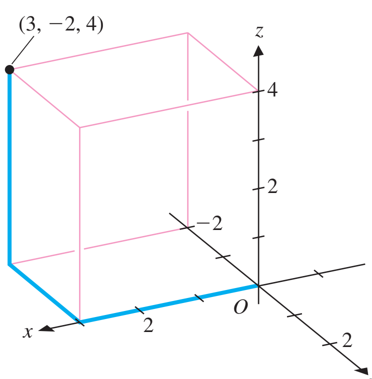


FIGURE 10.17b
The point (3, −2, 4)

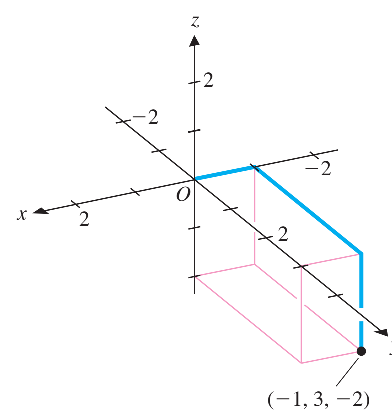


FIGURE 10.17c
The point (−1, 3, −2)

Recall that in $\mathbb{R}^2$, the coordinate axes divide the xy-plane into four quadrants. In a similar fashion, the three coordinate planes in $\mathbb{R}^3$ (the xy-plane, the yz-plane and the xz-plane) divide space into **eight octants** (see Figure 10.18 on the following page). The **first octant** is the one with $x > 0$, $y > 0$ and $z > 0$. We do not usually distinguish among the other seven octants.

We can compute the distance between two points in $\mathbb{R}^3$ by thinking of this as essentially a two-dimensional problem. For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbb{R}^3$, first locate the point $P_3(x_2, y_2, z_1)$ and observe that the three points are the vertices of a right triangle, with the right angle at the point P_3 (see Figure 10.19). The Pythagorean Theorem then says that the distance between P_1 and P_2, denoted $d\{P_1, P_2\}$, satisfies

$$d\{P_1, P_2\}^2 = d\{P_1, P_3\}^2 + d\{P_2, P_3\}^2. \tag{2.1}$$

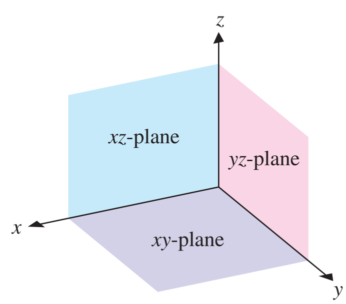


FIGURE 10.18
The coordinate planes

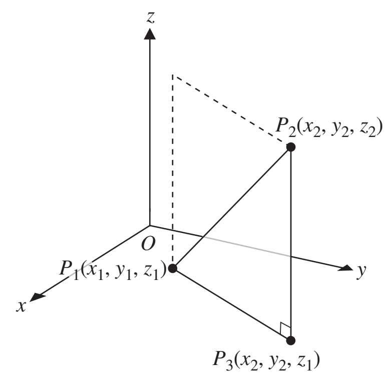


FIGURE 10.19
Distance in $\mathbb{R}^3$

Notice that P_2 lies directly above P_3 (or below, if $z_2 < z_1$), so that

$$d\{P_2, P_3\} = d\{(x_2, y_2, z_2), (x_2, y_2, z_1)\} = |z_2 - z_1|.$$

Since P_1 and P_3 both lie in the plane $z = z_1$, we can ignore the third coordinates of these points (since they're the same!) and use the usual two-dimensional distance formula:

$$d\{P_1, P_3\} = d\{(x_1, y_1, z_1), (x_2, y_2, z_1)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

From (2.1), we now have

$$\begin{aligned} d\{P_1, P_2\}^2 &= d\{P_1, P_3\}^2 + d\{P_2, P_3\}^2 \\ &= \left[\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right]^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2. \end{aligned}$$

Taking the square root of both sides gives us the **distance formula** for $\mathbb{R}^3$:

Distance in $\mathbb{R}^3$

$$\boxed{d\{(x_1, y_1, z_1), (x_2, y_2, z_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},} \tag{2.2}$$

which is a straightforward generalization of the familiar formula for the distance between two points in the plane.

EXAMPLE 2.2 Computing Distance in $\mathbb{R}^3$

Find the distance between the points $(1, -3, 5)$ and $(5, 2, -3)$.

Solution From (2.2), we have

$$\begin{aligned} d\{(1, -3, 5), (5, 2, -3)\} &= \sqrt{(5-1)^2 + [2-(-3)]^2 + (-3-5)^2} \\ &= \sqrt{4^2 + 5^2 + (-8)^2} = \sqrt{105}. \end{aligned}$$

Vectors in $\mathbb{R}^3$

As in two dimensions, vectors in three-dimensional space have both direction and magnitude. We again visualize vectors as directed line segments joining two points. A vector $\mathbf{v}$ is represented by any directed line segment with the appropriate magnitude and direction. The position vector $\mathbf{a}$ with terminal point at $A(a_1, a_2, a_3)$ (and initial point at the origin) is denoted by $\langle a_1, a_2, a_3 \rangle$ and is shown in Figure 10.20a.

We denote the set of all three-dimensional position vectors by

$$V_3 = \{\langle x, y, z \rangle \mid x, y, z \in \mathbb{R}\}.$$

The **magnitude** of the position vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ follows directly from the distance formula (2.2). We have

Magnitude of a vector

$$\boxed{\|\mathbf{a}\| = \|\langle a_1, a_2, a_3 \rangle\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.} \tag{2.3}$$

Note from Figure 10.20b that the vector with initial point at $P(a_1, a_2, a_3)$ and terminal point at $Q(b_1, b_2, b_3)$ corresponds to the position vector

$$\overrightarrow{PQ} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.$$

We define vector addition in V_3 just as we did in V_2, by drawing a parallelogram, as in Figure 10.20c.

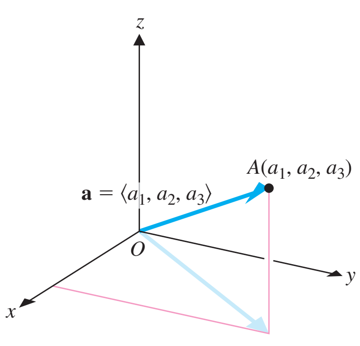


FIGURE 10.20a
Position vector in $\mathbb{R}^3$

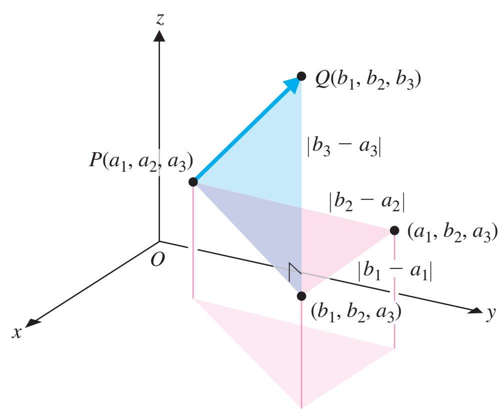


FIGURE 10.20b
Vector from P to Q

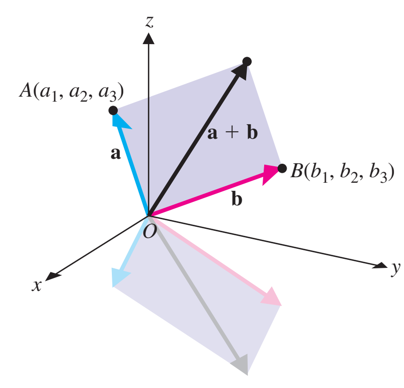


FIGURE 10.20c
Vector addition

Notice that for vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have

Vector addition

$$\boxed{\mathbf{a} + \mathbf{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.}$$

That is, as in V_2, addition of vectors in V_3 is done componentwise. Similarly, subtraction is done componentwise:

Vector subtraction

$$\boxed{\mathbf{a} - \mathbf{b} = \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle.}$$

Again as in V_2, for any scalar $c \in \mathbb{R}$, $c\mathbf{a}$ is a vector in the same direction as $\mathbf{a}$ when $c > 0$ and the opposite direction as $\mathbf{a}$ when $c < 0$. We have

Scalar multiplication

$$\boxed{c\mathbf{a} = c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle.}$$

Further, it's easy to show using (2.3), that

$$\|c\mathbf{a}\| = |c|\|\mathbf{a}\|.$$

We define the **zero vector 0** to be the vector in V_3 of length 0:

$$\mathbf{0} = \langle 0, 0, 0\rangle.$$

As in two dimensions, the zero vector has no particular direction. As we did in V_2, we define the **additive inverse** of a vector $\mathbf{a} \in V_3$ to be

$$-\mathbf{a} = -\langle a_1, a_2, a_3\rangle = \langle -a_1, -a_2, -a_3\rangle.$$

The rules of algebra established for vectors in V_2 hold verbatim in V_3, as seen in Theorem 2.1.

THEOREM 2.1

For any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in V_3, and any scalars d and e in $\mathbb{R}$, the following hold:

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
(ii) $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
(iii) $\mathbf{a} + \mathbf{0} = \mathbf{a}$ (zero vector)
(iv) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (additive inverse)
(v) $d(\mathbf{a} + \mathbf{b}) = d\mathbf{a} + d\mathbf{b}$ (distributive law)
(vi) $(d + e)\mathbf{a} = d\mathbf{a} + e\mathbf{a}$ (distributive law)
(vii) $(1)\mathbf{a} = \mathbf{a}$ (multiplication by 1) and
(viii) $(0)\mathbf{a} = \mathbf{0}$ (multiplication by 0).

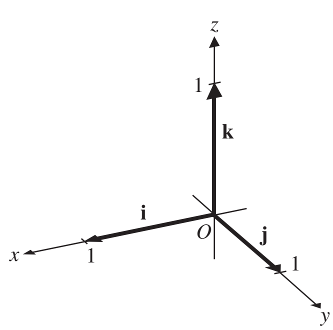


FIGURE 10.21
Standard basis for V_3

We leave the proof of Theorem 2.1 as an exercise.

Since V_3 is three-dimensional, the standard basis consists of three unit vectors, each lying along one of the three coordinate axes. We define these as a straightforward generalization of the standard basis for V_2 by

$$\boxed{\mathbf{i} = \langle 1, 0, 0\rangle, \quad \mathbf{j} = \langle 0, 1, 0\rangle \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1\rangle,}$$

as pictured in Figure 10.21. As in V_2, these basis vectors are unit vectors, since $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$. Also as in V_2, it is sometimes convenient to write position vectors in V_3 in terms of the standard basis. This is easily accomplished, as for any $\mathbf{a} \in V_3$, we can write

$$\boxed{\mathbf{a} = \langle a_1, a_2, a_3\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.}$$

If you're getting that déjà vu feeling that you've done all of this before, you're not imagining it. Vectors in V_3 follow all of the same rules as vectors in V_2. As a final note, observe that for any $\mathbf{a} = \langle a_1, a_2, a_3 \rangle \neq \mathbf{0}$, a unit vector in the same direction as $\mathbf{a}$ is given by

Unit vector

$$\boxed{\mathbf{u} = \frac{1}{\|\mathbf{a}\|}\mathbf{a}.} \tag{2.4}$$

The proof of this result is identical to the proof of the corresponding result for vectors in V_2, found in Theorem 1.2. Once again, it is often convenient to normalize a vector (i.e., produce a vector in the same direction, but with length 1).

EXAMPLE 2.3 Finding a Unit Vector

Find a unit vector in the same direction as $\langle 1, -2, 3 \rangle$ and write $\langle 1, -2, 3 \rangle$ as the product of its magnitude and a unit vector.

Solution First, we find the magnitude of the vector:

$$\|\langle 1, -2, 3 \rangle\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}.$$

From (2.4), we have that a unit vector having the same direction as $\langle 1, -2, 3 \rangle$ is given by

$$\mathbf{u} = \frac{1}{\sqrt{14}}\langle 1, -2, 3 \rangle = \left\langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle.$$

Further,
$$\langle 1, -2, 3 \rangle = \sqrt{14}\left\langle \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle.$$

Of course, going from two dimensions to three dimensions gives us a much richer geometry, with more interesting examples. For instance, we define a **sphere** to be the set of all points whose distance from a fixed point (the **center**) is constant.

EXAMPLE 2.4 Finding the Equation of a Sphere

Find the equation of the sphere of radius r centered at the point (a, b, c).

Solution The sphere consists of all points (x, y, z) whose distance from (a, b, c) is r, as illustrated in Figure 10.22. This says that

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = d\{(x, y, z), (a, b, c)\} = r.$$

Squaring both sides gives us

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2,}$$

the standard form of the **equation of a sphere.**

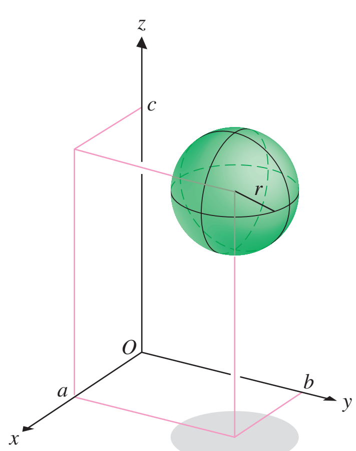


FIGURE 10.22
Sphere of radius r centered at (a, b, c)

You will occasionally need to recognize when a given equation represents a common geometric shape, as in example 2.5.

EXAMPLE 2.5 Finding the Center and Radius of a Sphere

Find the geometric shape described by the equation:

$$0 = x^2 + y^2 + z^2 - 4x + 8y - 10z + 36.$$

Solution Completing the squares in each variable, we have

$$\begin{aligned} 0 &= (x^2 - 4x + 4) - 4 + (y^2 + 8y + 16) - 16 + (z^2 - 10z + 25) - 25 + 36 \\ &= (x-2)^2 + (y+4)^2 + (z-5)^2 - 9. \end{aligned}$$

Adding 9 to both sides gives us

$$3^2 = (x-2)^2 + (y+4)^2 + (z-5)^2,$$

which is the equation of a sphere of radius 3 centered at the point $(2, -4, 5)$. ■

EXERCISES 10.2

WRITING EXERCISES

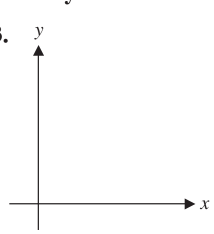
1. Visualize the circle $x^2 + y^2 = 1$. With three-dimensional axes oriented as in Figure 10.15a, describe how to sketch this circle in the plane $z = 0$. Then, describe how to sketch the parabola $y = x^2$ in the plane $z = 0$. In general, explain how to translate a two-dimensional curve into a three-dimensional sketch.
2. It is difficult, if not impossible, for most people to visualize what points in four dimensions would look like. Nevertheless, it is easy to generalize the distance formula to four dimensions. Describe what the distance formula looks like in general dimension n, for $n \geq 4$.
3. It is very important to be able to quickly and accurately visualize three-dimensional relationships. In three dimensions, describe how many lines are perpendicular to the unit vector **i**. Describe all lines that are perpendicular to **i** and that pass through the origin. In three dimensions, describe how many planes are perpendicular to the unit vector **i**. Describe all planes that are perpendicular to **i** and that contain the origin.
4. In three dimensions, describe all planes that contain a given vector **a**. Describe all planes that contain two given vectors **a** and **b** (where **a** and **b** are not parallel). Describe all planes that contain a given vector **a** and pass through the origin. Describe all planes that contain two given (nonparallel) vectors **a** and **b** and pass through the origin.

In exercises 1 and 2, plot the indicated points.

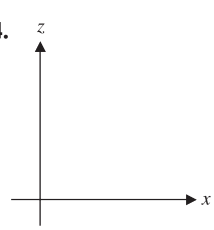
1. (a) $(2, 1, 5)$ (b) $(3, 1, -2)$ (c) $(-1, 2, -4)$
2. (a) $(-2, 1, 2)$ (b) $(2, -3, -1)$ (c) $(3, -2, 2)$

In exercises 3 and 4, sketch the third axis to make *xyz* a right-handed system.

3.



4.



In exercises 5–8, find the distance between the given points.

5. $(2, 1, 2), (5, 5, 2)$
6. $(1, 2, 0), (7, 10, 0)$
7. $(-1, 0, 2), (1, 2, 3)$
8. $(3, 1, 0), (1, 3, -4)$

In exercises 9–12, compute $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - 3\mathbf{b}$ and $\|4\mathbf{a} + 2\mathbf{b}\|$.

9. $\mathbf{a} = \langle 2, 1, -2 \rangle$, $\mathbf{b} = \langle 1, 3, 0 \rangle$
10. $\mathbf{a} = \langle -1, 0, 2 \rangle$, $\mathbf{b} = \langle 4, 3, 2 \rangle$
11. $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$
12. $\mathbf{a} = \mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

In exercises 13–18, (a) find two unit vectors parallel to the given vector and (b) write the given vector as the product of its magnitude and a unit vector.

13. $\langle 3, 1, 2 \rangle$
14. $\langle 2, -4, 6 \rangle$
15. $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
16. $4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
17. From $(1, 2, 3)$ to $(3, 2, 1)$
18. From $(1, 4, 1)$ to $(3, 2, 2)$

In exercises 19–22, find a vector with the given magnitude and in the same direction as the given vector.

19. Magnitude 6, $\mathbf{v} = \langle 2, 2, -1 \rangle$
20. Magnitude 10, $\mathbf{v} = \langle 3, 0, -4 \rangle$
21. Magnitude 4, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
22. Magnitude 3, $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

In exercises 23–26, find an equation of the sphere with radius r and center (a, b, c).

23. $r = 2, (a, b, c) = (3, 1, 4)$
24. $r = 3, (a, b, c) = (2, 0, 1)$
25. $r = \sqrt{5}, (a, b, c) = (\pi, 1, -3)$
26. $r = \sqrt{7}, (a, b, c) = (1, 3, 4)$

In exercises 27–30, identify the geometric shape described by the given equation.

27. $(x-1)^2 + y^2 + (z+2)^2 = 4$

28. $x^2 + (y-1)^2 + (z-4)^2 = 2$

29. $x^2 - 2x + y^2 + z^2 - 4z = 0$

30. $x^2 + x + y^2 - y + z^2 = \frac{7}{2}$

In exercises 31–34, identify the plane as parallel to the *xy*-plane, *xz*-plane or *yz*-plane and sketch a graph.

31. $y = 4$

32. $x = -2$

33. $z = -1$

34. $z = 3$

In exercises 35–38, give an equation (e.g., $z = 0$) for the indicated figure.

35. xz-plane

36. xy-plane

37. yz-plane

38. x-axis

39. Prove the commutative property of Theorem 2.1.

40. Prove the associative property of Theorem 2.1.

41. Prove the distributive properties of Theorem 2.1.

42. Prove the multiplicative properties of Theorem 2.1.

43. Find the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and determine whether the points $P = (2, 3, 1)$, $Q = (4, 2, 2)$ and $R = (8, 0, 4)$ are colinear (on the same line).

44. Find the displacement vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ and determine whether the points $P = (2, 3, 1)$, $Q = (0, 4, 2)$ and $R = (4, 1, 4)$ are colinear (on the same line).

45. Use vectors to determine whether the points $(0, 1, 1)$, $(2, 4, 2)$ and $(3, 1, 4)$ form an equilateral triangle.

46. Use vectors to determine whether the points $(2, 1, 0)$, $(4, 1, 2)$ and $(4, 3, 0)$ form an equilateral triangle.

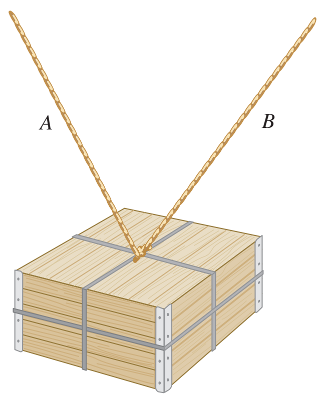
47. Use vectors and the Pythagorean Theorem to determine whether the points $(3, 1, -2)$, $(1, 0, 1)$ and $(4, 2, -1)$ form a right triangle.

48. Use vectors and the Pythagorean Theorem to determine whether the points $(1, -2, 1)$, $(4, 3, 2)$ and $(7, 1, 3)$ form a right triangle.

49. Use vectors to determine whether the points $(2, 1, 0)$, $(5, -1, 2)$, $(0, 3, 3)$ and $(3, 1, 5)$ form a square.

50. Use vectors to determine whether the points $(1, -2, 1)$, $(-2, -1, 2)$, $(2, 0, 2)$ and $(-1, 1, 3)$ form a square.

51. In the accompanying figure, two ropes are attached to a 500-pound crate. Rope A exerts a force of $\langle 10, -130, 200\rangle$ pounds on the crate, and rope B exerts a force of $\langle -20, 180, 160\rangle$ pounds on the crate. If no further ropes are added, find the net force on the crate and the direction it will move. If a third rope C is added to balance the crate, what force must this rope exert on the crate?



52. For the crate in exercise 51, suppose that the crate weighs only 300 pounds and the goal is to move the crate up and to the right with a constant force of $\langle 0, 30, 20\rangle$ pounds. If a third rope is added to accomplish this, what force must the rope exert on the crate?

53. The thrust of an airplane's engine produces a speed of 600 mph in still air. The plane is aimed in the direction of $\langle 2, 2, 1\rangle$ and the wind velocity is $\langle 10, -20, 0\rangle$ mph. Find the velocity vector of the plane with respect to the ground and find the speed.

54. The thrust of an airplane's engine produces a speed of 700 mph in still air. The plane is aimed in the direction of $\langle 6, -3, 2\rangle$ but its velocity with respect to the ground is $\langle 580, -330, 160\rangle$ mph. Find the wind velocity.

In exercises 55–62, you are asked to work with vectors of dimension higher than three. Use rules analogous to those introduced for two and three dimensions.

55. $\langle 2, 3, 1, 5\rangle + 2\langle 1, -2, 3, 1\rangle$

56. $2\langle 3, -2, 1, 0\rangle - \langle 2, 1, -2, 1\rangle$

57. $\langle 3, -2, 4, 1, 0, 2\rangle - 3\langle 1, 2, -2, 0, 3, 1\rangle$

58. $\langle 2, 1, 3, -2, 4, 1, 0, 2\rangle + 2\langle 3, 1, 1, 2, -2, 0, 3, 1\rangle$

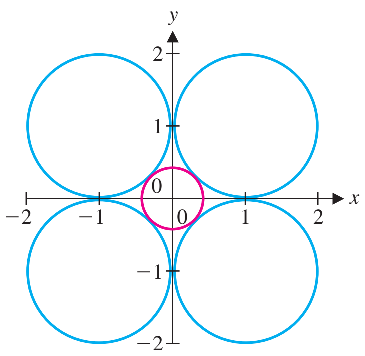
59. $\|\mathbf{a}\|$ for $\mathbf{a} = \langle 3, 1, -2, 4, 1\rangle$

60. $\|\mathbf{a}\|$ for $\mathbf{a} = \langle 1, 0, -3, -2, 4, 1\rangle$

61. $\|\mathbf{a} + \mathbf{b}\|$ for $\mathbf{a} = \langle 1, -2, 4, 1\rangle$ and $\mathbf{b} = \langle -1, 4, 2, -4\rangle$

62. $\|\mathbf{a} - 2\mathbf{b}\|$ for $\mathbf{a} = \langle 2, 1, -2, 4, 1\rangle$ and $\mathbf{b} = \langle 3, -1, 4, 2, -4\rangle$

63. Take four unit circles and place them tangent to the x- and y-axes as shown. Find the radius of the inscribed circle (shown in the accompanying figure in red).



64. Extend exercise 63 to three dimensions by finding the radius of the sphere inscribed by eight unit spheres that are tangent to the coordinate planes.

65. Generalize the results of exercises 63 and 64 to n dimensions. Show that for $n \geq 10$, the inscribed hypersphere is actually *not* contained in the "box" $-2 \leq x \leq 2, -2 \leq y \leq 2$ and so on, that contains all of the individual hyperspheres.

EXPLORATORY EXERCISES

1. Find an equation describing all points equidistant from $A = (0, 1, 0)$ and $B = (2, 4, 4)$ and sketch a graph. Based on your graph, describe the relationship between the displacement vector $\overrightarrow{AB} = \langle 2, 3, 4 \rangle$ and your graph. Simplify your equation for the three-dimensional surface until 2, 3 and 4 appear as coefficients of x, y and z. Use what you have learned to quickly write down an equation for the set of all points equidistant from $A = (0, 1, 0)$ and $C = (5, 2, 3)$.

2. In this exercise, you will try to identify the three-dimensional surface defined by the equation $a(x - 1) + b(y - 2) + c(z - 3) = 0$ for nonzero constants a, b and c. First, show that $(1, 2, 3)$ is one point on the surface. Then, show that any point that is equidistant from the points $(1 + a, 2 + b, 3 + c)$ and $(1 - a, 2 - b, 3 - c)$ is on the surface. Use this geometric fact to identify the surface.

10.3 THE DOT PRODUCT

In sections 10.1 and 10.2, we defined vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and examined many of the properties of vectors, including how to add and subtract two vectors. It turns out that two different kinds of products involving vectors have proved to be useful: the dot product (or scalar product) and the cross product (or vector product). We introduce the first of these two products in this section.

DEFINITION 3.1

The **dot product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in V_3 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{3.1}$$

Likewise, the dot product of two vectors in V_2 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2.$$

Be sure to notice that the dot product of two vectors is a *scalar* (i.e., a number, not a vector). For this reason, the dot product is also called the **scalar product.**

EXAMPLE 3.1 Computing a Dot Product in $\mathbb{R}^3$

Compute the dot product $\mathbf{a} \cdot \mathbf{b}$ for $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 5, -3, 4 \rangle$.

Solution We have

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, 3 \rangle \cdot \langle 5, -3, 4 \rangle = (1)(5) + (2)(-3) + (3)(4) = 11. \blacksquare$$

Certainly, dot products are very simple to compute, whether a vector is written in component form or written in terms of the standard basis vectors, as in example 3.2.

EXAMPLE 3.2 Computing a Dot Product in $\mathbb{R}^2$

Find the dot product of the two vectors $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 6\mathbf{j}$.

Solution We have

$$\mathbf{a} \cdot \mathbf{b} = (2)(3) + (-5)(6) = 6 - 30 = -24. \blacksquare$$

The dot product in V_2 or V_3 satisfies the following simple properties.

REMARK 3.1

Since vectors in V_2 can be thought of as special cases of vectors in V_3 (where the third component is zero), all of the results we prove for vectors in V_3 hold equally for vectors in V_2.

THEOREM 3.1

For vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and any scalar d, the following hold:

(i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutativity)
(ii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive law)
(iii) $(d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b})$
(iv) $\mathbf{0} \cdot \mathbf{a} = 0$ and
(v) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

PROOF

We prove (i) and (v) for $\mathbf{a}, \mathbf{b} \in V_3$. The remaining parts are left as exercises.

(i) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have from (3.1) that

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= b_1 a_1 + b_2 a_2 + b_3 a_3 = \mathbf{b} \cdot \mathbf{a}, \end{aligned}$$

since multiplication of real numbers is commutative.

(v) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, we have

$$\mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = \|\mathbf{a}\|^2. \blacksquare$$

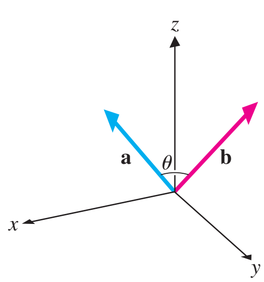


FIGURE 10.23a
The angle between two vectors

Notice that properties (i)–(iv) of Theorem 3.1 are also properties of multiplication of real numbers. This is why we use the word *product* in dot product. However, there are some properties of multiplication of real numbers not shared by the dot product. For instance, we will see that $\mathbf{a} \cdot \mathbf{b} = 0$ does not imply that either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

For two *nonzero* vectors $\mathbf{a}$ and $\mathbf{b}$ in V_3, we define the **angle** θ $(0 \le \theta \le \pi)$ **between the vectors** to be the smaller angle between $\mathbf{a}$ and $\mathbf{b}$, formed by placing their initial points at the same point, as illustrated in Figure 10.23a.

Notice that if $\mathbf{a}$ and $\mathbf{b}$ have the *same* direction, then $\theta = 0$. If $\mathbf{a}$ and $\mathbf{b}$ have *opposite* directions, then $\theta = \pi$. We say that $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** (or **perpendicular**) if $\theta = \frac{\pi}{2}$. We consider the zero vector $\mathbf{0}$ to be orthogonal to every vector. The general case is stated in Theorem 3.2.

THEOREM 3.2

Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta. \tag{3.2}$$

PROOF

We must prove the theorem for three separate cases.

(i) If $\mathbf{a}$ and $\mathbf{b}$ have the *same direction,* then $\mathbf{b} = c\mathbf{a}$, for some scalar $c > 0$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = 0$. This says that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{a}) = c\mathbf{a} \cdot \mathbf{a} = c\|\mathbf{a}\|^2.$$

Further,

$$\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta = \|\mathbf{a}\||c|\|\mathbf{a}\| \cos 0 = c\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{b},$$

since for $c > 0$, we have $|c| = c$.

(ii) If $\mathbf{a}$ and $\mathbf{b}$ have the opposite direction, the proof is nearly identical to case (i) above and we leave the details as an exercise.

(iii) If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then we have that $0 < \theta < \pi$, as shown in Figure 10.23b. Recall that the Law of Cosines allows us to relate the lengths of the sides of triangles like the one in Figure 10.23b. We have

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta. \tag{3.3}$$

Now, observe that

$$\begin{aligned}
\|\mathbf{a} - \mathbf{b}\|^2 &= \|\langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle\|^2 \\
&= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\
&= \left(a_1^2 - 2a_1b_1 + b_1^2\right) + \left(a_2^2 - 2a_2b_2 + b_2^2\right) + \left(a_3^2 - 2a_3b_3 + b_3^2\right) \\
&= \left(a_1^2 + a_2^2 + a_3^2\right) + \left(b_1^2 + b_2^2 + b_3^2\right) - 2(a_1b_1 + a_2b_2 + a_3b_3) \\
&= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b}
\end{aligned} \tag{3.4}$$

Equating the right-hand sides of (3.3) and (3.4), we get (3.2), as desired. ■

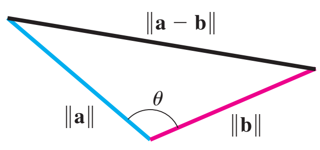


FIGURE 10.23b
The angle between two vectors

We can use (3.2) to find the angle between two vectors, as in example 3.3.

EXAMPLE 3.3 Finding the Angle between Two Vectors

Find the angle between the vectors $\mathbf{a} = \langle 2, 1, -3\rangle$ and $\mathbf{b} = \langle 1, 5, 6\rangle$.

Solution From (3.2), we have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{-11}{\sqrt{14}\sqrt{62}}.$$

It follows that $\theta = \cos^{-1}\left(\dfrac{-11}{\sqrt{14}\sqrt{62}}\right) \approx 1.953$ (radians)

(or about $112°$), since $0 \le \theta \le \pi$ and the inverse cosine function returns an angle in this range. ■

The following result is an immediate and important consequence of Theorem 3.2.

COROLLARY 3.1

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

PROOF

First, observe that if either $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, then $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, as the zero vector is considered orthogonal to every vector. If $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors and if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we have from Theorem 3.2 that

$$\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta = \mathbf{a} \cdot \mathbf{b} = 0$$

if and only if $\cos\theta = 0$ (since neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector). This occurs if and only if $\theta = \frac{\pi}{2}$, which is equivalent to having $\mathbf{a}$ and $\mathbf{b}$ orthogonal and so, the result follows. ■

EXAMPLE 3.4 Determining Whether Two Vectors Are Orthogonal

Determine whether the following pairs of vectors are orthogonal: (a) $\mathbf{a} = \langle 1, 3, -5\rangle$ and $\mathbf{b} = \langle 2, 3, 10\rangle$ and (b) $\mathbf{a} = \langle 4, 2, -1\rangle$ and $\mathbf{b} = \langle 2, 3, 14\rangle$.

Solution For (a), we have:

$$\mathbf{a} \cdot \mathbf{b} = 2 + 9 - 50 = -39 \neq 0,$$

so that $\mathbf{a}$ and $\mathbf{b}$ are *not* orthogonal.

For (b), we have

$$\mathbf{a} \cdot \mathbf{b} = 8 + 6 - 14 = 0,$$

so that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, in this case. ■

The following two results provide us with some powerful tools for comparing the magnitudes of vectors.

THEOREM 3.3 (Cauchy-Schwartz Inequality)

For any vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\|\,\|\mathbf{b}\|. \tag{3.5}$$

PROOF

If either $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, notice that (3.5) simply says that $0 \le 0$, which is certainly true. On the other hand, if neither $\mathbf{a}$ nor $\mathbf{b}$ is the zero vector, we have from (3.2) that

$$|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,|\cos\theta| \le \|\mathbf{a}\|\,\|\mathbf{b}\|,$$

since $|\cos\theta| \le 1$ for all values of θ. ■

One benefit of the Cauchy-Schwartz Inequality is that it allows us to prove the following very useful result. If you were going to learn only one inequality in your lifetime, this is probably the one you would want to learn.

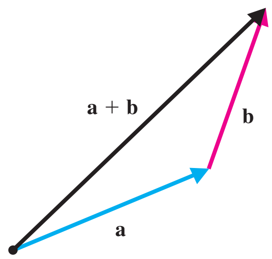


FIGURE 10.24
The Triangle Inequality

THEOREM 3.4 (The Triangle Inequality)

For any vectors **a** and **b**,

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|. \tag{3.6}$$

Before we prove the theorem, consider the triangle formed by the vectors **a**, **b** and $\mathbf{a}+\mathbf{b}$, shown in Figure 10.24. Notice that the Triangle Inequality says that the length of the vector $\mathbf{a}+\mathbf{b}$ never exceeds the sum of the individual lengths of **a** and **b**.

PROOF

From Theorem 3.1 (i), (ii) and (v), we have

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\ &= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2.\end{aligned}$$

From the Cauchy-Schwartz Inequality (3.5), we have $\mathbf{a}\cdot\mathbf{b} \leq |\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$ and so, we have

$$\begin{aligned}\|\mathbf{a}+\mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{b} + \|\mathbf{b}\|^2\\ &\leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.\end{aligned}$$

Taking square roots gives us (3.6). ■

TODAY IN MATHEMATICS

Lene Hau (1959–)
A Danish mathematician and physicist known for her experiments to slow down and stop light. Although neither of her parents had a background in science or mathematics, she says that as a student, "I loved mathematics. I would rather do mathematics than go to the movies in those days. But after awhile, I discovered quantum mechanics and I've been hooked ever since." Hau credits a culture of scientific achievement with her success. "I was lucky to be a Dane. Denmark has a long scientific tradition that included the great Niels Bohr. . . . In Denmark, physics is widely respected by laymen as well as scientists and laymen contribute to physics."

Components and Projections

Think about the case where a vector represents a force. Often, it's impractical to exert a force in the direction you'd like. For instance, in pulling a child's wagon, we exert a force in the direction determined by the position of the handle, instead of in the direction of motion. (See Figure 10.25.) An important question is whether there is a force of smaller magnitude that can be exerted in a different direction and still produce the same effect on the wagon. Notice that it is the horizontal portion of the force that most directly contributes to the motion of the wagon. (The vertical portion of the force only acts to reduce friction.) We now consider how to compute such a component of a force.

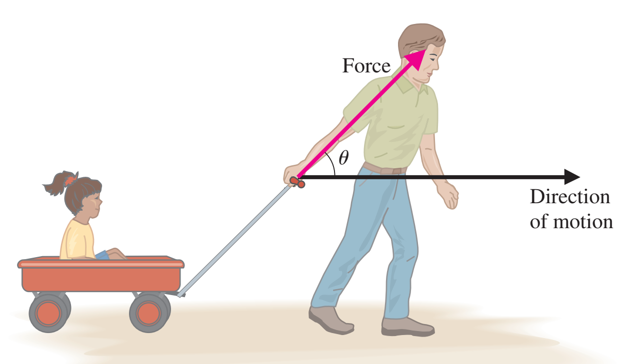


FIGURE 10.25
Pulling a wagon

For any two nonzero position vectors **a** and **b**, let the angle between the vectors be θ. If we drop a perpendicular line segment from the terminal point of **a** to the line containing the vector **b**, then from elementary trigonometry, the base of the triangle (in the case where $0 < \theta < \frac{\pi}{2}$) has length given by $\|\mathbf{a}\| \cos\theta$ (see Figure 10.26a).

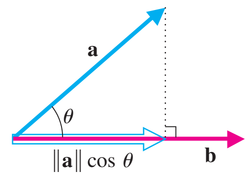


FIGURE 10.26a
$\text{comp}_{\mathbf{b}}\, \mathbf{a}$, for $0 < \theta < \frac{\pi}{2}$

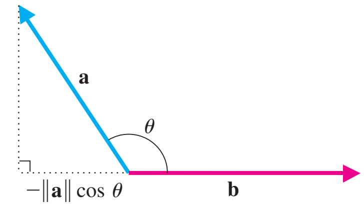


FIGURE 10.26b
$\text{comp}_{\mathbf{b}}\, \mathbf{a}$, for $\frac{\pi}{2} < \theta < \pi$

On the other hand, notice that if $\frac{\pi}{2} < \theta < \pi$, the length of the base is given by $-\|\mathbf{a}\| \cos\theta$ (see Figure 10.26b). In either case, we refer to $\|\mathbf{a}\| \cos\theta$ as the **component** of **a** along **b**, denoted $\text{comp}_{\mathbf{b}}\, \mathbf{a}$. Using (3.2), observe that we can rewrite this as

$$\begin{aligned} \text{comp}_{\mathbf{b}}\, \mathbf{a} &= \|\mathbf{a}\| \cos\theta = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\|}{\|\mathbf{b}\|} \cos\theta \\ &= \frac{1}{\|\mathbf{b}\|} \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta = \frac{1}{\|\mathbf{b}\|} \mathbf{a} \cdot \mathbf{b} \end{aligned}$$

Component of **a** along **b**

or

$$\boxed{\text{comp}_{\mathbf{b}}\, \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}.} \tag{3.7}$$

Notice that $\text{comp}_{\mathbf{b}}\, \mathbf{a}$ is a scalar and that we divide the dot product in (3.7) by $\|\mathbf{b}\|$ and not by $\|\mathbf{a}\|$. One way to keep this straight is to recognize that the components in Figures 10.26a and 10.26b depend on how long **a** is but not on how long **b** is. We can view (3.7) as the dot product of the vector **a** and a unit vector in the direction of **b**, given by $\dfrac{\mathbf{b}}{\|\mathbf{b}\|}$.

Once again, consider the case where the vector **a** represents a force. Rather than the component of **a** along **b**, we are often interested in finding a force vector parallel to **b** having the same component along **b** as **a**. We call this vector the **projection** of **a** onto **b**, denoted $\mathbf{proj_b\, a}$, as indicated in Figures 10.27a and 10.27b. Since the projection has magnitude $|\text{comp}_{\mathbf{b}}\, \mathbf{a}|$ and points in the direction of **b**, for $0 < \theta < \frac{\pi}{2}$ and opposite **b**, for $\frac{\pi}{2} < \theta < \pi$,

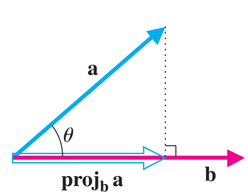


FIGURE 10.27a
$\mathbf{proj_b\, a}$, for $0 < \theta < \frac{\pi}{2}$

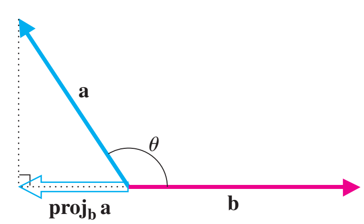


FIGURE 10.27b
$\mathbf{proj_b\, a}$, for $\frac{\pi}{2} < \theta < \pi$

we have from (3.7) that

$$\mathbf{proj_b\, a} = (\mathrm{comp}_\mathbf{b}\,\mathbf{a})\frac{\mathbf{b}}{\|\mathbf{b}\|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right)\frac{\mathbf{b}}{\|\mathbf{b}\|},$$

Projection of **a** onto **b**

or

$$\boxed{\mathbf{proj_b\, a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b},} \tag{3.8}$$

where $\dfrac{\mathbf{b}}{\|\mathbf{b}\|}$ represents a unit vector in the direction of **b**.

In example 3.5, we illustrate the process of finding components and projections.

CAUTION

Be careful to distinguish between the *projection* of **a** onto **b** (a vector) and the *component* of **a** along **b** (a scalar). It is very common to confuse the two.

EXAMPLE 3.5 Finding Components and Projections

For $\mathbf{a} = \langle 2, 3\rangle$ and $\mathbf{b} = \langle -1, 5\rangle$, find the component of **a** along **b** and the projection of **a** onto **b**.

Solution From (3.7), we have

$$\mathrm{comp}_\mathbf{b}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \frac{\langle 2,3\rangle\cdot\langle -1,5\rangle}{\|\langle -1,5\rangle\|} = \frac{-2+15}{\sqrt{1+5^2}} = \frac{13}{\sqrt{26}}.$$

Similarly, from (3.8), we have

$$\begin{aligned}\mathbf{proj_b\, a} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right)\frac{\mathbf{b}}{\|\mathbf{b}\|} = \left(\frac{13}{\sqrt{26}}\right)\frac{\langle -1,5\rangle}{\sqrt{26}}\\ &= \frac{13}{26}\langle -1,5\rangle = \frac{1}{2}\langle -1,5\rangle = \left\langle -\frac{1}{2}, \frac{5}{2}\right\rangle.\end{aligned}$$

We leave it as an exercise to show that, in general, $\mathrm{comp}_\mathbf{b}\,\mathbf{a} \neq \mathrm{comp}_\mathbf{a}\,\mathbf{b}$ and $\mathbf{proj_b\, a} \neq \mathbf{proj_a b}$. One reason for needing to consider components of a vector in a given direction is to compute work, as we see in example 3.6.

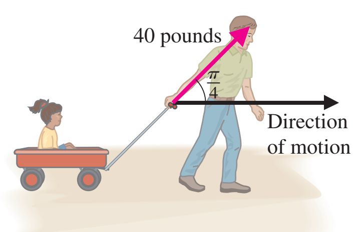


FIGURE 10.28
Pulling a wagon

EXAMPLE 3.6 Calculating Work

You exert a constant force of 40 pounds in the direction of the handle of the wagon pictured in Figure 10.28. If the handle makes an angle of $\frac{\pi}{4}$ with the horizontal and you pull the wagon along a flat surface for 1 mile (5280 feet), find the work done.

Solution First, recall from our discussion in Chapter 5 that if we apply a constant force F for a distance d, the work done is given by $W = Fd$. Unfortunately, the force exerted in the direction of motion is not given. Since the magnitude of the force is 40, the force vector must be

$$\mathbf{F} = 40\left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right\rangle = 40\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \langle 20\sqrt{2}, 20\sqrt{2}\rangle.$$

The force exerted in the direction of motion is simply the component of the force along the vector **i** (that is, the horizontal component of **F**) or $20\sqrt{2}$. The work done is then

$$W = Fd = 20\sqrt{2}\,(5280) \approx 149{,}341 \text{ foot-pounds}.$$

More generally, if a constant force **F** moves an object from point P to point Q, we refer to the vector $\mathbf{d} = \overrightarrow{PQ}$ as the **displacement vector.** The work done is the product of the component of **F** along **d** and the distance:

$$W = \text{comp}_{\mathbf{d}}\,\mathbf{F}\|\mathbf{d}\|$$
$$= \frac{\mathbf{F}\cdot\mathbf{d}}{\|\mathbf{d}\|}\cdot\|\mathbf{d}\| = \mathbf{F}\cdot\mathbf{d}.$$

Here, this gives us

$$W = \langle 20\sqrt{2}, 20\sqrt{2}\rangle \cdot \langle 5280, 0\rangle = 20\sqrt{2}\,(5280), \text{ as before.}$$

BEYOND FORMULAS

You can think of the dot product as a shortcut for computing components and projections. The dot product test for perpendicular vectors follows directly from this interpretation. In general, components and projections are used to isolate a particular portion of a large problem for detailed analysis. This sort of reductionism is central to much of modern science.

EXERCISES 10.3

WRITING EXERCISES

1. Explain in words why the Triangle Inequality is true.

2. The dot product is called a "product" because the properties listed in Theorem 3.1 are true for multiplication of real numbers. Two other properties of multiplication of real numbers involve factoring: (1) if $ab = ac$ $(a \neq 0)$ then $b = c$ and (2) if $ab = 0$ then $a = 0$ or $b = 0$. Discuss the extent to which these properties are true for the dot product.

3. On several occasions you have been asked to find unit vectors. To understand the importance of unit vectors, first identify the simplification in formulas for finding the angle between vectors and for finding the component of a vector, if the vectors are unit vectors. There is also a theoretical benefit to using unit vectors. Compare the number of vectors in a particular direction to the number of unit vectors in that direction. (For this reason, unit vectors are sometimes called **direction vectors.**)

4. It is important to understand why work is computed using only the component of force in the direction of motion. To take a simple example, suppose you are pushing on a door to try to close it. If you are pushing on the edge of the door straight at the door hinges, are you accomplishing anything useful? In this case, the work done would be zero. If you change the angle at which you push very slightly, what happens? Discuss what happens as you change that angle more and more (up to 90°). As the angle increases, discuss how the component of force in the direction of motion changes and how the work done changes.

In exercises 1–6, compute $\mathbf{a} \cdot \mathbf{b}$.

1. $\mathbf{a} = \langle 3, 1\rangle$, $\mathbf{b} = \langle 2, 4\rangle$
2. $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$
3. $\mathbf{a} = \langle 2, -1, 3\rangle$, $\mathbf{b} = \langle 0, 2, 4\rangle$
4. $\mathbf{a} = \langle 3, 2, 0\rangle$, $\mathbf{b} = \langle -2, 4, 3\rangle$
5. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} - \mathbf{k}$
6. $\mathbf{a} = 3\mathbf{i} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j}$

In exercises 7–10, compute the angle between the vectors.

7. $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$
8. $\mathbf{a} = \langle 2, 0, -2\rangle$, $\mathbf{b} = \langle 0, -2, 4\rangle$
9. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
10. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{k}$

In exercises 11–14, determine whether the vectors are orthogonal.

11. $\mathbf{a} = \langle 2, -1\rangle$, $\mathbf{b} = \langle 2, 4\rangle$
12. $\mathbf{a} = \langle 4, -1, 1\rangle$, $\mathbf{b} = \langle 2, 4, 4\rangle$
13. $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$
14. $\mathbf{a} = 3\mathbf{i}$, $\mathbf{b} = 6\mathbf{j} - 2\mathbf{k}$

In exercises 15–18, find a vector perpendicular to the given vector.

15. $\langle 2, -1\rangle$

16. $\langle 4, -1, 1\rangle$

17. $6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

18. $2\mathbf{i} - 3\mathbf{k}$

In exercises 19–24, find $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ and $\text{proj}_{\mathbf{b}}\,\mathbf{a}$.

19. $\mathbf{a} = \langle 2, 1\rangle$, $\mathbf{b} = \langle 3, 4\rangle$

20. $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$

21. $\mathbf{a} = \langle 2, -1, 3\rangle$, $\mathbf{b} = \langle 1, 2, 2\rangle$

22. $\mathbf{a} = \langle 1, 4, 5\rangle$, $\mathbf{b} = \langle -2, 1, 2\rangle$

23. $\mathbf{a} = \langle 2, 0, -2\rangle$, $\mathbf{b} = \langle 0, -3, 4\rangle$

24. $\mathbf{a} = \langle 3, 2, 0\rangle$, $\mathbf{b} = \langle -2, 2, 1\rangle$

25. Repeat example 3.6 with an angle of $\frac{\pi}{3}$ with the horizontal.

26. Repeat example 3.6 with an angle of $\frac{\pi}{6}$ with the horizontal.

27. Explain why the answers to exercises 25 and 26 aren't the same, even though the force exerted is the same. In this setting, explain why a larger amount of work corresponds to a more efficient use of the force.

28. Find the force needed in exercise 25 to produce the same amount of work as in example 3.6.

29. A constant force of $\langle 30, 20\rangle$ pounds moves an object in a straight line from the point (0, 0) to the point (24, 10). Compute the work done.

30. A constant force of $\langle 60, -30\rangle$ pounds moves an object in a straight line from the point (0, 0) to the point (10, −10). Compute the work done.

31. Label each statement as true or false. If it is true, briefly explain why; if it is false, give a counterexample.

(a) If $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

(b) If $\mathbf{b} = \mathbf{c}$, then $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$.

(c) $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2$.

(d) If $\|\mathbf{a}\| > \|\mathbf{b}\|$ then $\mathbf{a}\cdot\mathbf{c} > \mathbf{b}\cdot\mathbf{c}$.

(e) If $\|\mathbf{a}\| = \|\mathbf{b}\|$ then $\mathbf{a} = \mathbf{b}$.

32. To compute $\mathbf{a}\cdot\mathbf{b}$, where $\mathbf{a} = \langle 2, 5\rangle$ and $\mathbf{b} = \dfrac{\langle 4, 1\rangle}{\sqrt{17}}$, you can first compute $\langle 2, 5\rangle\cdot\langle 4, 1\rangle$ and then divide the result (13) by $\sqrt{17}$. Which property of Theorem 3.1 is being used?

33. By the Cauchy-Schwartz Inequality, $|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|$. What relationship must exist between $\mathbf{a}$ and $\mathbf{b}$ to have $|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\|\mathbf{b}\|$?

34. By the Triangle Inequality, $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$. What relationship must exist between $\mathbf{a}$ and $\mathbf{b}$ to have $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$?

35. Use the Triangle Inequality to prove that $\|\mathbf{a}-\mathbf{b}\| \ge \|\mathbf{a}\| - \|\mathbf{b}\|$.

36. Prove parts (ii) and (iii) of Theorem 3.1.

37. For vectors $\mathbf{a}$ and $\mathbf{b}$, use the Cauchy-Schwartz Inequality to find the maximum value of $\mathbf{a}\cdot\mathbf{b}$ if $\|\mathbf{a}\| = 3$ and $\|\mathbf{b}\| = 5$.

38. Find a formula for $\mathbf{a}$ in terms of $\mathbf{b}$ where $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 5$ and $\mathbf{a}\cdot\mathbf{b}$ is maximum.

39. Use the Cauchy-Schwartz Inequality in n dimensions to show that $\left(\sum_{k=1}^{n} |a_k b_k|\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$. If both $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ converge, what can be concluded? Apply the result to $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$.

40. Show that $\sum_{k=1}^{n} |a_k b_k| \le \frac{1}{2}\sum_{k=1}^{n} a_k^2 + \frac{1}{2}\sum_{k=1}^{n} b_k^2$. If both $\sum_{k=1}^{\infty} a_k^2$ and $\sum_{k=1}^{\infty} b_k^2$ converge, what can be concluded? Apply the result to $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$. Is this bound better or worse than the bound found in exercise 39?

41. Use the Cauchy-Schwartz Inequality in n dimensions to show that $\sum_{k=1}^{n} |a_k| \le \left(\sum_{k=1}^{n} |a_k|^{2/3}\right)^{1/2}\left(\sum_{k=1}^{n} |a_k|^{4/3}\right)^{1/2}$.

42. Use the Cauchy-Schwartz Inequality in n dimensions to show that $\sum_{k=1}^{n} |a_k| \le \sqrt{n}\left(\sum_{k=1}^{n} a_k^2\right)^{1/2}$.

43. If $p_1, p_2, \ldots, p_n$ are nonnegative numbers that sum to 1, show that $\sum_{k=1}^{n} p_k^2 \ge \dfrac{1}{n}$.

44. Among all sets of nonnegative numbers $p_1, p_2, \ldots, p_n$ that sum to 1, find the choice of $p_1, p_2, \ldots, p_n$ that minimizes $\sum_{k=1}^{n} p_k^2$.

45. Show that $\sum_{k=1}^{n} a_k^2 b_k^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$ and then $\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} c_k^2\right)$.

46. Show that $\sqrt{\dfrac{x+y}{x+y+z}} + \sqrt{\dfrac{y+z}{x+y+z}} + \sqrt{\dfrac{x+z}{x+y+z}} \le \sqrt{6}$.

47. In a methane molecule (CH_4), a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are at (0, 0, 0), (1, 1, 0), (1, 0, 1) and (0, 1, 1) and the carbon atom is at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Compute the **bond angle,** the angle from hydrogen atom to carbon atom to hydrogen atom.

48. Consider the parallelogram with vertices at (0, 0), (2, 0), (3, 2) and (1, 2). Find the angle at which the diagonals intersect.

49. Prove that $\text{comp}_{\mathbf{c}}(\mathbf{a}+\mathbf{b}) = \text{comp}_{\mathbf{c}}\,\mathbf{a} + \text{comp}_{\mathbf{c}}\,\mathbf{b}$ for any nonzero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

50. The **orthogonal projection** of vector **a** along vector **b** is defined as $\mathbf{orth_b\, a} = \mathbf{a} - \mathbf{proj_b\, a}$. Sketch a picture showing vectors **a**, **b**, $\mathbf{proj_b\, a}$ and $\mathbf{orth_b\, a}$, and explain what is orthogonal about $\mathbf{orth_b\, a}$.

51. Suppose that a beam of an oil rig is installed in a direction parallel to $\langle 10, 1, 5\rangle$. If a wave exerts a force of $\langle 0, -200, 0\rangle$ newtons, find the component of this force along the beam.

52. Repeat exercise 51 with a force of $\langle 13, -190, -61\rangle$ newtons. The forces here and in exercise 51 have nearly identical magnitudes. Explain why the force components are different.

53. A car makes a turn on a banked road. If the road is banked at 10°, show that a vector parallel to the road is $\langle \cos 10^\circ, \sin 10^\circ\rangle$. If the car has weight 2000 pounds, find the component of the weight vector along the road vector. This component of weight provides a force that helps the car turn.

54. Find the component of the weight vector along the road vector for a 2500-pound car on a 15° bank.

55. The racetrack at Bristol, Tennessee, is famous for being short with the steepest banked curves on the NASCAR circuit. The track is an oval of length 0.533 mile and the corners are banked at 36°. Circular motion at a constant speed v requires a centripetal force of $F = \dfrac{mv^2}{r}$, where r is the radius of the circle and m is the mass of the car. For a track banked at angle A, the weight of the car provides a centripetal force of $mg \sin A$, where g is the gravitational constant. Setting the two equal gives $\dfrac{v^2}{r} = g \sin A$. Assuming that the Bristol track is circular (it's not really) and using $g = 32$ ft/s^2, find the speed supported by the Bristol bank. Cars actually complete laps at over 120 mph. Discuss where the additional force for this higher speed might come from.

56. For a car driving on a 36° bank, compute the ratio of the component of weight along the road to the component of weight into the road. Discuss why it might be dangerous if this ratio is very small.

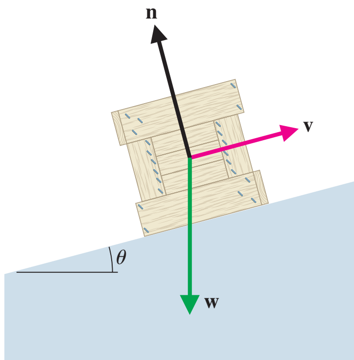
57. A small store sells CD players and DVD players. Suppose 32 CD players are sold at \$25 apiece and 12 DVD players are sold at \$125 apiece. The vector $\mathbf{a} = \langle 32, 12\rangle$ can be called the sales vector and $\mathbf{b} = \langle 25, 125\rangle$ the price vector. Interpret the meaning of $\mathbf{a} \cdot \mathbf{b}$.

58. Suppose that a company makes n products. The production vector $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$ records how many of each product are manufactured and the cost vector $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$ records how much each product costs to manufacture. Interpret the meaning of $\mathbf{a} \cdot \mathbf{b}$.

59. Parametric equations for one object are $x_1 = a \cos t$ and $y_1 = b \sin t$. The object travels along the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The parametric equations for a second object are $x_2 = a\cos(t + \frac{\pi}{2})$ and $y_2 = b \sin(t + \frac{\pi}{2})$. This object travels along the same ellipse but is $\frac{\pi}{2}$ time units ahead. If $a = b$, use the trigonometric identity $\cos u \cos v + \sin u \sin v = \cos(u - v)$ to show that the position vectors of the two objects are orthogonal. However, if $a \neq b$, the position vectors are not orthogonal.

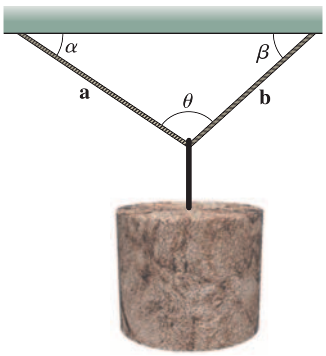
60. Show that the object with parametric equations $x_3 = b\cos(t + \frac{\pi}{2})$ and $y_3 = a\sin(t + \frac{\pi}{2})$ has position vector that is orthogonal to the first object of exercise 59.

61. In the diagram, a crate of weight w pounds is placed on a ramp inclined at angle θ above the horizontal. The vector **v** along the ramp is given by $\mathbf{v} = \langle \cos\theta, \sin\theta\rangle$ and the **normal** vector by $\mathbf{n} = \langle -\sin\theta, \cos\theta\rangle$. Show that **v** and **n** are perpendicular. Find the component of $\mathbf{w} = \langle 0, -w\rangle$ along **v** and the component of **w** along **n**.



62. If the coefficient of static friction between the crate and ramp in exercise 61 equals μ_s, physics tells us that the crate will slide down the ramp if the component of $\mathbf{w}$ along $\mathbf{v}$ is greater than the product of μ_s and the component of $\mathbf{w}$ along $\mathbf{n}$. Show that this occurs if the angle θ is steep enough that $\theta > \tan^{-1}\mu_s$.

63. A weight of 500 pounds is supported by two ropes that exert forces of $\mathbf{a} = \langle -100, 200\rangle$ pounds and $\mathbf{b} = \langle 100, 300\rangle$ pounds. Find the angle θ between the ropes.



64. In the diagram for exercise 63, find the angles α and β.

65. Suppose a small business sells three products. In a given month, if 3000 units of product A are sold, 2000 units of product B are sold and 4000 units of product C are sold, then the **sales vector** for that month is defined by $\mathbf{s} = \langle 3000, 2000, 4000\rangle$. If the prices of products A, B and C are \$20, \$15 and \$25, respectively, then the **price vector** is defined by $\mathbf{p} = \langle 20, 15, 25\rangle$. Compute $\mathbf{s}\cdot\mathbf{p}$ and discuss how it relates to monthly revenue.

66. Suppose that in a particular county, ice cream sales (in thousands of gallons) for a year is given by the vector $\mathbf{s} = \langle 3, 5, 12, 40, 60, 100, 120, 160, 110, 50, 10, 2\rangle$. That is, 3000 gallons were sold in January, 5000 gallons were sold in February, and so on. In the same county, suppose that murders for the year are given by the vector $\mathbf{m} = \langle 2, 0, 1, 6, 4, 8, 10, 13, 8, 2, 0, 6\rangle$. Show that the average monthly ice cream sales is $\bar{s} = 56{,}000$ gallons and that the average monthly number of murders is $\bar{m} = 5$. Compute the vectors $\mathbf{a}$ and $\mathbf{b}$, where the components of $\mathbf{a}$ equal the components of $\mathbf{s}$ with the mean 56 subtracted (so that $\mathbf{a} = \langle -53, -51, -44, \ldots\rangle$) and the components of $\mathbf{b}$ equal the components of $\mathbf{m}$ with the mean 5 subtracted. The correlation between ice cream sales and murders is defined as $\rho = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$. Often, a positive correlation is incorrectly interpreted as meaning that $\mathbf{a}$ "causes" $\mathbf{b}$. (In fact, correlation should *never* be used to infer a cause-and-effect relationship.) Explain why such a conclusion would be invalid in this case.

67. Use the Cauchy-Schwartz Inequality to show that if $a_k \ge 0$, then $\displaystyle\sum_{k=1}^{n}\frac{\sqrt{a_k}}{k^p} \le \sqrt{\sum_{k=1}^{n} a_k}\sqrt{\sum_{k=1}^{n}\frac{1}{k^{2p}}}$.

68. Show that if $a_k \ge 0$, $p > \frac{1}{2}$ and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty}\frac{\sqrt{a_k}}{k^p}$ converges.

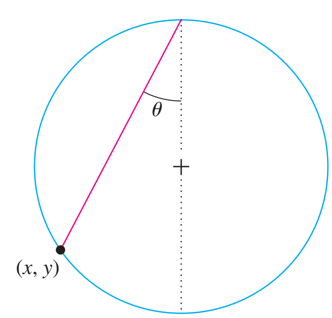
69. For the Mandelbrot set and associated Julia sets, functions of the form $f(x) = x^2 - c$ are analyzed for various constants c. The iterates of the function increase if $|x^2 - c| > |x|$. Show that this is true if $|x| > \frac{1}{2} + \sqrt{\frac{1}{4} + c}$.

70. Show that the vector analog of exercise 69 is also true. For vectors $\mathbf{x}$, $\mathbf{x}_2$ and $\mathbf{c}$, if $\|\mathbf{x}\| > \frac{1}{2} + \sqrt{\frac{1}{4} + \|\mathbf{c}\|}$ and $\|\mathbf{x}_2\| = \|\mathbf{x}\|^2$, then $\|\mathbf{x}_2 - \mathbf{c}\| > \|\mathbf{x}\|$.

EXPLORATORY EXERCISES

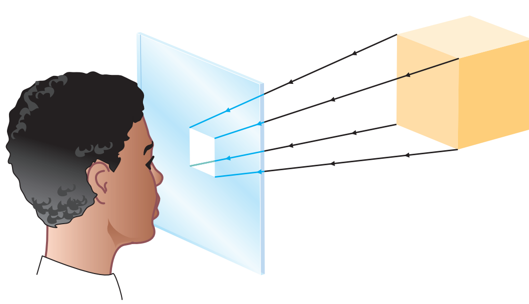
1. One of the basic problems throughout calculus is computing distances. In this exercise, we will find the distance between a point (x_1, y_1) and a line $ax + by + d = 0$. To start with a concrete example, take the point (5, 6) and the line $2x + 3y + 4 = 0$. First, show that the intercepts of the line are the points $\left(-\frac{4}{2}, 0\right)$ and $\left(0, -\frac{4}{3}\right)$. Show that the vector $\mathbf{b} = \langle 3, -2\rangle$ is parallel to the displacement vector between these points and hence, also to the line. Sketch a picture showing the point (5, 6), the line, the vector $\langle 3, -2\rangle$ and the displacement vector $\mathbf{v}$ from $(-2, 0)$ to (5, 6). Explain why the magnitude of the vector $\mathbf{v} - \mathbf{proj_b}\,\mathbf{v}$ equals the desired distance between point and line. Compute this distance. Show that, in general, the distance between the point (x_1, y_1) and the line $ax + by + d = 0$ equals $\dfrac{|ax_1 + by_1 + d|}{\sqrt{a^2 + b^2}}$.

2. In the accompanying figure, the circle $x^2 + y^2 = r^2$ is shown. In this exercise, we will compute the time required for an object to travel the length of a chord from the top of the circle to another point on the circle at an angle of θ from the vertical, assuming that gravity (acting downward) is the only force.

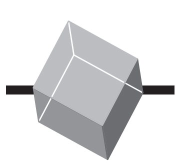


From our study of projectile motion in section 5.5, recall that an object traveling with a constant acceleration a covers a distance d in time $\sqrt{\frac{2d}{a}}$. Show that the component of gravity in the direction of the chord is $a = g\cos\theta$. If the chord ends at the point (x, y), show that the length of the chord is $d = \sqrt{2r^2 - 2ry}$. Also, show that $\cos\theta = \frac{r-y}{d}$. Putting this all together, compute the time it takes to travel the chord. Explain why it's surprising that the answer does not depend on the value of θ. Note that as θ increases, the distance d decreases but the effectiveness of gravity decreases. Discuss the balance between these two factors.

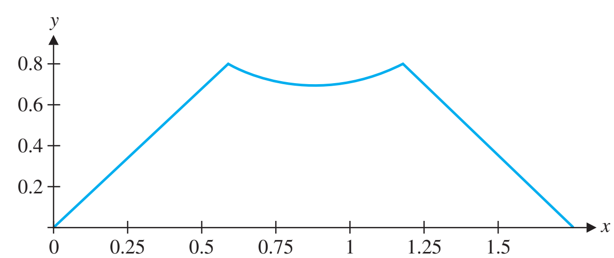
3. This exercise develops a basic principle used to create wireframe and other 3-D computer graphics. In the drawing, an artist traces the image of an object onto a pane of glass. Explain why the trace will be distorted unless the artist keeps the pane of glass perpendicular to the line of sight. The trace is thus a projection of the object onto the pane of glass. To make this precise, suppose that the artist is at the point $(100, 0, 0)$ and the point $P_1 = (2, 1, 3)$ is part of the object being traced. Find the projection $\mathbf{p}_1$ of the position vector $\langle 2, 1, 3\rangle$ along the artist's position vector $\langle 100, 0, 0\rangle$. Then find the vector $\mathbf{q}_1$ such that $\langle 2, 1, 3\rangle = \mathbf{p}_1 + \mathbf{q}_1$. Which of the vectors $\mathbf{p}_1$ and $\mathbf{q}_1$ does the artist actually see and which one is hidden? Repeat this with the point $P_2 = (-2, 1, 3)$ and find vectors $\mathbf{p}_2$ and $\mathbf{q}_2$ such that $\langle -2, 1, 3\rangle = \mathbf{p}_2 + \mathbf{q}_2$. The artist would plot both points P_1 and P_2 at the same point on the pane of glass. Identify which of the vectors $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2$ and $\mathbf{q}_2$ correspond to this point. From the artist's perspective, one of the points P_1 or P_2 is hidden behind the other. Identify which point is hidden and explain how the information in the vectors $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2$ and $\mathbf{q}_2$ can be used to determine which point is hidden.

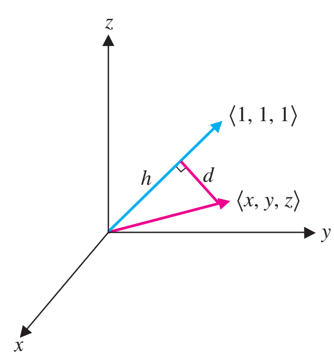


4. Take a cube and spin it around a diagonal.



If you spin it rapidly, you will see a curved outline appear in the middle. (See the figure below.) How does a cube become curved? This exercise answers that question. Suppose that the cube is a unit cube with $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$, and we rotate about the diagonal from $(0, 0, 0)$ to $(1, 1, 1)$. What we see on spinning the cube is the combination of points on the cube at their maximum distance from the diagonal. The points on the edge of the cube have the maximum distance, so we focus on them. If (x, y, z) is a point on an edge of the cube, define h to be the component of the vector $\langle x, y, z\rangle$ along the diagonal $\langle 1, 1, 1\rangle$. The distance d from (x, y, z) to the diagonal is then $d = \sqrt{\|\langle x, y, z\rangle\|^2 - h^2}$, as in the diagram below. The curve is produced by the edge from $(0, 0, 1)$ to $(0, 1, 1)$. Parametric equations for this segment are $x = 0$, $y = t$ and $z = 1$, for $0 \le t \le 1$. For the vector $\langle 0, t, 1\rangle$, compute h and then d. Graph $d(t)$. You should see a curve similar to the middle of the outline shown below. Show that this curve is actually part of a hyperbola. Then find the outline created by other sides of the cube. Which ones produce curves and which produce straight lines?





10.4 THE CROSS PRODUCT

In this section, we define a second type of product of vectors, the *cross product* or *vector product.* While the dot product of two vectors is a scalar, the cross product of two vectors is another vector. The cross product has many important applications, from physics and engineering mechanics to space travel. Before we define the cross product, we need a few definitions.

DEFINITION 4.1

The **determinant** of a 2×2 matrix of real numbers is defined by

$$\underbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}_{2 \times 2 \text{ matrix}} = a_1 b_2 - a_2 b_1. \tag{4.1}$$

EXAMPLE 4.1 Computing a 2 × 2 Determinant

Evaluate the determinant $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Solution From (4.1), we have

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2.$$

DEFINITION 4.2

The **determinant** of a 3×3 matrix of real numbers is defined as a combination of three 2×2 determinants, as follows:

$$\underbrace{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}_{3 \times 3 \text{ matrix}} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{4.2}$$

Equation (4.2) is referred to as an **expansion** of the determinant **along the first row.** Notice that the multipliers of each of the 2×2 determinants are the entries of the first row of the 3×3 matrix. Each 2×2 determinant is the determinant you get if you eliminate the row and column in which the corresponding multiplier lies. That is, for the *first* term, the multiplier is a_1 and the 2×2 determinant is found by eliminating the first row and *first* column from the 3×3 matrix:

$$\begin{vmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ \cancel{b_1} & b_2 & b_3 \\ \cancel{c_1} & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

Likewise, the *second* 2×2 determinant is found by eliminating the first row and the *second* column from the 3×3 determinant:

$$\begin{vmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & \cancel{b_2} & b_3 \\ c_1 & \cancel{c_2} & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}.$$

Be certain to notice the minus sign in front of this term. Finally, the *third* determinant is found by eliminating the first row and the *third* column from the 3×3 determinant:

$$\begin{vmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & b_2 & \cancel{b_3} \\ c_1 & c_2 & \cancel{c_3} \end{vmatrix} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

EXAMPLE 4.2 Evaluating a 3 × 3 Determinant

Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \\ 3 & -2 & 5 \end{vmatrix}$.

Solution Expanding along the first row, we have:

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \\ 3 & -2 & 5 \end{vmatrix} &= (1)\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} - (2)\begin{vmatrix} -3 & 1 \\ 3 & 5 \end{vmatrix} + (4)\begin{vmatrix} -3 & 3 \\ 3 & -2 \end{vmatrix} \\ &= (1)[(3)(5) - (1)(-2)] - (2)[(-3)(5) - (1)(3)] \\ &\quad + (4)[(-3)(-2) - (3)(3)] \\ &= 41.\end{aligned}$$

We use determinant notation as a convenient device for defining the cross product, as follows.

DEFINITION 4.3

For two vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ in V_3, we define the **cross product** (or **vector product**) of $\mathbf{a}$ and $\mathbf{b}$ to be

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}. \tag{4.3}$$

Notice that $\mathbf{a} \times \mathbf{b}$ is also a vector in V_3. To compute $\mathbf{a} \times \mathbf{b}$, you must write the components of $\mathbf{a}$ in the second row and the components of $\mathbf{b}$ in the third row; *the order is important!* Also note that while we've used the determinant notation, the 3 × 3 determinant indicated in (4.3) is not really a determinant, in the sense in which we defined them, since the entries in the first row are vectors instead of scalars. Nonetheless, we find this slight abuse of notation convenient for computing cross products and we use it routinely.

EXAMPLE 4.3 Computing a Cross Product

Compute $\langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle$.

Solution From (4.3), we have

$$\begin{aligned}\langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\mathbf{k} \\ &= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} = \langle -3, 6, -3\rangle.\end{aligned}$$

REMARK 4.1

The cross product is defined only for vectors in V_3. There is no corresponding operation for vectors in V_2.

THEOREM 4.1

For any vector $\mathbf{a} \in V_3$, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{0} = \mathbf{0}$.

PROOF

We prove the first of these two results. The second, we leave as an exercise. For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, we have from (4.3) that

$$\mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} = \mathbf{0}. \quad ■$$

Let's take a brief look back at the result of example 4.3. There, we saw that

$$\langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle = \langle -3, 6, -3 \rangle.$$

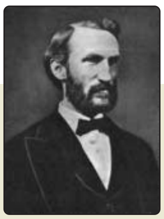
There is something rather interesting to observe here. Note that

$$\langle 1, 2, 3 \rangle \cdot \langle -3, 6, -3 \rangle = 0$$

and

$$\langle 4, 5, 6 \rangle \cdot \langle -3, 6, -3 \rangle = 0.$$

That is, both $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$ are orthogonal to their cross product. As it turns out, this is true in general, as we see in Theorem 4.2.



HISTORICAL NOTES

Josiah Willard Gibbs (1839–1903)
American physicist and mathematician who introduced and named the dot product and the cross product. A graduate of Yale, Gibbs published important papers in thermodynamics, statistical mechanics and the electromagnetic theory of light. Gibbs used vectors to determine the orbit of a comet from only three observations. Originally produced as printed notes for his students, Gibbs' vector system greatly simplified the original system developed by Hamilton. Gibbs was well liked but not famous in his lifetime. One biographer wrote of Gibbs that, "The greatness of his intellectual achievements will never overshadow the beauty and dignity of his life."

THEOREM 4.2

For any vectors **a** and **b** in V_3, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

PROOF

Recall that two vectors are orthogonal if and only if their dot product is zero. Now, using (4.3), we have

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= \langle a_1, a_2, a_3 \rangle \cdot \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \right] \\ &= a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= a_1[a_2b_3 - a_3b_2] - a_2[a_1b_3 - a_3b_1] + a_3[a_1b_2 - a_2b_1] \\ &= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0, \end{aligned}$$

so that **a** and $(\mathbf{a} \times \mathbf{b})$ are orthogonal. We leave it as an exercise to show that $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$, also. ■

Notice that since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**, it is also orthogonal to every vector lying in the plane containing **a** and **b**. (We also say that $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane, in this case.) But, given a plane, out of which side of the plane does $\mathbf{a} \times \mathbf{b}$ point? We can get an idea by computing some simple cross products.

Notice that

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{k}.$$

Likewise,

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

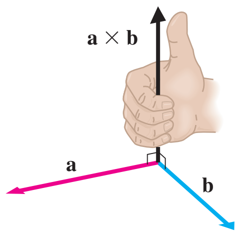


FIGURE 10.29a
$\mathbf{a} \times \mathbf{b}$

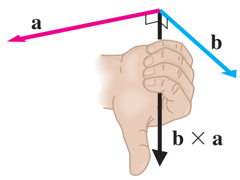


FIGURE 10.29b
$\mathbf{b} \times \mathbf{a}$

These are illustrations of the **right-hand rule:** If you align the fingers of your *right* hand along the vector $\mathbf{a}$ and bend your fingers around in the direction of rotation from $\mathbf{a}$ toward $\mathbf{b}$ (through an angle of less than 180°), your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$ (see Figure 10.29a). Now, following the right-hand rule, $\mathbf{b} \times \mathbf{a}$ will point in the direction opposite $\mathbf{a} \times \mathbf{b}$ (see Figure 10.29b). In particular, notice that

$$\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}.$$

We leave it as an exercise to show that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \text{and} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Take the time to think through the right-hand rule for each of these cross products.

There are several other unusual things to observe here. Notice that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i},$$

which says that the cross product is *not* commutative. Further, notice that

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$

while

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0},$$

so that the cross product is also *not* associative. That is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

Since the cross product does not follow several of the rules you might expect a product to satisfy, you might ask what rules the cross product *does* satisfy. We summarize these in Theorem 4.3.

THEOREM 4.3

For any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in V_3 and any scalar d, the following hold:

(i) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$ (anticommutativity)
(ii) $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$
(iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (distributive law)
(iv) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (distributive law)
(v) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ (scalar triple product) and
(vi) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ (vector triple product).

PROOF

We prove parts (i) and (iii) only. The remaining parts are left as exercises.

(i) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have from (4.3) that

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$
$$= -\begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} = -(\mathbf{b} \times \mathbf{a}),$$

since swapping two rows in a 2×2 matrix (or in a 3×3 matrix, for that matter) changes the sign of its determinant.

(iii) For $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, we have

$$\mathbf{b} + \mathbf{c} = \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

and so,
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}.$$

Looking only at the $\mathbf{i}$ component of this, we have

$$\begin{aligned} \begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} &= a_2(b_3 + c_3) - a_3(b_2 + c_2) \\ &= (a_2 b_3 - a_3 b_2) + (a_2 c_3 - a_3 c_2) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}, \end{aligned}$$

which you should note is also the $\mathbf{i}$ component of $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$. Similarly, you can show that the $\mathbf{j}$ and $\mathbf{k}$ components also match, which establishes the result. ■

Always keep in mind that vectors are specified by two things: magnitude and direction. We have now shown that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. In Theorem 4.4, we make a general (and quite useful) statement about $\|\mathbf{a} \times \mathbf{b}\|$.

THEOREM 4.4

For nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ in V_3, if θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ $(0 \le \theta \le \pi)$, then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta. \tag{4.4}$$

PROOF

From (4.3), we get

$$\begin{aligned} \|\mathbf{a} \times \mathbf{b}\|^2 &= [a_2 b_3 - a_3 b_2]^2 + [a_1 b_3 - a_3 b_1]^2 + [a_1 b_2 - a_2 b_1]^2 \\ &= a_2^2 b_3^2 - 2a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + a_1^2 b_3^2 - 2a_1 a_3 b_1 b_3 + a_3^2 b_1^2 \\ &\quad + a_1^2 b_2^2 - 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \cos^2 \theta \qquad \text{From Theorem 3.2} \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 (1 - \cos^2 \theta) \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \sin^2 \theta. \end{aligned}$$

Taking square roots, we get

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta,$$

since $\sin \theta \ge 0$, for $0 \le \theta \le \pi$. ■

The following characterization of parallel vectors is an immediate consequence of Theorem 4.4.

COROLLARY 4.1

Two nonzero vectors $\mathbf{a}, \mathbf{b} \in V_3$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

PROOF

Recall that $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$ and so, by Theorem 4.4,

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = \|\mathbf{a}\|\|\mathbf{b}\|(0) = 0.$$

The result then follows from the fact that the only vector with zero magnitude is the zero vector. ■

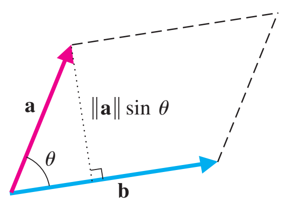


FIGURE 10.30
Parallelogram

Theorem 4.4 also provides us with the following interesting geometric interpretation of the cross product. For any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, as long as $\mathbf{a}$ and $\mathbf{b}$ are not parallel, they form two adjacent sides of a parallelogram, as seen in Figure 10.30. Notice that the area of the parallelogram is given by the product of the base and the altitude. We have

$$\begin{aligned}\text{Area} &= (\text{base})(\text{altitude})\\ &= \|\mathbf{b}\|\|\mathbf{a}\|\sin\theta = \|\mathbf{a}\times\mathbf{b}\|,\end{aligned} \tag{4.5}$$

from Theorem 4.4. That is, the magnitude of the cross product of two vectors gives the area of the parallelogram with two adjacent sides formed by the vectors.

EXAMPLE 4.4 Finding the Area of a Parallelogram Using the Cross Product

Find the area of the parallelogram with two adjacent sides formed by the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 4, 5, 6\rangle$.

Solution First notice that

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 3\\ 5 & 6\end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3\\ 4 & 6\end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2\\ 4 & 5\end{vmatrix} = \langle -3, 6, -3\rangle.$$

From (4.5), the area of the parallelogram is given by

$$\|\mathbf{a}\times\mathbf{b}\| = \|\langle -3, 6, -3\rangle\| = \sqrt{54} \approx 7.348. \blacksquare$$

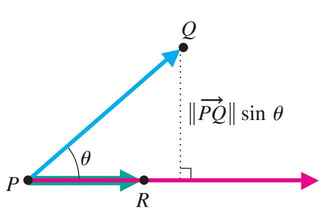


FIGURE 10.31
Distance from a point to a line

We can also use Theorem 4.4 to find the distance from a point to a line in $\mathbb{R}^3$, as follows. Let d represent the distance from the point Q to the line through the points P and R. From elementary trigonometry, we have that

$$d = \|\overrightarrow{PQ}\|\sin\theta,$$

where θ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ (see Figure 10.31). From (4.4), we have

$$\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|\sin\theta = \|\overrightarrow{PR}\|(d).$$

Solving this for d, we get
$$d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}. \tag{4.6}$$

EXAMPLE 4.5 Finding the Distance from a Point to a Line

Find the distance from the point $Q(1, 2, 1)$ to the line through the points $P(2, 1, -3)$ and $R(2, -1, 3)$.

Solution First, the position vectors corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are
$$\overrightarrow{PQ} = \langle -1, 1, 4\rangle \quad \text{and} \quad \overrightarrow{PR} = \langle 0, -2, 6\rangle,$$
and
$$\langle -1, 1, 4\rangle \times \langle 0, -2, 6\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 4 \\ 0 & -2 & 6 \end{vmatrix} = \langle 14, 6, 2\rangle.$$
We then have from (4.6) that
$$d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PR}\|} = \frac{\|\langle 14, 6, 2\rangle\|}{\|\langle 0, -2, 6\rangle\|} = \frac{\sqrt{236}}{\sqrt{40}} \approx 2.429.$$

For any three noncoplanar vectors **a**, **b** and **c** (i.e., three vectors that do not lie in a single plane), consider the parallelepiped formed using the vectors as three adjacent edges (see Figure 10.32). Recall that the volume of such a solid is given by
$$\text{Volume} = (\text{Area of base})(\text{altitude}).$$
Further, since two adjacent sides of the base are formed by the vectors **a** and **b**, we know that the area of the base is given by $\|\mathbf{a} \times \mathbf{b}\|$. Referring to Figure 10.32, notice that the altitude is given by
$$|\text{comp}_{\mathbf{a}\times\mathbf{b}}\, \mathbf{c}| = \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{\|\mathbf{a} \times \mathbf{b}\|},$$
from (3.7). The volume of the parallelepiped is then
$$\text{Volume} = \|\mathbf{a} \times \mathbf{b}\| \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{\|\mathbf{a} \times \mathbf{b}\|} = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|.$$
The scalar $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ is called the **scalar triple product** of the vectors **a**, **b** and **c**. As you can see from the following, you can evaluate the scalar triple product by computing a single

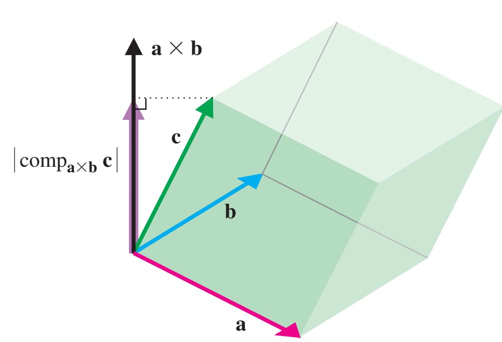


FIGURE 10.32
Parallelepiped formed by the vectors **a**, **b** and **c**

determinant. Note that for $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$, we have

$$\begin{aligned}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= \mathbf{c}\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &= \langle c_1, c_2, c_3\rangle\cdot\left(\mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - \mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + \mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right)\\ &= c_1\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix} - c_2\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix} + c_3\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\\ &= \begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.\end{aligned} \tag{4.7}$$

EXAMPLE 4.6 Finding the Volume of a Parallelepiped Using the Cross Product

Find the volume of the parallelepiped with three adjacent edges formed by the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle 4, 5, 6\rangle$ and $\mathbf{c} = \langle 7, 8, 0\rangle$.

Solution First, note that Volume $= |\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$. From (4.7), we have that

$$\begin{aligned}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= \begin{vmatrix}7 & 8 & 0\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix} = 7\begin{vmatrix}2 & 3\\ 5 & 6\end{vmatrix} - 8\begin{vmatrix}1 & 3\\ 4 & 6\end{vmatrix} + 0\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}\\ &= 7(-3) - 8(-6) = 27.\end{aligned}$$

So, the volume of the parallelepiped is Volume $= |\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = |27| = 27$. ■

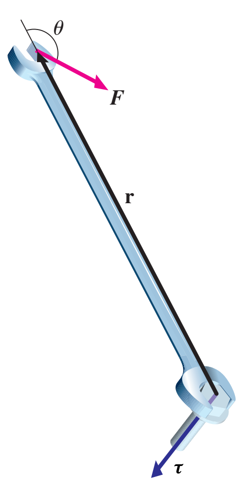


FIGURE 10.33
Torque, $\boldsymbol{\tau}$

Consider the action of a wrench on a bolt, as shown in Figure 10.33. In order to tighten the bolt, we apply a force $\mathbf{F}$ at the end of the handle, in the direction indicated in the figure. This force creates a **torque** $\boldsymbol{\tau}$ acting along the axis of the bolt, drawing it in tight. Notice that the torque acts in the direction perpendicular to both $\mathbf{F}$ and the position vector $\mathbf{r}$ for the handle as indicated in Figure 10.33. In fact, using the right-hand rule, the torque acts in the same direction as $\mathbf{r}\times\mathbf{F}$ and physicists define the torque vector to be

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}.$$

In particular, this says that

$$\|\boldsymbol{\tau}\| = \|\mathbf{r}\times\mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta, \tag{4.8}$$

from (4.4). There are several observations we can make from this. First, this says that the farther away from the axis of the bolt we apply the force (i.e., the larger $\|\mathbf{r}\|$ is), the greater the magnitude of the torque. So, a longer wrench produces a greater torque, for a given amount of force applied. Second, notice that $\sin\theta$ is maximized when $\theta = \frac{\pi}{2}$, so that from (4.8) the magnitude of the torque is maximized when $\theta = \frac{\pi}{2}$ (when the force vector $\mathbf{F}$ is orthogonal to the position vector $\mathbf{r}$). If you've ever spent any time using a wrench, this should fit well with your experience.

EXAMPLE 4.7 Finding the Torque Applied by a Wrench

If you apply a force of magnitude 25 pounds at the end of a 15-inch-long wrench, at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt. What is the maximum torque that a force of 25 pounds applied at that point can produce?

Solution From (4.8), we have

$$\|\boldsymbol{\tau}\| = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta = \left(\frac{15}{12}\right) 25 \sin\frac{\pi}{3}$$
$$= \left(\frac{15}{12}\right) 25 \frac{\sqrt{3}}{2} \approx 27.1 \text{ foot-pounds.}$$

Further, the maximum torque is obtained when the angle between the wrench and the force vector is $\frac{\pi}{2}$. This would give us a maximum torque of

$$\|\boldsymbol{\tau}\| = \|\mathbf{F}\|\|\mathbf{r}\|\sin\theta = 25\left(\frac{15}{12}\right)(1) = 31.25 \text{ foot-pounds.}$$

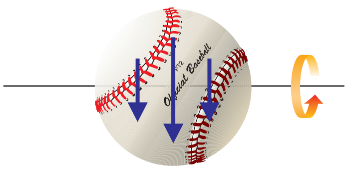


FIGURE 10.34
Spinning ball

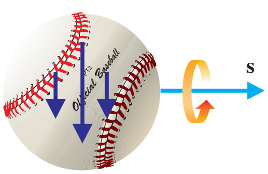


FIGURE 10.35a
Backspin

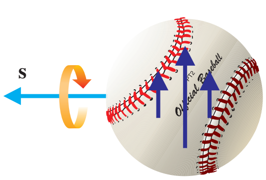


FIGURE 10.35b
Topspin

In many sports, the action is at least partially influenced by the motion of a spinning ball. For instance, in baseball, batters must contend with pitchers' curveballs and in golf, players try to control their slice. In tennis, players hit shots with topspin, while in basketball, players improve their shooting by using backspin. The list goes on and on. These are all examples of the **Magnus force,** which we describe below.

Suppose that a ball is spinning with angular velocity ω, measured in radians per second (i.e., ω is the rate of change of the rotational angle). The ball spins about an axis, as shown in Figure 10.34. We define the spin vector $\mathbf{s}$ to have magnitude ω and direction parallel to the spin axis. We use a right-hand rule to distinguish between the two directions parallel to the spin axis: curl the fingers of your right hand around the ball in the direction of the spin, and your thumb will point in the correct direction. Two examples are shown in Figures 10.35a and 10.35b. The motion of the ball disturbs the air through which it travels, creating a Magnus force $\mathbf{F}_m$ acting on the ball. For a ball moving with velocity $\mathbf{v}$ and spin vector $\mathbf{s}$, $\mathbf{F}_m$ is given by

$$\mathbf{F}_m = c(\mathbf{s} \times \mathbf{v}),$$

for some positive constant c. Suppose the balls in Figure 10.35a and Figure 10.35b are moving into the page and away from you. Using the usual sports terminology, the first ball has backspin and the second ball has topspin. Using the right-hand rule, we see that the Magnus force acting on the first ball acts in the upward direction, as shown in Figure 10.36a. This says that backspin (for example, on a basketball or golf shot) produces an upward force that helps the ball land more softly than a ball with no spin. Similarly, the Magnus force acting on the second ball acts in the downward direction (see Figure 10.36b), so that topspin (for example, on a tennis shot or baseball hit) produces a downward force that causes the ball to drop to the ground more quickly than a ball with no spin.

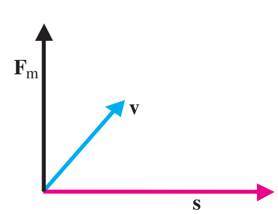


FIGURE 10.36a
Magnus force for a ball with backspin

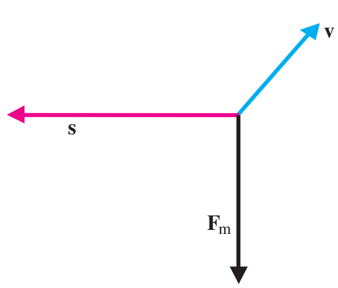


FIGURE 10.36b
Magnus force for a ball with topspin

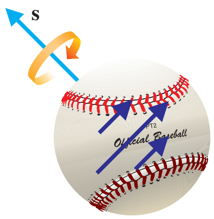


FIGURE 10.37a
Right-hand curveball

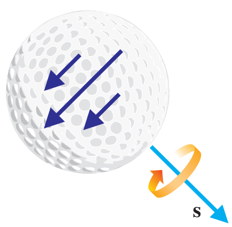


FIGURE 10.37b
Right-hand golf shot

EXAMPLE 4.8 Finding the Direction of a Magnus Force

The balls shown in Figures 10.37a and 10.37b are moving into the page and away from you with spin as indicated. The first ball represents a right-handed baseball pitcher's curveball, while the second ball represents a right-handed golfer's shot. Determine the direction of the Magnus force and discuss the effects on the ball.

Solution For the first ball, notice that the spin vector points up and to the left, so that $\mathbf{s} \times \mathbf{v}$ points down and to the left as shown in Figure 10.38a. Such a ball will curve to the left and drop faster than a ball that is not spinning, making it more difficult to hit. For the second ball, the spin vector points down and to the right, so $\mathbf{s} \times \mathbf{v}$ points up and to the right. Such a ball will move to the right (a "slice") and stay in the air longer than a ball that is not spinning (see Figure 10.38b).

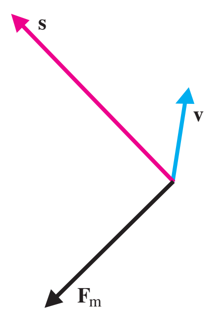


FIGURE 10.38a
Magnus force for a right-handed curveball

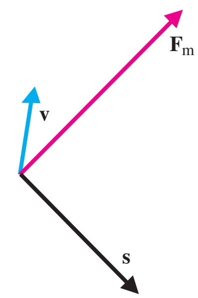


FIGURE 10.38b
Magnus force for a right-handed golf shot

EXERCISES 10.4

WRITING EXERCISES

1. In this chapter, we have developed several tests for geometric relationships. Briefly describe how to test whether two vectors are (a) parallel; (b) perpendicular. Briefly describe how to test whether (c) three points are colinear; (d) four points are coplanar.

2. The flip side of the problems in exercise 1 is to construct vectors with desired properties. Briefly describe how to construct a vector (a) parallel to a given vector; (b) perpendicular to a given vector. Given a vector, describe how to construct two other vectors such that the three vectors are mutually perpendicular.

3. Recall that torque is defined as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{F}$ is the force applied to the end of the handle and $\mathbf{r}$ is the position vector for the end of the handle. In example 4.7, how would the torque change if the force $\mathbf{F}$ were replaced with the force $-\mathbf{F}$? Answer both in mathematical terms and in physical terms.

4. Explain in geometric terms why $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

In exercises 1–4, compute the given determinant.

1. $\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 3 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 2 \end{vmatrix}$

In exercises 5–10, compute the cross product a × b.

5. $\mathbf{a} = \langle 1, 2, -1 \rangle$, $\mathbf{b} = \langle 1, 0, 2 \rangle$

6. $\mathbf{a} = \langle 3, 0, -1 \rangle$, $\mathbf{b} = \langle 1, 2, 2 \rangle$

7. $\mathbf{a} = \langle 0, 1, 4 \rangle$, $\mathbf{b} = \langle -1, 2, -1 \rangle$

8. $\mathbf{a} = \langle 2, -2, 0 \rangle$, $\mathbf{b} = \langle 3, 0, 1 \rangle$

9. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

10. $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$

In exercises 11–16, find two unit vectors orthogonal to the two given vectors.

11. $\mathbf{a} = \langle 1, 0, 4 \rangle$, $\mathbf{b} = \langle 1, -4, 2 \rangle$

12. $\mathbf{a} = \langle 2, -2, 1 \rangle$, $\mathbf{b} = \langle 0, 0, -2 \rangle$

13. $\mathbf{a} = \langle 2, -1, 0 \rangle$, $\mathbf{b} = \langle 1, 0, 3 \rangle$

14. $\mathbf{a} = \langle 0, 2, 1 \rangle$, $\mathbf{b} = \langle 1, 0, -1 \rangle$

15. $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

16. $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

In exercises 17–20, use the cross product to determine the angle between the vectors, assuming that $0 \le \theta \le \frac{\pi}{2}$.

17. $\mathbf{a} = \langle 1, 0, 4 \rangle$, $\mathbf{b} = \langle 2, 0, 1 \rangle$

18. $\mathbf{a} = \langle 2, 2, 1 \rangle$, $\mathbf{b} = \langle 0, 0, 2 \rangle$

19. $\mathbf{a} = 3\mathbf{i} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

20. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$

In exercises 21–24, find the distance from the point Q to the given line.

21. $Q = (1, 2, 0)$, line through $(0, 1, 2)$ and $(3, 1, 1)$

22. $Q = (2, 0, 1)$, line through $(1, -2, 2)$ and $(3, 0, 2)$

23. $Q = (3, -2, 1)$, line through $(2, 1, -1)$ and $(1, 1, 1)$

24. $Q = (1, 3, 1)$, line through $(1, 3, -2)$ and $(1, 0, -2)$

25. If you apply a force of magnitude 20 pounds at the end of an 8-inch-long wrench at an angle of $\frac{\pi}{4}$ to the wrench, find the magnitude of the torque applied to the bolt.

26. If you apply a force of magnitude 40 pounds at the end of an 18-inch-long wrench at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt.

27. If you apply a force of magnitude 30 pounds at the end of an 8-inch-long wrench at an angle of $\frac{\pi}{6}$ to the wrench, find the magnitude of the torque applied to the bolt.

28. If you apply a force of magnitude 30 pounds at the end of an 8-inch-long wrench at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt.

In exercises 29–32, assume that the balls are moving into the page (and away from you) with the indicated spin. Determine the direction of the Magnus force.

29. (a)

(b)

30. (a)

(b)

31. (a)

(b)

32. (a)

(b)

In exercises 33–40, a sports situation is described, with the typical ball spin shown in the indicated exercise. Discuss the effects on the ball and how the game is affected.

33. Baseball overhand fastball, spin in exercise 29(a)

34. Baseball right-handed curveball, spin in exercise 31(a)

35. Tennis topspin groundstroke, spin in exercise 32(a)

36. Tennis left-handed slice serve, spin in exercise 30(b)

37. Football spiral pass, spin in exercise 32(b)

38. Soccer left-footed "curl" kick, spin in exercise 29(b)

39. Golf "pure" hit, spin in exercise 29(a)

40. Golf right-handed "hook" shot, spin in exercise 31(b)

In exercises 41–46, label each statement as true or false. If it is true, briefly explain why. If it is false, give a counterexample.

41. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

42. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

43. $\mathbf{a} \times \mathbf{a} = \|\mathbf{a}\|^2$

44. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

45. If the force is doubled, the torque doubles.

46. If the spin rate is doubled, the Magnus force is doubled.

In exercises 47–52, find the indicated area or volume.

47. Area of the parallelogram with two adjacent sides formed by $\langle 2, 3 \rangle$ and $\langle 1, 4 \rangle$

48. Area of the parallelogram with two adjacent sides formed by $\langle -2, 1 \rangle$ and $\langle 1, -3 \rangle$

49. Area of the triangle with vertices $(0, 0, 0)$, $(2, 3, -1)$ and $(3, -1, 4)$

50. Area of the triangle with vertices $(0, 0, 0)$, $(0, -2, 1)$ and $(1, -3, 0)$

51. Volume of the parallelepiped with three adjacent edges formed by $\langle 2, 1, 0\rangle$, $\langle -1, 2, 0\rangle$ and $\langle 1, 1, 2\rangle$

52. Volume of the parallelepiped with three adjacent edges formed by $\langle 0, -1, 0\rangle$, $\langle 0, 2, -1\rangle$ and $\langle 1, 0, 2\rangle$

In exercises 53–58, use geometry to identify the cross product (do not compute!).

53. $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$

54. $\mathbf{j} \times (\mathbf{j} \times \mathbf{k})$

55. $\mathbf{j} \times (\mathbf{j} \times \mathbf{i})$

56. $(\mathbf{j} \times \mathbf{i}) \times \mathbf{k}$

57. $\mathbf{i} \times (3\mathbf{k})$

58. $\mathbf{k} \times (2\mathbf{i})$

In exercises 59–62, use the parallelepiped volume formula to determine whether the vectors are coplanar.

59. $\langle 2, 3, 1\rangle$, $\langle 1, 0, 2\rangle$ and $\langle 0, 3, -3\rangle$

60. $\langle 1, -3, 1\rangle$, $\langle 2, -1, 0\rangle$ and $\langle 0, -5, 2\rangle$

61. $\langle 1, 0, -2\rangle$, $\langle 3, 0, 1\rangle$ and $\langle 2, 1, 0\rangle$

62. $\langle 1, 1, 2\rangle$, $\langle 0, -1, 0\rangle$ and $\langle 3, 2, 4\rangle$

63. Show that $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

64. Prove parts (ii), (iv), (v) and (vi) of Theorem 4.3.

65. In each of the situations shown here, $\|\mathbf{a}\| = 3$ and $\|\mathbf{b}\| = 4$. In which case is $\|\mathbf{a} \times \mathbf{b}\|$ larger? What is the maximum possible value for $\|\mathbf{a} \times \mathbf{b}\|$?

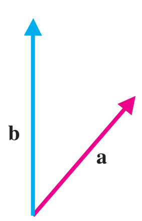


FIGURE A

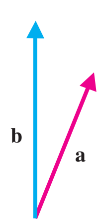


FIGURE B

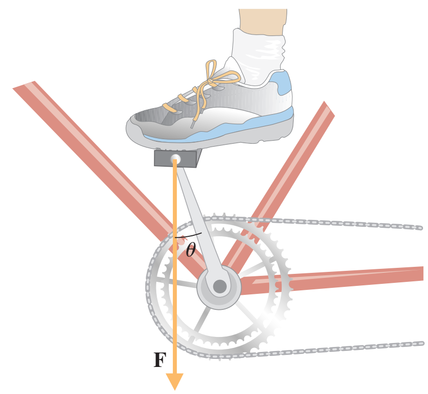
66. Show that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

67. Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$.

EXPLORATORY EXERCISES

1. Use the torque formula $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ to explain the positioning of doorknobs. In particular, explain why the knob is placed as far as possible from the hinges and at a height that makes it possible for most people to push or pull on the door at a right angle to the door.

2. In the diagram, a foot applies a force $\mathbf{F}$ vertically to a bicycle pedal. Compute the torque on the sprocket in terms of θ and $\mathbf{F}$. Determine the angle θ at which the torque is maximized. When helping a young person to learn to ride a bicycle, most people rotate the sprocket so that the pedal sticks straight out to the front. Explain why this is helpful.



3. Devise a test that quickly determines whether $\|\mathbf{a} \times \mathbf{b}\| < |\mathbf{a} \cdot \mathbf{b}|$, $\|\mathbf{a} \times \mathbf{b}\| > |\mathbf{a} \cdot \mathbf{b}|$ or $\|\mathbf{a} \times \mathbf{b}\| = |\mathbf{a} \cdot \mathbf{b}|$. Apply your test to the following vectors: (a) $\langle 2, 1, 1\rangle$ and $\langle 3, 1, 2\rangle$ and (b) $\langle 2, 1, -1\rangle$ and $\langle -1, -2, 1\rangle$. For randomly chosen vectors, which of the three cases is the most likely?

4. In this exercise, we explore the equation of motion for a general projectile in three dimensions. Newton's second law remains $\mathbf{F} = m\mathbf{a}$, but now force and acceleration are vectors. Three forces that could affect the motion of the projectile are gravity, air drag and the Magnus force. Orient the axes such that positive z is up, positive x is right and positive y is straight ahead. The force due to gravity is weight, given by $\mathbf{F}_g = \langle 0, 0, -mg\rangle$. Air drag has magnitude proportional to the square of speed and direction opposite that of velocity. Show that if $\mathbf{v}$ is the velocity vector, then $\mathbf{F}_d = -\|\mathbf{v}\|\mathbf{v}$ satisfies both properties. That is, $\|\mathbf{F}_d\|^2 = \|\mathbf{v}\|^2$ and the angle between $\mathbf{F}_d$ and $\mathbf{v}$ is π. Finally, the Magnus force is proportional to $\mathbf{s} \times \mathbf{v}$, where $\mathbf{s}$ is the spin vector. The full model is then

$$\frac{d\mathbf{v}}{dt} = \langle 0, 0, -g\rangle - c_d\|\mathbf{v}\|\mathbf{v} + c_m(\mathbf{s} \times \mathbf{v}),$$

for positive constants c_d and c_m. With $\mathbf{v} = \langle v_x, v_y, v_z\rangle$ and $\mathbf{s} = \langle s_x, s_y, s_z\rangle$, expand this equation into separate differential equations for v_x, v_y and v_z. We can't solve these equations, but we can get some information by considering signs. For a golf drive, the spin produced could be pure backspin, in which case the spin vector is $\mathbf{s} = \langle \omega, 0, 0\rangle$ for some large $\omega > 0$. (A golf shot can have spins of 4000 rpm.) The initial velocity of a good shot would be straight ahead with some loft, $\mathbf{v}(0) = \langle 0, b, c\rangle$ for positive constants b and c. At the beginning of the flight, show that $v_y' < 0$ and thus, v_y decreases. If the ball spends approximately the same amount of time going up as coming down, conclude that the ball will travel further downrange going up than coming down. In fact, the ball does *not* spend equal amounts of time going up and coming down. By examining the sign of v_z' going up ($v_z > 0$) versus coming down ($v_z < 0$), determine whether the ball spends more time going up or coming down. Next, consider the case of a ball with some sidespin, so that $s_x > 0$ and $s_y > 0$. By examining the sign of v_x', determine whether this ball will curve to the right or left. Examine the other equations and determine what other effects this sidespin may have.

10.5 LINES AND PLANES IN SPACE

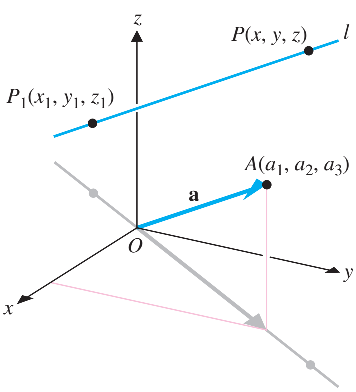


FIGURE 10.39
Line in space

Normally, you specify a line in the xy-plane by selecting any two points on the line or a single point on the line and its direction, as indicated by the *slope* of the line. In three dimensions, specifying two points on a line will still determine the line. An alternative is to specify a single point on the line and its *direction*. In three dimensions, direction should make you think about vectors right away.

Let's look for the line that passes through the point $P_1(x_1, y_1, z_1)$ and that is parallel to the position vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ (see Figure 10.39). For any other point $P(x, y, z)$ on the line, observe that the vector $\overrightarrow{P_1P}$ will be parallel to $\mathbf{a}$. Further, two vectors are parallel if and only if one is a scalar multiple of the other, so that

$$\overrightarrow{P_1P} = t\mathbf{a}, \tag{5.1}$$

for some scalar t. The line then consists of all points $P(x, y, z)$ for which (5.1) holds. Since

$$\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle,$$

we have from (5.1) that

$$\langle x - x_1, y - y_1, z - z_1 \rangle = t\mathbf{a} = t\langle a_1, a_2, a_3 \rangle.$$

Finally, since two vectors are equal if and only if all of their components are equal, we have

Parametric equations of a line

$$\boxed{x - x_1 = a_1 t, \quad y - y_1 = a_2 t \quad \text{and} \quad z - z_1 = a_3 t.} \tag{5.2}$$

We call (5.2) **parametric equations** for the line, where t is the **parameter.** As in the two-dimensional case, a line in space can be represented by many different sets of parametric equations. Provided none of a_1, a_2 or a_3 are zero, we can solve for the parameter in each of the three equations, to obtain

Symmetric equations of a line

$$\boxed{\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}.} \tag{5.3}$$

We refer to (5.3) as **symmetric equations** of the line.

EXAMPLE 5.1 Finding Equations of a Line Given a Point and a Vector

Find an equation of the line through the point (1, 5, 2) and parallel to the vector $\langle 4, 3, 7 \rangle$. Also, determine where the line intersects the yz-plane.

Solution From (5.2), parametric equations for the line are

$$x - 1 = 4t, \quad y - 5 = 3t \quad \text{and} \quad z - 2 = 7t.$$

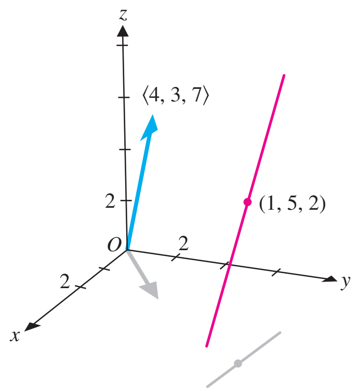


FIGURE 10.40
The line $x = 1 + 4t$, $y = 5 + 3t$, $z = 2 + 7t$

From (5.3), symmetric equations of the line are

$$\frac{x-1}{4} = \frac{y-5}{3} = \frac{z-2}{7}. \tag{5.4}$$

We show the graph of the line in Figure 10.40. Note that the line intersects the yz-plane where $x = 0$. Setting $x = 0$ in (5.4), we solve for y and z to obtain

$$y = \frac{17}{4} \quad \text{and} \quad z = \frac{1}{4}.$$

Alternatively, observe that we could solve $x - 1 = 4t$ for t (again where $x = 0$) and substitute this into the parametric equations for y and z. So, the line intersects the yz-plane at the point $\left(0, \frac{17}{4}, \frac{1}{4}\right)$. ■

Given two points, we can easily find the equations of the line passing through them, as in example 5.2.

EXAMPLE 5.2 Finding Equations of a Line Given Two Points

Find an equation of the line passing through the points $P(1, 2, -1)$ and $Q(5, -3, 4)$.

Solution First, a vector that is parallel to the line is

$$\overrightarrow{PQ} = \langle 5 - 1, -3 - 2, 4 - (-1) \rangle = \langle 4, -5, 5 \rangle.$$

Picking either point will give us equations for the line. Here, we use P, so that parametric equations for the line are

$$x - 1 = 4t, \quad y - 2 = -5t \quad \text{and} \quad z + 1 = 5t.$$

Similarly, symmetric equations of the line are

$$\frac{x-1}{4} = \frac{y-2}{-5} = \frac{z+1}{5}.$$

We show the graph of the line in Figure 10.41. ■

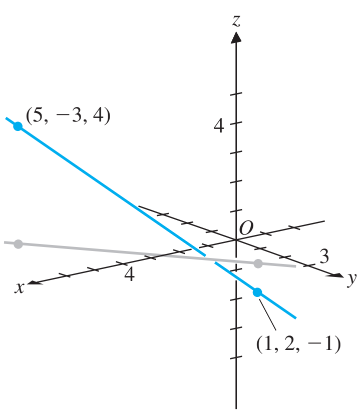


FIGURE 10.41
The line $\dfrac{x-1}{4} = \dfrac{y-2}{-5} = \dfrac{z+1}{5}$

Since we have specified a line by choosing a point on the line and a vector with the same direction, Definition 5.1 should be no surprise.

DEFINITION 5.1

Let l_1 and l_2 be two lines in $\mathbb{R}^3$, with parallel vectors **a** and **b**, respectively, and let θ be the angle between **a** and **b**.

(i) The lines l_1 and l_2 are **parallel** whenever **a** and **b** are parallel.
(ii) If l_1 and l_2 intersect, then
  (a) the angle between l_1 and l_2 is θ and
  (b) the lines l_1 and l_2 are **orthogonal** whenever **a** and **b** are orthogonal.

In two dimensions, two lines are either parallel or they intersect. This is not true in three dimensions, as we see in example 5.3.

EXAMPLE 5.3 Showing Two Lines Are Not Parallel but Do Not Intersect

Show that the lines

$$l_1 : x - 2 = -t, \quad y - 1 = 2t \quad \text{and} \quad z - 5 = 2t$$

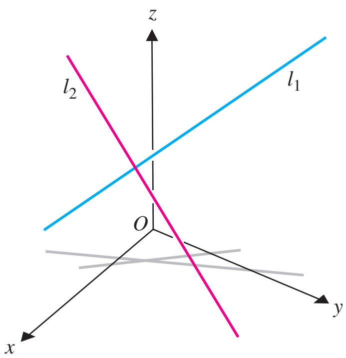


FIGURE 10.42
Skew lines

and

$$l_2 : x - 1 = s, \quad y - 2 = -s \quad \text{and} \quad z - 1 = 3s$$

are not parallel, yet do not intersect.

Solution Notice immediately that we have used different letters (t and s) as parameters for the two lines. In this setting, the parameter is a dummy variable, so the letter used is not significant. However, solving the first parametric equation of each line for the parameter in terms of x, we get

$$t = 2 - x \quad \text{and} \quad s = x - 1,$$

respectively. This says that the parameter represents something different in each line; so we must use different letters. Notice from the graph in Figure 10.42 that the lines are most certainly not parallel, but it is unclear whether or not they intersect. (Remember, the graph is a two-dimensional rendering of lines in three dimensions and so, while the two-dimensional lines drawn do intersect, it's unclear whether or not the three-dimensional lines that they represent intersect.)

You can read from the parametric equations that a vector parallel to l_1 is $\mathbf{a}_1 = \langle -1, 2, 2 \rangle$, while a vector parallel to l_2 is $\mathbf{a}_2 = \langle 1, -1, 3 \rangle$. Since $\mathbf{a}_1$ is not a scalar multiple of $\mathbf{a}_2$, the vectors are not parallel and so, the lines l_1 and l_2 are not parallel. The lines intersect if there's a choice of the parameters s and t that produces the same point, that is, that produces the same values for all of x, y and z. Setting the x-values equal, we get

$$2 - t = 1 + s,$$

so that $s = 1 - t$. Setting the y-values equal and setting $s = 1 - t$, we get

$$1 + 2t = 2 - s = 2 - (1 - t) = 1 + t.$$

Solving this for t yields $t = 0$, which further implies that $s = 1$. Setting the z-components equal gives

$$5 + 2t = 3s + 1,$$

but this is not satisfied when $t = 0$ and $s = 1$. So, l_1 and l_2 are not parallel, yet do not intersect. ■

DEFINITION 5.2

Nonparallel, nonintersecting lines are called **skew** lines.

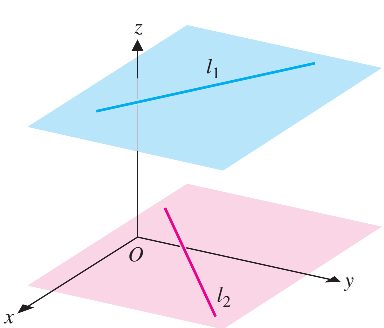


FIGURE 10.43
Skew lines

Note that it's fairly easy to visualize skew lines. Draw two planes that are parallel and draw a line in each plane (so that it lies completely in the plane). As long as the two lines are not parallel, these are skew lines (see Figure 10.43).

Planes in $\mathbb{R}^3$

Think about what information you might need to specify a plane in space. As a simple example, observe that the yz-plane is a set of points in space such that every vector connecting two points in the set is orthogonal to $\mathbf{i}$. However, every plane parallel to the yz-plane satisfies this criterion (see Figure 10.44). In order to select the one that corresponds to the yz-plane, you need to specify a point through which it passes (any one will do).

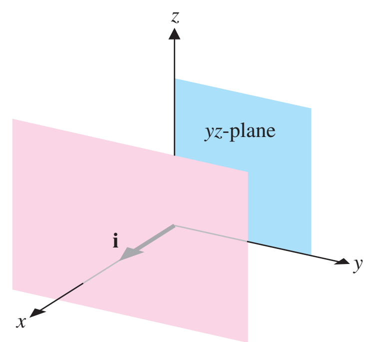


FIGURE 10.44
Parallel planes

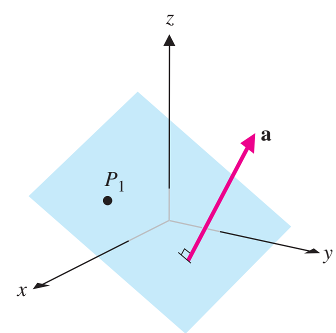


FIGURE 10.45
Plane in $\mathbb{R}^3$

In general, a plane in space is determined by specifying a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ that is **normal** to the plane (i.e., orthogonal to every vector lying in the plane) and a point $P_1(x_1, y_1, z_1)$ lying in the plane (see Figure 10.45). In order to find an equation of the plane, let $P(x, y, z)$ represent any point in the plane. Then, since P and P_1 are both points in the plane, the vector $\overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle$ lies in the plane and so, must be orthogonal to $\mathbf{a}$. By Corollary 3.1, we have that

$$0 = \mathbf{a} \cdot \overrightarrow{P_1P} = \langle a_1, a_2, a_3 \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle$$

Equation of a plane

or

$$\boxed{0 = a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1).} \tag{5.5}$$

Equation (5.5) is an equation for the plane passing through the point (x_1, y_1, z_1) with normal vector $\langle a_1, a_2, a_3 \rangle$. It's a simple matter to use this to find the equation of any particular plane. We illustrate this in example 5.4.

EXAMPLE 5.4 The Equation and Graph of a Plane Given a Point and a Normal Vector

Find an equation of the plane containing the point (1, 2, 3) with normal vector $\langle 4, 5, 6 \rangle$, and sketch the plane.

Solution From (5.5), we have the equation

$$0 = 4(x - 1) + 5(y - 2) + 6(z - 3). \tag{5.6}$$

To draw the plane, we locate three points lying in the plane. In this case, the simplest way to do this is to look at the intersections of the plane with each of the coordinate axes. When $y = z = 0$, we get from (5.6) that

$$0 = 4(x - 1) + 5(0 - 2) + 6(0 - 3) = 4x - 4 - 10 - 18,$$

so that $4x = 32$ or $x = 8$. The intersection of the plane with the x-axis is then the point (8, 0, 0). Similarly, you can find the intersections of the plane with the y- and z-axes: $\left(0, \frac{32}{5}, 0\right)$ and $\left(0, 0, \frac{16}{3}\right)$, respectively. Using these three points, we can draw the plane

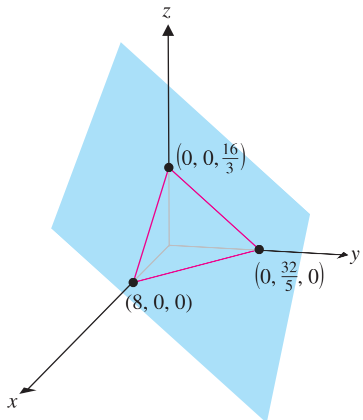


FIGURE 10.46
The plane through (8, 0, 0), $\left(0, \frac{32}{5}, 0\right)$ and $\left(0, 0, \frac{16}{3}\right)$

seen in Figure 10.46. We start by drawing the triangle with vertices at the three points; the plane we want is the one containing this triangle. Notice that since the plane intersects all three of the coordinate axes, the portion of the plane in the first octant is the indicated triangle. ■

Note that if we expand out the expression in (5.5), we get

$$\begin{aligned} 0 &= a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) \\ &= a_1 x + a_2 y + a_3 z + \underbrace{(-a_1 x_1 - a_2 y_1 - a_3 z_1)}_{\text{constant}}. \end{aligned}$$

We refer to this last equation as a **linear equation** in the three variables x, y and z. In particular, this says that every linear equation of the form

$$0 = ax + by + cz + d,$$

where a, b, c and d are constants, is the equation of a plane with normal vector $\langle a, b, c\rangle$.

We observed earlier that three points determine a plane. But, how can you find an equation of a plane given only three points? If you are to use (5.5), you'll first need to find a normal vector. We can easily resolve this, as in example 5.5.

EXAMPLE 5.5 Finding the Equation of a Plane Given Three Points

Find the plane containing the three points $P(1, 2, 2)$, $Q(2, -1, 4)$ and $R(3, 5, -2)$.

Solution First, we'll need to find a vector normal to the plane. Notice that two vectors lying in the plane are

$$\overrightarrow{PQ} = \langle 1, -3, 2\rangle \quad \text{and} \quad \overrightarrow{QR} = \langle 1, 6, -6\rangle.$$

Consequently, a vector orthogonal to both of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ is the cross product

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 1 & 6 & -6 \end{vmatrix} = \langle 6, 8, 9\rangle.$$

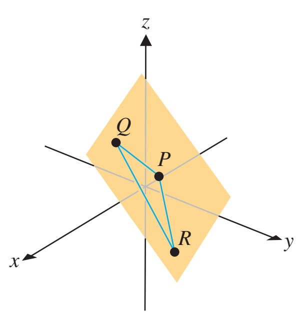


FIGURE 10.47
Plane containing three points

Since $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are not parallel, $\overrightarrow{PQ} \times \overrightarrow{QR}$ must be orthogonal to the plane, as well. (Why is that?) From (5.5), an equation for the plane is then

$$0 = 6(x - 1) + 8(y - 2) + 9(z - 2).$$

In Figure 10.47, we show the triangle with vertices at the three points. The plane in question is the one containing the indicated triangle. ■

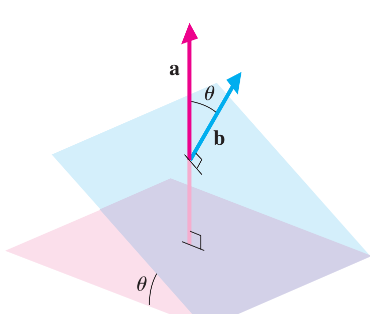


FIGURE 10.48
Angle between planes

In three dimensions, two planes are either parallel or they intersect in a straight line. (Think about this some.) Suppose that two planes having normal vectors **a** and **b**, respectively, intersect. Then the angle between the planes is the same as the angle between **a** and **b** (see Figure 10.48). With this in mind, we say that the two planes are **parallel** whenever their normal vectors are parallel and the planes are **orthogonal** whenever their normal vectors are orthogonal.

EXAMPLE 5.6 The Equation of a Plane Given a Point and a Parallel Plane

Find an equation for the plane through the point $(1, 4, -5)$ and parallel to the plane defined by $2x - 5y + 7z = 12$.

Solution First, notice that a normal vector to the given plane is $\langle 2, -5, 7\rangle$. Since the two planes are to be parallel, this vector is also normal to the new plane. From (5.5), we can write down the equation of the plane:

$$0 = 2(x-1) - 5(y-4) + 7(z+5).$$

It's particularly easy to see that some planes are parallel to the coordinate planes.

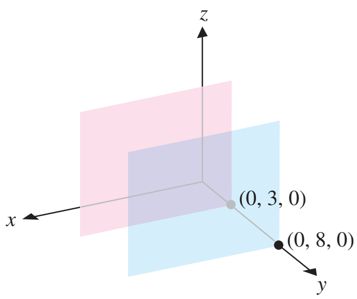


FIGURE 10.49
The planes $y = 3$ and $y = 8$

EXAMPLE 5.7 Drawing Some Simple Planes

Draw the planes $y = 3$ and $y = 8$.

Solution First, notice that both equations represent planes with the same normal vector, $\langle 0, 1, 0\rangle = \mathbf{j}$. This says that the planes are both parallel to the xz-plane, the first one passing through the point (0, 3, 0) and the second one passing through (0, 8, 0), as seen in Figure 10.49.

You should recognize that the intersection of two nonparallel planes will be a line. (Think about this some!) In example 5.8, we see how to find an equation of the line of intersection.

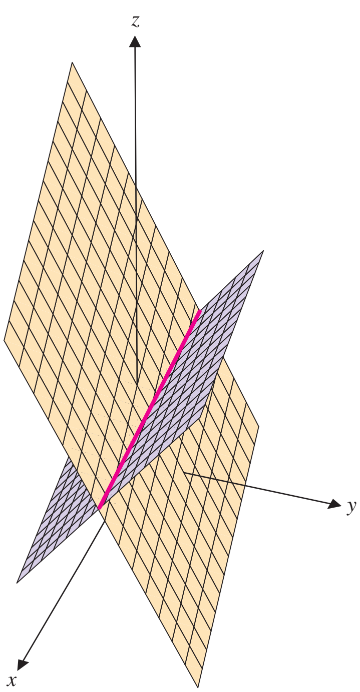


FIGURE 10.50
Intersection of planes

EXAMPLE 5.8 Finding the Intersection of Two Planes

Find the line of intersection of the planes $x + 2y + z = 3$ and $x - 4y + 3z = 5$.

Solution Solving both equations for x, we get

$$x = 3 - 2y - z \quad \text{and} \quad x = 5 + 4y - 3z. \tag{5.7}$$

Setting these expressions for x equal gives us

$$3 - 2y - z = 5 + 4y - 3z.$$

Solving this for z gives us

$$2z = 6y + 2 \quad \text{or} \quad z = 3y + 1.$$

Returning to either equation in (5.7), we can solve for x (also in terms of y). We have

$$x = 3 - 2y - z = 3 - 2y - (3y + 1) = -5y + 2.$$

Taking y as the parameter (i.e., letting $y = t$), we obtain parametric equations for the line of intersection:

$$x = -5t + 2, \quad y = t \quad \text{and} \quad z = 3t + 1.$$

You can see the line of intersection in the computer-generated graph of the two planes seen in Figure 10.50.

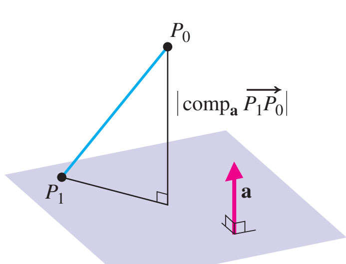


FIGURE 10.51
Distance from a point to a plane

Observe that the distance from the plane $ax + by + cz + d = 0$ to a point $P_0(x_0, y_0, z_0)$ not on the plane is measured along a line segment connecting the point to the plane that is orthogonal to the plane (see Figure 10.51). To compute this distance, pick any point $P_1(x_1, y_1, z_1)$ lying in the plane and let $\mathbf{a} = \langle a, b, c\rangle$ denote a vector normal to the plane. From Figure 10.51 notice that the distance from P_0 to the plane is simply $|\text{comp}_{\mathbf{a}} \overrightarrow{P_1P_0}|$, where

$$\overrightarrow{P_1P_0} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle.$$

From (3.7), the distance is

$$\begin{aligned}\left|\text{comp}_{\mathbf{a}}\,\overrightarrow{P_1P_0}\right| &= \left|\overrightarrow{P_1P_0}\cdot\frac{\mathbf{a}}{\|\mathbf{a}\|}\right|\\ &= \left|\langle x_0-x_1,\,y_0-y_1,\,z_0-z_1\rangle\cdot\frac{\langle a,b,c\rangle}{\|\langle a,b,c\rangle\|}\right|\\ &= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}}\\ &= \frac{|ax_0+by_0+cz_0-(ax_1+by_1+cz_1)|}{\sqrt{a^2+b^2+c^2}}\\ &= \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}},\end{aligned} \tag{5.8}$$

since (x_1, y_1, z_1) lies in the plane and $ax + by + cz = -d$, for every point (x, y, z) in the plane.

EXAMPLE 5.9 Finding the Distance between Parallel Planes

Find the distance between the parallel planes:

$$P_1\colon 2x - 3y + z = 6$$

and

$$P_2\colon 4x - 6y + 2z = 8.$$

Solution First, observe that the planes are parallel, since their normal vectors $\langle 2, -3, 1\rangle$ and $\langle 4, -6, 2\rangle$ are parallel. Further, since the planes are parallel, the distance from the plane P_1 to every point in the plane P_2 is the same. So, pick any point in P_2, say $(0, 0, 4)$. (This is certainly convenient.) The distance d from the point $(0, 0, 4)$ to the plane P_1 is then given by (5.8) to be

$$d = \frac{|(2)(0) - (3)(0) + (1)(4) - 6|}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{2}{\sqrt{14}}. \blacksquare$$

BEYOND FORMULAS

Both lines and planes are defined in this section in terms of a point and a vector. To avoid confusing the equations of lines and planes, focus on understanding the derivation of each equation. Parametric equations of a line simply express the line in terms of a starting point and a direction of motion. The equation of a plane is simply an expanded version of the dot product equation for the normal vector being perpendicular to the plane. Hopefully, you have discovered that a formula is easier to memorize if you understand the logic behind the result.

EXERCISES 10.5

WRITING EXERCISES

1. Explain how to shift back and forth between the parametric and symmetric equations of a line. Describe one situation in which you would prefer to have parametric equations to work with and one situation in which symmetric equations would be more convenient.

2. Lines and planes can both be specified with a point and a vector. Discuss the differences in the vectors used, and explain why the normal vector of the plane specifies an entire plane, while the direction vector of the line merely specifies a line.

3. Notice that if $c = 0$ in the general equation $ax + by + cz + d = 0$ of a plane, you have an equation that would describe a line in the xy-plane. Describe how this line relates to the plane.

4. Our hint about visualizing skew lines was to place the lines in parallel planes. Discuss whether every pair of skew lines must necessarily lie in parallel planes. (Hint: Discuss how the cross product of the direction vectors of the lines would relate to the parallel planes.)

In exercises 1–10, find (a) parametric equations and (b) symmetric equations of the line.

1. The line through $(1, 2, -3)$ and parallel to $\langle 2, -1, 4\rangle$
2. The line through $(3, -2, 4)$ and parallel to $\langle 3, 2, -1\rangle$
3. The line through $(2, 1, 3)$ and $(4, 0, 4)$
4. The line through $(0, 2, 1)$ and $(2, 0, 2)$
5. The line through $(1, 4, 1)$ and parallel to the line $x = 2 - 3t$, $y = 4$, $z = 6 + t$
6. The line through $(-1, 0, 0)$ and parallel to the line $\dfrac{x+1}{-2} = \dfrac{y}{3} = z - 2$
7. The line through $(2, 0, 1)$ and perpendicular to both $\langle 1, 0, 2\rangle$ and $\langle 0, 2, 1\rangle$
8. The line through $(-3, 1, 0)$ and perpendicular to both $\langle 0, -3, 1\rangle$ and $\langle 4, 2, -1\rangle$
9. The line through $(1, 2, -1)$ and normal to the plane $2x - y + 3z = 12$
10. The line through $(0, -2, 1)$ and normal to the plane $y + 3z = 4$

In exercises 11–16, state whether the lines are parallel or perpendicular and find the angle between the lines.

11. $\begin{cases} x = 1 - 3t \\ y = 2 + 4t \\ z = -6 + t \end{cases}$ and $\begin{cases} x = 1 + 2s \\ y = 2 - 2s \\ z = -6 + s \end{cases}$

12. $\begin{cases} x = 4 - 2t \\ y = 3t \\ z = -1 + 2t \end{cases}$ and $\begin{cases} x = 4 + s \\ y = -2s \\ z = -1 + 3s \end{cases}$

13. $\begin{cases} x = 1 + 2t \\ y = 3 \\ z = -1 + t \end{cases}$ and $\begin{cases} x = 2 - s \\ y = 10 + 5s \\ z = 3 + 2s \end{cases}$

14. $\begin{cases} x = 1 - 2t \\ y = 2t \\ z = 5 - t \end{cases}$ and $\begin{cases} x = 3 + 2s \\ y = -2 - 2s \\ z = 6 + s \end{cases}$

15. $\begin{cases} x = -1 + 2t \\ y = 3 + 4t \\ z = -6t \end{cases}$ and $\begin{cases} x = 3 - s \\ y = 1 - 2s \\ z = 3s \end{cases}$

16. $\begin{cases} x = 3 - t \\ y = 4 \\ z = -2 + 2t \end{cases}$ and $\begin{cases} x = 1 + 2s \\ y = 7 - 3s \\ z = -3 + s \end{cases}$

In exercises 17–20, determine whether the lines are parallel, skew or intersect.

17. $\begin{cases} x = 4 + t \\ y = 2 \\ z = 3 + 2t \end{cases}$ and $\begin{cases} x = 2 + 2s \\ y = 2s \\ z = -1 + 4s \end{cases}$

18. $\begin{cases} x = 3 + t \\ y = 3 + 3t \\ z = 4 - t \end{cases}$ and $\begin{cases} x = 2 - s \\ y = 1 - 2s \\ z = 6 + 2s \end{cases}$

19. $\begin{cases} x = 1 + 2t \\ y = 3 \\ z = -1 - 4t \end{cases}$ and $\begin{cases} x = 2 - s \\ y = 2 \\ z = 3 + 2s \end{cases}$

20. $\begin{cases} x = 1 - 2t \\ y = 2t \\ z = 5 - t \end{cases}$ and $\begin{cases} x = 3 + 2s \\ y = -2 \\ z = 3 + 2s \end{cases}$

In exercises 21–30, find an equation of the given plane.

21. The plane containing the point $(1, 3, 2)$ with normal vector $\langle 2, -1, 5\rangle$
22. The plane containing the point $(-2, 1, 0)$ with normal vector $\langle -3, 0, 2\rangle$
23. The plane containing the points $(2, 0, 3)$, $(1, 1, 0)$ and $(3, 2, -1)$
24. The plane containing the points $(1, -2, 1)$, $(2, -1, 0)$ and $(3, -2, 2)$
25. The plane containing the points $(-2, 2, 0)$, $(-2, 3, 2)$ and $(1, 2, 2)$
26. The plane containing the point $(3, -2, 1)$ and parallel to the plane $x + 3y - 4z = 2$
27. The plane containing the point $(0, -2, -1)$ and parallel to the plane $-2x + 4y = 3$
28. The plane containing the point $(3, 1, 0)$ and parallel to the plane $-3x - 3y + 2z = 4$
29. The plane containing the point $(1, 2, 1)$ and perpendicular to the planes $x + y = 2$ and $2x + y - z = 1$

30. The plane containing the point $(3, 0, -1)$ and perpendicular to the planes $x + 2y - z = 2$ and $2x - z = 1$

In exercises 31–40, sketch the given plane.

31. $x + y + z = 4$

32. $2x - y + 4z = 4$

33. $3x + 6y - z = 6$

34. $2x + y + 3z = 6$

35. $x = 4$

36. $y = 3$

37. $z = 2$

38. $x + y = 1$

39. $2x - z = 2$

40. $y = x + 2$

In exercises 41–44, find the intersection of the planes.

41. $2x - y - z = 4$ and $3x - 2y + z = 0$

42. $3x + y - z = 2$ and $2x - 3y + z = -1$

43. $3x + 4y = 1$ and $x + y - z = 3$

44. $x - 2y + z = 2$ and $x + 3y - 2z = 0$

In exercises 45–50, find the distance between the given objects.

45. The point $(2, 0, 1)$ and the plane $2x - y + 2z = 4$

46. The point $(1, 3, 0)$ and the plane $3x + y - 5z = 2$

47. The point $(2, -1, -1)$ and the plane $x - y + z = 4$

48. The point $(0, -1, 1)$ and the plane $2x - 3y = 2$

49. The planes $2x - y - z = 1$ and $2x - y - z = 4$

50. The planes $x + 3y - 2z = 3$ and $x + 3y - 2z = 1$

51. Show that the distance between planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is given by $\dfrac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$.

52. Suppose that $\langle 2, 1, 3\rangle$ is a normal vector for a plane containing the point $(2, -3, 4)$. Show that an equation of the plane is $2x + y + 3z = 13$. Explain why another normal vector for this plane is $\langle -4, -2, -6\rangle$. Use this normal vector to find an equation of the plane and show that the equation reduces to the same equation, $2x + y + 3z = 13$.

53. Find an equation of the plane containing the lines $\begin{cases} x = 4 + t \\ y = 2 \\ z = 3 + 2t \end{cases}$ and $\begin{cases} x = 2 + 2s \\ y = 2s \\ z = -1 + 4s \end{cases}$.

54. Find an equation of the plane containing the lines $\begin{cases} x = 1 - t \\ y = 2 + 3t \\ z = 2t \end{cases}$ and $\begin{cases} x = 1 - s \\ y = 5 \\ z = 4 - 2s \end{cases}$.

In exercises 55–62, state whether the statement is true or false (not always true).

55. Two planes either are parallel or intersect.

56. The intersection of two planes is a line.

57. The intersection of three planes is a point.

58. Lines that lie in parallel planes are always skew.

59. The set of all lines perpendicular to a given line forms a plane.

60. There is one line perpendicular to a given plane.

61. The set of all points equidistant from two given points forms a plane.

62. The set of all points equidistant from two given planes forms a plane.

In exercises 63–66, determine whether the given lines or planes are the same.

63. $x = 3 - 2t, y = 3t, z = t - 2$ and $x = 1 + 4t, y = 3 - 6t, z = -1 - 2t$

64. $x = 1 + 4t, y = 2 - 2t, z = 2 + 6t$ and $x = 9 - 2t, y = -2 + t, z = 8 - 3t$

65. $2(x - 1) - (y + 2) + (z - 3) = 0$ and $4x - 2y + 2z = 2$

66. $3(x + 1) + 2(y - 2) - 3(z + 1) = 0$ and $6(x - 2) + 4(y + 1) - 6z = 0$

67. Suppose two airplanes fly paths described by the parametric equations $P_1: \begin{cases} x = 3 \\ y = 6 - 2t \\ z = 3t + 1 \end{cases}$ and $P_2: \begin{cases} x = 1 + 2s \\ y = 3 + s \\ z = 2 + 2s \end{cases}$. Describe the shape of the flight paths. If $t = s$ represents time, determine whether the paths intersect. Determine if the planes collide.

EXPLORATORY EXERCISES

1. Compare the equations that we have developed for the distance between a (two-dimensional) point and a line and for a (three-dimensional) point and a plane. Based on these equations, hypothesize a formula for the distance between the (four-dimensional) point (x_1, y_1, z_1, w_1) and the hyperplane $ax + by + cz + dw + e = 0$.

2. In this exercise, we will explore the geometrical object determined by the parametric equations $\begin{cases} x = 2s + 3t \\ y = 3s + 2t \\ z = s + t \end{cases}$. Given that there are two parameters, what dimension do you expect the object to have? Given that the individual parametric equations are linear, what do you expect the object to be? Show that the points $(0, 0, 0)$, $(2, 3, 1)$ and $(3, 2, 1)$ are on the object. Find an equation of the plane containing these three points. Substitute in the equations for x, y and z and show that the object lies in the plane. Argue that the object is, in fact, the entire plane.

10.6 SURFACES IN SPACE

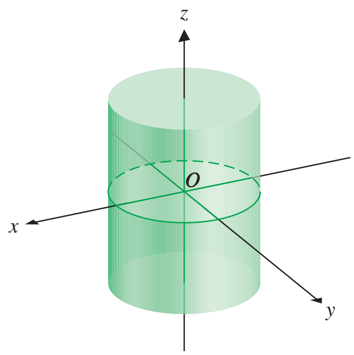


FIGURE 10.52
Right circular cylinder

Now that we have discussed lines and planes in $\mathbb{R}^3$, we continue our graphical development by drawing more complicated objects in three dimensions. Don't expect a general theory like we developed for two-dimensional graphs. Drawing curves and surfaces in three dimensions by hand or correctly interpreting computer-generated graphics is something of an art. After all, you must draw a two-dimensional image that somehow represents an object in three dimensions. Our goal here is not to produce artists, but rather to leave you with the ability to deal with a small group of surfaces in three dimensions. In numerous exercises throughout the rest of the book, taking a few extra minutes to draw a better graph will often result in a huge savings of time and effort.

Cylindrical Surfaces

We begin with a simple type of three-dimensional surface. When you see the word *cylinder,* you probably think of a right circular cylinder. For instance, consider the graph of the equation $x^2 + y^2 = 9$ in *three* dimensions. While the graph of $x^2 + y^2 = 9$ in *two* dimensions is the circle of radius 3, centered at the origin, what is its graph in *three* dimensions? Consider the intersection of the surface with the plane $z = k$, for some constant k. Since the equation has no z's in it, the intersection with every such plane (called the **trace** of the surface in the plane $z = k$) is the same: a circle of radius 3, centered at the origin. Think about it: whatever this three-dimensional surface is, its intersection with every plane parallel to the xy-plane is a circle of radius 3, centered at the origin. This describes a right circular cylinder, in this case one of radius 3, whose axis is the z-axis (see Figure 10.52).

More generally, the term **cylinder** is used to refer to any surface whose traces in every plane parallel to a given plane are the same. With this definition, many surfaces qualify as cylinders.

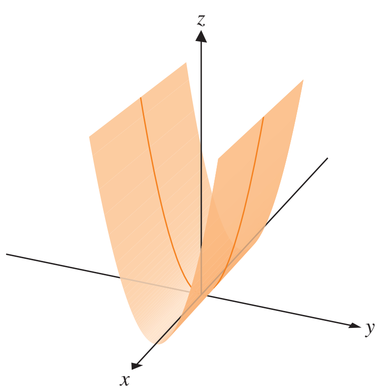


FIGURE 10.53a
$z = y^2$

EXAMPLE 6.1 Sketching a Surface

Draw a graph of the surface $z = y^2$ in $\mathbb{R}^3$.

Solution Since there are no x's in the equation, the trace of the graph in the plane $x = k$ is the same for every k. This is then a cylinder whose trace in every plane parallel to the yz-plane is the parabola $z = y^2$. To draw this, we first draw the trace in the yz-plane and then make several copies of the trace, locating the vertices at various points along the x-axis. Finally, we connect the traces with lines parallel to the x-axis to give the drawing its three-dimensional look (see Figure 10.53a). A computer-generated wireframe graph of the same surface is seen in Figure 10.53b. Notice that the wireframe consists of numerous traces for fixed values of x or y. ■

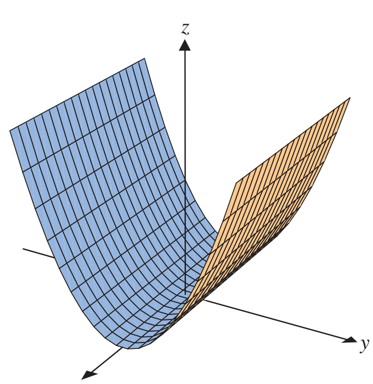


FIGURE 10.53b
Wireframe of $z = y^2$

EXAMPLE 6.2 Sketching an Unusual Cylinder

Draw a graph of the surface $z = \sin x$ in $\mathbb{R}^3$.

Solution Once again, one of the variables is missing; in this case, there are no y's. Consequently, traces of the surface in any plane parallel to the xz-plane are the same; they all look like the two-dimensional graph of $z = \sin x$. We draw one of these in the xz-plane and then make copies in planes parallel to the xz-plane, finally connecting the traces with lines parallel to the y-axis (see Figure 10.54a). In Figure 10.54b, we show a computer-generated wireframe plot of the same surface. In this case, the cylinder looks like a plane with ripples in it.

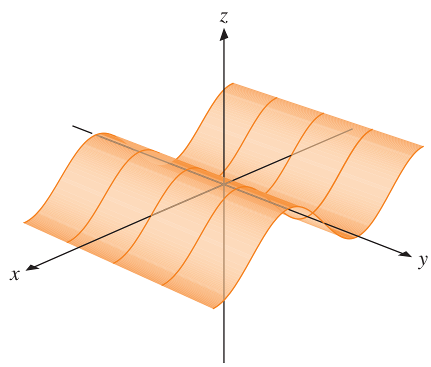


FIGURE 10.54a
The surface $z = \sin x$

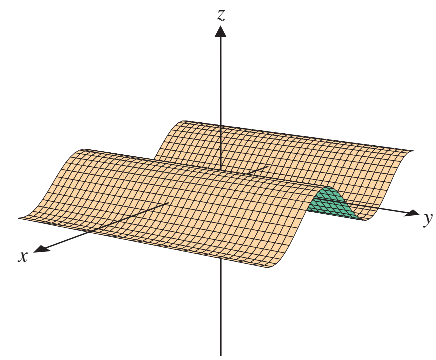


FIGURE 10.54b
Wireframe: $z = \sin x$

Quadric Surfaces

The graph of the equation

$$ax^2 + by^2 + cz^2 + dxy + eyz + fxz + gx + hy + jz + k = 0$$

in three-dimensional space (where $a, b, c, d, e, f, g, h, j$ and k are all constants and at least one of a, b, c, d, e or f is nonzero) is referred to as a **quadric surface.**

The most familiar quadric surface is the **sphere:**

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

of radius r centered at the point (a, b, c). To draw the sphere centered at $(0, 0, 0)$, first draw a circle of radius r, centered at the origin in the yz-plane. Then, to give the surface its three-dimensional look, draw circles of radius r centered at the origin, in both the xz- and xy-planes, as in Figure 10.55. Note that due to the perspective, these circles will look like ellipses and will be only partially visible. (We indicate the hidden parts of the circles with dashed lines.)

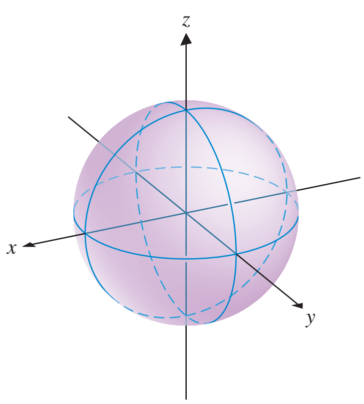


FIGURE 10.55
Sphere

A generalization of the sphere is the **ellipsoid:**

$$\frac{(x - a)^2}{d^2} + \frac{(y - b)^2}{e^2} + \frac{(z - c)^2}{f^2} = 1.$$

(Notice that when $d = e = f$, the surface is a sphere.)

EXAMPLE 6.3 Sketching an Ellipsoid

Graph the ellipsoid

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1.$$

Solution First draw the traces in the three coordinate planes. (In general, you may need to look at the traces in planes parallel to the three coordinate planes, but the traces in the three coordinate planes will suffice, here.) In the yz-plane, $x = 0$, which gives us the ellipse

$$\frac{y^2}{4} + \frac{z^2}{9} = 1,$$

shown in Figure 10.56a. Next, add to Figure 10.56a the traces in the xy- and xz-planes. These are

$$\frac{x^2}{1} + \frac{y^2}{4} = 1 \quad \text{and} \quad \frac{x^2}{1} + \frac{z^2}{9} = 1,$$

respectively, which are both ellipses (see Figure 10.56b).

CASs have the capability of plotting functions of several variables in three dimensions. Many graphing calculators with three-dimensional plotting capabilities produce three-dimensional plots only when given z as a function of x and y. For the problem at hand, notice that we can solve for z and plot the two functions $z = 3\sqrt{1 - x^2 - \frac{y^2}{4}}$ and $z = -3\sqrt{1 - x^2 - \frac{y^2}{4}}$, to obtain the graph of the surface. Observe that the wireframe graph in Figure 10.56c (on the following page) is not particularly smooth and appears to have some gaps. To correctly interpret such a graph, you must mentally fill in the gaps. This requires an understanding of how the graph should look, which we obtained drawing Figure 10.56b.

As an alternative, many CASs enable you to graph the equation $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ using **implicit plot** mode. In this mode, the CAS numerically solves the equation for the value of z corresponding to each one of a large number of sample values of x and y and plots the resulting points. The graph obtained in Figure 10.56d is an improvement over Figure 10.56c, but doesn't show the elliptical traces that we used to construct Figure 10.56b.

The best option, when available, is often a **parametric plot.** In three dimensions, this involves writing each of the three variables x, y and z in terms of two parameters, with the resulting surface plotted by plotting points corresponding to a sample of values of the two parameters. (A more extensive discussion of the mathematics of parametric surfaces is given in section 11.6.) As we develop in the exercises, parametric equations for the ellipsoid are $x = \sin s \cos t$, $y = 2 \sin s \sin t$ and $z = 3 \cos s$, with the parameters taken to be in the intervals $0 \le s \le 2\pi$ and $0 \le t \le 2\pi$. Notice how Figure 10.56e shows a nice smooth plot and clearly shows the elliptical traces.

TODAY IN MATHEMATICS

Grigori Perelman (1966–)
A Russian mathematician whose work on a famous problem known as Poincaré's conjecture is still (as of 2005) undergoing mathematical scrutiny to verify its correctness. Perelman showed his mathematical prowess as a high school student, winning a gold medal at the 1982 International Mathematical Olympiad. Eight years of isolated work produced what he believes proves a deep result known as Thurston's Geometrization Conjecture, which is more general than the 100-year-old Poincaré conjecture. Perelman's techniques have already enabled other researchers to make breakthroughs on less famous problems and are important advances in our understanding of the geometry of three-dimensional space.

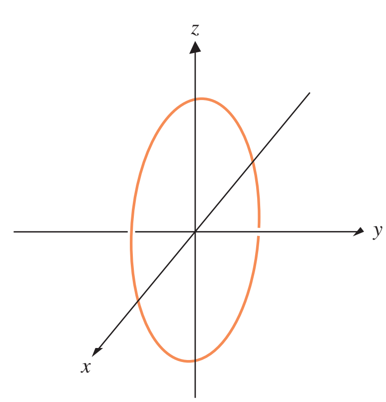


FIGURE 10.56a
Ellipse in yz-plane

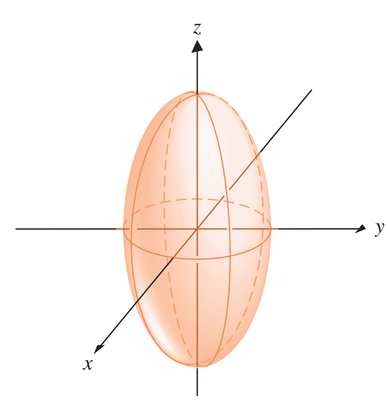


FIGURE 10.56b
Ellipsoid

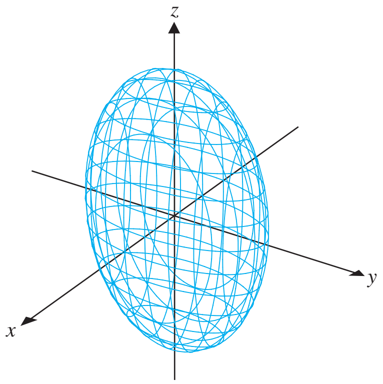


FIGURE 10.56c
Wireframe ellipsoid

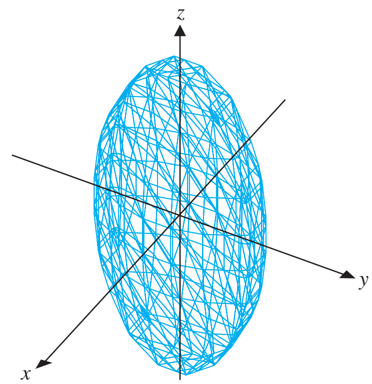


FIGURE 10.56d
Implicit wireframe plot

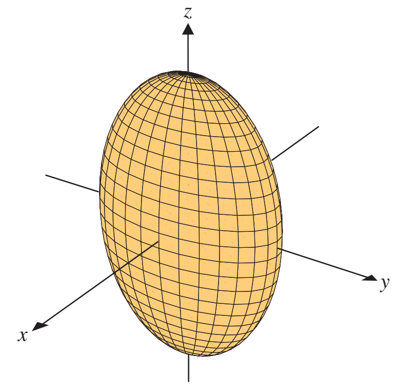


FIGURE 10.56e
Parametric plot

EXAMPLE 6.4 Sketching a Paraboloid

Draw a graph of the quadric surface

$$x^2 + y^2 = z.$$

Solution To get an idea of what the graph looks like, first draw its traces in the three coordinate planes. In the yz-plane, we have $x = 0$ and so, $y^2 = z$ (a parabola). In the xz-plane, we have $y = 0$ and so, $x^2 = z$ (a parabola). In the xy-plane, we have $z = 0$ and so, $x^2 + y^2 = 0$ (a point—the origin). We sketch the traces in Figure 10.57a. Finally, since the trace in the xy-plane is just a point, we consider the traces in the planes $z = k$ (for $k > 0$). Notice that these are the circles $x^2 + y^2 = k$, where for larger values of z (i.e., larger values of k), we get circles of larger radius. We sketch the surface in Figure 10.57b. Such surfaces are called **paraboloids** and since the traces in planes parallel to the xy-plane are circles, this is called a **circular paraboloid.**

Graphing utilities with three-dimensional capabilities generally produce a graph like Figure 10.57c for $z = x^2 + y^2$. Notice that the parabolic traces are visible, but not the circular cross sections we drew in Figure 10.57b. The four peaks visible in Figure 10.57c are due to the rectangular domain used for the plot (in this case, $-5 \le x \le 5$ and $-5 \le y \le 5$).

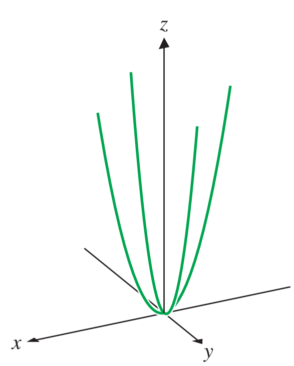


FIGURE 10.57a
Traces

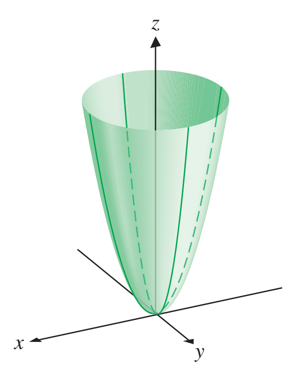


FIGURE 10.57b
Paraboloid

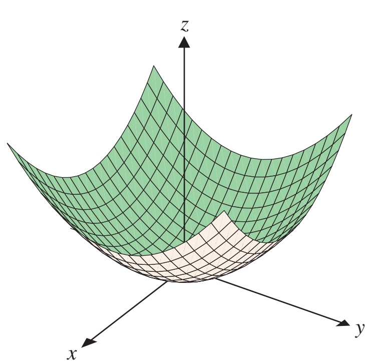


FIGURE 10.57c
Wireframe paraboloid

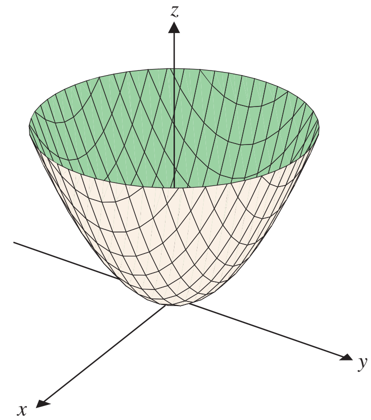


FIGURE 10.57d
Wireframe paraboloid for $0 \leq z \leq 15$

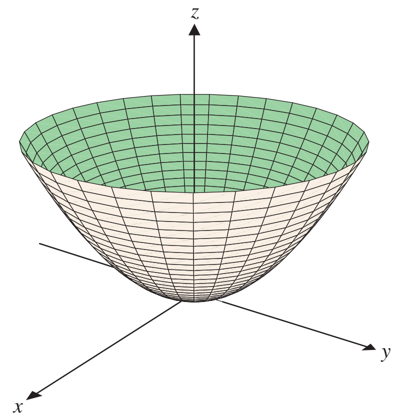


FIGURE 10.57e
Parametric plot paraboloid

We can improve this by restricting the range of the z-values. With $0 \leq z \leq 15$, you can clearly see the circular cross section in the plane $z = 15$ in Figure 10.57d.

As in example 6.3, a parametric surface plot is even better. Here, we have $x = s\cos t$, $y = s\sin t$ and $z = s^2$, with $0 \leq s \leq 5$ and $0 \leq t \leq 2\pi$. Figure 10.57e clearly shows the circular cross sections in the planes $z = k$, for $k > 0$. ■

Notice that in each of the last several examples, we have had to use some thought to produce computer-generated graphs that adequately show the important features of the given quadric surface. We want to encourage you to use your graphing calculator or CAS for drawing three-dimensional plots, because computer graphics are powerful tools for visualization and problem solving. However, be aware that you will need a basic understanding of the geometry of quadric surfaces to effectively produce and interpret computer-generated graphs.

EXAMPLE 6.5 Sketching an Elliptic Cone

Draw a graph of the quadric surface

$$x^2 + \frac{y^2}{4} = z^2.$$

Solution While this equation may look a lot like that of an ellipsoid, there is a significant difference. (Look where the z^2 term is!) Again, we start by looking at the traces in the coordinate planes. For the yz-plane, we have $x = 0$ and so, $\frac{y^2}{4} = z^2$ or $y^2 = 4z^2$, so that $y = \pm 2z$. That is, the trace is a pair of lines: $y = 2z$ and $y = -2z$. We show these in Figure 10.58a. Likewise, the trace in the xz-plane is a pair of lines: $x = \pm z$. The trace in the xy-plane is simply the origin. (Why?) Finally, the traces in the planes $z = k$ $(k \neq 0)$, parallel to the xy-plane, are the ellipses $x^2 + \frac{y^2}{4} = k^2$. Adding these to the drawing gives us the double-cone seen in Figure 10.58b (on the following page).

Since the traces in planes parallel to the xy-plane are ellipses, we refer to this as an **elliptic cone.** One way to plot this with a CAS is to graph the two functions $z = \sqrt{x^2 + \frac{y^2}{4}}$ and $z = -\sqrt{x^2 + \frac{y^2}{4}}$. In Figure 10.58c, we restrict the z-range to

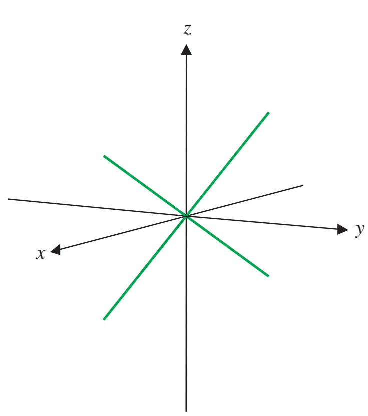


FIGURE 10.58a
Trace in yz-plane

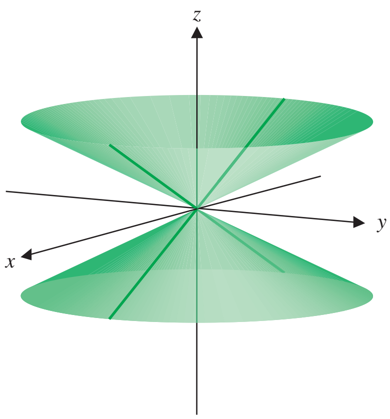


FIGURE 10.58b
Elliptic cone

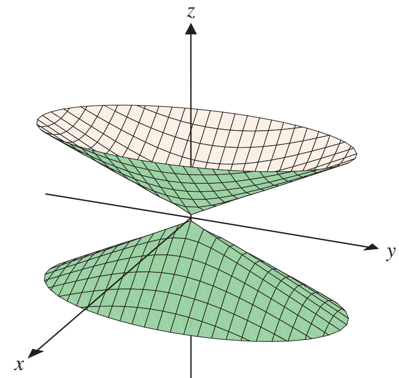


FIGURE 10.58c
Wireframe cone

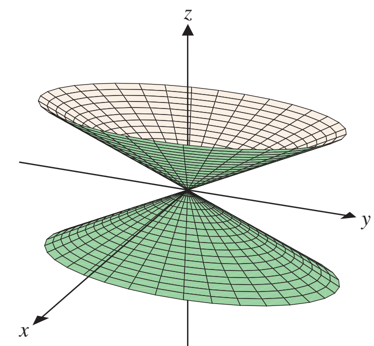


FIGURE 10.58d
Parametric plot

$-10 \le z \le 10$ to show the elliptical cross sections. Notice that this plot shows a gap between the two halves of the cone. If you have drawn Figure 10.58b yourself, this plotting deficiency won't fool you. Alternatively, the parametric plot shown in Figure 10.58d, with $x = \sqrt{s^2}\cos t$, $y = 2\sqrt{s^2}\sin t$ and $z = s$, with $-5 \le s \le 5$ and $0 \le t \le 2\pi$, shows the full cone with its elliptical and linear traces. ■

EXAMPLE 6.6 Sketching a Hyperboloid of One Sheet

Draw a graph of the quadric surface

$$\frac{x^2}{4} + y^2 - \frac{z^2}{2} = 1.$$

Solution The traces in the coordinate plane are as follows:

$$yz\text{-plane } (x = 0)\text{: } y^2 - \frac{z^2}{2} = 1 \text{ (hyperbola)}$$

(see Figure 10.59a),

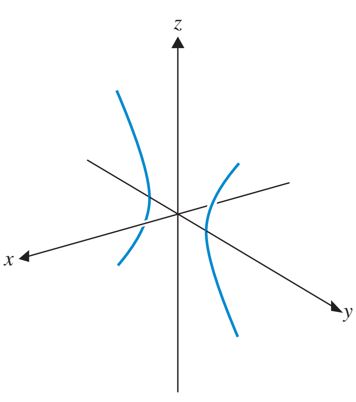


FIGURE 10.59a
Trace in yz-plane

$$xy\text{-plane } (z = 0)\text{: } \frac{x^2}{4} + y^2 = 1 \text{ (ellipse)}$$

and

$$xz\text{-plane } (y = 0)\text{: } \frac{x^2}{4} - \frac{z^2}{2} = 1 \text{ (hyperbola).}$$

Further, notice that the trace of the surface in each plane $z = k$ (parallel to the xy-plane) is also an ellipse:

$$\frac{x^2}{4} + y^2 = \frac{k^2}{2} + 1.$$

Finally, observe that the larger k is, the larger the axes of the ellipses are. Adding this information to Figure 10.59a, we draw the surface seen in Figure 10.59b. We call this surface a **hyperboloid of one sheet.**

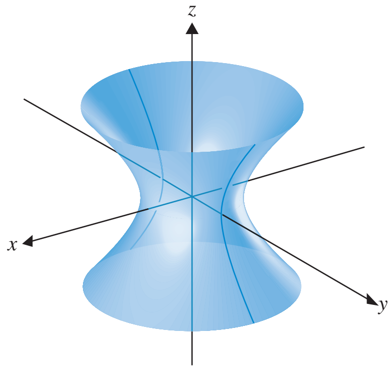


FIGURE 10.59b
Hyperboloid of one sheet

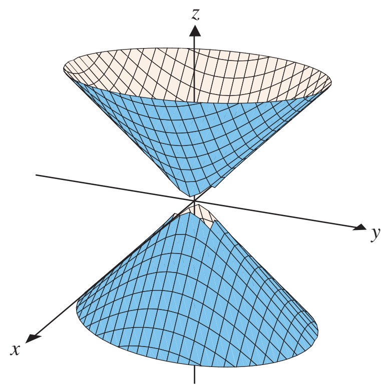


FIGURE 10.59c
Wireframe hyperboloid

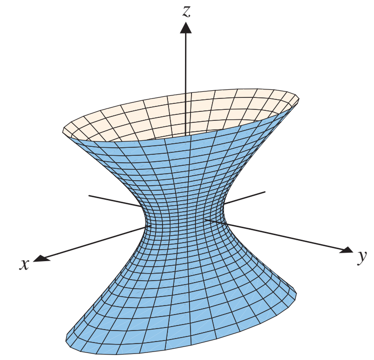


FIGURE 10.59d
Parametric plot

To plot this with a CAS, you could graph the two functions $z = \sqrt{2\left(\frac{x^2}{4} + y^2 - 1\right)}$ and $z = -\sqrt{2\left(\frac{x^2}{4} + y^2 - 1\right)}$. (See Figure 10.59c, where we have restricted the z-range to $-10 \le z \le 10$, to show the elliptical cross sections.) Notice that this plot looks more like a cone than the hyperboloid in Figure 10.59b. If you have drawn Figure 10.59b yourself, this plotting problem won't fool you.

Alternatively, the parametric plot seen in Figure 10.59d, with $x = 2\cos s \cosh t$, $y = \sin s \cosh t$ and $z = \sqrt{2}\sinh t$, with $0 \le s \le 2\pi$ and $-5 \le t \le 5$, shows the full hyperboloid with its elliptical and hyperbolic traces. ■

EXAMPLE 6.7 Sketching a Hyperboloid of Two Sheets

Draw a graph of the quadric surface

$$\frac{x^2}{4} - y^2 - \frac{z^2}{2} = 1.$$

Solution Notice that this is the same equation as in example 6.6, except for the sign of the y-term. As we have done before, we first look at the traces in the three coordinate planes. The trace in the yz-plane $(x = 0)$ is defined by

$$-y^2 - \frac{z^2}{2} = 1.$$

Since it is clearly impossible for two negative numbers to add up to something positive, this is a contradiction and there is no trace in the yz-plane. That is, the surface does not intersect the yz-plane. The traces in the other two coordinate planes are as follows:

$$xy\text{-plane } (z = 0)\text{: } \frac{x^2}{4} - y^2 = 1 \text{ (hyperbola)}$$

and

$$xz\text{-plane } (y = 0)\text{: } \frac{x^2}{4} - \frac{z^2}{2} = 1 \text{ (hyperbola).}$$

We show these traces in Figure 10.60a. Finally, notice that for $x = k$, we have that

$$y^2 + \frac{z^2}{2} = \frac{k^2}{4} - 1,$$

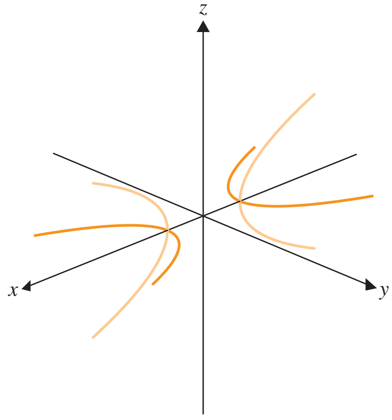


FIGURE 10.60a
Traces in xy- and xz-planes

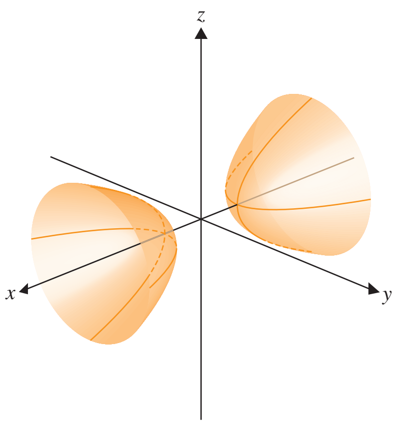


FIGURE 10.60b
Hyperboloid of two sheets

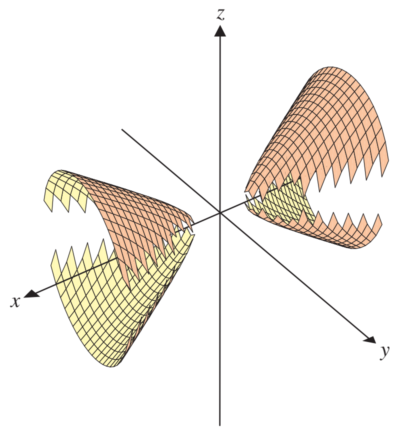


FIGURE 10.60c
Wireframe hyperboloid

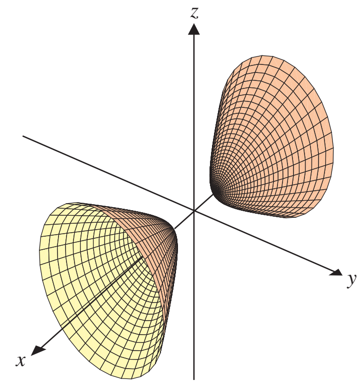


FIGURE 10.60d
Parametric plot

so that the traces in the plane $x = k$ are ellipses for $k^2 > 4$. It is important to notice here that if $k^2 < 4$, the equation $y^2 + \frac{z^2}{9} = \frac{k^2}{4} - 1$ has no solution. (Why is that?) So, for $-2 < k < 2$, the surface has no trace at all in the plane $x = k$, leaving a gap which separates the hyperbola into two *sheets*. Putting this all together, we have the surface seen in Figure 10.60b. We call this surface a **hyperboloid of two sheets.**

We can plot this on a CAS by graphing the two functions $z = \sqrt{2\left(\frac{x^2}{4} - y^2 - 1\right)}$ and $z = -\sqrt{2\left(\frac{x^2}{4} - y^2 - 1\right)}$. (See Figure 10.60c, where we have restricted the z-range to $-10 \leq z \leq 10$, to show the elliptical cross sections.) Notice that this plot shows large gaps between the two halves of the hyperboloid. If you have drawn Figure 10.60b yourself, this plotting deficiency won't fool you.

Alternatively, the parametric plot with $x = 2\cosh s$, $y = \sinh s \cos t$ and $z = \sqrt{2}\sinh s \sin t$, for $-4 \leq s \leq 4$ and $0 \leq t \leq 2\pi$, produces the left half of the hyperboloid with its elliptical and hyperbolic traces. The right half of the hyperboloid has parametric equations $x = -2\cosh s$, $y = \sinh s \cos t$ and $z = \sqrt{2}\sinh s \sin t$, with $-4 \leq s \leq 4$ and $0 \leq t \leq 2\pi$. We show both halves in Figure 10.60d. ■

As our final example, we offer one of the more interesting quadric surfaces. It is also one of the more difficult surfaces to sketch.

EXAMPLE 6.8 Sketching a Hyperbolic Paraboloid

Sketch the graph of the quadric surface defined by the equation

$$z = 2y^2 - x^2.$$

Solution We first consider the traces in planes parallel to each of the coordinate planes:

$$\text{parallel to } xy\text{-plane } (z = k)\text{: } 2y^2 - x^2 = k \text{ (hyperbola, for } k \neq 0\text{)},$$
$$\text{parallel to } xz\text{-plane } (y = k)\text{: } z = -x^2 + 2k^2 \text{ (parabola opening down)}$$

and $$\text{parallel to } yz\text{-plane } (x = k)\text{: } z = 2y^2 - k^2 \text{ (parabola opening up).}$$

We begin by drawing the traces in the xz- and yz-planes, as seen in Figure 10.61a. Since the trace in the xy-plane is the degenerate hyperbola $2y^2 = x^2$ (two lines: $x = \pm\sqrt{2}y$), we instead draw the trace in several of the planes $z = k$. Notice that for $k > 0$, these are hyperbolas opening toward the positive and negative y-direction and for $k < 0$, these are hyperbolas opening toward the positive and negative x-direction. We indicate one of these for $k > 0$ and one for $k < 0$ in Figure 10.61b, where we show a sketch of the surface. We refer to this surface as a **hyperbolic paraboloid.** More than anything else, the surface resembles a saddle. In fact, we refer to the origin as a **saddle point** for this graph. (We'll discuss the significance of saddle points in Chapter 12.)

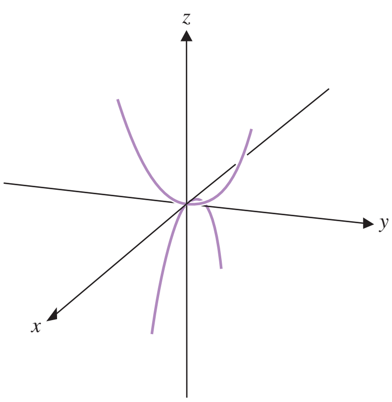


FIGURE 10.61a
Traces in the xz- and yz-planes

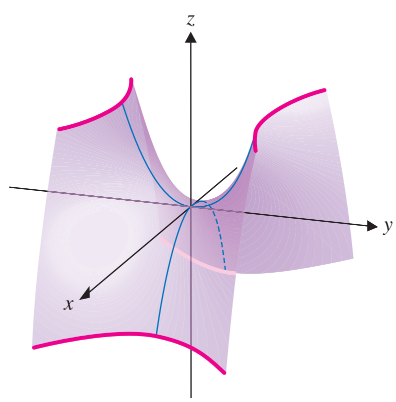


FIGURE 10.61b
The surface $z = 2y^2 - x^2$

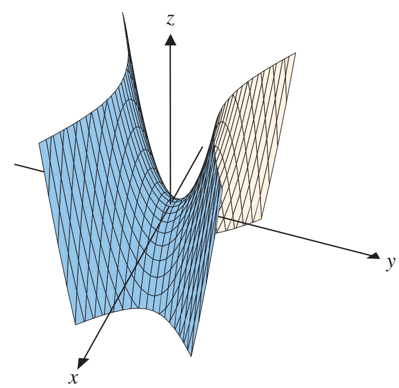


FIGURE 10.61c
Wireframe plot of $z = 2y^2 - x^2$

A wireframe graph of $z = 2y^2 - x^2$ is shown in Figure 10.61c (with $-5 \le x \le 5$ and $-5 \le y \le 5$ and where we limited the z-range to $-8 \le z \le 12$). Note that only the parabolic cross sections are drawn, but the graph shows all the features of Figure 10.61b. Plotting this surface parametrically is fairly tedious (requiring four different sets of equations) and doesn't improve the graph noticeably. ■

○ An Application

You may have noticed the large number of paraboloids around you. For instance, radiotelescopes and even home television satellite dishes have the shape of a portion of a paraboloid. Reflecting telescopes have parabolic mirrors that again, are a portion of a paraboloid. There is a very good reason for this. It turns out that in all of these cases, light waves and radio waves striking *any* point on the parabolic dish or mirror are reflected toward *one* point, the focus of each parabolic cross section through the vertex of the paraboloid. This remarkable fact means that all light waves and radio waves end up being concentrated at just one point. In the case of a radiotelescope, placing a small receiver just in front of the focus can take a very faint signal and increase its effective strength immensely (see Figure 10.62). The same principle is used in optical telescopes to concentrate the light from a faint source (e.g., a distant star). In this case, a small mirror is mounted in a line from the parabolic mirror to the focus. The small mirror then reflects the concentrated light to an eyepiece for viewing (see Figure 10.63).

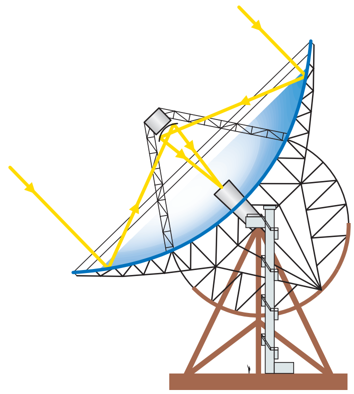


FIGURE 10.62
Radiotelescope

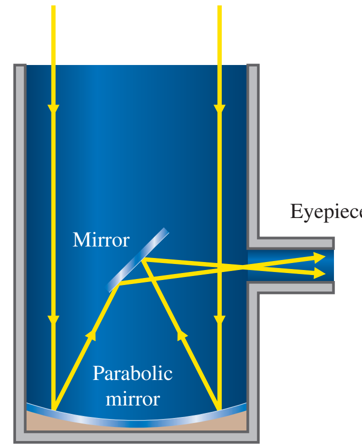
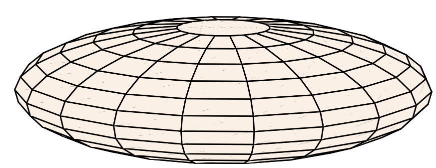
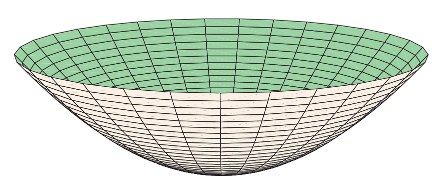
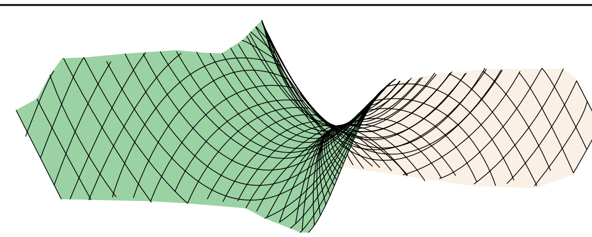
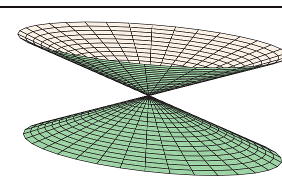
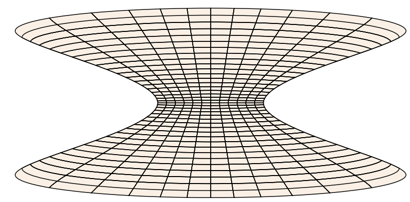
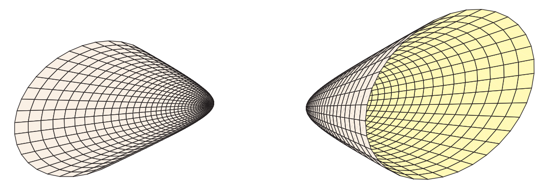


FIGURE 10.63
Reflecting telescope

The following table summarizes the graphs of quadric surfaces.

Name	*Generic Equation*	*Graph*
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	
Elliptic paraboloid	$z=ax^2+by^2+c$ $(a, b>0)$	
Hyperbolic paraboloid	$z=ax^2-by^2+c$ $(a, b>0)$	
Cone	$z^2=ax^2+by^2$ $(a, b>0)$	

Continued

Name	*Generic Equation*	*Graph*
Hyperboloid of one sheet	$ax^2 + by^2 - cz^2 = 1$ $(a, b, c > 0)$	
Hyperboloid of two sheets	$ax^2 - by^2 - cz^2 = 1$ $(a, b, c > 0)$	

EXERCISES 10.6

WRITING EXERCISES

1. In the text, different hints were given for graphing cylinders as opposed to quadric surfaces. Explain how to tell from the equation whether you have a cylinder, a quadric surface, a plane or some other surface.
2. The first step in graphing a quadric surface is identifying traces. Given the traces, explain how to tell whether you have an ellipsoid, elliptical cone, paraboloid or hyperboloid. (Hint: For a paraboloid, how many traces are parabolas?)
3. Suppose you have identified that a given equation represents a hyperboloid. Explain how to determine whether the hyperboloid has one sheet or two sheets.
4. Circular paraboloids have a bowl-like shape. However, the paraboloids $z = x^2 + y^2$, $z = 4 - x^2 - y^2$, $y = x^2 + z^2$ and $x = y^2 + z^2$ all open up in different directions. Explain why these paraboloids are different and how to determine in which direction a paraboloid opens.

In exercises 1–40, sketch the appropriate traces, and then sketch and identify the surface.

1. $z = x^2$
2. $z = 4 - y^2$
3. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$
4. $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} = 1$
5. $z = 4x^2 + 4y^2$
6. $z = x^2 + 4y^2$
7. $z^2 = 4x^2 + y^2$
8. $z^2 = \frac{x^2}{4} + \frac{y^2}{9}$
9. $z = x^2 - y^2$
10. $z = y^2 - x^2$
11. $x^2 - y^2 + z^2 = 1$
12. $x^2 + \frac{y^2}{4} - z^2 = 1$
13. $x^2 - \frac{y^2}{9} - z^2 = 1$
14. $x^2 - y^2 - \frac{z^2}{4} = 1$
15. $z = \cos x$
16. $z = \sqrt{x^2 + 4y^2}$
17. $z = 4 - x^2 - y^2$
18. $x = y^2 + z^2$
19. $z = x^3$
20. $z = 4 - y^2$
21. $z = \sqrt{x^2 + y^2}$
22. $z = \sin y$
23. $y = x^2$
24. $x = 2 - y^2$
25. $y = x^2 + z^2$
26. $z = 9 - x^2 - y^2$
27. $x^2 + 4y^2 + 16z^2 = 16$
28. $2x - z = 4$
29. $4x^2 - y^2 - z = 0$
30. $-x^2 - y^2 + 9z^2 = 9$
31. $4x^2 + y^2 - z^2 = 4$
32. $x^2 - y^2 + 9z^2 = 9$
33. $-4x^2 + y^2 - z^2 = 4$
34. $x^2 - 4y^2 + z = 0$
35. $x + y = 1$
36. $9x^2 + y^2 + 9z^2 = 9$
37. $x^2 + y^2 = 4$
38. $9x^2 + z^2 = 9$
39. $x^2 + y^2 - z = 4$
40. $x + y^2 + z^2 = 2$

In exercises 41–44, sketch the given traces on a single three-dimensional coordinate system.

41. $z = x^2 + y^2$; $x = 0, x = 1, x = 2$
42. $z = x^2 + y^2$; $y = 0, y = 1, y = 2$
43. $z = x^2 - y^2$; $x = 0, x = 1, x = 2$
44. $z = x^2 - y^2$; $y = 0, y = 1, y = 2$
45. Hyperbolic paraboloids are sometimes called “saddle” graphs. The architect of the Saddle Dome in the Canadian city of

Calgary used this shape to create an attractive and symbolically meaningful structure.

One issue in using this shape is water drainage from the roof. If the Saddle Dome roof is described by $z = x^2 - y^2$, $-1 \le x \le 1$, $-1 \le y \le 1$, in which direction would the water drain? First, consider traces for which y is constant. Show that the trace has a minimum at $x = 0$. Identify the plane $x = 0$ in the picture. Next, show that the trace at $x = 0$ has an absolute maximum at $y = 0$. Use this information to identify the two primary points at which the water would drain.

46. Cooling towers for nuclear reactors are often constructed as hyperboloids of one sheet because of the structural stability of that surface. (See the accompanying photo.) Suppose all horizontal cross sections are circular, with a minimum radius of 200 feet occurring at a height of 600 feet. The tower is to be 800 feet tall with a maximum cross-sectional radius of 300 feet. Find an equation for the structure.

47. If $x = a \sin s \cos t$, $y = b \sin s \sin t$ and $z = c \cos s$, show that (x, y, z) lies on the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

48. If $x = as \cos t$, $y = bs \sin t$ and $z = s^2$, show that (x, y, z) lies on the paraboloid $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.

49. If $x = as \cos t$, $y = bs \sin t$ and $z = s$, show that (x, y, z) lies on the cone $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$.

50. If $x = a \cos s \cosh t$, $y = b \sin s \cosh t$ and $z = c \sinh t$, show that (x, y, z) lies on the hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$.

51. If $a > 0$ and $x = a \cosh s$, $y = b \sinh s \cos t$ and $z = c \sinh s \sin t$, show that (x, y, z) lies on the right half of the hyperboloid of two sheets $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$.

52. If $a < 0$ and $x = a \cosh s$, $y = b \sinh s \cos t$ and $z = c \sinh s \sin t$, show that (x, y, z) lies on the left half of the hyperboloid of two sheets $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$.

53. Find parametric equations as in exercises 47–52 for the surfaces in exercises 3, 5 and 7. Use a CAS to graph the parametric surfaces.

54. Find parametric equations as in exercises 47–52 for the surfaces in exercises 11 and 13. Use a CAS to graph the parametric surfaces.

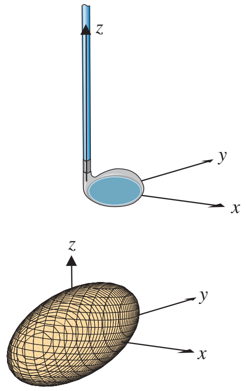
55. Find parametric equations for the surface in exercise 17.

56. Find parametric equations for the surface in exercise 33.

57. You can improve the appearance of a wireframe graph by carefully choosing the viewing window. We commented on the curved edge in Figure 10.57c. Graph this function with domain $-5 \le x \le 5$ and $-5 \le y \le 5$, but limit the z-range to $-1 \le z \le 20$. Does this look more like Figure 10.57b?

EXPLORATORY EXERCISES

1. Golf club manufacturers use ellipsoids (called **inertia ellipsoids**) to visualize important characteristics of golf clubs. A three-dimensional coordinate system is set up as shown in the figure. The (second) moments of inertia are then computed for the clubhead about each coordinate axis. The inertia ellipsoid is defined as $I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 + 2I_{xy}xy + 2I_{yz}yz + 2I_{xz}xz = 1$. The graph of this ellipsoid provides important information to the club designer. For comparison purposes, a homogeneous spherical shell would have a perfect sphere as its inertia ellipsoid. In *Science and Golf II,* the data given here are provided for a 6-iron and driver, respectively. Graph the ellipsoids and compare the shapes. (Recall that the larger the moment of inertia of an object, the harder it is to rotate.)



For the 6-iron, $89.4x^2 + 195.8y^2 + 124.9z^2 - 48.6xy - 111.8xz + 0.4yz = 1{,}000{,}000$ and for the driver, $119.3x^2 + 243.9y^2 + 139.4z^2 - 1.2xy - 71.4xz - 25.8yz = 1{,}000{,}000$.

2. Sketch the graphs of $x^2 + cy^2 - z^2 = 1$ for a variety of positive and negative constants c. If your CAS allows you to animate a sequence of graphs, set up an animation that shows a sequence of hyperboloids of one sheet morphing into hyperboloids of two sheets.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Vector	Scalar	Magnitude
Position vector	Unit vector	Displacement vector
First octant	Sphere	
Angle between vectors	Triangle Inequality	Dot product
	Cross product	Component
Projection	Parametric equations of line	Torque
Magnus force		Symmetric equations of line
Parallel planes	Orthogonal planes	
Traces	Cylinder	Skew lines
Circular paraboloid	Hyperbolic paraboloid	Ellipsoid
Hyperboloid of one sheet	Hyperboloid of two sheets	Cone
		Saddle

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. Two vectors are parallel if one vector equals the other divided by a constant.

2. For a given vector, there is one unit vector parallel to it.

3. A sphere is the set of all points at a given distance from a fixed point.

4. The dot product $\mathbf{a} \cdot \mathbf{b} = 0$ implies that either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

5. If $\mathbf{a} \cdot \mathbf{b} > 0$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is less than $\frac{\pi}{2}$.

6. $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

7. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

8. $\mathbf{a} \times \mathbf{b}$ is the unique vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

9. The cross product can be used to determine the angle between vectors.

10. Two planes are parallel if and only if their normal vectors are parallel.

11. The distance between parallel planes equals the distance between any two points in the planes.

12. The equation of a hyperboloid of two sheets has two minus signs in it.

13. In an equation of a quadric surface, if one variable is linear and the other two are squared, then the surface is a paraboloid wrapping around the axis corresponding to the linear variable.

In exercises 1–4, compute $\mathbf{a} + \mathbf{b}$, $4\mathbf{b}$ and $\|2\mathbf{b} - \mathbf{a}\|$.

1. $\mathbf{a} = \langle -2, 3 \rangle$, $\mathbf{b} = \langle 1, 0 \rangle$

2. $\mathbf{a} = \langle -1, -2 \rangle$, $\mathbf{b} = \langle 2, 3 \rangle$

3. $\mathbf{a} = 10\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = -4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

4. $\mathbf{a} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

In exercises 5–8, determine whether a and b are parallel, orthogonal or neither.

5. $\mathbf{a} = \langle 2, 3 \rangle$, $\mathbf{b} = \langle 4, 5 \rangle$

6. $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$

7. $\mathbf{a} = \langle -2, 3, 1 \rangle$, $\mathbf{b} = \langle 4, -6, -2 \rangle$

8. $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

In exercises 9 and 10, find the displacement vector $\overrightarrow{PQ}$.

9. $P = (3, 1, -2)$, $Q = (2, -1, 1)$

10. $P = (3, 1)$, $Q = (1, 4)$

Review Exercises

In exercises 11–16, find a unit vector in the same direction as the given vector.

11. $\langle 3, 6\rangle$
12. $\langle -2, 3\rangle$
13. $10\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
14. $-\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
15. from $(4, 1, 2)$ to $(1, 1, 6)$
16. from $(2, -1, 0)$ to $(0, 3, -2)$

In exercises 17 and 18, find the distance between the given points.

17. $(0, -2, 2), (3, 4, 1)$
18. $(3, 1, 0), (1, 4, 1)$

In exercises 19 and 20, find a vector with the given magnitude and in the same direction as the given vector.

19. magnitude 2, $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$
20. magnitude $\frac{1}{2}$, $\mathbf{v} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$
21. The thrust of an airplane's engine produces a speed of 500 mph in still air. The wind velocity is given by $\langle 20, -80\rangle$. In what direction should the plane head to fly due east?
22. Two ropes are attached to a crate. The ropes exert forces of $\langle -160, 120\rangle$ and $\langle 160, 160\rangle$, respectively. If the crate weighs 300 pounds, what is the net force on the crate?

In exercises 23 and 24, find an equation of the sphere with radius r and center (a, b, c).

23. $r = 6, (a, b, c) = (0, -2, 0)$
24. $r = \sqrt{3}, (a, b, c) = (-3, 1, 2)$

In exercises 25–28, compute $\mathbf{a} \cdot \mathbf{b}$.

25. $\mathbf{a} = \langle 2, -1\rangle, \mathbf{b} = \langle 2, 4\rangle$
26. $\mathbf{a} = \mathbf{i} - 2\mathbf{j}, \mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$
27. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}, \mathbf{b} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
28. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - 3\mathbf{k}$

In exercises 29 and 30, find the angle between the vectors.

29. $\langle 3, 2, 1\rangle$ and $\langle -1, 1, 2\rangle$
30. $\langle 3, 4\rangle$ and $\langle 2, -1\rangle$

In exercises 31 and 32, find $\text{comp}_{\mathbf{b}}\, \mathbf{a}$ and $\text{proj}_{\mathbf{b}}\, \mathbf{a}$.

31. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$
32. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - 3\mathbf{k}$

In exercises 33–36, compute the cross product $\mathbf{a} \times \mathbf{b}$.

33. $\mathbf{a} = \langle 1, -2, 1\rangle, \mathbf{b} = \langle 2, 0, 1\rangle$
34. $\mathbf{a} = \langle 1, -2, 0\rangle, \mathbf{b} = \langle 1, 0, -2\rangle$
35. $\mathbf{a} = 2\mathbf{j} + \mathbf{k}, \mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
36. $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}, \mathbf{b} = 2\mathbf{i} - \mathbf{j}$

In exercises 37 and 38, find two unit vectors orthogonal to both given vectors.

37. $\mathbf{a} = 2\mathbf{i} + \mathbf{k}, \mathbf{b} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
38. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}, \mathbf{b} = 2\mathbf{i} - \mathbf{j}$
39. A force of $\langle 40, -30\rangle$ pounds moves an object in a straight line from $(1, 0)$ to $(60, 22)$. Compute the work done.
40. Use vectors to find the angles in the triangle with vertices $(0, 0)$, $(3, 1)$ and $(1, 4)$.

In exercises 41 and 42, find the distance from the point Q to the given line.

41. $Q = (1, -1, 0)$, line $\begin{cases} x = t + 1 \\ y = 2t - 1 \\ z = 3 \end{cases}$
42. $Q = (0, 1, 0)$, line $\begin{cases} x = 2t - 1 \\ y = 4t \\ z = 3t + 2 \end{cases}$

In exercises 43 and 44, find the indicated area or volume.

43. Area of the parallelogram with adjacent edges formed by $\langle 2, 0, 1\rangle$ and $\langle 0, 1, -3\rangle$
44. Volume of the parallelepiped with three adjacent edges formed by $\langle 1, -1, 2\rangle$, $\langle 0, 0, 4\rangle$ and $\langle 3, 0, 1\rangle$
45. A force of magnitude 50 pounds is applied at the end of a 6-inch-long wrench at an angle of $\frac{\pi}{6}$ to the wrench. Find the magnitude of the torque applied to the bolt.
46. A ball is struck with backspin. Find the direction of the Magnus force and describe the effect on the ball.

In exercises 47–50, find (a) parametric equations and (b) symmetric equations of the line.

47. The line through $(2, -1, -3)$ and $(0, 2, -3)$
48. The line through $(-1, 0, 2)$ and $(-3, 0, -2)$
49. The line through $(2, -1, 1)$ and parallel to $\frac{x-1}{2} = 2y = \frac{z+2}{-3}$
50. The line through $(0, 2, 1)$ and normal to the plane $2x - 3y + z = 4$

In exercises 51 and 52, find the angle between the lines.

51. $\begin{cases} x = 4 + t \\ y = 2 \\ z = 3 + 2t \end{cases}$ and $\begin{cases} x = 4 + 2s \\ y = 2 + 2s \\ z = 3 + 4s \end{cases}$

Review Exercises

52. $\begin{cases} x = 3 + t \\ y = 3 + 3t \\ z = 4 - t \end{cases}$ and $\begin{cases} x = 3 - s \\ y = 3 - 2s \\ z = 4 + 2s \end{cases}$

In exercises 53 and 54, determine whether the lines are parallel, skew or intersect.

53. $\begin{cases} x = 2t \\ y = 3 + t \\ z = -1 + 4t \end{cases}$ and $\begin{cases} x = 4 \\ y = 4 + s \\ z = 3 + s \end{cases}$

54. $\begin{cases} x = 1 - t \\ y = 2t \\ z = 5 - t \end{cases}$ and $\begin{cases} x = 3 + 3s \\ y = 2 \\ z = 1 - 3s \end{cases}$

In exercises 55–58, find an equation of the given plane.

55. The plane containing the point $(-5, 0, 1)$ with normal vector $\langle 4, 1, -2\rangle$

56. The plane containing the point $(2, -1, 2)$ with normal vector $\langle 3, -1, 0\rangle$

57. The plane containing the points $(2, 1, 3)$, $(2, -1, 2)$ and $(3, 3, 2)$

58. The plane containing the points $(2, -1, 2)$, $(1, -1, 4)$ and $(3, -1, 2)$

In exercises 59–72, sketch and identify the surface.

59. $9x^2 + y^2 + z = 9$

60. $x^2 + y + z^2 = 1$

61. $y^2 + z^2 = 1$

62. $x^2 + 4y^2 = 4$

63. $x^2 - 2x + y^2 + z^2 = 3$

64. $x^2 + (y + 2)^2 + z^2 = 6$

65. $y = 2$

66. $z = 5$

67. $2x - y + z = 4$

68. $3x + 2y - z = 6$

69. $x^2 - y^2 + 4z^2 = 4$

70. $x^2 - y^2 - z = 1$

71. $x^2 - y^2 - 4z^2 = 4$

72. $x^2 + y^2 - z = 1$

EXPLORATORY EXERCISES

1. Suppose that a piece of pottery is made in the shape of $z = 4 - x^2 - y^2$ for $z \geq 0$. A light source is placed at $(2, 2, 100)$. Draw a sketch showing the pottery and the light source. Based on this picture, which parts of the pottery would be brightly lit and which parts would be poorly lit? This can be quantified, as follows. For several points of your choice on the pottery, find the vector **a** that connects the point to the light source and the normal vector **n** to the tangent plane at that point, and then find the angle between the vectors. For points with relatively large angles, are the points well lit or poorly lit? Develop a rule for using the angle between vectors to determine the lighting level. Find the point that is best lit and the point that is most poorly lit.

2. As we focus on three-dimensional geometry throughout the balance of the book, some projections will be difficult but important to visualize. In this exercise, we contrast the curves C_1 and C_2 defined parametrically by $\begin{cases} x = \cos t \\ y = \cos t \\ z = \sin t \end{cases}$ and $\begin{cases} x = \cos t \\ y = \cos t \\ z = \sqrt{2}\sin t \end{cases}$, respectively. If you have access to three-dimensional graphics, try sketching each curve from a variety of perspectives. Our question will be whether either curve is a circle. For both curves, note that $x = y$. Describe in words and sketch a graph of the plane $x = y$. Next, note that the projection of C_1 back into the yz-plane is a circle ($y = \cos t$, $z = \sin t$). If C_1 is actually a circle in the plane $x = y$, discuss what its projection (shadow) in the yz-plane would look like. Given this, explain whether C_1 is actually a circle or an ellipse. Compare your description of the projection of a circle into the yz-plane to the projection of C_2 into the yz-plane. To make this more quantitative, we can use the general rule that for a two-dimensional region, the area of its projection onto a plane equals the area of the region multiplied by $\cos\theta$, where θ is the angle between the plane in which the region lies and the plane into which it is being projected. Given this, compute the radius of the circle C_2.