Functions of Several Variables and Partial Differentiation

CHAPTER

12

Few things in baseball are as exciting as a home run. In the summer of 2004, Barry Bonds chased the all-time home run record held by Hank Aaron. Every time Bonds hit the ball, fans watched in anxious anticipation as the ball reached its peak height and then slowly dropped back to the field. Would the ball clear the fence and stay fair for a home run? Through years of experience, players can usually tell exactly where the ball will land. However, since baseballs do not follow simple parabolic paths, most spectators must wait for the ball to land to see whether a given fly ball is a home run.

Think for a minute about the factors that determine how far the ball goes. In our studies of projectile motion, we identified three forces that affect the path: gravity, drag and the Magnus force. If we know the initial velocity (both speed and angle) and initial spin, we can write down a differential equation whose solution closely approximates the flight of the ball. This gives us distance as a function of speed, angle and spin. In this chapter, we introduce some of the basic techniques needed to analyze functions of two or more variables. While many of the ideas are familiar, the details change as we move from one to two or more variables.

You have probably realized that our situation is really far more complicated than outlined here. Air drag depends on environmental factors such as temperature and humidity. Other factors include the type of pitch thrown, the wind velocity and the type of bat used. With all of these factors, we would need a function of ten or more variables! Fortunately, the calculus of functions of ten variables is very similar to the calculus of functions of two or three variables. The theory presented in this chapter is easily extended to as many variables as are needed in a particular application.

After studying the basic calculus for functions of several variables, you should be able to find extrema of relatively simple functions. Perhaps more importantly, you should understand enough about such functions to be able to approximate extrema of more complicated functions. Of course, in real applications, you are rarely given a convenient formula. Even so, the understanding of multivariable calculus that you develop here will help you to make sense of a broad range of complex phenomena.

12.1 FUNCTIONS OF SEVERAL VARIABLES

The first ten chapters of this book focused on functions $f(x)$ whose domain and range were subsets of the real numbers. In Chapter 11, we studied vector-valued functions $\mathbf{F}(t)$ whose domain was a subset of the real numbers, but whose range was a set of vectors in two or more dimensions. In this section, we expand our concept of function to include functions that depend on more than one variable, that is, functions whose *domain* is multidimensional.

A **function of two variables** is a rule that assigns a real number $f(x, y)$ to each ordered pair of real numbers (x, y) in the domain of the function. For a function f defined on the domain $D \subset \mathbb{R}^2$, we sometimes write $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$, to indicate that f maps points in two dimensions to real numbers. You may think of such a function as a rule whose input is a pair of real numbers and whose output is a single real number. For instance, $f(x, y) = xy^2$ and $g(x, y) = x^2 - e^y$ are both functions of the two variables x and y.

Likewise, a **function of three variables** is a rule that assigns a real number $f(x, y, z)$ to each ordered triple of real numbers (x, y, z) in the domain $D \subset \mathbb{R}^3$ of the function. We sometimes write $f\colon D \subset \mathbb{R}^3 \to \mathbb{R}$ to indicate that f maps points in three dimensions to real numbers. For instance, $f(x, y, z) = xy^2 \cos z$ and $g(x, y, z) = 3zx^2 - e^y$ are both functions of the three variables x, y and z.

We can similarly define functions of four (or five or more) variables. Our focus here is on functions of two and three variables, although most of our results can be easily extended to higher dimensions.

Unless specifically stated otherwise, the domain of a function of several variables is taken to be the set of all values of the variables for which the given expression is defined.

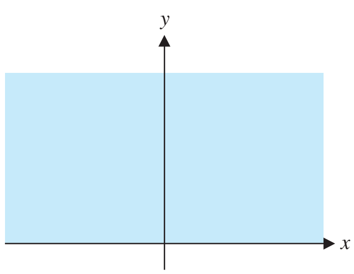


FIGURE 12.1a
The domain of $f(x, y) = x \ln y$

EXAMPLE 1.1 Finding the Domain of a Function of Two Variables

Find and sketch the domain for (a) $f(x, y) = x \ln y$ and (b) $g(x, y) = \dfrac{2x}{y - x^2}$.

Solution (a) For $f(x, y) = x \ln y$, recall that $\ln y$ is defined only for $y > 0$. The domain of f is then the set $D = \{(x, y) | y > 0\}$, that is, the half-plane lying above the x-axis (see Figure 12.1a).

(b) For $g(x, y) = \dfrac{2x}{y - x^2}$, note that g is defined unless there is a division by zero, which occurs when $y - x^2 = 0$. The domain of g is then $\{(x, y) | y \neq x^2\}$, which is the entire xy-plane with the parabola $y = x^2$ removed (see Figure 12.1b). ■

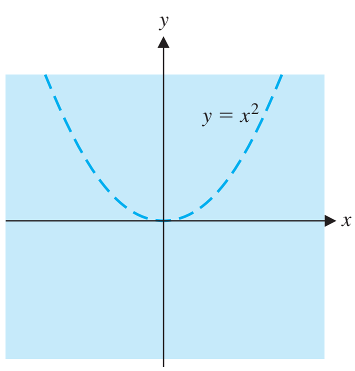


FIGURE 12.1b
The domain of $g(x, y) = \dfrac{2x}{y - x^2}$

EXAMPLE 1.2 Finding the Domain of a Function of Three Variables

Find and describe in graphical terms the domains of (a) $f(x, y, z) = \dfrac{\cos(x + z)}{xy}$ and (b) $g(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$.

Solution (a) For $f(x, y, z) = \dfrac{\cos(x+z)}{xy}$, there is a division by zero if $xy = 0$, which occurs if $x = 0$ or $y = 0$. The domain is then $\{(x, y, z) | x \neq 0 \textit{ and } y \neq 0\}$, which is all of three-dimensional space, excluding the yz-plane ($x = 0$) and the xz-plane ($y = 0$).

(b) Notice that for $g(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$ to be defined, you'll need to have $9 - x^2 - y^2 - z^2 \geq 0$, or $x^2 + y^2 + z^2 \leq 9$. The domain of g is then the sphere of radius 3 centered at the origin and its interior. ■

In many applications, you won't have a formula representing a function of interest. Rather, you may know values of the function at only a relatively small number of points, as in example 1.3.

EXAMPLE 1.3 A Function Defined by a Table of Data

A computer simulation of the flight of a baseball provided the data displayed in the accompanying table for the range in feet of a ball hit with initial velocity v ft/s and backspin rate of ω rpm. Each ball is struck at an angle of 30° above the horizontal.

v \ ω	0	1000	2000	3000	4000
150	294	312	333	350	367
160	314	334	354	373	391
170	335	356	375	395	414
180	355	376	397	417	436

Range of baseball in feet

Thinking of the range as a function $R(v, \omega)$, find $R(180, 0)$, $R(160, 0)$, $R(160, 4000)$ and $R(160, 2000)$. Discuss the results in baseball terms.

Solution The function values are found by looking in the row with the given value of v and the column with the given value of ω. Thus, $R(180, 0) = 355$, $R(160, 0) = 314$, $R(160, 4000) = 391$ and $R(160, 2000) = 354$. This says that a ball with no backspin and initial velocity 180 ft/s flies 41 ft farther than one with initial velocity 160 ft/s (no surprise there). However, observe that if a 160 ft/s ball also has backspin of 4000 rpm, it actually flies 36 ft farther than the 180 ft/s ball with no backspin. (The backspin gives the ball a lift force that keeps it in the air longer.) The combination of 160 ft/s and 2000 rpm produces almost exactly the same distance as 180 ft/s with no spin. (Watts and Bahill estimate that hitting the ball $\frac{1}{4}''$ below center produces 2000 rpm.) Thus, both initial velocity and spin have significant effects on the distance the ball flies. ■

The **graph** of the function $f(x, y)$ is the graph of the equation $z = f(x, y)$. This is not new, as you have already graphed a number of quadric surfaces that represent functions of two variables.

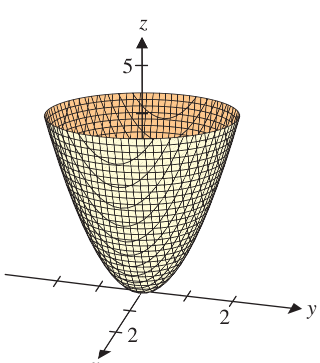


FIGURE 12.2a
$z = x^2 + y^2$

EXAMPLE 1.4 Graphing Functions of Two Variables

Graph (a) $f(x, y) = x^2 + y^2$ and (b) $g(x, y) = \sqrt{4 - x^2 + y^2}$.

Solution (a) For $f(x, y) = x^2 + y^2$, you may recognize the surface $z = x^2 + y^2$ as a circular paraboloid. Notice that the traces in the planes $z = k > 0$ are circles, while the traces in the planes $x = k$ and $y = k$ are parabolas. A graph is shown in Figure 12.2a.

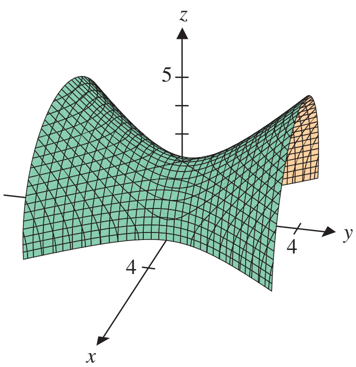


FIGURE 12.2b
$z = \sqrt{4 - x^2 + y^2}$

(b) For $g(x, y) = \sqrt{4 - x^2 + y^2}$, note that the surface $z = \sqrt{4 - x^2 + y^2}$ is the top half of the surface $z^2 = 4 - x^2 + y^2$ or $x^2 - y^2 + z^2 = 4$. Here, observe that the traces in the planes $x = k$ and $z = k$ are hyperbolas, while the traces in the planes $y = k$ are circles. This gives us a hyperboloid of one sheet, wrapped around the y-axis. The graph of $z = g(x, y)$ is the top half of the hyperboloid, as shown in Figure 12.2b. ■

Recall from your earlier experience drawing surfaces in three dimensions that an analysis of traces is helpful in sketching many graphs.

EXAMPLE 1.5 Graphing Functions in Three Dimensions

Graph (a) $f(x, y) = \sin x \cos y$ and (b) $g(x, y) = e^{-x^2}(y^2 + 1)$.

Solution (a) For $f(x, y) = \sin x \cos y$, notice that the traces in the planes $y = k$ are the sine curves $z = \sin x \cos k$, while its traces in the planes $x = k$ are the cosine curves $z = \sin k \cos y$. The traces in the planes $z = k$ are the curves $k = \sin x \cos y$. These are a bit more unusual, as seen in Figure 12.3a (which is computer-generated) for $k = 0.5$. The surface should look like a sine wave in all directions, as shown in the computer-generated plot in Figure 12.3b.

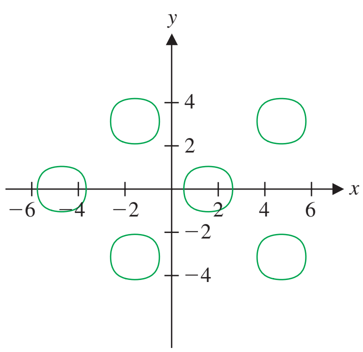


FIGURE 12.3a
The traces of the surface
$z = \sin x \cos y$ in the plane $z = 0.5$

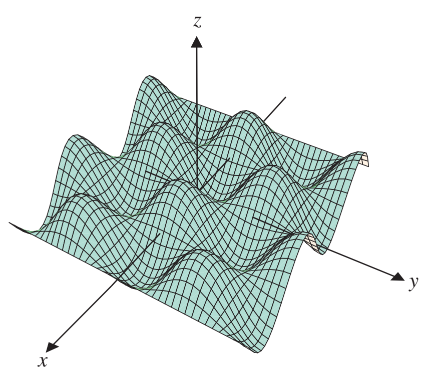


FIGURE 12.3b
$z = \sin x \cos y$

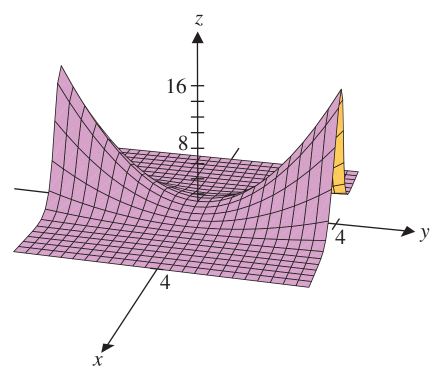


FIGURE 12.3c
$z = e^{-x^2}(y^2 + 1)$

(b) For $g(x, y) = e^{-x^2}(y^2 + 1)$, observe that the traces of the surface in the planes $x = k$ are parabolic, while the traces in the planes $y = k$ are proportional to $z = e^{-x^2}$, which are bell-shaped curves. The traces in the planes $z = k$ are not particularly helpful here. A sketch of the surface is shown in Figure 12.3c. ■

Graphing functions of more than one variable is not a simple business. For most functions of two variables, you must take hints from the expressions and try to piece together the clues to identify the surface. Your knowledge of functions of one variable is critical here.

EXAMPLE 1.6 Matching a Function of Two Variables to Its Graph

Match the functions $f_1(x, y) = \cos(x^2 + y^2)$, $f_2(x, y) = \cos(e^x + e^y)$, $f_3(x, y) = \ln(x^2 + y^2)$ and $f_4(x, y) = e^{-xy}$ to the surfaces shown in Figures 12.4a–12.4d.

Solution There are two properties of $f_1(x, y)$ that you should immediately notice. First, since the cosine of any angle lies between -1 and 1, $z = f_1(x, y)$ must always lie between -1 and 1. Second, the expression $x^2 + y^2$ is significant. Given any value of r, and any point (x, y) on the circle $x^2 + y^2 = r^2$, the height of the surface at the point (x, y) is a constant, given by $z = f_1(x, y) = \cos(r^2)$. Look for a surface that is bounded (this rules out Figure 12.4a) and has circular cross sections parallel to the xy-plane (ruling out Figures 12.4b and 12.4d). That leaves Figure 12.4c for the graph of $z = f_1(x, y)$.

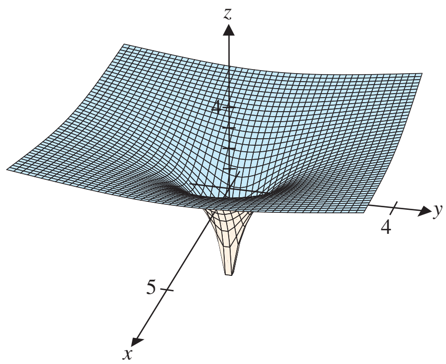


FIGURE 12.4a

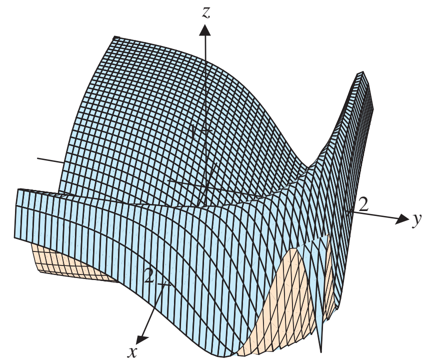


FIGURE 12.4b

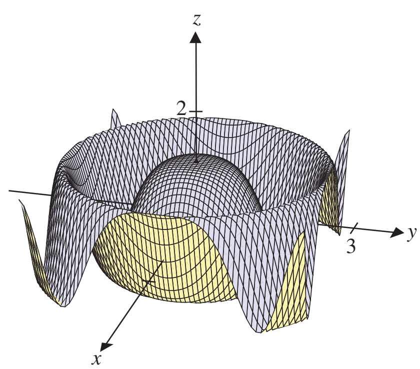


FIGURE 12.4c

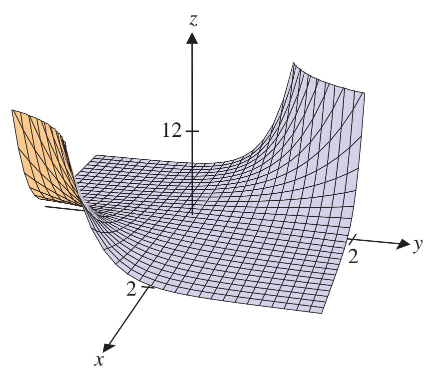


FIGURE 12.4d

You should notice that $y = f_3(x, y)$ also has circular cross sections parallel to the xy-plane, again because of the expression $x^2 + y^2$. (Think of polar coordinates.) Another important property of $f_3(x, y)$ for you to recognize is that the logarithm tends to $-\infty$ as its argument (in this case, $x^2 + y^2$) approaches 0. This appears to be what is indicated in Figure 12.4a, with the surface dropping sharply toward the center of the sketch. So, $z = f_3(x, y)$ corresponds to Figure 12.4a.

The remaining two functions involve exponentials. The most important distinction between them is that $f_2(x, y)$ lies between -1 and 1, due to the cosine term. This suggests that the graph of $f_2(x, y)$ is given in Figure 12.4b. To avoid jumping to a decision prematurely (after all, the domains used to produce these figures are all slightly different and could be misleading), make sure that the properties of $f_4(x, y)$ correspond

NOTES

You may be tempted to use computer-generated graphs throughout this chapter. However, we must emphasize that our goal is an *understanding* of three-dimensional graphs, which can best be obtained by sketching many graphs by hand. Doing this will help you know whether a computer-generated graph is accurate or misleading. Even when you use a graphing utility to produce a three-dimensional graph, we urge you to think through the traces, as we did in example 1.5.

to Figure 12.4d. Note that $e^{-xy} \to 0$ as $xy \to \infty$ and $e^{-xy} \to \infty$ as $xy \to -\infty$. As you move away from the origin in regions where x and y have the same sign, the surface should approach the xy-plane ($z = 0$). In regions where x and y have opposite signs, the surface should rise sharply. Notice that this behavior is exactly what you are seeing in Figure 12.4d. ■

REMARK 1.1

The analysis we used in example 1.6 may seem a bit slow, but we urge you to practice this on your own. The more you think (carefully) about how the properties of functions correspond to the structures of surfaces in three dimensions, the easier this chapter will be.

As with any use of technology, the creation of informative three-dimensional graphs can require a significant amount of knowledge and trial-and-error exploration. Even when you have an idea of what a graph should look like (and most often you won't), you may need to change the viewing window several times before you can clearly see a particular feature. The wireframe graph in Figure 12.5a is a poor representation of $f(x, y) = x^2 + y^2$. Notice that this graph shows numerous traces in the planes $x = c$ and $y = c$ for $-5 \le c \le 5$. However, no traces are drawn in planes parallel to the xy-plane, so you get no sense that the figure has circular cross sections. One way to improve this is to limit the range of z-values to $0 \le z \le 20$, as in Figure 12.5b. Observe that cutting off the graph here (i.e., not displaying all values of z for the displayed values of x and y) reveals the circular cross section at $z = 20$. An even better plot is obtained by using the parametric representation $x = u \cos v$, $y = u \sin v$, $z = u^2$, as shown in Figure 12.5c.

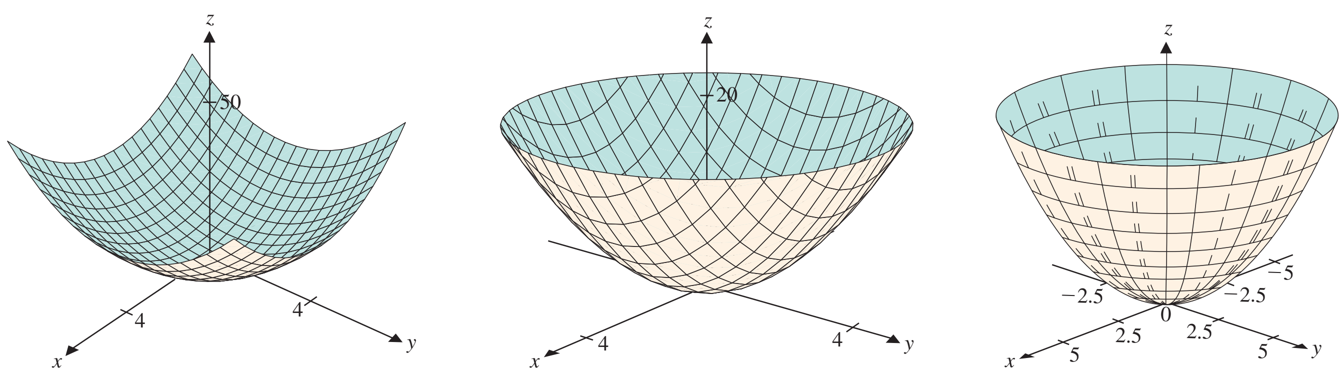


FIGURE 12.5a
$z = x^2 + y^2$

FIGURE 12.5b
$z = x^2 + y^2$

FIGURE 12.5c
$z = x^2 + y^2$

An important feature of three-dimensional graphs that is not present in two-dimensional graphs is the **viewpoint** from which the graph is drawn. In Figures 12.5a and 12.5b, we are looking at the paraboloid from a viewpoint that is above the xy-plane and between the positive x- and y-axes. This is the default viewpoint for many graphing utilities and is very similar to the way we have drawn graphs by hand. Figure 12.3c shows the default viewpoint of $f(x, y) = e^{-x^2}(y^2 + 1)$. In Figure 12.6a, we switch the viewpoint to the positive y-axis, from which we can see the bell-shaped profile of the graph. This viewpoint shows us several traces with $y = c$, so that we see a number of curves of the form $z = ke^{-x^2}$. In Figure 12.6b, the viewpoint is the positive x-axis, so that we see parabolic traces of the form $z = k(y^2 + 1)$. Figure 12.6c shows the view from high above the x-axis.

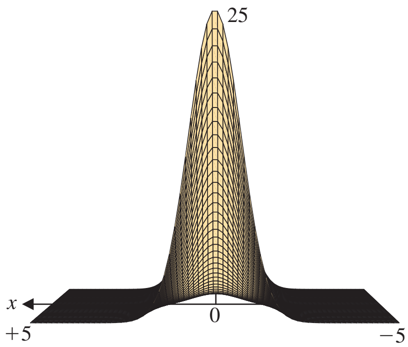


FIGURE 12.6a
$z = e^{-x^2}(y^2 + 1)$

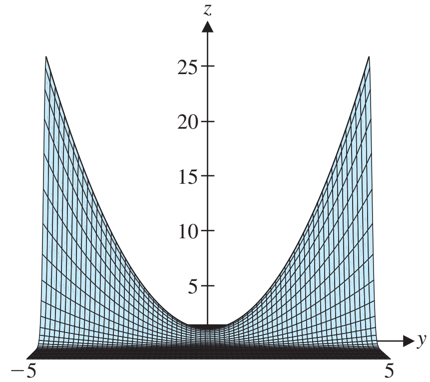


FIGURE 12.6b
$z = e^{-x^2}(y^2 + 1)$

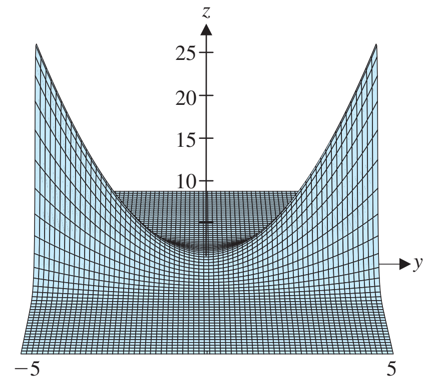


FIGURE 12.6c
$z = e^{-x^2}(y^2 + 1)$

Many graphing utilities offer alternatives to wireframe graphs. One deficiency of wireframe graphs is the lack of traces parallel to the xy-plane. This is not a problem in Figures 12.6a to 12.6c, where traces in the planes $z = c$ are too complicated to be helpful. However, in Figures 12.5a and 12.5b, the circular cross sections provide valuable information about the structure of the graph. To see such traces, many graphing utilities provide a "contour mode" or "parametric surface" option. These are shown in Figures 12.7a and 12.7b for $f(x, y) = x^2 + y^2$ and are explored further in the exercises.

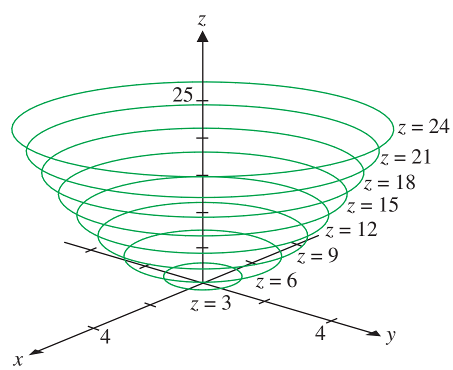


FIGURE 12.7a
$z = x^2 + y^2$ (contour mode)

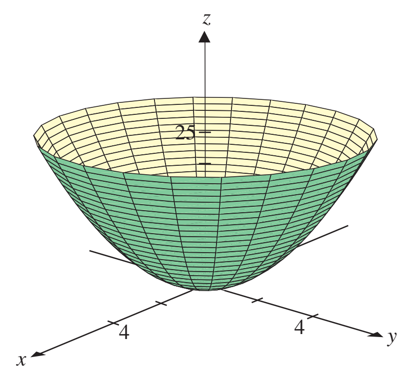


FIGURE 12.7b
$z = x^2 + y^2$ (parametric plot)

Two other types of graphs, the **contour plot** and the **density plot,** provide the same information condensed into a two-dimensional picture. Recall that for two of the surfaces in example 1.6, it was important to recognize that the surface had circular cross sections, since x and y appeared only in the combination $x^2 + y^2$. The contour plot and the density plot will aid in identifying features such as this.

A **level curve** of the function $f(x, y)$ is the (two-dimensional) graph of the equation $f(x, y) = c$, for some constant c. (So, the level curve $f(x, y) = c$ is a two-dimensional graph of the trace of the surface $z = f(x, y)$ in the plane $z = c$.) A **contour plot** of $f(x, y)$ is a graph of numerous level curves $f(x, y) = c$, for representative values of c.

EXAMPLE 1.7 Sketching Contour Plots

Sketch contour plots for (a) $f(x, y) = -x^2 + y$ and (b) $g(x, y) = x^2 + y^2$.

Solution (a) First, note that the level curves of $f(x, y)$ are defined by $-x^2 + y = c$, where c is a constant. Solving for y, you can identify the level curves as the parabolas $y = x^2 + c$. A contour plot with $c = -4, -2, 0, 2$ and 4 is shown in Figure 12.8a.

(b) The level curves for $g(x, y)$ are the circles $x^2 + y^2 = c$. In this case, note that there are level curves *only* for $c \geq 0$. A contour plot with $c = 1, 4, 7$ and 10 is shown in Figure 12.8b. ■

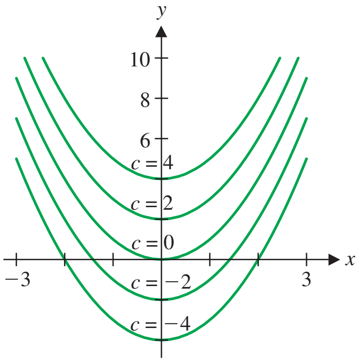


FIGURE 12.8a
Contour plot of $f(x, y) = -x^2 + y$

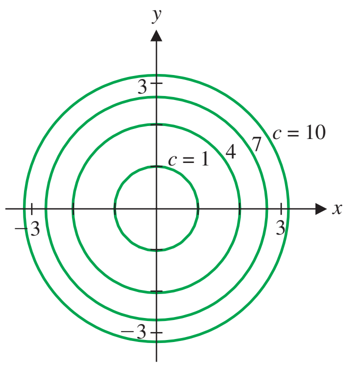


FIGURE 12.8b
Contour plot of $g(x, y) = x^2 + y^2$

Note that in example 1.7, we used values for c that were equally spaced. There is no requirement that you do so, but it can help you to get a sense for how the level curves would "stack up" to produce the three-dimensional graph. We show a more extensive contour plot for $g(x, y) = x^2 + y^2$ in Figure 12.9a. In Figure 12.9b, we show a plot of the surface, with a number of traces drawn (in planes parallel to the xy-plane). Notice that the projections of these traces onto the xy-plane correspond to the contour plot in Figure 12.9a.

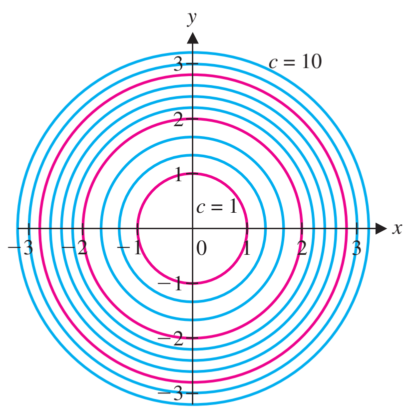


FIGURE 12.9a
Contour plot of $g(x, y) = x^2 + y^2$

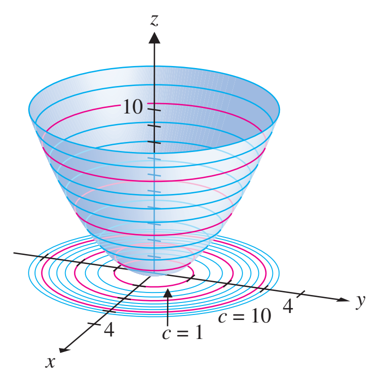


FIGURE 12.9b
$z = x^2 + y^2$

Look carefully at Figure 12.9a and observe that the contour plot indicates that the increase in the radii of the circles is not constant as c increases.

As you might expect, for more complicated functions, the process of matching contour plots with surfaces becomes more challenging.

EXAMPLE 1.8 Matching Surfaces to Contour Plots

Match the surfaces of example 1.6 to the contour plots shown in Figures 12.10a–12.10d.

Solution In Figures 12.4a and 12.4c, the level curves are circular, so these surfaces correspond to the contour plots in Figures 12.10a and 12.10b, but, which is which? The principal feature of the surface in Figure 12.4a is the behavior near the z-axis. Because of the rapid change in the function near the z-axis, there will be a large number of level curves near the origin. (Think about this.) By contrast, the oscillations in Figure 12.4c would produce level curves that alternately get closer together and farther apart. We can conclude that Figure 12.4a matches with Figure 12.10a, while Figure 12.4c matches with Figure 12.10b. Now, consider the two remaining surfaces and level curves. Imagine intersecting the surface in Figure 12.4d with the plane $z = 4$. You

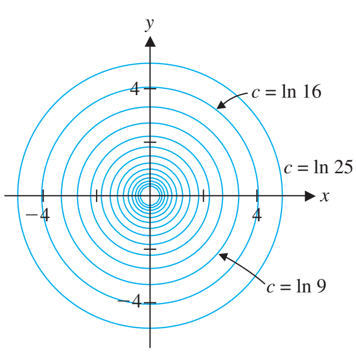


FIGURE 12.10a

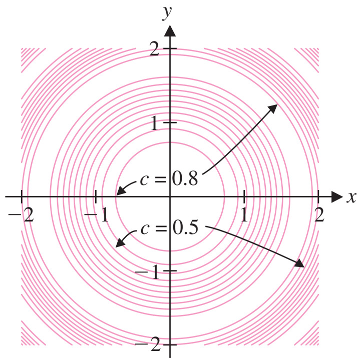


FIGURE 12.10b

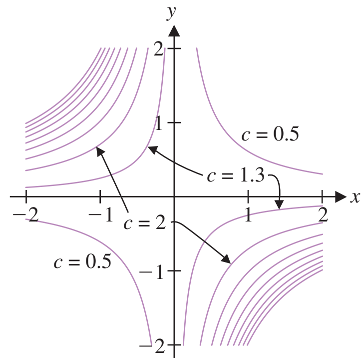


FIGURE 12.10c

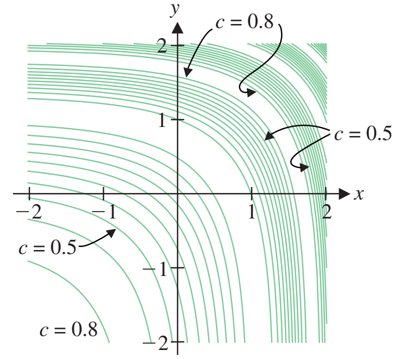


FIGURE 12.10d

would get two separate curves that open in opposite directions (to the lower left and upper right of Figure 12.4d). These correspond to the hyperbolas seen in Figure 12.10c. The final match of Figure 12.10d to Figure 12.4b is more difficult to see, but notice how the curves of Figure 12.10d correspond to the curve of the peaks in Figure 12.4b. (To see this, you will need to adjust for the y-axis pointing up in Figure 12.10d and to the right in Figure 12.4b.) As an additional means of distinguishing the last two graphs, notice that Figure 12.4d is very flat near the origin. This corresponds to the lack of level curves near the origin in Figure 12.10c. By contrast, Figure 12.4b shows oscillation near the origin and there are several level curves near the origin in Figure 12.10d. ■

REMARK 1.2

If the level curves in a contour plot are plotted for equally spaced values of z, observe that a tightly packed region of the contour plot will correspond to a region of rapid change in the function. Alternatively, blank space in the contour plot corresponds to a region of slow change in the function. For this reason, we typically draw contour plots using equally spaced values of z.

A **density plot** is closely related to a contour plot, in that they are both two-dimensional representations of a surface in three dimensions. For a density plot, each pixel is shaded according to the size of the function value at a point representing the pixel, with different colors and shades indicating different function values. In a density plot, notice that level curves can be seen as curves formed by a specific shade.

EXAMPLE 1.9 Matching Functions and Density Plots

Match the density plots in Figures 12.11a–12.11c with the functions

$$f_1(x, y) = \frac{1}{y^2 - x^2},\ f_2(x, y) = \frac{2x}{y - x^2} \text{ and } f_3(x, y) = \cos(x^2 + y^2).$$

Solution As we did with contour plots, we start with the most obvious properties of the functions and try to identify the corresponding properties in the density plots. Both $f_1(x, y)$ and $f_2(x, y)$ have gaps in their domains due to divisions by zero. Near the discontinuities, you should expect large function values. Notice that Figure 12.11b shows a lighter color band in the shape of a hyperbola (like $y^2 - x^2 = c$ for a small

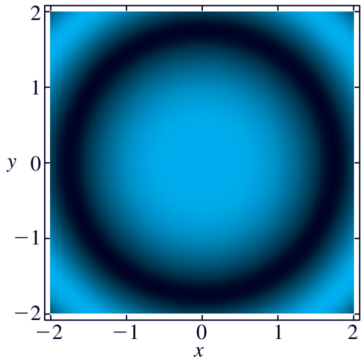


FIGURE 12.11a

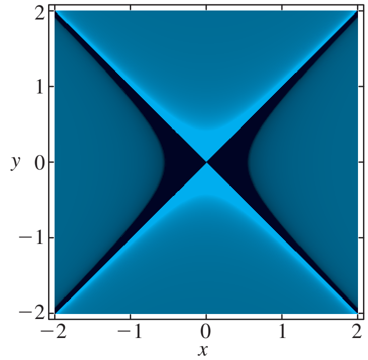


FIGURE 12.11b

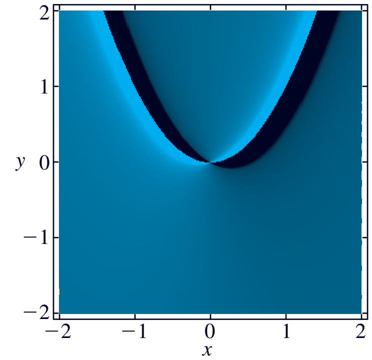


FIGURE 12.11c

number c) and Figure 12.11c shows a lighter color band in the shape of a parabola (like $y - x^2 = 0$). This tells you that the density plot for $f_1(x, y)$ is Figure 12.11b and the density plot for $f_2(x, y)$ is Figure 12.11c. That leaves Figure 12.11a for $f_3(x, y)$. You should be able to see the circular bands in the density plot arising from the $x^2 + y^2$ term in $f_3(x, y)$. ■

There are many examples of contour plots and density plots that you see every day. Weather maps often show level curves of atmospheric pressure (see Figure 12.12a). In this setting, the level curves are called **isobars** (that is, curves along which the barometric pressure is constant). Other weather maps represent temperature or degree of wetness with color coding (see Figure 12.12b), which are essentially density plots.

Scientists also use density plots while studying other climatic phenomena. For instance, in Figures 12.12c and 12.12d, we show two density plots indicating sea-surface height (which correlates with ocean heat content) indicating changes in the El Niño phenomenon over a period of several weeks.

We close this section by briefly looking at the graphs of functions of three variables, $f(x, y, z)$. We won't actually graph any such functions, since a true graph would require

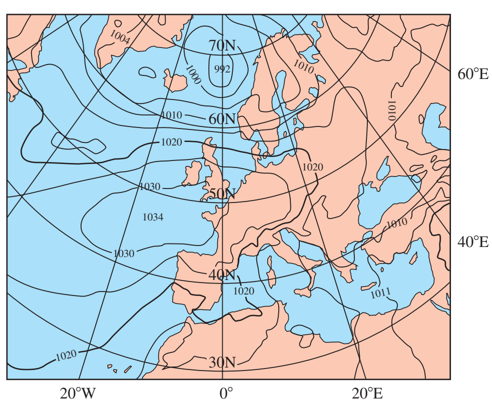


FIGURE 12.12a
Weather map showing barometric pressure

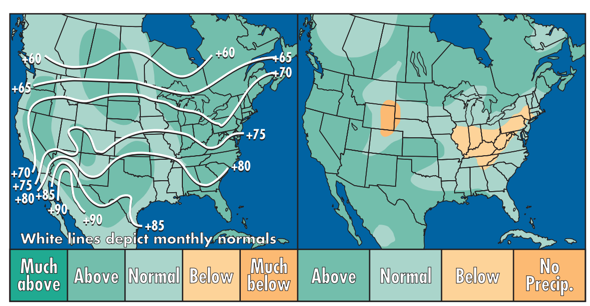


FIGURE 12.12b
Weather maps showing bands of temperature and precipitation

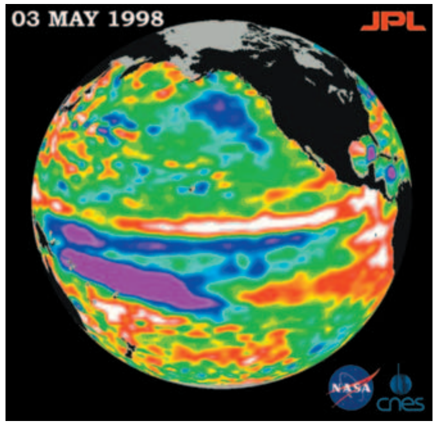


FIGURE 12.12c
Ocean heat content

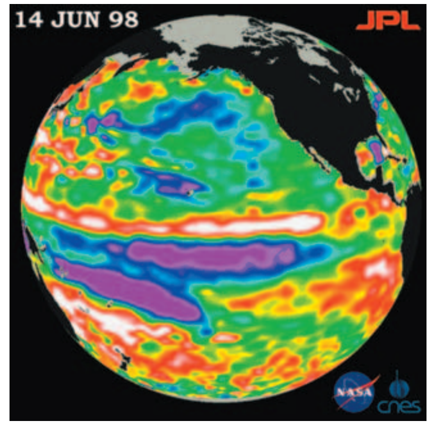


FIGURE 12.12d
Ocean heat content

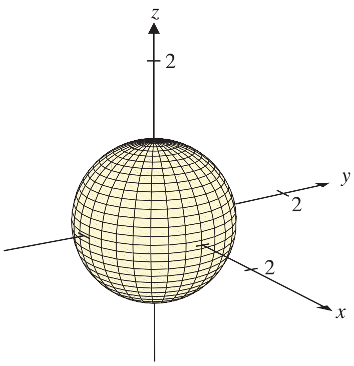


FIGURE 12.13a
$x^2 + y^2 + z^2 = 1$

four dimensions (three independent variables plus one dependent variable). We can, however, gain important information from looking at graphs of the **level surfaces** of a function f. These are the graphs of the equation $f(x, y, z) = c$, for different choices of the constant c. Much as level curves do for functions of two variables, level surfaces can help you identify symmetries and regions of rapid or slow change in a function of three variables.

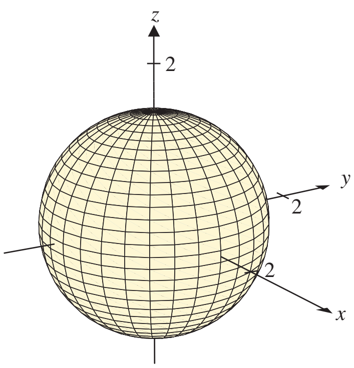


FIGURE 12.13b
$x^2 + y^2 + z^2 = 2$

EXAMPLE 1.10 Sketching Level Surfaces

Sketch several level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

Solution The level surfaces are described by the equation $x^2 + y^2 + z^2 = c$. Of course, these are spheres of radius $\sqrt{c}$ for $c > 0$. Surfaces with $c = 1$ and $c = 2$ are shown in Figures 12.13a and 12.13b, respectively. ■

Notice that the function in example 1.10 measures the square of the distance from the origin. If you didn't recognize this at first, carefully plotting a few of the level surfaces would clearly show you the symmetry and gradual increase of the function.

BEYOND FORMULAS

Our main way of thinking about surfaces in three dimensions is to analyze two-dimensional cross sections and build them up into a three-dimensional image. This allows us to use our experience with equations and graphs in two dimensions to determine properties of the graphs. Contour plots and density plots do essentially the same thing, with the one restriction that the cross sections represented are in parallel planes (for example, all parallel to the xy-plane). These two-dimensional plots do not show the distortions that result from trying to represent a three-dimensional object on two-dimensional paper. Thus, we can often draw better conclusions from a contour plot than from a three-dimensional graph.

EXERCISES 12.1

WRITING EXERCISES

1. In example 1.4, we sketched a paraboloid and the top half of a hyperboloid as examples of graphs of functions of two variables. Explain why neither a full hyperboloid nor an ellipsoid would be the graph of a function of two variables. Develop a "vertical line test" for determining whether a given surface is the graph of a function of two variables.

2. In example 1.4, we used traces to help sketch the surface, but in example 1.5 the traces were less helpful. Discuss the differences in the functions involved and how you can tell whether or not traces will be helpful.

3. In examples 1.7 and 1.8, we discussed how to identify a contour plot given the formula for a function. In this exercise, you will discuss the inverse problem. That is, given a contour plot, what can be said about the function? For example, explain why a contour plot without labels (identifying the value of z) could correspond to more than one function. If the contour plot shows a set of concentric circles around a point, explain why you would expect that point to be the location of a local extremum. Explain why, without labels, you could not distinguish a local maximum from a local minimum.

4. For this exercise, imagine a contour plot that shows level curves for equally spaced z-values (e.g., $z = 0$, $z = 2$ and $z = 4$). Near point A, the level curves are very close together, but near point B, there are no level curves showing at all. Discuss the behavior of the function near points A and B, especially commenting on whether the function is changing rapidly or slowly.

In exercises 1–6, describe and sketch the domain of the function.

1. $f(x, y) = \dfrac{1}{x + y}$

2. $f(x, y) = \dfrac{3xy}{y - x^2}$

3. $f(x, y) = \ln(2 + x + y)$

4. $f(x, y) = \sqrt{1 - x^2 - y^2}$

5. $f(x, y, z) = \dfrac{2xz}{\sqrt{4 - x^2 - y^2 - z^2}}$

6. $f(x, y, z) = \dfrac{e^{yz}}{z - x^2 - y^2}$

In exercises 7–10, describe the range of the function.

7. $f(x, y) = \sqrt{2 + x - y}$

8. $f(x, y) = \cos(x^2 + y^2)$

9. $f(x, y) = x^2 + y^2 - 1$

10. $f(x, y) = e^{x-y}$

In exercises 11–14, compute the indicated function values.

11. $f(x, y) = x^2 + y$; $f(1, 2)$, $f(0, 3)$

12. $f(x, y, z) = \dfrac{x + y}{z}$; $f(1, 2, 3)$, $f(5, -4, 3)$

13. $f(w, x, y, z) = \cos w - \dfrac{2xz}{y + z}$; $f(0, 1, 2, 3)$, $f(\pi, 2, 0, -1)$

14. $f(x_1, x_2, x_3, x_4, x_5) = \dfrac{x_1 + x_2 + x_3}{x_4^2 + x_5^2}$; $f(1, -1, 2, 3, 4)$, $f(5, -4, 3, -2, 1)$

In exercises 15 and 16, use the table in example 1.3.

15. Find (a) $R(150, 1000)$, (b) $R(150, 2000)$ and (c) $R(150, 3000)$. (d) Based on your answers, how much extra distance is gained from an additional 1000 rpm of backspin?

16. Find (a) $R(150, 2000)$, (b) $R(160, 2000)$ and (c) $R(170, 2000)$. (d) Based on your answers, how much extra distance is gained from an additional 10 ft/s of initial velocity?

In exercises 17–20, sketch the indicated traces and graph $z = f(x, y)$.

17. $f(x, y) = x^2 + y^2$; $z = 1$, $z = 4$, $z = 9$, $x = 0$

18. $f(x, y) = x^2 - y^2$; $z = 0$, $z = 1$, $y = 0$, $y = 2$

19. $f(x, y) = \sqrt{x^2 + y^2}$; $z = 1$, $z = 2$, $z = 3$, $y = 0$

20. $f(x, y) = x - 2y$; $z = 0$, $z = 1$, $x = 0$, $y = 0$

In exercises 21–30, use a graphing utility to sketch graphs of $z = f(x, y)$ from two different viewpoints, showing different features of the graphs.

21. $f(x, y) = x^2 + y^3$

22. $f(x, y) = x^2 + y^4$

23. $f(x, y) = x^2 + y^2 - x^4$

24. $f(x, y) = \dfrac{x^2}{x^2 + y^2 + 1}$

25. $f(x, y) = \cos\sqrt{x^2 + y^2}$

26. $f(x, y) = \sin^2 x + \cos^2 y$

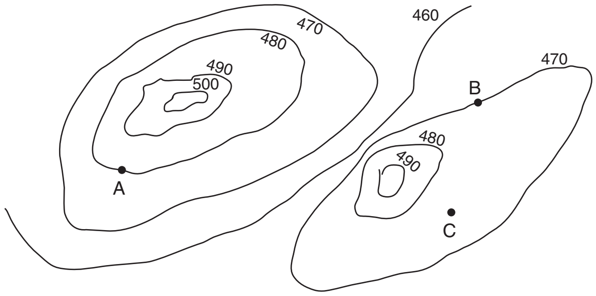
27. $f(x, y) = xye^{-x^2-y^2}$

28. $f(x, y) = ye^x$

29. $f(x, y) = \ln(x^2 + y^2 - 1)$

30. $f(x, y) = 2x \sin xy \ln y$

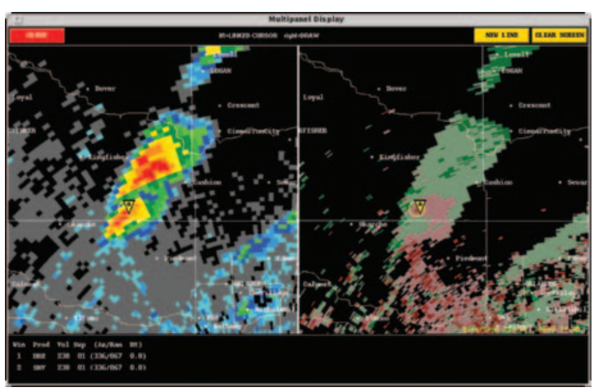
31. The **topographical map** seen at the top of the following page shows level curves for the height of a hill. For each point indicated, identify the height and sketch a short arrow indicating the direction from that point that corresponds to "straight up" the hill; that is, show the direction of the largest rate of increase in height.



For exercise 31

32. For the topographical map from exercise 31, there are two peaks shown. Identify the locations of the peaks and use the labels to approximate the height of each peak.

33. **Doppler radar** is used by meteorologists to track storms. The radar can measure the position and velocity of water and dust particles with enough accuracy to identify characteristics of the storm. The image on the left shows reflectivity, the amount of microwave energy reflected. The red represents the highest levels of reflectivity. In this case, a tornado is on-screen and the particles with the highest energy are debris from the tornado. From this image, conjecture the center of the tornado and the direction of movement of the tornado.



34. Referring to exercise 31, the image on the right shows the same tornado as the image on the left. In this image, velocity is color-coded with positive velocity in green and negative velocity in red (positive and negative mean toward the radar location and away from the radar, respectively). The area where red and green spiral together is considered a **tornado signature.** Explain why this set of velocity measurements indicates a tornado.

35. The **heat index** is a combination of temperature and humidity that measures how effectively the human body is able to dissipate heat; in other words, the heat index is a measure of how hot it feels. The more humidity there is, the harder it is for the body to evaporate moisture and cool off, so the hotter you feel. The table shows the heat index for selected temperatures and humidities in shade with a light breeze. For the function $H(t, h)$, find $H(80, 20)$, $H(80, 40)$ and $H(80, 60)$. At $80°$, approximately how many degrees does an extra 20% humidity add to the heat index?

	20%	**40%**	**60%**	**80%**
70°	65.1	66.9	68.8	70.7
80°	77.4	80.4	82.8	85.9
90°	86.5	92.3	100.5	112.0
100°	98.8	111.2	129.5	154.0

36. Use the preceding heat index table to find $H(90, 20)$, $H(90, 40)$ and $H(90, 60)$. At $90°$, approximately how many degrees does an extra 20% humidity add to the heat index? This answer is larger than the answer to exercise 35. Discuss what this means in terms of the danger of high humidity.

In exercises 37–42, sketch a contour plot.

37. $f(x, y) = x^2 + 4y^2$ **38.** $f(x, y) = \cos\sqrt{x^2 + y^2}$

39. $f(x, y) = y - 4x^2$ **40.** $f(x, y) = y^3 - 2x$

41. $f(x, y) = e^{y-x^3}$ **42.** $f(x, y) = ye^x$

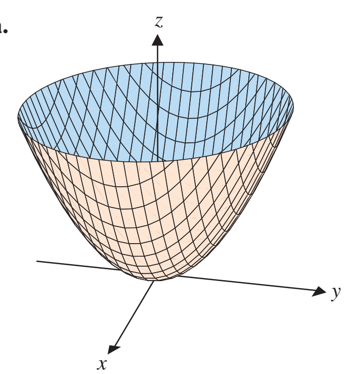
In exercises 43–46, use a CAS to sketch a contour plot.

43. $f(x, y) = xye^{-x^2-y^2}$ **44.** $f(x, y) = x^3 - 3xy + y^2$

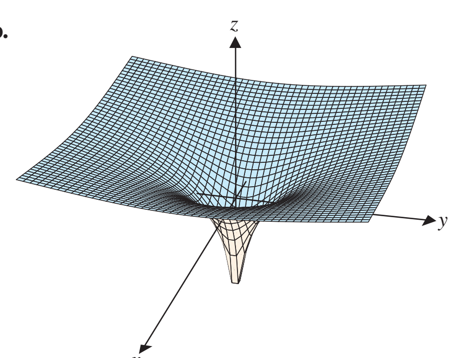
45. $f(x, y) = \sin x \sin y$ **46.** $f(x, y) = \sin(y - x^2)$

47. In parts a–d, match the surfaces to the contour plots.

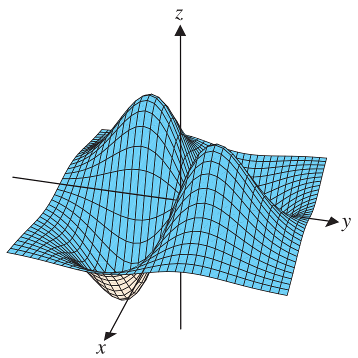
a.



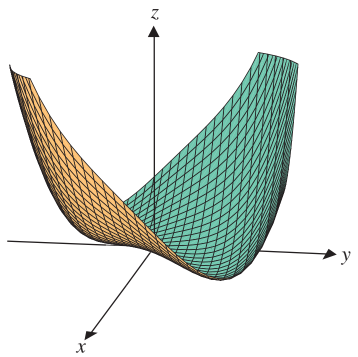
b.

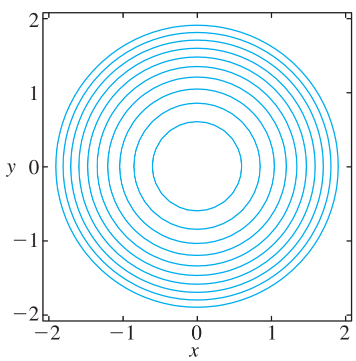


c.

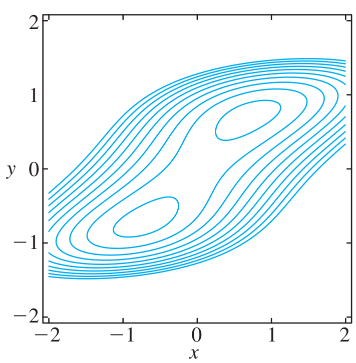


d.

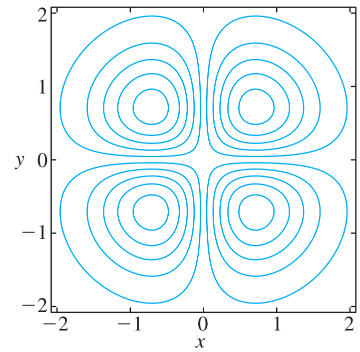




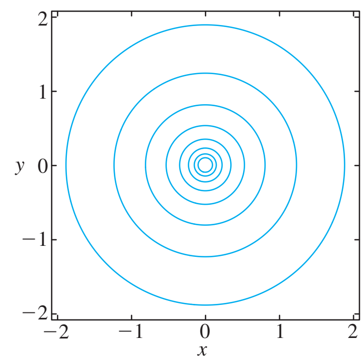
CONTOUR A



CONTOUR B



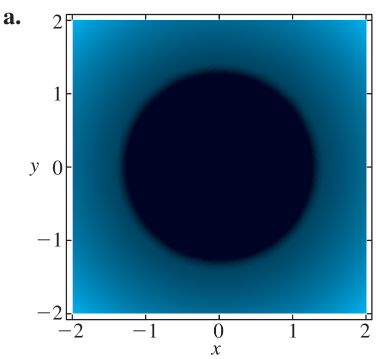
CONTOUR C



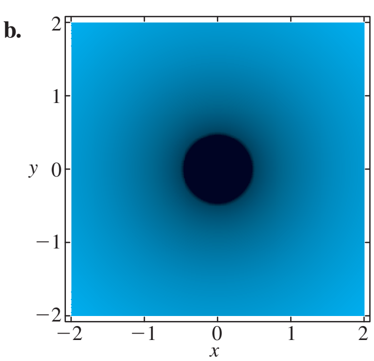
CONTOUR D

48. In parts a–d, match the density plots to the contour plots of exercise 47.

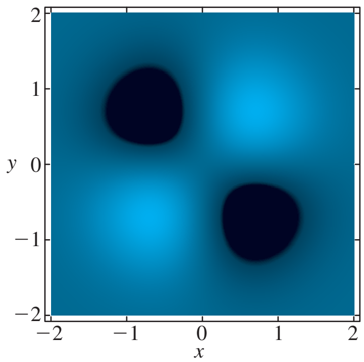
a.



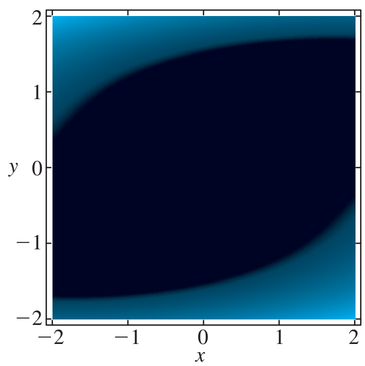
b.



c.



d.



49. In parts a–f, match the functions to the surfaces.

a. $f(x, y) = x^2 + 3x^7$

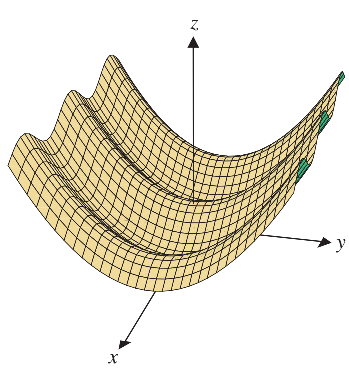
b. $f(x, y) = x^2 - y^3$

c. $f(x, y) = \cos^2 x + y^2$

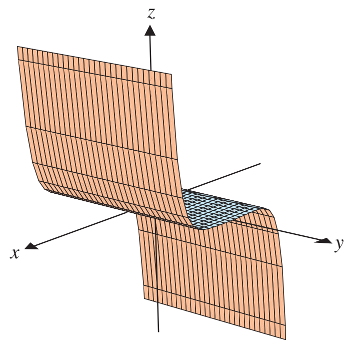
d. $f(x, y) = \cos(x^2 + y^2)$

e. $f(x, y) = \sin(x^2 + y^2)$

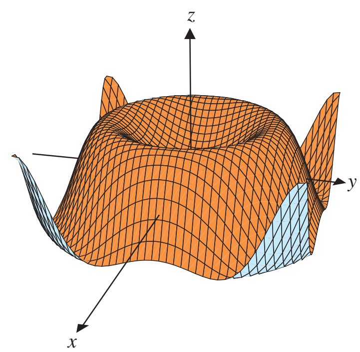
f. $f(x, y) = e^{-x^2-y^2}$



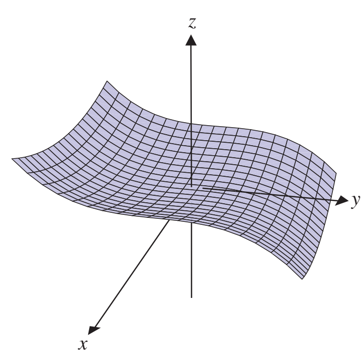
SURFACE A



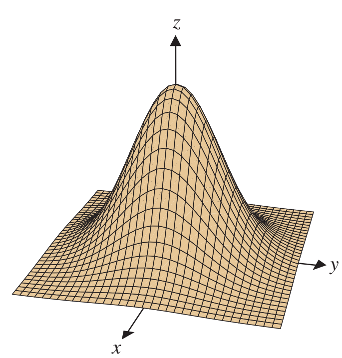
SURFACE B



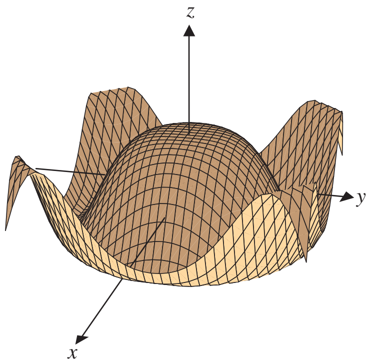
SURFACE C



SURFACE D



SURFACE E



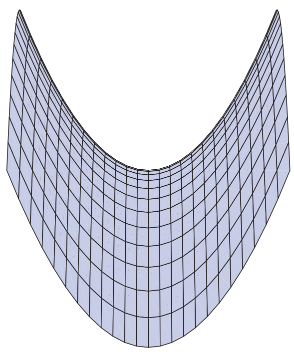
SURFACE F

In exercises 50–52, sketch several level surfaces of the given function.

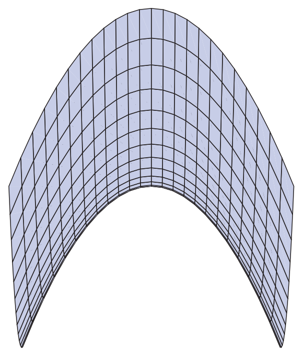
50. $f(x, y, z) = x^2 - y^2 + z^2$ **51.** $f(x, y, z) = x^2 + y^2 - z$

52. $f(x, y, z) = z - \sqrt{x^2 + y^2}$

53. The graph of $f(x, y) = x^2 - y^2$ is shown from two different viewpoints. Identify which is viewed from (a) the positive x-axis and (b) the positive y-axis.

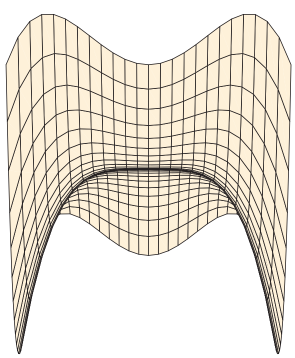


VIEW A

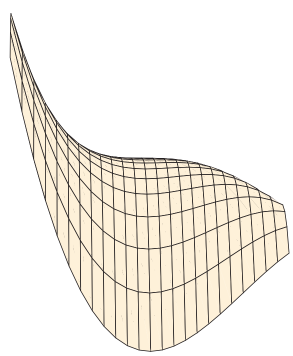


VIEW B

54. The graph of $f(x, y) = x^2y^2 - y^4 + x^3$ is shown from two different viewpoints. Identify which is viewed from (a) the positive x-axis and (b) the positive y-axis.



VIEW A



VIEW B

55. For the graphs in exercises 53 and 54, most software that produces wireframe graphs will show the view from the z-axis as a square grid. Explain why this is an accurate (although not very helpful) representation.

56. Suppose that you are shining a flashlight down at a surface from the positive z-axis. Explain why the result will be similar to a density plot.

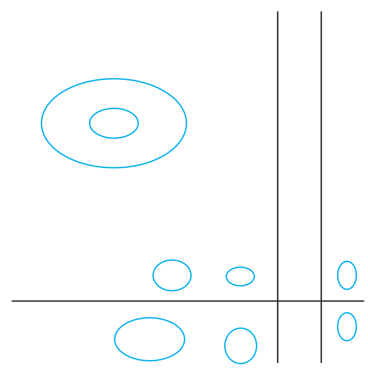
57. Describe in words the graph of $z = \sin(x + y)$. In which direction is the "wave" traveling? Explain why the wireframe graph as viewed from the point (100, 100, 0) appears to be rectangular.

58. Find a viewpoint from which a wireframe graph of $z = \sin(x + y)$ shows only a single sine wave.

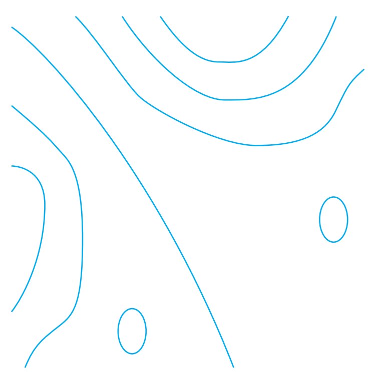
59. Find a viewpoint from which a wireframe graph of $z = (y - \sqrt{3}x)^2$ shows only a single parabola.

60. Find all viewpoints from which a wireframe graph of $z = e^{-x^2-y^2}$ shows a bell-shaped curve.

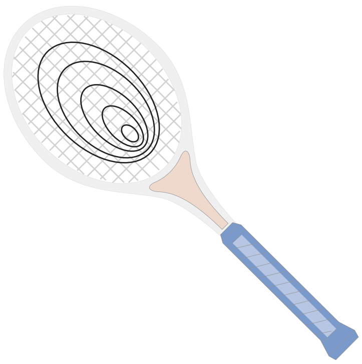
61. Suppose that the accompanying contour plot represents the population density in a city at a particular time in the evening. If there is a large rock concert that evening, locate the stadium. Speculate on what might account for other circular level curves and the linear level curves.



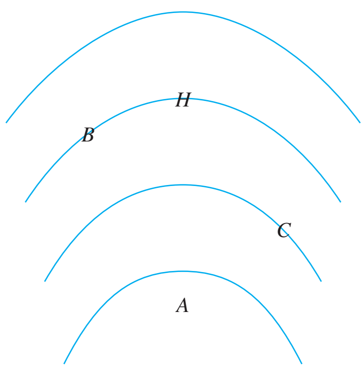
62. Suppose that the accompanying contour plot represents the temperature in a room. If it is winter, identify likely positions for a heating vent and a window. Speculate on what the circular level curves might represent.



63. Suppose that the accompanying contour plot represents the coefficient of restitution (the "bounciness") at various locations on a tennis racket. Locate the point of maximum power for the racket, and explain why you know it's *maximum* power and not minimum power. Racket manufacturers sometimes call one of the level curves the "sweet spot" of the racket. Explain why this is reasonable.



64. Suppose that the accompanying contour plot represents the elevation on a golf putting green. Assume that the elevation increases as you move up the contour plot. If the hole is at point H, describe what putts from points A, B and C would be like.



65. A well-known college uses the following formula to predict the grade average of prospective students:

$$\text{PGA} = 0.708(\text{HS}) + 0.0018(\text{SATV}) + 0.001(\text{SATM}) - 1.13$$

Here, PGA is the predicted grade average, HS is the student's high school grade average (in core academic courses, on a four-point scale), SATV is the student's SAT verbal score and SATM is the student's SAT math score. Use your scores to compute your own predicted grade average. Determine whether it is possible to have a predicted average of 4.0, or a negative predicted grade average. In this formula, the predicted grade average is a function of three variables. State which variable you think is the most important and explain why you think so.

66. In *The Hidden Game of Football,* Carroll, Palmer and Thorn give the following formula for the probability p that the team with the ball will win the game:

$$\ln\left(\frac{p}{1-p}\right) = 0.6s + 0.084\frac{s}{\sqrt{t/60}} - 0.0073(y-74).$$

Here, s is the current score differential (+ if you're winning, – if you're losing), t is the number of minutes remaining and y is the number of yards to the goal line. For the function $p(s, t, y)$, compute $p(2, 10, 40)$, $p(3, 10, 40)$, $p(3, 10, 80)$ and $p(3, 20, 40)$, and interpret the differences in football terms.

67. Suppose that you drive x mph for d miles and then y mph for d miles. Show that your average speed S is given by $S(x, y) = \dfrac{2xy}{x+y}$ mph. On a 40-mile trip, if you average 30 mph for the first 20 miles, how fast must you go to average 40 mph for the entire trip? How fast must you go to average 60 mph for the entire trip?

68. The **price-to-earnings ratio** of a stock is defined by $R = \frac{P}{E}$, where P is the price per share of the stock and E is the earnings. The yield of the stock is defined by $Y = \frac{d}{P}$, where d is the dividends per share. Find the yield as a function of R, d and E.

69. If your graphing utility can draw three-dimensional parametric graphs, compare the wireframe graph of $z = x^2 + y^2$ with the parametric graph of $x(r, t) = r\cos t$, $y(r, t) = r\sin t$ and $z(r, t) = r^2$. (Change parameter letters from r and t to whichever letters your utility uses.)

70. If your graphing utility can draw three-dimensional parametric graphs, compare the wireframe graph of $z = \ln(x^2 + y^2)$ with the parametric graph of $x(r, t) = r\cos t$, $y(r, t) = r\sin t$ and $z(r, t) = \ln(r^2)$.

71. If your graphing utility can draw three-dimensional parametric graphs, find parametric equations for $z = \cos(x^2 + y^2)$ and compare the wireframe and parametric graphs.

72. If your graphing utility can draw three-dimensional parametric graphs, compare the wireframe graphs of $z = \pm\sqrt{1 - x^2 - y^2}$ with the parametric graph of $x(u, v) = \cos u \sin v$, $y(u, v) = \sin u \sin v$ and $z(u, v) = \cos v$.

EXPLORATORY EXERCISES

1. Graphically explore the results of the transformations $g_1(x, y) = f(x, y) + c$, $g_2(x, y) = f(x, y + c)$ and $g_3(x, y) = f(x + c, y)$. [Hint: Take a specific function like $f(x, y) = x^2 + y^2$ and look at the graphs of the transformed functions $x^2 + y^2 + 2$, $x^2 + (y+2)^2$ and $(x+2)^2 + y^2$.] Determine what changes occur when the constant is added. Test your hypothesis for other constants (be sure to try negative constants, too). Then, explore the transformations $g_4(x, y) = cf(x, y)$ and $g_5(x, y) = f(c_1x, c_2y)$.

2. One common use of functions of two or more variables is in image processing. For instance, to digitize a black-and-white photograph, you can superimpose a rectangular grid and label each subrectangle with a number representing the brightness of that portion of the photograph. The grid defines the x- and y-values and the brightness numbers are the function values. Briefly describe how this function differs from other functions in this section. (Hint: How many x- and y-values are there?) Near the soccer jersey in the photograph shown, describe how the brightness function behaves. To "sharpen" the picture by increasing the contrast, should you transform the function values to make them closer together or farther apart?

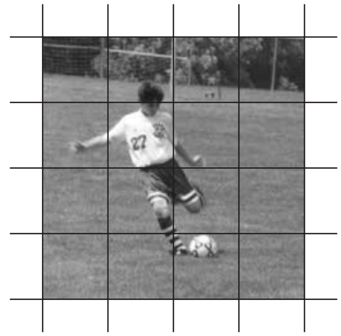


PHOTO WITH GRID

B & W PHOTO

7	8	8	9
5	2	5	6
6	7	7	6
6	6	4	6

DIGITIZED PHOTO

12.2 LIMITS AND CONTINUITY

At the beginning of our study of the calculus and again when we introduced vector-valued functions, we have followed the same progression of topics, beginning with graphs of functions, then limits, continuity, derivatives and integrals. We continue this progression now by extending the concept of limit to functions of two (and then three) variables. As you will see, the increase in dimension causes some interesting complications.

First, recall that for a function of a single variable, if we write $\lim_{x \to a} f(x) = L$, we mean that as x gets closer and closer to a, $f(x)$ gets closer and closer to the number L. Here, for functions of several variables, the idea is very similar. When we write

$$\lim_{(x,y) \to (a,b)} f(x, y) = L,$$

we mean that as (x, y) gets closer and closer to (a, b), $f(x, y)$ is getting closer and closer to the number L. In this case, (x, y) may approach (a, b) along any of the infinitely through (a, b) many different paths passing.

For instance, $\lim_{(x,y) \to (2,3)} (xy - 2)$ asks us to identify what happens to the function $xy - 2$ as x approaches 2 and y approaches 3. Clearly, $xy - 2$ approaches $2(3) - 2 = 4$ and we write

$$\lim_{(x,y) \to (2,3)} (xy - 2) = 4.$$

Similarly, you can reason that

$$\lim_{(x,y)\to(-1,\pi)} (\sin xy - x^2 y) = \sin(-\pi) - \pi = -\pi.$$

In other words, for many (nice) functions, we can compute limits simply by substituting into the function.

However, as with functions of a single variable, the limits in which we're most interested cannot be computed by simply substituting values for x and y. For instance, for

$$\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1},$$

substituting in $x = 1$ and $y = 0$ gives the indeterminate form $\frac{0}{0}$. To evaluate this limit, we must investigate further.

You may recall from our discussion in section 1.6 that for a function f of a single variable defined on an open interval containing a (but not necessarily at a), we say that $\lim_{x\to a} f(x) = L$ if given any number $\varepsilon > 0$, there is another number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. In other words, no matter how close you wish to make $f(x)$ to L (we represent this distance by ε), you can make it that close, just by making x sufficiently close to a (i.e., within a distance δ of a).

The definition of the limit of a function of two variables is completely analogous to the definition for a function of a single variable. We say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$, if we can make $f(x, y)$ as close as desired to L by making the point (x, y) sufficiently close to (a, b). We make this more precise in Definition 2.1.

DEFINITION 2.1 (Formal Definition of Limit)

Let f be defined on the interior of a circle centered at the point (a, b), except possibly at (a, b) itself. We say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

We illustrate the definition in Figure 12.14.

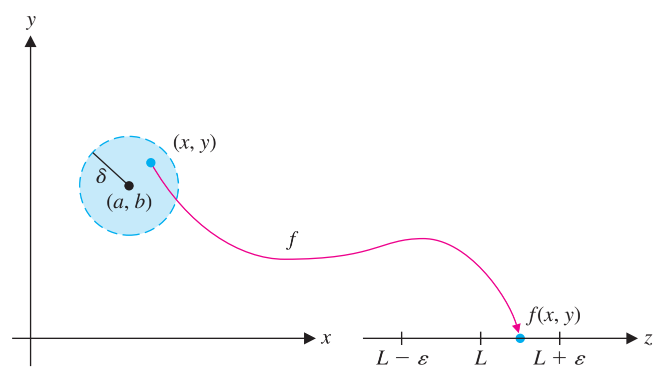


FIGURE 12.14
The definition of limit

Notice that the definition says that given any desired degree of closeness $\varepsilon > 0$, you must be able to find another number $\delta > 0$, so that *all* points lying within a distance δ of (a, b) are mapped by f to points within distance ε of L on the real line.

EXAMPLE 2.1 Using the Definition of Limit

Verify that $\lim_{(x,y)\to(a,b)} x = a$ and $\lim_{(x,y)\to(a,b)} y = b$.

Solution Certainly, both of these limits are intuitively quite clear. We can use Definition 2.1 to verify them, however. Given any number $\varepsilon > 0$, we must find another number $\delta > 0$ so that $|x-a| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Notice that

$$\sqrt{(x-a)^2 + (y-b)^2} \geq \sqrt{(x-a)^2} = |x-a|,$$

and so, taking $\delta = \varepsilon$, we have that

$$|x-a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon,$$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Likewise, we can show that $\lim_{(x,y)\to(a,b)} y = b$. ■

With this definition of limit, we can prove the usual results for limits of sums, products and quotients. That is, if $f(x, y)$ and $g(x, y)$ both have limits as (x, y) approaches (a, b), we have

$$\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$$

(i.e., the limit of a sum or difference is the sum or difference of the limits),

$$\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right]$$

(i.e., the limit of a product is the product of the limits) and

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$

(i.e., the limit of a quotient is the quotient of the limits), *provided* $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$.

A **polynomial** in the two variables x and y is any sum of terms of the form $cx^n y^m$, where c is a constant and n and m are nonnegative integers. Using the preceding results and example 2.1, we can show that the limit of any polynomial always exists and is found simply by substitution.

EXAMPLE 2.2 Finding a Simple Limit

Evaluate $\lim_{(x,y)\to(2,1)} \dfrac{2x^2y + 3xy}{5xy^2 + 3y}$.

Solution First, note that this is the limit of a rational function (i.e., the quotient of two polynomials). Since the limit in the denominator is

$$\lim_{(x,y)\to(2,1)} (5xy^2 + 3y) = 10 + 3 = 13 \neq 0,$$

we have

$$\lim_{(x,y)\to(2,1)} \frac{2x^2y + 3xy}{5xy^2 + 3y} = \frac{\lim_{(x,y)\to(2,1)} (2x^2y + 3xy)}{\lim_{(x,y)\to(2,1)} (5xy^2 + 3y)} = \frac{14}{13}.$$ ■

Think about the implications of Definition 2.1 (even if you are a little unsure of the role of ε and δ). If there is *any* way to approach the point (a, b) without the function values approaching the value L (e.g., by virtue of the function values blowing up, oscillating or by approaching some other value), then the limit will not equal L. For the limit to equal L, the function has to approach L along *every* possible path. This gives us a simple method for determining that a limit does not exist.

REMARK 2.1

If $f(x, y)$ approaches L_1 as (x, y) approaches (a, b) along a path P_1 and $f(x, y)$ approaches $L_2 \neq L_1$ as (x, y) approaches (a, b) along a path P_2, then $\lim_{(x,y)\to(a,b)} f(x, y)$ *does not exist.*

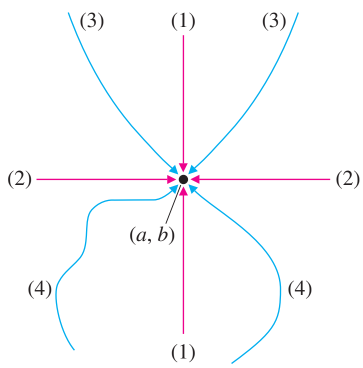


FIGURE 12.15
Various paths to (a, b)

Unlike the case for functions of a single variable where we must consider left- and right-hand limits in two dimensions, instead of just two paths approaching a given point, there are infinitely many (and you obviously can't check each one individually). In practice, when you suspect that a limit does not exist, you should check the limit along the simplest paths first. We will use the following guidelines.

REMARK 2.2

The simplest paths to try are (1) $x = a$, $y \to b$ (vertical lines); (2) $y = b$, $x \to a$ (horizontal lines); (3) $y = g(x)$, $x \to a$ [where $b = g(a)$] and (4) $x = g(y)$, $y \to b$ [where $a = g(b)$]. Several of these paths are illustrated in Figure 12.15.

EXAMPLE 2.3 A Limit That Does Not Exist

Evaluate $\lim_{(x,y)\to(1,0)} \dfrac{y}{x + y - 1}$.

Solution First, we consider the vertical line path along the line $x = 1$ and compute the limit as y approaches 0. If $(x, y) \to (1, 0)$ along the line $x = 1$, we have

$$\lim_{(1,y)\to(1,0)} \frac{y}{1 + y - 1} = \lim_{y\to 0} 1 = 1.$$

We next consider the path along the horizontal line $y = 0$ and compute the limit as x approaches 1. Here, we have

$$\lim_{(x,0)\to(1,0)} \frac{0}{x + 0 - 1} = \lim_{x\to 0} 0 = 0.$$

Since the function approaches two different values along two different paths to the point $(1, 0)$, the limit does not exist. ■

Many of our examples and exercises have (x, y) approaching $(0, 0)$. In this case, notice that another simple path passing through $(0, 0)$ is the line $y = x$.

EXAMPLE 2.4 A Limit That Is the Same Along Two Paths but Does Not Exist

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$.

Solution First, we consider the limit along the path $x = 0$. We have

$$\lim_{(0,y)\to(0,0)} \frac{0}{0 + y^2} = \lim_{y\to 0} 0 = 0.$$

Similarly, for the path $y = 0$, we have

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^2 + 0} = \lim_{x\to 0} 0 = 0.$$

Be careful; just because the limits along the first two paths you try are the same does *not* mean that the limit exists. For a limit to exist, the limit must be the same along *all* paths through (0, 0) (not just along two). Here, we may simply need to look at more paths. Notice that for the path $y = x$, we have

$$\lim_{(x,x)\to(0,0)} \frac{x(x)}{x^2 + x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Since the limit along this path doesn't match the limit along the first two paths, the limit does not exist. ■

As you've seen in examples 2.3 and 2.4, substitutions for particular paths often result in the function reducing to a constant. When choosing paths, you should look for substitutions that will simplify the function dramatically.

EXAMPLE 2.5 A Limit Problem Requiring a More Complicated Choice of Path

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$.

Solution First, we consider the path $x = 0$ and get

$$\lim_{(0,y)\to(0,0)} \frac{0}{0 + y^4} = \lim_{y\to 0} 0 = 0.$$

Similarly, following the path $y = 0$, we get

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^2 + 0} = \lim_{x\to 0} 0 = 0.$$

Since the limits along the first two paths are the same, we try another path. As in example 2.4, we next try the line $y = x$. As it turns out, this limit is

$$\lim_{(x,x)\to(0,0)} \frac{x^3}{x^2 + x^4} = \lim_{x\to 0} \frac{x}{1 + x^2} = 0,$$

also. In exercise 51, you will show that the limit along *every* straight line through the origin is 0. However, we still cannot conclude that the limit is 0. For this to happen, the limit along *all* paths (not just along all straight-line paths) must be 0. At this point, there are two possibilities: either the limit exists (and equals 0) or the limit does not exist, in which case, we must discover a path through (0, 0) along which the limit is not 0. Notice that along the path $x = y^2$, the terms x^2 and y^4 will be equal. We then have

$$\lim_{(y^2,y)\to(0,0)} \frac{y^2(y^2)}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

Since this limit does not agree with the limits along the earlier paths, the original limit does not exist. ■

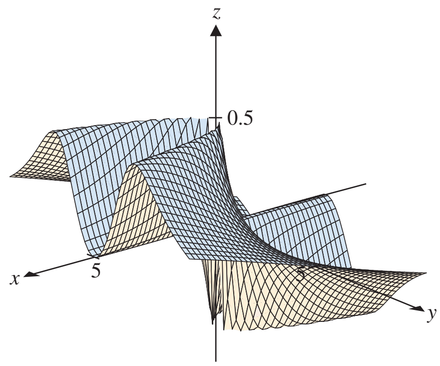


FIGURE 12.16a

$z = \dfrac{xy^2}{x^2+y^4}$, for $-5 \le x \le 5$, $-5 \le y \le 5$

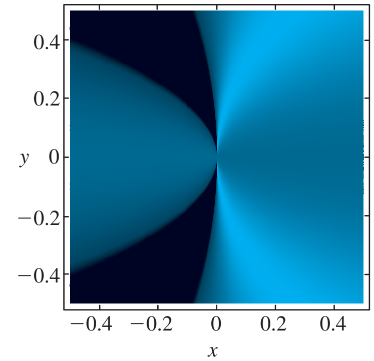


FIGURE 12.16b

Density plot of $f(x, y) = \dfrac{xy^2}{x^2+y^4}$

Before discussing how to show that a limit *does* exist, we pause to explore example 2.5 graphically. First, try to imagine what the graph of $f(x, y) = \dfrac{xy^2}{x^2+y^4}$ might look like. The function is defined except at the origin, it approaches 0 along the x-axis, y-axis and along any line $y = kx$ through the origin. Yet, $f(x, y)$ approaches $\frac{1}{2}$ along the parabola $x = y^2$. A standard sketch of the surface $z = f(x, y)$ with $-5 \le x \le 5$ and $-5 \le y \le 5$ is helpful, but you need to know what you're looking for. You can see part of the ridge at $z = 0.5$, as well as a trough at $z = -0.5$ corresponding to $x = -y^2$, in Figure 12.16a. A density plot clearly shows the parabola of large function values in light blue and a parabola of small function values in black (see Figure 12.16b). Near the origin, the surface has a ridge at $x = y^2$, $z = \frac{1}{2}$, dropping off quickly to a smooth surface that approaches the origin. The ridge is in two pieces ($y > 0$ and $y < 0$) separated by the origin.

The procedure we followed in examples 2.3, 2.4 and 2.5 was used to show that a limit does *not* exist. What if a limit does exist? Of course, you'll never be able to establish that a limit exists by computing limits along specific paths. There are infinitely many paths through any given point and you'll never be able to exhaust all of the possibilities. However, after following a number of paths and getting the same limit along each of them, you should begin to suspect that the limit just might exist. One tool you can use is the following generalization of the Squeeze Theorem presented in section 1.3.

THEOREM 2.1

Suppose that $|f(x, y) - L| \le g(x, y)$ for all (x, y) in the interior of some circle centered at (a, b), except possibly at (a, b). If $\lim\limits_{(x,y)\to(a,b)} g(x, y) = 0$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$.

PROOF

For any given $\varepsilon > 0$, we know from the definition of $\lim\limits_{(x,y)\to(a,b)} g(x, y) = 0$, that there is a number $\delta > 0$ such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ guarantees that $|g(x, y) - 0| < \varepsilon$.

For any such points (x, y), we have

$$|f(x, y) - L| \le g(x, y) < \varepsilon.$$

It now follows from the definition of limit that $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$. ■

In other words, the theorem simply states that if $|f(x, y) - L|$ is trapped between 0 (the absolute value is never negative) and a function (g) that approaches 0, then $|f(x, y) - L|$ must also have a limit of 0.

To use Theorem 2.1, you start with a conjecture for the limit L (obtained for instance, by calculating the limit along several simple paths). Then, look for a simpler function that is larger than $|f(x, y) - L|$ and that tends to zero as (x, y) approaches (a, b).

EXAMPLE 2.6 Proving That a Limit Exists

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2}$.

Solution As we did in earlier examples, we start by looking at the limit along several paths through (0, 0). Along the path $x = 0$, we have

$$\lim_{(0,y)\to(0,0)} \frac{0}{0 + y^2} = 0.$$

Similarly, along the path $y = 0$, we have

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^2 + 0} = 0.$$

Further, along the path $y = x$, we have

$$\lim_{(x,x)\to(0,0)} \frac{x^3}{x^2 + x^2} = \lim_{x\to 0} \frac{x}{2} = 0.$$

We know that if the limit exists, it must equal 0. Our last calculation gives an important clue that the limit does exist. After simplifying the expression, there remained an extra power of x in the numerator forcing the limit to 0. To show that the limit equals 0, consider

$$|f(x, y) - L| = |f(x, y) - 0| = \left|\frac{x^2 y}{x^2 + y^2}\right|.$$

Notice that without the y^2 term in the denominator, we could cancel the x^2 terms. Since $x^2 + y^2 \ge x^2$, we have that for $x \ne 0$

$$|f(x, y) - L| = \left|\frac{x^2 y}{x^2 + y^2}\right| \le \left|\frac{x^2 y}{x^2}\right| = |y|.$$

Since $\lim\limits_{(x,y)\to(0,0)} |y| = 0$, Theorem 2.1 gives us $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2 + y^2} = 0$, also. ■

When (x, y) approaches a point other than (0, 0), the idea is the same as in example 2.6, but the algebra may get messier, as we see in example 2.7.

EXAMPLE 2.7 Finding a Limit of a Function of Two Variables

Evaluate $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2}$.

Solution Along the path $x = 1$, we have

$$\lim_{(1,y)\to(1,0)} \frac{0}{y^2} = 0.$$

Along the path $y = 0$, we have

$$\lim_{(x,0)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2} = \lim_{x\to 1} \ln x = 0.$$

A third path through $(1, 0)$ is the line $y = x - 1$ (note that in this case, we must have $y \to 0$ as $x \to 1$). We have

$$\lim_{(x,x-1)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2 + (x-1)^2} = \lim_{x\to 1} \frac{(x-1)^2 \ln x}{2(x-1)^2} = \lim_{x\to 1} \frac{\ln x}{2} = 0.$$

At this point, you should begin to suspect that the limit just might be 0. You never know, though, until you find another path along which the limit is different or until you prove that the limit actually is 0. To show this, we consider

$$|f(x, y) - L| = \left| \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2} \right|.$$

Notice that if the y^2 term were not present in the denominator, then we could cancel the $(x-1)^2$ terms. We have

$$|f(x, y) - L| = \left| \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2} \right| \le \left| \frac{(x-1)^2 \ln x}{(x-1)^2} \right| = |\ln x|$$

Since $\displaystyle\lim_{(x,y)\to(1,0)} |\ln x| = 0$, it follows from Theorem 2.1 that

$\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2 + y^2} = 0$, also. ■

As with functions of one variable and (more recently) vector-valued functions, the concept of continuity is closely connected to limits. Recall that in these cases, a function (or vector-valued function) is continuous at a point whenever the limit and the value of the function are the same. This same characterization applies to continuous functions of several variables, as we see in Definition 2.2.

DEFINITION 2.2

Suppose that $f(x, y)$ is defined in the interior of a circle centered at the point (a, b). We say that f is **continuous** at (a, b) if $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

If $f(x, y)$ is not continuous at (a, b), then we call (a, b) a **discontinuity** of f.

This definition is completely analogous to our previous definitions of continuity for the cases of functions of one variable and vector-valued functions. The graphical interpretation is similar, although three-dimensional graphs can be more complicated. Still, the idea is

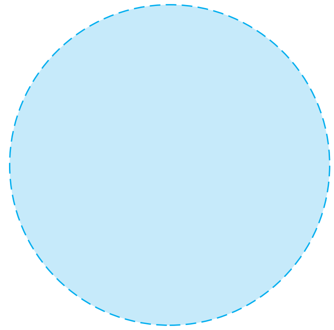


FIGURE 12.17a
Open disk

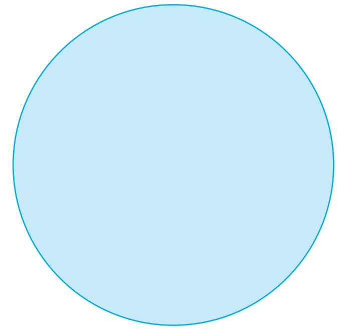


FIGURE 12.17b
Closed disk

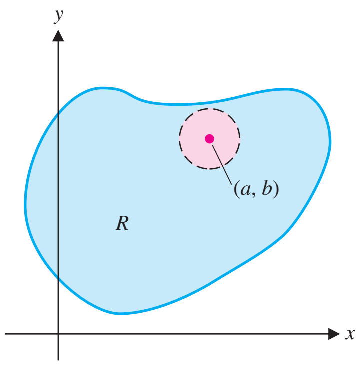


FIGURE 12.18a
Interior point

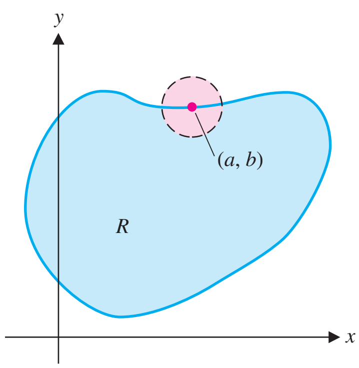


FIGURE 12.18b
Boundary point

that for a continuous function $f(x, y)$, if (x, y) changes slightly, then $f(x, y)$ also changes slightly.

Before we define the concept of continuity on a region $R \subset \mathbb{R}^2$, we first need to define open and closed regions in two dimensions. We refer to the interior of a circle (i.e., the set of all points inside but not on the circle) as an **open disk** (see Figure 12.17a). A **closed disk** consists of the circle and its interior (see Figure 12.17b). These are the two-dimensional analogs of open and closed intervals, respectively, of the real line. For a given two-dimensional region R, a point (a, b) in R is called an **interior point** of R if there is an open disk centered at (a, b) that lies *completely* inside of R (see Figure 12.18a). A point (a, b) in R is called a **boundary point** of R if *every* open disk centered at (a, b) contains points in R *and* points outside R (see Figure 12.18b). A set R is **closed** if it contains *all* of its boundary points. Alternatively, R is **open** if it contains *none* of its boundary points. Note that these are analogous to closed and open intervals of the real line: closed intervals include all (both) of their boundary points (endpoints), while open intervals include none (neither) of their boundary points.

If the domain of a function contains any of its boundary points, we will need to modify our definition of continuity slightly, to ensure that the limit is calculated over paths that lie inside the domain only. (Recall that this is essentially what we did to define continuity of a function of a single variable on a closed interval.) If (a, b) is a boundary point of the domain D of a function f, we say that f is continuous at (a, b) if

$$\lim_{\substack{(x,y)\to(a,b)\\(x,y)\in D}} f(x, y) = f(a, b).$$

This notation indicates that the limit is taken only along paths lying completely inside D. Note that this limit requires a slight modification of Definition 2.1, as follows.

We say that

$$\lim_{\substack{(x,y)\to(a,b)\\(x,y)\in D}} f(x, y) = L$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

We say that a function $f(x, y)$ is **continuous on a region** R if it is continuous at each point in R.

Notice that because we define continuity in terms of limits, we immediately have the following results, which follow directly from the corresponding results for limits. If $f(x, y)$

and $g(x, y)$ are continuous at (a, b), then $f + g$, $f - g$ and $f \cdot g$ are all continuous at (a, b). Further, f/g is continuous at (a, b), if, in addition, $g(a, b) \neq 0$. We leave the proofs of these statements as exercises.

In many cases, determining where a function is continuous involves identifying where the function isn't defined and using our continuity results for functions of a single variable.

EXAMPLE 2.8 Determining Where a Function of Two Variables Is Continuous

Find all points where the given function is continuous: (a) $f(x, y) = \dfrac{x}{x^2 - y}$ and

(b) $g(x, y) = \begin{cases} \dfrac{x^4}{x(x^2 + y^2)}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$.

Solution For (a), notice that $f(x, y)$ is a quotient of two polynomials (i.e., a rational function) and so, it is continuous at any point where we don't divide by 0. Since division by zero occurs only when $y = x^2$, we have that f is continuous at all points (x, y) with $y \neq x^2$. For (b), the function g is also a quotient of polynomials, except at the origin. Notice that there is a division by 0 whenever $x = 0$. We must consider the point $(0, 0)$ separately, however, since the function is not defined by the rational expression there. We can verify that $\lim\limits_{(x,y)\to(0,0)} g(x, y) = 0$ using the following string of inequalities. Notice that for $(x, y) \neq (0, 0)$,

$$|g(x, y)| = \left|\frac{x^4}{x(x^2 + y^2)}\right| \leq \left|\frac{x^4}{x(x^2)}\right| = |x|$$

and $|x| \to 0$ as $(x, y) \to (0, 0)$. By Theorem 2.1, we have that

$$\lim_{(x,y)\to(0,0)} g(x, y) = 0 = g(0, 0),$$

so that g is continuous at $(0, 0)$. Putting this all together, we get that g is continuous at the origin and also at all points (x, y) with $x \neq 0$. ■

Theorem 2.2 shows that we can use all of our established continuity results for functions of a single variable when considering functions of several variables.

THEOREM 2.2

Suppose that $f(x, y)$ is continuous at (a, b) and $g(x)$ is continuous at the point $f(a, b)$. Then

$$h(x, y) = (g \circ f)(x, y) = g(f(x, y))$$

is continuous at (a, b).

SKETCH OF THE PROOF

We leave the proof as an exercise, but it goes something like this. Notice that if (x, y) is close to (a, b), then by the continuity of f at (a, b), $f(x, y)$ will be close to $f(a, b)$. By the continuity of g at the point $f(a, b)$, it follows that $g(f(x, y))$ will be close to $g(f(a, b))$, so that $g \circ f$ is also continuous at (a, b). ■

EXAMPLE 2.9 Determining Where a Composition of Functions Is Continuous

Determine where $f(x, y) = e^{x^2 y}$ is continuous.

Solution Notice that $f(x, y) = g(h(x, y))$, where $g(t) = e^t$ and $h(x, y) = x^2 y$. Since g is continuous for all values of t and h is a polynomial in x and y (and hence continuous for all x and y), it follows from Theorem 2.2 that f is continuous for all x and y. ■

REMARK 2.3

All of the foregoing analysis is extended to functions of three (or more) variables in the obvious fashion.

DEFINITION 2.3

Let the function $f(x, y, z)$ be defined on the interior of a sphere, centered at the point (a, b, c), except possibly at (a, b, c) itself. We say that $\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x, y, z) - L| < \varepsilon$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta.$$

Observe that, as with limits of functions of two variables, Definition 2.3 says that in order to have $\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L$, we must have that $f(x, y, z)$ approaches L along every possible path through the point (a, b, c). Just as with functions of two variables, notice that if a function of three variables approaches different limits along two particular paths, then the limit does not exist.

EXAMPLE 2.10 A Limit in Three Dimensions That Does Not Exist

Evaluate $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2}$.

Solution First, we consider the path $x = y = 0$ (the z-axis). There, we have

$$\lim_{(0,0,z)\to(0,0,0)} \frac{0^2 + 0^2 - z^2}{0^2 + 0^2 + z^2} = \lim_{z\to 0} \frac{-z^2}{z^2} = -1.$$

Along the path $x = z = 0$ (the y-axis), we have

$$\lim_{(0,y,0)\to(0,0,0)} \frac{0^2 + y^2 - 0^2}{0^2 + y^2 + 0^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1.$$

Since the limits along these two specific paths do not agree, the limit does not exist. ■

We extend the definition of continuity to functions of three variables in the obvious way, as follows.

DEFINITION 2.4

Suppose that $f(x, y, z)$ is defined in the interior of a sphere centered at (a, b, c). We say that f is **continuous** at (a, b, c) if $\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$.

If $f(x, y, z)$ is not continuous at (a, b, c), then we call (a, b, c) a **discontinuity** of f.

As you can see, limits and continuity for functions of three variables work essentially the same as they do for functions of two variables. You will examine these in more detail in the exercises.

EXAMPLE 2.11 Continuity for a Function of Three Variables

Find all points where $f(x, y, z) = \ln(9 - x^2 - y^2 - z^2)$ is continuous.

Solution Notice that $f(x, y, z)$ is defined only for $9 - x^2 - y^2 - z^2 > 0$. On this domain, f is a composition of continuous functions, which is also continuous. So, f is continuous for $9 > x^2 + y^2 + z^2$, which you should recognize as the interior of the sphere of radius 3 centered at (0, 0, 0). ■

BEYOND FORMULAS

The examples in this section illustrate an important principle of logic. In this case, a limit exists if you get the same limiting value *for all possible paths* through the point. To disprove such a "for all" statement, you need only to find one specific counterexample. Finding two paths with different limits *proves* that the limit does not exist. However, to prove a "for all" statement, you must demonstrate that a general (arbitrary) example produces the desired result. This is typically a more elaborate task than finding a counterexample. To see what we mean, compare examples 2.3 and 2.6. To understand the different methods for proving that a limit does or does not exist, you need the more basic understanding of the logic of a universal "for all" statement.

EXERCISES 12.2

WRITING EXERCISES

1. Choosing between the paths $y = x$ and $x = y^2$, explain why $y = x$ is a better choice in example 2.4 but $x = y^2$ is a better choice in example 2.5.
2. In terms of Definition 2.1, explain why the limit in example 2.5 does not exist. That is, explain why making (x, y) close to (0, 0) doesn't guarantee that $f(x, y)$ is close to 0.
3. A friend claims that a limit equals 0, but you found that it does not exist. Looking over your friend's work, you see that the path with $x = 0$ and the path with $y = 0$ both produce a limit of 0. No other work is shown. Explain to your friend why other paths must be checked.
4. Explain why the path $y = x$ is not a valid path for the limit in example 2.7.

In exercises 1–6, compute the indicated limit.

1. $\lim\limits_{(x,y)\to(1,3)} \dfrac{x^2y}{4x^2 - y}$
2. $\lim\limits_{(x,y)\to(2,-1)} \dfrac{x + y}{x^2 - 2xy}$
3. $\lim\limits_{(x,y)\to(\pi,1)} \dfrac{\cos xy}{y^2 + 1}$
4. $\lim\limits_{(x,y)\to(-3,0)} \dfrac{e^{xy}}{x^2 + y^2}$
5. $\lim\limits_{(x,y,z)\to(1,0,2)} \dfrac{4xz}{y^2 + z^2}$
6. $\lim\limits_{(x,y,z)\to(1,1,2)} \dfrac{e^{x+y-z}}{x - z}$

In exercises 7–22, show that the indicated limit does not exist.

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2}{x^2+y^2}$

8. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2y^2}{2x^2-y^2}$

9. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{3y^2-x^2}$

10. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+2y^2}$

11. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$

12. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^3\sqrt{y}}{x^4+y^2}$

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt[3]{x}y^2}{x+y^3}$

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^3}{x^2+8y^6}$

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{y\sin x}{x^2+y^2}$

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x(\cos y-1)}{x^3+y^3}$

17. $\lim\limits_{(x,y)\to(1,2)} \dfrac{xy-2x-y+2}{x^2-2x+y^2-4y+5}$

18. $\lim\limits_{(x,y)\to(2,0)} \dfrac{2y^2}{(x-2)^2+y^2}$

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{3x^2}{x^2+y^2+z^2}$

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2+y^2+z^2}{x^2-y^2+z^2}$

21. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^3+y^3+z^3}$

22. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2yz}{x^4+y^4+z^4}$

In exercises 23–30, show that the indicated limit exists.

23. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

24. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

25. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2\sin y}{2x^2+y^2}$

26. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y+x^2y^3}{x^2+y^2}$

27. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+4x^2+2y^2}{2x^2+y^2}$

28. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y-x^2-y^2}{x^2+y^2}$

29. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{3x^3}{x^2+y^2+z^2}$

30. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^2+y^2+z^2}$

In exercises 31–34, use graphs and density plots to explain why the limit in the indicated exercise does not exist.

31. Exercise 7

32. Exercise 8

33. Exercise 9

34. Exercise 10

In exercises 35–44, determine all points at which the given function is continuous.

35. $f(x,y)=\sqrt{9-x^2-y^2}$

36. $f(x,y,z)=\dfrac{x^3}{y}+\sin z$

37. $f(x,y)=\ln(3-x^2+y)$

38. $f(x,y)=\tan(x+y)$

39. $f(x,y,z)=\sqrt{x^2+y^2+z^2-4}$

40. $f(x,y,z)=\sqrt{z-x^2-y^2}$

41. $f(x,y)=\begin{cases}(y-2)\cos\left(\frac{1}{x^2}\right), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$

42. $f(x,y)=\begin{cases}\dfrac{\sin\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}, & \text{if } x^2+y^2\neq 1\\ 1, & \text{if } x^2+y^2=1\end{cases}$

43. $f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x-y}, & \text{if } x\neq y\\ 2x, & \text{if } x=y\end{cases}$

44. $f(x,y)=\begin{cases}\cos\left(\dfrac{1}{x^2+y^2}\right), & \text{if } (x,y)\neq(0,0)\\ 1, & \text{if } (x,y)=(0,0)\end{cases}$

In exercises 45 and 46, estimate the indicated limit numerically.

45. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos xy}{x^2y^2+x^2y^3}$

46. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3\sin xy^2}{x^2y^2+xy^2}$

In exercises 47–50, label the statement as true or false and explain.

47. If $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$, then $\lim\limits_{x\to a} f(x,b)=L$.

48. If $\lim\limits_{x\to a} f(x,b)=L$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$.

49. If $\lim\limits_{x\to a} f(x,b)=\lim\limits_{y\to b} f(a,y)=L$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$.

50. If $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$, then $\lim\limits_{(x,y)\to(0,0)} f(cx,y)=0$ for any constant c.

51. In example 2.5, show that for any k, if the limit is evaluated along the line $y=kx$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}=0$.

52. Repeat exercise 51 for the limit in exercise 14.

53. Show that the function $f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$ is not continuous at $(0,0)$. Notice that this function is closely related to that of example 2.5.

54. Show that the function in exercise 53 "acts" continuous at the origin along any straight line through the origin, in the sense that for any such line l with the limit restricted to points (x,y) on l, $\lim\limits_{(x,y)\to(0,0)} f(x,y)=f(0,0)$.

In exercises 55–58, use polar coordinates to find the indicated limit, if it exists. Note that $(x,y)\to(0,0)$ is equivalent to $r\to 0$.

55. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2+y^2}}{\sin\sqrt{x^2+y^2}}$

56. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{x^2+y^2}-1}{x^2+y^2}$

57. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

58. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

EXPLORATORY EXERCISES

1. In this exercise, you will explore how the patterns of contour plots relate to the existence of limits. Start by showing that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ doesn't exist and $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$. Then sketch several contour plots for each function while zooming in on the point (0, 0). For a function whose limit exists as (x, y) approaches (a, b), what should be happening to the range of function values as you zoom in on the point (a, b)? Describe the appearance of each contour plot for $\frac{x^2 y}{x^2+y^2}$ near (0, 0). By contrast, what should be happening to the range of function values as you zoom in on a point at which the limit doesn't exist? Explain how this appears in the contour plots for $\frac{x^2}{x^2+y^2}$. Use contour plots to conjecture whether or not the following limits exist: $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y}$ and $\lim_{(x,y)\to(0,0)} \frac{x \sin y}{x^2+y^2}$.

2. Find a function $g(y)$ such that $f(x, y)$ is continuous for

$$f(x,y) = \begin{cases} (1+xy)^{1/x}, & \text{if } x \neq 0 \\ g(y), & \text{if } x = 0 \end{cases}.$$

12.3 PARTIAL DERIVATIVES

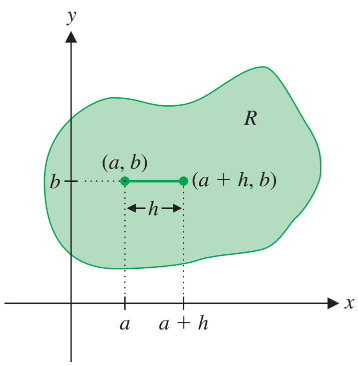


FIGURE 12.19
Average temperature on a horizontal line segment

In this section, we generalize the notion of derivative to functions of more than one variable. First, recall that for a function f of a single variable, we define the derivative function as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

for any values of x for which the limit exists. At any particular value $x = a$, we interpret $f'(a)$ as the instantaneous rate of change of the function with respect to x at that point.

Consider a flat metal plate in the shape of the region $R \subset \mathbb{R}^2$. Suppose that the temperature at any point $(x, y) \in R$ is given by $f(x, y)$. If you move along the horizontal line segment from (a, b) to $(a + h, b)$, what is the average rate of change of the temperature with respect to the horizontal distance x (see Figure 12.19)? Notice that on this line segment, y is a constant ($y = b$). So, the average rate of change on this line segment is given by

$$\frac{f(a+h,b) - f(a,b)}{h}.$$

To get the instantaneous rate of change of f in the x-direction at the point (a, b), we take the limit as $h \to 0$:

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

You should recognize this limit as a derivative. Since f is a function of two variables and we have held the one variable fixed ($y = b$), we call this the **partial derivative of f with respect to x** at the point (a, b), denoted

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

This says that $\frac{\partial f}{\partial x}(a, b)$ gives the instantaneous rate of change of f with respect to x (i.e., in the x-direction) at the point (a, b). Graphically, observe that in defining $\frac{\partial f}{\partial x}(a, b)$, we are looking only at points in the plane $y = b$. The intersection of $z = f(x, y)$ and $y = b$ is a curve, as shown in Figures 12.20a and 12.20b (on the following page). The partial derivative $\frac{\partial f}{\partial x}(a, b)$ then gives the slope of the tangent line to this curve at $x = a$, as indicated in Figure 12.20b.

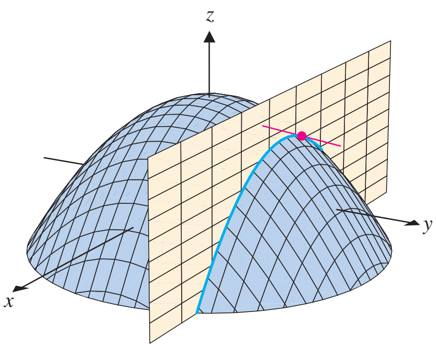


FIGURE 12.20a
Intersection of the surface $z = f(x, y)$ with the plane $y = b$

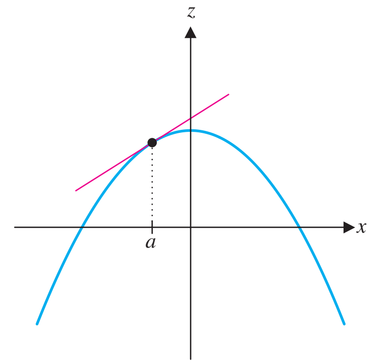


FIGURE 12.20b
The curve $z = f(x, b)$

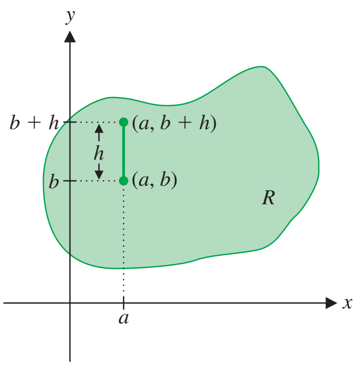


FIGURE 12.21
Average temperature on a vertical line segment

Alternatively, if we move along a vertical line segment from (a, b) to $(a, b + h)$ (see Figure 12.21), the average rate of change of f along this segment is given by

$$\frac{f(a, b + h) - f(a, b)}{h}.$$

The instantaneous rate of change of f in the y-direction at the point (a, b) is then given by

$$\lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

which you should again recognize as a derivative. In this case, however, we have held the value of x fixed ($x = a$) and refer to this as the **partial derivative of f with respect to y** at the point (a, b), denoted

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

Graphically, observe that in defining $\frac{\partial f}{\partial y}(a, b)$, we are looking only at points in the plane $x = a$. The intersection of $z = f(x, y)$ and $x = a$ is a curve, as shown in Figures 12.22a and 12.22b. In this case, notice that the partial derivative $\frac{\partial f}{\partial y}(a, b)$ gives the slope of the tangent line to the curve at $y = b$, as shown in Figure 12.22b.

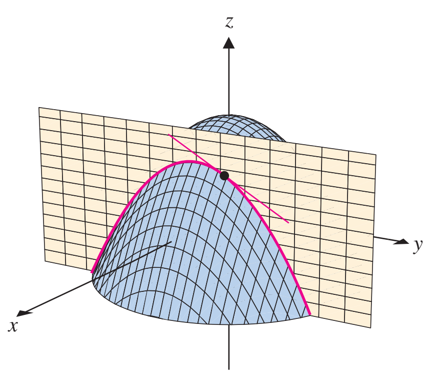


FIGURE 12.22a
The intersection of the surface $z = f(x, y)$ with the plane $x = a$

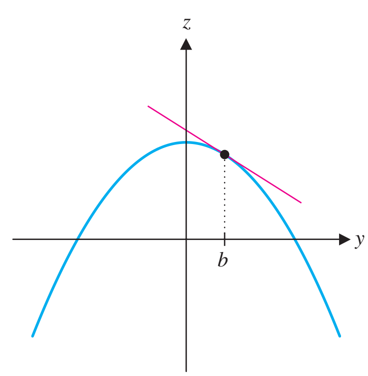


FIGURE 12.22b
The curve $z = f(a, y)$

More generally, we define the partial derivative functions as follows.

DEFINITION 3.1

The **partial derivative of $f(x, y)$ with respect to x,** written $\frac{\partial f}{\partial x}$, is defined by

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

for any values of x and y for which the limit exists.

The **partial derivative of $f(x, y)$ with respect to y,** written $\frac{\partial f}{\partial y}$, is defined by

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

for any values of x and y for which the limit exists.

Since we are now dealing with functions of several variables, we can no longer use the prime notation for denoting partial derivatives. [Which partial derivative would $f'(x, y)$ denote?] We introduce several convenient types of notation here. For $z = f(x, y)$, we write

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \frac{\partial z}{\partial x}(x, y) = \frac{\partial}{\partial x}[f(x, y)].$$

The expression $\frac{\partial}{\partial x}$ is a **partial differential operator.** It tells you to take the partial derivative (with respect to x) of whatever expression follows it. Similarly, we have

$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \frac{\partial z}{\partial y}(x, y) = \frac{\partial}{\partial y}[f(x, y)].$$

Look carefully at how we defined these derivatives and you'll see that we can compute partial derivatives using all of our usual rules for computing ordinary derivatives. Notice that in the definition of $\frac{\partial f}{\partial x}$, the value of y is held constant, say at $y = b$. If we define $g(x) = f(x, b)$, then

$$\frac{\partial f}{\partial x}(x, b) = \lim_{h \to 0} \frac{f(x+h, b) - f(x, b)}{h} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x).$$

That is, to compute the partial derivative $\frac{\partial f}{\partial x}$, you simply take an ordinary derivative with respect to x, while treating y as a constant. Similarly, you can compute $\frac{\partial f}{\partial y}$ by taking an ordinary derivative with respect to y, while treating x as a constant.

EXAMPLE 3.1 Computing Partial Derivatives

For $f(x, y) = 3x^2 + x^3y + 4y^2$, compute $\frac{\partial f}{\partial x}(x, y)$, $\frac{\partial f}{\partial y}(x, y)$, $f_x(1, 0)$ and $f_y(2, -1)$.

Solution Compute $\frac{\partial f}{\partial x}$ by treating y as a constant. We have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^2 + x^3y + 4y^2) = 6x + (3x^2)y + 0 = 6x + 3x^2y.$$

The partial derivative of $4y^2$ with respect to x is 0, since $4y^2$ is treated as if it were a constant when differentiating with respect to x. Next, we compute $\frac{\partial f}{\partial y}$ by treating x as a constant. We have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^2 + x^3 y + 4y^2) = 0 + x^3(1) + 8y = x^3 + 8y.$$

Substituting values for x and y, we get

$$f_x(1, 0) = \frac{\partial f}{\partial x}(1, 0) = 6 + 0 = 6$$

and

$$f_y(2, -1) = \frac{\partial f}{\partial y}(2, -1) = 8 - 8 = 0.$$

Since we are holding one of the variables fixed when we compute a partial derivative, we have the product rules:

$$\frac{\partial}{\partial x}(uv) = \frac{\partial u}{\partial x}v + u\frac{\partial v}{\partial x}$$

and

$$\frac{\partial}{\partial y}(uv) = \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y}$$

and the quotient rule:

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{\frac{\partial u}{\partial x}v - u\frac{\partial v}{\partial x}}{v^2},$$

with a corresponding quotient rule holding for $\frac{\partial}{\partial y}\left(\frac{u}{v}\right)$.

EXAMPLE 3.2 Computing Partial Derivatives

For $f(x, y) = e^{xy} + \frac{x}{y}$, compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution Recall that if $g(x) = e^{4x} + \frac{x}{4}$, then $g'(x) = 4e^{4x} + \frac{1}{4}$, from the chain rule. Replacing the 4's with y and treating them as we would any other constant, we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{xy} + \frac{x}{y}\right) = ye^{xy} + \frac{1}{y}.$$

For the y-partial derivative, recall that if $h(y) = \frac{4}{y}$, then $h'(y) = -\frac{4}{y^2}$. Replacing the 4 with x and treating it as you would any other constant, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{xy} + \frac{x}{y}\right) = xe^{xy} - \frac{x}{y^2}.$$

We interpret partial derivatives as rates of change, in the same way as we interpret ordinary derivatives of functions of a single variable.

EXAMPLE 3.3 An Application of Partial Derivatives to Thermodynamics

For a real gas, van der Waals' equation states that

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT.$$

Here, P is the pressure of the gas, V is the volume of the gas, T is the temperature (in degrees Kelvin), n is the number of moles of gas, R is the universal gas constant and a and b are constants. Compute and interpret $\frac{\partial P}{\partial V}$ and $\frac{\partial T}{\partial P}$.

Solution We first solve for P to get

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}$$

and compute

$$\frac{\partial P}{\partial V} = \frac{\partial}{\partial V}\left(\frac{nRT}{V - nb} - \frac{n^2 a}{V^2}\right) = -\frac{nRT}{(V - nb)^2} + 2\frac{n^2 a}{V^3}.$$

Notice that this gives the rate of change of pressure relative to a change in volume (with temperature held constant). Next, solving van der Waals' equation for T, we get

$$T = \frac{1}{nR}\left(P + \frac{n^2 a}{V^2}\right)(V - nb)$$

and compute

$$\frac{\partial T}{\partial P} = \frac{\partial}{\partial P}\left[\frac{1}{nR}\left(P + \frac{n^2 a}{V^2}\right)(V - nb)\right] = \frac{1}{nR}(V - nb).$$

This gives the rate of change of temperature relative to a change in pressure (with volume held constant). In exercise 22, you will have an opportunity to discover an interesting fact about these partial derivatives. ■

TODAY IN MATHEMATICS

Shing-Tung Yau (1949–)
A Chinese-born mathematician who earned a Fields Medal for his contributions to algebraic geometry and partial differential equations. A strong supporter of mathematics education in China, he established the Institute of Mathematical Science in Hong Kong, the Morningside Center of Mathematics of the Chinese Academy of Sciences and the Center of Mathematical Sciences at Zhejiang University. His work has been described by colleagues as "extremely deep and powerful" and as showing "enormous technical power and insight. He has cracked problems on which progress has been stopped for years."

Notice that the partial derivatives found in the preceding examples are themselves functions of two variables. We have seen that second- and higher-order derivatives of functions of a single variable provide much significant information. Not surprisingly, **higher-order partial derivatives** are also very important in applications.

For functions of two variables, there are four different second-order partial derivatives. The partial derivative with respect to x of $\frac{\partial f}{\partial x}$ is $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, usually abbreviated as $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}. Similarly, taking two successive partial derivatives with respect to y gives us $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$. For **mixed second-order partial derivatives,** one derivative is taken with respect to each variable. If the first partial derivative is taken with respect to x, we have $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, abbreviated as $\frac{\partial^2 f}{\partial y \partial x}$, or $(f_x)_y = f_{xy}$. If the first partial derivative is taken with respect to y, we have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, abbreviated as $\frac{\partial^2 f}{\partial x \partial y}$, or $(f_y)_x = f_{yx}$.

EXAMPLE 3.4 Computing Second-Order Partial Derivatives

Find all second-order partial derivatives of $f(x, y) = x^2y - y^3 + \ln x$.

Solution We start by computing the first-order partial derivatives: $\dfrac{\partial f}{\partial x} = 2xy + \dfrac{1}{x}$ and $\dfrac{\partial f}{\partial y} = x^2 - 3y^2$. We then have

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy + \frac{1}{x}\right) = 2y - \frac{1}{x^2},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy + \frac{1}{x}\right) = 2x,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 - 3y^2\right) = 2x$$

and finally,
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 - 3y^2\right) = -6y.$$

Notice in example 3.4 that $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$. It turns out that this is true for most, but *not all*, of the functions that you will encounter. (See exercise 61 for a counterexample.) The proof of the following result can be found in most texts on advanced calculus.

THEOREM 3.1

If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous on an open set containing (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

We can, of course, compute third-, fourth- or even higher-order partial derivatives. Theorem 3.1 can be extended to show that as long as the partial derivatives are all continuous in an open set, the order of differentiation doesn't matter. With higher-order partial derivatives, notations such as $\dfrac{\partial^3 f}{\partial x \partial y \partial x}$ become quite awkward and so, we usually use f_{xyx} instead.

EXAMPLE 3.5 Computing Higher-Order Partial Derivatives

For $f(x, y) = \cos(xy) - x^3 + y^4$, compute f_{xyy} and f_{xyyy}.

Solution We have

$$f_x = \frac{\partial}{\partial x}\left[\cos(xy) - x^3 + y^4\right] = -y\sin(xy) - 3x^2.$$

Differentiating f_x with respect to y gives us

$$f_{xy} = \frac{\partial}{\partial y}[-y\sin(xy) - 3x^2] = -\sin(xy) - xy\cos(xy)$$

and
$$f_{xyy} = \frac{\partial}{\partial y}[-\sin(xy) - xy\cos(xy)] = -2x\cos(xy) + x^2y\sin(xy).$$

Finally, we have

$$\begin{aligned} f_{xyyy} &= \frac{\partial}{\partial y}[-2x\cos(xy) + x^2 y\sin(xy)] \\ &= 2x^2\sin(xy) + x^2\sin(xy) + x^3 y\cos(xy) = 3x^2\sin(xy) + x^3 y\cos(xy). \end{aligned}$$

Thus far, we have worked with partial derivatives of functions of two variables. The extensions to functions of three or more variables are completely analogous to what we have discussed here. In example 3.6, you can see that the calculations proceed just as you would expect.

EXAMPLE 3.6 Partial Derivatives of Functions of Three Variables

For $f(x, y, z) = \sqrt{xy^3z} + 4x^2y$, defined for $x, y, z \geq 0$, compute f_x, f_{xy} and f_{xyz}.

Solution To keep x, y and z as separate as possible, we first rewrite f as

$$f(x, y, z) = x^{1/2}y^{3/2}z^{1/2} + 4x^2y.$$

To compute the partial derivative with respect to x, we treat y and z as constants and obtain

$$f_x = \frac{\partial}{\partial x}(x^{1/2}y^{3/2}z^{1/2} + 4x^2y) = \left(\frac{1}{2}x^{-1/2}\right)y^{3/2}z^{1/2} + 8xy.$$

Next, treating x and z as constants, we get

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{1}{2}x^{-1/2}y^{3/2}z^{1/2} + 8xy\right) = \left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right)z^{1/2} + 8x.$$

Finally, treating x and y as constants, we get

$$f_{xyz} = \frac{\partial}{\partial z}\left[\left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right)z^{1/2} + 8x\right] = \left(\frac{1}{2}x^{-1/2}\right)\left(\frac{3}{2}y^{1/2}\right)\left(\frac{1}{2}z^{-1/2}\right).$$

Notice that this derivative is defined for $x, z > 0$ and $y \geq 0$. Further, you can show that all first-, second- and third-order partial derivatives are continuous for $x, y, z > 0$, so that the order in which we take the partial derivatives is irrelevant in this case.

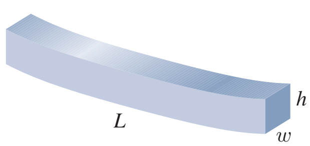


FIGURE 12.23
A horizontal beam

EXAMPLE 3.7 An Application of Partial Derivatives to a Sagging Beam

The sag in a beam of length L, width w and height h (see Figure 12.23) is given by $S(L, w, h) = c\dfrac{L^4}{wh^3}$ for some constant c. Show that $\dfrac{\partial S}{\partial L} = \dfrac{4}{L}S$, $\dfrac{\partial S}{\partial w} = -\dfrac{1}{w}S$ and $\dfrac{\partial S}{\partial h} = -\dfrac{3}{h}S$. Use this result to determine which variable has the greatest proportional effect on the sag.

Solution We start by computing

$$\frac{\partial S}{\partial L} = \frac{\partial}{\partial L}\left(c\frac{L^4}{wh^3}\right) = c\frac{4L^3}{wh^3}.$$

To rewrite this in terms of S, multiply top and bottom by L to get

$$\frac{\partial S}{\partial L} = c\frac{4L^3}{wh^3} = c\frac{4L^4}{wh^3L} = \frac{4}{L}c\frac{L^4}{wh^3} = \frac{4}{L}S.$$

The other calculations are similar and are left as exercises. To interpret the results, suppose that a small change ΔL in length produces a small change ΔS in the sag. We now have that $\frac{\Delta S}{\Delta L} \approx \frac{\partial S}{\partial L} = \frac{4}{L} S$. Rearranging the terms, we have

$$\frac{\Delta S}{S} \approx 4 \frac{\Delta L}{L}.$$

That is, the proportional change in S is approximately four times the proportional change in L. Similarly, we have that in absolute value, the proportional change in S is approximately the proportional change in w and three times the proportional change in h. Proportionally then, a change in the length has the greatest effect on the amount of sag. In this sense, length is the most *important* of the three dimensions. ■

In many applications, no formula for the function is available and we can only estimate the value of the partial derivatives from a small collection of data points.

EXAMPLE 3.8 Estimating Partial Derivatives from a Table of Data

A computer simulation of the flight of a baseball provided the data displayed in the table for the range $f(v, \omega)$ in feet of a ball hit with initial velocity v ft/s and backspin rate of ω rpm. Each ball is struck at an angle of 30° above the horizontal.

v \ ω	**0**	**1000**	**2000**	**3000**	**4000**
150	294	312	333	350	367
160	314	334	354	373	391
170	335	356	375	395	414
180	355	376	397	417	436

Use the data to estimate $\frac{\partial f}{\partial v}(160, 2000)$ and $\frac{\partial f}{\partial \omega}(160, 2000)$. Interpret both quantities in baseball terms.

Solution From the definition of partial derivative, we know that

$$\frac{\partial f}{\partial v}(160, 2000) = \lim_{h \to 0} \frac{f(160 + h, 2000) - f(160, 2000)}{h},$$

so we can approximate the value of the partial derivative by computing the difference quotient $\frac{f(160 + h, 2000) - f(160, 2000)}{h}$ for as small a value of h as possible. Since data points are provided for $v = 150$, we can compute the difference quotient for $h = -10$, to get

$$\frac{\partial f}{\partial v}(160, 2000) \approx \frac{f(150, 2000) - f(160, 2000)}{150 - 160} = \frac{333 - 354}{150 - 160} = 2.1.$$

We can also use the data point for $v = 170$, to get

$$\frac{\partial f}{\partial v}(160, 2000) \approx \frac{f(170, 2000) - f(160, 2000)}{170 - 160} = \frac{375 - 354}{170 - 160} = 2.1.$$

Since both estimates equal 2.1, we make the estimate $\frac{\partial f}{\partial v}(160, 2000) \approx 2.1$. The data point $f(160, 2000) = 354$ tells us that a ball struck with initial velocity 160 ft/s and backspin 2000 rpm will fly 354 feet. The partial derivative tells us that increasing the initial velocity by 1 ft/s will add approximately 2.1 feet to the distance.

Similarly, to estimate $\frac{\partial f}{\partial \omega}(160, 2000)$, we note that the closest data values to $\omega = 2000$ are $\omega = 1000$ and $\omega = 3000$. We get

$$\frac{\partial f}{\partial \omega}(160, 2000) \approx \frac{f(160,1000) - f(160,2000)}{1000 - 2000} = \frac{334 - 354}{1000 - 2000} = 0.02$$

and $$\frac{\partial f}{\partial \omega}(160, 2000) \approx \frac{f(160,3000) - f(160,2000)}{3000 - 2000} = \frac{373 - 354}{3000 - 2000} = 0.019.$$

Reasonable estimates for $\frac{\partial f}{\partial \omega}(160, 2000)$ are then 0.02, 0.019 or 0.0195 (the average of the two calculations). Using 0.02 as our approximation, we can interpret this to mean that an increase in backspin of 1 rpm will add approximately 0.02 ft to the distance. A simpler way to interpret this is to say that an increase of 100 rpm will add approximately 2 ft to the distance. ■

BEYOND FORMULAS

When you think about partial derivatives, it helps to use the Rule of Three, which suggests that mathematical topics should be explored from symbolic, graphical and numerical viewpoints, where appropriate. Symbolically, you have all of the usual derivative formulas at your disposal. Graphically, you can view the value of a partial derivative at a particular point as the slope of the tangent line to a cross section of the surface $z = f(x, y)$. Numerically, you can approximate the value of a partial derivative at a point using a difference quotient, as in example 3.8.

EXERCISES 12.3

WRITING EXERCISES

1. Suppose that the function $f(x, y)$ is a sum of terms where each term contains x or y but not both. Explain why $f_{xy} = 0$.

2. In Definition 3.1, explain how to remember which partial derivative involves the term $f(x + h, y)$ and which involves the term $f(x, y + h)$.

3. In section 2.8, we computed derivatives implicitly, by using the chain rule and differentiating both sides of an equation with respect to x. In the process of doing so, we made calculations such as $(x^2y^2)' = 2xy^2 + 2x^2yy'$. Explain why this derivative is computed differently than the partial derivatives of this section.

4. For $f(x, y, z) = x^3e^{4x \sin y} + y^2 \sin xy + 4xyz$, you could compute f_{xyz} in a variety of orders. Discuss how many different orders are possible and which order(s) would be the easiest.

In exercises 1–8, find all first-order partial derivatives.

1. $f(x, y) = x^3 - 4xy^2 + y^4$

2. $f(x, y) = x^2y^3 - 3x$

3. $f(x, y) = x^2 \sin xy - 3y^3$

4. $f(x, y) = 3e^{x^2y} - \sqrt{x - 1}$

5. $f(x, y) = 4e^{x/y} - \frac{y}{x}$

6. $f(x, y) = \frac{x - 3}{y} + x^2 \tan y$

7. $f(x, y, z) = 3x \sin y + 4x^3y^2z$

8. $f(x, y, z) = \frac{2}{\sqrt{x^2 + y^2 + z^2}}$

In exercises 9–16, find the indicated partial derivatives.

9. $f(x, y) = x^3 - 4xy^2 + 3y$; $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}$

10. $f(x, y) = x^2 y - 4x + 3 \sin y$; $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}$

11. $f(x, y) = x^4 - 3x^2 y^3 + 5y$; f_{xx}, f_{xy}, f_{xyy}

12. $f(x, y) = e^{4x} - \sin y^2 - \sqrt{xy}$; f_{xx}, f_{xy}, f_{yyx}

13. $f(x, y, z) = x^3 y^2 - \sin yz$; f_{xx}, f_{yz}, f_{xyz}

14. $f(x, y, z) = e^{2xy} - \frac{z^2}{y} + xz \sin y$; f_{xx}, f_{yy}, f_{yyzz}

15. $f(w, x, y, z) = w^2 xy - e^{wz}$; $f_{ww}, f_{wxy}, f_{wwxyz}$

16. $f(w, x, y, z) = \sqrt{wyz} - x^3 \sin w$; f_{xx}, f_{yy}, f_{wxyz}

In exercises 17–20, (a) sketch the graph of $z = f(x, y)$ and (b) on this graph, highlight the appropriate two-dimensional trace and interpret the partial derivative as a slope.

17. $f(x, y) = 4 - x^2 - y^2$, $\frac{\partial f}{\partial x}(1, 1)$

18. $f(x, y) = \sqrt{x^2 + y^2}$, $\frac{\partial f}{\partial x}(1, 0)$

19. $f(x, y) = 4 - x^2 - y^2$, $\frac{\partial f}{\partial y}(2, 0)$

20. $f(x, y) = \sqrt{x^2 + y^2}$, $\frac{\partial f}{\partial y}(0, 2)$

21. Compute and interpret $\frac{\partial V}{\partial T}$ for van der Waals' equation (see example 3.3).

22. For van der Waals' equation, show that $\frac{\partial T}{\partial P} \frac{\partial P}{\partial V} \frac{\partial V}{\partial T} = -1$. If you misunderstood the chain rule, why might you expect this product to equal 1?

23. For the specific case of van der Waals' equation given by $\left(P + \frac{14}{V^2}\right)(V - 0.004) = 12T$, use the partial derivative $\frac{\partial P}{\partial T}$ to estimate the change in pressure due to an increase of one degree.

24. For the specific case of van der Waals' equation given by $\left(P + \frac{14}{V^2}\right)(V - 0.004) = 12T$, use the partial derivative $\frac{\partial T}{\partial V}$ to estimate the change in temperature due to an increase in volume of one unit.

25. In example 3.7, show that $\frac{\partial S}{\partial w} = -\frac{1}{w} S$.

26. In example 3.7, show that $\frac{\partial S}{\partial h} = -\frac{3}{h} S$.

27. If the sag in the beam of example 3.7 were given by $S(L, w, h) = c \frac{L^3}{wh^4}$, determine which variable would have the greatest proportional effect.

28. Based on example 3.7 and your result in exercise 27, state a simple rule for determining which variable has the greatest proportional effect.

In exercises 29–32, find all points at which $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ and interpret the significance of the points graphically.

29. $f(x, y) = x^2 + y^2$

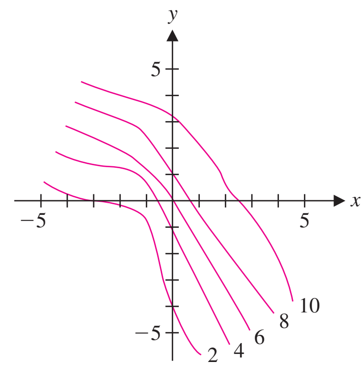
30. $f(x, y) = x^2 + y^2 - x^4$

31. $f(x, y) = \sin x \sin y$

32. $f(x, y) = e^{-x^2 - y^2}$

In exercises 33–36, use the contour plot to estimate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the origin.

33.



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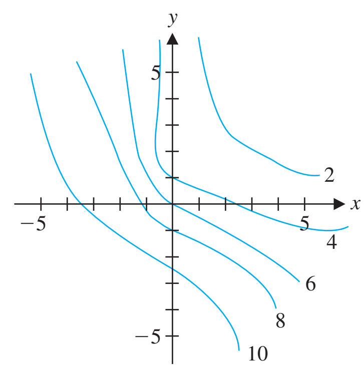

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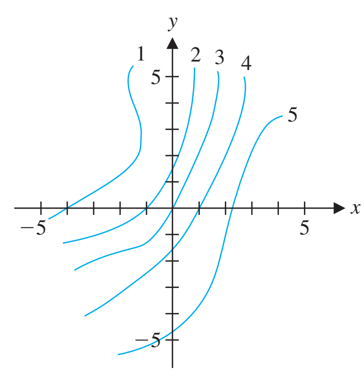
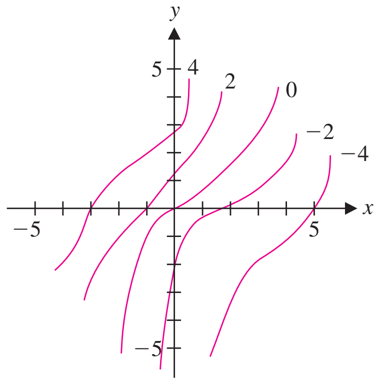


36.



37. The table shows wind chill (how cold it "feels" outside) as a function of temperature (degrees Fahrenheit) and wind speed (mph). We can think of this as a function $C(t, s)$. Estimate the partial derivatives $\frac{\partial C}{\partial t}(10, 10)$ and $\frac{\partial C}{\partial s}(10, 10)$. Interpret each partial derivative and explain why it is surprising that $\frac{\partial C}{\partial t}(10, 10) \neq 1$.

Speed \ *Temp*	30	20	10	0	−10
0	30	20	10	0	−10
5	27	16	6	−5	−15
10	16	4	−9	−24	−33
15	9	−5	−18	−32	−45
20	4	−10	−25	−39	−53
25	0	−15	−29	−44	−59
30	−2	−18	−33	−48	−63

38. Rework exercise 37 using the point (10, 20). Explain the significance of the inequality $\left|\frac{\partial C}{\partial s}(10, 10)\right| > \left|\frac{\partial C}{\partial s}(10, 20)\right|$.

39. Using the baseball data in example 3.8, estimate and interpret $\frac{\partial f}{\partial v}(170, 3000)$ and $\frac{\partial f}{\partial \omega}(170, 3000)$.

40. According to the data in example 3.8, a baseball with initial velocity 170 ft/s and backspin 3000 rpm flies 395 ft. Suppose that the ball must go 400 ft to clear the fence for a home run. Based on your answers to exercise 39, how much extra backspin is needed for a home run?

41. Carefully write down a definition for the three first-order partial derivatives of a function of three variables $f(x, y, z)$.

42. Determine how many second-order partial derivatives there are of $f(x, y, z)$. Assuming a result analogous to Theorem 3.1, how many of these second-order partial derivatives are actually different?

43. Show that the functions $f_n(x, t) = \sin n\pi x \cos n\pi ct$ satisfy the **wave equation** $c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$, for any positive integer n and any constant c.

44. Show that if $f(x)$ is a function with a continuous second derivative, then $f(x - ct)$ and $f(x + ct)$ are both solutions of the wave equation of exercise 43. If x represents position and t represents time, explain why c can be interpreted as the velocity of the wave.

45. The value of an investment of \$1000 invested at a constant 10% rate for 5 years is $V = 1000\left[\frac{1 + 0.1(1 - T)}{1 + I}\right]^5$, where T is the tax rate and I is the inflation rate. Compute $\frac{\partial V}{\partial I}$ and $\frac{\partial V}{\partial T}$, and discuss whether the tax rate or the inflation rate has a greater influence on the value of the investment.

46. The value of an investment of \$1000 invested at a rate r for 5 years with a tax rate of 28% is $V = 1000\left[\frac{1 + 0.72r}{1 + I}\right]^5$, where I is the inflation rate. Compute $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial I}$, and discuss whether the investment rate or the inflation rate has a greater influence on the value of the investment.

47. Suppose that the position of a guitar string of length L varies according to $p(x, t) = \sin x \cos t$, where x represents the distance along the string, $0 \le x \le L$, and t represents time. Compute and interpret $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial t}$.

48. Suppose that the concentration of some pollutant in a river as a function of position x and time t is given by $p(x, t) = p_0(x - ct)e^{-\mu t}$ for constants p_0, c and μ. Show that $\frac{\partial p}{\partial t} = -c\frac{\partial p}{\partial x} - \mu p$. Interpret both $\frac{\partial p}{\partial t}$ and $\frac{\partial p}{\partial x}$, and explain how this equation relates the change in pollution at a specific location to the current of the river and the rate at which the pollutant decays.

49. In a chemical reaction, the temperature T, entropy S, Gibbs free energy G and enthalpy H are related by $G = H - TS$. Show that $\frac{\partial(G/T)}{\partial T} = -\frac{H}{T^2}$.

50. For the chemical reaction of exercise 49, show that $\frac{\partial(G/T)}{\partial(1/T)} = H$. Chemists measure the enthalpy of a reaction by measuring this rate of change.

51. Suppose that three resistors are in parallel in an electrical circuit. If the resistances are R_1, R_2 and R_3 ohms, respectively, then the net resistance in the circuit equals $R = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$. Compute and interpret the partial derivative $\frac{\partial R}{\partial R_1}$. Given this partial derivative, explain how to quickly write down the partial derivatives $\frac{\partial R}{\partial R_2}$ and $\frac{\partial R}{\partial R_3}$.

52. The ideal gas law relating pressure, temperature and volume is $P = \frac{cT}{V}$, for some constant c. Show that $T\frac{\partial P}{\partial T}\frac{\partial V}{\partial T} = c$.

53. A process called **tag-and-recapture** is used to estimate populations of animals in the wild. First, some number T of the animals are captured, tagged and released into the wild. Later, a number S of the animals are captured, of which t are observed to be tagged. The estimate of the total population is then $P(T, S, t) = \frac{TS}{t}$. Compute $P(100, 60, 15)$; the proportion of tagged animals in the recapture is $\frac{15}{60} = \frac{1}{4}$. Based on your estimate of the total population, what proportion of the total population has been tagged? Now compute $\frac{\partial P}{\partial t}(100, 60, 15)$ and use it to estimate how much your population estimate would change if one more recaptured animal were tagged.

54. Let $T(x, y)$ be the temperature at longitude x and latitude y in the United States. In general, explain why you would expect to have $\frac{\partial T}{\partial y} < 0$. If a cold front is moving from east to west, would you expect $\frac{\partial T}{\partial x}$ to be positive or negative?

55. Suppose that L hours of labor and K dollars of investment by a company result in a productivity of $P = L^{0.75}K^{0.25}$. Compute the marginal productivity of labor, defined by $\frac{\partial P}{\partial L}$ and the marginal productivity of capital, defined by $\frac{\partial P}{\partial K}$.

56. For the production function in exercise 55, show that $\frac{\partial^2 P}{\partial L^2} < 0$ and $\frac{\partial^2 P}{\partial K^2} < 0$. Interpret this in terms of diminishing returns on investments in labor and capital. Show that $\frac{\partial^2 P}{\partial L \partial K} > 0$ and interpret it in economic terms.

57. Suppose that the demand for flour is given by $D_1 = 300 + \frac{10}{p_1+4} - 5p_2$ and the demand for bread is given by $D_2 = 250 + \frac{6}{p_2+2} - 6p_1$, where p_1 is the price of a pound of flour and p_2 is the price of a loaf of bread. Show that $\frac{\partial D_1}{\partial p_2}$ and $\frac{\partial D_2}{\partial p_1}$ are both negative. This is the definition of **complementary commodities.** Interpret the partial derivatives and explain why the word *complementary* is appropriate.

58. Suppose that $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ are demand functions for commodities with prices p_1 and p_2, respectively. If $\frac{\partial D_1}{\partial p_2}$ and $\frac{\partial D_2}{\partial p_1}$ are both positive, explain why the commodities are called **substitute commodities.**

59. Suppose that the output of a factory is given by $P = 20K^{1/3}L^{1/2}$, where K is the capital investment in thousands of dollars and L is the labor force in thousands of workers. If $K = 125$ and $L = 900$, use a partial derivative to estimate the effect of adding a thousand workers.

60. Suppose that the output of a factory is given by $P = 80K^{1/4}L^{3/4}$, where K is the capital investment in thousands of dollars and L is the labor force in thousands of workers. If $K = 256$ and $L = 10,000$, use a partial derivative to estimate the effect of increasing capital by one thousand dollars.

61. For the function

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

use the limit definitions of partial derivatives to show that $f_{xy}(0, 0) = -1$ but $f_{yx}(0, 0) = 1$. Determine which assumption in Theorem 3.1 is not true.

62. For $f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$, show that $\frac{\partial f}{\partial x}(0, 0) = \frac{\partial f}{\partial y}(0, 0) = 0$. [Note that we have previously shown that this function is not continuous at $(0, 0)$.]

63. Sometimes the order of differentiation makes a practical difference. For $f(x, y) = \frac{1}{x}\sin(xy^2)$, show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ but that the ease of calculations is not the same.

64. For a rectangle of length L and perimeter P, show that the area is given by $A = \frac{1}{2}LP - L^2$. Compute $\frac{\partial A}{\partial L}$. A simpler formula for area is $A = LW$, where W is the width of the rectangle. Compute $\frac{\partial A}{\partial L}$ and show that your answer is not equivalent to the previous derivative. Explain the difference by noting that in one case the width is held constant while L changes, whereas in the other case the perimeter is held constant while L changes.

65. Suppose that $f(x, y)$ is a function with continuous second-order partial derivatives. Consider the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$. Explain how the slope of this curve at the point $x = x_0$ relates to $\frac{\partial f}{\partial x}(x_0, y_0)$. Relate the concavity of this curve at the point $x = x_0$ to $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$.

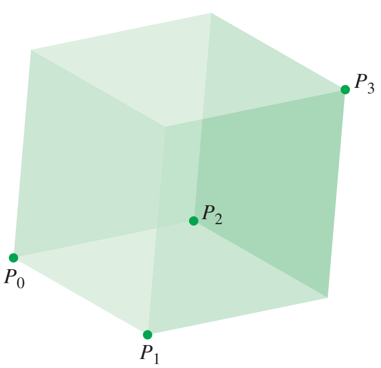
66. As in exercise 65, develop a graphical interpretation of $\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$.

EXPLORATORY EXERCISES

1. In exercises 65 and 66, you interpreted the second-order partial derivatives f_{xx} and f_{yy} in terms of concavity. In this exercise, you will develop a geometric interpretation of the mixed partial derivative f_{xy}. (More information can be found in the article "What is f_{xy}?" by Brian McCartin in the March 1998 issue of the journal *PRIMUS*.) Start by using Taylor's Theorem (see section 8.7) to show that

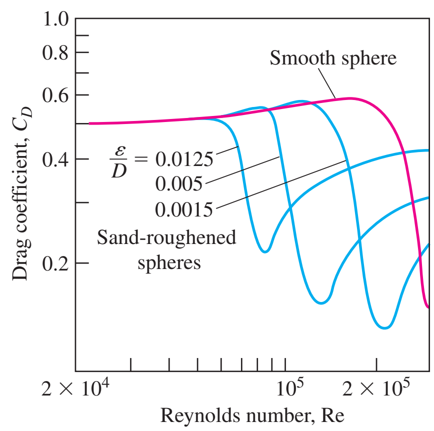
$$\lim_{k\to 0}\lim_{h\to 0}\frac{f(x,y)-f(x+h,y)-f(x,y+k)+f(x+h,y+k)}{hk}=f_{xy}(x,y).$$

[Hint: Treating y as a constant, you have $f(x+h,y)=f(x,y)+hf_x(x,y)+h^2g(x,y)$, for some function $g(x,y)$. Similarly, expand the other terms in the numerator.] Therefore, for small h and k, $f_{xy}(x,y)\approx\dfrac{f_0-f_1-f_2+f_3}{hk}$, where $f_0=f(x,y)$, $f_1=f(x+h,y)$, $f_2=f(x,y+k)$ and $f_3=f(x+h,y+k)$. The four points $P_0=(x,y,f_0)$, $P_1=(x+h,y,f_1)$, $P_2=(x,y+k,f_2)$ and $P_3=(x+h,y+k,f_3)$ determine a parallelepiped, as shown in the figure below.



Recalling that the volume of a parallelepiped formed by vectors **a**, **b** and **c** is given by $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$, show that the volume of this box equals $|(f_0-f_1-f_2+f_3)hk|$. That is, the volume is approximately equal to $|f_{xy}(x,y)|(hk)^2$. Conclude that the larger $|f_{xy}(x,y)|$ is, the greater the volume of the box and hence, the farther the point P_3 is from the plane determined by the points P_0, P_1 and P_2. To see what this means graphically, start with the function $f(x,y)=x^2+y^2$ at the point (1, 1, 2). With $h=k=0.1$, show that the points (1, 1, 2), (1.1, 1, 2.21), (1, 1.1, 2.21) and (1.1, 1.1, 2.42) all lie in the same plane. The derivative $f_{xy}(1,1)=0$ indicates that at the point (1.1, 1.1, 2.42), the graph does not curve away from the plane of the points (1, 1, 2), (1.1, 1, 2.21) and (1, 1.1, 2.21). Contrast this to the behavior of the function $f(x,y)=x^2+xy$ at the point (1, 1, 2). This says that f_{xy} measures the amount of curving of the surface as you sequentially change x and y by small amounts.

2. A ball, such as a baseball, flying through the air encounters air resistance in the form of **air drag.** The magnitude of the drag force is typically the product of a number (called the drag coefficient) and the square of the velocity. The drag coefficient is not actually a constant. The figure (reprinted from *Keep Your Eye on the Ball* by Watts and Bahill) shows experimental data for the drag coefficient as a function of the roughness of the ball (measured by ε/D, where ε is the size of the bumps on the ball and D is the diameter of the ball) and the Reynolds number (Re, which is proportional to velocity). We'll call the drag coefficient f, rename $u=\varepsilon/D$ and $v=\text{Re}$ and consider $f(u,v)$. Use the graph to estimate $\dfrac{\partial f}{\partial u}(0.005, 1.5\times10^5)$ and $\dfrac{\partial f}{\partial v}(0.005, 1.5\times10^5)$ and interpret each partial derivative. All golf balls have "dimples" that make the surface of the golf ball rougher. Explain why a golf ball with dimples, traveling at a velocity corresponding to a Reynolds number of about 0.9×10^5, will fly much farther than a ball with no dimples.



3. For a function $g(x,y)$, define $F(x)=\int_a^b g(x,y)\,dy$. In this exercise, you will explore the question of whether or not $F'(x)=\int_a^b\frac{\partial g}{\partial x}(x,y)\,dy$. (a) Show that this is true for $g(x,y)=e^{xy}$. (b) Show that it is true for $g(x,y)=h(x)k(y)$ if k is continuous and h is differentiable. (c) Show that it is true for $g(x,y)=\frac{1}{x}e^{xy}$ on the interval [0, 2]. (d) Find numerically that it is not true for $g(x,y)=\frac{1}{y}e^{xy}$. (e) Conjecture conditions on the function $g(x,y)$ for which the statement is true. (f) A mathematician would say that the underlying issue in this problem is the interchangeability of limits and integrals. Explain how limits are involved.

12.4 TANGENT PLANES AND LINEAR APPROXIMATIONS

Recall that the tangent line to the curve $y = f(x)$ at $x = a$ stays close to the curve near the point of tangency. This enables us to use the tangent line to approximate values of the function close to the point of tangency (see Figure 12.24a). The equation of the tangent line is given by

$$y = f(a) + f'(a)(x - a) . \tag{4.1}$$

In section 3.1, we called this the *linear approximation* to $f(x)$ at $x = a$.

In much the same way, we can approximate the value of a function of two variables near a given point using the tangent *plane* to the surface at that point. For instance, the graph of $z = 6 - x^2 - y^2$ and its tangent plane at the point (1, 2, 1) are shown in Figure 12.24b. Notice that near the point (1, 2, 1), the surface and the tangent plane are very close together.

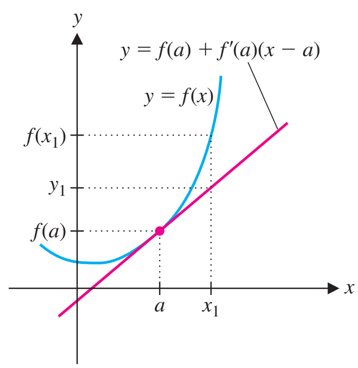


FIGURE 12.24a
Linear approximation

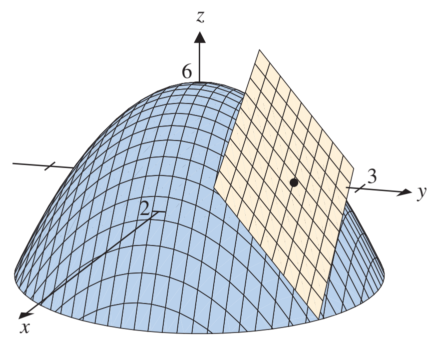


FIGURE 12.24b
$z = 6 - x^2 - y^2$ and the tangent plane at (1, 2, 1)

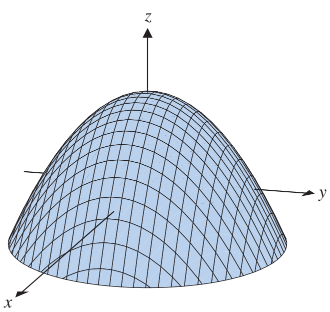


FIGURE 12.25a
$z = 6 - x^2 - y^2$, with $-3 \le x \le 3$ and $-3 \le y \le 3$

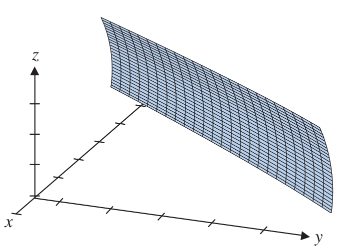


FIGURE 12.25b
$z = 6 - x^2 - y^2$, with $0.9 \le x \le 1.1$ and $1.9 \le y \le 2.1$

Refer to Figures 12.25a and 12.25b to visualize the process. Starting from a standard graphing window (Figure 12.25a shows $z = 6 - x^2 - y^2$ with $-3 \le x \le 3$ and $-3 \le y \le 3$), zoom in on the point (1, 2, 1), as in Figure 12.25b (showing $z = 6 - x^2 - y^2$ with $0.9 \le x \le 1.1$ and $1.9 \le y \le 2.1$). The surface in Figure 12.25b looks like a plane, since we have zoomed in sufficiently far that the surface and its tangent plane are difficult to distinguish visually. This suggests that for points (x, y) close to the point of tangency, we can use the corresponding z-value on the tangent plane as an approximation to the value of the function at that point. More generally, we begin by looking for an equation of the tangent plane to $z = f(x, y)$ at the point $(a, b, f(a, b))$, where f_x and f_y are continuous at (a, b). For this, we'll need a point in the plane and a vector normal to the plane. One point lying in the tangent plane is, of course, the point of tangency $(a, b, f(a, b))$. To find a normal vector, we will find two vectors lying in the plane and then take their cross product to find a vector orthogonal to both (and thus, orthogonal to the plane).

Imagine intersecting the surface $z = f(x, y)$ with the plane $y = b$, as shown in Figure 12.26a. As we observed in section 12.3, the result is a curve in the plane $y = b$ whose slope at $x = a$ is given by $f_x(a, b)$. Along the tangent line at $x = a$, a change of 1 unit in x corresponds to a change of $f_x(a, b)$ in z. Since we're looking at a curve that lies in the plane $y = b$, the value of y doesn't change at all along the curve. A vector with the same direction as the tangent line is then $\langle 1, 0, f_x(a, b)\rangle$. This vector must then be parallel to

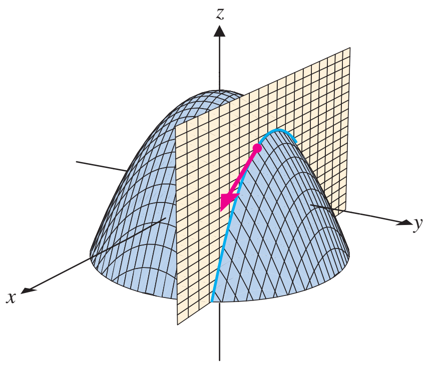


FIGURE 12.26a
The intersection of the surface $z = f(x, y)$ with the plane $y = b$

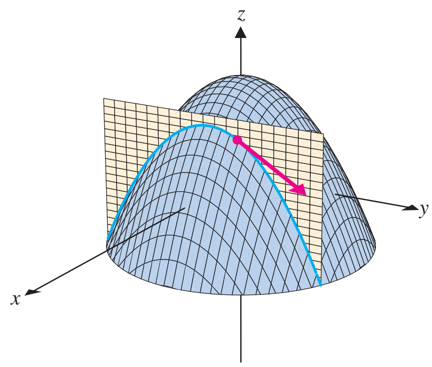


FIGURE 12.26b
The intersection of the surface $z = f(x, y)$ with the plane $x = a$

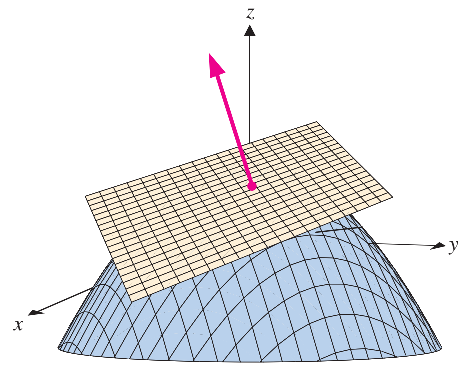


FIGURE 12.26c
Tangent plane and normal vector

the tangent plane. (Think about this some.) Similarly, intersecting the surface $z = f(x, y)$ with the plane $x = a$, as shown in Figure 12.26b, we get a curve lying in the plane $x = a$, whose slope at $y = b$ is given by $f_y(a, b)$. A vector with the same direction as the tangent line at $y = b$ is then $\langle 0, 1, f_y(a, b)\rangle$.

We have now found two vectors that are parallel to the tangent plane: $\langle 1, 0, f_x(a, b)\rangle$ and $\langle 0, 1, f_y(a, b)\rangle$. A vector normal to the plane is then given by the cross product:

$$\langle 0, 1, f_y(a, b)\rangle \times \langle 1, 0, f_x(a, b)\rangle = \langle f_x(a, b), f_y(a, b), -1\rangle.$$

We indicate the tangent plane and normal vector at a point in Figure 12.26c. We have the following result.

REMARK 4.1

Notice the similarity between the equation of the tangent plane given in (4.2) and the equation of the tangent line to $y = f(x)$ given in (4.1).

THEOREM 4.1

Suppose that $f(x, y)$ has continuous first partial derivatives at (a, b). A normal vector to the tangent plane to $z = f(x, y)$ at (a, b) is then $\langle f_x(a, b), f_y(a, b), -1\rangle$. Further, an equation of the tangent plane is given by

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \tag{4.2}$$

Observe that since we now know a normal vector to the tangent plane, the line orthogonal to the tangent plane and passing through the point $(a, b, f(a, b))$ is given by

$$x = a + f_x(a, b)t, \quad y = b + f_y(a, b)t, \quad z = f(a, b) - t. \tag{4.3}$$

This line is called the **normal line** to the surface at the point $(a, b, f(a, b))$.

It's now a simple matter to use Theorem 4.1 to construct the equations of a tangent plane and normal line to nearly any surface, as we illustrate in examples 4.1 and 4.2.

EXAMPLE 4.1 Finding Equations of the Tangent Plane and the Normal Line

Find equations of the tangent plane and the normal line to $z = 6 - x^2 - y^2$ at the point $(1, 2, 1)$.

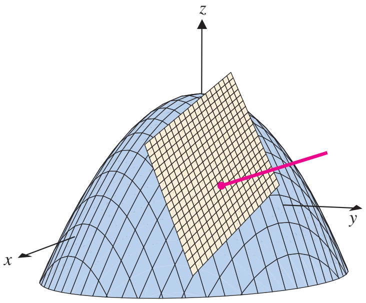


FIGURE 12.27
Surface, tangent plane and normal line at the point (1, 2, 1)

Solution For $f(x, y) = 6 - x^2 - y^2$, we have $f_x = -2x$ and $f_y = -2y$. This gives us $f_x(1, 2) = -2$ and $f_y(1, 2) = -4$. A normal vector is then $\langle -2, -4, -1 \rangle$ and from (4.2), an equation of the tangent plane is

$$z = 1 - 2(x - 1) - 4(y - 2).$$

From (4.3), equations of the normal line are

$$x = 1 - 2t, \quad y = 2 - 4t, \quad z = 1 - t.$$

A sketch of the surface, the tangent plane and the normal line is shown in Figure 12.27. ■

EXAMPLE 4.2 Finding Equations of the Tangent Plane and the Normal Line

Find equations of the tangent plane and the normal line to $z = x^3 + y^3 + \dfrac{x^2}{y}$ at (2, 1, 13).

Solution Here, $f_x = 3x^2 + \dfrac{2x}{y}$ and $f_y = 3y^2 - \dfrac{x^2}{y^2}$, so that $f_x(2, 1) = 12 + 4 = 16$ and $f_y(2, 1) = 3 - 4 = -1$. A normal vector is then $\langle 16, -1, -1 \rangle$ and from (4.2), an equation of the tangent plane is

$$z = 13 + 16(x - 2) - (y - 1).$$

From (4.3), equations of the normal line are

$$x = 2 + 16t, \quad y = 1 - t, \quad z = 13 - t.$$

A sketch of the surface, the tangent plane and the normal line is shown in Figure 12.28.

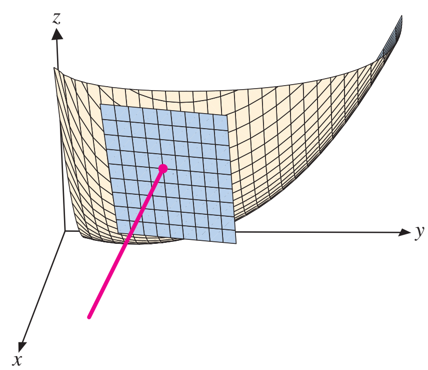


FIGURE 12.28
Surface, tangent plane and normal line at the point (2, 1, 13)

■

In Figures 12.27 and 12.28, the tangent plane appears to stay close to the surface near the point of tangency. This says that the z-values on the tangent plane should be close to the corresponding z-values on the surface, which are given by the function values $f(x, y)$, at least for (x, y) close to the point of tangency. Further, the simple form of the equation for the tangent plane makes it ideal for approximating the value of complicated functions.

We define the **linear approximation** $L(x, y)$ of $f(x, y)$ at the point (a, b) to be the function defining the z-values on the tangent plane, namely,

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b), \qquad (4.4)$$

from (4.2). We illustrate this with example 4.3.

EXAMPLE 4.3 Finding a Linear Approximation

Compute the linear approximation of $f(x, y) = 2x + e^{x^2-y}$ at (0, 0). Compare the linear approximation to the actual function values for (a) $x = 0$ and y near 0; (b) $y = 0$ and x near 0; (c) $y = x$, with both x and y near 0 and (d) $y = 2x$, with both x and y near 0.

Solution Here, $f_x = 2 + 2xe^{x^2-y}$ and $f_y = -e^{x^2-y}$, so that $f_x(0, 0) = 2$ and $f_y(0, 0) = -1$. Also, $f(0, 0) = 1$. From (4.4), the linear approximation is then given by

$$L(x, y) = 1 + 2(x - 0) - (y - 0) = 1 + 2x - y.$$

The following table compares values of $L(x, y)$ and $f(x, y)$ for a number of points of the form $(0, y)$, $(x, 0)$, (x, x) and $(x, 2x)$.

(x, y)	$f(x, y)$	$L(x, y)$
(0, 0.1)	0.905	0.9
(0, 0.01)	0.99005	0.99
(0, −0.1)	1.105	1.1
(0, −0.01)	1.01005	1.01
(0.1, 0)	1.21005	1.2
(0.01, 0)	1.02010	1.02
(−0.1, 0)	0.81005	0.8
(−0.01, 0)	0.98010	0.98

(x, y)	$f(x, y)$	$L(x, y)$
(0.1, 0.1)	1.11393	1.1
(0.01, 0.01)	1.01015	1.01
(−0.1, −0.1)	0.91628	0.9
(−0.01, −0.01)	0.99015	0.99
(0.1, 0.2)	1.02696	1.0
(0.01, 0.02)	1.00030	1.0
(−0.1, −0.2)	1.03368	1.0
(−0.01, −0.02)	1.00030	1.0

Notice that the closer a given point is to the point of tangency, the more accurate the linear approximation tends to be at that point. This is typical of this type of approximation. We will explore this further in the exercises. ■

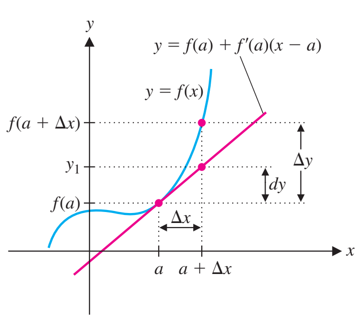


FIGURE 12.29
Increments and differentials for a function of one variable

Increments and Differentials

Now that we have examined linear approximations from a graphical perspective, we will examine them in a symbolic fashion. First, we remind you of the notation and some alternative language that we used in section 3.1 for functions of a single variable. We defined the *increment* Δy of the function $f(x)$ at $x = a$ to be

$$\Delta y = f(a + \Delta x) - f(a).$$

Referring to Figure 12.29, notice that for Δx small,

$$\Delta y \approx dy = f'(a)\,\Delta x,$$

where we referred to dy as the *differential* of y. Further, observe that if f is differentiable at $x = a$ and $\varepsilon = \dfrac{\Delta y - dy}{\Delta x}$, then we have

$$\varepsilon = \frac{\Delta y - dy}{\Delta x} = \frac{f(a+\Delta x) - f(a) - f'(a)\,\Delta x}{\Delta x}$$
$$= \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \to 0,$$

as $\Delta x \to 0$. (You'll need to recognize the definition of derivative here!) Finally, solving for Δy in terms of ε, we have

$$\Delta y = dy + \varepsilon\,\Delta x,$$

where $\varepsilon \to 0$, as $\Delta x \to 0$. We can make a similar observation for functions of several variables, as follows.

For $z = f(x, y)$, we define the **increment** of f at (a, b) to be

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a, b).$$

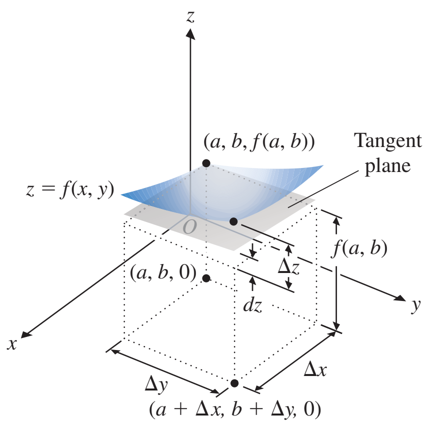


FIGURE 12.30
Linear approximation

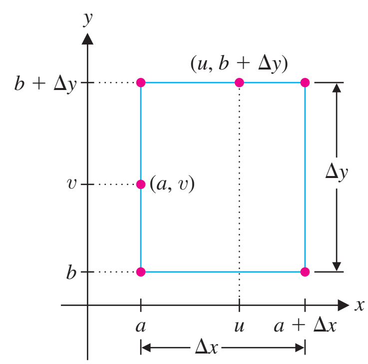


FIGURE 12.31
Intermediate points from the Mean Value Theorem

That is, Δz is the change in z that occurs when a is incremented by Δx and b is incremented by Δy, as illustrated in Figure 12.30. Notice that as long as f is continuous in some open region containing (a, b) and f has first partial derivatives on that region, we can write

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a, b)$$
$$= [f(a+\Delta x, b+\Delta y) - f(a, b+\Delta y)] + [f(a, b+\Delta y) - f(a, b)]$$

Adding and subtracting $f(a, b+\Delta y)$.

$$= f_x(u, b+\Delta y)[(a+\Delta x) - a] + f_y(a, v)[(b+\Delta y) - b]$$

Applying the Mean Value Theorem to both terms.

$$= f_x(u, b+\Delta y)\,\Delta x + f_y(a, v)\,\Delta y,$$

by the Mean Value Theorem. Here, u is some value between a and $a + \Delta x$, and v is some value between b and $b + \Delta y$ (see Figure 12.31). This gives us

$$\Delta z = f_x(u, b+\Delta y)\,\Delta x + f_y(a, v)\,\Delta y$$
$$= \{f_x(a, b) + [f_x(u, b+\Delta y) - f_x(a, b)]\}\,\Delta x$$
$$+ \{f_y(a, b) + [f_y(a, v) - f_y(a, b)]\}\,\Delta y,$$

which we rewrite as

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2\,\Delta y,$$

where $\qquad \varepsilon_1 = f_x(u, b + \Delta y) - f_x(a, b) \quad$ and $\quad \varepsilon_2 = f_y(a, v) - f_y(a, b).$

Finally, observe that if f_x and f_y are both continuous in some open region containing (a, b), then ε_1 and ε_2 will both tend to 0, as $(\Delta x, \Delta y) \to (0, 0)$. In fact, you should recognize that since $\varepsilon_1, \varepsilon_2 \to 0$, as $(\Delta x, \Delta y) \to (0, 0)$, the products $\varepsilon_1\,\Delta x$ and $\varepsilon_2\,\Delta y$ both tend to 0 even faster than do ε_1, ε_2, Δx or Δy individually. (Think about this!)

We have now established the following result.

THEOREM 4.2

Suppose that $z = f(x, y)$ is defined on the rectangular region $R = \{(x, y) | x_0 < x < x_1, y_0 < y < y_1\}$ and f_x and f_y are defined on R and are continuous at $(a, b) \in R$. Then for $(a + \Delta x, b + \Delta y) \in R$,

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y, \tag{4.5}$$

where ε_1 and ε_2 are functions of Δx and Δy that both tend to zero, as $(\Delta x, \Delta y) \to (0, 0)$.

For some very simple functions, we can compute Δz by hand, as illustrated in example 4.4.

EXAMPLE 4.4 Computing the Increment Δz

For $z = f(x, y) = x^2 - 5xy$, find Δz and write it in the form indicated in Theorem 4.2.

Solution We have

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [(x + \Delta x)^2 - 5(x + \Delta x)(y + \Delta y)] - (x^2 - 5xy)\\ &= x^2 + 2x\,\Delta x + (\Delta x)^2 - 5(xy + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y) - x^2 + 5xy\\ &= \underbrace{(2x - 5y)}_{f_x}\,\Delta x + \underbrace{(-5x)}_{f_y}\,\Delta y + \underbrace{(\Delta x)}_{\varepsilon_1}\,\Delta x + \underbrace{(-5\Delta x)}_{\varepsilon_2}\,\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,\end{aligned}$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = -5\Delta x$ both tend to zero, as $(\Delta x, \Delta y) \to (0, 0)$, as indicated in Theorem 4.2. You should observe here that by grouping the terms differently, we would get different choices for ε_1 and ε_2. ■

Look closely at the first two terms in the expansion of the increment Δz given in (4.5). If we take $\Delta x = x - a$ and $\Delta y = y - b$, then they correspond to the linear approximation of $f(x, y)$. In this context, we give this a special name. If we increment x by the amount $dx = \Delta x$ and increment y by $dy = \Delta y$, then we define the **differential** of z to be

$$\boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy.}$$

This is sometimes referred to as a **total differential.** Notice that for dx and dy small, we have from (4.5) that

$$\Delta z \approx dz.$$

You should recognize that this is the same approximation as the linear approximation developed in the beginning of this section. In this case, though, we have developed this from an analytical perspective, rather than the geometrical one used in the beginning of the section.

In definition 4.1, we give a special name to functions that can be approximated linearly in the above fashion.

DEFINITION 4.1

Let $z = f(x, y)$. We say that f is **differentiable** at (a, b) if we can write

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y,$$

where ε_1 and ε_2 are both functions of Δx and Δy and $\varepsilon_1, \varepsilon_2 \to 0$, as $(\Delta x, \Delta y) \to (0, 0)$. We say that f is differentiable on a region $R \subset \mathbb{R}^2$ whenever f is differentiable at every point in R.

Although this definition of a differentiable function may not appear to be an obvious generalization of the corresponding definition for a function of a single variable, in fact, it is. We explore this in the exercises.

Note that from Theorem 4.2, if f_x and f_y are defined on some open rectangle R containing the point (a, b) and if f_x and f_y are continuous at (a, b), then f will be differentiable at (a, b). Just as with functions of a single variable, it can be shown that if f is differentiable at a point (a, b), then it is also continuous at (a, b). Further, owing to Theorem 4.2, if a function is differentiable at a point, then the linear approximation (differential) at that point provides a good approximation to the function near that point. Be very careful of what this does *not* say, however. If a function has partial derivatives at a point, it need *not* be differentiable or even continuous at that point. (In exercises 31 and 32, you will see examples of a function with partial derivatives defined everywhere, but that is not continuous at a point.)

The idea of a linear approximation extends easily to three or more dimensions. We lose the graphical interpretation of a tangent plane approximating a surface, but the definition should make sense.

DEFINITION 4.2

The **linear approximation** to $f(x, y, z)$ at the point (a, b, c) is given by

$$\begin{aligned} L(x, y, z) = {} & f(a, b, c) + f_x(a, b, c)(x - a) \\ & + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c). \end{aligned}$$

We can write the linear approximation in the context of increments and differentials, as follows. If we increment x by Δx, y by Δy and z by Δz, then the increment of $w = f(x, y, z)$

is given by

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$
$$\approx dw = f_x(x, y, z)\,\Delta x + f_y(x, y, z)\,\Delta y + f_z(x, y, z)\,\Delta z.$$

A good way to interpret (and remember!) the linear approximation is that each partial derivative represents the change in the function relative to the change in that variable. The linear approximation starts with the function value at the known point and adds in the approximate changes corresponding to each of the independent variables.

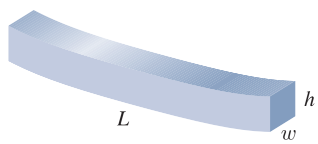


FIGURE 12.32
A typical beam

EXAMPLE 4.5 Approximating the Sag in a Beam

Suppose that the sag in a beam of length L, width w and height h is given by $S(L, w, h) = 0.0004\dfrac{L^4}{wh^3}$, with all lengths measured in inches. We illustrate the beam in Figure 12.32. A beam is supposed to measure $L = 36$, $w = 2$ and $h = 6$ with a corresponding sag of 1.5552 inches. Due to weathering and other factors, the manufacturer only guarantees measurements with error tolerances $L = 36 \pm 1$, $w = 2 \pm 0.4$ and $h = 6 \pm 0.8$. Use a linear approximation to estimate the possible range of sags in the beam.

Solution We first compute $\dfrac{\partial S}{\partial L} = 0.0016\dfrac{L^3}{wh^3}$, $\dfrac{\partial S}{\partial w} = -0.0004\dfrac{L^4}{w^2h^3}$ and $\dfrac{\partial S}{\partial h} = -0.0012\dfrac{L^4}{wh^4}$. At the point (36, 2, 6), we then have $\dfrac{\partial S}{\partial L}(36, 2, 6) = 0.1728$, $\dfrac{\partial S}{\partial w}(36, 2, 6) = -0.7776$ and $\dfrac{\partial S}{\partial h}(36, 2, 6) = -0.7776$. From Definition 4.2, the linear approximation of the sag is then given by

$$S \approx 1.5552 + 0.1728(L - 36) - 0.7776(w - 2) - 0.7776(h - 6).$$

Notice that we could have written this in differential form using Definition 4.1.

From the stated tolerances, $L - 36$ must be between -1 and 1, $w - 2$ must be between -0.4 and 0.4 and $h - 6$ must be between -0.8 and 0.8. Notice that the maximum sag then occurs with $L - 36 = 1$, $w - 2 = -0.4$ and $h - 6 = -0.8$. The linear approximation predicts that

$$S - 1.5552 \approx 0.1728 + 0.31104 + 0.62208 = 1.10592.$$

Similarly, the minimum sag occurs with $L - 36 = -1$, $w - 2 = 0.4$ and $h - 6 = 0.8$. The linear approximation predicts that

$$S - 1.5552 \approx -0.1728 - 0.31104 - 0.62208 = -1.10592.$$

Based on the linear approximation, the sag is 1.5552 ± 1.10592, or between 0.44928 and 2.66112. As you can see, in this case, the uncertainty in the sag is substantial. ■

In many real-world situations, we do not have a formula for the quantity we are interested in computing. Even so, given sufficient information, we can still use linear approximations to estimate the desired quantity.

EXAMPLE 4.6 Estimating the Gauge of a Sheet of Metal

Manufacturing plants create rolls of metal of a desired gauge (thickness) by feeding the metal through very large rollers. The thickness of the resulting metal depends on the gap between the working rollers, the speed at which the rollers turn and the temperature of the metal. Suppose that for a certain metal, a gauge of 4 mm is produced by a gap of 4 mm, a speed of 10 m/s and a temperature of 900°. Experiments show that an increase in speed of 0.2 m/s increases the gauge by 0.06 mm and an increase in temperature of 10° decreases the gauge by 0.04 mm. Use a linear approximation to estimate the gauge at 10.1 m/s and 880°.

Solution With no change in gap, we assume that the gauge is a function $g(s, t)$ of the speed s and the temperature t. Based on our data, $\frac{\partial g}{\partial s} \approx \frac{0.06}{0.2} = 0.3$ and $\frac{\partial g}{\partial t} \approx \frac{-0.04}{10} = -0.004$. From Definition 4.2, the linear approximation of $g(s, t)$ is given by

$$g(s, t) \approx 4 + 0.3(s - 10) - 0.004(t - 900).$$

With $s = 10.1$ and $t = 880$, we get the estimate

$$g(10.1, 880) \approx 4 + 0.3(0.1) - 0.004(-20) = 4.11. \blacksquare$$

BEYOND FORMULAS

You should think of linear approximations more in terms of example 4.6 than example 4.5. That is, linear approximations are most commonly used when there is no known formula for the function *f*. You can then read equation (4.4) or Definition 4.2 as a recipe that tells you which ingredients (i.e., function values and derivatives) you need to approximate a function value. The visual image behind this formula, shown in Figure 12.30, gives you information about how good your approximation is.

EXERCISES 12.4

WRITING EXERCISES

1. Describe which graphical properties of the surface $z = f(x, y)$ would cause the linear approximation of $f(x, y)$ at (a, b) to be particularly accurate or inaccurate.

2. Temperature varies with longitude (x), latitude (y) and altitude (z). Speculate whether or not the temperature function would be differentiable and what significance the answer would have for weather prediction.

3. Imagine a surface $z = f(x, y)$ with a ridge of discontinuities along the line $y = x$. Explain in graphical terms why $f(x, y)$ would not be differentiable at (0, 0) or any other point on the line $y = x$.

4. The function in exercise 3 might have first partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$. Explain why the slopes along $x = 0$ and $y = 0$ could have limits as x and y approach 0. If *differentiable* is intended to describe functions with smooth graphs, explain why differentiability is not defined in terms of the existence of partial derivatives.

In exercises 1–6, find equations of the tangent plane and normal line to the surface at the given point.

1. $z = x^2 + y^2 - 1$ at (a) (2, 1, 4) and (b) (0, 2, 3)
2. $z = e^{-x^2-y^2}$ at (a) (0, 0, 1) and (b) $(1, 1, e^{-2})$
3. $z = \sin x \cos y$ at (a) $(0, \pi, 0)$ and (b) $(\frac{\pi}{2}, \pi, -1)$
4. $z = x^3 - 2xy$ at (a) (−2, 3, 4) and (b) (1, −1, 3)
5. $z = \sqrt{x^2 + y^2}$ at (a) (−3, 4, 5) and (b) (8, −6, 10)
6. $z = \frac{4x}{y}$ at (a) (1, 2, 2) and (b) (−1, 4, −1)

In exercises 7–12, compute the linear approximation of the function at the given point.

7. $f(x, y) = \sqrt{x^2 + y^2}$ at (a) (3, 0) and (b) (0, −3)
8. $f(x, y) = \sin x \cos y$ at (a) $(0, \pi)$ and (b) $(\frac{\pi}{2}, \pi)$
9. $f(x, y) = xe^{xy^2} + 3y^2$ at (a) (0, 1) and (b) (2, 0)

10. $f(x, y, z) = xe^{yz} - \sqrt{x - y^2}$ at (a) (4, 1, 0) and (b) (1, 0, 2)

11. $f(w, x, y, z) = w^2xy - e^{wyz}$ at (a) $(-2, 3, 1, 0)$ and (b) $(0, 1, -1, 2)$

12. $f(w, x, y, z) = \cos xyz - w^3x^2$ at (a) $(2, -1, 4, 0)$ and (b) $(2, 1, 0, 1)$

In exercises 13–16, compare the linear approximation from the indicated exercise to the exact function value at the given points.

13. Exercise 7 part (a) at $(3, -0.1)$, $(3.1, 0)$, $(3.1, -0.1)$

14. Exercise 7 part (b) at $(0.1, -3)$, $(0, -3.1)$, $(0.1, -3.1)$

15. Exercise 8 part (a) at $(0, 3)$, $(0.1, \pi)$, $(0.1, 3)$

16. Exercise 9 part (b) at $(2.1, 0)$, $(2, 0.2)$, $(1, -1)$

17. Use a linear approximation to estimate the range of sags in the beam of example 4.5 if the error tolerances are $L = 36 \pm 0.5$, $w = 2 \pm 0.2$ and $h = 6 \pm 0.5$.

18. Use a linear approximation to estimate the range of sags in the beam of example 4.5 if the error tolerances are $L = 32 \pm 0.4$, $w = 2 \pm 0.3$ and $h = 8 \pm 0.4$.

19. Use a linear approximation to estimate the gauge of the metal in example 4.6 at 9.9 m/s and 930°.

20. Use a linear approximation to estimate the gauge of the metal in example 4.6 at 10.2 m/s and 910°.

21. Suppose that for a metal similar to that of example 4.6, an increase in speed of 0.3 m/s increases the gauge by 0.03 mm and an increase in temperature of 20° decreases the gauge by 0.02 mm. Use a linear approximation to estimate the gauge at 10.2 m/s and 890°.

22. Suppose that for the metal in example 4.6, a decrease of 0.05 mm in the gap between the working rolls decreases the gauge by 0.04 mm. Use a linear approximation in three variables to estimate the gauge at 10.15 m/s, 905° and a gap of 3.98 mm.

In exercises 23–26, find the increment Δz and write it in the form given in Theorem 4.2.

23. $f(x, y) = 2xy + y^2$

24. $f(x, y) = (x + y)^2$

25. $f(x, y) = x^2 + y^2$

26. $f(x, y) = x^3 - 3xy$

27. Determine whether or not $f(x, y) = x^2 + 3xy$ is differentiable.

28. Determine whether or not $f(x, y) = xy^2$ is differentiable.

In exercises 29 and 30, find the total differential of $f(x, y)$.

29. $f(x, y) = ye^x + \sin x$

30. $f(x, y) = \sqrt{x + y}$

In exercises 31 and 32, show that the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but the function $f(x, y)$ is not differentiable at (0, 0).

31. $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

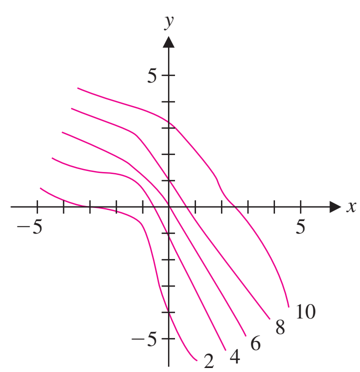
32. $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

33. In this exercise, we visualize the linear approximation of example 4.3. Start with a contour plot of $f(x, y) = 2x + e^{x^2 - y}$ with $-1 \le x \le 1$ and $-1 \le y \le 1$. Then zoom in on the point (0, 0) of the contour plot until the level curves appear straight and equally spaced. (Level curves for z-values between 0.9 and 1.1 with a graphing window of $-0.1 \le x \le 0.1$ and $-0.1 \le y \le 0.1$ should work.) You will need the z-values for the level curves. Notice that to move from the $z = 1$ level curve to the $z = 1.05$ level curve you move 0.025 unit to the right. Then $\dfrac{\partial f}{\partial x}(0, 0) \approx \dfrac{\Delta z}{\Delta x} = \dfrac{0.05}{0.025} = 2$. Verify graphically that $\dfrac{\partial f}{\partial y}(0, 0) \approx -1$. Explain how to use the contour plot to reproduce the linear approximation $1 + 2x - y$.

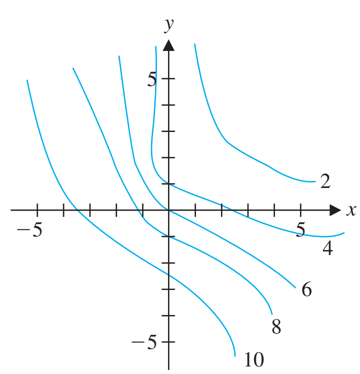
34. Use the graphical method of exercise 33 to find the linear approximation of $f(x, y) = \sin(x^2 + 2xy)$ at the point (1, 3).

In exercises 35–38, use the given contour plot to estimate the linear approximation of $f(x, y)$ at (0, 0).

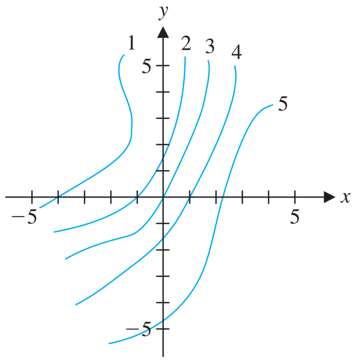
35.



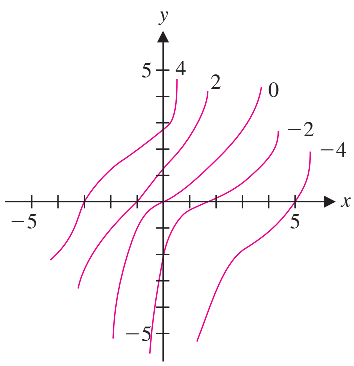
36.



37.



38.



39. The table here gives wind chill (how cold it "feels" outside) as a function of temperature (degrees Fahrenheit) and wind speed (mph). We can think of this as a function $w(t, s)$. Estimate the partial derivatives $\frac{\partial w}{\partial t}(10, 10)$ and $\frac{\partial w}{\partial s}(10, 10)$ and the linear approximation of $w(t, s)$ at (10, 10). Use the linear approximation to estimate the wind chill at (12, 13).

Speed \ *Temp*	**30**	**20**	**10**	**0**	**−10**
0	30	20	10	0	−10
5	27	16	6	−5	−15
10	16	4	−9	−24	−33
15	9	−5	−18	−32	−45
20	4	−10	−25	−39	−53
25	0	−15	−29	−44	−59
30	−2	−18	−33	−48	−63

40. Estimate the linear approximation of wind chill at (10, 15) and use it to estimate the wind chill at (12, 13). Explain any differences between this answer and that of exercise 39.

41. In exercise 33, we specified that you zoom in on the contour plot until the level curves appear linear *and* equally spaced. To see why the second condition is necessary, sketch a contour plot of $f(x, y) = e^{x-y}$ with $-1 \le x \le 1$ and $-1 \le y \le 1$. Use this plot to estimate $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ and compare to the exact values. Zoom in until the level curves are equally spaced and estimate again. Explain why this estimate is much better.

42. Show that $\left\langle 0, 1, \frac{\partial f}{\partial y}(a, b)\right\rangle \times \left\langle 1, 0, \frac{\partial f}{\partial x}(a, b)\right\rangle = \left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\right\rangle$.

43. Show that $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$ is continuous but not differentiable at (0, 0).

44. Let S be a surface defined parametrically by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$. Define $\mathbf{r}_u(u, v) = \left\langle \frac{\partial x}{\partial u}(u, v), \frac{\partial y}{\partial u}(u, v), \frac{\partial z}{\partial u}(u, v)\right\rangle$ and $\mathbf{r}_v(u, v) = \left\langle \frac{\partial x}{\partial v}(u, v), \frac{\partial y}{\partial v}(u, v), \frac{\partial z}{\partial v}(u, v)\right\rangle$. Show that $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane at the point $(x(u, v), y(u, v), z(u, v))$.

In exercises 45–48, use the result of exercise 44 to find an equation of the tangent plane to the parametric surface at the indicated point.

45. S is defined by $x = 2u$, $y = v$ and $z = 4uv$; at $u = 1$ and $v = 2$.

46. S is defined by $x = 2u^2$, $y = uv$ and $z = 4uv^2$; at $u = -1$ and $v = 1$.

47. S is the cylinder $x^2 + y^2 = 1$ with $0 \le z \le 2$; at (1, 0, 1).

48. S is the cylinder $y^2 = 2x$ with $0 \le z \le 2$; at (2, 2, 1).

EXPLORATORY EXERCISES

For exercises 1 and 2, we need to use the notation of matrix algebra. First, we define the 2 × 2 matrix A to be a two-dimensional array of real numbers, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define a column vector x to be a one-dimensional array of real numbers, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and define the product of a matrix and column vector to be the (column) vector $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$. You can learn much more about matrix algebra by looking at one of the many textbooks on the subject, including those bearing the title *Linear Algebra*.

1. We can extend the linear approximation of this section to quadratic approximations. Define the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$, the **gradient vector** $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$, the column vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, the vector $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and the transpose vector

$\mathbf{x}^T = [x\ y]$. The **quadratic approximation** of $f(x, y)$ at the point (x_0, y_0) is defined by

$$Q(x, y) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T H(x_0, y_0)(\mathbf{x} - \mathbf{x}_0).$$

Find the quadratic approximation of $f(x, y) = 2x + e^{x^2-y}$ and compute $Q(x, y)$ for the points in the table of example 4.3.

2. An important application of linear approximations of functions $f(x)$ is Newton's method for finding solutions of equations of the form $f(x) = 0$. In this exercise, we extend Newton's method to functions of several variables. To be specific, suppose that $f_1(x, y)$ and $f_2(x, y)$ are functions of two variables with continuous partial derivatives. To solve the equations $f_1(x, y) = 0$ and $f_2(x, y) = 0$ simultaneously, start with a guess $x = x_0$ and $y = y_0$. The idea is to replace $f_1(x, y)$ and $f_2(x, y)$ with their linear approximations $L_1(x, y)$ and $L_2(x, y)$ and solve the (simpler) equations $L_1(x, y) = 0$ and $L_2(x, y) = 0$ simultaneously. Write out the linear approximations and show that we want

$$\frac{\partial f_1}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f_1}{\partial y}(x_0, y_0)(y - y_0) = -f_1(x_0, y_0)$$
$$\frac{\partial f_2}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f_2}{\partial y}(x_0, y_0)(y - y_0) = -f_2(x_0, y_0).$$

Recall that there are several ways (substitution and elimination are popular) to solve two linear equations in two unknowns. The simplest way symbolically is to use matrices. If we define the **Jacobian matrix**

$$J(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(\mathbf{x}_0) & \frac{\partial f_1}{\partial y}(\mathbf{x}_0) \\ \frac{\partial f_2}{\partial x}(\mathbf{x}_0) & \frac{\partial f_2}{\partial y}(\mathbf{x}_0) \end{bmatrix}$$

where $\mathbf{x}_0$ represents the point (x_0, y_0), the preceding equations can be written as $J(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = -\mathbf{f}(\mathbf{x}_0)$, which has solution $\mathbf{x} - \mathbf{x}_0 = -J^{-1}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)$ or $\mathbf{x} = \mathbf{x}_0 - J^{-1}(\mathbf{x}_0)\mathbf{f}(\mathbf{x}_0)$. Here, the matrix $J^{-1}(\mathbf{x}_0)$ is called the **inverse** of the matrix $J(\mathbf{x}_0)$ and $\mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} f_1(\mathbf{x}_0) \\ f_2(\mathbf{x}_0) \end{bmatrix}$. The inverse A^{-1} of a matrix A (when it is defined) is the matrix for which $\mathbf{a} = A\mathbf{b}$ if and only if $\mathbf{b} = A^{-1}\mathbf{a}$, for all column vectors $\mathbf{a}$ and $\mathbf{b}$. In general, Newton's method is defined by the iteration

$$\mathbf{x}_{n+1} = \mathbf{x}_n - J^{-1}(\mathbf{x}_n)\mathbf{f}(\mathbf{x}_n).$$

Use Newton's method with an initial guess of $\mathbf{x}_0 = (-1, 0.5)$ to approximate a solution of the equations $x^2 - 2y = 0$ and $x^2 y - \sin y = 0$.

12.5 THE CHAIN RULE

You already are quite familiar with the chain rule for functions of a single variable. For instance, to differentiate the function $e^{\sin(x^2)}$, we have

$$\frac{d}{dx}[e^{\sin(x^2)}] = e^{\sin(x^2)} \underbrace{\frac{d}{dx}[\sin(x^2)]}_{\text{the derivative of the } inside}$$
$$= e^{\sin(x^2)} \cos(x^2) \underbrace{\frac{d}{dx}(x^2)}_{\text{the derivative of the } inside}$$
$$= e^{\sin(x^2)} \cos(x^2)(2x).$$

The general form of the chain rule says that for differentiable functions f and g,

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \underbrace{g'(x)}_{\text{the derivative of the } inside}.$$

We now extend the chain rule to functions of several variables. This takes several slightly different forms, depending on the number of independent variables, but each is a variation of the already familiar chain rule for functions of a single variable.

For a differentiable function $f(x, y)$, where x and y are both in turn, differentiable functions of a single variable t, to find the derivative of $f(x, y)$ with respect to t, we first write $g(t) = f(x(t), y(t))$. Then, from the definition of (an ordinary) derivative, we have

$$\begin{aligned}\frac{d}{dt}[f(x(t), y(t))] = g'(t) &= \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))}{\Delta t}.\end{aligned}$$

For simplicity, we write $\Delta x = x(t + \Delta t) - x(t)$, $\Delta y = y(t + \Delta t) - y(t)$ and $\Delta z = f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))$. This gives us

$$\frac{d}{dt}[f(x(t), y(t))] = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}.$$

Since f is a differentiable function of x and y, we have (from the definition of differentiability) that

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where ε_1 and ε_2 both tend to 0, as $(\Delta x, \Delta y) \to (0, 0)$. Dividing through by Δt gives us

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$ now gives us

$$\begin{aligned}\frac{d}{dt}[f(x(t), y(t))] &= \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} \\ &= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &\quad + \lim_{\Delta t \to 0} \varepsilon_1 \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \varepsilon_2 \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}. \qquad (5.1)\end{aligned}$$

Notice that
$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

and
$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} = \frac{dy}{dt}.$$

Further, notice that since $x(t)$ and $y(t)$ are differentiable, they are also continuous and so,

$$\lim_{\Delta t \to 0} \Delta x = \lim_{\Delta t \to 0} [x(t + \Delta t) - x(t)] = 0.$$

Likewise, $\lim_{\Delta t \to 0} \Delta y = 0$, also. Consequently, since $(\Delta x, \Delta y) \to (0, 0)$, as $\Delta t \to 0$, we have

$$\lim_{\Delta t \to 0} \varepsilon_1 = \lim_{\Delta t \to 0} \varepsilon_2 = 0.$$

From (5.1), we now have

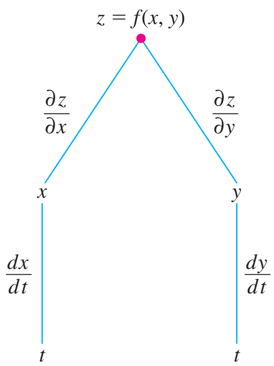
$$\begin{aligned}\frac{d}{dt}[f(x(t), y(t))] &= \frac{\partial f}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &\quad + \lim_{\Delta t \to 0} \varepsilon_1 \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \varepsilon_2 \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.\end{aligned}$$

We summarize the chain rule for the derivative of $f(x(t), y(t))$ in Theorem 5.1.

THEOREM 5.1 (Chain Rule)

If $z = f(x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable and $f(x, y)$ is a differentiable function of x and y, then

$$\frac{dz}{dt} = \frac{d}{dt}[f(x(t), y(t))] = \frac{\partial f}{\partial x}(x(t), y(t))\,\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\,\frac{dy}{dt}.$$



As a convenient device for remembering the chain rule, we sometimes use a **tree diagram** like the one shown in the margin. Notice that if $z = f(x, y)$ and x and y are both functions of the variable t, then t is the independent variable. We consider x and y to be **intermediate variables,** since they both depend on t. In the tree diagram, we list the dependent variable z at the top, followed by each of the intermediate variables x and y, with the independent variable t at the bottom level, with each of the variables connected by a path. Next to each of the paths, we indicate the corresponding derivative $\left(\text{i.e., between } z \text{ and } x, \text{ we indicate } \frac{\partial z}{\partial x}\right)$. The chain rule then gives $\frac{dz}{dt}$ as the sum of all of the products of the derivatives along each path to t. That is,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

This device is especially useful for functions of several variables that are in turn functions of several other variables, as we will see shortly.

We illustrate the use of this new chain rule in example 5.1.

EXAMPLE 5.1 Using the Chain Rule

For $z = f(x, y) = x^2e^y$, $x(t) = t^2 - 1$ and $y(t) = \sin t$, find the derivative of $g(t) = f(x(t), y(t))$.

Solution We first compute the derivatives $\frac{\partial z}{\partial x} = 2xe^y$, $\frac{\partial z}{\partial y} = x^2e^y$, $x'(t) = 2t$ and $y'(t) = \cos t$. The chain rule (Theorem 5.1) then gives us

$$\begin{aligned} g'(t) &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2xe^y(2t) + x^2e^y\cos t \\ &= 2(t^2-1)e^{\sin t}(2t) + (t^2-1)^2e^{\sin t}\cos t. \ \blacksquare \end{aligned}$$

In example 5.1, notice that you could have first substituted for x and y and then computed the derivative of $g(t) = (t^2 - 1)^2e^{\sin t}$, using the usual rules of differentiation. In example 5.2, you don't have any alternative but to use the chain rule.

EXAMPLE 5.2 A Case Where the Chain Rule Is Needed

Suppose the production of a firm is modeled by the **Cobb-Douglas production** function $P(k, l) = 20k^{1/4}l^{3/4}$, where k measures capital (in millions of dollars) and l measures the labor force (in thousands of workers). Suppose that when $l = 2$ and $k = 6$, the labor force is decreasing at the rate of 20 workers per year and capital is growing at the rate of \$400,000 per year. Determine the rate of change of production.

Solution Suppose that $g(t) = P(k(t), l(t))$. From the chain rule, we have

$$g'(t) = \frac{\partial P}{\partial k}k'(t) + \frac{\partial P}{\partial l}l'(t).$$

Notice that $\frac{\partial P}{\partial k} = 5k^{-3/4}l^{3/4}$ and $\frac{\partial P}{\partial l} = 15k^{1/4}l^{-1/4}$. With $l = 2$ and $k = 6$, this gives us $\frac{\partial P}{\partial k}(6, 2) \approx 2.1935$ and $\frac{\partial P}{\partial l}(6, 2) \approx 19.7411$. Since k is measured in millions of dollars and l is measured in thousands of workers, we have $k'(t) = 0.4$ and $l'(t) = -0.02$. From the chain rule, we now have

$$\begin{aligned} g'(t) &= \frac{\partial P}{\partial k}k'(t) + \frac{\partial P}{\partial l}l'(t) \\ &\approx 2.1935(0.4) + 19.7411(-0.02) = 0.48258. \end{aligned}$$

This indicates that the production is increasing at the rate of approximately one-half unit per year. ■

We can easily extend Theorem 5.1 to the case of a function $f(x, y)$, where x and y are both functions of the two independent variables s and t, $x = x(s, t)$ and $y = y(s, t)$. Notice that if we differentiate with respect to s, we treat t as a constant. Applying Theorem 5.1 (while holding t fixed), we have

$$\frac{\partial}{\partial s}[f(x, y)] = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

Similarly, we can find a chain rule for $\frac{\partial}{\partial t}[f(x, y)]$. This gives us the following more general form of the chain rule.

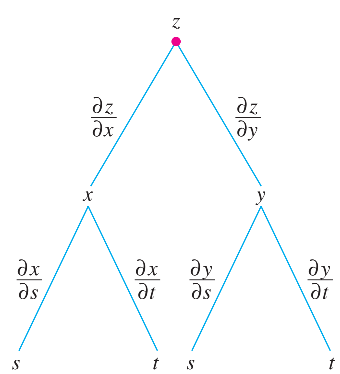
THEOREM 5.2 (Chain Rule)

Suppose that $z = f(x, y)$, where f is a differentiable function of x and y and where $x = x(s, t)$ and $y = y(s, t)$ both have first-order partial derivatives. Then we have the chain rules:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$



The tree diagram shown in the margin serves as a convenient reminder of the chain rules indicated in Theorem 5.2, again by summing the products of the indicated partial derivatives along each path from z to s or t, respectively.

The chain rule is easily extended to functions of three or more variables. You will explore this in the exercises.

EXAMPLE 5.3 Using the Chain Rule

Suppose that $f(x, y) = e^{xy}$, $x(u, v) = 3u \sin v$ and $y(u, v) = 4v^2u$. For $g(u, v) = f(x(u, v), y(u, v))$, find the partial derivatives $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$.

Solution We first compute the partial derivatives $\frac{\partial f}{\partial x} = ye^{xy}$, $\frac{\partial f}{\partial y} = xe^{xy}$, $\frac{\partial x}{\partial u} = 3 \sin v$ and $\frac{\partial y}{\partial u} = 4v^2$. The chain rule (Theorem 5.2) gives us

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = ye^{xy}(3\sin v) + xe^{xy}(4v^2).$$

Substituting for x and y, we get

$$\frac{\partial g}{\partial u} = 4v^2ue^{12u^2v^2\sin v}(3\sin v) + 3u\sin ve^{12u^2v^2\sin v}(4v^2).$$

For the partial derivative of g with respect to v, we compute $\frac{\partial x}{\partial v} = 3u\cos v$ and $\frac{\partial y}{\partial v} = 8vu$. Here, the chain rule gives us

$$\frac{\partial g}{\partial v} = ye^{xy}(3u\cos v) + xe^{xy}(8vu).$$

Substituting for x and y, we have

$$\frac{\partial g}{\partial v} = 4v^2ue^{12u^2v^2\sin v}(3u\cos v) + 3u\sin ve^{12u^2v^2\sin v}(8vu).$$ ■

Once again, it is often simpler to first substitute in the expressions for x and y. We leave it as an exercise to show that you get the same derivatives either way. On the other hand, there are plenty of times where the general forms of the chain rule seen in Theorems 5.1 and 5.2 are indispensable. You will see some of these in the exercises, while we present several important uses next.

EXAMPLE 5.4 Converting from Rectangular to Polar Coordinates

For a differentiable function $f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_r = f_x\cos\theta + f_y\sin\theta$ and $f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$.

Solution First, notice that $\frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$. From Theorem 5.2, we now have

$$f_r = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta.$$

Be very careful when computing the second partial derivative. Using the expression we have already found for f_r and Theorem 5.2, we have

$$\begin{aligned}
f_{rr} = \frac{\partial(f_r)}{\partial r} &= \frac{\partial}{\partial r}(f_x\cos\theta + f_y\sin\theta) = \frac{\partial}{\partial r}(f_x)\cos\theta + \frac{\partial}{\partial r}(f_y)\sin\theta \\
&= \left[\frac{\partial}{\partial x}(f_x)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(f_x)\frac{\partial y}{\partial r}\right]\cos\theta + \left[\frac{\partial}{\partial x}(f_y)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}(f_y)\frac{\partial y}{\partial r}\right]\sin\theta \\
&= (f_{xx}\cos\theta + f_{xy}\sin\theta)\cos\theta + (f_{yx}\cos\theta + f_{yy}\sin\theta)\sin\theta \\
&= f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta,
\end{aligned}$$

as desired. ■

In the exercises, you will use the chain rule to compute other partial derivatives in polar coordinates. One important exercise is to show that we can write (for $r \neq 0$)

$$f_{xx} + f_{yy} = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}.$$

This particular combination of second partial derivatives, $f_{xx} + f_{yy}$, is called the **Laplacian** of f and appears frequently in equations describing heat conduction and wave propagation, among others.

A slightly different use of a change of variables is demonstrated in example 5.5. An important strategy in solving some equations is to first rewrite and solve them in the most general form possible. One convenient approach to this is to convert to **dimensionless variables.** As the name implies, these are typically combinations of variables such that all the units cancel. One example would be for an object with (one-dimensional) velocity v ft/s and initial velocity $v(0) = v_0$ ft/s. The variable $V = \dfrac{v}{v_0}$ is dimensionless because the units of V are ft/s divided by ft/s, leaving no units. Often, a change to dimensionless variables will simplify an equation.

EXAMPLE 5.5 Dimensionless Variables

An object moves in two dimensions according to the equations of motion: $x''(t) = 0$, $y''(t) = -g$, with initial velocity $x'(0) = v_0 \cos\theta$ and $y'(0) = v_0 \sin\theta$ and initial position $x(0) = y(0) = 0$. Rewrite the equations and initial conditions in terms of the variables $X = \dfrac{g}{v_0^2}x$, $Y = \dfrac{g}{v_0^2}y$ and $T = \dfrac{g}{v_0}t$. Show that the variables X, Y and T are dimensionless, assuming that x and y are given in feet and t in seconds.

Solution To transform the equations, we first need to rewrite the derivatives $x'' = \dfrac{d^2x}{dt^2}$ and $y'' = \dfrac{d^2y}{dt^2}$ in terms of X, Y and T. From the chain rule, we have

$$\frac{dx}{dt} = \frac{dx}{dT}\frac{dT}{dt} = \frac{d\left(v_0^2 X/g\right)}{dT}\frac{d(gt/v_0)}{dt} = \frac{v_0^2}{g}\frac{dX}{dT}\frac{g}{v_0} = v_0\frac{dX}{dT}.$$

Again, we must be careful computing the second derivative. We have

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(v_0\frac{dX}{dT}\right) = \frac{d}{dT}\left(v_0\frac{dX}{dT}\right)\frac{dT}{dt} \\ &= v_0\frac{d^2X}{dT^2}\frac{g}{v_0} = g\frac{d^2X}{dT^2}.\end{aligned}$$

You should verify that similar calculations give $\dfrac{dy}{dt} = v_0\dfrac{dY}{dT}$ and $\dfrac{d^2y}{dt^2} = g\dfrac{d^2Y}{dT^2}$. The differential equation $x''(t) = 0$ then becomes $g\dfrac{d^2X}{dT^2} = 0$ or simply, $\dfrac{d^2X}{dT^2} = 0$. Similarly, the differential equation $y''(t) = -g$ becomes $g\dfrac{d^2Y}{dT^2} = -g$ or $\dfrac{d^2Y}{dT^2} = -1$. Further, the initial condition $x'(0) = v_0\cos\theta$ becomes $v_0\dfrac{dX}{dT}(0) = v_0\cos\theta$ or $\dfrac{dX}{dT}(0) = \cos\theta$ and the initial condition $y'(0) = v_0\sin\theta$ becomes $v_0\dfrac{dY}{dT}(0) = v_0\sin\theta$ or $\dfrac{dY}{dT}(0) = \sin\theta$. The initial value problem is now

$$\frac{d^2X}{dT^2} = 0,\ \frac{d^2Y}{dT^2} = -1,\ \frac{dX}{dT}(0) = \cos\theta,\ \frac{dY}{dT}(0) = \sin\theta,\ X(0) = 0,\ Y(0) = 0.$$

Notice that the only parameter left in the entire set of equations is θ, which is measured in radians (which is considered unitless). So, we would not need to know which unit system is being used to solve this initial value problem. Finally, to show that the variables are indeed dimensionless, we look at the units. In the English system, the initial speed v_0 has units ft/s and g has units ft/s^2 (the same as acceleration). Then $X = \frac{g}{v_0^2}x$ has units

$$\frac{\text{ft/s}^2}{(\text{ft/s})^2}(\text{ft}) = \frac{\text{ft}^2/\text{s}^2}{\text{ft}^2/\text{s}^2} = 1.$$

Similarly, Y has no units. Finally $T = \frac{g}{v_0}t$ has units

$$\frac{\text{ft/s}^2}{\text{ft/s}}(\text{s}) = \frac{\text{ft/s}}{\text{ft/s}} = 1.$$

Implicit Differentiation

Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x, say $y = f(x)$. In section 2.8, we saw how to calculate $\frac{dy}{dx}$ in such a case. We can use the chain rule for functions of several variables to obtain an alternative method for calculating this. Moreover, this will provide us with new insights into when this can be done and, more important yet, this will generalize to functions of several variables defined implicitly by an equation.

We let $z = F(x, y)$, where $x = t$ and $y = f(t)$. From Theorem 5.1, we have

$$\frac{dz}{dt} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt}.$$

But, since $z = F(x, y) = 0$, we have $\frac{dz}{dt} = 0$, too. Further, since $x = t$, we have $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = \frac{dy}{dx}$. This leaves us with

$$0 = F_x + F_y\frac{dy}{dx}.$$

Notice that we can solve this for $\frac{dy}{dx}$, provided $F_y \neq 0$. In this case, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Recognize that we already know how to calculate $\frac{dy}{dx}$ implicitly, so this doesn't appear to give us anything new. However, it turns out that the **Implicit Function Theorem** (proved in a course in advanced calculus) says that if F_x and F_y are continuous on an open disk containing the point (a, b) where $F(a, b) = 0$ and $F_y(a, b) \neq 0$, then the equation $F(x, y) = 0$ implicitly defines y as a function of x nearby the point (a, b).

More significantly, we can extend this notion to functions of several variables defined implicitly, as follows. Suppose that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$, where f is differentiable. Then, we can find the partial derivatives f_x and f_y using the chain rule, as follows. We first let $w = F(x, y, z)$. From the chain rule, we have

$$\frac{\partial w}{\partial x} = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x}.$$

Notice that since $w = F(x, y, z) = 0$, $\dfrac{\partial w}{\partial x} = 0$. Also, $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$, since x and y are independent variables. This gives us

$$0 = F_x + F_z \frac{\partial z}{\partial x}.$$

We can solve this for $\dfrac{\partial z}{\partial x}$, as long as $F_z \neq 0$, to obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}. \tag{5.2}$$

Likewise, differentiating w with respect to y leads us to

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \tag{5.3}$$

again, as long as $F_z \neq 0$. Much as in the two-variable case, the Implicit Function Theorem for functions of three variables says that if F_x, F_y and F_z are continuous inside a sphere containing the point (a, b, c) where $F(a, b, c) = 0$ and $F_z(a, b, c) \neq 0$, then the equation $F(x, y, z) = 0$ implicitly defines z as a function of x and y nearby the point (a, b, c).

EXAMPLE 5.6 Finding Partial Derivatives Implicitly

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, given that $F(x, y, z) = xy^2 + z^3 + \sin(xyz) = 0$.

Solution First, note that using the usual chain rule, we have

$$F_x = y^2 + yz\cos(xyz),$$
$$F_y = 2xy + xz\cos(xyz)$$

and

$$F_z = 3z^2 + xy\cos(xyz).$$

From (5.2), we now have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)}.$$

Likewise, from (5.3), we have

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}.$$ ■

Notice that, much like implicit differentiation with two variables, implicit differentiation with three variables yields expressions for the derivatives that depend on all three variables.

BEYOND FORMULAS

There are many more examples of the chain rule than the two written out in Theorems 5.1 and 5.2. We ask you to write out other forms of the chain rule in the exercises. All of these variations would be impossible to memorize, but all you need to do to reproduce whichever rule you need is construct the appropriate tree diagram and remember the general format, "derivative of the outside times derivative of the inside."

EXERCISES 12.5

WRITING EXERCISES

1. In example 5.1, we mentioned that direct substitution followed by differentiation was an option (see exercises 1 and 2 below) and is often preferable. Discuss the advantages and disadvantages of direct substitution versus the method of example 5.1.
2. In example 5.6, we treated z as a function of x and y. Explain how to modify our results from the Implicit Function Theorem for treating x as a function of y and z.

---

1. Repeat example 5.1 by first substituting $x = t^2 - 1$ and $y = \sin t$ and then computing $g'(t)$.
2. Repeat example 5.3 by first substituting $x = 3u \sin v$ and $y = 4v^2 u$ and then computing $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$.

In exercises 3–6, use the chain rule to find the indicated derivative(s).

3. $g'(t)$, where $g(t) = f(x(t), y(t))$, $f(x, y) = x^2 y - \sin y$, $x(t) = \sqrt{t^2 + 1}$, $y(t) = e^t$
4. $g'(t)$, where $g(t) = f(x(t), y(t))$, $f(x, y) = \sqrt{x^2 + y^2}$, $x(t) = \sin t$, $y(t) = t^2 + 2$
5. $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$, where $g(u, v) = f(x(u, v), y(u, v))$, $f(x, y) = 4x^2 y^3$, $x(u, v) = u^3 - v \sin u$, $y(u, v) = 4u^2$
6. $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$, where $g(u, v) = f(x(u, v), y(u, v))$, $f(x, y) = xy^3 - 4x^2$, $x(u, v) = e^{u^2}$, $y(u, v) = \sqrt{v^2 + 1} \sin u$

In exercises 7–10, state the chain rule for the general composite function.

7. $g(t) = f(x(t), y(t), z(t))$
8. $g(u, v) = f(x(u, v), y(u, v), z(u, v))$
9. $g(u, v, w) = f(x(u, v, w), y(u, v, w))$
10. $g(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w))$
11. In example 5.2, suppose that $l = 4$ and $k = 6$, the labor force is decreasing at the rate of 60 workers per year and capital is growing at the rate of \$100,000 per year. Determine the rate of change of production.
12. In example 5.2, suppose that $l = 3$ and $k = 4$, the labor force is increasing at the rate of 80 workers per year and capital is decreasing at the rate of \$200,000 per year. Determine the rate of change of production.
13. Suppose the production of a firm is modeled by $P(k, l) = 16k^{1/3} l^{2/3}$, with k and l defined as in example 5.2. Suppose that $l = 3$ and $k = 4$, the labor force is increasing at the rate of 80 workers per year and capital is decreasing at the rate of \$200,000 per year. Determine the rate of change of production.
14. Suppose the production of a firm is modeled by $P(k, l) = 16k^{1/3} l^{2/3}$, with k and l defined as in example 5.2. Suppose that $l = 2$ and $k = 5$, the labor force is increasing at the rate of 40 workers per year and capital is decreasing at the rate of \$100,000 per year. Determine the rate of change of production.
15. For a business product, income is the product of the quantity sold and the price, which we can write as $I = qp$. If the quantity sold increases at a rate of 5% and the price increases at a rate of 3%, show that income increases at a rate of 8%.
16. Assume that $I = qp$ as in exercise 15. If the quantity sold decreases at a rate of 3% and price increases at a rate of 5%, determine the rate of increase or decrease in income.

In exercises 17–20, use the chain rule twice to find the indicated derivative.

17. $g(t) = f(x(t)), y(t))$, find $g''(t)$
18. $g(t) = f(x(t), y(t), z(t))$, find $g''(t)$
19. $g(u, v) = f(x(u, v), y(u, v))$, find $\dfrac{\partial^2 g}{\partial u^2}$
20. $g(u, v) = f(x(u, v), y(u, v))$, find $\dfrac{\partial^2 g}{\partial u \partial v}$

In exercises 21–24, use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

21. $3x^2 z + 2z^3 - 3yz = 0$
22. $xyz - 4y^2 z^2 + \cos xy = 0$
23. $3e^{xyz} - 4xz^2 + x \cos y = 2$
24. $3yz^2 - e^{4x} \cos 4z - 3y^2 = 4$
25. For a differentiable function $f(x, y)$ with $x = r \cos \theta$ and $y = r \sin \theta$, show that $f_\theta = -f_x r \sin \theta + f_y r \cos \theta$.
26. For a differentiable function $f(x, y)$ with $x = r \cos \theta$ and $y = r \sin \theta$, show that $f_{\theta\theta} = f_{xx} r^2 \sin^2 \theta - 2 f_{xy} r^2 \cos \theta \sin \theta + f_{yy} r^2 \cos^2 \theta - f_x r \cos \theta - f_y r \sin \theta$.
27. For a differentiable function $f(x, y)$ with $x = r \cos \theta$ and $y = r \sin \theta$, use the results of exercises 25 and 26 and example 5.4 to show that $f_{xx} + f_{yy} = f_{rr} + \dfrac{1}{r} f_r + \dfrac{1}{r^2} f_{\theta\theta}$. This expression is called the **Laplacian** of f.

28. Given that $r = \sqrt{x^2 + y^2}$, show that $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r} = \cos\theta$. Starting from $r = \dfrac{x}{\cos\theta}$, does it follow that $\dfrac{\partial r}{\partial x} = \dfrac{1}{\cos\theta}$? Explain why it's not possible for both calculations to be correct. Find all mistakes.

29. The **heat equation** for the temperature $u(x, t)$ of a thin rod of length L is $\alpha^2 u_{xx} = u_t$, $0 < x < L$, for some constant α^2, called the **thermal diffusivity.** Make the change of variables $X = \dfrac{x}{L}$ and $T = \dfrac{\alpha^2}{L^2}t$ to simplify the equation. Show that X and T are dimensionless, given that the units of α^2 are ft^2/s.

30. The **wave equation** for the displacement $u(x, t)$ of a vibrating string of length L is $a^2 u_{xx} = u_{tt}$, $0 < x < L$, for some constant a^2. Make the change of variables $X = \dfrac{x}{L}$ and $T = \dfrac{a}{L}t$ to simplify the equation. Assuming that X and T are dimensionless, find the dimensions of a^2.

Exercises 31–40 relate to Taylor series for functions of two or more variables.

31. Suppose that $f(x, y)$ is a function with all partial derivatives continuous. For constants u_1 and u_2, define $g(h) = f(x + hu_1, y + hu_2)$. We will construct the Taylor series for $g(h)$ about $h = 0$. First, show that $g(0) = f(x, y)$. Then show that $g'(0) = f_x(x, y)u_1 + f_y(x, y)u_2$. Next, show that $g''(0) = f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2$, where the functions f_{xx}, f_{xy} and f_{yy} are all evaluated at (x, y). Evaluate $g'''(0)$ and $g^{(4)}(0)$, and briefly describe the pattern of terms that emerges.

32. Use the result of exercise 31 with $hu_1 = \Delta x$ and $hu_2 = \Delta y$ to show that

$$\begin{aligned}&f(x + \Delta x, y + \Delta y) = f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y\\&+ \tfrac{1}{2}[f_{xx}(x, y)\Delta x^2 + 2f_{xy}(x, y)\Delta x\Delta y + f_{yy}(x, y)\Delta y^2]\\&+ \tfrac{1}{3!}[f_{xxx}(x, y)\Delta x^3 + 3f_{xxy}(x, y)\Delta x^2\Delta y + 3f_{xyy}(x, y)\\&\Delta x\Delta y^2 + f_{yyy}(x, y)\Delta y^3] + \cdots,\end{aligned}$$

which is the form of Taylor series for functions of two variables about the center (x, y).

33. Use the result of exercise 32 to write out the third-order Taylor polynomial for $f(x, y) = \sin x \cos y$ about $(0, 0)$.

34. Compare your answer in exercise 33 to a term-by-term multiplication of the Maclaurin series (Taylor series with center 0) for $\sin x$ and $\cos y$. Write out the fourth-order and fifth-order terms for this product.

35. Write out the third-order polynomial for $f(x, y) = \sin xy$ about $(0, 0)$.

36. Compare your answer in exercise 35 to the Maclaurin series for $\sin u$ with the substitution $u = xy$.

37. Write out the third-order polynomial for $f(x, y) = e^{2x+y}$ about $(0, 0)$.

38. Compare your answer in exercise 37 to the Maclaurin series for e^u with the substitution $u = 2x + y$.

39. The Environmental Protection Agency uses the 55/45 rule for combining a car's highway gas mileage rating h and its city gas mileage rating c into a single rating R for fuel efficiency using the formula $R = \dfrac{1}{0.55/c + 0.45/h}$. Find the first-order Taylor series (terms for Δc and Δh but not Δc^2) for $R(c, h)$ about $(1, 1)$.

40. Given the answer to exercise 39, explain why it's surprising that the EPA would use the complicated formula it does. To see why, consider a car with $h = 40$ and graphically compare the actual rating R to the Taylor approximation for $0 \le c \le 40$. If c is approximately the same as h, is there much difference in the graphs? As c approaches 0, how do the graphs compare? The EPA wants to convey useful information to the public. If a car got 40 mpg on the highway and 5 mpg in the city, would you want the overall rating to be (relatively) high or low?

41. The pressure, temperature, volume and enthalpy of a gas are all interrelated. Enthalpy is determined by pressure and temperature, so $E = f(P, T)$, for some function f. Pressure is determined by temperature and volume, so $P = g(T, V)$, for some function g. Show that $E = h(T, V)$ where h is a composition of f and g. Chemists write $\dfrac{\partial f}{\partial T}$ as $\left(\dfrac{\partial E}{\partial T}\right)_P$ to show that P is being held constant. Similarly, $\left(\dfrac{\partial E}{\partial T}\right)_V$ would refer to $\dfrac{\partial h}{\partial T}$. Using this convention, show that $\left(\dfrac{\partial E}{\partial T}\right)_V = \left(\dfrac{\partial E}{\partial T}\right)_P + \left(\dfrac{\partial E}{\partial P}\right)_T \left(\dfrac{\partial P}{\partial T}\right)_V$.

42. An economist analyzing the relationship among capital expenditure, labor and production in an industry might start with production $p(x, y)$ as a function of capital x and labor y. An additional assumption is that if labor and capital are doubled, the production should double. This translates to $p(2x, 2y) = 2p(x, y)$. This can be generalized to the relationship $p(kx, ky) = kp(x, y)$, for any positive constant k. Differentiate both sides of this equation with respect to k and show that $p(x, y) = xp_x(x, y) + yp_y(x, y)$. This would be stated by the economist as, "The total production equals the sum of the costs of capital and labor paid at their level of marginal product." Match each term in the quote with the corresponding term in the equation.

43. A baseball player who has h hits in b at bats has a batting average of $a = \dfrac{h}{b}$. For example, 100 hits in 400 at bats would be an average of 0.250. It is traditional to carry three decimal places and to describe this average as being "250 points." To use the chain rule to estimate the change in batting average after a player gets a hit, assume that h and b are functions of time and that getting a hit means $h' = b' = 1$. Show that $a' = \dfrac{b-h}{b^2}$.

Early in a season, a typical batter might have 50 hits in 200 at bats. Show that getting a hit will increase batting average by about 4 points. Find the approximate increase in batting average later in the season for a player with 100 hits in 400 at bats. In general, if b and h are both doubled, how does a' change?

44. For the baseball players of exercise 43, approximate the number of points that the batting average will decrease by making an out.

45. Find the general form for the derivative of $g(t) = u(t)^{v(t)}$ for differentiable functions u and v. (Hint: Start with $f(u, v) = u^v$.) Apply the result to find the derivative of $(2t + 1)^{3t^2}$.

EXPLORATORY EXERCISES

1. We have previously done calculations of the amount of work done by some force. Recall that if a scalar force $F(x)$ is applied as x increases from $x = a$ to $x = b$, then the work done equals $W = \int_a^b F(x)\,dx$. If the position x is a differentiable function of time, then we can write $W = \int_0^T F(x(t))x'(t)\,dt$, where $x(0) = a$ and $x(T) = b$. **Power** is defined as the time derivative of work. Work is sometimes measured in foot-pounds, so power could be measured in foot-pounds per second (ft-lb/s). One horsepower is equal to 550 ft-lb/s. Show that if force and velocity are constant, then power is the product of force and velocity, Determine how many pounds of force are required to maintain 400 hp at 80 mph. For a variable force and velocity, use the chain rule to compute power.

2. Engineers and physicists (and therefore mathematicians) spend countless hours studying the properties of forced oscillators. Two physical situations that are well modeled by the same mathematical equations are a spring oscillating due to some force and a simple electrical circuit with a voltage source. A general solution of a forced oscillator can have the form $u(t) = g(t) - \int_0^t g(u)e^{-(t-u)/2}[\cos\frac{\sqrt{3}}{2}(t-u) + \frac{2}{3}\sin\frac{\sqrt{3}}{2}(t-u)]\,du$. If $g(0) = 1$ and $g'(0) = 2$, compute $u(0)$ and $u'(0)$.

12.6 THE GRADIENT AND DIRECTIONAL DERIVATIVES

Suppose that you are hiking in rugged terrain. You can think of your altitude at the point given by longitude x and latitude y as defining a function $f(x, y)$. Although you are unlikely to have a handy formula for this function, you can learn more about this function than you might expect. If you face due east (in the direction of the positive x-axis), the slope of the terrain is given by the partial derivative $\frac{\partial f}{\partial x}(x, y)$. Similarly, facing due north, the slope of the terrain is given by $\frac{\partial f}{\partial y}(x, y)$. However, in terms of $f(x, y)$, how would you compute the slope in some other direction, say north-by-northwest? In this section, we develop the notion of *directional derivative,* which will answer this question.

Suppose that we want to find the instantaneous rate of change of $f(x, y)$ at the point $P(a, b)$ and in the direction given by the *unit* vector $\mathbf{u} = \langle u_1, u_2\rangle$. Let $Q(x, y)$ be any point on the line through $P(a, b)$ in the direction of $\mathbf{u}$. Notice that the vector $\overrightarrow{PQ}$ is then parallel to $\mathbf{u}$. Since two vectors are parallel if and only if one is a scalar multiple of the other, we have that $\overrightarrow{PQ} = h\mathbf{u}$, for some scalar h, so that

$$\overrightarrow{PQ} = \langle x - a, y - b\rangle = h\mathbf{u} = h\langle u_1, u_2\rangle = \langle hu_1, hu_2\rangle.$$

It then follows that $x - a = hu_1$ and $y - b = hu_2$, so that

$$x = a + hu_1 \quad \text{and} \quad y = b + hu_2.$$

The point Q is then described by $(a + hu_1, b + hu_2)$, as indicated in Figure 12.33 (on the following page). Notice that the average rate of change of $z = f(x, y)$ along the line from P to Q is then

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

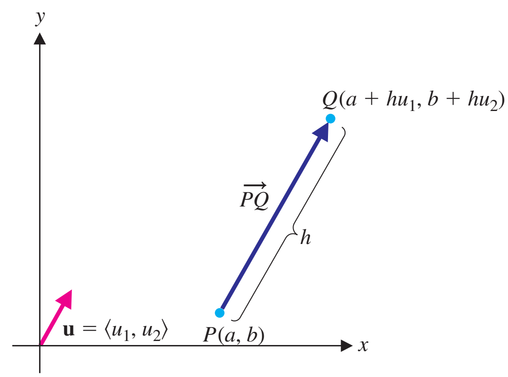


FIGURE 12.33
The vector $\overrightarrow{PQ}$

The instantaneous rate of change of $f(x, y)$ at the point $P(a, b)$ and in the direction of the unit vector $\mathbf{u}$ is then found by taking the limit as $h \to 0$. We give this limit a special name in Definition 6.1.

DEFINITION 6.1

The **directional derivative of** $f(x, y)$ at the point (a, b) and in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is given by

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

Notice that this limit resembles the definition of partial derivative, except that in this case, both variables may change. Further, you should observe that the directional derivative in the direction of the positive x-axis (i.e., in the direction of the unit vector $\mathbf{u} = \langle 1, 0 \rangle$) is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},$$

which you should recognize as the partial derivative $\frac{\partial f}{\partial x}$. Likewise, the directional derivative in the direction of the positive y-axis (i.e., in the direction of the unit vector $\mathbf{u} = \langle 0, 1 \rangle$) is $\frac{\partial f}{\partial y}$. It turns out that any directional derivative can be calculated simply, in terms of the first partial derivatives, as we see in Theorm 6.1.

THEOREM 6.1

Suppose that f is differentiable at (a, b) and $\mathbf{u} = \langle u_1, u_2 \rangle$ is any unit vector. Then, we can write

$$D_{\mathbf{u}} f(a, b) = f_x(a, b) u_1 + f_y(a, b) u_2.$$

PROOF

Let $g(h) = f(a + hu_1, b + hu_2)$. Then, $g(0) = f(a, b)$ and so, from Definition 6.1, we have

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0).$$

If we define $x = a + hu_1$ and $y = b + hu_2$, we have $g(h) = f(x, y)$. From the chain rule (Theorem 5.1), we have

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2.$$

Finally, taking $h = 0$ gives us

$$D_{\mathbf{u}} f(a, b) = g'(0) = \frac{\partial f}{\partial x}(a, b)u_1 + \frac{\partial f}{\partial y}(a, b)u_2,$$

as desired. ■

EXAMPLE 6.1 Computing Directional Derivatives

For $f(x, y) = x^2y - 4y^3$, compute $D_{\mathbf{u}} f(2, 1)$ for the directions (a) $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and (b) $\mathbf{u}$ in the direction from (2, 1) to (4, 0).

Solution Regardless of the direction, we first need to compute the first partial derivatives $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 - 12y^2$. Then, $f_x(2, 1) = 4$ and $f_y(2, 1) = -8$. For (a), the unit vector is given as $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and so, from Theorem 6.1 we have

$$D_{\mathbf{u}} f(2, 1) = f_x(2, 1)u_1 + f_y(2, 1)u_2 = 4\frac{\sqrt{3}}{2} - 8\left(\frac{1}{2}\right) = 2\sqrt{3} - 4 \approx -0.5.$$

Notice that this says that the function is decreasing in this direction.

For (b), we must first find the unit vector $\mathbf{u}$ in the indicated direction. Observe that the vector from (2, 1) to (4, 0) corresponds to the position vector $\langle 2, -1 \rangle$ and so, the unit vector in that direction is $\mathbf{u} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$. We then have from Theorem 6.1 that

$$D_{\mathbf{u}} f(2, 1) = f_x(2, 1)u_1 + f_y(2, 1)u_2 = 4\frac{2}{\sqrt{5}} - 8\left(-\frac{1}{\sqrt{5}}\right) = \frac{16}{\sqrt{5}}.$$

So, the function is increasing rapidly in this direction. ■

For convenience, we define the **gradient** of a function to be the vector-valued function whose components are the first-order partial derivatives of f, as specified in Definition 6.2. We denote the gradient of a function f by **grad** f or ∇f (read "del f").

DEFINITION 6.2

The **gradient** of $f(x, y)$ is the vector-valued function

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j},$$

provided both partial derivatives exist.

Using the gradient, we can write a directional derivative as the dot product of the gradient and the unit vector in the direction of interest, as follows. For any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$,

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)u_1 + f_y(x, y)u_2 \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle u_1, u_2 \rangle \\ &= \nabla f(x, y) \cdot \mathbf{u}. \end{aligned}$$

We state this result in Theorem 6.2.

THEOREM 6.2

If f is a differentiable function of x and y and $\mathbf{u}$ is any unit vector, then

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Writing directional derivatives as a dot product has many important consequences, one of which we see in example 6.2.

EXAMPLE 6.2 Finding Directional Derivatives

For $f(x, y) = x^2 + y^2$, find $D_{\mathbf{u}} f(1, -1)$ for (a) $\mathbf{u}$ in the direction of $\mathbf{v} = \langle -3, 4 \rangle$ and (b) $\mathbf{u}$ in the direction of $\mathbf{v} = \langle 3, -4 \rangle$.

Solution First, note that

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y \rangle.$$

At the point $(1, -1)$, we have $\nabla f(1, -1) = \langle 2, -2 \rangle$. For (a), a unit vector in the same direction as $\mathbf{v}$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$. The directional derivative of f in this direction at the point $(1, -1)$ is then

$$D_{\mathbf{u}} f(1, -1) = \langle 2, -2 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = \frac{-6 - 8}{5} = -\frac{14}{5}.$$

For (b), the unit vector is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$ and so, the directional derivative of f in this direction at $(1, -1)$ is

$$D_{\mathbf{u}} f(1, -1) = \langle 2, -2 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{6 + 8}{5} = \frac{14}{5}.$$

A graphical interpretation of the directional derivatives in example 6.2 is given in Figure 12.34a. Suppose we intersect the surface $z = f(x, y)$ with a plane passing through the point $(1, -1, 2)$, which is perpendicular to the xy-plane and parallel to the vector $\mathbf{u}$ (see Figure 12.34a). Notice that the intersection is a curve in two dimensions. Sketch this curve on a new set of coordinate axes, chosen so that the new origin corresponds to the point $(1, -1, 2)$, the new vertical axis is in the z-direction and the new positive horizontal axis points in the direction of the vector $\mathbf{u}$. In Figure 12.34b, we show the case for $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$ and in Figure 12.34c, we show the case for $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$. In each case, the directional derivative gives the slope of the curve at the origin (in the new coordinate system). Notice that the direction vectors in example 6.2 parts (a) and (b) differ only by sign and the resulting curves in Figures 12.34b and 12.34c are exact mirror images of each other.

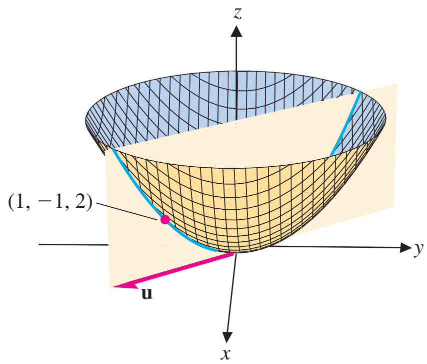


FIGURE 12.34a
Intersection of surface with plane

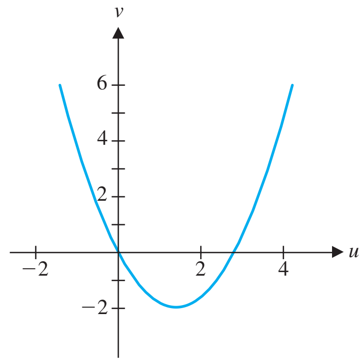


FIGURE 12.34b
$\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

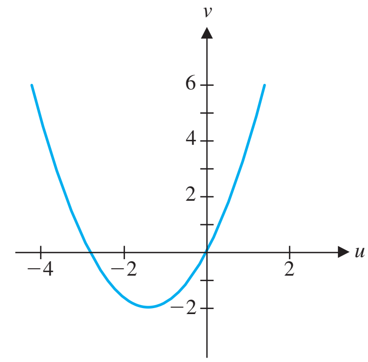


FIGURE 12.34c
$\mathbf{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

We can use a contour plot to estimate the value of a directional derivative, as we illustrate in example 6.3.

EXAMPLE 6.3 Directional Derivatives and Level Curves

Use a contour plot of $z = x^2 + y^2$ to estimate $D_{\mathbf{u}} f(1, -1)$ for $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$.

Solution A contour plot of $z = x^2 + y^2$ is shown in Figure 12.35 with the direction vector $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$ sketched in with its initial point located at the point $(1, -1)$. The level curves shown correspond to $z = 0.2, 0.5, 1, 2$ and 3. From the graph, you can approximate the directional derivative by estimating $\frac{\Delta z}{\Delta u}$, where Δu is the distance traveled along the unit vector $\mathbf{u}$. For the unit vector shown, $\Delta u = 1$. Further, the vector appears to extend from the $z = 2$ level curve to the $z = 0.2$ level curve. In this case, $\Delta z = 0.2 - 2 = -1.8$ and our estimate of the directional derivative is $\frac{\Delta z}{\Delta u} = -1.8$. Compared to the actual directional derivative of $-\frac{14}{5} = -2.8$ (found in example 6.2), this is not very accurate. A better estimate could be obtained with a smaller Δu. For example, to get from the $z = 2$ level curve to the $z = 1$ level curve, it appears that we travel along about 40% of the unit vector. Then $\frac{\Delta z}{\Delta u} \approx \frac{1-2}{0.4} = -2.5$. You could continue this process by drawing more level curves, corresponding to values of z closer to $z = 2$. ■

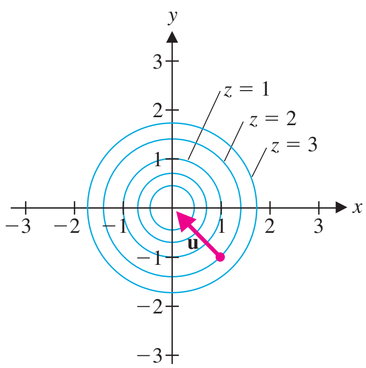


FIGURE 12.35
Contour plot of $z = x^2 + y^2$

Keep in mind that a directional derivative gives the rate of change of a function in a given direction. So, it's reasonable to ask in what direction a given function has its maximum or minimum rate of increase. First, recall from Theorem 3.2 in Chapter 10 that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Applying this to the form of the directional derivative given in Theorem 6.2, we have

$$\begin{aligned} D_{\mathbf{u}} f(a, b) &= \nabla f(a, b) \cdot \mathbf{u} \\ &= \|\nabla f(a, b)\|\|\mathbf{u}\| \cos\theta = \|\nabla f(a, b)\| \cos\theta, \end{aligned}$$

where θ is the angle between the gradient vector at (a, b) and the direction vector $\mathbf{u}$.

Notice now that $\|\nabla f(a, b)\| \cos\theta$ has its maximum value when $\theta = 0$, so that $\cos\theta = 1$. The directional derivative is then $\|\nabla f(a, b)\|$. Further, observe that the angle $\theta = 0$ when $\nabla f(a, b)$ and $\mathbf{u}$ are in the *same* direction, so that $\mathbf{u} = \frac{\nabla f(a, b)}{\|\nabla f(a, b)\|}$. Similarly, the minimum value of the directional derivative occurs when $\theta = \pi$, so that $\cos\theta = -1$. In this case, $\nabla f(a, b)$ and $\mathbf{u}$ have *opposite* directions, so that $\mathbf{u} = -\frac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$. Finally, observe that when $\theta = \frac{\pi}{2}$, $\mathbf{u}$ is perpendicular to $\nabla f(a, b)$ and the directional derivative in this direction is zero. Since the level curves are curves in the xy-plane on which f is constant, notice that a zero directional derivative at a point indicates that $\mathbf{u}$ is tangent to a level curve. We summarize these observations in Theorem 6.3.

THEOREM 6.3

Suppose that f is a differentiable function of x and y at the point (a, b). Then

(i) the maximum rate of change of f at (a, b) is $\|\nabla f(a, b)\|$, occuring in the direction of the gradient;
(ii) the minimum rate of change of f at (a, b) is $-\|\nabla f(a, b)\|$, occuring in the direction opposite the gradient;
(iii) the rate of change of f at (a, b) is 0 in the directions orthogonal to $\nabla f(a, b)$ and
(iv) the gradient $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = c$ at the point (a, b), where $c = f(a, b)$.

In using Theorem 6.3, remember that the directional derivative corresponds to the rate of change of the function $f(x, y)$ in the given direction.

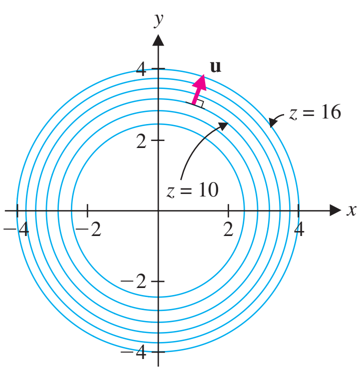


FIGURE 12.36
Contour plot of $z = x^2 + y^2$

EXAMPLE 6.4 Finding Maximum and Minimum Rates of Change

Find the maximum and minimum rates of change of the function $f(x, y) = x^2 + y^2$ at the point $(1, 3)$.

Solution We first compute the gradient $\nabla f = \langle 2x, 2y\rangle$ and evaluate it at the point $(1, 3)$: $\nabla f(1, 3) = \langle 2, 6\rangle$. From Theorem 6.3, the maximum rate of change of f at $(1, 3)$ is $\|\nabla f(1, 3)\| = \|\langle 2, 6\rangle\| = \sqrt{40}$ and occurs in the direction of

$$\mathbf{u} = \frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = \frac{\langle 2, 6\rangle}{\sqrt{40}}.$$

Similarly, the minimum rate of change of f at $(1, 3)$ is $-\|\nabla f(1, 3)\| = -\|\langle 2, 6\rangle\| = -\sqrt{40}$, which occurs in the direction of

$$\mathbf{u} = -\frac{\nabla f(1,3)}{\|\nabla f(1,3)\|} = \frac{-\langle 2, 6\rangle}{\sqrt{40}}.$$

Notice that the direction of maximum increase in example 6.4 points away from the origin, since the displacement vector from $(0, 0)$ to $(1, 3)$ is parallel to $\mathbf{u} = \langle 2, 6\rangle/\sqrt{40}$. This should make sense given the familiar shape of the paraboloid. The contour plot of $f(x, y)$ shown in Figure 12.36 indicates that the gradient is perpendicular to the level curves. We expand on this idea in example 6.5.

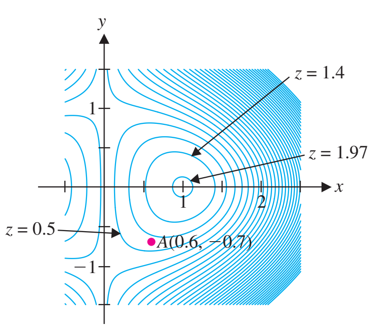


FIGURE 12.37
Contour plot of $z = 3x - x^3 - 3xy^2$

EXAMPLE 6.5 Finding the Direction of Steepest Ascent

The contour plot of $f(x, y) = 3x - x^3 - 3xy^2$ shown in Figure 12.37 indicates several level curves near a relative maximum at $(1, 0)$. Find the direction of maximum increase from the point $A(0.6, -0.7)$ and sketch in the path of steepest ascent.

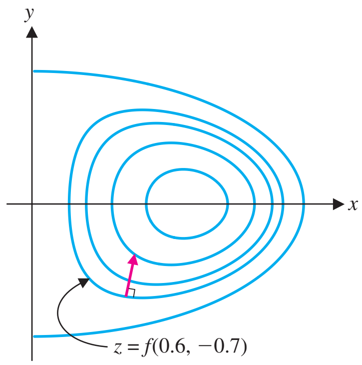


FIGURE 12.38a
Direction of steepest ascent at (0.6, −0.7)

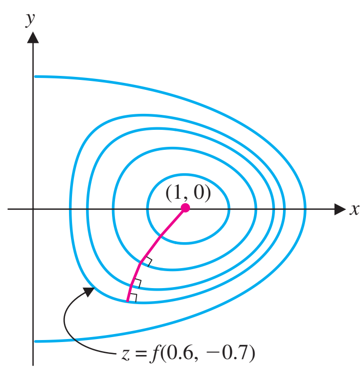


FIGURE 12.38b
Path of steepest ascent

Solution From Theorem 6.3, the direction of maximum increase at $(0.6, -0.7)$ is given by the gradient $\nabla f(0.6, -0.7)$. We have $\nabla f = \langle 3 - 3x^2 - 3y^2, -6xy \rangle$ and so, $\nabla f(0.6, -0.7) = \langle 0.45, 2.52 \rangle$. The unit vector in this direction is then $\mathbf{u} = \langle 0.176, 0.984 \rangle$. A vector in this direction (not drawn to scale) at the point $(0.6, -0.7)$ is shown in Figure 12.38a. Notice that this vector *does not* point toward the maximum at $(1, 0)$. (By analogy, on a mountain, the steepest path from a given point will not always point toward the actual peak.) The **path of steepest ascent** is a curve that remains perpendicular to each level curve through which it passes. Notice that at the tip of the vector drawn in Figure 12.38a, the vector is no longer perpendicular to the level curve. Finding an equation for the path of steepest ascent is challenging. In Figure 12.38b, we sketch in a plausible path of steepest ascent. ■

Most of the results of this section extend easily to functions of any number of variables.

DEFINITION 6.3

The **directional derivative** of $f(x, y, z)$ at the point (a, b, c) and in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is given by

$$D_{\mathbf{u}} f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h},$$

provided the limit exists.

The **gradient** of $f(x, y, z)$ is the vector-valued function

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

provided all the partial derivatives are defined.

As was the case for functions of two variables, the gradient gives us a simple representation of directional derivatives in three dimensions.

THEOREM 6.4

If f is a differentiable function of x, y and z and $\mathbf{u}$ is any unit vector, then

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}. \tag{6.1}$$

As in two dimensions, we have that

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= \nabla f(x, y, z) \cdot \mathbf{u} = \|\nabla f(x, y, z)\| \|\mathbf{u}\| \cos\theta \\ &= \|\nabla f(x, y, z)\| \cos\theta, \end{aligned}$$

where θ is the angle between the vectors $\nabla f(x, y, z)$ and $\mathbf{u}$. For precisely the same reasons as in two dimensions, it follows that the direction of maximum increase at any given point is given by the gradient at that point.

EXAMPLE 6.6 Finding the Direction of Maximum Increase

If the temperature at point (x, y, z) is given by $T(x, y, z) = 85 + (1 - z/100)e^{-(x^2+y^2)}$, find the direction from the point $(2, 0, 99)$ in which the temperature increases most rapidly.

Solution We first compute the gradient

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$
$$= \left\langle -2x\left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}, -2y\left(1 - \frac{z}{100}\right)e^{-(x^2+y^2)}, -\left(\frac{1}{100}\right)e^{-(x^2+y^2)} \right\rangle$$

and $\nabla f(2, 0, 99) = \left\langle -\frac{1}{25}e^{-4}, 0, -\frac{1}{100}e^{-4} \right\rangle$. To find a unit vector in this direction, you can simplify the algebra by canceling the common factor of e^{-4} (think about why this makes sense) and multiplying by 100. A unit vector in the direction of $\langle -4, 0, -1 \rangle$ and also in the direction of $\nabla f(2, 0, 99)$, is then $\dfrac{\langle -4,\ 0,\ -1 \rangle}{\sqrt{17}}$. ■

Recall that for any constant k, the equation $f(x, y, z) = k$ defines a level surface of the function $f(x, y, z)$. Now, suppose that $\mathbf{u}$ is any unit vector lying in the tangent plane to the level surface $f(x, y, z) = k$ at a point (a, b, c) on the level surface. Then, it follows that the rate of change of f in the direction of $\mathbf{u}$ at (a, b, c) [given by the directional derivative $D_{\mathbf{u}} f(a, b, c)$] is zero, since f is constant on a level surface. From (6.1), we now have that

$$0 = D_{\mathbf{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{u}.$$

This occurs only when the vectors $\nabla f(a, b, c)$ and $\mathbf{u}$ are orthogonal. Since $\mathbf{u}$ was taken to be any vector lying in the tangent plane, we now have that $\nabla f(a, b, c)$ is orthogonal to every vector lying in the tangent plane at the point (a, b, c). Observe that this says that $\nabla f(a, b, c)$ is a normal vector to the tangent plane to the surface $f(x, y, z) = k$ at the point (a, b, c). This proves Theorem 6.5.

THEOREM 6.5

Suppose that $f(x, y, z)$ has continuous partial derivatives at the point (a, b, c) and $\nabla f(a, b, c) \neq \mathbf{0}$. Then, $\nabla f(a, b, c)$ is a normal vector to the tangent plane to the surface $f(x, y, z) = k$, at the point (a, b, c). Further, the equation of the tangent plane is

$$0 = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

We refer to the line through (a, b, c) in the direction of $\nabla f(a, b, c)$ as the **normal line** to the surface at the point (a, b, c). Observe that this has parametric equations

$$x = a + f_x(a, b, c)t, \quad y = b + f_y(a, b, c)t, \quad z = c + f_z(a, b, c)t.$$

In example 6.7, we illustrate the use of the gradient at a point to find the tangent plane and normal line to a surface at that point.

EXAMPLE 6.7 Using a Gradient to Find a Tangent Plane and Normal Line to a Surface

Find equations of the tangent plane and the normal line to $x^3y - y^2 + z^2 = 7$ at the point $(1, 2, 3)$.

Solution If we interpret the surface as a level surface of the function $f(x, y, z) = x^3y - y^2 + z^2$, a normal vector to the tangent plane at the point $(1, 2, 3)$ is given by $\nabla f(1, 2, 3)$. We have $\nabla f = \langle 3x^2y, x^3 - 2y, 2z \rangle$ and

$\nabla f(1, 2, 3) = \langle 6, -3, 6\rangle$. Given the normal vector $\langle 6, -3, 6\rangle$ and point $(1, 2, 3)$, an equation of the tangent plane is

$$6(x - 1) - 3(y - 2) + 6(z - 3) = 0.$$

The normal line has parametric equations

$$x = 1 + 6t, \quad y = 2 - 3t, \quad z = 3 + 6t. \blacksquare$$

Recall that in section 12.4, we found that a normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is given by $\left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\right\rangle$. Note that this is simply a special case of the gradient formula of Theorem 6.5, as follows. First, observe that we can rewrite the equation $z = f(x, y)$ as $f(x, y) - z = 0$. We can then think of this surface as a level surface of the function $g(x, y, z) = f(x, y) - z$, which at the point $(a, b, f(a, b))$ has normal vector

$$\nabla g(a, b, f(a, b)) = \left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1\right\rangle.$$

Just as it is important to constantly think of ordinary derivatives as slopes of tangent lines and as instantaneous rates of change, it is crucial to keep in mind at all times the interpretations of gradients. *Always* think of gradients as vector-valued functions whose values specify the direction of maximum increase of a function and whose values provide normal vectors (to the level curves in two dimensions and to the level surfaces in three dimensions).

EXAMPLE 6.8 Using a Gradient to Find a Tangent Plane to a Surface

Find an equation of the tangent plane to $z = \sin(x + y)$ at the point $(\pi, \pi, 0)$.

Solution We rewrite the equation of the surface as $g(x, y, z) = \sin(x + y) - z = 0$ and compute $\nabla g(x, y, z) = \langle\cos(x + y), \cos(x + y), -1\rangle$. At the point $(\pi, \pi, 0)$, the normal vector to the surface is given by $\nabla g(\pi, \pi, 0) = \langle 1, 1, -1\rangle$. An equation of the tangent plane is then

$$(x - \pi) + (y - \pi) - z = 0. \blacksquare$$

BEYOND FORMULAS

The term *gradient* shows up in a large number of applications. Standard usage of the term is very close to our development in this section. By its use in directional derivatives, the gradient gives all the information you need to determine the change in a quantity as you move in some direction from your current position. Because the gradient gives the direction of maximum increase, any process that depends on maximizing or minimizing some quantity may be described with the gradient. When you see *gradient* in an application, think of these properties.

EXERCISES 12.6

WRITING EXERCISES

1. Pick an area outside your classroom that has a small hill. Starting at the bottom of the hill, describe how to follow the gradient path to the top. In particular, describe how to determine the direction in which the gradient points at a given point on the hill. In general, should you be looking ahead or down at the ground? Should individual blades of grass count? What should you do if you encounter a wall?

2. Discuss whether the gradient path described in exercise 1 is guaranteed to get you to the top of the hill. Discuss whether the gradient path is the shortest path, the quickest path or the easiest path.

3. Use the sketch in Figure 12.34a to explain why the curves in Figures 12.34b and 12.34c are different.

4. Suppose the function $f(x, y)$ represents the altitude at various points on a ski slope. Explain in physical terms why the direction of maximum increase is 180° opposite the direction of maximum decrease, with the direction of zero change halfway in between. If $f(x, y)$ represents altitude on a rugged mountain instead of a ski slope, explain why the results (which are still true) are harder to visualize.

In exercises 1–4, find the gradient of the given function.

1. $f(x, y) = x^2 + 4xy^2 - y^5$
2. $f(x, y) = x^3e^{3y} - y^4$
3. $f(x, y) = xe^{xy^2} + \cos y^2$
4. $f(x, y) = e^{3y/x} - x^2y^3$

In exercises 5–10, find the gradient of the given function at the indicated point.

5. $f(x, y) = 2e^{4x/y} - 2x$, $(2, -1)$
6. $f(x, y) = \sin 3xy + y^2$, $(\pi, 1)$
7. $f(x, y, z) = 3x^2y - z\cos x$, $(0, 2, -1)$
8. $f(x, y, z) = z^2e^{2x-y} - 4xz^2$, $(1, 2, 2)$
9. $f(w, x, y, z) = w^2\cos x + 3ye^{xz}$, $(2, \pi, 1, 4)$
10. $f(x_1, x_2, x_3, x_4, x_5) = \sin\left(\frac{x_1}{x_2}\right) - 3x_3^2x_4x_5 - 2\sqrt{x_1x_3}$, $(2, 1, 2, -1, 4)$

In exercises 11–26, compute the directional derivative of f at the given point in the direction of the indicated vector.

11. $f(x, y) = x^2y + 4y^2$, $(2, 1)$, $\mathbf{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$
12. $f(x, y) = x^3y - 4y^2$, $(2, -1)$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$
13. $f(x, y) = \sqrt{x^2 + y^2}$, $(3, -4)$, $\mathbf{u}$ in the direction of $\langle 3, -2 \rangle$
14. $f(x, y) = e^{4x^2-y}$, $(1, 4)$, $\mathbf{u}$ in the direction of $\langle -2, -1 \rangle$
15. $f(x, y) = \cos(2x - y)$, $(\pi, 0)$, $\mathbf{u}$ in the direction from $(\pi, 0)$ to $(2\pi, \pi)$
16. $f(x, y) = x^2 \sin 4y$, $(-2, \frac{\pi}{8})$, $\mathbf{u}$ in the direction from $(-2, \frac{\pi}{8})$ to $(0, 0)$
17. $f(x, y) = x^2 - 2xy + y^2$, $(-2, -1)$, $\mathbf{u}$ in the direction from $(-2, -1)$ to $(2, -3)$
18. $f(x, y) = y^2 + 2ye^{4x}$, $(0, -2)$, $\mathbf{u}$ in the direction from $(0, -2)$ to $(-4, 4)$
19. $f(x, y, z) = x^3yz^2 - 4xy$, $(1, -1, 2)$, $\mathbf{u}$ in the direction of $\langle 2, 0, -1 \rangle$
20. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(1, -4, 8)$, $\mathbf{u}$ in the direction of $\langle 1, 1, -2 \rangle$
21. $f(x, y, z) = e^{xy+z}$, $(1, -1, 1)$, $\mathbf{u}$ in the direction of $\langle 4, -2, 3 \rangle$
22. $f(x, y, z) = \cos xy + z$, $(0, -2, 4)$, $\mathbf{u}$ in the direction of $\langle 0, 3, -4 \rangle$
23. $f(w, x, y, z) = w^2\sqrt{x^2 + 1} + 3ze^{xz}$, $(2, 0, 1, 0)$, $\mathbf{u}$ in the direction of $\langle 1, 3, 4, -2 \rangle$
24. $f(w, x, y, z) = \cos(w^2xy) + 3z - \tan 2z$, $(2, -1, 1, 0)$, $\mathbf{u}$ in the direction of $\langle -2, 0, 1, 4 \rangle$
25. $f(x_1, x_2, x_3, x_4, x_5) = \frac{x_1^2}{x_2} - \sin^{-1} 2x_3 + 3\sqrt{x_4x_5}$, $(2, 1, 0, 1, 4)$, $\mathbf{u}$ in the direction of $\langle 1, 0, -2, 4, -2 \rangle$
26. $f(x_1, x_2, x_3, x_4, x_5) = 3x_1x_2^3x_3 - e^{4x_3} + \ln\sqrt{x_4x_5}$, $(-1, 2, 0, 4, 1)$, $\mathbf{u}$ in the direction of $\langle 2, -1, 0, 1, -2 \rangle$

In exercises 27–36, find the directions of maximum and minimum change of f at the given point, and the values of the maximum and minimum rates of change.

27. $f(x, y) = x^2 - y^3$, $(2, 1)$
28. $f(x, y) = x^2 - y^3$, $(-1, -2)$
29. $f(x, y) = y^2e^{4x}$, $(0, -2)$
30. $f(x, y) = y^2e^{4x}$, $(3, -1)$
31. $f(x, y) = x\cos 3y$, $(2, 0)$
32. $f(x, y) = x\cos 3y$, $(-2, \pi)$
33. $f(x, y) = \sqrt{2x^2 - y}$, $(3, 2)$
34. $f(x, y) = \sqrt{x^2 + y^2}$, $(3, -4)$
35. $f(x, y, z) = 4x^2yz^3$, $(1, 2, 1)$
36. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(1, 2, -2)$
37. In exercises 34 and 36, compare the gradient direction to the position vector from the origin to the given point. Explain in terms of the graph of f why this relationship should hold.
38. Suppose that $g(x)$ is a differentiable function and $f(x, y) = g(x^2 + y^2)$. Show that $\nabla f(a, b)$ is parallel to $\langle a, b \rangle$. Explain this in graphical terms.
39. Graph $z = \sin(x + y)$. Compute $\nabla \sin(x + y)$ and explain why the gradient gives you the direction that the sine wave travels. In which direction would the sine wave travel for $z = \sin(2x - y)$?
40. Show that the vector $\langle 100, -100 \rangle$ is perpendicular to $\nabla \sin(x + y)$. Explain why the directional derivative of $\sin(x + y)$ in the direction of $\langle 100, -100 \rangle$ must be zero.

Sketch a wireframe graph of $z = \sin(x + y)$ from the viewpoint $(100, -100, 0)$. Explain why you only see one trace. Find a viewpoint from which $z = \sin(2x - y)$ only shows one trace.

In exercises 41–44, find equations of the tangent plane and normal line to the surface at the given point.

41. $z = x^2 + y^3$ at $(1, -1, 0)$

42. $z = \sqrt{x^2 + y^2}$ at $(3, -4, 5)$

43. $x^2 + y^2 + z^2 = 6$ at $(-1, 2, 1)$

44. $z^2 = x^2 - y^2$ at $(5, -3, -4)$

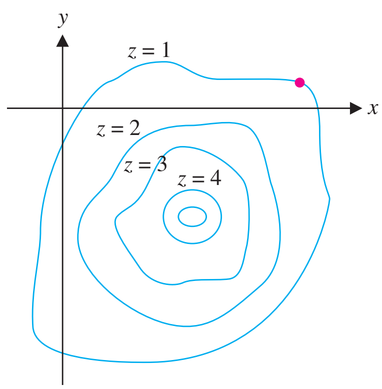
In exercises 45 and 46, find all points at which the tangent plane to the surface is parallel to the xy-plane. Discuss the graphical significance of each point.

45. $z = 2x^2 - 4xy + y^4$

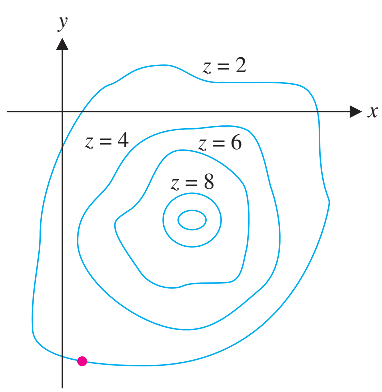
46. $z = \sin x \cos y$

In exercises 47–50, sketch in the path of steepest ascent from the indicated point.

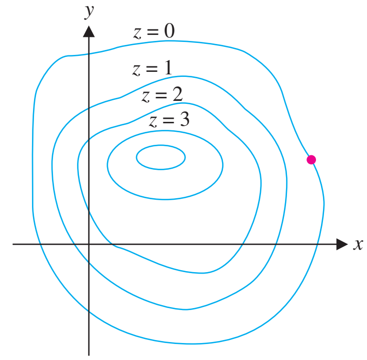
47.



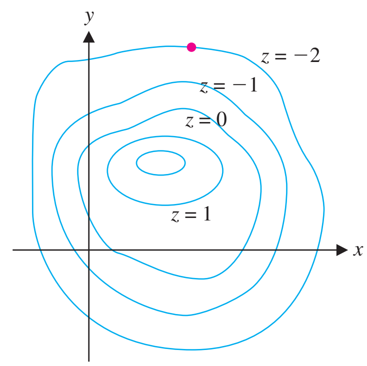
48.



49.

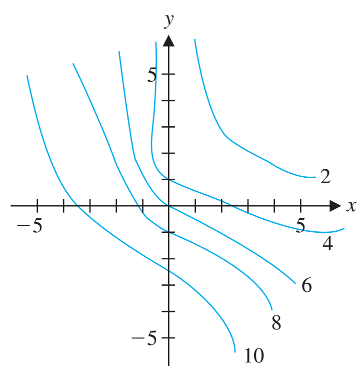


50.



In exercises 51 and 52, use the contour plot to estimate $\nabla f(0, 0)$.

51.



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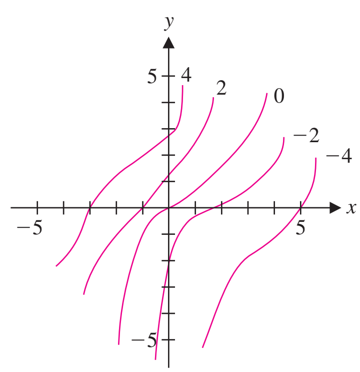


In exercises 53 and 54, use the table to estimate $\nabla f(0, 0)$.

53.

$y \backslash x$	−0.2	−0.1	0	0.1	0.2
−0.4	2.1	2.5	2.8	3.1	3.4
−0.2	1.9	2.2	2.4	2.6	2.9
0	1.6	1.8	2.0	2.2	2.5
0.2	1.3	1.4	1.6	1.8	2.1
0.4	1.1	1.2	1.1	1.4	1.7

54.

$y \backslash x$	−0.4	−0.2	0	0.2	0.4
−0.6	2.4	2.1	1.8	1.3	1.0
−0.3	2.6	2.2	1.9	1.5	1.2
0	2.7	2.4	2.0	1.6	1.3
0.3	2.9	2.5	2.1	1.7	1.5
0.6	3.1	2.7	2.3	1.9	1.7

55. At a certain point on a mountain, a surveyor sights due east and measures a 10° drop-off, then sights due north and measures a 6° rise. Find the direction of steepest ascent and compute the degree rise in that direction.

56. At a certain point on a mountain, a surveyor sights due west and measures a 4° rise, then sights due north and measures a 3° rise. Find the direction of steepest ascent and compute the degree rise in that direction.

57. Suppose that the elevation on a hill is given by $f(x, y) = 200 - y^2 - 4x^2$. From the site at (1, 2), in which direction will the rain run off?

58. For the hill of exercise 57, if a level road is to be built at elevation 100, find the shape of the road.

59. Suppose the temperature at each point (x, y, z) on a surface S is given by the function $T(x, y, z)$. Physics tells us heat flows from hot to cold and that the greater the temperature difference, the greater the flow. Explain why these facts would lead you to conclude that the maximum heat flow occurs in the direction $-\nabla T$ and, by Fourier's Law of Heat Flow, that the maximum heat flow is proportional to $\|\nabla T\|$.

60. If the temperature at the point (x, y, z) is given by $T(x, y, z) = 80 + 5e^{-z}(x^{-2} + y^{-1})$, find the direction from the point (1, 4, 8) in which the temperature decreases most rapidly.

61. Suppose that a spacecraft is slightly off course. The function f is an error function that measures how far off course the spacecraft is as a function of its position $\langle x, y, z\rangle$ and velocity $\langle v_x, v_y, v_z\rangle$. That is, f is a function of six variables. Writing $f(x, y, z, v_x, v_y, v_z)$, if the gradient of f at a particular time is $\nabla f = \langle 0, 2, 0, -3, 0, 0\rangle$, identify the change in position and change in velocity needed to correct the error.

62. Suppose that a person has money invested in five stocks. Let x_i be the number of shares held in stock i and let $f(x_1, x_2, x_3, x_4, x_5)$ equal the total value of the stocks. If $\nabla f = \langle 2, -1, 6, 0, -2\rangle$, indicate which stocks should be sold and which should be bought, and indicate the relative amounts of each sale or buy.

63. Sharks find their prey through a keen sense of smell and an ability to detect small electrical impluses. If $f(x, y, z)$ indicates the electrical charge in the water at position (x, y, z) and a shark senses that $\nabla f = \langle 12, -20, 5\rangle$, in which direction should the shark swim to find its prey?

64. The speed S of a tennis serve depends on the speed v of the tennis racket, the tension t of the strings of the racket, the liveliness e of the ball and the angle θ at which the racket is held. Writing $S(v, t, e, \theta)$, if $\nabla S = \langle 12, -2, 3, -3\rangle$, discuss the relative contributions of each factor. That is, for each variable, if the variable is increased, does the ball speed increase or decrease?

65. Label each as true or false and explain why. (a) $\nabla(f + g) = \nabla f + \nabla g$, (b) $\nabla(fg) = (\nabla f)g + f(\nabla g)$

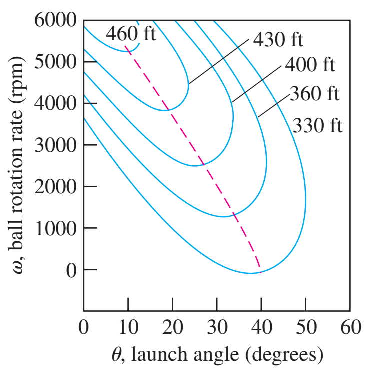
66. Show that for $f(x, y) = \begin{cases} \frac{x^2 y}{x^6 + 2y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$ and any $\mathbf{u}$, the directional derivative $D_{\mathbf{u}} f(0, 0)$ exists, but f is not continuous at (0, 0).

67. In example 4.6 of this chapter, we looked at a manufacturing process. Suppose that a gauge of 4 mm results from a gap of 4 mm, a speed of 10 m/s and a temperature of 900°. Further, suppose that an increase in gap of 0.05 mm increases the gauge by 0.04 mm, an increase in speed of 0.2 m/s increases the gauge by 0.06 mm and an increase in temperature of 10° decreases the gauge by 0.04 mm. Thinking of gauge as a function of gap, speed and temperature, find the direction of maximum increase of gauge.

68. The **Laplacian** of a function $f(x, y)$ is defined by $\nabla^2 f(x, y) = f_{xx}(x, y) + f_{yy}(x, y)$. Compute $\nabla^2 f(x, y)$ for $f(x, y) = x^3 - 2xy + y^2$.

EXPLORATORY EXERCISES

1. The horizontal range of a baseball that has been hit depends on its launch angle and the rate of backspin on the ball. The accompanying figure (reprinted from *Keep Your Eye on the Ball* by Watts and Bahill) shows level curves for the range as a function of angle and spin rate for an initial speed of 110 mph. Watts and Bahill suggest using the dashed line to find the best launch angle for a given spin rate. For example, start at $\omega = 2000$, move horizontally to the dashed line and then vertically down to $\theta = 30$. For a spin rate of 2000 rpm, the greatest range is achieved with a launch angle of 30°. To understand why, note that the dashed line intersects level curves at points where the level curves have horizontal tangents. Start at a point where the dashed line intersects a level curve and explain why you can conclude from the graph that changing the angle would decrease the range. Therefore, the dashed line indicates optimal angles. As ω increases, does the optimal angle increase or decrease? Explain in physical terms why this makes sense. Explain why you know that the dashed line does not follow a gradient path and explain what a gradient path would represent.



2. With the computer revolution of the 1990s came a new need to generate realistic-looking graphics. In this exercise, we look at one of the basic principles of three-dimensional graphics. We have often used wireframe graphs such as Figure A to visualize surfaces in three dimensions. Certainly, the graphic in Figure A is crude, but even this sketch is quite informative to us, as we can clearly see a local maximum. By having the computer plot more points, as in Figure B, we can smooth out some of the rough edges. Still, there is something missing, isn't there?

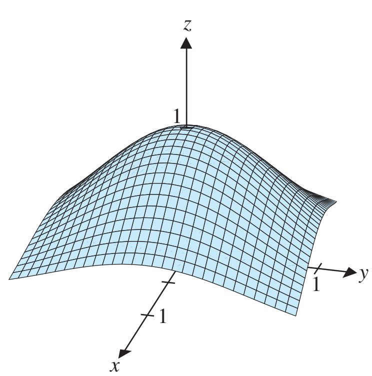


FIGURE A

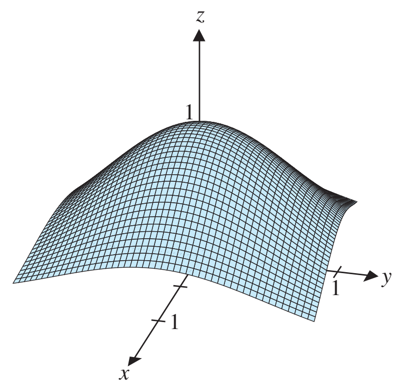


FIGURE B

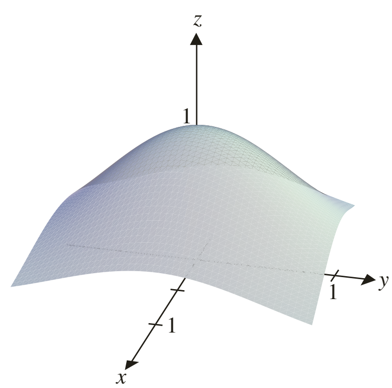


FIGURE C

Almost everything we see in real life is shaded by a light source from above. This shading gives us very important clues about the three-dimensional structure of the surface. In Figure C, we have simply added some shading to Figure B. There is more work to be done in smoothing out Figure C, but for now we want to understand how the shading works. In particular, we'll discuss a basic type of shading called **Lambert shading.** The idea is to shade a portion of the picture based on the size of the angle between the normal to the surface and the line to the light source. The larger the angle, the darker the portion of the picture should be. Explain why this works. For the surface $z = e^{-x^2-y^2}$ (shown in Figures A–C with $-1 \le x \le 1$ and $-1 \le y \le 1$) and a light source at $(0, 0, 100)$, compute the angle at the points $(0, 0, 1)$, $(0, 1, e^{-1})$ and $(1, 0, e^{-1})$. Show that all points with $x^2 + y^2 = 1$ have the same angle and explain why, in terms of the symmetry of the surface. If the position of the light source is changed, will these points remain equally well lit? Based on Figure C, try to determine where the light source is located.

12.7 EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

You have seen optimization problems reappear in a number of places, since we first introduced the idea in section 3.7. In this section, we introduce the mathematical basis for optimizing functions of several variables.

Carefully examine the surface $z = xe^{-x^2/2-y^3/3+y}$, shown in Figure 12.39a for $-2 \le x \le 4$ and $-1 \le y \le 4$. From the graph, notice that you can identify both a peak and a valley. We can zoom in to get a better view of the peak. (See Figure 12.39b for $0.9 \le x \le 1.1$ and $0.9 \le y \le 1.1$.) Referring to Figure 12.39a, we can zoom in to get a better view of the valley. (See Figure 12.39c for $-1.1 \le x \le -0.9$ and $0.9 \le y \le 1.1$.) Such points are referred to as local extrema, which we define as follows.

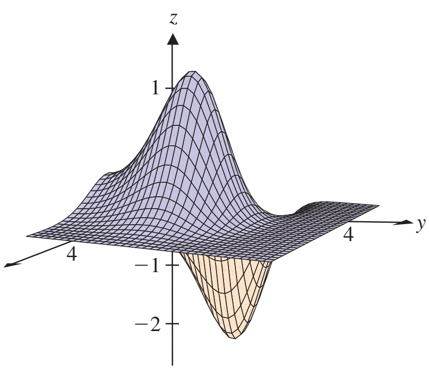


FIGURE 12.39a
$z = xe^{-x^2/2-y^3/3+y}$

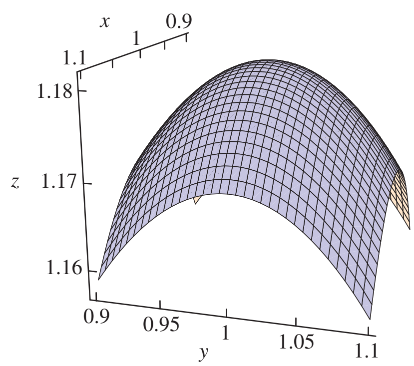


FIGURE 12.39b
Local maximum

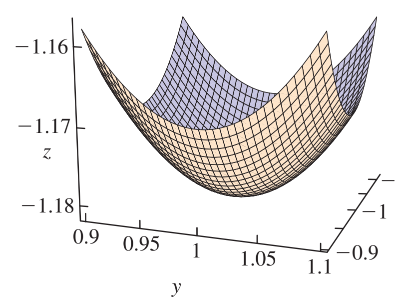


FIGURE 12.39c
Local minimum

DEFINITION 7.1

We call $f(a, b)$ a **local maximum** of f if there is an open disk R centered at (a, b), for which $f(a, b) \ge f(x, y)$ for all $(x, y) \in R$. Similarly, $f(a, b)$ is called a **local minimum** of f if there is an open disk R centered at (a, b), for which $f(a, b) \le f(x, y)$ for all $(x, y) \in R$. In either case, $f(a, b)$ is called a **local extremum** of f.

Note the similarity between Definition 7.1 and the definition of local extrema given in section 3.3. The idea here is the same as it was in Chapter 3. That is, if $f(a, b) \ge f(x, y)$ for all (x, y) "near" (a, b), we call $f(a, b)$ a local maximum.

Look carefully at Figures 12.39b and 12.39c; it appears that at both local extrema, the tangent plane is horizontal. Think about this for a moment and convince yourself that if the tangent plane was tilted, the function would be increasing in one direction and decreasing in another direction, which can't happen at a local extremum (maximum or minimum). Much as with functions of one variable, it turns out that local extrema can occur only where the first (partial) derivatives are zero or do not exist.

DEFINITION 7.2

The point (a, b) is a **critical point** of the function $f(x, y)$ if (a, b) is in the domain of f and either $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$ or one or both of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not exist at (a, b).

Recall that for a function $f(x)$ of a single variable, if f has a local extremum at $x = a$, then a must be a critical number of f [i.e., $f'(a) = 0$ or $f'(a)$ is undefined]. Similarly, if $f(a, b)$ is a local extremum (local maximum or local minimum), then (a, b) must be a critical point of f. Be careful, though; although local extrema can occur only at critical points, every critical point need *not* correspond to a local extremum. For this reason, we refer to critical points as *candidates* for local extrema.

THEOREM 7.1

If $f(x, y)$ has a local extremum at (a, b), then (a, b) must be a critical point of f.

PROOF

Suppose that $f(x, y)$ has a local extremum at (a, b). Holding y constant at $y = b$, notice that the function $g(x) = f(x, b)$ has a local extremum at $x = a$. By Fermat's Theorem (Theorem 3.2 in Chapter 3), either $g'(a) = 0$ or $g'(a)$ doesn't exist. Note that $g'(a) = \frac{\partial f}{\partial x}(a, b)$. Likewise, holding x constant at $x = a$, observe that the function $h(y) = f(a, y)$ has a local extremum at $y = b$. It follows that $h'(b) = 0$ or $h'(b)$ doesn't exist. Note that $h'(b) = \frac{\partial f}{\partial y}(a, b)$. Combining these two observations, we have that each of $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ equals 0 or doesn't exist. We can then conclude that (a, b) must be a critical point of f. ■

When looking for local extrema, you must first find all critical points, since local extrema can occur only at critical points. Then, analyze each critical point to determine whether it is the location of a local maximum, local minimum or neither. We now return to the function $f(x, y) = xe^{-x^2/2-y^3/3+y}$ discussed in the introduction to the section.

EXAMPLE 7.1 Finding Local Extrema Graphically

Find all critical points of $f(x, y) = xe^{-x^2/2-y^3/3+y}$ and analyze each critical point graphically.

Solution First, we compute the first partial derivatives:

$$\frac{\partial f}{\partial x} = e^{-x^2/2-y^3/3+y} + x(-x)e^{-x^2/2-y^3/3+y} = (1 - x^2)e^{-x^2/2-y^3/3+y}$$

and

$$\frac{\partial f}{\partial y} = x(-y^2 + 1)e^{-x^2/2-y^3/3+y}.$$

Since exponentials are always positive, we have $\frac{\partial f}{\partial x} = 0$ if and only if $1 - x^2 = 0$, that is, when $x = \pm 1$. We have $\frac{\partial f}{\partial y} = 0$ if and only if $x(-y^2 + 1) = 0$, that is, when $x = 0$ or $y = \pm 1$. Notice that both partial derivatives exist for all (x, y) and so, the only critical points are solutions of $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. For this to occur, we need $x = \pm 1$ and either $x = 0$ or $y = \pm 1$. However, if $x = 0$, then $\frac{\partial f}{\partial x} \neq 0$, so there are no critical points with $x = 0$. This leaves all combinations of $x = \pm 1$ and $y = \pm 1$ as critical points: $(1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$. Keep in mind that the critical points are only candidates for local extrema; we must look further to determine whether they correspond to extrema. We zoom in on each critical point in turn, to graphically identify any local extrema. We have already seen (see Figures 12.39b and 12.39c) that $f(x, y)$ has a local maximum at $(1, 1)$ and a local minimum at $(-1, 1)$. Figures 12.40a and 12.40b show $z = f(x, y)$ zoomed in on $(1, -1)$ and $(-1, -1)$, respectively. In Figure 12.40a, notice that in the plane $x = 1$ (extending left to right), the point at $(1, -1)$ is a local minimum. However, in the plane $y = -1$ (extending back to front), the point at $(1, -1)$ is a local maximum. This point is therefore not a local extremum. We refer to such a point as a *saddle point*. (It looks like a saddle, doesn't it?) Similarly, in Figure 12.40b, notice that in the plane $x = -1$ (extending left to right), the point at $(-1, -1)$ is a local maximum. However, in the plane $y = -1$ (extending back to front), the point at $(-1, -1)$ is a local minimum. Again, at $(-1, -1)$ we have a saddle point.

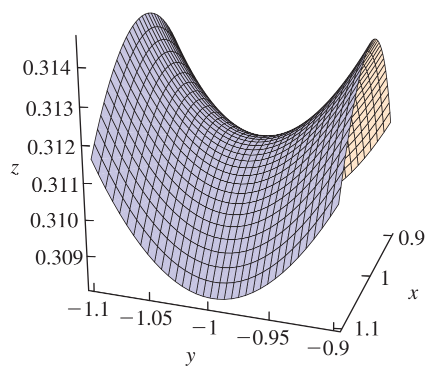


FIGURE 12.40a
Saddle point at $(1, -1)$

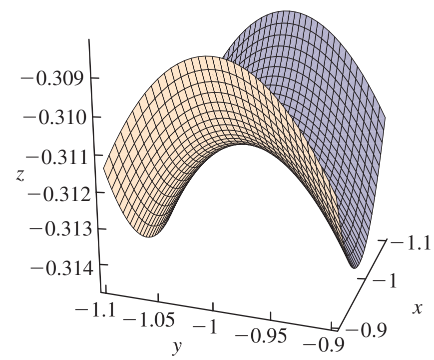


FIGURE 12.40b
Saddle point at $(-1, -1)$

We now pause to carefully define saddle points.

DEFINITION 7.3

The point $P(a, b, f(a, b))$ is a **saddle point** of $z = f(x, y)$ if (a, b) is a critical point of f and if every open disk centered at (a, b) contains points (x, y) in the domain of f for which $f(x, y) < f(a, b)$ and points (x, y) in the domain of f for which $f(x, y) > f(a, b)$.

To further explore example 7.1 graphically, we show a contour plot of $f(x, y) = xe^{-x^2/2 - y^3/3 + y}$ in Figure 12.41. Notice that near the local maximum at $(1, 1)$

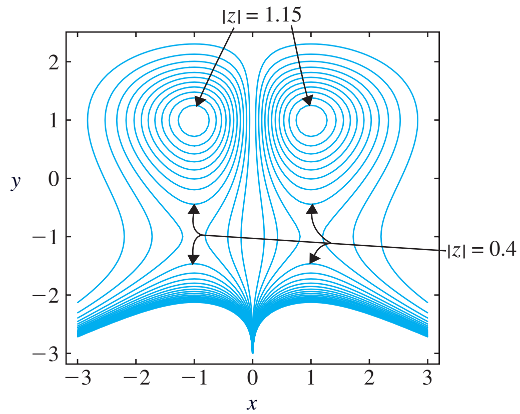


FIGURE 12.41
Contour plot of $f(x, y) = xe^{-x^2/2-y^3/3+y}$

and the local minimum at $(-1, 1)$ the level curves resemble concentric circles. This corresponds to the paraboloid-like shape of the surface near these points (see Figures 12.39b and 12.39c). Concentric ovals are characteristic of local extrema. Notice that, without the level curves labeled, there is no way to tell from the contour plot which is the maximum and which is the minimum. Saddle points are typically characterized by the hyperbolic-looking curves seen near $(-1, -1)$ and $(1, -1)$.

Of course, we can't rely on interpreting three-dimensional graphs for finding local extrema. Recall that for functions of a single variable, we developed two tests (the first derivative test and the second derivative test) for determining when a given critical number corresponds to a local maximum or a local minimum or neither. The following result, which we prove at the end of the section, is surprisingly simple and is a generalization of the second derivative test for functions of a single variable.

THEOREM 7.2 (Second Derivatives Test)

Suppose that $f(x, y)$ has continuous second-order partial derivatives in some open disk containing the point (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Define the **discriminant** D for the point (a, b) by

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(i) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
(ii) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
(iii) If $D(a, b) < 0$, then f has a saddle point at (a, b).
(iv) If $D(a, b) = 0$, then no conclusion can be drawn.

It's important to make some sense of this result (in other words, to understand it and not just memorize it). Note that to have $D(a, b) > 0$, we must have *both* $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$ *or* both $f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$. In the first case, notice that the surface $z = f(x, y)$ will be concave up in the plane $y = b$ and concave up in the plane $x = a$. In this case, the surface will look like an upward-opening paraboloid near the point (a, b). Consequently, f has a local minimum at (a, b). In the second case, both $f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$. This says that the surface $z = f(x, y)$ will be concave down in the plane

$y = b$ and concave down in the plane $x = a$. So, in this case, the surface looks like a downward-opening paraboloid near the point (a, b) and hence, f has a local maximum at (a, b). Observe that one way to get $D(a, b) < 0$ is for $f_{xx}(a, b)$ and $f_{yy}(a, b)$ to have opposite signs (one positive and one negative). To have opposite concavities in the planes $x = a$ and $y = b$ means that there is a saddle point at (a, b), as in Figures 12.40a and 12.40b. We note that having $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$, without having $D(a, b) > 0$ does not say that $f(a, b)$ is a local minimum. We explore this in the exercises.

EXAMPLE 7.2 Using the Discriminant to Find Local Extrema

Locate and classify all critical points for $f(x, y) = 2x^2 - y^3 - 2xy$.

Solution We first compute the first partial derivatives: $f_x = 4x - 2y$ and $f_y = -3y^2 - 2x$. Since both f_x and f_y are defined for all (x, y), the critical points are solutions of the two equations:

$$f_x = 4x - 2y = 0$$

and

$$f_y = -3y^2 - 2x = 0.$$

Solving the first equation for y, we get $y = 2x$. Substituting this into the second equation, we have

$$\begin{aligned} 0 &= -3(4x^2) - 2x = -12x^2 - 2x \\ &= -2x(6x + 1), \end{aligned}$$

so that $x = 0$ or $x = -\frac{1}{6}$. The corresponding y-values are $y = 0$ and $y = -\frac{1}{3}$. The only two critical points are then $(0, 0)$ and $\left(-\frac{1}{6}, -\frac{1}{3}\right)$. To classify these points, we first compute the second partial derivatives: $f_{xx} = 4$, $f_{yy} = -6y$ and $f_{xy} = -2$, and then test the discriminant. We have

$$D(0, 0) = (4)(0) - (-2)^2 = -4 < 0$$

and

$$D\left(-\tfrac{1}{6}, -\tfrac{1}{3}\right) = (4)(2) - (-2)^2 = 4 > 0.$$

From Theorem 7.2, we conclude that there is a saddle point of f at $(0, 0)$, since $D(0, 0) < 0$. Further, there is a local minimum at $\left(-\frac{1}{6}, -\frac{1}{3}\right)$ since $D\left(-\frac{1}{6}, -\frac{1}{3}\right) > 0$ and $f_{xx}\left(-\frac{1}{6}, -\frac{1}{3}\right) > 0$. The surface is shown in Figure 12.42. ■

Point	$(0, 0)$	$\left(-\frac{1}{6}, -\frac{1}{3}\right)$
$f_{xx} = 4$	4	4
$f_{yy} = -6y$	0	2
$f_{xy} = -2$	−2	−2
$D(a, b)$	−4	4

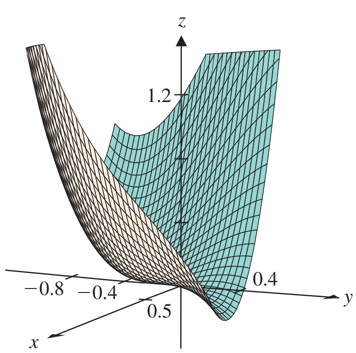


FIGURE 12.42
$z = 2x^2 - y^3 - 2xy$

As we see in example 7.3, the second derivatives test does not always help us to classify a critical point.

EXAMPLE 7.3 Classifying Critical Points

Locate and classify all critical points for $f(x, y) = x^3 - 2y^2 - 2y^4 + 3x^2y$.

Solution Here, we have $f_x = 3x^2 + 6xy$ and $f_y = -4y - 8y^3 + 3x^2$. Since both f_x and f_y exist for all (x, y), the critical points are solutions of the two equations:

$$f_x = 3x^2 + 6xy = 0$$

and

$$f_y = -4y - 8y^3 + 3x^2 = 0.$$

From the first equation, we have

$$0 = 3x^2 + 6xy = 3x(x + 2y),$$

so that at a critical point, $x = 0$ or $x = -2y$. Substituting $x = 0$ into the second equation, we have

$$0 = -4y - 8y^3 = -4y(1 + 2y^2).$$

The only (real) solution of this equation is $y = 0$. This says that for $x = 0$, we have only one critical point: (0, 0). Substituting $x = -2y$ into the second equation, we have

$$0 = -4y - 8y^3 + 3(4y^2) = -4y(1 + 2y^2 - 3y) = -4y(2y - 1)(y - 1).$$

The solutions of this equation are $y = 0$, $y = \frac{1}{2}$ and $y = 1$, with corresponding critical points (0, 0), $\left(-1, \frac{1}{2}\right)$ and $(-2, 1)$. To classify the critical points, we compute the second partial derivatives, $f_{xx} = 6x + 6y$, $f_{yy} = -4 - 24y^2$ and $f_{xy} = 6x$, and evaluate the discriminant at each critical point. We have

$$D(0, 0) = (0)(-4) - (0)^2 = 0,$$

$$D\left(-1, \frac{1}{2}\right) = (-3)(-10) - (-6)^2 = -6 < 0$$

and

$$D(-2, 1) = (-6)(-28) - (-12)^2 = 24 > 0.$$

From Theorem 7.2, we conclude that f has a saddle point at $\left(-1, \frac{1}{2}\right)$, since $D\left(-1, \frac{1}{2}\right) < 0$. Further, f has a local maximum at $(-2, 1)$ since $D(-2, 1) > 0$ and $f_{xx}(-2, 1) < 0$. Unfortunately, Theorem 7.2 gives us no information about the critical point (0, 0), since $D(0, 0) = 0$. However, notice that in the plane $y = 0$ we have $f(x, y) = x^3$. In two dimensions, the curve $z = x^3$ has an inflection point at $x = 0$. This shows that there is no local extremum at (0, 0). The surface near (0, 0) is shown in Figure 12.43a. The surface near $(-2, 1)$ and $\left(-1, \frac{1}{2}\right)$ is shown in Figures 12.43b and 12.43c, respectively. Since the graphs are not especially clear, it's good that we have done the analysis!

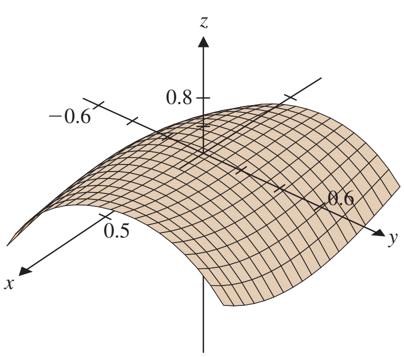


FIGURE 12.43a
$z = f(x, y)$ near (0, 0)

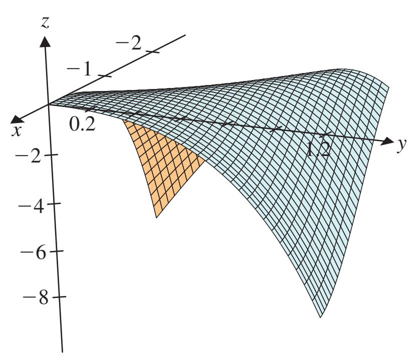


FIGURE 12.43b
$z = f(x, y)$ near $(-2, 1)$

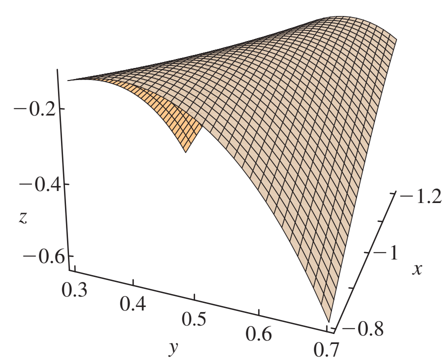


FIGURE 12.43c
$z = f(x, y)$ near $\left(-1, \frac{1}{2}\right)$

One commonly used application of the theory of local extrema is the statistical technique of **least squares.** This technique (or, more accurately, this criterion) is essential to many commonly used curve-fitting and data analysis procedures. The following example illustrates the use of least squares in **linear regression.**

EXAMPLE 7.4 Linear Regression

Population data from the U.S. census are shown in the following table.

Year	*Population*
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,709,873

Find the straight line that "best" fits the data.

x	y
0	179
1	203
2	227
3	249

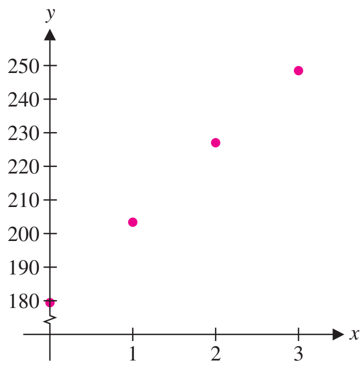


FIGURE 12.44
U.S. population since 1960 (in millions)

Solution To make the data more manageable, we first transform the raw data into variables x (the number of decades since 1960) and y (population, in millions of people, rounded off to the nearest whole number). We display the transformed data in the table in the margin. A plot of x and y is shown in Figure 12.44. From the plot, it would appear that the population data are nearly linear. Our goal is to find the line that "best" fits the data. (This is called the **regression line.**) The criterion for "best" fit is the least-squares criterion, as defined below. We take the equation of the line to be $y = ax + b$, with constants a and b to be determined. For a value of x represented in the data, the error (or **residual**) is given by the difference between the actual y-value and the predicted value $ax + b$. The least-squares criterion is to choose a and b to minimize the sum of the squares of all the residuals. (In a sense, this minimizes the total error.) For the given data, the residuals are shown in the following table.

x	$ax + b$	y	*Residual*
0	b	179	$b - 179$
1	$a + b$	203	$a + b - 203$
2	$2a + b$	227	$2a + b - 227$
3	$3a + b$	249	$3a + b - 249$

The sum of the squares of the residuals is then given by the function

$$f(a, b) = (b - 179)^2 + (a + b - 203)^2 + (2a + b - 227)^2 + (3a + b - 249)^2.$$

From Theorem 7.1, we must have $\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0$ at the minimum point, since f_a and f_b are defined everywhere. We have

$$0 = \frac{\partial f}{\partial a} = 2(a + b - 203) + 4(2a + b - 227) + 6(3a + b - 249)$$

and

$$0 = \frac{\partial f}{\partial b} = 2(b - 179) + 2(a + b - 203) + 2(2a + b - 227) + 2(3a + b - 249).$$

After multiplying out all terms, we have

$$28a + 12b = 2808$$

and

$$12a + 8b = 1716.$$

The second equation reduces to $3a + 2b = 429$, so that $a = 143 - \frac{2}{3}b$. Substituting this into the first equation, we have

$$28\left(143 - \frac{2}{3}b\right) + 12b = 2808,$$

or

$$4004 - 2808 = \left(\frac{56}{3} - 12\right)b.$$

This gives us $b = \frac{897}{5} = 179.4$, so that

$$a = 143 - \frac{2}{3}\left(\frac{897}{5}\right) = \frac{117}{5} = 23.4.$$

The regression line with these coefficients is

$$y = 23.4x + 179.4.$$

Realize that all we have determined so far is that (a, b) is a critical point, a candidate for a local extremum. To verify that our choice of a and b gives the *minimum* function value, note that the surface $z = f(x, y)$ is a paraboloid opening toward the positive z-axis (see Figure 12.45) and the only critical point of an upward-curving paraboloid is an absolute minimum. Alternatively, you can show that $D(a, b) = 80 > 0$ and $f_{aa} > 0$. A plot of the regression line $y = 23.4x + 179.4$ with the data points is shown in Figure 12.46. Look carefully and notice that the line matches the data quite well. This also gives us confidence that we have found the minimum sum of the squared residuals. ■

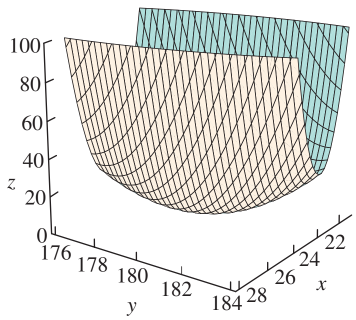


FIGURE 12.45
$z = f(x, y)$

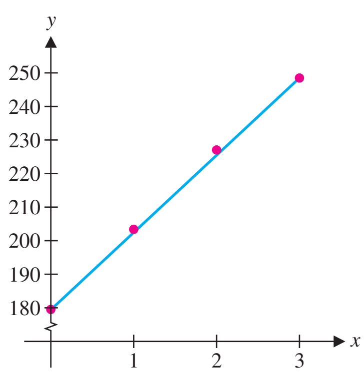


FIGURE 12.46
The regression line

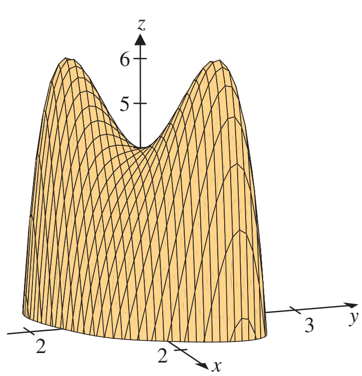


FIGURE 12.47
$z = 4xy - x^4 - y^4 + 4$

As you will see in the exercises, finding critical points of even simple functions of several variables can be challenging. For the complicated functions that often arise in applications, finding critical points by hand can be nearly impossible. Because of this, numerical procedures for estimating maxima and minima are essential. We briefly introduce one such method here.

Given a function $f(x, y)$, make your best guess (x_0, y_0) of the location of a local maximum (or minimum). We call this your **initial guess** and want to use this to obtain a more precise estimate of the location of the maximum (or minimum). How might we do that? Well, recall that the direction of maximum increase of the function from the point (x_0, y_0) is given by the gradient $\nabla f(x_0, y_0)$. So, starting at (x_0, y_0), if we move in the direction of $\nabla f(x_0, y_0)$, f should be increasing, but how far should we go in this direction? One strategy (the method of **steepest ascent**) is to continue moving in the direction of the gradient until the function stops increasing. We call this stopping point (x_1, y_1). Starting anew from (x_1, y_1), we repeat the process, by computing a new gradient $\nabla f(x_1, y_1)$ and following it until $f(x, y)$ stops increasing, at some point (x_2, y_2). We then continue this process until the change in function values from $f(x_n, y_n)$ to $f(x_{n+1}, y_{n+1})$ is insignificant. Likewise, to find a local minimum, follow the path of **steepest descent,** by moving in the direction opposite the gradient, $-\nabla f(x_0, y_0)$ (the direction of maximum decrease of the function). We illustrate the steepest ascent algorithm in example 7.5.

EXAMPLE 7.5 Method of Steepest Ascent

Use the steepest ascent algorithm to estimate the maximum of $f(x, y) = 4xy - x^4 - y^4 + 4$ in the first octant.

Solution A sketch of the surface is shown in Figure 12.47. We will estimate the maximum on the right by starting with an initial guess of (2, 3), where $f(2, 3) = -69$.

(Note that this is obviously not the maximum, but it will suffice as a crude initial guess.) From this point, we want to follow the path of steepest ascent and move in the direction of $\nabla f(2, 3)$. We have

$$\nabla f(x, y) = \langle 4y - 4x^3, 4x - 4y^3 \rangle$$

and so, $\nabla f(2, 3) = \langle -20, -100 \rangle$. Note that every point lying on the line through (2, 3) in the direction of $\langle -20, -100 \rangle$ will have the form $(2 - 20h, 3 - 100h)$, for some value of $h > 0$. (Think about this!) Our goal is to move in this direction until $f(x, y)$ stops increasing. Notice that this puts us at a critical point for function values on the line of points $(2 - 20h, 3 - 100h)$. Since the function values along this line are given by $g(h) = f(2 - 20h, 3 - 100h)$, we find the smallest positive h such that $g'(h) = 0$. From the chain rule, we have

$$\begin{aligned} g'(h) &= -20\frac{\partial f}{\partial x}(2 - 20h, 3 - 100h) - 100\frac{\partial f}{\partial y}(2 - 20h, 3 - 100h) \\ &= -20[4(3 - 100h) - 4(2 - 20h)^3] - 100[4(2 - 20h) - 4(3 - 100h)^3]. \end{aligned}$$

Solving the equation $g'(h) = 0$ (we did it numerically), we get $h \approx 0.02$. This moves us to the point $(x_1, y_1) = (2 - 20h, 3 - 100h) = (1.6, 1)$, with function value $f(x_1, y_1) = 2.8464$. A contour plot of $f(x, y)$ with this first step is shown in Figure 12.48a. Notice that since $f(x_1, y_1) > f(x_0, y_0)$, we have found an improved approximation of the local maximum. To improve this further, we repeat the process starting with the new point. In this case, we have $\nabla f(1.6, 1) = \langle -12.384, 2.4 \rangle$ and we look for a critical point for the new function $g(h) = f(1.6 - 12.384h, 1 + 2.4h)$, for $h > 0$. Again, from the chain rule, we have

$$g'(h) = -12.384\frac{\partial f}{\partial x}(1.6 - 12.384h, 1 + 2.4h) + 2.4\frac{\partial f}{\partial y}(1.6 - 12.384h, 1 + 2.4h).$$

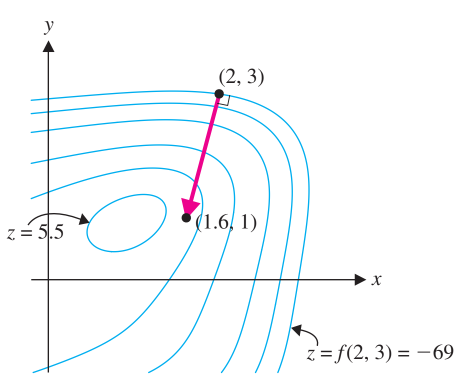


FIGURE 12.48a
First step of steepest ascent

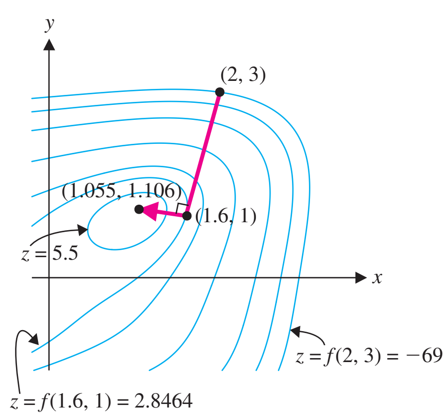


FIGURE 12.48b
Second step of steepest ascent

Solving $g'(h) = 0$ numerically gives us $h \approx 0.044$. This moves us to the point $(x_2, y_2) = (1.6 - 12.384h, 1 + 2.4h) = (1.055, 1.106)$, with function value $f(x_2, y_2) = 5.932$. Notice that we have again improved our approximation of the local maximum. A contour plot of $f(x, y)$ with the first two steps is shown in Figure 12.48b. From the contour plot, it appears that we are now very near a local maximum. In practice, you continue this process until you are no longer improving the approximation significantly. (This is easily implemented on a computer.) In the accompanying table, we show the first seven steps of steepest ascent. We leave it as an exercise to show that the local maximum is actually at (1, 1) with function value $f(1, 1) = 6$.

n	x_n	y_n	$f(x_n, y_n)$
0	2	3	−69
1	1.6	1	2.846
2	1.055	1.106	5.932
3	1.0315	1.0035	5.994
4	1.0049	1.0094	5.9995
5	1.0029	1.0003	5.99995
6	1.0005	1.0009	5.999995
7	1.0003	1.0003	5.9999993

We define absolute extrema in a similar fashion to local extrema.

DEFINITION 7.4

We call $f(a, b)$ the **absolute maximum** of f on the region R if $f(a, b) \geq f(x, y)$ for all $(x, y) \in R$. Similarly, $f(a, b)$ is called the **absolute minimum** of f on R if $f(a, b) \leq f(x, y)$ for all $(x, y) \in R$. In either case, $f(a, b)$ is called an **absolute extremum** of f.

Recall that for a function f of a single variable, we observed that whenever f is continuous on the closed interval $[a, b]$, it will assume a maximum and minimum value on $[a, b]$. Further, we proved that absolute extrema must occur at either critical numbers of f or at the endpoints of the interval $[a, b]$. The situation for absolute extrema of functions of two variables is very similar. First, we need some terminology. We say that a region $R \subset \mathbb{R}^2$ is **bounded** if there is a disk that completely contains R. We now have the following result (whose proof can be found in more advanced texts).

THEOREM 7.3 (Extreme Value Theorem)

Suppose that $f(x, y)$ is continuous on the closed and bounded region $R \subset \mathbb{R}^2$. Then f has both an absolute maximum and an absolute minimum on R. Further, an absolute extremum may only occur at a critical point in R or at a point on the boundary of R.

Note that if $f(a, b)$ is an absolute extremum of f in R and (a, b) is in the interior of R, then (a, b) is also a local extremum of f, in which case, (a, b) must be a critical point. This says that all of the absolute extrema of a function f in a region R occur either at critical points (and we already know how to find these) or on the boundary of the region. Observe that this also provides us with a method for locating absolute extrema of continuous functions on closed and bounded regions. That is, we find the extrema on the boundary and compare these against the local extrema. We examine this in example 7.6, where the basic steps are as follows:

- Find all critical points of f in the region R.
- Find the maximum and minimum values of f on the boundary of R.
- Compare the values of f at the critical points with the maximum and minimum values of f on the boundary of R.

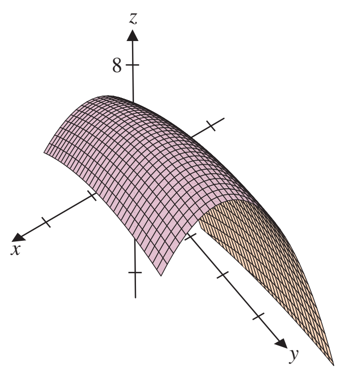


FIGURE 12.49a
The surface
$z = 5 + 4x - 2x^2 + 3y - y^2$

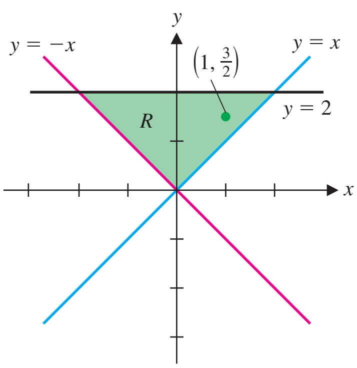


FIGURE 12.49b
The region R

EXAMPLE 7.6 Finding Absolute Extrema

Find the absolute extrema of $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$ on the region R bounded by the lines $y = 2$, $y = x$ and $y = -x$.

Solution We show a sketch of the surface in Figure 12.49a and a sketch of the region R in Figure 12.49b. From the sketch of the surface, notice that the absolute minimum appears to occur on the line $x = -2$ and the absolute maximum occurs somewhere near the line $x = 1$. Since an extremum can occur only at a critical point or at a point on the boundary of R, we first check to see whether there are any interior critical points. We have $f_x = 4 - 4x = 0$ for $x = 1$ and $f_y = 3 - 2y = 0$ for $y = \frac{3}{2}$. So, there is only one critical point $\left(1, \frac{3}{2}\right)$ and it is located in the interior of R. Next, we look for the maximum and minimum values of f on the boundary of R. In this case, R consists of three separate pieces: the portion of the line $y = 2$ for $-2 \le x \le 2$, the portion of the line $y = x$ for $0 \le x \le 2$ and the portion of the line $y = -x$ for $-2 \le x \le 0$. We look for the maximum value of f on each of these separately. On the portion of the line $y = 2$ for $-2 \le x \le 2$, we have

$$f(x, y) = f(x, 2) = 5 + 4x - 2x^2 + 6 - 4 = 7 + 4x - 2x^2 = g(x).$$

To find the maximum and minimum values of f on this portion of the boundary, we need only find the maximum and minimum values of g on the interval $[-2, 2]$. We have $g'(x) = 4 - 4x = 0$ only for $x = 1$. Comparing the value of g at the endpoints and at the only critical number in the interval, we have: $g(-2) = -9$, $g(2) = 7$ and $g(1) = 9$. So, the maximum value of f on this portion of the boundary is 9 and the minimum value is -9.

On the portion of the line $y = x$ for $0 \le x \le 2$, we have

$$f(x, y) = f(x, x) = 5 + 7x - 3x^2 = h(x).$$

We have $h'(x) = 7 - 6x = 0$, only for $x = \frac{7}{6}$, which is in the interval. Comparing the values of h at the endpoints and the critical number, we have: $h(0) = 5$, $h(2) = 7$ and $h\left(\frac{7}{6}\right) \approx 9.08$. So, the maximum value of f on this portion of the boundary is approximately 9.08 and its minimum value is 5.

On the portion of the line $y = -x$ for $-2 \le x \le 0$, we have

$$f(x, y) = f(x, -x) = 5 + x - 3x^2 = k(x).$$

We have $k'(x) = 1 - 6x = 0$, only for $x = \frac{1}{6}$, which is *not* in the interval $[-2, 0]$ under consideration. Comparing the values of k at the endpoints, we have $k(-2) = -9$ and $k(0) = 5$, so that the maximum value of f on this portion of the boundary is 5 and its minimum value is -9.

Finally, we compute the value of f at the lone critical point in the interior of R: $f\left(1, \frac{3}{2}\right) = \frac{37}{4} = 9.25$. The largest of all these values we have computed is the absolute maximum in R and the smallest is the absolute minimum. So, the absolute maximum is $f\left(1, \frac{3}{2}\right) = 9.25$ and the absolute minimum is $f(-2, 2) = -9$. Note that these are also consistent with what we observed in Figure 12.49a. ■

We close this section with a proof of the Second Derivatives Test (Theorem 7.2). To keep the notation to a minimum, we will assume that the critical point to be tested is (0, 0). The proof can be extended to any critical point by a change of variables.

Proof of the Second Derivatives Test

Suppose that (0, 0) is a critical point of $f(x, y)$ with $f_x(0, 0) = f_y(0, 0) = 0$. We will look at the change in $f(x, y)$ from (0, 0) in the direction of the unit vector $\mathbf{u} = \dfrac{\langle k, 1\rangle}{\sqrt{k^2+1}}$, for some constant k. (Note that $\mathbf{u}$ can point in any direction *except* the direction of $\mathbf{i}$.) In this direction, notice that $x = ky$. If we define $g(x) = f(kx, x)$, then by the chain rule, we have

$$g'(x) = kf_x(kx, x) + f_y(kx, x) \tag{7.1}$$

and $$g''(x) = k^2 f_{xx}(kx, x) + kf_{xy}(kx, x) + kf_{yx}(kx, x) + f_{yy}(kx, x).$$

At $x = 0$, this gives us

$$g''(0) = k^2 f_{xx}(0, 0) + 2kf_{xy}(0, 0) + f_{yy}(0, 0), \tag{7.2}$$

where we have $f_{xy} = f_{yx}$, since f was assumed to have continuous second partial derivatives. Since $f_x(0, 0) = f_y(0, 0) = 0$, we have from (7.1) that

$$g'(0) = kf_x(0, 0) + f_y(0, 0) = 0.$$

Using the second derivative test for functions of a single variable, the sign of $g''(0)$ can tell us whether there is a local maximum or a local minimum of g at $x = 0$. Observe that using (7.2), we can write $g''(0)$ as

$$g''(0) = ak^2 + 2bk + c = p(k),$$

where a, b and c are the constants $a = f_{xx}(0, 0)$, $b = f_{xy}(0, 0)$ and $c = f_{yy}(0, 0)$. Of course, the graph of $p(k)$ is a parabola. Recall that for any parabola, if $a > 0$, then $p(k)$ has a minimum at $k = -\frac{b}{a}$, given by $p\left(-\frac{b}{a}\right) = -\frac{b^2}{a} + c$. (Hint: Complete the square.) In case (i) of the theorem, we assume that the discriminant satisfies

$$0 < D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = ac - b^2,$$

so that $-\frac{b^2}{a} + c > 0$. In this case,

$$p(k) \geq p\left(-\frac{b}{a}\right) = -\frac{b^2}{a} + c > 0.$$

We have shown that, in case (i), when $D(0, 0) > 0$ and $f_{xx}(0, 0) > 0$, $g''(0) = p(k) > 0$ for all k. So, g has a local minimum at 0 and consequently, in all directions, the point at (0, 0) is a local minimum of f. For case (ii), where $D(0, 0) > 0$ and $f_{xx}(0, 0) < 0$, we consider $p(k)$ with $a < 0$. In a similar fashion, we can show that here, $p(k) \leq -\frac{b^2}{a} + c < 0$. Given that we have $g''(0) = p(k) < 0$ for all k, we conclude that the point at (0, 0) is a local maximum of f. For case (iii), where the discriminant $D(0, 0) < 0$, the parabola $p(k)$ will assume both positive and negative values. For some values of k, we have $g''(0) > 0$ and the point (0, 0) is a local minimum along the path $x = ky$, while for other values of k, we have $g''(0) < 0$ and the point (0, 0) is a local maximum along the path $x = ky$. Taken together, this says that the point at (0, 0) must be a saddle point of f. To complete the proof, we must only consider the case where $\mathbf{u} = \mathbf{i}$. In this case, the preceding proof is easily revised to show the same results and we leave the details as an exercise.

BEYOND FORMULAS

You can think about local extrema for functions of n variables in the same way for any value of n. Critical points are points where either all of the first partial derivatives are zero or where one is undefined. These provide the candidates for local extrema, in the sense that a local extremum may occur only at a critical point. However, further testing is needed to determine whether a function has a local maximum, a local minimum or neither at a given critical point.

EXERCISES 12.7

WRITING EXERCISES

1. If $f(x, y)$ has a local minimum at (a, b), explain why the point $(a, b, f(a, b))$ is a local minimum in the intersection of $z = f(x, y)$ with any vertical plane passing through the point. Explain why the condition $f_x(a, b) = f_y(a, b) = 0$ guarantees that (a, b) is a critical point in any such plane.
2. Suppose that $f_x(a, b) \neq 0$. Explain why the tangent plane to $z = f(x, y)$ at (a, b) must be "tilted", so that there is not a local extremum at (a, b).
3. Suppose that $f_x(a, b) = f_y(a, b) = 0$ and $f_{xx}(a, b) f_{yy}(a, b) < 0$. Explain why there must be a saddle point at (a, b).
4. Explain why the center of a set of concentric circles in a contour plot will often represent a local extremum.

In exercises 1–8, locate all critical points and classify them using Theorem 7.2.

1. $f(x, y) = e^{-x^2}(y^2 + 1)$
2. $f(x, y) = \cos^2 x + y^2$
3. $f(x, y) = x^3 - 3xy + y^3$
4. $f(x, y) = 4xy - x^4 - y^4 + 4$
5. $f(x, y) = y^2 + x^2 y + x^2 - 2y$
6. $f(x, y) = 2x^2 + y^3 - x^2 y - 3y$
7. $f(x, y) = e^{-x^2-y^2}$
8. $f(x, y) = x \sin y$

In exercises 9–14, locate all critical points and analyze each graphically. If you have a CAS, use Theorem 7.2 to classify each point.

9. $f(x, y) = x^2 - \dfrac{4xy}{y^2 + 1}$
10. $f(x, y) = \dfrac{x + y}{x^2 + y^2 + 1}$
11. $f(x, y) = xe^{-x^2-y^2}$
12. $f(x, y) = x^2 e^{-x^2-y^2}$
13. $f(x, y) = xye^{-x^2-y^2}$
14. $f(x, y) = xye^{-x^2-y^4}$

In exercises 15–18, numerically approximate all critical points. Classify each point graphically or with Theorem 7.2.

15. $f(x, y) = xy^2 - x^2 - y + \frac{1}{16}x^4$
16. $f(x, y) = 2y(x + 2) - x^2 + y^4 - 9y^2$
17. $f(x, y) = (x^2 - y^3)e^{-x^2-y^2}$
18. $f(x, y) = (x^2 - 3x)e^{-x^2-y^2}$
19. Show that for data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the least-squares equations become

$$\left(\sum_{k=1}^{n} x_k\right) a + \left(\sum_{k=1}^{n} 1\right) b = \sum_{k=1}^{n} y_k$$
$$\left(\sum_{k=1}^{n} x_k^2\right) a + \left(\sum_{k=1}^{n} x_k\right) b = \sum_{k=1}^{n} x_k y_k$$

20. Solve the equations in exercise 19 for a and b.

In exercises 21–27, use least squares as in example 7.4 to find a linear model of the data.

21. A famous mental calculation prodigy named Jacques Inaudi was timed in 1894 at various mental arithmetic problems. His times are shown below. (Think about what your times might be!) The data are taken from *The Number Sense* by Stanislas Dehaene. Treat the number of operations as the independent variable (x) and time as the dependent variable (y).

Number of operations	1	4	9	16
Time (sec)	0.6	2.0	6.4	21
Example	$3 \cdot 7$	$63 \cdot 58$	$638 \cdot 823$	$7286 \cdot 5397$

22. Repeat exercise 21 with the following data point added: 36 operations in 240 seconds (an example is $729{,}856 \cdot 297{,}143$). How much effect does this last point have on the linear model?

23. The Dow Jones Industrial averages for several days starting in June 1998 are shown. Use your linear model to predict the average on day 12. Linear models of similar data can be found in information supplied by financial consulting firms, typically with the warning to not use the linear model for forecasting. Explain why this warning is appropriate.

Date (number of days)	0	2	4	6	8
Dow Jones average	8910	8800	9040	9040	9050

24. The following data show the average price of a gallon of regular gasoline in California. Use the linear model to predict the price in 1990 and 1995. The actual prices were \$1.09 and \$1.23. Explain why your forecasts were not accurate.

Year	1970	1975	1980	1985
Price	\$0.34	\$0.59	\$1.23	\$1.11

25. The following data show the height and weight of a small number of people. Use the linear model to predict the weight of a 6′8″ person and a 5′0″ person. Comment on how accurate you think the model is.

Height (inches)	68	70	70	71
Weight (pounds)	160	172	184	180

26. The following data show the age and income for a small number of people. Use the linear model to predict the income of a 45-year-old and of an 80-year-old. Comment on how accurate you think the model is.

Age (years)	24	32	40	56
Income (\$)	30,000	34,000	52,000	82,000

27. The accompanying data show the average number of points professional football teams score when starting different distances from the opponents' goal line. (For more information, see Hal Stern's "A Statistician Reads the Sports Pages" in *Chance,* Summer 1998. The number of points is determined by the next score, so that if the opponent scores next, the number of points is negative.) Use the linear model to predict the average number of points starting (a) 60 yards from the goal line and (b) 40 yards from the goal line.

Yards from goal	15	35	55	75	95
Average points	4.57	3.17	1.54	0.24	−1.25

28. In *The Hidden Game of Pro Football,* authors Carroll, Palmer and Thorn claim that the data presented in exercise 27 support the conclusion that when a team loses a fumble they lose an average of 4 points *regardless of where they are on the field*. That is, a fumble at the 50-yard line costs the same number of points as a fumble at the opponents' 10-yard line. Use your result from exercise 27 to verify this claim.

In exercises 29–32, calculate the first two steps of the steepest ascent algorithm from the given starting point.

29. $f(x, y) = 2xy - 2x^2 + y^3$, $(0, -1)$

30. $f(x, y) = 3xy - x^3 - y^2$, $(1, 1)$

31. $f(x, y) = x - x^2y^4 + y^2$, $(1, 1)$

32. $f(x, y) = xy^2 - x^2 - y$, $(1, 0)$

33. Calculate one step of the steepest ascent algorithm for $f(x, y) = 2xy - 2x^2 + y^3$, starting at $(0, 0)$. Explain in graphical terms what goes wrong.

34. Define a **steepest descent algorithm** for finding local minima.

In exercises 35–38, find the absolute extrema of the function on the region.

35. $f(x, y) = x^2 + 3y - 3xy$, region bounded by $y = x$, $y = 0$ and $x = 2$

36. $f(x, y) = x^2 + y^2 - 4xy$, region bounded by $y = x$, $y = -3$ and $x = 3$

37. $f(x, y) = x^2 + y^2$, region bounded by $(x - 1)^2 + y^2 = 4$

38. $f(x, y) = x^2 + y^2 - 2x - 4y$, region bounded by $y = x$, $y = 3$ and $x = 0$

39. A box is to be constructed out of 96 square feet of material. Find the dimensions x, y and z that maximize the volume of the box.

40. If the bottom of the box in exercise 39 must be reinforced by doubling up the material (essentially, there are two bottoms of the box), find the dimensions that maximize the volume of the box.

41. Heron's formula gives the area of a triangle with sides of lengths a, b and c as $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$. For a given perimeter, find the triangle of maximum area.

42. Find the maximum of $x^2 + y^2$ on the square with $-1 \le x \le 1$ and $-1 \le y \le 1$. Use your result to explain why a computer graph of $z = x^2 + y^2$ with the graphing window $-1 \le x \le 1$ and $-1 \le y \le 1$ does not show a circular cross section at the top.

43. Find all critical points of $f(x, y) = x^2y^2$ and show that Theorem 7.2 fails to identify any of them. Use the form of the function to determine what each critical point represents. Repeat for $f(x, y) = x^{2/3}y^2$.

44. Complete the square to identify all local extrema of (a) $f(x, y) = x^2 + 2x + y^2 - 4y + 1$, (b) $f(x, y) = x^4 - 6x^2 + y^4 + 2y^2 - 1$.

45. In exercise 3, there is a saddle point at (0, 0). This means that there is (at least) one trace of $z = x^3 - 3xy + y^3$ with a local minimum at (0, 0) and (at least) one trace with a local maximum at (0, 0). To analyze traces in the planes $y = kx$ (for some constant k), substitute $y = kx$ and show that $z = (1 + k^3)x^3 - 3kx^2$. Show that $f(x) = (1 + k^3)x^3 - 3kx^2$ has a local minimum at $x = 0$ if $k < 0$ and a local maximum at $x = 0$ if $k > 0$. (Hint: Use the Second Derivative Test from section 3.5.)

46. In exercise 4, there is a saddle point at (0, 0). As in exercise 45, find traces such that there is a local maximum at (0, 0) and traces such that there is a local minimum at (0, 0).

47. In example 7.3, (0, 0) is a critical point but is not classified by Theorem 7.2. Use the technique of exercise 45 to analyze this saddle point.

48. Repeat exercise 47 for $f(x, y) = x^2 - 3xy^2 + 4x^3y$.

49. For $f(x, y, z) = xz - x + y^3 - 3y$, show that (0, 1, 1) is a critical point. To classify this critical point, show that $f(0 + \Delta x, 1 + \Delta y, 1 + \Delta z) = \Delta x \Delta z + 3\Delta y^2 + \Delta y^3 + f(0, 1, 1)$. Setting $\Delta y = 0$ and $\Delta x \Delta z > 0$, conclude that $f(0, 1, 1)$ is not a local maximum. Setting $\Delta y = 0$ and $\Delta x \Delta z < 0$, conclude that $f(0, 1, 1)$ is not a local minimum.

50. Repeat exercise 49 for the point (0, −1, 1).

In exercises 51–54, label the statement as true or false and explain why.

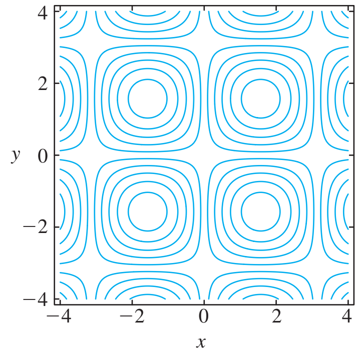
51. If $f(x, y)$ has a local maximum at (a, b), then $\dfrac{\partial f}{\partial x}(a, b) = \dfrac{\partial f}{\partial y}(a, b) = 0$.

52. If $\dfrac{\partial f}{\partial x}(a, b) = \dfrac{\partial f}{\partial y}(a, b) = 0$, then $f(x, y)$ has a local maximum at (a, b).

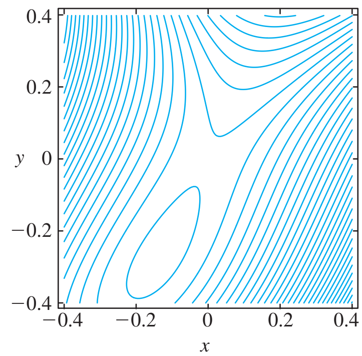
53. In between any two local maxima of $f(x, y)$ there must be at least one local minimum of $f(x, y)$.

54. If $f(x, y)$ has exactly two critical points, they can't both be local maxima.

55. In the contour plot, the locations of four local extrema and nine saddle points are visible. Identify these critical points.

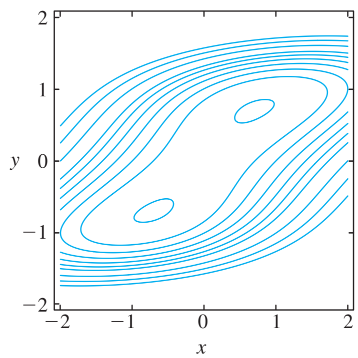


56. In the contour plot, the locations of one local extremum and one saddle point are visible. Identify each critical point.



In exercises 57–60, use the contour plot to conjecture the locations of all local extrema and saddle points.

57.



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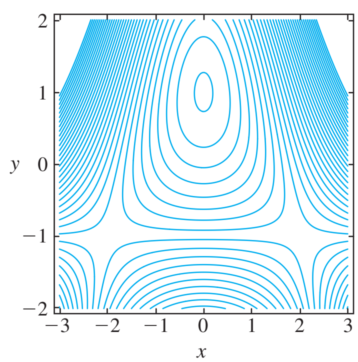

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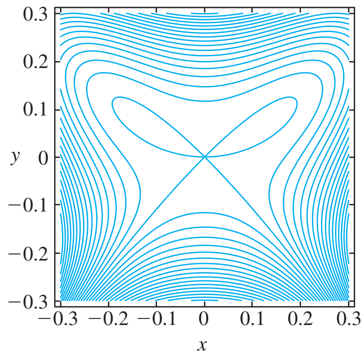
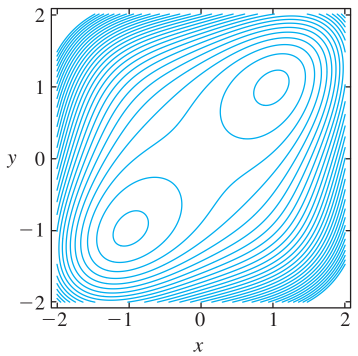


60.



61. Construct the function $d(x, y)$ giving the distance from a point (x, y, z) on the paraboloid $z = 4 - x^2 - y^2$ to the point $(3, -2, 1)$. Then determine the point that minimizes $d(x, y)$.

62. Use the method of exercise 61 to find the closest point on the cone $z = \sqrt{x^2 + y^2}$ to the point $(2, -3, 0)$.

63. Use the method of exercise 61 to find the closest point on the sphere $x^2 + y^2 + z^2 = 9$ to the point $(2, 1, -3)$.

64. Use the method of exercise 61 to find the closest point on the plane $3x - 4y + 3z = 12$ to the origin.

65. Show that the function $f(x, y) = 5xe^y - x^5 - e^{5y}$ has exactly one critical point, which is a local maximum but not an absolute maximum.

66. Show that the function $f(x, y) = 2x^4 + e^{4y} - 4x^2e^y$ has exactly two critical points, both of which are local minima.

67. Prove that the situation of exercise 66 (two local minima without a local maximum) can never occur for differentiable functions of one variable.

68. The **Hardy-Weinberg law** of genetics describes the relationship between the proportions of different genes in populations. Suppose that a certain gene has three types (e.g., blood types of A, B and O). If the three types have proportions p, q and r, respectively, in the population, then the Hardy-Weinberg law states that the proportion of people who carry two different types of genes equals $f(p, q, r) = 2pq + 2pr + 2qr$. Explain why $p + q + r = 1$ and then show that the maximum value of $f(p, q, r)$ is $\frac{2}{3}$.

EXPLORATORY EXERCISES

1. In example 7.4, we found the "best" linear fit to population data using the least-squares criterion. Use the least-squares criterion to find the best quadratic fit to the data. That is, for functions of the form $ax^2 + bx + c$, find the values of the constants a, b and c that minimize the sum of the squares of the residuals. For the given data, show that the sum of the squares of the residuals for the quadratic model is less than for the linear model. Explain why this has to be true mathematically. In spite of this, explain why the linear model might be preferable to the quadratic model. (Hint: Use both models to predict 100 years into the future and backtrack 100 years into the past.)

2. Use the least-squares criterion to find the best exponential fit ($y = ae^{bx}$) to the data of example 7.4. Compare the actual residuals of this model to the actual residuals of the linear model. Explain why there is some theoretical justification for using an exponential model, and then discuss the advantages and disadvantages of the exponential and linear models.

3. A practical flaw with the method of steepest ascent presented in example 7.5 is that the equation $g'(h) = 0$ may be difficult to solve. An alternative is to use Newton's method to approximate a solution. A method commonly used in practice is to approximate h using one iteration of Newton's method with initial guess $h = 0$. We derive the resulting formula here. Recall that $g(h) = f(x_k + ah, y_k + bh)$, where (x_k, y_k) is the current point and $\langle a, b\rangle = \nabla f(x_k, y_k)$. Newton's method applied to $g'(h) = 0$ with $h_0 = 0$ is given by $h_1 = -\dfrac{g'(0)}{g''(0)}$. Show that $g'(0) = af_x(x_k, y_k) + bf_y(x_k, y_k) = a^2 + b^2 = \nabla f(x_k, y_k) \cdot \nabla f(x_k, y_k)$. Also, show that $g''(0) = a^2 f_{xx}(x_k, y_k) + 2abf_{xy}(x_k, y_k) + b^2 f_{yy}(x_k, y_k) = \nabla f(x_k, y_k) \cdot H(x_k, y_k)\nabla f(x_k, y_k)$, where the Hessian matrix is defined by $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. Putting this together with the work in example 7.5, the method of steepest ascent becomes $\mathbf{v}_{k+1} = \mathbf{v}_k - \dfrac{\nabla f(\mathbf{v}_k) \cdot \nabla f(\mathbf{v}_k)}{\nabla f(\mathbf{v}_k) \cdot H(\mathbf{v}_k)\nabla f(\mathbf{v}_k)} \nabla f(\mathbf{v}_k)$, where $\mathbf{v}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$.

12.8 CONSTRAINED OPTIMIZATION AND LAGRANGE MULTIPLIERS

The local extrema we found in section 12.7 form only one piece of the optimization puzzle. In many applications, the goal is not to identify theoretical maximum or minimum values, but to achieve the absolute best possible product given a large set of constraints such as limited resources or technology. For example, an automotive engineer's objective might be to minimize wind drag in the design of a car. However, new designs are severely limited by customer demands of luxury and attractiveness and particularly by manufacturing constraints such as cost. In this section, we develop a technique for finding the maximum or minimum of a function, given one or more constraints on the function's domain.

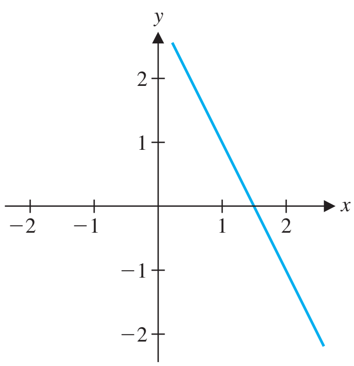


FIGURE 12.50a
$y = 3 - 2x$

We first consider the two-dimensional geometric problem of finding the point on the line $y = 3 - 2x$ that is closest to the origin. A graph of the line is shown in Figure 12.50a. Notice that the set of points that are 1 unit from the origin form the circle $x^2 + y^2 = 1$. In Figure 12.50b, you can see that the line $y = 3 - 2x$ lies entirely outside this circle. This tells us that every point on the line $y = 3 - 2x$ lies more than 1 unit from the origin. Looking at the circle $x^2 + y^2 = 4$ in Figure 12.50c, you can clearly see that there are infinitely many points on the line that are less than 2 units from the origin. If we shrink the circle in Figure 12.50c (or enlarge the circle in Figure 12.50b), it will eventually reach a size at which the line is tangent to the circle (see Figure 12.50d). The point of tangency is the closest point on the line to the origin, since all other points on the line are outside the circle and hence, are farther away from the origin.

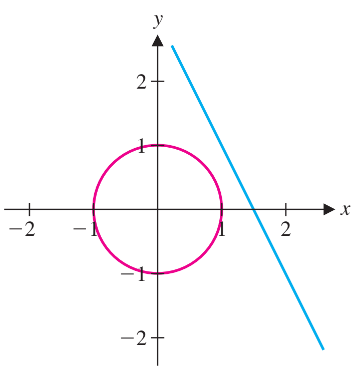


FIGURE 12.50b
$y = 3 - 2x$ and the circle of radius 1 centered at (0, 0)

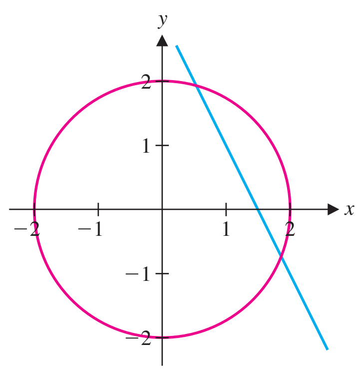


FIGURE 12.50c
$y = 3 - 2x$ and the circle of radius 2 centered at (0, 0)

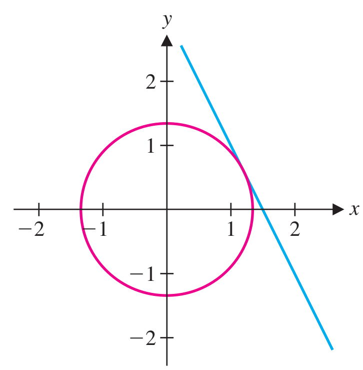


FIGURE 12.50d
$y = 3 - 2x$ and a circle tangent to the line

Translating the preceding geometric argument into the language of calculus, we want to minimize the distance from the point (x, y) to the origin, given by $\sqrt{x^2 + y^2}$. Before we continue, observe that the distance is minimized at exactly the same point at which the square of the distance is minimized. Minimizing the square of the distance, given by $x^2 + y^2$, avoids the mess created by the square root in the distance formula. So, instead, we minimize $f(x, y) = x^2 + y^2$, subject to the constraint that the point lie on the line (i.e., that $y = 3 - 2x$) or $g(x, y) = 2x + y - 3 = 0$. We have already argued that at the closest point, the line and circle are tangent. Since the gradient vector for a given function is orthogonal to its level curves at any given point, for a level curve of f to be tangent to the constraint curve $g(x, y) = 0$, the gradients of f and g must be parallel. That is, at the closest point (x, y) on the line to the origin, we must have $\nabla f(x, y) = \lambda \nabla g(x, y)$, for some constant λ. We illustrate this in example 8.1.

EXAMPLE 8.1 Finding a Minimum Distance

Use the relationship $\nabla f(x, y) = \lambda \nabla g(x, y)$ and the constraint $y = 3 - 2x$ to find the point on the line $y = 3 - 2x$ that is closest to the origin.

Solution For $f(x, y) = x^2 + y^2$, we have $\nabla f(x, y) = \langle 2x, 2y \rangle$ and for $g(x, y) = 2x + y - 3$, we have $\nabla g(x, y) = \langle 2, 1 \rangle$. The vector equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ becomes

$$\langle 2x, 2y \rangle = \lambda \langle 2, 1 \rangle,$$

from which it follows that

$$2x = 2\lambda \quad \text{and} \quad 2y = \lambda.$$

The second equation gives us $\lambda = 2y$. The first equation then gives us $x = \lambda = 2y$. Substituting $x = 2y$ into the constraint equation $y = 3 - 2x$, we have $y = 3 - 2(2y)$, or $5y = 3$. The solution is $y = \frac{3}{5}$, giving us $x = 2y = \frac{6}{5}$. The closest point is then $\left(\frac{6}{5}, \frac{3}{5}\right)$. Look carefully at Figure 12.50d and recognize that this is consistent with our graphical solution. Also, note that the line described parametrically by $x = \lambda$, $y = \frac{\lambda}{2}$ is the line through the origin and perpendicular to $y = 3 - 2x$. ■

HISTORICAL NOTES

Joseph-Louis Lagrange (1736–1813) Mathematician who developed many fundamental techniques in the calculus of variations, including the method that bears his name. Lagrange was largely self-taught, but quickly attracted the attention of the great mathematician Leonhard Euler. At age 19, Lagrange was appointed Professor of Mathematics at the Royal Artillery School in his native Turin. Over a long and outstanding career, Lagrange made contributions to probability, differential equations and fluid mechanics, for which he introduced what is now known as the Lagrangian function.

The technique illustrated in example 8.1 can be applied to a wide variety of constrained optimization problems. We will now develop this method, referred to as the **method of Lagrange multipliers.**

Suppose that we want to find maximum or minimum values of the function $f(x, y, z)$, subject to the constraint that $g(x, y, z) = 0$. We assume that both f and g have continuous first partial derivatives. Now, suppose that f has an extremum at (x_0, y_0, z_0) lying on the level surface S defined by $g(x, y, z) = 0$. Let C be any curve lying on the level surface and passing through the point (x_0, y_0, z_0). Assume that C is traced out by the terminal point of the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Define a function of the single variable t by

$$h(t) = f(x(t), y(t), z(t)).$$

Notice that if $f(x, y, z)$ has an extremum at (x_0, y_0, z_0), then $h(t)$ must have an extremum at t_0 and so, $h'(t_0) = 0$. From the chain rule, we get that

$$\begin{aligned} 0 = h'(t_0) &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle \cdot \langle x'(t_0), y'(t_0), z'(t_0) \rangle \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0). \end{aligned}$$

That is, if $f(x_0, y_0, z_0)$ is an extremum, the gradient of f at (x_0, y_0, z_0) is orthogonal to the tangent vector $\mathbf{r}'(t_0)$. Since C was an arbitrary curve lying on the level surface S, it follows that $\nabla f(x_0, y_0, z_0)$ must be orthogonal to every curve lying on the level surface S and so, too is orthogonal to S. Recall from Theorem 6.5 that ∇g is also orthogonal to the level surface $g(x, y, z) = 0$, so that $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. This proves the following result.

THEOREM 8.1

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are functions with continuous first partial derivatives and $\nabla g(x, y, z) \neq \mathbf{0}$ on the surface $g(x, y, z) = 0$. Suppose that either

(i) the minimum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0); or
(ii) the maximum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0).

Then $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$, for some constant λ (called a **Lagrange multiplier**).

Note that Theorem 8.1 says that if $f(x, y, z)$ has an extremum at a point (x_0, y_0, z_0) on the surface $g(x, y, z) = 0$, we will have for $(x, y, z) = (x_0, y_0, z_0)$,

$$f_x(x, y, z) = \lambda g_x(x, y, z),$$
$$f_y(x, y, z) = \lambda g_y(x, y, z),$$
$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

and $$g(x, y, z) = 0.$$

Finding such extrema then boils down to solving these four equations for the four unknowns x, y, z and λ. (Actually, we need only find the values of x, y and z.)

It's important to recognize that this method only produces *candidates* for extrema. Along with finding a solution(s) to the above four equations, you need to verify (graphically as we did in example 8.1 or by some other means) that the solution you found in fact represents the desired optimal point.

Notice that the Lagrange multiplier method we have just developed can also be applied to functions of two variables, by ignoring the third variable in Theorem 8.1. That is, if $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives and $f(x_0, y_0)$ is an extremum of f, subject to the constraint $g(x, y) = 0$, then we must have

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$

for some constant λ. Graphically, this says that if $f(x_0, y_0)$ is an extremum, the level curve of f passing through (x_0, y_0) is tangent to the constraint curve $g(x, y) = 0$ at (x_0, y_0). We illustrate this in Figure 12.51. In this case, we end up with the three equations

$$f_x(x, y) = \lambda g_x(x, y), \quad f_y(x, y) = \lambda g_y(x, y) \quad \text{and} \quad g(x, y) = 0,$$

for the three unknowns x, y and λ. We illustrate this in example 8.2.

EXAMPLE 8.2 Finding the Optimal Thrust of a Rocket

A rocket is launched with a constant thrust corresponding to an acceleration of u ft/s^2. Ignoring air resistance, the rocket's height after t seconds is given by $f(t, u) = \frac{1}{2}(u - 32)t^2$ feet. Fuel usage for t seconds is proportional to $u^2 t$ and the limited fuel capacity of the rocket satisfies the equation $u^2 t = 10{,}000$. Find the value of u that maximizes the height that the rocket reaches when the fuel runs out.

Solution From Theorem 8.1, we look for solutions of $\nabla f(t, u) = \lambda \nabla g(t, u)$, where $g(t, u) = u^2 t - 10{,}000 = 0$ is the constraint equation. We have $\nabla f(t, u) = \left\langle (u - 32)t, \frac{1}{2}t^2 \right\rangle$

(the constraint curve) and finally, compare the function values. We illustrate this in example 8.3.

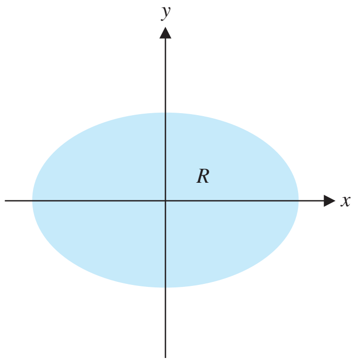


FIGURE 12.52
A metal plate

EXAMPLE 8.3 Optimization with an Inequality Constraint

Suppose that the temperature of a metal plate is given by $T(x, y) = x^2 + 2x + y^2$, for points (x, y) on the elliptical plate defined by $x^2 + 4y^2 \le 24$. Find the maximum and minimum temperatures on the plate.

Solution The plate corresponds to the shaded region R shown in Figure 12.52. We first look for critical points of $T(x, y)$ inside the region R. We have $\nabla T(x, y) = \langle 2x + 2, 2y\rangle = \langle 0, 0\rangle$ if $(x, y) = (-1, 0)$, which is in R. At this point, $T(-1, 0) = -1$. We next look for the extrema of $T(x, y)$ on the ellipse $x^2 + 4y^2 = 24$. We first rewrite the constraint equation as $g(x, y) = x^2 + 4y^2 - 24 = 0$. From Theorem 8.1, any extrema on the ellipse will satisfy the Lagrange multiplier equation: $\nabla T(x, y) = \lambda \nabla g(x, y)$ or

$$\langle 2x + 2, 2y\rangle = \lambda\langle 2x, 8y\rangle = \langle 2\lambda x, 8\lambda y\rangle.$$

This occurs when $$2x + 2 = 2\lambda x \quad \text{and} \quad 2y = 8\lambda y.$$

Notice that the second equation holds when $y = 0$ or $\lambda = \frac{1}{4}$. If $y = 0$, the constraint $x^2 + 4y^2 = 24$ gives $x = \pm\sqrt{24}$. If $\lambda = \frac{1}{4}$, the first equation becomes $2x + 2 = \frac{1}{2}x$ so that $x = -\frac{4}{3}$. The constraint $x^2 + 4y^2 = 24$ now gives $y = \pm\frac{\sqrt{50}}{3}$. Finally, we compare the function values at all of these points (the one interior critical point and the candidates for boundary extrema):

$$T(-1, 0) = -1,$$
$$T(\sqrt{24}, 0) = 24 + 2\sqrt{24} \approx 33.8,$$
$$T(-\sqrt{24}, 0) = 24 - 2\sqrt{24} \approx 14.2,$$
$$T\left(-\frac{4}{3}, \frac{\sqrt{50}}{3}\right) = \frac{14}{3} \approx 4.7$$

and $$T\left(-\frac{4}{3}, -\frac{\sqrt{50}}{3}\right) = \frac{14}{3} \approx 4.7.$$

From this list, it's easy to identify the minimum value of -1 at the point $(-1, 0)$ and the maximum value of $24 + 2\sqrt{24}$ at the point $(\sqrt{24}, 0)$. ■

In example 8.4, we illustrate the use of Lagrange multipliers for functions of three variables. In the course of doing so, we develop an interpretation of the Lagrange multiplier λ.

EXAMPLE 8.4 Finding an Optimal Level of Production

For a business that produces three products, suppose that when producing x, y and z thousand units of the products, the profit of the company (in thousands of dollars) can be modeled by $P(x, y, z) = 4x + 8y + 6z$. Manufacturing constraints force $x^2 + 4y^2 + 2z^2 \le 800$. Find the maximum profit for the company. Rework the problem with the constraint $x^2 + 4y^2 + 2z^2 \le 801$ and use the result to interpret the meaning of λ.

Solution We start with $\nabla P(x, y, z) = \langle 4, 8, 6\rangle$ and note that there are no critical points. This says that the extrema must lie on the boundary of the constraint region. That is, they must satisfy the constraint equation $g(x, y, z) = x^2 + 4y^2 + 2z^2 - 800 = 0$. From Theorem 8.1, the Lagrange multiplier equation is $\nabla P(x, y, z) = \lambda\nabla g(x, y, z)$ or

$$\langle 4, 8, 6\rangle = \lambda\langle 2x, 8y, 4z\rangle = \langle 2\lambda x, 8\lambda y, 4\lambda z\rangle.$$

This occurs when $\quad 4 = 2\lambda x, \quad 8 = 8\lambda y \quad \text{and} \quad 6 = 4\lambda z.$

From the first equation, we get $x = \frac{2}{\lambda}$. The second equation gives us $y = \frac{1}{\lambda}$ and the third equation gives us $z = \frac{3}{2\lambda}$. From the constraint equation $x^2 + 4y^2 + 2z^2 = 800$, we now have

$$800 = \left(\frac{2}{\lambda}\right)^2 + 4\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{3}{2\lambda}\right)^2 = \frac{25}{2\lambda^2},$$

so that $\lambda^2 = \frac{25}{1600}$ and $\qquad \lambda = \frac{1}{8}.$

(Why did we choose the positive sign for λ? Hint: Think about what x, y and z represent. Since $x > 0$, we must have $\lambda = \frac{2}{x} > 0$.) The only candidate for an extremum is then

$$x = \frac{2}{\lambda} = 16, \quad y = \frac{1}{\lambda} = 8 \quad \text{and} \quad z = \frac{3}{2\lambda} = 12,$$

and the corresponding profit is

$$P(16, 8, 9) = 4(16) + 8(8) + 6(12) = 200.$$

Observe that this is the maximum profit. Notice that if the constant on the right-hand side of the constraint equation is changed to 801, the first difference occurs in solving for λ, where we now get

$$801 = \frac{25}{2\lambda^2},$$

so that $\lambda \approx 0.12492$, $x = \frac{2}{\lambda} \approx 16.009997$, $y = \frac{1}{\lambda} \approx 8.004998$ and $z = \frac{3}{2\lambda} \approx 12.007498$. In this case, the maximum profit is

$$P\left(\frac{2}{\lambda}, \frac{1}{\lambda}, \frac{3}{2\lambda}\right) \approx 200.12496.$$

It is interesting to observe that the increase in profit is

$$P\left(\frac{2}{\lambda}, \frac{1}{\lambda}, \frac{3}{2\lambda}\right) - P(16, 8, 9) \approx 200.12496 - 200 = 0.12496 \approx \lambda.$$

As you might suspect from this observation, the Lagrange multiplier λ actually gives you the instantaneous rate of change of the profit with respect to a change in the production constraint. ■

We close this section by considering the case of finding the minimum or maximum value of a differentiable function $f(x, y, z)$ subject to two constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, where g and h are also differentiable. Notice that for both constraints to be satisfied at a point (x, y, z), the point must lie on both surfaces defined by the constraints. Consequently, in order for there to be a solution, we must assume that the two surfaces intersect. We further assume that ∇g and ∇h are nonzero and are not parallel, so that the two surfaces intersect in a curve C and are not tangent to one another. As we have already

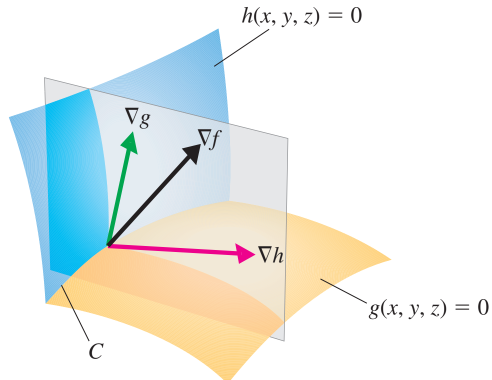


FIGURE 12.53
Constraint surfaces and the plane determined by the normal vectors ∇g and ∇h

seen, if f has an extremum at a point (x_0, y_0, z_0) on a curve C, then $\nabla f(x_0, y_0, z_0)$ must be normal to the curve. Notice that since C lies on both constraint surfaces, $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$ are both orthogonal to C at (x_0, y_0, z_0). This says that $\nabla f(x_0, y_0, z_0)$ must lie in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$ (see Figure 12.53). That is, for $(x, y, z) = (x_0, y_0, z_0)$ and some constants λ and μ (Lagrange multipliers),

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z).$$

The method of Lagrange multipliers for the case of two constraints then consists of finding the point (x, y, z) and the Lagrange multipliers λ and μ (for a total of five unknowns) satisfying the five equations defined by:

$$f_x(x, y, z) = \lambda g_x(x, y, z) + \mu h_x(x, y, z),$$
$$f_y(x, y, z) = \lambda g_y(x, y, z) + \mu h_y(x, y, z),$$
$$f_z(x, y, z) = \lambda g_z(x, y, z) + \mu h_z(x, y, z),$$
$$g(x, y, z) = 0$$

and

$$h(x, y, z) = 0.$$

We illustrate the use of Lagrange multipliers for the case of two constraints in example 8.5.

EXAMPLE 8.5 Optimization with Two Constraints

The plane $x + y + z = 12$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the point on the ellipse that is closest to the origin.

Solution We illustrate the intersection of the plane with the paraboloid in Figure 12.54. Observe that minimizing the distance to the origin is equivalent to minimizing $f(x, y, z) = x^2 + y^2 + z^2$ [the *square* of the distance from the point (x, y, z) to the origin]. Further, the constraints may be written as $g(x, y, z) = x + y + z - 12 = 0$ and $h(x, y, z) = x^2 + y^2 - z = 0$. At any extremum, we must have that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

or

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle + \mu \langle 2x, 2y, -1 \rangle.$$

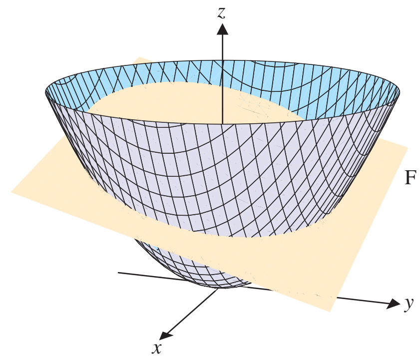


FIGURE 12.54
Intersection of a paraboloid and a plane

Together with the constraint equations, we now have the system of equations

$$2x = \lambda + 2\mu x, \tag{8.1}$$
$$2y = \lambda + 2\mu y, \tag{8.2}$$
$$2z = \lambda - \mu, \tag{8.3}$$
$$x + y + z - 12 = 0 \tag{8.4}$$

and

$$x^2 + y^2 - z = 0. \tag{8.5}$$

From (8.1), we have $$\lambda = 2x(1 - \mu),$$

while from (8.2), we have $$\lambda = 2y(1 - \mu).$$

Setting these two expressions for λ equal gives us

$$2x(1 - \mu) = 2y(1 - \mu),$$

from which it follows that either $\mu = 1$ (in which case $\lambda = 0$) or $x = y$. However, if $\mu = 1$ and $\lambda = 0$, we have from (8.3) that $z = -\frac{1}{2}$, which contradicts (8.5). Consequently, the only possibility is to have $x = y$, from which it follows from (8.5) that $z = 2x^2$. Substituting this into (8.4) gives us

$$\begin{aligned} 0 &= x + y + z - 12 = x + x + 2x^2 - 12 \\ &= 2x^2 + 2x - 12 = 2(x^2 + x - 6) = 2(x + 3)(x - 2), \end{aligned}$$

so that $x = -3$ or $x = 2$. Since $y = x$ and $z = 2x^2$, we have that (2, 2, 8) and (−3, −3, 18) are the only candidates for extrema. Finally, since

$$f(2, 2, 8) = 72 \quad \text{and} \quad f(-3, -3, 18) = 342,$$

the closest point on the intersection of the two surfaces to the origin is (2, 2, 8). By the same reasoning, observe that the farthest point on the intersection of the two surfaces from the origin is (−3, −3, 18). Notice that these are also consistent with what you can see in Figure 12.54. ■

The method of Lagrange multipliers can be extended in a straightforward fashion to the case of minimizing or maximizing a function of any number of variables subject to any number of constraints.

EXERCISES 12.8

WRITING EXERCISES

1. Explain why the point of tangency in Figure 12.50d must be the closest point to the origin.
2. Explain why in example 8.1 you know that the critical point found corresponds to the minimum distance and not the maximum distance or a saddle point.
3. In example 8.2, explain in physical terms why there would be a value of u that would maximize the rocket's height. In particular, explain why a larger value of u wouldn't *always* produce a larger height.
4. In example 8.4, we showed that the Lagrange multiplier λ corresponds to the rate of change of profit with respect to a change in production level. Explain how knowledge of this value (positive, negative, small, large) would be useful to a plant manager.

In exercises 1–8, use Lagrange multipliers to find the closest point on the given curve to the indicated point.

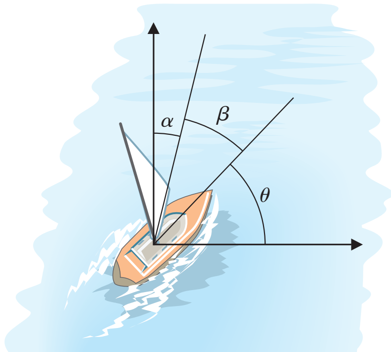
1. $y = 3x - 4$, origin
2. $y = 2x + 1$, origin
3. $y = 3 - 2x$, $(4, 0)$
4. $y = x - 2$, $(0, 2)$
5. $y = x^2$, $(3, 0)$
6. $y = x^2$, $(0, 2)$
7. $y = x^2$, $\left(2, \frac{1}{2}\right)$
8. $y = x^2 - 1$, $(1, 2)$

In exercises 9–16, use Lagrange multipliers to find the maximum and minimum of the function $f(x, y)$ subject to the constraint $g(x, y) = c$.

9. $f(x, y) = 4xy$ subject to $x^2 + y^2 = 8$
10. $f(x, y) = 4xy$ subject to $4x^2 + y^2 = 8$
11. $f(x, y) = 4x^2y$ subject to $x^2 + y^2 = 3$
12. $f(x, y) = 2x^3y$ subject to $x^2 + y^2 = 4$
13. $f(x, y) = xe^y$ subject to $x^2 + y^2 = 2$
14. $f(x, y) = e^{2x+y}$ subject to $x^2 + y^2 = 5$
15. $f(x, y) = x^2e^y$ subject to $x^2 + y^2 = 3$
16. $f(x, y) = x^2y^2$ subject to $x^2 + 4y^2 = 24$

In exercises 17–20, find the maximum and minimum of the function $f(x, y)$ subject to the constraint $g(x, y) \le c$.

17. $f(x, y) = 4xy$ subject to $x^2 + y^2 \le 8$
18. $f(x, y) = 4xy$ subject to $4x^2 + y^2 \le 8$
19. $f(x, y) = 4x^2y$ subject to $x^2 + y^2 \le 3$
20. $f(x, y) = 2x^3y$ subject to $x^2 + y^2 \le 4$
21. Rework example 8.2 with extra fuel, so that $u^2t = 11{,}000$.
22. In exercise 21, compute λ. Comparing solutions to example 8.2 and exercise 21, compute the change in z divided by the change in u^2t.
23. Solve example 8.2 by substituting $t = 10{,}000/u^2$ into the height equation. Be sure to show that your solution represents a *maximum* height.
24. In example 8.2, the general constraint is $u^2t = k$ and the resulting maximum height is $h(k)$. Use the technique of exercise 23 and the results of example 8.2 to show that $\lambda = h'(k)$.
25. Suppose that the business in example 8.4 has profit function $P(x, y, z) = 3x + 6y + 6z$ and manufacturing constraint $2x^2 + y^2 + 4z^2 \le 8800$. Maximize the profits.
26. Suppose that the business in example 8.4 has profit function $P(x, y, z) = 3xz + 6y$ and manufacturing constraint $x^2 + 2y^2 + z^2 \le 6$. Maximize the profits.
27. In exercise 25, show that the Lagrange multiplier gives the rate of change of the profit relative to a change in the production constraint.
28. Use the value of λ (do not solve any equations) to determine the amount of profit if the constraint in exercise 26 is changed to $x^2 + 2y^2 + z^2 \le 7$.
29. Minimize $2x + 2y$ subject to the constraint $xy = c$ for some constant $c > 0$ and conclude that for a given area, the rectangle with smallest perimeter is the square.
30. As in exercise 29, find the rectangular box of a given volume that has the minimum surface area.
31. Maximize $y - x$ subject to the constraint $x^2 + y^2 = 1$.
32. Maximize e^{x+y} subject to the constraint $x^2 + y^2 = 2$.
33. In the picture, a sailboat is sailing into a crosswind. The wind is blowing out of the north; the sail is at an angle α to the east of due north and at an angle β north of the hull of the boat. The hull, in turn, is at an angle θ to the north of due east. Explain why $\alpha + \beta + \theta = \frac{\pi}{2}$. If the wind is blowing with speed w, then the northward component of the wind's force on the boat is given by $w \sin\alpha \sin\beta \sin\theta$. If this component is positive, the boat can travel "against the wind." Taking $w = 1$ for convenience, maximize $\sin\alpha \sin\beta \sin\theta$ subject to the constraint $\alpha + \beta + \theta = \frac{\pi}{2}$.



34. Suppose a music company sells two types of speakers. The profit for selling x speakers of style A and y speakers of style B is modeled by $f(x, y) = x^3 + y^3 - 5xy$. The company can't manufacture more than k speakers total in a given month for some constant $k > 5$. Show that the maximum profit is $\frac{k^2(k-5)}{4}$ and show that $\lambda = \frac{df}{dk}$.

35. Consider the problem of finding extreme values of xy^2 subject to $x + y = 0$. Show that the Lagrange multiplier method identifies $(0, 0)$ as a critical point. Show that this point is neither a local minimum nor a local maximum.

36. Make the substitution $y = -x$ in the function $f(x, y) = xy^2$. Show that $x = 0$ is a critical point and determine what type point is at $x = 0$. Explain why the Lagrange multiplier method fails in exercise 35.

37. The production of a company is given by the Cobb-Douglas function $P = 200L^{2/3}K^{1/3}$. Cost constraints on the business force $2L + 5K \le 150$. Find the values of the labor L and capital K to maximize production.

38. Maximize the profit $P = 4x + 5y$ of a business given the production possibilities constraint curve $2x^2 + 5y^2 \le 32{,}500$.

In exercises 39 and 40, you will illustrate the least-cost rule.

39. Minimize the cost function $C = 25L + 100K$, given the production constraint $P = 60L^{2/3}K^{1/3} = 1920$.

40. In exercise 39, show that the minimum cost occurs when the ratio of marginal productivity of labor, $\frac{\partial P}{\partial L}$, to the marginal productivity of capital, $\frac{\partial P}{\partial K}$, equals the ratio of the price of labor, $\frac{\partial C}{\partial L}$, to the price of capital, $\frac{\partial C}{\partial K}$.

Exercises 41–46 involve optimization with two constraints.

41. A person has \$300 to spend on entertainment. Assume that CDs cost \$10 apiece, DVDs cost \$15 apiece and the person's utility function is $10c^{0.4}d^{0.6}$ for buying c CDs and d DVDs. Find c and d to maximize the utility function.

42. To generalize exercise 41, suppose that on a fixed budget of \$$k$ you buy x units of product A purchased at \$$a$ apiece and y units of product B purchased at \$$b$ apiece. For the utility function $x^p y^q$ with $p + q = 1$ and $0 < p < 1$, show that the utility function is maximized with $x = \frac{kp}{a}$ and $y = \frac{kq}{b}$.

43. Minimize $f(x, y, z) = x^2 + y^2 + z^2$, subject to the constraints $x + 2y + 3z = 6$ and $y + z = 0$.

44. Interpret the function $f(x, y, z)$ of exercise 43 in terms of the distance from a point (x, y, z) to the origin. Sketch the two planes given in exercise 43. Interpret exercise 43 as finding the closest point on a line to the origin.

45. Maximize $f(x, y, z) = xyz$, subject to the constraints $x + y + z = 4$ and $x + y - z = 0$.

46. Maximize $f(x, y, z) = 3x + y + 2z$, subject to the constraints $y^2 + z^2 = 1$ and $x + y - z = 1$.

47. Find the points on the intersection of $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ that are (a) closest to and (b) farthest from the origin.

48. Find the point on the intersection of $x + 2y + z = 2$ and $y = x$ that is closest to the origin.

49. Use Lagrange multipliers to explore the problem of finding the closest point on $y = x^n$ to the point $(0, 1)$, for some positive integer n. Show that $(0, 0)$ is always a solution to the Lagrange multiplier equation. Show that $(0, 0)$ is the location of a local maximum for $n = 2$, but a local minimum for $n > 2$. As $n \to \infty$, show that the difference between the absolute minimum and the local minimum at $(0, 0)$ goes to 0.

50. Repeat example 8.4 with constraints $x \ge 0$, $y \ge 0$ and $z \ge 0$. Note that you can find the maximum on the boundary $x = 0$ by maximizing $8y + 6z$ subject to $4y^2 + 2z^2 \le 800$.

51. Estimate the closest point on the paraboloid $z = x^2 + y^2$ to the point $(1, 0, 0)$.

52. Estimate the closest point on the hyperboloid $x^2 + y^2 - z^2 = 1$ to the point $(0, 2, 0)$.

EXPLORATORY EXERCISES

1. (This exercise was suggested by Adel Faridani of Oregon State University.) The maximum height found in example 8.2 is actually the height at the time the fuel runs out. A different problem is to find the maximum total height, including the extra height gained after the fuel runs out. In this exercise, we find the value of u that maximizes the total height. First, find the velocity and height when the fuel runs out. (Hint: This should be a function of u only.) Then find the total height of a rocket with that initial height and initial velocity, again assuming that gravity is the only force. Find u to maximize this function. Compare this u-value with that found in example 8.2. Explain in physical terms why this one is larger.

2. Find the maximum value of $f(x, y, z) = xyz$ subject to $x + y + z = 1$ with $x > 0$, $y > 0$ and $z > 0$. Conclude that $\sqrt[3]{xyz} \le \frac{x+y+z}{3}$. The expression on the left is called the **geometric mean** of x, y and z while the expression on the right is the more familiar **arithmetic mean.** Generalize this result in two ways. First, show that the geometric mean of any three positive numbers does not exceed the arithmetic mean. Then, show that the geometric mean of any number of positive numbers does not exceed the arithmetic mean.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Level curve	Limit of $f(x, y)$	Continuous
Level surface	Tangent plane	Normal line
Partial derivative	Differential	Differentiable
Linear approximation	Laplacian	Implicit differentiation
Chain rule	Gradient	
Directional derivative	Saddle point	Local extremum
Critical point	Extreme Value Theorem	Second Derivatives Test
Linear regression		
Contour plot	Density plot	Lagrange multiplier

TRUE OR FALSE

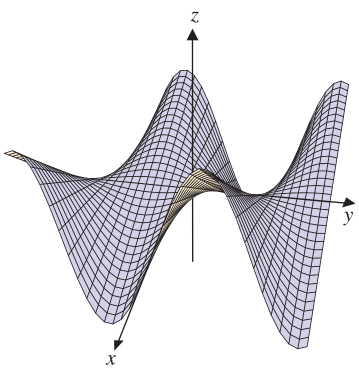
State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. Quadric surfaces are examples of graphs of functions of two variables.
2. Level curves are traces in planes $z = c$ of the surface $z = f(x, y)$.
3. If a function is continuous on every line through (a, b), then it is continuous at (a, b).
4. $\frac{\partial f}{\partial x}(a, b)$ equals the slope of the tangent line to $z = f(x, y)$ at (a, b).
5. For the partial derivative $\frac{\partial^2 f}{\partial x \partial y}$, the order of partial derivatives does not matter.
6. The normal vector to the tangent plane is given by the partial derivatives.
7. A linear approximation is an equation for a tangent plane.
8. The gradient vector is perpendicular to all level curves.
9. If $D_{\mathbf{u}} f(a, b) < 0$, then $f(a + u_1, b + u_2) < f(a, b)$.
10. A normal vector to the tangent plane to $z = f(x, y)$ is $\nabla f(x, y)$.
11. If $\frac{\partial f}{\partial x}(a, b) > 0$ and $\frac{\partial f}{\partial y}(a, b) < 0$, then there is a saddle point at (a, b).
12. The maximum of f on a region R occurs either at a critical point or on the boundary of R.
13. Solving the equation $\nabla f = \lambda \nabla g$ gives the maximum of f subject to $g = 0$.

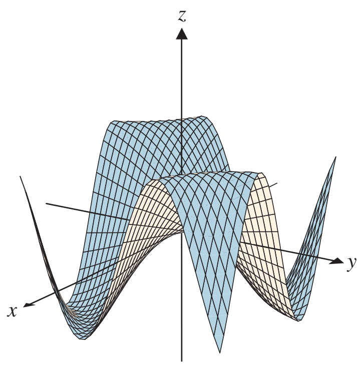
In exercises 1–10, sketch the graph of $z = f(x, y)$.

1. $f(x, y) = x^2 - y^2$
2. $f(x, y) = \sqrt{x^2 + y^2}$
3. $f(x, y) = 2 - x^2 - y^2$
4. $f(x, y) = \sqrt{2 - x^2 - y^2}$
5. $f(x, y) = \frac{3}{x^2} + \frac{2}{y^2}$
6. $f(x, y) = \frac{x^5}{y}$
7. $f(x, y) = \sin(x^2 y)$
8. $f(x, y) = \sin(y - x^2)$
9. $f(x, y) = 3xe^y - x^3 - e^{3y}$
10. $f(x, y) = 4x^2 e^y - 2x^4 - e^{4y}$
11. In parts a–f, match the functions to the surfaces.
    a. $f(x, y) = \sin xy$ **b.** $f(x, y) = \sin(x/y)$
    c. $f(x, y) = \sin\sqrt{x^2 + y^2}$ **d.** $f(x, y) = x \sin y$
    e. $f(x, y) = \frac{4}{2x^2 + 3y^2 - 1}$
    f. $f(x, y) = \frac{4}{2x^2 + 3y^2 + 1}$

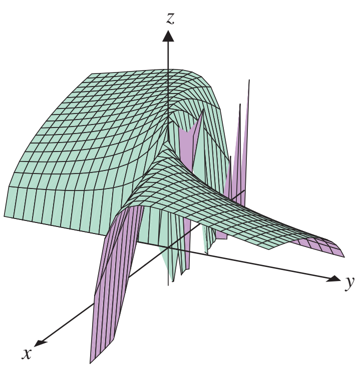
Review Exercises



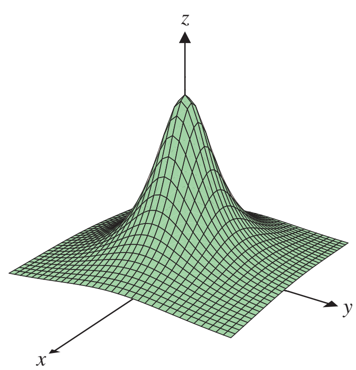
SURFACE A



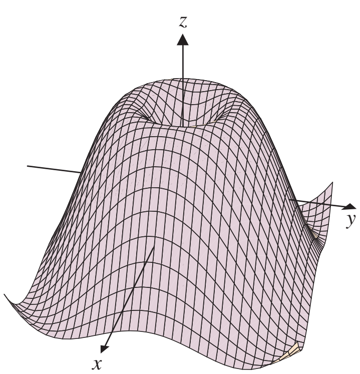
SURFACE D



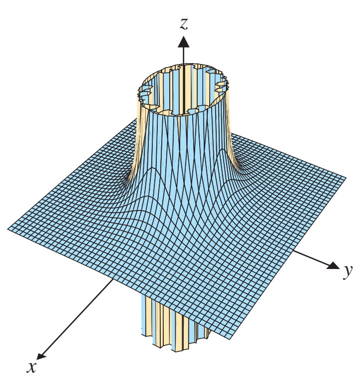
SURFACE B



SURFACE E



SURFACE C

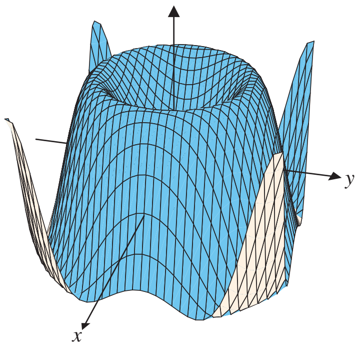


SURFACE F

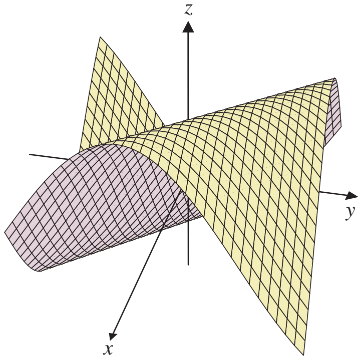
Review Exercises

12. In parts a–d, match the surfaces to the contour plots.

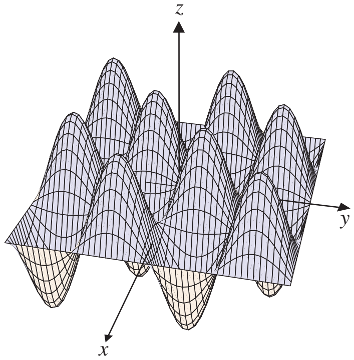
a.



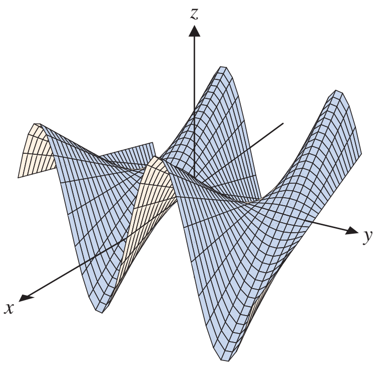
b.

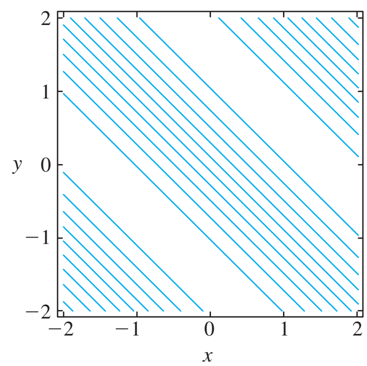


c.

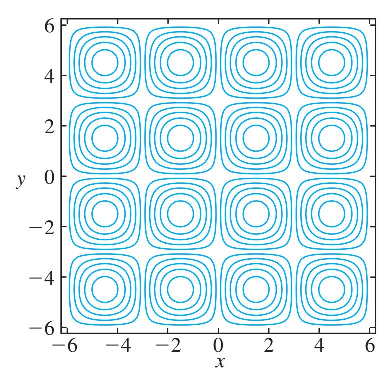


d.



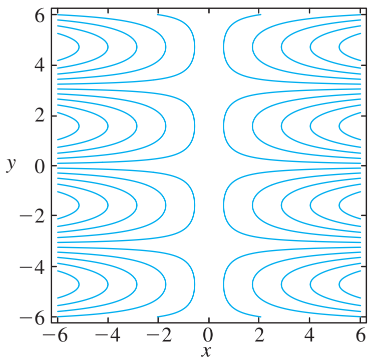


CONTOUR A

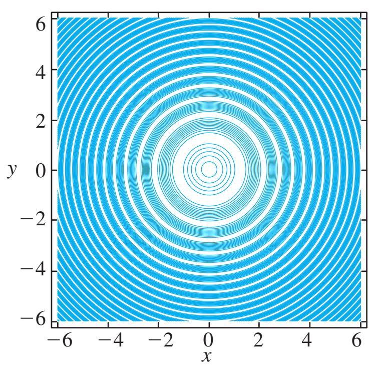


CONTOUR B

Review Exercises



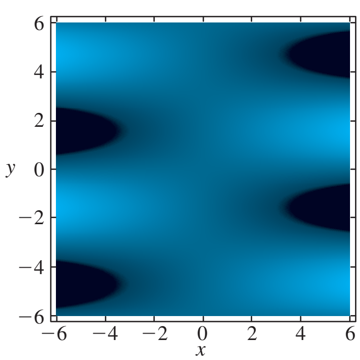
CONTOUR C



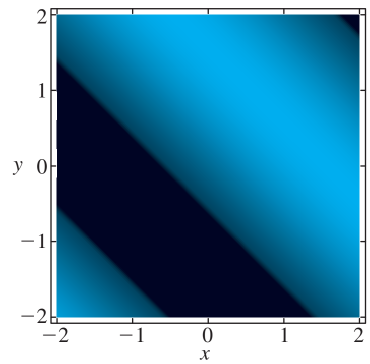
CONTOUR D

13. In parts a–d, match the density plots to the contour plots of exercise 12.

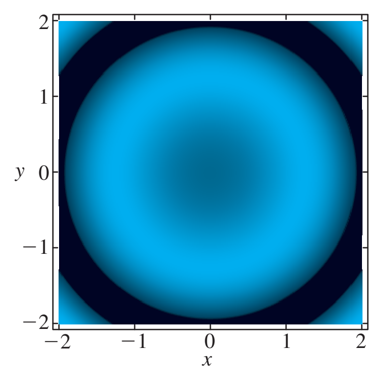
a.



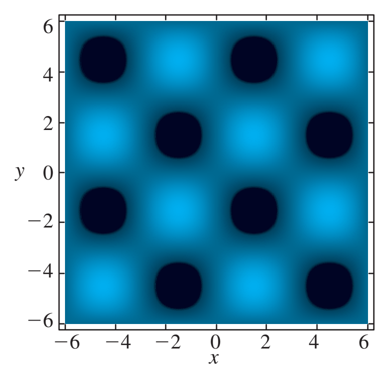
b.



c.



d.



14. Compute the indicated limit.

a. $\lim_{(x,y)\to(0,2)} \frac{3x}{y^2+1}$ **b.** $\lim_{(x,y)\to(1,\pi)} \frac{xy-1}{\cos xy}$

Review Exercises

In exercises 15–18, show that the indicated limit does not exist.

15. $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$

16. $\lim_{(x,y)\to(0,0)} \frac{2xy^{3/2}}{x^2+y^3}$

17. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+xy+y^2}$

18. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+xy+y^2}$

In exercises 19 and 20, show that the indicated limit exists.

19. $\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2}{x^2+y^2}$

20. $\lim_{(x,y)\to(0,0)} \frac{3y^2\ln(x+1)}{x^2+3y^2}$

In exercises 21 and 22, find the region on which the function is continuous.

21. $f(x,y) = 3x^2e^{4y} - \frac{3y}{x}$

22. $f(x,y) = \sqrt{4-4x^2-y^2}$

In exercises 23–26, find both first-order partial derivatives.

23. $f(x,y) = \frac{4x}{y} + xe^{xy}$

24. $f(x,y) = xe^{xy} + 3y^2$

25. $f(x,y) = 3x^2y\cos y - \sqrt{x}$

26. $f(x,y) = \sqrt{x^3y} + 3x - 5$

27. Show that the function $f(x,y) = e^x\sin y$ satisfies **Laplace's equation** $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

28. Show that the function $f(x,y) = e^x\cos y$ satisfies Laplace's equation. (See exercise 27.)

In exercises 29 and 30, use the chart to estimate the partial derivatives.

29. $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$

30. $\frac{\partial f}{\partial x}(10,0)$ and $\frac{\partial f}{\partial y}(10,0)$

y \ x	−20	−10	0	10	20
−20	2.4	2.1	0.8	0.5	1.0
−10	2.6	2.2	1.4	1.0	1.2
0	2.7	2.4	2.0	1.6	1.2
10	2.9	2.5	2.6	2.2	1.8
20	3.1	2.7	3.0	2.9	2.7

In exercises 31–34, compute the linear approximation of the function at the given point.

31. $f(x,y) = 3y\sqrt{x^2+5}$ at $(-2,5)$

32. $f(x,y) = \frac{x+2}{4y-2}$ at $(2,3)$

33. $f(x,y) = \tan(x+2y)$ at $(\pi, \frac{\pi}{2})$

34. $f(x,y) = \ln(x^2+3y)$ at $(4,2)$

In exercises 35 and 36, find the indicated derivatives.

35. $f(x,y) = 2x^4y + 3x^2y^2$; f_{xx}, f_{yy}, f_{xy}

36. $f(x,y) = x^2e^{3y} - \sin y$; f_{xx}, f_{yy}, f_{yyx}

In exercises 37–40, find an equation of the tangent plane.

37. $z = x^2y + 2x - y^2$ at $(1,-1,0)$

38. $z = \sqrt{x^2+y^2}$ at $(3,-4,5)$

39. $x^2 + 2xy + y^2 + z^2 = 5$ at $(0,2,1)$

40. $x^2z - y^2x + 3y - z = -4$ at $(1,-1,2)$

In exercises 41 and 42, use the chain rule to find the indicated derivative(s).

41. $g'(t)$ where $g(t) = f(x(t), y(t))$, $f(x,y) = x^2y + y^2$, $x(t) = e^{4t}$ and $y(t) = \sin t$

42. $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial v}$ where $g(u,v) = f(x(u,v), y(u,v))$, $f(x,y) = 4x^2 - y$, $x(u,v) = u^3v + \sin u$ and $y(u,v) = 4v^2$

In exercises 43 and 44, state the chain rule for the general composite function.

43. $g(t) = f(x(t), y(t), z(t), w(t))$

44. $g(u,v) = f(x(u,v), y(u,v))$

In exercises 45 and 46, use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

45. $x^2 + 2xy + y^2 + z^2 = 1$

46. $x^2z - y^2x + 3y - z = -4$

In exercises 47 and 48, find the gradient of the given function at the indicated point.

47. $f(x,y) = 3x\sin 4y - \sqrt{xy}$, (π, π)

48. $f(x,y,z) = 4xz^2 - 3\cos x + 4y^2$, $(0,1,-1)$

In exercises 49–52, compute the directional derivative of f at the given point in the direction of the indicated vector.

49. $f(x,y) = x^3y - 4y^2$, $(-2,3)$, $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

50. $f(x,y) = x^2 + xy^2$, $(2,1)$, $\mathbf{u}$ in the direction of $\langle 3,-2 \rangle$

Review Exercises

51. $f(x, y) = e^{3xy} - y^2$, $(0, -1)$, **u** in the direction from $(2, 3)$ to $(3, 1)$

52. $f(x, y) = \sqrt{x^2 + xy^2}$, $(2, 1)$, **u** in the direction of $\langle 1, -2 \rangle$

In exercises 53–56, find the directions of maximum and minimum change of *f* at the given point, and the values of the maximum and minimum rates of change.

53. $f(x, y) = x^3y - 4y^2$, $(-2, 3)$

54. $f(x, y) = x^2 + xy^2$, $(2, 1)$

55. $f(x, y) = \sqrt{x^4 + y^4}$, $(2, 0)$

56. $f(x, y) = x^2 + xy^2$, $(1, 2)$

57. Suppose that the elevation on a hill is given by $f(x, y) = 100 - 4x^2 - 2y$. From the site at $(2, 1)$, in which direction will the rain run off?

58. If the temperature at the point (x, y, z) is given by $T(x, y, z) = 70 + 5e^{-z^2}(4x + 3y^{-1})$, find the direction from the point $(1, 2, 1)$ in which the temperature decreases most rapidly.

In exercises 59–62, find all critical points and use Theorem 7.2 (if applicable) to classify them.

59. $f(x, y) = 2x^4 - xy^2 + 2y^2$

60. $f(x, y) = 2x^4 + y^3 - x^2y$

61. $f(x, y) = 4xy - x^3 - 2y^2$

62. $f(x, y) = 3xy - x^3y + y^2 - y$

63. The following data show the height and weight of a small number of people. Use the linear model to predict the weight of a 6′2″ person and a 5′0″ person. Comment on how accurate you think the model is.

Height (inches)	64	66	70	71
Weight (pounds)	140	156	184	190

64. The following data show the age and income for a small number of people. Use the linear model to predict the income of a 20-year-old and of a 60-year-old. Comment on how accurate you think the model is.

Age (years)	28	32	40	56
Income ($)	36,000	34,000	88,000	104,000

In exercises 65 and 66, find the absolute extrema of the function on the given region.

65. $f(x, y) = 2x^4 - xy^2 + 2y^2$, $0 \le x \le 4$, $0 \le y \le 2$

66. $f(x, y) = 2x^4 + y^3 - x^2y$, region bounded by $y = 0$, $y = x$ and $x = 2$

In exercises 67–70, use Lagrange multipliers to find the maximum and minimum of the function $f(x, y)$, subject to the constraint $g(x, y) = c$.

67. $f(x, y) = x + 2y$, subject to $x^2 + y^2 = 5$

68. $f(x, y) = 2x^2y$, subject to $x^2 + y^2 = 4$

69. $f(x, y) = xy$, subject to $x^2 + y^2 = 1$

70. $f(x, y) = x^2 + 2y^2 - 2x$, subject to $x^2 + y^2 = 1$

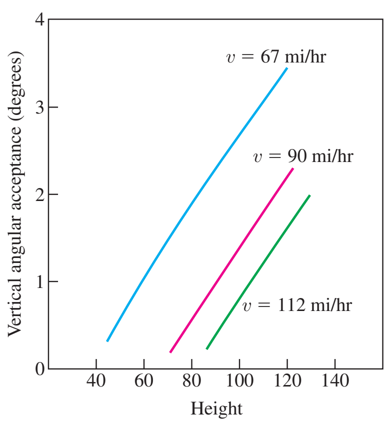
In exercises 71 and 72, use Lagrange multipliers to find the closest point on the given curve to the indicated point.

71. $y = x^3$, $(4, 0)$

72. $y = x^3$, $(2, 1)$

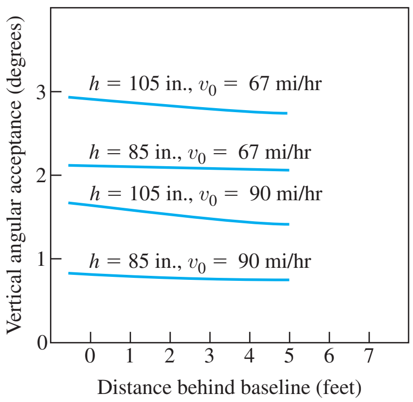
EXPLORATORY EXERCISES

1. The graph (from the excellent book *Tennis Science for Tennis Players* by Howard Brody) shows the vertical angular acceptance of a tennis serve as a function of velocity and the height at which the ball is hit. With vertical angular acceptance, Brody is measuring a margin of error. For example, if serves with angles ranging from 5° to 8° will land in the service box (for a given height and velocity), the vertical angular acceptance is 3°. For a given height, does the angular acceptance increase or decrease as velocity increases? Explain why this is reasonable. For a given velocity, does the angular acceptance increase or decrease as height increases? Explain why this is reasonable.



Review Exercises

2. The graphic in exercise 1 is somewhat like a contour plot. Assuming that angular acceptance is the dependent variable, explain what is different about this plot from the contour plots drawn in this chapter. Which type of plot do you think is easier to read? The accompanying plot shows angular acceptance in terms of a number of variables. Identify the independent variables and compare this plot to the level surfaces drawn in this chapter.



3. The horizontal range of a golf ball or baseball depends on the launch angle and the rate of backspin on the ball. The accompanying figure, reprinted from *Keep Your Eye on the Ball* by Watts and Bahill, shows level curves for this relationship for an initial velocity of 110 mph, although the dependent variable (range) is graphed vertically and the level curves represent constant values of one of the independent variables. Estimate the partial derivatives of range at 30° and 1910 rpm and use them to find a linear approximation of range. Predict the range at 25° and 2500 rpm, and also at 40° and 4000 rpm. Discuss the accuracy of each prediction.