CHAPTER

0

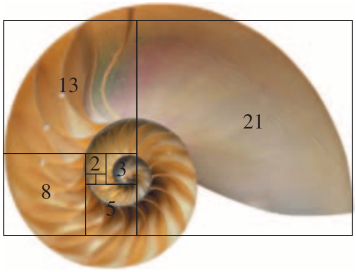
Preliminaries

In this chapter you will find a collection of familiar topics. You need not spend a great deal of time here. Rather, review the material as necessary, until you are comfortable with all of the topics discussed. We have primarily included material that we consider *essential* for the study of calculus that you are about to begin. We must emphasize that understanding is always built upon a solid foundation. While we do not intend this chapter to be a comprehensive review of precalculus mathematics, we have tried to hit the highlights and provide you with some standard notation and language that we will use throughout the text.

As it grows, a chambered nautilus creates a spiral shell. Behind this beautiful geometry is a surprising amount of mathematics. The nautilus grows in such a way that the overall proportions of its shell remain constant. That is, if you draw a rectangle to circumscribe the shell, the ratio of height to width of the rectangle remains nearly constant.

There are several ways to represent this property mathematically. In polar coordinates (which we present in Chapter 9), we study logarithmic spirals that have the property that the angle of growth is constant, producing the constant proportions of a nautilus shell. Using basic geometry, you can divide the circumscribing rectangle into a sequence of squares as in the figure. The relative sizes of the squares form the famous Fibonacci sequence 1, 1, 2, 3, 5, 8, . . . , where each number in the sequence is the sum of the preceding two numbers.

The Fibonacci sequence has an amazing list of interesting properties. (Search on the Internet to see what we mean!) Numbers in the sequence have a surprising habit of showing up in nature, such as the number of petals on a lily (3), buttercup (5), marigold (13), black-eyed Susan (21) and pyrethrum (34). Although we have a very simple description of how to generate the Fibonacci sequence, think about how you might describe it as an algebraic function. A plot of the first several numbers in the sequence (shown in Figure 0.1) should give you the impression of a graph curving up, perhaps a parabola or an exponential function.



A nautilus shell

In this chapter, we discuss methods for deciding exactly which function provides the best description of these numbers.

Two aspects of this problem are important themes throughout the calculus. One of

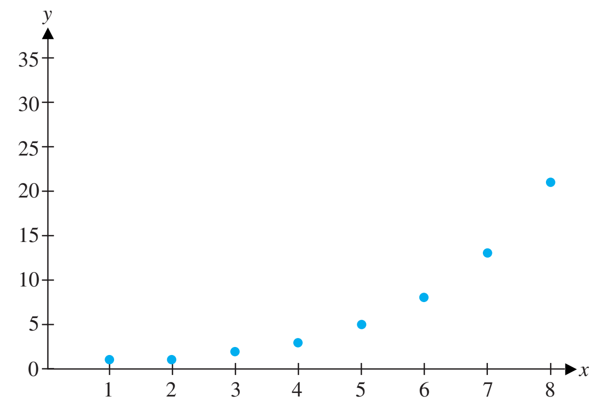


FIGURE 0.1
The Fibonacci sequence

these is the attempt to find patterns to help us better describe the world. The other theme is the interplay between graphs and functions. By connecting the powerful equation-solving techniques of algebra with the visual images provided by graphs, you will significantly improve your ability to make use of your mathematical skills in solving real-world problems.

0.1 POLYNOMIALS AND RATIONAL FUNCTIONS

The Real Number System and Inequalities

Although mathematics is far more than just a study of numbers, our journey into calculus begins with the real number system. While this may seem to be a fairly mundane starting place, we want to give you the opportunity to brush up on those properties that are of particular interest for calculus.

The most familiar set of numbers is the set of **integers,** consisting of the whole numbers and their additive inverses: $0, \pm1, \pm2, \pm3, \ldots$. A **rational number** is any number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{2}{3}$, $-\frac{7}{3}$ and $\frac{27}{125}$ are all rational numbers. Notice that every integer n is also a rational number, since we can write it as the quotient of two integers: $n = \frac{n}{1}$.

The **irrational numbers** are all those real numbers that cannot be written in the form $\frac{p}{q}$, where p and q are integers. Recall that rational numbers have decimal expansions that either terminate or repeat. For instance, $\frac{1}{2} = 0.5$, $\frac{1}{3} = 0.3333\bar{3}$, $\frac{1}{8} = 0.125$ and $\frac{1}{6} = 0.16666\bar{6}$ are all rational numbers. By contrast, irrational numbers have decimal expansions that do not repeat or terminate. For instance, three familiar irrational numbers and their decimal expansions are

$$\sqrt{2} = 1.41421\,35623\ldots,$$
$$\pi = 3.14159\,26535\ldots$$

and

$$e = 2.71828\,18284\ldots.$$

We picture the real numbers arranged along the number line displayed in Figure 0.2 (the **real line**). The set of real numbers is denoted by the symbol $\mathbb{R}$.

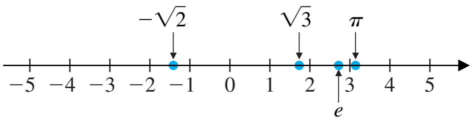


FIGURE 0.2
The real line

For real numbers a and b, where $a < b$, we define the **closed interval** $[a, b]$ to be the set of numbers between a and b, including a and b (the **endpoints**), that is,

$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\},$$

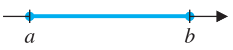


FIGURE 0.3
A closed interval

as illustrated in Figure 0.3, where the solid circles indicate that a and b are included in $[a, b]$.

Similarly, the **open interval** (a, b) is the set of numbers between a and b, but *not* including the endpoints a and b, that is,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

FIGURE 0.4
An open interval

as illustrated in Figure 0.4, where the open circles indicate that a and b are not included in (a, b).

You should already be very familiar with the following properties of real numbers.

THEOREM 1.1

If a and b are real numbers and $a < b$, then

(i) For any real number c, $a + c < b + c$.
(ii) For real numbers c and d, if $c < d$, then $a + c < b + d$.
(iii) For any real number $c > 0$, $a \cdot c < b \cdot c$.
(iv) For any real number $c < 0$, $a \cdot c > b \cdot c$.

REMARK 1.1

We need the properties given in Theorem 1.1 to solve inequalities. Notice that (i) says that you can add the same quantity to both sides of an inequality. Part (iii) says that you can multiply both sides of an inequality by a positive number. Finally, (iv) says that if you multiply both sides of an inequality by a negative number, the inequality is reversed.

We illustrate the use of Theorem 1.1 by solving a simple inequality.

EXAMPLE 1.1 Solving a Linear Inequality

Solve the linear inequality $2x + 5 < 13$.

Solution We can use the properties in Theorem 1.1 to isolate the x. First, subtract 5 from both sides to obtain

$$(2x + 5) - 5 < 13 - 5$$

or $$2x < 8.$$

Finally, divide both sides by 2 (since $2 > 0$, the inequality is not reversed) to obtain

$$x < 4.$$

We often write the solution of an inequality in interval notation. In this case, we get the interval $(-\infty, 4)$. ■

You can deal with more complicated inequalities in the same way.

EXAMPLE 1.2 Solving a Two-Sided Inequality

Solve the two-sided inequality $6 < 1 - 3x \leq 10$.

Solution First, recognize that this problem requires that we find values of x such that

$$6 < 1 - 3x \quad \textit{and} \quad 1 - 3x \leq 10.$$

Here, we can use the properties in Theorem 1.1 to isolate the x by working on both inequalities simultaneously. First, subtract 1 from each term, to get

$$6 - 1 < (1 - 3x) - 1 \leq 10 - 1$$

or $$5 < -3x \leq 9.$$

Now, divide by -3, but be careful. Since $-3 < 0$, the inequalities are reversed. We have

$$\frac{5}{-3} > \frac{-3x}{-3} \geq \frac{9}{-3}$$

or $$-\frac{5}{3} > x \geq -3.$$

We usually write this as $$-3 \leq x < -\frac{5}{3},$$

or in interval notation as $[-3, -\frac{5}{3})$. ■

You will often need to solve inequalities involving fractions. We present a typical example in the following.

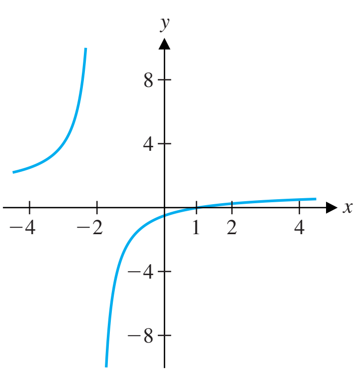


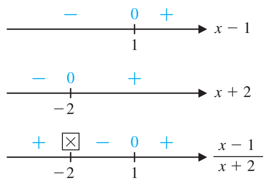
FIGURE 0.5

$$y = \frac{x-1}{x+2}$$

EXAMPLE 1.3 Solving an Inequality Involving a Fraction

Solve the inequality $\dfrac{x-1}{x+2} \geq 0$.

Solution In Figure 0.5, we show a graph of the function, which appears to indicate that the solution includes all $x < -2$ and $x \geq 1$. Carefully read the inequality and observe that there are only three ways to satisfy this: either both numerator and denominator are positive, both are negative or the numerator is zero. To visualize this, we draw number lines for each of the individual terms, indicating where each is positive, negative or zero and use these to draw a third number line indicating the value of the quotient, as shown in the margin. In the third number line, we have placed an "☒"



above the -2 to indicate that the quotient is undefined at $x = -2$. From this last number line, you can see that the quotient is nonnegative whenever $x < -2$ *or* $x \geq 1$. We write the solution in interval notation as $(-\infty, -2) \cup [1, \infty)$. ■

For inequalities involving a polynomial of degree 2 or higher, factoring the polynomial and determining where the individual factors are positive and negative, as in example 1.4, will lead to a solution.

EXAMPLE 1.4 Solving a Quadratic Inequality

Solve the quadratic inequality

$$x^2 + x - 6 > 0. \tag{1.1}$$

Solution In Figure 0.6, we show a graph of the polynomial on the left side of the inequality. Since this polynomial factors, (1.1) is equivalent to

$$(x + 3)(x - 2) > 0. \tag{1.2}$$

This can happen in only two ways: when both factors are positive or when both factors are negative. As in example 1.3, we draw number lines for both of the individual factors, indicating where each is positive, negative or zero and use these to draw a number line representing the product. We show these in the margin. Notice that the third number line indicates that the product is positive whenever $x < -3$ *or* $x > 2$. We write this in interval notation as $(-\infty, -3) \cup (2, \infty)$. ■

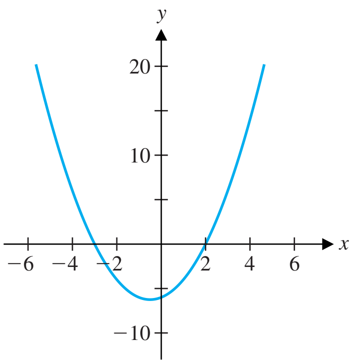
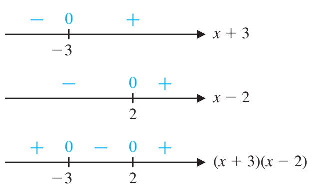


FIGURE 0.6
$y = x^2 + x - 6$



No doubt, you will recall the following standard definition.

DEFINITION 1.1

The **absolute value** of a real number x is $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$.

Make certain that you read Definition 1.1 correctly. If x is negative, then $-x$ is positive. This says that $|x| \geq 0$ for all real numbers x. For instance, using the definition,

$$|-4| = -(-4) = 4,$$

notice that for any real numbers a and b,

$$|a \cdot b| = |a| \cdot |b|.$$

However, $$|a + b| \neq |a| + |b|,$$

in general. (To verify this, simply take $a = 5$ and $b = -2$ and compute both quantities.)

However, it is always true that

$$|a + b| \leq |a| + |b|.$$

This is referred to as the **triangle inequality.**

The interpretation of $|a - b|$ as the distance between a and b (see the note in the margin) is particularly useful for solving inequalities involving absolute values. Wherever possible, we suggest that you use this interpretation to read what the inequality means, rather than merely following a procedure to produce a solution.

NOTES

For any two real numbers a and b, $|a - b|$ gives the *distance* between a and b. (See Figure 0.7.)

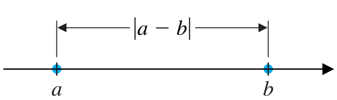


FIGURE 0.7
The distance between a and b

EXAMPLE 1.5 Solving an Inequality Containing an Absolute Value

Solve the inequality

$$|x - 2| < 5. \tag{1.3}$$

Solution Before you start trying to solve this, take a few moments to read what it *says*. Since $|x - 2|$ gives the distance from x to 2, (1.3) says that the *distance* from x to 2 must be *less than* 5. So, find all numbers x whose distance from 2 is less than 5. We indicate the set of all numbers within a distance 5 of 2 in Figure 0.8. You can now read the solution directly from the figure: $-3 < x < 7$ or in interval notation: $(-3, 7)$. ■

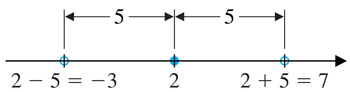


FIGURE 0.8
$|x - 2| < 5$

Many inequalities involving absolute values can be solved simply by reading the inequality correctly, as in example 1.6.

EXAMPLE 1.6 Solving an Inequality with a Sum Inside an Absolute Value

Solve the inequality

$$|x + 4| \le 7. \tag{1.4}$$

Solution To use our distance interpretation, we must first rewrite (1.4) as

$$|x - (-4)| \le 7.$$

This now says that the distance from x to -4 is less than or equal to 7. We illustrate the solution in Figure 0.9, from which it follows that $-11 \le x \le 3$ or $[-11, 3]$. ■

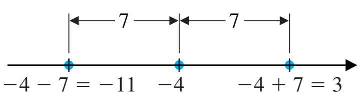


FIGURE 0.9
$|x + 4| \le 7$

Recall that for any real number $r > 0$, $|x| < r$ is equivalent to the following inequality not involving absolute values:

$$-r < x < r.$$

In example 1.7, we use this to revisit the inequality from example 1.5.

EXAMPLE 1.7 An Alternative Method for Solving Inequalities

Solve the inequality $|x - 2| < 5$.

Solution This is equivalent to the two-sided inequality

$$-5 < x - 2 < 5.$$

Adding 2 to each term, we get the solution

$$-3 < x < 7,$$

or in interval notation $(-3, 7)$, as before. ■

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is a simple consequence of the Pythagorean Theorem and is given by

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We illustrate this in Figure 0.10.

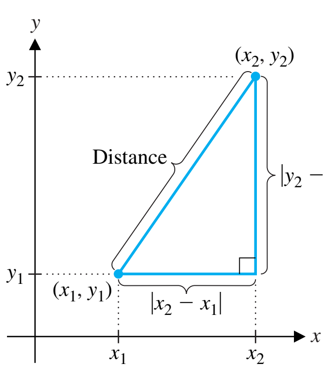


FIGURE 0.10
Distance

EXAMPLE 1.8 Using the Distance Formula

Find the distance between the points (1, 2) and (3, 4).

Solution The distance between (1, 2) and (3, 4) is

$$d\{(1,2),(3,4)\} = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8}.$$

Equations of Lines

Year	*U.S. Population*
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,709,873

x	y
0	179
10	203
20	227
30	249

Transformed data

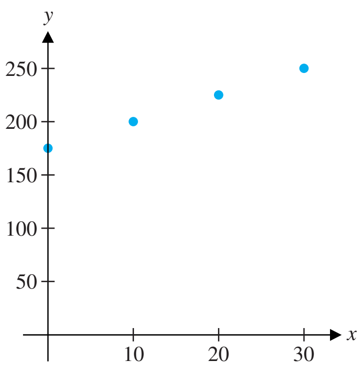


FIGURE 0.11
Population data

The federal government conducts a nationwide census every 10 years to determine the population. Population data for the last several decades are shown in the accompanying table.

One difficulty with analyzing these data is that the numbers are so large. This problem is remedied by **transforming** the data. We can simplify the year data by defining x to be the number of years since 1960. Then, 1960 corresponds to $x = 0$, 1970 corresponds to $x = 10$ and so on. The population data can be simplified by rounding the numbers to the nearest million. The transformed data are shown in the accompanying table and a scatter plot of these data points is shown in Figure 0.11.

Most people would say that the points in Figure 0.11 appear to form a straight line. (Use a ruler and see if you agree.) To determine whether the points are, in fact, on the same line (such points are called **colinear**), we might consider the population growth in each of the indicated decades. From 1960 to 1970, the growth was 24 million. (That is, to move from the first point to the second, you increase x by 10 and increase y by 24.) Likewise, from 1970 to 1980, the growth was 24 million. However, from 1980 to 1990, the growth was only 22 million. Since the rate of growth is not constant, the data points do not fall on a line. Notice that to stay on the same line, y would have had to increase by 24 again. The preceding argument involves the familiar concept of *slope*.

DEFINITION 1.2

For $x_1 \neq x_2$, the **slope** of the straight line through the points (x_1, y_1) and (x_2, y_2) is the number

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1.5}$$

When $x_1 = x_2$, the line through (x_1, y_1) and (x_2, y_2) is **vertical** and the slope is undefined.

We often describe slope as "the change in y divided by the change in x," written $\frac{\Delta y}{\Delta x}$, or more simply as $\frac{\text{Rise}}{\text{Run}}$ (see Figure 0.12a on the following page).

The slope of a straight line is the same no matter which two points on the line you select. Referring to Figure 0.12b (where the line has positive slope), notice that for any four points A, B, D and E on the line, the two right triangles ΔABC and ΔDEF are similar. Recall that for similar triangles, the ratios of corresponding sides must be the same. In this case, this says that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x'}$$

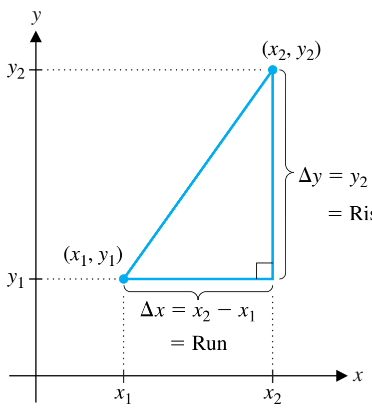


FIGURE 0.12a
Slope

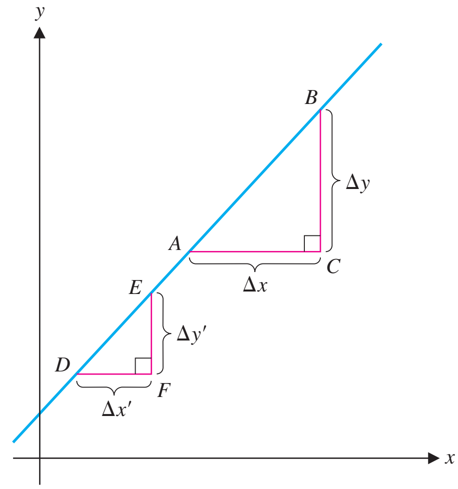


FIGURE 0.12b
Similar triangles and slope

and so, the slope is the same no matter which two points on the line are selected. Furthermore, a line is the only curve with constant slope. Notice that a line is **horizontal** if and only if its slope is zero.

EXAMPLE 1.9 Finding the Slope of a Line

Find the slope of the line through the points (4, 3) and (2, 5).

Solution From (1.5), we get

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-4} = \frac{2}{-2} = -1. \blacksquare$$

EXAMPLE 1.10 Using Slope to Determine if Points Are Colinear

Use slope to determine whether the points (1, 2), (3, 10) and (4, 14) are colinear.

Solution First, notice that the slope of the line joining (1, 2) and (3, 10) is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-2}{3-1} = \frac{8}{2} = 4.$$

Similarly, the slope through the line joining (3, 10) and (4, 14) is

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14-10}{4-3} = 4.$$

Since the slopes are the same, the points must be colinear. ■

Recall that if you know the slope and a point through which the line must pass, you have enough information to graph the line. The easiest way to graph a line is to plot two points and then draw the line through them. In this case, you need only to find a second point.

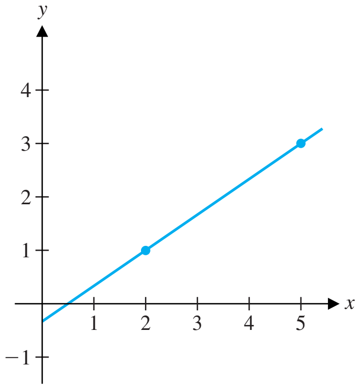


FIGURE 0.13a
Graph of straight line

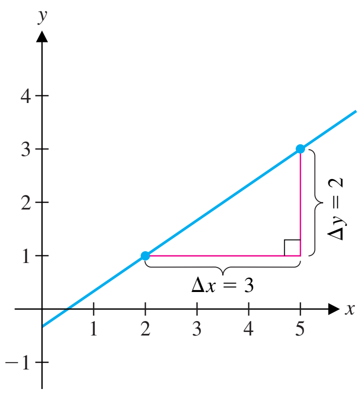


FIGURE 0.13b
Using slope to find a second point

EXAMPLE 1.11 Graphing a Line

If a line passes through the point (2, 1) with slope $\frac{2}{3}$, find a second point on the line and then graph the line.

Solution Since slope is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we take $m = \frac{2}{3}$, $y_1 = 1$ and $x_1 = 2$, to obtain

$$\frac{2}{3} = \frac{y_2 - 1}{x_2 - 2}.$$

You are free to choose the x-coordinate of the second point. For instance, to find the point at $x_2 = 5$, substitute this in and solve. From

$$\frac{2}{3} = \frac{y_2 - 1}{5 - 2} = \frac{y_2 - 1}{3},$$

we get $2 = y_2 - 1$ or $y_2 = 3$. A second point is then (5, 3). The graph of the line is shown in Figure 0.13a. An alternative method for finding a second point is to use the slope

$$m = \frac{2}{3} = \frac{\Delta y}{\Delta x}.$$

The slope of $\frac{2}{3}$ says that if we move three units to the right, we must move two units up to stay on the line, as illustrated in Figure 0.13b. ■

In example 1.11, the choice of $x = 5$ was entirely arbitrary; you can choose any x-value you want to find a second point. Further, since x can be any real number, you can leave x as a variable and write out an equation satisfied by any point (x, y) on the line. In the general case of the line through the point (x_0, y_0) with slope m, we have from (1.5) that

$$m = \frac{y - y_0}{x - x_0}. \tag{1.6}$$

Multiplying both sides of (1.6) by $(x - x_0)$, we get

$$y - y_0 = m(x - x_0)$$

or

POINT-SLOPE FORM OF A LINE

$$y = m(x - x_0) + y_0. \tag{1.7}$$

Equation (1.7) is called the **point-slope form** of the line.

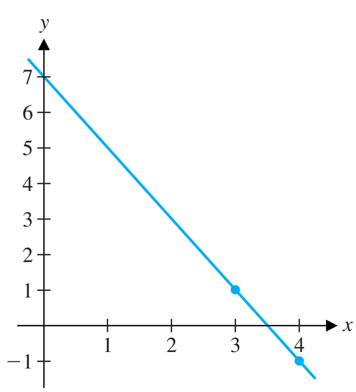


FIGURE 0.14
$y = -2(x - 3) + 1$

EXAMPLE 1.12 Finding the Equation of a Line Given Two Points

Find an equation of the line through the points (3, 1) and (4, −1), and graph the line.

Solution From (1.5), the slope is $m = \dfrac{-1 - 1}{4 - 3} = \dfrac{-2}{1} = -2$. Using (1.7) with slope $m = -2$, x-coordinate $x_0 = 3$ and y-coordinate $y_0 = 1$, we get the equation of the line:

$$y = -2(x - 3) + 1. \tag{1.8}$$

To graph the line, plot the points (3, 1) and (4, −1), and you can easily draw the line seen in Figure 0.14. ■

In example 1.12, you may be tempted to simplify the expression for y given in (1.8). As it turns out, the point-slope form of the equation is often the most convenient to work with. So, we will typically not ask you to rewrite this expression in other forms. At times, a form of the equation called the **slope-intercept form** is more convenient. This has the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept (i.e., the place where the graph crosses the y-axis). In example 1.12, you simply multiply out (1.8) to get $y = -2x + 6 + 1$ or

$$y = -2x + 7.$$

As you can see from Figure 0.14, the graph crosses the y-axis at $y = 7$.

Theorem 1.2 presents a familiar result on parallel and perpendicular lines.

THEOREM 1.2

Two (nonvertical) lines are **parallel** if they have the same slope. Further, any two vertical lines are parallel. Two (nonvertical) lines of slope m_1 and m_2 are **perpendicular** whenever the product of their slopes is -1 (i.e., $m_1 \cdot m_2 = -1$). Also, any vertical line and any horizontal line are perpendicular.

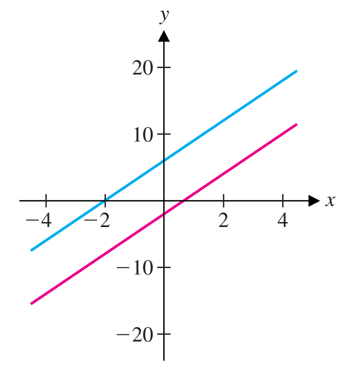


FIGURE 0.15
Parallel lines

Since we can read the slope from the equation of a line, it's a simple matter to determine when two lines are parallel or perpendicular. We illustrate this in examples 1.13 and 1.14.

EXAMPLE 1.13 Finding the Equation of a Parallel Line

Find an equation of the line parallel to $y = 3x - 2$ and through the point $(-1, 3)$.

Solution It's easy to read the slope of the line from the equation: $m = 3$. The equation of the parallel line is then

$$y = 3[x - (-1)] + 3$$

or simply $y = 3(x + 1) + 3$. We show a graph of both lines in Figure 0.15. ■

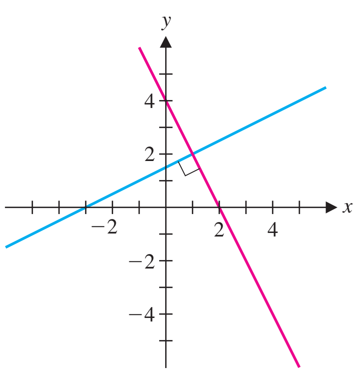


FIGURE 0.16
Perpendicular lines

EXAMPLE 1.14 Finding the Equation of a Perpendicular Line

Find an equation of the line perpendicular to $y = -2x + 4$ and intersecting the line at the point $(1, 2)$.

Solution The slope of $y = -2x + 4$ is -2. The slope of the perpendicular line is then $-1/(-2) = \frac{1}{2}$. Since the line must pass through the point $(1, 2)$, the equation of the perpendicular line is

$$y = \frac{1}{2}(x - 1) + 2.$$

We show a graph of the two lines in Figure 0.16. ■

We now return to this subsection's introductory example and use the equation of a line to estimate the population in the year 2000.

EXAMPLE 1.15 Using a Line to Estimate Population

Given the population data for the census years 1960, 1970, 1980 and 1990, estimate the population for the year 2000.

Solution We began this subsection by showing that the points in the corresponding table are not colinear. Nonetheless, they are *nearly* colinear. So, why not use the straight line connecting the last two points (20, 227) and (30, 249) (corresponding to the populations in the years 1980 and 1990) to estimate the population in 2000? (This is a simple example of a more general procedure called **extrapolation.**) The slope of the line joining the two data points is

$$m = \frac{249 - 227}{30 - 20} = \frac{22}{10} = \frac{11}{5}.$$

The equation of the line is then

$$y = \frac{11}{5}(x - 30) + 249.$$

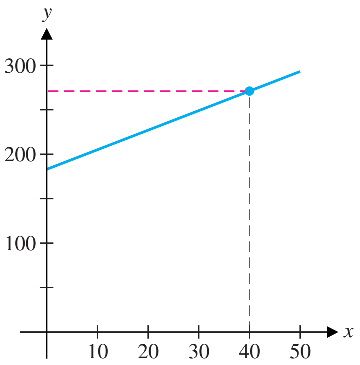


FIGURE 0.17
Population

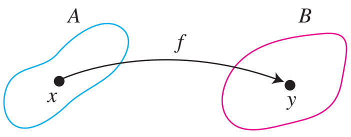
See Figure 0.17 for a graph of the line. If we follow this line to the point corresponding to $x = 40$ (the year 2000), we have the estimated population

$$\frac{11}{5}(40 - 30) + 249 = 271.$$

That is, the estimated population is 271 million people. The actual census figure for 2000 was 281 million, which indicates that the U.S. population has grown at a rate that is faster than linear. ■

Functions

For any two subsets A and B of the real line, we make the following familiar definition.



DEFINITION 1.3

A **function** f is a rule that assigns *exactly one* element y in a set B to each element x in a set A. In this case, we write $y = f(x)$.

We call the set A the **domain** of f. The set of all values $f(x)$ in B is called the **range** of f. That is, the range of f is $\{f(x) \mid x \in A\}$. Unless explicitly stated otherwise, the domain of a function f is the largest set of real numbers for which the function is defined. We refer to x as the **independent variable** and to y as the **dependent variable.**

REMARK 1.2

Functions can be defined by simple formulas, such as $f(x) = 3x + 2$, but in general, any correspondence meeting the requirement of matching exactly *one* y to each x defines a function.

By the **graph** of a function f, we mean the graph of the equation $y = f(x)$. That is, the graph consists of all points (x, y), where x is in the domain of f and where $y = f(x)$.

Notice that not every curve is the graph of a function, since for a function, only one y-value corresponds to a given value of x. You can graphically determine whether a curve is the graph of a function by using the **vertical line test:** if any vertical line intersects the graph in more than one point, the curve is not the graph of a function.

EXAMPLE 1.16 Using the Vertical Line Test

Determine which of the curves in Figures 0.18a and 0.18b correspond to functions.

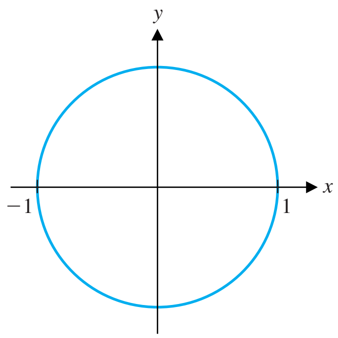


FIGURE 0.18a

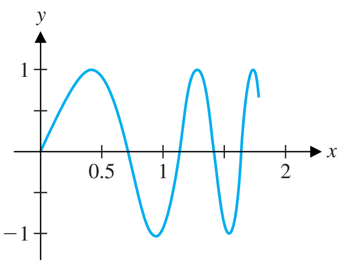


FIGURE 0.18b

Solution Notice that the circle in Figure 0.18a is not the graph of a function, since a vertical line at $x = 0.5$ intersects the circle twice (see Figure 0.19a). The graph in Figure 0.18b is the graph of a function, even though it swings up and down repeatedly. Although horizontal lines intersect the graph repeatedly, vertical lines, such as the one at $x = 1.2$, intersect only once (see Figure 0.19b). ■

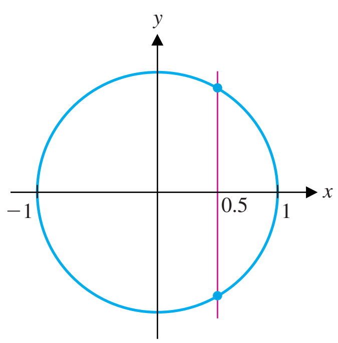


FIGURE 0.19a
Curve fails vertical line test

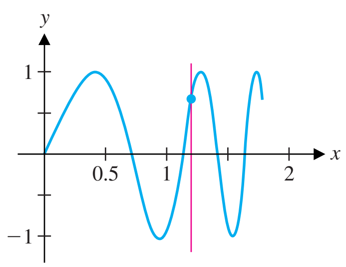


FIGURE 0.19b
Curve passes vertical line test

You are already familiar with a number of different types of functions, and we will only briefly review these here and in sections 0.4 and 0.5. The functions that you are probably most familiar with are *polynomials*. These are the simplest functions to work with because they are defined entirely in terms of arithmetic.

DEFINITION 1.4

A **polynomial** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and $n \geq 0$ is an integer (the **degree** of the polynomial).

Note that the domain of every polynomial function is the entire real line. Further, recognize that the graph of the linear (degree 1) polynomial $f(x) = ax + b$ is a straight line.

EXAMPLE 1.17 Sample Polynomials

The following are all examples of polynomials:

$f(x) = 2$ (polynomial of degree 0 or **constant**),

$f(x) = 3x + 2$ (polynomial of degree 1 or **linear** polynomial),

$f(x) = 5x^2 - 2x + 1$ (polynomial of degree 2 or **quadratic** polynomial),

$f(x) = x^3 - 2x + 1$ (polynomial of degree 3 or **cubic** polynomial),

$f(x) = -6x^4 + 12x^2 - 3x + 13$ (polynomial of degree 4 or **quartic** polynomial), and

$$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3 \text{ (polynomial of degree 5 or \textbf{quintic} polynomial).}$$

We show graphs of these six functions in Figures 0.20a–0.20f.

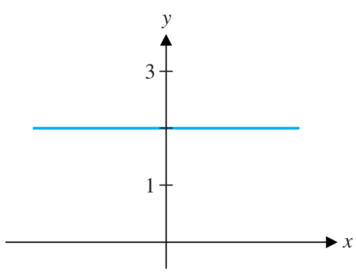


FIGURE 0.20a
$f(x) = 2$

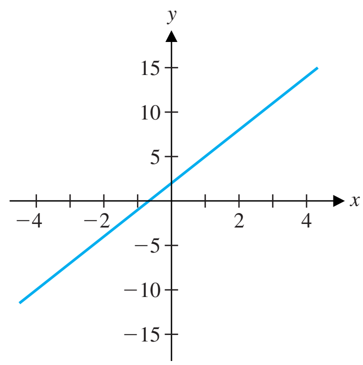


FIGURE 0.20b
$f(x) = 3x + 2$

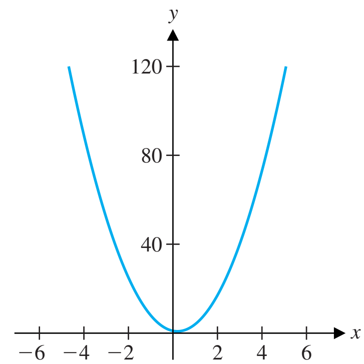


FIGURE 0.20c
$f(x) = 5x^2 - 2x + 1$

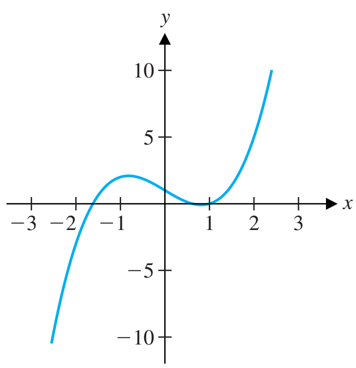


FIGURE 0.20d
$f(x) = x^3 - 2x + 1$

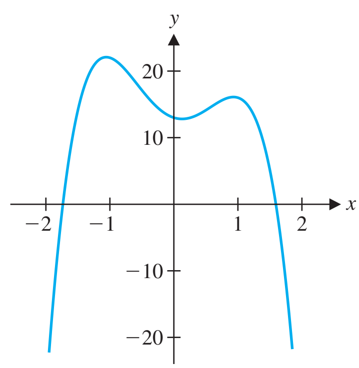


FIGURE 0.20e
$f(x) = -6x^4 + 12x^2 - 3x + 13$

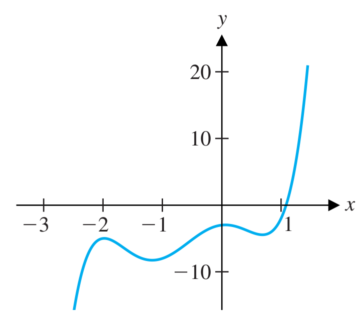


FIGURE 0.20f
$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3$

DEFINITION 1.5

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials, is called a **rational** function.

Notice that since $p(x)$ and $q(x)$ are polynomials, they are both defined for all x, and so, the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is defined for all x for which $q(x) \neq 0$.

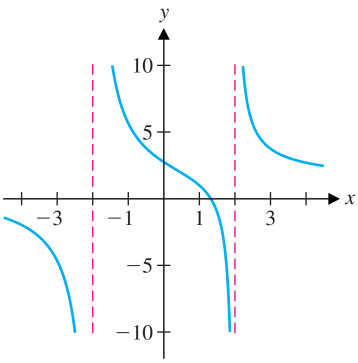


FIGURE 0.21
$f(x) = \dfrac{x^2 + 7x - 11}{x^2 - 4}$

EXAMPLE 1.18 A Sample Rational Function

Find the domain of the function

$$f(x) = \frac{x^2 + 7x - 11}{x^2 - 4}.$$

Solution Here, $f(x)$ is a rational function. We show a graph in Figure 0.21. Its domain consists of those values of x for which the denominator is nonzero. Notice that

$$x^2 - 4 = (x - 2)(x + 2)$$

and so, the denominator is zero if and only if $x = \pm 2$. This says that the domain of f is

$$\{x \in \mathbb{R} \mid x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$ ■

The **square root** function is defined in the usual way. When we write $y = \sqrt{x}$, we mean that y is that number for which $y^2 = x$ and $y \geq 0$. In particular, $\sqrt{4} = 2$. Be careful not to write erroneous statements such as $\sqrt{4} = \pm 2$. In particular, be careful to write

$$\sqrt{x^2} = |x|.$$

Since $\sqrt{x^2}$ is asking for the *nonnegative* number whose square is x^2, we are looking for $|x|$ and not x. We can say

$$\sqrt{x^2} = x, \textit{ only } \text{for } x \geq 0.$$

Similarly, for any integer $n \geq 2$, $y = \sqrt[n]{x}$ whenever $y^n = x$, where for n even, $x \geq 0$ and $y \geq 0$.

EXAMPLE 1.19 Finding the Domain of a Function Involving a Square Root or a Cube Root

Find the domains of $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \sqrt[3]{x^2 - 4}$.

Solution Since even roots are defined only for nonnegative values, $f(x)$ is defined only for $x^2 - 4 \geq 0$. Notice that this is equivalent to having $x^2 \geq 4$, which occurs when $x \geq 2$ or $x \leq -2$. The domain of f is then $(-\infty, -2] \cup [2, \infty)$. On the other hand, odd roots are defined for both positive and negative values. Consequently, the domain of $g(x)$ is the entire real line, $(-\infty, \infty)$. ■

We often find it useful to label intercepts and other significant points on a graph. Finding these points typically involves solving equations. A solution of the equation $f(x) = 0$ is called a **zero** of the function f or a **root** of the equation $f(x) = 0$. Notice that a zero of the function f corresponds to an x-intercept of the graph of $y = f(x)$.

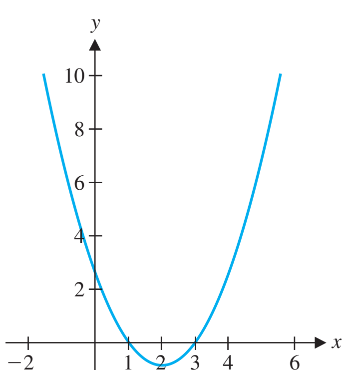


FIGURE 0.22
$y = x^2 - 4x + 3$

EXAMPLE 1.20 Finding Zeros by Factoring

Find all x- and y-intercepts of $f(x) = x^2 - 4x + 3$.

Solution To find the y-intercept, set $x = 0$ to obtain

$$y = 0 - 0 + 3 = 3.$$

To find the x-intercepts, solve the equation $f(x) = 0$. In this case, we can factor to get

$$f(x) = x^2 - 4x + 3 = (x - 1)(x - 3) = 0.$$

You can now read off the zeros: $x = 1$ and $x = 3$, as indicated in Figure 0.22. ■

Unfortunately, factoring is not always so easy. Of course, for the quadratic equation

$$ax^2 + bx + c = 0$$

(for $a \neq 0$), the solution(s) are given by the familiar **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 1.21 Finding Zeros Using the Quadratic Formula

Find the zeros of $f(x) = x^2 - 5x - 12$.

Solution You probably won't have much luck trying to factor this. However, from the quadratic formula, we have

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 + 48}}{2} = \frac{5 \pm \sqrt{73}}{2}.$$

So, the two solutions are given by $x = \frac{5}{2} + \frac{\sqrt{73}}{2} \approx 6.772$ and $x = \frac{5}{2} - \frac{\sqrt{73}}{2} \approx -1.772$. (No wonder you couldn't factor the polynomial!) ■

Finding zeros of polynomials of degree higher than 2 and other functions is usually trickier and is sometimes impossible. At the least, you can always find an approximation of any zero(s) by using a graph to zoom in closer to the point(s) where the graph crosses the x-axis, as we'll illustrate shortly. A more basic question, though, is to determine *how many* zeros a given function has. In general, there is no way to answer this question without the use of calculus. For the case of polynomials, however, Theorem 1.3 (a consequence of the Fundamental Theorem of Algebra) provides a clue.

THEOREM 1.3

A polynomial of degree n has *at most* n distinct zeros.

REMARK 1.3

Polynomials may also have complex zeros. For instance, $f(x) = x^2 + 1$ has only the complex zeros $x = \pm i$, where i is the imaginary number defined by $i = \sqrt{-1}$.

Notice that Theorem 1.3 does not say how many zeros a given polynomial has, but rather, that the *maximum* number of distinct (i.e., different) zeros is the same as the degree. A polynomial of degree n may have anywhere from 0 to n distinct real zeros. However, polynomials of odd degree must have *at least one* real zero. For instance, for the case of a cubic polynomial, we have one of the three possibilities illustrated in Figures 0.23a, 0.23b and 0.23c on the following page.

In these three figures, we show the graphs of cubic polynomials with 1, 2 and 3 distinct, real zeros, respectively. These are the graphs of the functions

$$f(x) = x^3 - 2x^2 + 3 = (x+1)(x^2 - 3x + 3),$$
$$g(x) = x^3 - x^2 - x + 1 = (x+1)(x-1)^2$$

and

$$h(x) = x^3 - 3x^2 - x + 3 = (x+1)(x-1)(x-3),$$

respectively. Note that you can see from the factored form where the zeros are (and how many there are).

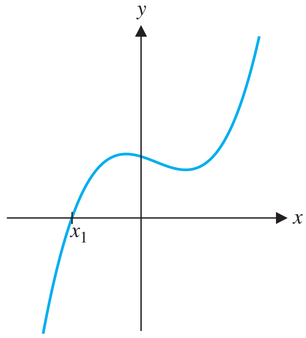


FIGURE 0.23a
One zero

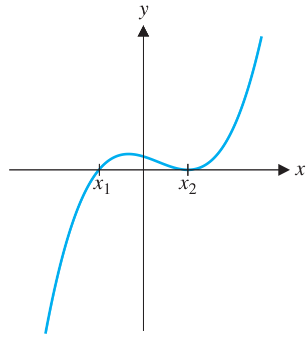


FIGURE 0.23b
Two zeros

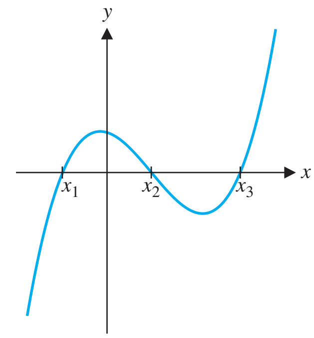


FIGURE 0.23c
Three zeros

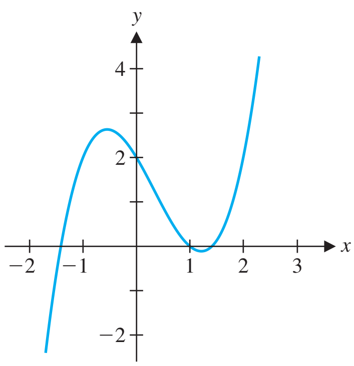


FIGURE 0.24a
$y = x^3 - x^2 - 2x + 2$

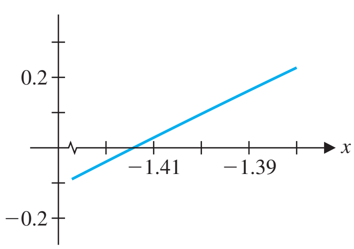


FIGURE 0.24b
Zoomed in on zero near $x = -1.4$

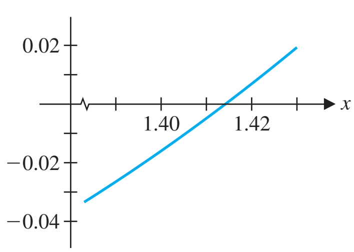


FIGURE 0.24c
Zoomed in on zero near $x = 1.4$

Theorem 1.4 provides an important connection between factors and zeros of polynomials.

THEOREM 1.4 (Factor Theorem)

For any polynomial f, $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$.

EXAMPLE 1.22 Finding the Zeros of a Cubic Polynomial

Find the zeros of $f(x) = x^3 - x^2 - 2x + 2$.

Solution By calculating $f(1)$, you can see that one zero of this function is $x = 1$, but how many other zeros are there? A graph of the function (see Figure 0.24a) shows that there are two other zeros of f, one near $x = -1.5$ and one near $x = 1.5$. You can find these zeros more precisely by using your graphing calculator or computer algebra system to zoom in on the locations of these zeros (as shown in Figures 0.24b and 0.24c). From these zoomed graphs it is clear that the two remaining zeros of f are near $x = 1.41$ and $x = -1.41$. You can make these estimates more precise by zooming in even more closely. Most graphing calculators and computer algebra systems can also find approximate zeros, using a built-in "solve" program. In Chapter 3, we present a versatile method (called Newton's method) for obtaining accurate approximations to zeros. The only way to find the exact solutions is to factor the expression (using either long division or synthetic division). Here, we have

$$f(x) = x^3 - x^2 - 2x + 2 = (x - 1)(x^2 - 2) = (x - 1)(x - \sqrt{2})(x + \sqrt{2}),$$

from which you can see that the zeros are $x = 1$, $x = \sqrt{2}$ and $x = -\sqrt{2}$. ■

Recall that to find the points of intersection of two curves defined by $y = f(x)$ and $y = g(x)$, we set $f(x) = g(x)$ to find the x-coordinates of any points of intersection.

EXAMPLE 1.23 Finding the Intersections of a Line and a Parabola

Find the points of intersection of the parabola $y = x^2 - x - 5$ and the line $y = x + 3$.

Solution A sketch of the two curves (see Figure 0.25) shows that there are two intersections, one near $x = -2$ and the other near $x = 4$. To determine these precisely,

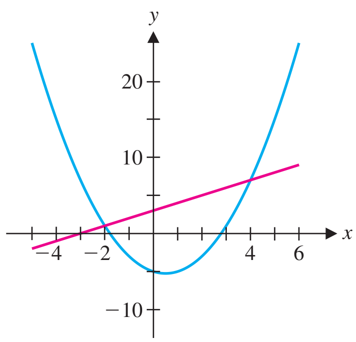


FIGURE 0.25
$y = x + 3$ and $y = x^2 - x - 5$

we set the two functions equal and solve for x:

$$x^2 - x - 5 = x + 3.$$

Subtracting $(x + 3)$ from both sides leaves us with

$$0 = x^2 - 2x - 8 = (x - 4)(x + 2).$$

This says that the solutions are exactly $x = -2$ and $x = 4$. We compute the corresponding y-values from the equation of the line $y = x + 3$ (or the equation of the parabola). The points of intersection are then $(-2, 1)$ and $(4, 7)$. Notice that these are consistent with the intersections seen in Figure 0.25. ■

Unfortunately, you won't always be able to solve equations exactly, as we did in examples 1.20–1.23. We explore some options for dealing with more difficult equations in section 0.2.

EXERCISES 0.1

WRITING EXERCISES

1. If the slope of the line passing through points A and B equals the slope of the line passing through points B and C, explain why the points A, B and C are colinear.
2. If a graph fails the vertical line test, it is not the graph of a function. Explain this result in terms of the definition of a function.
3. You should not automatically write the equation of a line in slope-intercept form. Compare the following forms of the same line: $y = 2.4(x - 1.8) + 0.4$ and $y = 2.4x - 3.92$. Given $x = 1.8$, which equation would you rather use to compute y? How about if you are given $x = 0$? For $x = 8$, is there any advantage to one equation over the other? Can you quickly read off the slope from either equation? Explain why neither form of the equation is "better."
4. To understand Definition 1.1, you must believe that $|x| = -x$ for negative x's. Using $x = -3$ as an example, explain in words why multiplying x by -1 produces the same result as taking the absolute value of x.

In exercises 1–4, determine if the points are colinear.

1. $(2, 1), (0, 2), (4, 0)$
2. $(3, 1), (4, 4), (5, 8)$
3. $(4, 1), (3, 2), (1, 3)$
4. $(1, 2), (2, 5), (4, 8)$

In exercises 5–10, find the slope of the line through the given points.

5. $(1, 2), (3, 6)$
6. $(1, 2), (3, 3)$
7. $(3, -6), (1, -1)$
8. $(1, -2), (-1, -3)$
9. $(0.3, -1.4), (-1.1, -0.4)$
10. $(1.2, 2.1), (3.1, 2.4)$

In exercises 11–16, find a second point on the line with slope *m* and point *P*, graph the line and find an equation of the line.

11. $m = 2$, $P = (1, 3)$
12. $m = -2$, $P = (1, 4)$
13. $m = 0$, $P = (-1, 1)$
14. $m = \frac{1}{2}$, $P = (2, 1)$
15. $m = 1.2$, $P = (2.3, 1.1)$
16. $m = -\frac{1}{4}$, $P = (-2, 1)$

In exercises 17–22, determine if the lines are parallel, perpendicular, or neither.

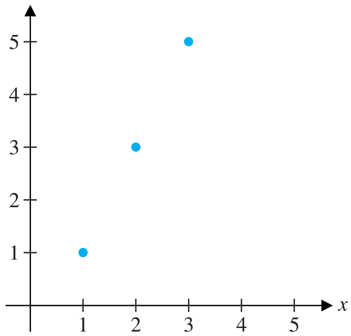
17. $y = 3(x - 1) + 2$ and $y = 3(x + 4) - 1$
18. $y = 2(x - 3) + 1$ and $y = 4(x - 3) + 1$
19. $y = -2(x + 1) - 1$ and $y = \frac{1}{2}(x - 2) + 3$
20. $y = 2x - 1$ and $y = -2x + 2$
21. $y = 3x + 1$ and $y = -\frac{1}{3}x + 2$
22. $x + 2y = 1$ and $2x + 4y = 3$

In exercises 23–26, find an equation of a line through the given point and (a) parallel to and (b) perpendicular to the given line.

23. $y = 2(x + 1) - 2$ at $(2, 1)$
24. $y = 3(x - 2) + 1$ at $(0, 3)$
25. $y = 2x + 1$ at $(3, 1)$
26. $y = 1$ at $(0, -1)$

In exercises 27–30, find an equation of the line through the given points and compute the y-coordinate of the point on the line corresponding to $x = 4$.

27.



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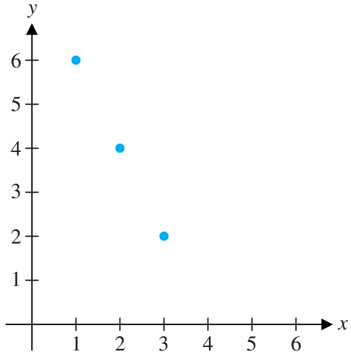

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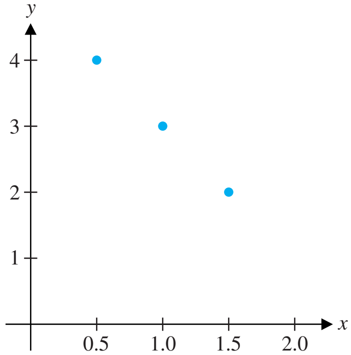

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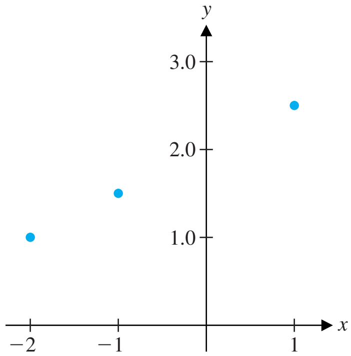
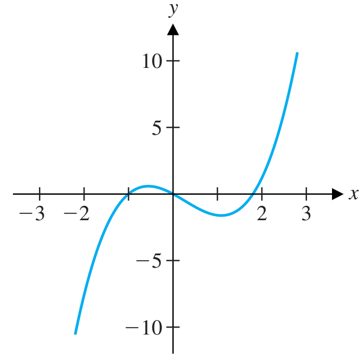
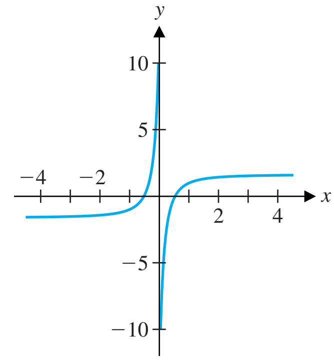


In exercises 31–34, use the vertical line test to determine whether the curve is the graph of a function.

31.



32.



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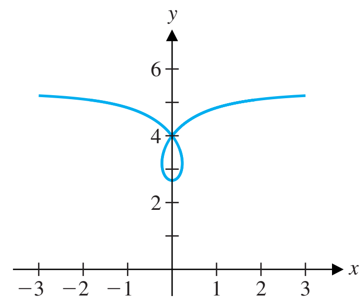

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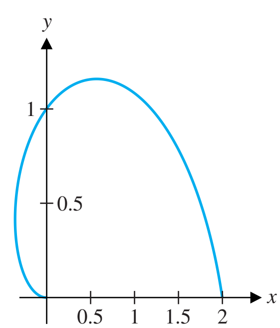


In exercises 35–40, identify the given function as polynomial, rational, both or neither.

35. $f(x) = x^3 - 4x + 1$

36. $f(x) = 3 - 2x + x^4$

37. $f(x) = \dfrac{x^2 + 2x - 1}{x + 1}$

38. $f(x) = \dfrac{x^3 + 4x - 1}{x^4 - 1}$

39. $f(x) = \sqrt{x^2 + 1}$

40. $f(x) = 2x - x^{2/3} - 6$

In exercises 41–46, find the domain of the function.

41. $f(x) = \sqrt{x + 2}$

42. $f(x) = \sqrt{2x + 1}$

43. $f(x) = \sqrt[3]{x - 1}$

44. $f(x) = \sqrt{x^2 - 4}$

45. $f(x) = \dfrac{4}{x^2 - 1}$

46. $f(x) = \dfrac{4x}{x^2 + 2x - 6}$

In exercises 47–50, find the indicated function values.

47. $f(x) = x^2 - x - 1$; $f(0)$, $f(2)$, $f(-3)$, $f(1/2)$

48. $f(x) = \dfrac{x + 1}{x - 1}$; $f(0)$, $f(2)$, $f(-2)$, $f(1/2)$

49. $f(x) = \sqrt{x + 1}$; $f(0)$, $f(3)$, $f(-1)$, $f(1/2)$

50. $f(x) = \dfrac{3}{x}$; $f(1)$, $f(10)$, $f(100)$, $f(1/3)$

In exercises 51–54, a brief description is given of a physical situation. For the indicated variable, state a reasonable domain.

51. A parking deck is to be built; x = width of deck (in feet).

52. A parking deck is to be built on a 200′-by-200′ lot; x = width of deck (in feet).

53. A new candy bar is to be sold; x = number of candy bars sold in the first month.

54. A new candy bar is to be sold; x = cost of candy bar (in cents).

In exercises 55–58, discuss whether you think y would be a function of x.

55. y = grade you get on an exam, x = number of hours you study

56. y = probability of getting lung cancer, x = number of cigarettes smoked per day

57. y = a person's weight, x = number of minutes exercising per day

58. y = speed at which an object falls, x = weight of object

59. Figure A shows the speed of a bicyclist as a function of time. For the portions of this graph that are flat, what is happening to the bicyclist's speed? What is happening to the bicyclist's speed when the graph goes up? down? Identify the portions of the graph that correspond to the bicyclist going uphill; downhill.

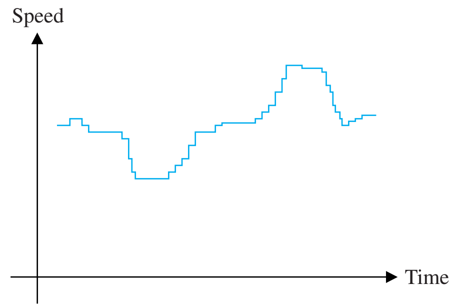


FIGURE A
Bicycle speed

60. Figure B shows the population of a small country as a function of time. During the time period shown, the country experienced two influxes of immigrants, a war and a plague. Identify these important events.

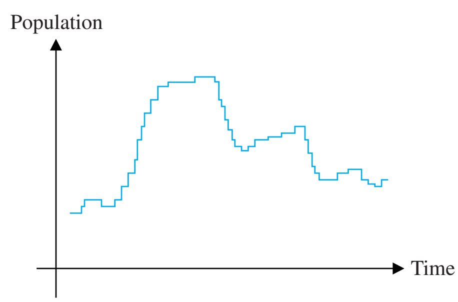


FIGURE B
Population

In exercises 61–66, find all intercepts of the given graph.

61. $y = x^2 - 2x - 8$

62. $y = x^2 + 4x + 4$

63. $y = x^3 - 8$

64. $y = x^3 - 3x^2 + 3x - 1$

65. $y = \dfrac{x^2 - 4}{x + 1}$

66. $y = \dfrac{2x - 1}{x^2 - 4}$

In exercises 67–74, factor and/or use the quadratic formula to find all zeros of the given function.

67. $f(x) = x^2 - 4x + 3$

68. $f(x) = x^2 + x - 12$

69. $f(x) = x^2 - 4x + 2$

70. $f(x) = 2x^2 + 4x - 1$

71. $f(x) = x^3 - 3x^2 + 2x$

72. $f(x) = x^3 - 2x^2 - x + 2$

73. $f(x) = x^6 + x^3 - 2$

74. $f(x) = x^3 + x^2 - 4x - 4$

75. The boiling point of water (in degrees Fahrenheit) at elevation h (in thousands of feet above sea level) is given by $B(h) = -1.8h + 212$. Find h such that water boils at 98.6°. Why would this altitude be dangerous to humans?

76. The spin rate of a golf ball hit with a 9 iron has been measured at 9100 rpm for a 120-compression ball and at 10,000 rpm for a 60-compression ball. Most golfers use 90-compression balls. If the spin rate is a linear function of compression, find the spin rate for a 90-compression ball. Professional golfers often use 100-compression balls. Estimate the spin rate of a 100-compression ball.

77. The chirping rate of a cricket depends on the temperature. A species of tree cricket chirps 160 times per minute at 79°F and 100 times per minute at 64°F. Find a linear function relating temperature to chirping rate.

78. When describing how to measure temperature by counting cricket chirps, most guides suggest that you count the number of chirps in a 15-second time period. Use exercise 77 to explain why this is a convenient period of time.

79. A person has played a computer game many times. The statistics show that she has won 415 times and lost 120 times, and the winning percentage is listed as 78%. How many times in a row must she win to raise the reported winning percentage to 80%?

EXPLORATORY EXERCISES

1. Suppose you have a machine that will proportionally enlarge a photograph. For example, it could enlarge a 4×6 photograph to 8×12 by doubling the width and height. You could make an 8×10 picture by cropping 1 inch off each side. Explain how you would enlarge a $3\frac{1}{2} \times 5$ picture to an 8×10. A friend returns from Scotland with a $3\frac{1}{2} \times 5$ picture showing the Loch Ness monster in the outer $\frac{1}{4}''$ on the right. If you use your procedure to make an 8×10 enlargement, does Nessie make the cut?

2. Solve the equation $|x-2| + |x-3| = 1$. (Hint: It's an unusual solution, in that it's more than just a couple of numbers.) Then, solve the equation $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$. (Hint: If you make the correct substitution, you can use your solution to the previous equation.)

0.2 GRAPHING CALCULATORS AND COMPUTER ALGEBRA SYSTEMS

The relationships between functions and their graphs are central topics in calculus. Graphing calculators and user-friendly computer software allow you to explore these relationships for a much wider variety of functions than you could with pencil and paper alone. This section presents a general framework for using technology to explore the graphs of functions.

Recall that the graphs of linear functions are straight lines and the graphs of quadratic polynomials are parabolas. One of the goals of this section is for you to become more familiar with the graphs of other functions. The best way to become familiar is through experience, by working example after example.

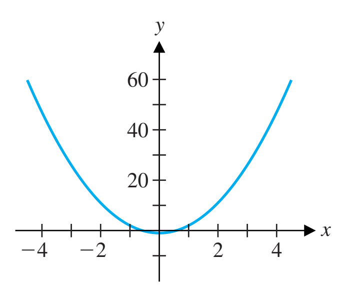


FIGURE 0.26a
$y = 3x^2 - 1$

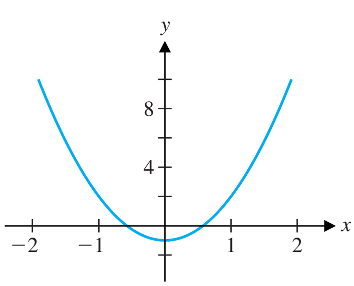


FIGURE 0.26b
$y = 3x^2 - 1$

EXAMPLE 2.1 Generating a Calculator Graph

Use your calculator or computer to sketch a graph of $f(x) = 3x^2 - 1$.

Solution You should get an initial graph that looks something like that in Figure 0.26a. This is simply a parabola opening upward. A graph is often used to search for important points, such as x-intercepts, y-intercepts or peaks and troughs. In this case, we could see these points better if we zoom in, that is, display a smaller range of x- and y-values than the technology has initially chosen for us. The graph in Figure 0.26b shows x-values from $x = -2$ to $x = 2$ and y-values from $y = -2$ to $y = 10$.

You can see more clearly in Figure 0.26b that the parabola bottoms out roughly at the point $(0, -1)$ and crosses the x-axis at approximately $x = -0.5$ and $x = 0.5$. You can make this more precise by doing some algebra. Recall that an x-intercept is a point where $y = 0$ or $f(x) = 0$. Solving $3x^2 - 1 = 0$ gives $3x^2 = 1$ or $x^2 = \frac{1}{3}$, so that $x = \pm\sqrt{\frac{1}{3}} \approx \pm 0.57735$. ■

Notice that in example 2.1, the graph suggested approximate values for the two x-intercepts, but we needed the algebra to find the values exactly. We then used those values to obtain a view of the graph that highlighted the features that we wanted.

Before investigating other graphs, we should say a few words about what a computer- or calculator-generated graph really is. Although we call them graphs, what the computer actually does is light up some tiny screen elements called **pixels.** If the pixels are small enough, the image appears to be a continuous curve or graph.

By **graphing window,** we mean the rectangle defined by the range of x- and y-values displayed. The graphing window can dramatically affect the look of a graph. For example, suppose the x's run from $x = -2$ to $x = 2$. If the computer or calculator screen is wide enough for 400 columns of pixels from left to right, then points will be displayed for $x = -2, x = -1.99, x = -1.98, \ldots$. If there is an interesting feature of this function located between $x = -1.99$ and $x = -1.98$, you will not see it unless you zoom in some. In this case, zooming in would reduce the difference between adjacent x's. Similarly, suppose that the y's run from $y = 0$ to $y = 3$ and that there are 600 rows of pixels from top to bottom. Then, there will be pixels corresponding to $y = 0$, $y = 0.005$, $y = 0.01, \ldots$. Now, suppose that $f(-2) = 0.0049$ and $f(-1.99) = 0.0051$. Before points are plotted, function values are rounded to the nearest y-value, in this case 0.005. You won't be able to see any difference in the y-values of these points. If the actual difference is important, you will have to zoom in some to see it.

REMARK 2.1

Most calculators and computer drawing packages use one of the following two schemes for defining the graphing window for a given function.

- **Fixed graphing window:** Most calculators follow this method. Graphs are plotted in a preselected range of x- and y-values, unless you specify otherwise. For example, the Texas Instruments graphing calculators' default graphing window plots points in the rectangle defined by $-10 \le x \le 10$ and $-10 \le y \le 10$.
- **Automatic graphing window:** Most computer drawing packages and some calculators use this method. Graphs are plotted for a preselected range of x-values and the computer calculates the range of y-values so that all of the calculated points will fit in the window.

Get to know how your calculator or computer software operates, and use it routinely as you progress through this course. You should always be able to reproduce the computer-generated graphs used in this text by adjusting your graphing window appropriately.

Graphs are drawn to provide visual displays of the significant features of a function. What qualifies as *significant* will vary from problem to problem, but often the x- and y-intercepts and points known as **extrema** are of interest. The function value $f(M)$ is called a **local maximum** of the function f if $f(M) \ge f(x)$ for all x's "nearby" $x = M$.

Similarly, the function value $f(m)$ is a **local minimum** of the function f if $f(m) \le f(x)$ for all x's "nearby" $x = m$. A **local extremum** is a function value that is either a local maximum or local minimum. Whenever possible, you should produce graphs that show all intercepts and extrema.

REMARK 2.2

To be precise, $f(M)$ is a local maximum of f if there exist numbers a and b with $a < M < b$ such that $f(M) \ge f(x)$ for all x such that $a < x < b$.

EXAMPLE 2.2 Sketching a Graph

Sketch a graph of $f(x) = x^3 + 4x^2 - 5x - 1$ showing all intercepts and extrema.

Solution Depending on your calculator or computer software, you may initially get a graph that looks like one of those in Figure 0.27a or 0.27b.

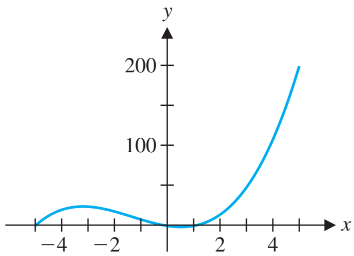


FIGURE 0.27a
$y = x^3 + 4x^2 - 5x - 1$

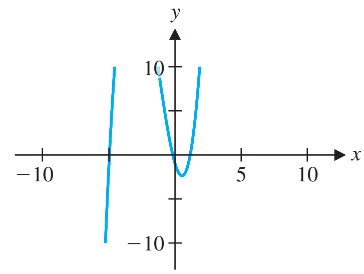


FIGURE 0.27b
$y = x^3 + 4x^2 - 5x - 1$

Neither graph is completely satisfactory, although both should give you the idea of a graph that (reading left to right) rises to a local maximum near $x = -3$, drops to a local minimum near $x = 1$, and then rises again. To get a better graph, notice the scales on the x- and y-axes. The graphing window for Figure 0.27a is the rectangle defined by $-5 \leq x \leq 5$ and $-6 \leq y \leq 203$. The graphing window for Figure 0.27b is defined by the rectangle $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. From either graph, it appears that we need to show y-values larger than 10, but not nearly as large as 203, to see the local maximum. Since all of the significant features appear to lie between $x = -6$ and $x = 6$, one choice for a better window is $-5 \leq x \leq 5$ and $-6 \leq y \leq 30$, as seen in Figure 0.28. There, you can clearly see the three x-intercepts, the local maximum and the local minimum. ■

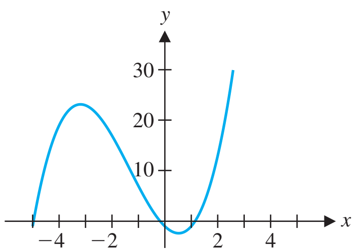


FIGURE 0.28
$y = x^3 + 4x^2 - 5x - 1$

The graph in example 2.2 was produced by a process of trial and error with thoughtful corrections. You are unlikely to get a perfect picture on the first try. However, you can enlarge the graphing window (i.e., *zoom out*) if you need to see more, or shrink the graphing window (i.e., *zoom in*) if the details are hard to see. You should get comfortable enough with your technology that this revision process is routine (and even fun!).

In the exercises, you will be asked to graph a variety of functions and discuss the shapes of the graphs of polynomials of different degrees. Having some knowledge of the general shapes will help you decide whether you have found an acceptable graph. To get you started, we now summarize the different shapes of linear, quadratic and cubic polynomials. Of course, the graphs of linear functions $[f(x) = a_1x + a_0]$ are simply straight lines of slope a_1. Two possibilities are shown in Figures 0.29a and 0.29b.

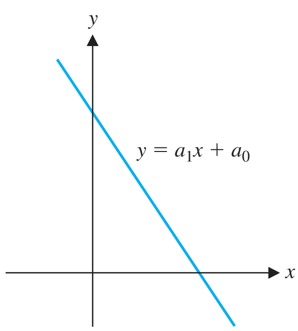


FIGURE 0.29a
Line, $a_1 < 0$

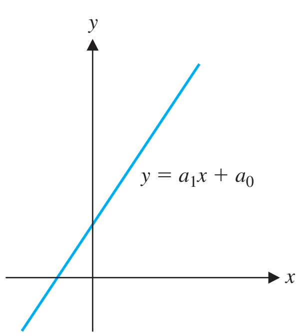


FIGURE 0.29b
Line, $a_1 > 0$

The graphs of quadratic polynomials $[f(x) = a_2x^2 + a_1x + a_0]$ are parabolas. The parabola opens upward if $a_2 > 0$ and opens downward if $a_2 < 0$. We show typical parabolas in Figures 0.30a and 0.30b.

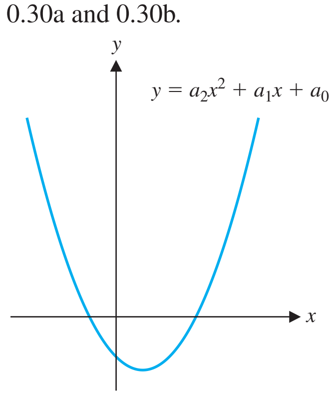


FIGURE 0.30a
Parabola, $a_2 > 0$

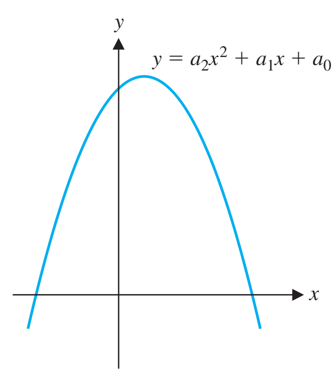


FIGURE 0.30b
Parabola, $a_2 < 0$

The graphs of cubic functions $[f(x) = a_3x^3 + a_2x^2 + a_1x + a_0]$ are somewhat S-shaped. Reading from left to right, the function begins negative and ends positive if $a_3 > 0$, and begins positive and ends negative if $a_3 < 0$, as indicated in Figures 0.31a and 0.31b, respectively.

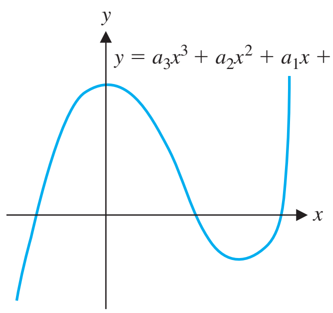


FIGURE 0.31a
Cubic: one max, min, $a_3 > 0$

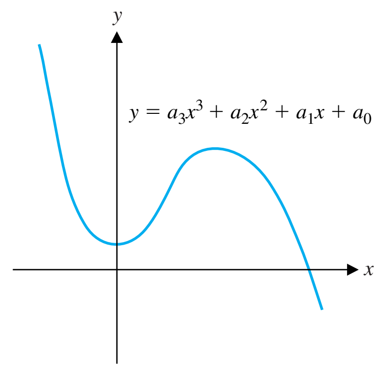


FIGURE 0.31b
Cubic: one max, min, $a_3 < 0$

Some cubics have one local maximum and one local minimum, as do those in Figures 0.31a and 0.31b. Many curves (including all cubics) have what's called an **inflection point,** where the curve changes its shape (from being bent upward, to being bent downward, or vice versa), as indicated in Figures 0.32a and 0.32b.

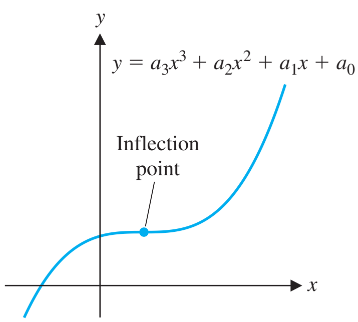


FIGURE 0.32a
Cubic: no max or min, $a_3 > 0$

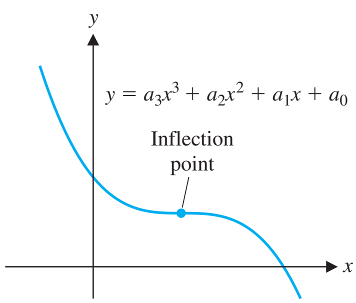


FIGURE 0.32b
Cubic: no max or min, $a_3 < 0$

You can already use your knowledge of the general shapes of certain functions to see how to adjust the graphing window, as in example 2.3.

EXAMPLE 2.3 Sketching the Graph of a Cubic Polynomial

Sketch a graph of the cubic polynomial $f(x) = x^3 - 20x^2 - x + 20$.

Solution Your initial graph probably looks like Figure 0.33a or 0.33b.

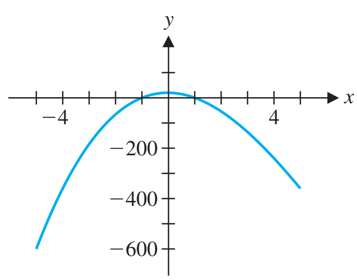


FIGURE 0.33a
$f(x) = x^3 - 20x^2 - x + 20$

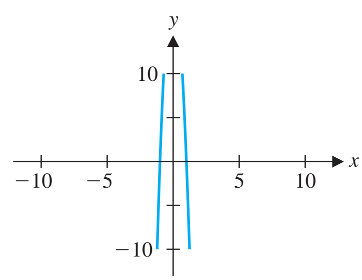


FIGURE 0.33b
$f(x) = x^3 - 20x^2 - x + 20$

However, you should recognize that neither of these graphs looks like a cubic; they look more like parabolas. To see the S-shape behavior in the graph, we need to consider a larger range of x-values. To determine how much larger, we need some of the concepts

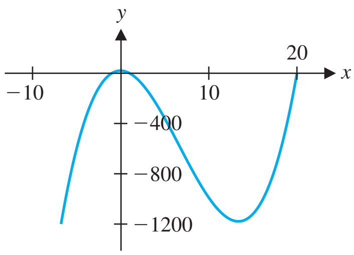


FIGURE 0.33c
$f(x) = x^3 - 20x^2 - x + 20$

of calculus. For the moment, we use trial and error, until the graph resembles the shape of a cubic. You should recognize the characteristic shape of a cubic in Figure 0.33c. Although we now see more of the big picture (often referred to as the **global** behavior of the function), we have lost some of the details (such as the x-intercepts), which we could clearly see in Figures 0.33a and 0.33b (often referred to as the **local** behavior of the function). ■

Rational functions have some properties not found in polynomials, as we see in examples 2.4, 2.5 and 2.6.

EXAMPLE 2.4 Sketching the Graph of a Rational Function

Sketch a graph of $f(x) = \dfrac{x-1}{x-2}$ and describe the behavior of the graph near $x = 2$.

Solution Your initial graph should look something like Figure 0.34a or 0.34b. From either graph, it should be clear that something unusual is happening near $x = 2$. Zooming in closer to $x = 2$ should yield a graph like that in Figure 0.35.

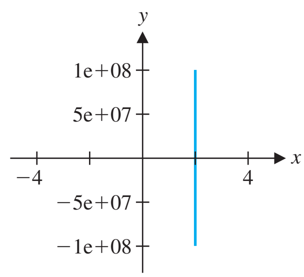


FIGURE 0.34a
$y = \dfrac{x-1}{x-2}$

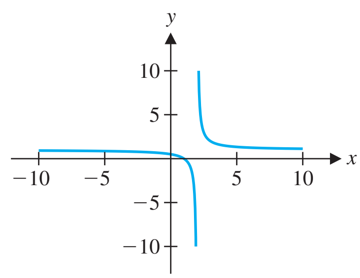


FIGURE 0.34b
$y = \dfrac{x-1}{x-2}$

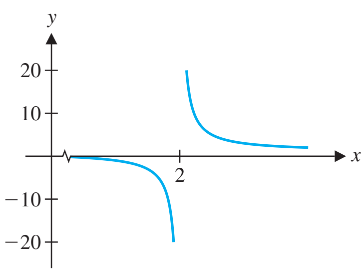


FIGURE 0.35
$y = \dfrac{x-1}{x-2}$

In Figure 0.35, it appears that as x increases up to 2, the function values get more and more negative, while as x decreases down to 2, the function values get more and more positive. This is also observed in the following table of function values.

x	$f(x)$
1.8	−4
1.9	−9
1.99	−99
1.999	−999
1.9999	−9999

x	$f(x)$
2.2	6
2.1	11
2.01	101
2.001	1001
2.0001	10,001

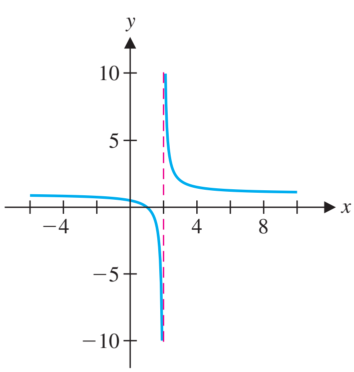


FIGURE 0.36
Vertical asymptote

Note that at $x = 2$, $f(x)$ is undefined. However, as x approaches 2 from the left, the graph veers down sharply. In this case, we say that $f(x)$ tends to $-\infty$. Likewise, as x approaches 2 from the right, the graph rises sharply. Here, we say that $f(x)$ tends to ∞ and there is a **vertical asymptote** at $x = 2$. (We'll define this more carefully in Chapter 1.) It is common to draw a vertical dashed line at $x = 2$ to indicate this (see Figure 0.36). Since $f(2)$ is undefined, there is no point plotted at $x = 2$. ■

Many rational functions have vertical asymptotes. Notice that there is no point plotted on the vertical asymptote since the function is undefined at such an x-value (due to the division by zero when that value of x is substituted in). Given a rational function, you can locate possible vertical asymptotes by finding where the denominator is zero. It turns out that if the numerator is not zero at that point, there is a vertical asymptote at that point.

EXAMPLE 2.5 A Graph with Several Vertical Asymptotes

Find all vertical asymptotes for $f(x) = \dfrac{x-1}{x^2-5x+6}$.

Solution Note that the denominator factors as

$$x^2 - 5x + 6 = (x-2)(x-3),$$

so that the only possible locations for vertical asymptotes are $x = 2$ and $x = 3$. Since neither x-value makes the numerator $(x-1)$ equal to zero, there are vertical asymptotes at both $x = 2$ and $x = 3$. A computer-generated graph gives little indication of how the function behaves near the asymptotes. (See Figure 0.37a and note the scale on the y-axis.)

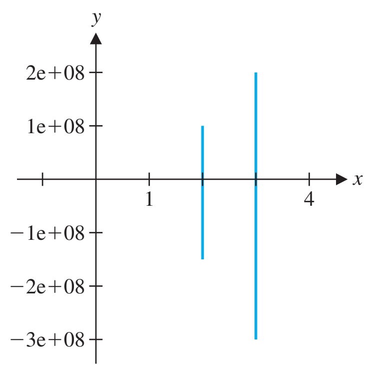


FIGURE 0.37a
$y = \dfrac{x-1}{x^2-5x+6}$

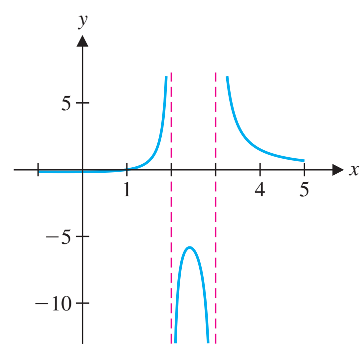


FIGURE 0.37b
$y = \dfrac{x-1}{x^2-5x+6}$

We can improve the graph by zooming in in both the x- and y-directions. Figure 0.37b shows a graph of the same function using the graphing window defined by the rectangle $-1 \leq x \leq 5$ and $-13 \leq y \leq 7$. This graph clearly shows the vertical asymptotes at $x = 2$ and $x = 3$. ■

As we see in example 2.6, not all rational functions have vertical asymptotes.

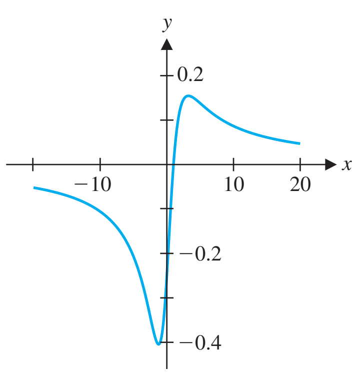


FIGURE 0.38
$y = \dfrac{x-1}{x^2+4}$

EXAMPLE 2.6 A Rational Function with No Vertical Asymptotes

Find all vertical asymptotes of $\dfrac{x-1}{x^2+4}$.

Solution Notice that $x^2 + 4 = 0$ has no (real) solutions, since $x^2 + 4 > 0$ for all real numbers, x. So, there are no vertical asymptotes. The graph in Figure 0.38 is consistent with this observation. ■

Graphs are useful for finding approximate solutions of difficult equations, as we see in examples 2.7 and 2.8.

EXAMPLE 2.7 Finding Zeros Approximately

Find approximate solutions of the equation $x^2 = \sqrt{x+3}$.

Solution You could rewrite this equation as $x^2 - \sqrt{x+3} = 0$ and then look for zeros in the graph of $f(x) = x^2 - \sqrt{x+3}$, seen in Figure 0.39a. Note that two zeros are clearly indicated: one near -1, the other near 1.5. However, since you know very little of the nature of the function $x^2 - \sqrt{x+3}$, you cannot say whether or not there are any other zeros, ones that don't show up in the window seen in Figure 0.39a. On the other hand, if you graph the two functions on either side of the equation on the same set of axes, as in Figure 0.39b, you can clearly see two points where the graphs intersect (corresponding to the two zeros seen in Figure 0.39a). Further, since you know the general shapes of both of the graphs, you can infer from Figure 0.39b that there are no other intersections (i.e., there are no other zeros of f). This is important information that you cannot obtain from Figure 0.39a. Now that you know how many solutions there are, you need to estimate their values. One method is to zoom in on the zeros graphically. We leave it as an exercise to verify that the zeros are approximately $x = 1.4$ and $x = -1.2$. If your calculator or computer algebra system has a solve command, you can use it to quickly obtain an accurate approximation. In this case, we get $x \approx 1.452626878$ and $x \approx -1.164035140$. ■

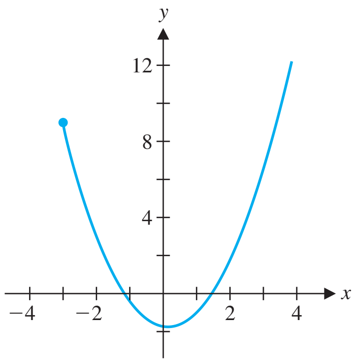


FIGURE 0.39a
$y = x^2 - \sqrt{x+3}$

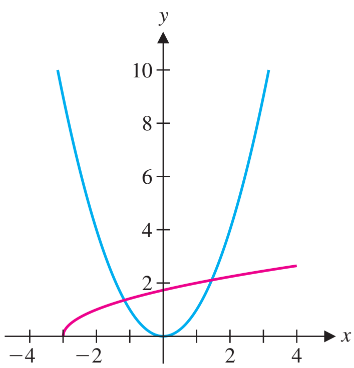


FIGURE 0.39b
$y = x^2$ and $y = \sqrt{x+3}$

When using the solve command on your calculator or computer algebra system, be sure to check that the solutions make sense. If the results don't match what you've seen in your preliminary sketches and zooms, beware! Even high-tech equation solvers make mistakes occasionally.

EXAMPLE 2.8 Finding Intersections by Calculator: An Oversight

Find all points of intersection of the graphs of $y = 2\cos x$ and $y = 2 - x$.

Solution Notice that the intersections correspond to solutions of the equation $2\cos x = 2 - x$. Using the solve command on the TI-92 graphing calculator, we found intersections at $x \approx 3.69815$ and $x = 0$. So, what's the problem? A sketch of the graphs of $y = 2 - x$ and $y = 2\cos x$ (we discuss this function in the next section) clearly shows three intersections (see Figure 0.40).

The middle solution, $x \approx 1.10914$, was somehow passed over by the calculator's solve routine. The lesson here is to use graphical evidence to support your solutions, especially when using software and/or functions with which you are less than completely familiar. ■

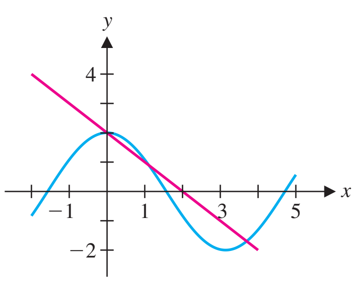


FIGURE 0.40
$y = 2\cos x$ and $y = 2 - x$

You need to look skeptically at the answers provided by your calculator's solver program. While such solvers provide a quick means of approximating solutions of equations, these programs will sometimes return incorrect answers, as we illustrate with example 2.9. So, how do you know if your solver is giving you an accurate answer or one that's incorrect? The only answer to this is that you must carefully test your calculator's solution, by separately calculating both sides of the equation (*by hand*) at the calculated solution.

EXAMPLE 2.9 Solving an Equation by Calculator: An Erroneous Answer

Use your calculator's solver program to solve the equation $x + \frac{1}{x} = \frac{1}{x}$.

Solution Certainly, you don't need a calculator to solve this equation, but consider what happens when you use one. Most calculators report a solution that is very close to zero, while others report that the solution is $x = 0$. Not only are these answers incorrect, but the given equation has *no* solution, as follows. First, notice that the equation makes sense only when $x \neq 0$. Subtracting $\frac{1}{x}$ from both sides of the equation leaves us with $x = 0$, which can't possibly be a solution, since it does not satisfy the original equation. Notice further that, if your calculator returns the approximate solution $x = 1 \times 10^{-7}$ and you use your calculator to compute the values on both sides of the equation, the calculator will compute

$$x + \frac{1}{x} = 1 \times 10^{-7} + 1 \times 10^{7},$$

which it approximates as $1 \times 10^{7} = \frac{1}{x}$, since calculators carry only a finite number of digits. In other words, although

$$1 \times 10^{-7} + 1 \times 10^{7} \neq 1 \times 10^{7},$$

your calculator treats these numbers as the same and so incorrectly reports that the equation is satisfied. The moral of this story is to be an intelligent user of technology and don't blindly accept everything a calculator tells you. ■

We want to emphasize again that graphing should be the first step in the equation-solving process. A good graph will show you how many solutions to expect, as well as give their approximate locations. Whenever possible, you should factor or use the quadratic formula to get exact solutions. When this is impossible, approximate the solutions by zooming in on them graphically or by using your calculator's solve command. *Always* compare your results to a graph to see if there's anything you've missed.

EXERCISES 0.2

WRITING EXERCISES

1. Explain why there is a significant difference among Figures 0.33a, 0.33b and 0.33c.

2. In Figure 0.36, the graph approaches the lower portion of the vertical asymptote from the left, whereas the graph approaches the upper portion of the vertical asymptote from the right. Use the table of function values found in example 2.4 to explain how to determine whether a graph approaches a vertical asymptote by dropping down or rising up.

3. In the text, we discussed the difference between graphing with a fixed window versus an automatic window. Discuss the advantages and disadvantages of each. (Hint: Consider the case of a first graph of a function you know nothing about and the case of hoping to see the important details of a graph for which you know the general shape.)

4. Examine the graph of $y = \frac{x^3 + 1}{x}$ with each of the following graphing windows: (a) $-10 \le x \le 10$, (b) $-1000 \le x \le 1000$. Explain why the graph in (b) doesn't show the details that the graph in (a) does.

In exercises 1–30, sketch a graph of the function showing all extrema, intercepts and asymptotes.

1. $f(x) = x^2 - 1$
2. $f(x) = 3 - x^2$
3. $f(x) = x^2 + 2x + 8$
4. $f(x) = x^2 - 20x + 11$
5. $f(x) = x^3 + 1$
6. $f(x) = 10 - x^3$
7. $f(x) = x^3 + 2x - 1$
8. $f(x) = x^3 - 3x + 1$

9. $f(x) = x^4 - 1$

10. $f(x) = 2 - x^4$

11. $f(x) = x^4 + 2x - 1$

12. $f(x) = x^4 - 6x^2 + 3$

13. $f(x) = x^5 + 2$

14. $f(x) = 12 - x^5$

15. $f(x) = x^5 - 8x^3 + 20x - 1$

16. $f(x) = x^5 + 5x^4 + 2x^3 + 1$

17. $f(x) = \dfrac{3}{x-1}$

18. $f(x) = \dfrac{4}{x+2}$

19. $f(x) = \dfrac{3x}{x-1}$

20. $f(x) = \dfrac{4x}{x+2}$

21. $f(x) = \dfrac{3x^2}{x-1}$

22. $f(x) = \dfrac{4x^2}{x+2}$

23. $f(x) = \dfrac{2}{x^2-4}$

24. $f(x) = \dfrac{6}{x^2-9}$

25. $f(x) = \dfrac{3}{x^2+4}$

26. $f(x) = \dfrac{6}{x^2+9}$

27. $f(x) = \dfrac{x+2}{x^2+x-6}$

28. $f(x) = \dfrac{x-1}{x^2+4x+3}$

29. $f(x) = \dfrac{3x}{\sqrt{x^2+4}}$

30. $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

In exercises 31–38, find all vertical asymptotes.

31. $f(x) = \dfrac{3x}{x^2-4}$

32. $f(x) = \dfrac{x+4}{x^2-9}$

33. $f(x) = \dfrac{4x}{x^2+3x-10}$

34. $f(x) = \dfrac{x+2}{x^2-2x-15}$

35. $f(x) = \dfrac{4x}{x^2+4}$

36. $f(x) = \dfrac{3x}{\sqrt{x^2-9}}$

37. $f(x) = \dfrac{x^2+1}{x^3+3x^2+2x}$

38. $f(x) = \dfrac{3x}{x^4-1}$

In exercises 39–42, a standard graphing window will not reveal all of the important details of the graph. Adjust the graphing window to find the missing details.

39. $f(x) = \frac{1}{3}x^3 - \frac{1}{400}x$

40. $f(x) = x^4 - 11x^3 + 5x - 2$

41. $f(x) = x\sqrt{144 - x^2}$

42. $f(x) = \frac{1}{5}x^5 - \frac{7}{8}x^4 + \frac{1}{3}x^3 + \frac{7}{2}x^2 - 6x$

In exercises 43–48, adjust the graphing window to identify all vertical asympotes.

43. $f(x) = \dfrac{3}{x-1}$

44. $f(x) = \dfrac{4x}{x^2-1}$

45. $f(x) = \dfrac{3x^2}{x^2-1}$

46. $f(x) = \dfrac{2x}{x+4}$

47. $f(x) = \dfrac{x^2-1}{\sqrt{x^4+x}}$

48. $f(x) = \dfrac{2x}{\sqrt{x^2+x}}$

In exercises 49–56, determine the number of (real) solutions. Solve for the intersection points exactly if possible and estimate the points if necessary.

49. $\sqrt{x-1} = x^2 - 1$

50. $\sqrt{x^2+4} = x^2 + 2$

51. $x^3 - 3x^2 = 1 - 3x$

52. $x^3 + 1 = -3x^2 - 3x$

53. $(x^2 - 1)^{2/3} = 2x + 1$

54. $(x + 1)^{2/3} = 2 - x$

55. $\cos x = x^2 - 1$

56. $\sin x = x^2 + 1$

In exercises 57–62, use a graphing calculator or computer graphing utility to estimate all zeros.

57. $f(x) = x^3 - 3x + 1$

58. $f(x) = x^3 - 4x^2 + 2$

59. $f(x) = x^4 - 3x^3 - x + 1$

60. $f(x) = x^4 - 2x + 1$

61. $f(x) = x^4 - 7x^3 - 15x^2 - 10x - 1410$

62. $f(x) = x^6 - 4x^4 + 2x^3 - 8x - 2$

63. Graph $y = x^2$ in the graphing window $-10 \le x \le 10$, $-10 \le y \le 10$, without drawing the x- and y-axes. Adjust the graphing window for $y = 2(x-1)^2 + 3$ so that (without the axes showing) the graph looks identical to that of $y = x^2$.

64. Graph $y = x^2$ in the graphing window $-10 \le x \le 10$, $-10 \le y \le 10$. Separately graph $y = x^4$ with the same graphing window. Compare and contrast the graphs. Then graph the two functions on the same axes and carefully examine the differences in the intervals $-1 < x < 1$ and $x > 1$.

65. In this exercise, you will find an equation describing all points equidistant from the x-axis and the point $(0, 2)$. First, see if you can sketch a picture of what this curve ought to look like. For a point (x, y) that is on the curve, explain why $\sqrt{y^2} = \sqrt{x^2 + (y-2)^2}$. Square both sides of this equation and solve for y. Identify the curve.

66. Find an equation describing all points equidistant from the x-axis and $(1, 4)$ (see exercise 65).

EXPLORATORY EXERCISES

1. Suppose that a graphing calculator is set up with pixels corresponding to $x = 0, 0.1, 0.2, 0.3, \ldots, 2.0$ and $y = 0, 0.1, 0.2, 0.3, \ldots, 4.0$. For the function $f(x) = x^2$, compute the indicated function values and round off to give pixel coordinates [e.g., the point $(1.19, 1.4161)$ has pixel coordinates $(1.2, 1.4)$].

(a) $f(0.4)$, (b) $f(0.39)$, (c) $f(1.17)$, (d) $f(1.20)$, (e) $f(1.8)$, (f) $f(1.81)$. Repeat (c)–(d) if the graphing window is zoomed in so that $x = 1.00, 1.01, \ldots, 1.20$ and $y = 1.30, 1.31, \ldots, 1.50$. Repeat (e)–(f) if the graphing window is zoomed in so that $x = 1.800, 1.801, \ldots, 1.820$ and $y = 3.200, 3.205, \ldots, 3.300$.

2. Graph $y = x^2 - 1$, $y = x^2 + x - 1$, $y = x^2 + 2x - 1$, $y = x^2 - x - 1$, $y = x^2 - 2x - 1$ and other functions of the form $y = x^2 + cx - 1$. Describe the effect(s) a change in c has on the graph.

3. Figures 0.31 and 0.32 provide a catalog of the possible types of graphs of cubic polynomials. In this exercise, you will compile a catalog of graphs of fourth-order polynomials (i.e., $y = ax^4 + bx^3 + cx^2 + dx + e$). Start by using your calculator or computer to sketch graphs with different values of a, b, c, d and e. Try $y = x^4$, $y = 2x^4$, $y = -2x^4$, $y = x^4 + x^3$, $y = x^4 + 2x^3$, $y = x^4 - 2x^3$, $y = x^4 + x^2$, $y = x^4 - x^2$, $y = x^4 - 2x^2$, $y = x^4 + x$, $y = x^4 - x$ and so on. Try to determine what effect each constant has.

0.3 INVERSE FUNCTIONS

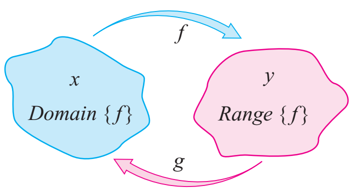


FIGURE 0.41
$g(x) = f^{-1}(x)$

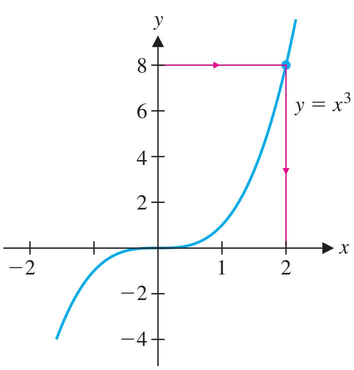


FIGURE 0.42
Finding the x-value corresponding to $y = 8$

The notion of an *inverse* relationship is basic to many areas of science. The number of common inverse problems is immense. As only one example, take the case of the electrocardiogram (EKG). In an EKG, technicians connect a series of electrodes to a patient's chest and use measurements of electrical activity on the surface of the body to infer something about the electrical activity on the surface of the heart. This is referred to as an *inverse* problem, since physicians are attempting to determine what *inputs* (i.e., the electrical activity on the surface of the heart) cause an observed *output* (the measured electrical activity on the surface of the chest).

The mathematical notion of inverse is much the same as that just described. Given an output (in this case, a value in the range of a given function), we wish to find the input (the value in the domain) that produced that output. That is, given a $y \in Range\{f\}$, find the $x \in Domain\{f\}$ for which $y = f(x)$. (See the illustration of the inverse function g shown in Figure 0.41.)

For instance, suppose that $f(x) = x^3$ and $y = 8$. Can you find an x such that $x^3 = 8$? That is, can you find the x-value corresponding to $y = 8$? (See Figure 0.42.) Of course, the solution of this particular equation is $x = \sqrt[3]{8} = 2$. In general, if $x^3 = y$, then $x = \sqrt[3]{y}$. In light of this, we say that the cube root function is the *inverse* of $f(x) = x^3$.

EXAMPLE 3.1 Two Functions That Reverse the Action of Each Other

If $f(x) = x^3$ and $g(x) = x^{1/3}$, show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

for all x.

Solution For all real numbers x, we have

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

and

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x. \blacksquare$$

Notice in example 3.1 that the action of f undoes the action of g and vice versa. We take this as the definition of an inverse function. (Again, think of Figure 0.41.)

REMARK 3.1

Pay close attention to the notation. Notice that $f^{-1}(x)$ does *not* mean $\dfrac{1}{f(x)}$. We write the reciprocal of $f(x)$ as

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

DEFINITION 3.1

Assume that f and g have domains A and B, respectively, and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$. If

$$f(g(x)) = x, \quad \text{for all } x \in B \quad \text{and}$$
$$g(f(x)) = x, \quad \text{for all } x \in A,$$

we say that g is the **inverse** of f, written $g = f^{-1}$. Equivalently, f is the inverse of g, $f = g^{-1}$.

Observe that many familiar functions have no inverse.

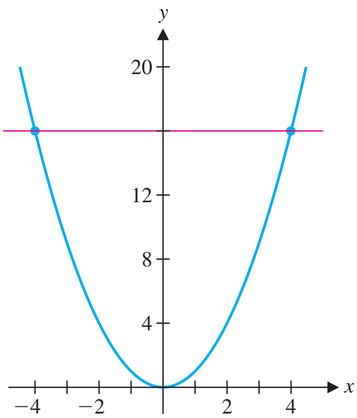


FIGURE 0.43
$y = x^2$

EXAMPLE 3.2 A Function with No Inverse

Show that $f(x) = x^2$ has no inverse on the interval $(-\infty, \infty)$.

Solution Notice that $f(4) = 16$ and $f(-4) = 16$. That is, there are *two* x-values that produce the same y-value. So, if we were to try to define an inverse of f, how would we define $f^{-1}(16)$? Look at the graph of $y = x^2$ (see Figure 0.43) to see what the problem is. For each $y > 0$, there are *two* x-values for which $y = x^2$. Because of this, the function does not have an inverse. ■

For $f(x) = x^2$, it is tempting to jump to the conclusion that $g(x) = \sqrt{x}$ is the inverse of $f(x)$. Notice that although $f(g(x)) = (\sqrt{x})^2 = x$ for all $x \geq 0$ (i.e., for all x in the domain of $g(x)$), it is *not* generally true that $g(f(x)) = \sqrt{x^2} = x$. In fact, this last equality holds *only* for $x \geq 0$. However, for $f(x) = x^2$ restricted to the domain $x \geq 0$, we do have that $f^{-1}(x) = \sqrt{x}$.

DEFINITION 3.2

A function f is called **one-to-one** when for every $y \in \mathit{Range}\{f\}$, there is *exactly one* $x \in \mathit{Domain}\{f\}$ for which $y = f(x)$.

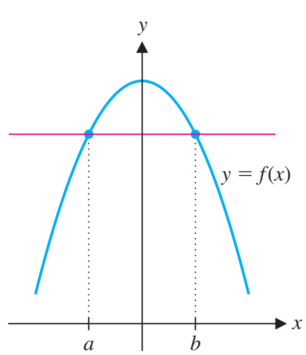


FIGURE 0.44a
$f(a) = f(b)$, for $a \neq b$
So, f does not pass the horizontal line test and is not one-to-one.

REMARK 3.2

Observe that an equivalent definition of one-to-one is the following. A function $f(x)$ is one-to-one if and only if the equality $f(a) = f(b)$ implies $a = b$. This version of the definition is often useful for proofs involving one-to-one functions.

It is helpful to think of the concept of one-to-one in graphical terms. Notice that a function f is one-to-one if and only if every horizontal line intersects the graph in at most one point. This is referred to as the **horizontal line test.** We illustrate this in Figures 0.44a and 0.44b. The following result should now make sense.

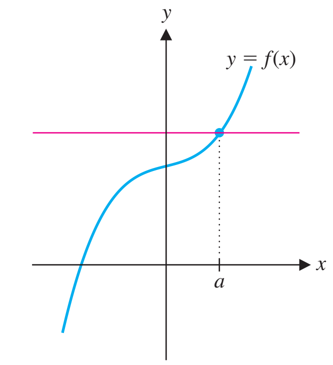


FIGURE 0.44b
Every horizontal line intersects the curve in at most one point. So, f passes the horizontal line test and is one-to-one.

THEOREM 3.1

A function f has an inverse if and only if it is one-to-one.

This theorem simply says that every one-to-one function has an inverse and every function that has an inverse is one-to-one. However, it says nothing about how to find an inverse. For very simple functions, we can find inverses by solving equations.

EXAMPLE 3.3 Finding an Inverse Function

Find the inverse of $f(x) = x^3 - 5$.

Solution Note that it is not entirely clear from the graph (see Figure 0.45) whether f passes the horizontal line test. To find the inverse function, write $y = f(x)$ and solve for x (i.e., solve for the input x that produced the observed output y). We have

$$y = x^3 - 5.$$

Adding 5 to both sides and taking the cube root gives us

$$(y + 5)^{1/3} = (x^3)^{1/3} = x.$$

So, $x = f^{-1}(y) = (y + 5)^{1/3}$. Reversing the variables x and y gives us

$$f^{-1}(x) = (x + 5)^{1/3}. \quad \blacksquare$$

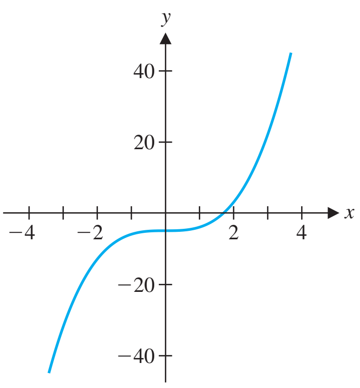


FIGURE 0.45
$y = x^3 - 5$

EXAMPLE 3.4 A Function That Is Not One-to-One

Show that $f(x) = 10 - x^4$ does not have an inverse.

Solution You can see from a graph (see Figure 0.46) that f is not one-to-one; for instance, $f(1) = f(-1) = 9$. Consequently, f does not have an inverse. $\blacksquare$

Most often, we cannot find a formula for an inverse function and must be satisfied with simply knowing that the inverse function exists. Example 3.5 is typical of this situation.

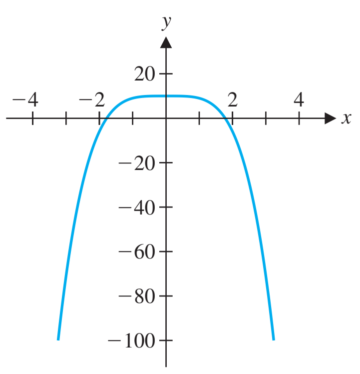


FIGURE 0.46
$y = 10 - x^4$

EXAMPLE 3.5 Finding Values of an Inverse Function

Given that $f(x) = x^5 + 8x^3 + x + 1$ has an inverse, find $f^{-1}(1)$ and $f^{-1}(11)$.

Solution First, notice that from the graph shown in Figure 0.47 (on the following page), the function looks like it might be one-to-one, but how can we be certain of this? (Remember that graphs can be deceptive!) Until we develop some calculus, we will be unable to verify this. Ideally, we would show that f has an inverse by finding a formula for f^{-1}, as in example 3.3. However, in this case, we must solve the equation

$$y = x^5 + 8x^3 + x + 1$$

for x. Think about this for a moment: you should realize that we can't solve for x in terms of y here. We need to assume that the inverse exists, as indicated in the instructions.

Turning to the problem of finding $f^{-1}(1)$ and $f^{-1}(11)$, you might wonder if this is possible, since we were unable to find a formula for $f^{-1}(x)$. While it's certainly true that we have no such formula, you might observe that $f(0) = 1$, so that $f^{-1}(1) = 0$. By trial and error, you might also discover that $f(1) = 11$ and so, $f^{-1}(11) = 1$. ■

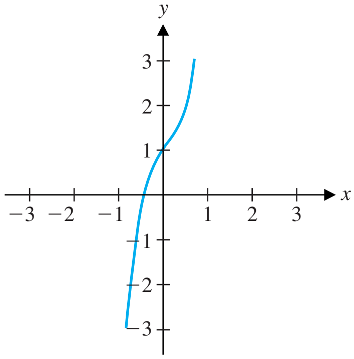


FIGURE 0.47
$y = x^5 + 8x^3 + x + 1$

In example 3.5, we examined a function that has an inverse, although we could not find that inverse algebraically. Even when we can't find an inverse function explicitly, we can say something graphically. Notice that if (a, b) is a point on the graph of $y = f(x)$ and f has an inverse, f^{-1}, then since

$$b = f(a),$$

we have that

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

That is, (b, a) is a point on the graph of $y = f^{-1}(x)$. This tells us a great deal about the inverse function. In particular, we can immediately obtain any number of points on the graph of $y = f^{-1}(x)$, simply by inspection. Further, notice that the point (b, a) is the reflection of the point (a, b) through the line $y = x$ (see Figure 0.48). It now follows that given the graph of any one-to-one function, you can draw the graph of its inverse simply by reflecting the entire graph through the line $y = x$.

In example 3.6, we illustrate the symmetry of a function and its inverse.

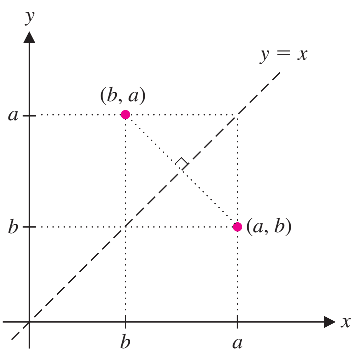


FIGURE 0.48
Reflection through $y = x$

EXAMPLE 3.6 The Graph of a Function and Its Inverse

Draw a graph of $f(x) = x^3$ and its inverse.

Solution From example 3.1, the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$. Notice the symmetry of their graphs shown in Figure 0.49. ■

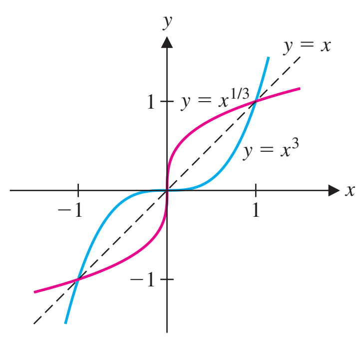


FIGURE 0.49
$y = x^3$ and $y = x^{1/3}$

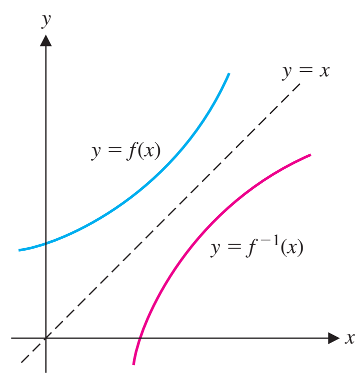


FIGURE 0.50
Graphs of f and f^{-1}

Observe that we can use this symmetry principle to draw the graph of an inverse function, even when we don't have a formula for that function (see Figure 0.50).

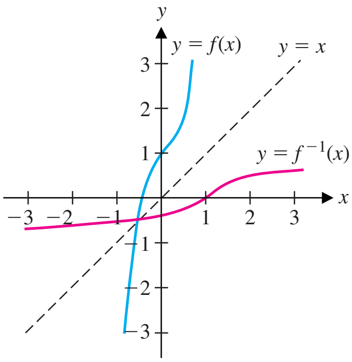


FIGURE 0.51
$y = f(x)$ and $y = f^{-1}(x)$

EXAMPLE 3.7 Drawing the Graph of an Unknown Inverse Function

Draw a graph of $f(x) = x^5 + 8x^3 + x + 1$ and its inverse.

Solution In example 3.5, we were unable to find a formula for the inverse function. Despite this, we can draw a graph of f^{-1} with ease. We simply take the graph of $y = f(x)$ seen in Figure 0.47 and reflect it across the line $y = x$, as shown in Figure 0.51. (When we introduce parametric equations in section 9.1, we will see a clever way to draw this graph with a graphing calculator.) ■

In example 3.8, we apply our theoretical knowledge of inverse functions in a medical setting.

EXAMPLE 3.8 Determining the Proper Dosage of a Drug

Suppose that the injection of a certain drug raises the level of a key hormone in the body. Physicians want to determine the dosage that produces a healthy hormone level. Dosages of 1, 2, 3 and 4 mg produce hormone levels of 12, 20, 40 and 76, respectively. If the desired hormone level is 30, what is the proper dosage?

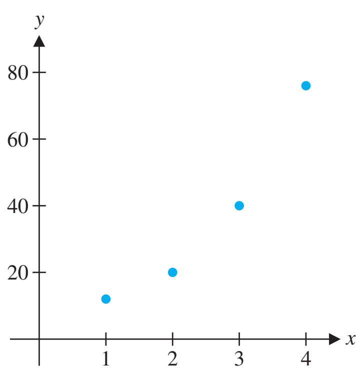


FIGURE 0.52a
Hormone data

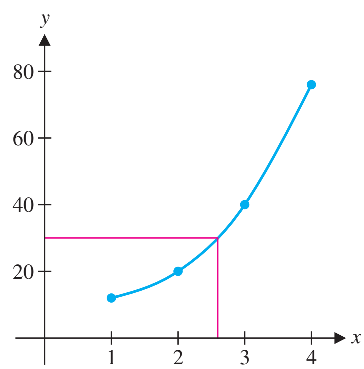


FIGURE 0.52b
Approximate curve

Solution A plot of the points (1, 12), (2, 20), (3, 40) and (4, 76) summarizes the data (see Figure 0.52a). The problem is an inverse problem: given $y = 30$, what is x? It is tempting to argue the following: since 30 is halfway between 20 and 40, the x-value should be halfway between 2 and 3: $x = 2.5$. This method of solution is called **linear interpolation,** since the point $(x, y) = (2.5, 30)$ lies on the line through the points (2, 20) and (3, 40). However, this estimate does not take into account all of the information we have. The points in Figure 0.52a suggest a graph that is curving up. If this is the case, $x = 2.6$ or $x = 2.7$ may be a better estimate of the required dosage. In Figure 0.52b, we have sketched a smooth curve through the data points and indicated a graphical solution of the problem. More advanced techniques (e.g., polynomial interpolation) have been developed by mathematicians to make the estimate of such quantities as accurate as possible. ■

TODAY IN MATHEMATICS

Kim Rossmo (1955–)
A Canadian criminologist who developed the Criminal Geographic Targeting algorithm that indicates the most probable area of residence for serial murderers, rapists and other criminals. Rossmo served 21 years with the Vancouver Police Department. His mentors were Professors Paul and Patricia Brantingham of Simon Fraser University. The Brantinghams developed Crime Pattern Theory, which predicts crime locations from where criminals live, work and play. Rossmo inverted their model and used the crime sites to determine where the criminal most likely lives. The premiere episode of the television drama *Numb3rs* was based on Rossmo's work.

EXERCISES 0.3

WRITING EXERCISES

1. Explain in words (and a picture) why the following is true: if $f(x)$ is increasing for all x, then f has an inverse.
2. Suppose the graph of a function passes the horizontal line test. Explain why you know that the function has an inverse (defined on the range of the function).
3. Radar works by bouncing a high-frequency electromagnetic pulse off of a moving object, then measuring the disturbance in the pulse as it is bounced back. Explain why this is an inverse problem by identifying the input and output.
4. Each human disease has a set of symptoms associated with it. Physicians attempt to solve an inverse problem: given the symptoms, they try to identify the disease causing the symptoms. Explain why this is not a well-defined inverse problem (i.e., logically it is not always possible to correctly identify diseases from symptoms alone).

In exercises 1–4, show that $f(g(x)) = x$ and $g(f(x)) = x$ for all x:

1. $f(x) = x^5$ and $g(x) = x^{1/5}$
2. $f(x) = 4x^3$ and $g(x) = \left(\frac{1}{4}x\right)^{1/3}$
3. $f(x) = 2x^3 + 1$ or $g(x) = \sqrt[3]{\dfrac{x-1}{2}}$
4. $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1-2x}{x}$ $(x \neq 0, x \neq -2)$

In exercises 5–12, determine whether the function is one-to-one. If it is, find the inverse and graph both the function and its inverse.

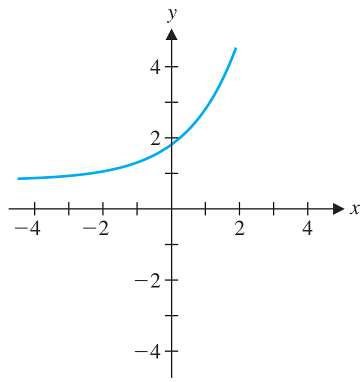
5. $f(x) = x^3 - 2$
6. $f(x) = x^3 + 4$
7. $f(x) = x^5 - 1$
8. $f(x) = x^5 + 4$
9. $f(x) = x^4 + 2$
10. $f(x) = x^4 - 2x - 1$
11. $f(x) = \sqrt{x^3 + 1}$
12. $f(x) = \sqrt{x^2 + 1}$

In exercises 13–18, assume that the function has an inverse. Without solving for the inverse, find the indicated function values.

13. $f(x) = x^3 + 4x - 1$, (a) $f^{-1}(-1)$, (b) $f^{-1}(4)$
14. $f(x) = x^3 + 2x + 1$, (a) $f^{-1}(1)$, (b) $f^{-1}(13)$
15. $f(x) = x^5 + 3x^3 + x$, (a) $f^{-1}(-5)$, (b) $f^{-1}(5)$
16. $f(x) = x^5 + 4x - 2$, (a) $f^{-1}(38)$, (b) $f^{-1}(3)$
17. $f(x) = \sqrt{x^3 + 2x + 4}$, (a) $f^{-1}(4)$, (b) $f^{-1}(2)$
18. $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}$, (a) $f^{-1}(3)$, (b) $f^{-1}(1)$

In exercises 19–22, use the given graph to graph the inverse function.

19.



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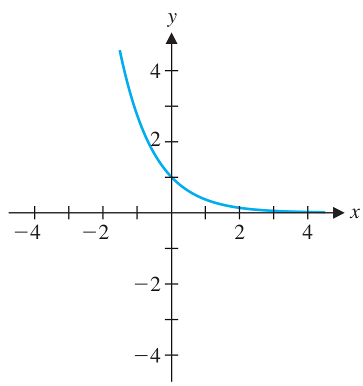

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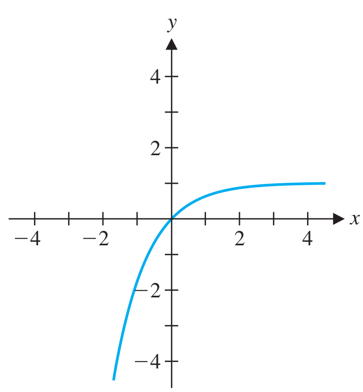
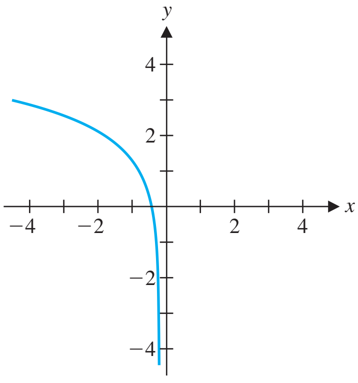


22.



In exercises 23–26, use linear interpolation (see example 3.8) to estimate $f^{-1}(b)$. Use the apparent curving of the graph to conjecture whether the estimate is too high or too low.

23. (1, 12), (2, 20), (3, 26), (4, 30), $b = 23$

24. (1, 12), (2, 10), (3, 6), (4, 0), $b = 8$

25. (1, 12), (2, 6), (3, 2), (4, 0), $b = 5$

26. (1, 12), (2, 20), (3, 36), (4, 50), $b = 32$

In exercises 27–36, use a graph to determine whether the function is one-to-one. If it is, graph the inverse function.

27. $f(x) = x^3 - 5$

28. $f(x) = x^2 - 3$

29. $f(x) = x^3 + 2x - 1$

30. $f(x) = x^3 - 2x - 1$

31. $f(x) = x^5 - 3x^3 - 1$

32. $f(x) = x^5 + 4x^3 - 2$

33. $f(x) = \dfrac{1}{x+1}$

34. $f(x) = \dfrac{4}{x^2+1}$

35. $f(x) = \dfrac{x}{x+4}$

36. $f(x) = \dfrac{x}{\sqrt{x^2+4}}$

Exercises 37–46 involve inverse functions on restricted domains.

37. Show that $f(x) = x^2$ $(x \geq 0)$ and $g(x) = \sqrt{x}$ $(x \geq 0)$ are inverse functions. Graph both functions.

38. Show that $f(x) = x^2 - 1$ $(x \geq 0)$ and $g(x) = \sqrt{x+1}$ $(x \geq -1)$ are inverse functions. Graph both functions.

39. Graph $f(x) = x^2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

40. Graph $f(x) = x^2 + 2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

41. Graph $f(x) = (x - 2)^2$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

42. Graph $f(x) = (x + 1)^4$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

43. Graph $f(x) = \sqrt{x^2 - 2x}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

44. Graph $f(x) = \dfrac{x}{x^2 - 4}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

45. Graph $f(x) = \sin x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

46. Graph $f(x) = \cos x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

In exercises 47–52, discuss whether the function described has an inverse.

47. The income of a company varies with time.

48. The height of a person varies with time.

49. For a dropped ball, its height varies with time.

50. For a ball thrown upward, its height varies with time.

51. The shadow made by an object depends on its three-dimensional shape.

52. The number of calories burned depends on how fast a person runs.

53. Suppose that your boss informs you that you have been awarded a 10% raise. The next week, your boss announces that due to circumstances beyond her control, all employees will have their salaries cut by 10%. Are you as well off now as you were two weeks ago? Show that increasing by 10% and decreasing by 10% are not inverse processes. Find the inverse for adding 10%. (Hint: To add 10% to a quantity you can multiply the quantity by 1.10.)

EXPLORATORY EXERCISES

1. Find all values of k such that $f(x) = x^3 + kx + 1$ is one-to-one.

2. Find all values of k such that $f(x) = x^3 + 2x^2 + kx - 1$ is one-to-one.

0.4 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Many phenomena encountered in your daily life involve *waves.* For instance, music is transmitted from radio stations in the form of electromagnetic waves. Your radio receiver decodes these electromagnetic waves and causes a thin membrane inside the speakers to vibrate, which, in turn, creates pressure waves in the air. When these waves reach your ears, you hear the music from your radio (see Figure 0.53). Each of these waves is *periodic,* meaning that the basic shape of the wave is repeated over and over again. The mathematical description of such phenomena involves periodic functions, the most familiar of which are the trigonometric functions. First, we remind you of a basic definition.

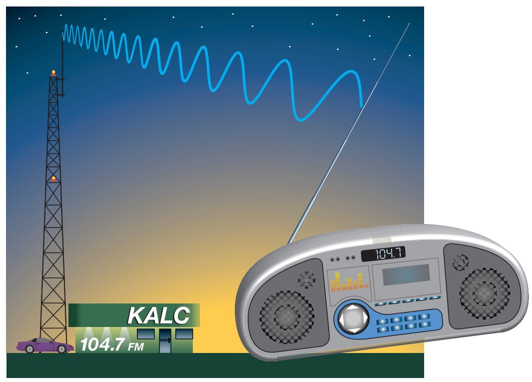


FIGURE 0.53
Radio and sound waves

NOTES

When we discuss the period of a function, we most often focus on the fundamental period.

DEFINITION 4.1

A function f is **periodic** of **period** T if

$$f(x+T) = f(x)$$

for all x such that x and $x + T$ are in the domain of f. The smallest such number $T > 0$ is called the **fundamental period.**

There are several equivalent ways of defining the sine and cosine functions. We want to emphasize a simple definition from which you can easily reproduce many of the basic properties of these functions. Referring to Figure 0.54, begin by drawing the unit circle $x^2 + y^2 = 1$. Let θ be the angle measured (counterclockwise) from the positive x-axis to the line segment connecting the origin to the point (x, y) on the circle. Here, we measure θ in **radians,** where the radian measure of the angle θ is the length of the arc indicated in the figure. Again referring to Figure 0.54, we define $\sin\theta$ to be the y-coordinate of the point on the circle and $\cos\theta$ to be the x-coordinate of the point. From this definition, it follows that $\sin\theta$ and $\cos\theta$ are defined for all values of θ, so that each has domain $-\infty < \theta < \infty$, while the range for each of these functions is the interval $[-1, 1]$.

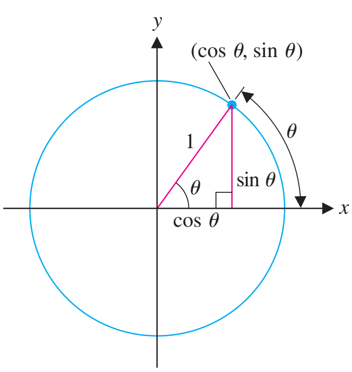


FIGURE 0.54
Definition of $\sin\theta$ and $\cos\theta$: $\cos\theta = x$ and $\sin\theta = y$

REMARK 4.1

Unless otherwise noted, we always measure angles in radians.

Note that since the circumference of a circle ($C = 2\pi r$) of radius 1 is 2π, we have that $360°$ corresponds to 2π radians. Similarly, $180°$ corresponds to π radians, $90°$ corresponds to $\pi/2$ radians, and so on. In the accompanying table, we list some common angles as measured in degrees, together with the corresponding radian measures.

Angle in degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$135°$	$180°$	$270°$	$360°$
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

THEOREM 4.1

The functions $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ are periodic, of period 2π.

PROOF

Referring to Figure 0.54, since a complete circle is 2π radians, adding 2π to any angle takes you all the way around the circle and back to the same point (x, y). This says that

$$\sin(\theta + 2\pi) = \sin\theta$$

and

$$\cos(\theta + 2\pi) = \cos\theta,$$

for all values of θ. Furthermore, 2π is the smallest angle for which this is true. ■

You are likely already familiar with the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ shown in Figures 0.55a and 0.55b, respectively.

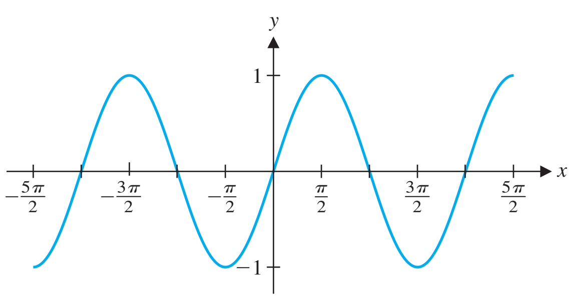


FIGURE 0.55a
$y = \sin x$

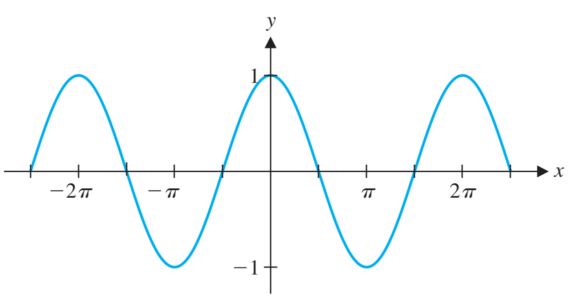


FIGURE 0.55b
$y = \cos x$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

Notice that you could slide the graph of $y = \sin x$ slightly to the left or right and get an exact copy of the graph of $y = \cos x$. Specifically, we have the relationship

$$\boxed{\sin\left(x + \frac{\pi}{2}\right) = \cos x.}$$

The accompanying table lists some common values of sine and cosine. Notice that many of these can be read directly from Figure 0.54.

EXAMPLE 4.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations (a) $2\sin x - 1 = 0$ and (b) $\cos^2 x - 3\cos x + 2 = 0$.

Solution For (a), notice that $2\sin x - 1 = 0$ if $2\sin x = 1$ or $\sin x = \frac{1}{2}$. From the unit circle, we find that $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Since $\sin x$ has period 2π, additional solutions are $\frac{\pi}{6} + 2\pi$, $\frac{5\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$ and so on. A convenient way of indicating that *any* integer multiple of 2π can be added to either solution is to write $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, for any integer n. Part (b) may look rather difficult at first. However, notice that it looks like a quadratic equation using $\cos x$ instead of x. With this clue, you can factor the left-hand side to get

$$0 = \cos^2 x - 3\cos x + 2 = (\cos x - 1)(\cos x - 2),$$

from which it follows that either $\cos x = 1$ or $\cos x = 2$. Since $-1 \le \cos x \le 1$ for all x, the equation $\cos x = 2$ has no solution. However, we get $\cos x = 1$ if $x = 0, 2\pi$ or any integer multiple of 2π. We can summarize all the solutions by writing $x = 2n\pi$, for any integer n. ■

REMARK 4.2

Instead of writing $(\sin\theta)^2$ or $(\cos\theta)^2$, we usually use the notation $\sin^2\theta$ and $\cos^2\theta$, respectively.

We now give definitions of the remaining four trigonometric functions.

DEFINITION 4.2

The **tangent** function is defined by $\tan x = \dfrac{\sin x}{\cos x}$.

The **cotangent** function is defined by $\cot x = \dfrac{\cos x}{\sin x}$.

The **secant** function is defined by $\sec x = \dfrac{1}{\cos x}$.

The **cosecant** function is defined by $\csc x = \dfrac{1}{\sin x}$.

REMARK 4.3

Most calculators have keys for the functions $\sin x$, $\cos x$ and $\tan x$, but not for the other three trigonometric functions. This reflects the central role that $\sin x$, $\cos x$ and $\tan x$ play in applications. To calculate function values for the other three trigonometric functions, you can simply use the identities

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}$$

$$\text{and} \quad \csc x = \frac{1}{\sin x}.$$

We show graphs of these functions in Figures 0.56a, 0.56b, 0.56c and 0.56d. Notice in each graph the locations of the vertical asymptotes. For the "co" functions $\cot x$ and $\csc x$, the division by $\sin x$ causes vertical asymptotes at $0, \pm\pi, \pm 2\pi$ and so on (where $\sin x = 0$). For $\tan x$ and $\sec x$, the division by $\cos x$ produces vertical asymptotes at $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$ and so on (where $\cos x = 0$). Once you have determined the vertical asymptotes, the graphs are relatively easy to draw.

Notice that $\tan x$ and $\cot x$ are periodic, of period π, while $\sec x$ and $\csc x$ are periodic, of period 2π.

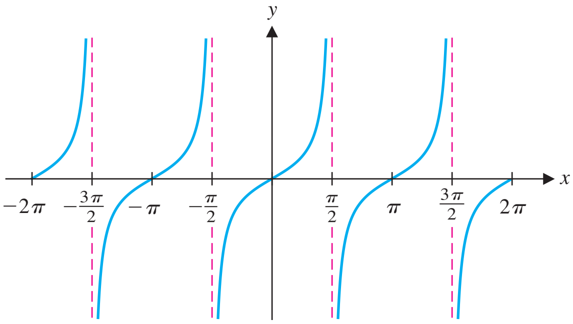


FIGURE 0.56a
$y = \tan x$

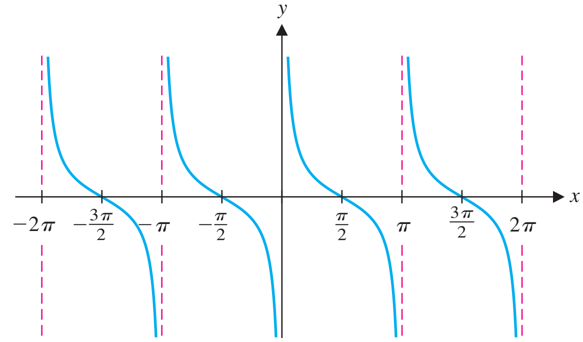


FIGURE 0.56b
$y = \cot x$

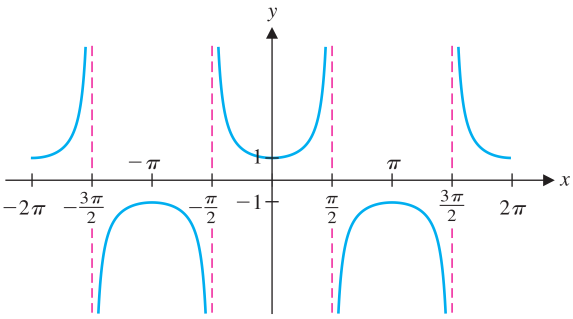


FIGURE 0.56c
$y = \sec x$

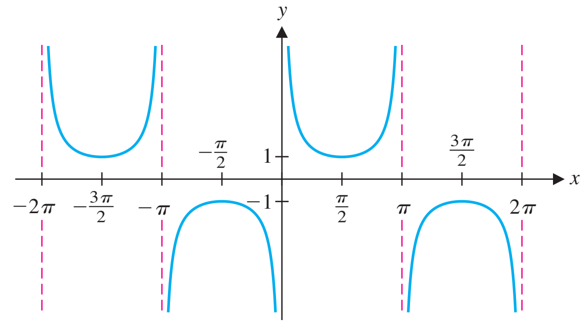


FIGURE 0.56d
$y = \csc x$

It is important to learn the effect of slight modifications of these functions. We present a few ideas here and in the exercises.

EXAMPLE 4.2 Altering Amplitude and Period

Graph $y = 2 \sin x$ and $y = \sin 2x$, and describe how each differs from the graph of $y = \sin x$ (see Figure 0.57a).

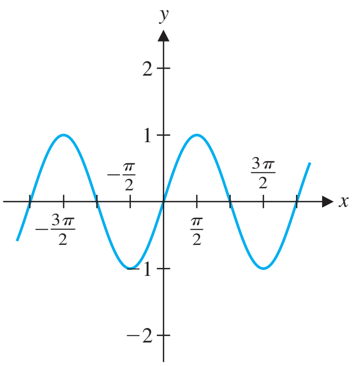


FIGURE 0.57a
$y = \sin x$

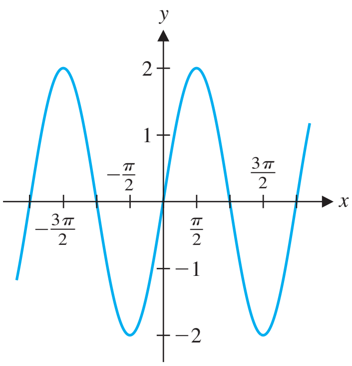


FIGURE 0.57b
$y = 2 \sin x$

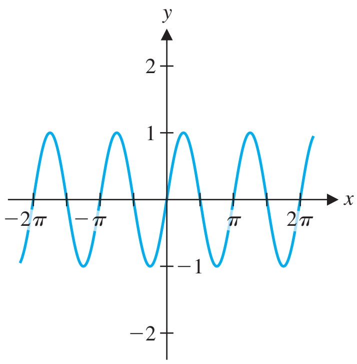


FIGURE 0.57c
$y = \sin(2x)$

Solution The graph of $y = 2 \sin x$ is given in Figure 0.57b. Notice that this graph is similar to the graph of $y = \sin x$, except that the y-values oscillate between -2 and 2 instead of -1 and 1. Next, the graph of $y = \sin 2x$ is given in Figure 0.57c. In this case, the graph is similar to the graph of $y = \sin x$ except that the period is π instead of 2π (so that the oscillations occur twice as fast). ■

The results in example 4.2 can be generalized. For $A > 0$, the graph of $y = A \sin x$ oscillates between $y = -A$ and $y = A$. In this case, we call A the **amplitude** of the sine curve. Notice that for any positive constant c, the period of $y = \sin cx$ is $2\pi/c$. Similarly, for the function $A \cos cx$, the amplitude is A and the period is $2\pi/c$.

The sine and cosine functions can be used to model sound waves. A pure tone (think of a single flute note) is a pressure wave described by the sinusoidal function $A \sin ct$. (Here, we are using the variable t, since the air pressure is a function of *time*.) The amplitude A determines how loud the tone is perceived to be and the period determines the pitch of the note. In this setting, it is convenient to talk about the **frequency** $f = c/2\pi$. The higher the frequency is, the higher the pitch of the note will be. (Frequency is measured in hertz, where 1 hertz equals 1 cycle per second.) Note that the frequency is simply the reciprocal of the period.

EXAMPLE 4.3 Finding Amplitude, Period and Frequency

Find the amplitude, period and frequency of (a) $f(x) = 4 \cos 3x$ and (b) $g(x) = 2 \sin(x/3)$.

Solution (a) For $f(x)$, the amplitude is 4, the period is $2\pi/3$ and the frequency is $3/(2\pi)$ (see Figure 0.58a). (b) For $g(x)$, the amplitude is 2, the period is $2\pi/(1/3) = 6\pi$ and the frequency is $1/(6\pi)$ (see Figure 0.58b).

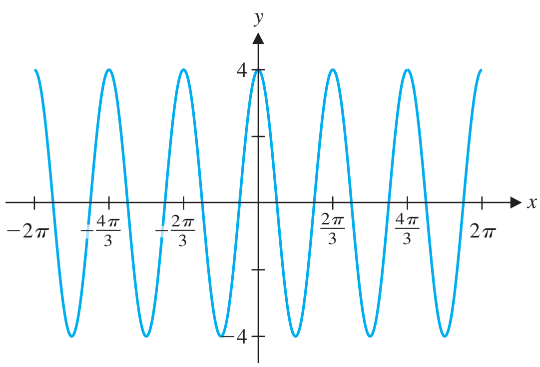


FIGURE 0.58a
$y = 4 \cos 3x$

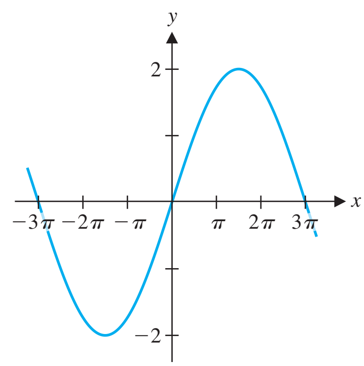


FIGURE 0.58b
$y = 2 \sin(x/3)$

■

There are numerous formulas or **identities** that are helpful in manipulating the trigonometric functions. You should observe that, from the definition of $\sin\theta$ and $\cos\theta$ (see Figure 0.54), the Pythagorean Theorem gives us the familiar identity

$$\sin^2\theta + \cos^2\theta = 1,$$

since the hypotenuse of the indicated triangle is 1. This is true for any angle θ. In addition,

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta$$

We list several important identities in Theorem 4.2.

THEOREM 4.2

For any real numbers α and β, the following identities hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \tag{4.1}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{4.2}$$

$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha) \tag{4.3}$$

$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha). \tag{4.4}$$

From the basic identities summarized in Theorem 4.2, numerous other useful identities can be derived. We derive two of these in example 4.4.

EXAMPLE 4.4 Deriving New Trigonometric Identities

Derive the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Solution These can be obtained from formulas (4.1) and (4.2), respectively, by substituting $\alpha = \theta$ and $\beta = \theta$. Alternatively, the identity for $\cos 2\theta$ can be obtained by subtracting equation (4.3) from equation (4.4). ■

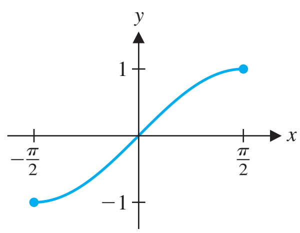


FIGURE 0.59
$y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Inverse Trigonometric Functions

We now expand the set of functions available to you by defining inverses to the trigonometric functions. To get started, look at a graph of $y = \sin x$ (see Figure 0.57a). Notice that we cannot define an inverse function, since $\sin x$ is not one-to-one. Although the sine function does not have an inverse function, we can define one by modifying the domain of the sine. We do this by choosing a portion of the sine curve that passes the horizontal line test. If we restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $y = \sin x$ is one-to-one there (see Figure 0.59) and, hence, has an inverse. We thus define the **inverse sine** function by

$$\boxed{y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \textit{ and } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}.} \tag{4.5}$$

Think of this definition as follows: if $y = \sin^{-1} x$, then y is the angle $\left(\text{between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2}\right)$ for which $\sin y = x$. Note that we could have selected any interval on which $\sin x$ is one-to-one, but $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the most convenient. To verify that these are inverse functions, observe that

$$\sin(\sin^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\sin^{-1}(\sin x) = x, \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{4.6}$$

Read equation (4.6) very carefully. It *does not* say that $\sin^{-1}(\sin x) = x$ for *all* x, but rather, *only* for those in the restricted domain, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For instance, $\sin^{-1}(\sin\pi) \neq \pi$, since

$$\sin^{-1}(\sin\pi) = \sin^{-1}(0) = 0.$$

REMARK 4.4

Mathematicians often use the notation **arcsin *x*** in place of $\sin^{-1} x$. People read $\sin^{-1} x$ interchangeably as "inverse sine of x" or "arcsine of x."

EXAMPLE 4.5 Evaluating the Inverse Sine Function

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution For (a), we look for the angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin\theta = \frac{\sqrt{3}}{2}$. Note that since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. For (b), note that $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus,

$$\sin^{-1}\left(-\tfrac{1}{2}\right) = -\tfrac{\pi}{6}.$$

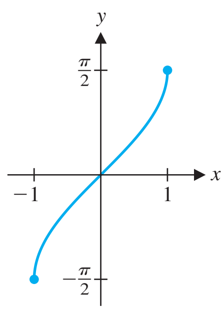


FIGURE 0.60
$y = \sin^{-1} x$

Judging by example 4.5, you might think that (4.5) is a roundabout way of defining a function. If so, you've got the idea exactly. In fact, we want to emphasize that what we know about the inverse sine function is principally through reference to the sine function.

Recall from our discussion in section 0.3 that we can draw a graph of $y = \sin^{-1} x$ simply by reflecting the graph of $y = \sin x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (from Figure 0.59) through the line $y = x$ (see Figure 0.60).

Turning to $y = \cos x$, observe that restricting the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as we did for the inverse sine function, will not work here. (Why not?) The simplest way to make $\cos x$ one-to-one is to restrict its domain to the interval $[0, \pi]$ (see Figure 0.61). Consequently, we define the **inverse cosine** function by

$$\boxed{y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \textit{ and } 0 \le y \le \pi.}$$

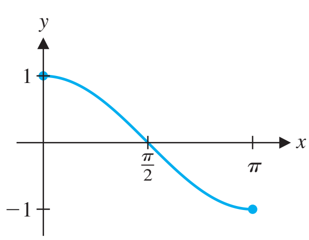


FIGURE 0.61
$y = \cos x$ on $[0, \pi]$

Note that here, we have

$$\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\cos^{-1}(\cos x) = x, \quad \text{for all } x \in [0, \pi].$$

As with the definition of arcsine, it is helpful to think of $\cos^{-1} x$ as that angle θ in $[0, \pi]$ for which $\cos\theta = x$. As with $\sin^{-1} x$, it is common to use $\cos^{-1} x$ and $\arccos x$ interchangeably.

EXAMPLE 4.6 Evaluating the Inverse Cosine Function

Evaluate (a) $\cos^{-1}(0)$ and (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution For (a), you will need to find that angle θ in $[0, \pi]$ for which $\cos\theta = 0$. It's not hard to see that $\cos^{-1}(0) = \frac{\pi}{2}$. (If you calculate this on your calculator and get 90, your calculator is in degrees mode. In this event, you should immediately change it to radians mode.) For (b), look for the angle $\theta \in [0, \pi]$ for which $\cos\theta = -\frac{\sqrt{2}}{2}$. Notice that $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4} \in [0, \pi]$. Consequently,

$$\cos^{-1}\left(-\tfrac{\sqrt{2}}{2}\right) = \tfrac{3\pi}{4}.$$

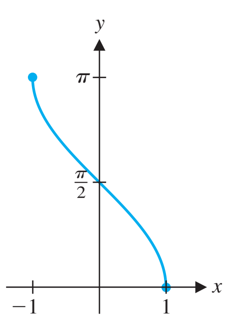


FIGURE 0.62
$y = \cos^{-1} x$

Once again, we obtain the graph of this inverse function by reflecting the graph of $y = \cos x$ on the interval $[0, \pi]$ (seen in Figure 0.61) through the line $y = x$ (see Figure 0.62).

We can define inverses for each of the four remaining trigonometric functions in similar ways. For $y = \tan x$, we restrict the domain to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Think about why the endpoints of this interval are not included (see Figure 0.63). Having done this, you should readily see that we define the **inverse tangent** function by

$$\boxed{y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \ \textit{and} \ -\frac{\pi}{2} < y < \frac{\pi}{2}.}$$

The graph of $y = \tan^{-1} x$ is then as seen in Figure 0.64, found by reflecting the graph in Figure 0.63 through the line $y = x$.

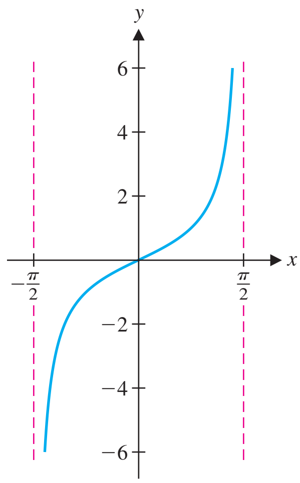


FIGURE 0.63
$y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

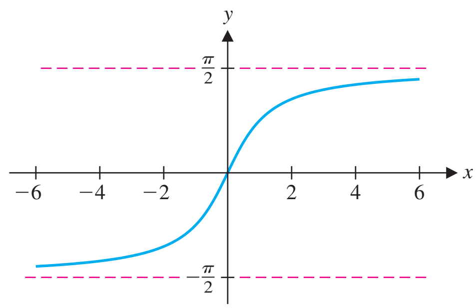


FIGURE 0.64
$y = \tan^{-1} x$

EXAMPLE 4.7 Evaluating an Inverse Tangent

Evaluate $\tan^{-1}(1)$.

Solution You must look for the angle θ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan\theta = 1$. This is easy enough. Since $\tan\left(\frac{\pi}{4}\right) = 1$ and $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have that $\tan^{-1}(1) = \frac{\pi}{4}$. ■

We now turn to defining an inverse for $\sec x$. First, we must issue a disclaimer. There are several reasonable ways in which to suitably restrict the domain and different authors restrict it differently. We have (somewhat arbitrarily) chosen to restrict the domain to be $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Why not use all of $[0, \pi]$? You need only think about the definition of $\sec x$ to see why we needed to exclude the value $x = \frac{\pi}{2}$. See Figure 0.65 for a graph of $\sec x$ on this domain. (Note the vertical asymptote at $x = \frac{\pi}{2}$.) Consequently, we define the **inverse secant** function by

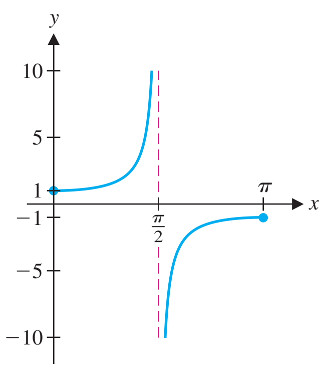


FIGURE 0.65
$y = \sec x$ on $[0, \pi]$

$$\boxed{y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \ \textit{and} \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].}$$

A graph of $\sec^{-1} x$ is shown in Figure 0.66.

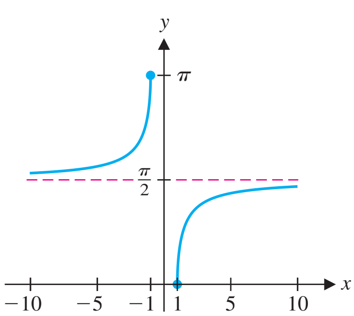


FIGURE 0.66
$y = \sec^{-1} x$

EXAMPLE 4.8 Evaluating an Inverse Secant

Evaluate $\sec^{-1}(-\sqrt{2})$.

REMARK 4.5

We can likewise define inverses to $\cot x$ and $\csc x$. As these functions are used only infrequently, we will omit them here and examine them in the exercises.

Function	*Domain*	*Range*
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

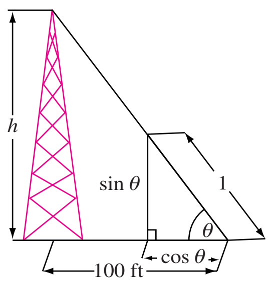


FIGURE 0.67
Height of a tower

Solution You must look for the angle θ with $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, for which $\sec\theta = -\sqrt{2}$. Notice that if $\sec\theta = -\sqrt{2}$, then $\cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. Recall from example 4.6 that $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$. Further, the angle $\frac{3\pi}{4}$ is in the interval $\left(\frac{\pi}{2}, \pi\right]$ and so, $\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$. ■

Calculators do not usually have built-in functions for $\sec x$ or $\sec^{-1} x$. In this case, you must convert the desired secant value to a cosine value and use the inverse cosine function, as we did in example 4.8.

We summarize the domains and ranges of the three main inverse trigonometric functions in the margin.

In many applications, we need to calculate the length of one side of a right triangle using the length of another side and an **acute** angle (i.e., an angle between 0 and $\frac{\pi}{2}$ radians). We can do this rather easily, as in example 4.9.

EXAMPLE 4.9 Finding the Height of a Tower

A person 100 feet from the base of a tower measures an angle of 60° from the ground to the top of the tower (see Figure 0.67). (a) Find the height of the tower. (b) What angle is measured if the person is 200 feet from the base?

Solution For (a), we first convert 60° to radians:

$$60^\circ = 60\frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

We are given that the base of the triangle in Figure 0.67 is 100 feet. We must now compute the height h of the tower. Using the similar triangles indicated in Figure 0.67, we have

$$\frac{\sin\theta}{\cos\theta} = \frac{h}{100},$$

so that the height of the tower is

$$h = 100\frac{\sin\theta}{\cos\theta} = 100\tan\theta = 100\tan\frac{\pi}{3} = 100\sqrt{3} \approx 173 \text{ feet.}$$

For part (b), the similar triangles in Figure 0.67 give us

$$\tan\theta = \frac{h}{200} = \frac{100\sqrt{3}}{200} = \frac{\sqrt{3}}{2}.$$

Since $0 < \theta < \dfrac{\pi}{2}$, we have

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 \text{ radians (about 41 degrees).}$$ ■

In example 4.10, we simplify expressions involving both trigonometric and inverse trigonometric functions.

EXAMPLE 4.10 Simplifying Expressions Involving Inverse Trigonometric Functions

Simplify (a) $\sin(\cos^{-1} x)$ and (b) $\tan(\cos^{-1} x)$.

Solution Do not look for some arcane formula to help you out. *Think* first: $\cos^{-1} x$ is the angle (call it θ) for which $x = \cos\theta$. First, consider the case where $x > 0$. Looking

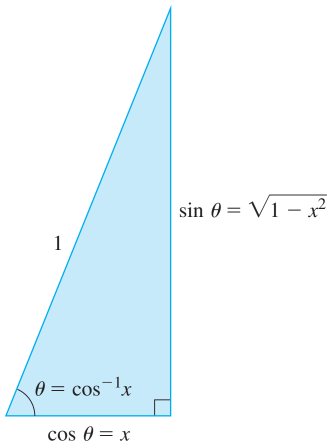


FIGURE 0.68
$\theta = \cos^{-1} x$

at Figure 0.68, we have drawn a right triangle, with hypotenuse 1 and adjacent angle θ. From the definition of the sine and cosine, then, we have that the base of the triangle is $\cos\theta = x$ and the altitude is $\sin\theta$, which by the Pythagorean Theorem is

$$\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}.$$

Wait! We have not yet finished part (a). Figure 0.68 shows $0 < \theta < \frac{\pi}{2}$, but by definition, $\theta = \cos^{-1} x$ could range from 0 to π. Does our answer change if $\frac{\pi}{2} < \theta < \pi$? To see that it doesn't change, note that if $0 \le \theta \le \pi$, then $\sin\theta \ge 0$. From the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we get

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-x^2}.$$

Since $\sin\theta \ge 0$, we must have

$$\sin\theta = \sqrt{1-x^2},$$

for *all* values of x.

For part (b), you can read from Figure 0.68 that

$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}.$$

Note that this last identity is valid, regardless of whether $x = \cos\theta$ is positive or negative. ■

EXERCISES 0.4

WRITING EXERCISES

1. Many students are comfortable using degrees to measure angles and don't understand why they must learn radian measures. As discussed in the text, radians directly measure distance along the unit circle. Distance is an important aspect of many applications. In addition, we will see later that many calculus formulas are simpler in radians form than in degrees. Aside from familiarity, discuss any and all advantages of degrees over radians. On balance, which is better?

2. A student graphs $f(x) = \cos x$ on a graphing calculator and gets what appears to be a straight line at height $y = 1$ instead of the usual cosine curve. Upon investigation, you discover that the calculator has graphing window $-10 \le x \le 10$, $-10 \le y \le 10$ and is in degrees mode. Explain what went wrong and how to correct it.

3. Inverse functions are necessary for solving equations. The restricted range we had to use to define inverses of the trigonometric functions also restricts their usefulness in equation solving. Explain how to use $\sin^{-1} x$ to find all solutions of the equation $\sin u = x$.

4. Discuss how to compute $\sec^{-1} x$, $\csc^{-1} x$ and $\cot^{-1} x$ on a calculator that has built-in functions only for $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.

5. In example 4.3, $f(x) = 4\cos 3x$ has period $2\pi/3$ and $g(x) = 2\sin(x/3)$ has period 6π. Explain why the sum $h(x) = 4\cos 3x + 2\sin(x/3)$ has period 6π.

6. Give a different range for $\sec^{-1} x$ than that given in the text. For which x's would the value of $\sec^{-1} x$ change? Using the calculator discussion in exercise 4, give one reason why we might have chosen the range that we did.

In exercises 1 and 2, convert the given radians measure to degrees.

1. (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{4\pi}{3}$
2. (a) $\frac{3\pi}{5}$ (b) $\frac{\pi}{7}$ (c) 2 (d) 3

In exercises 3 and 4, convert the given degrees measure to radians.

3. (a) 180° (b) 270° (c) 120° (d) 30°
4. (a) 40° (b) 80° (c) 450° (d) 390°

In exercises 5–14, find all solutions of the given equation.

5. $2\cos x - 1 = 0$
6. $2\sin x + 1 = 0$
7. $\sqrt{2}\cos x - 1 = 0$
8. $2\sin x - \sqrt{3} = 0$

9. $\sin^2 x - 4\sin x + 3 = 0$

10. $\sin^2 x - 2\sin x - 3 = 0$

11. $\sin^2 x + \cos x - 1 = 0$

12. $\sin 2x - \cos x = 0$

13. $\cos^2 x + \cos x = 0$

14. $\sin^2 x - \sin x = 0$

In exercises 15–24, sketch a graph of the function.

15. $f(x) = \sin 2x$

16. $f(x) = \cos 3x$

17. $f(x) = \tan 2x$

18. $f(x) = \sec 3x$

19. $f(x) = 3\cos(x - \pi/2)$

20. $f(x) = 4\cos(x + \pi)$

21. $f(x) = \sin 2x - 2\cos 2x$

22. $f(x) = \cos 3x - \sin 3x$

23. $f(x) = \sin x \sin 12x$

24. $f(x) = \sin x \cos 12x$

In exercises 25–32, identify the amplitude, period and frequency.

25. $f(x) = 3\sin 2x$

26. $f(x) = 2\cos 3x$

27. $f(x) = 5\cos 3x$

28. $f(x) = 3\sin 5x$

29. $f(x) = 3\cos(2x - \pi/2)$

30. $f(x) = 4\sin(3x + \pi)$

31. $f(x) = -4\sin x$

32. $f(x) = -2\cos 3x$

In exercises 33–36, prove that the given trigonometric identity is true.

33. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

34. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

35. (a) $\cos(2\theta) = 2\cos^2\theta - 1$ (b) $\cos(2\theta) = 1 - 2\sin^2\theta$

36. (a) $\sec^2\theta = \tan^2\theta + 1$ (b) $\csc^2\theta = \cot^2\theta + 1$

In exercises 37–46, evaluate the inverse function by sketching a unit circle and locating the correct angle on the circle.

37. $\cos^{-1} 0$

38. $\tan^{-1} 0$

39. $\sin^{-1}(-1)$

40. $\cos^{-1}(1)$

41. $\sec^{-1} 1$

42. $\tan^{-1}(-1)$

43. $\sec^{-1} 2$

44. $\csc^{-1} 2$

45. $\cot^{-1} 1$

46. $\tan^{-1}\sqrt{3}$

47. Prove that, for some constant β,

$$4\cos x - 3\sin x = 5\cos(x + \beta).$$

Then, estimate the value of β.

48. Prove that, for some constant β,

$$2\sin x + \cos x = \sqrt{5}\sin(x + \beta).$$

Then, estimate the value of β.

In exercises 49–52, determine whether the function is periodic. If it is periodic, find the smallest (fundamental) period.

49. $f(x) = \cos 2x + 3\sin \pi x$

50. $f(x) = \sin x - \cos\sqrt{2}x$

51. $f(x) = \sin 2x - \cos 5x$

52. $f(x) = \cos 3x - \sin 7x$

In exercises 53–56, use the range for θ to determine the indicated function value.

53. $\sin\theta = \frac{1}{3}, 0 \le \theta \le \frac{\pi}{2}$; find $\cos\theta$.

54. $\cos\theta = \frac{4}{5}, 0 \le \theta \le \frac{\pi}{2}$; find $\sin\theta$.

55. $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \le \theta \le \pi$; find $\cos\theta$.

56. $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \le \theta \le \pi$; find $\tan\theta$.

In exercises 57–64, use a triangle to simplify each expression. Where applicable, state the range of x's for which the simplification holds.

57. $\cos(\sin^{-1} x)$

58. $\cos(\tan^{-1} x)$

59. $\tan(\sec^{-1} x)$

60. $\cot(\cos^{-1} x)$

61. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

62. $\cos\left(\sin^{-1}\frac{1}{2}\right)$

63. $\tan\left(\cos^{-1}\frac{3}{5}\right)$

64. $\csc\left(\sin^{-1}\frac{2}{3}\right)$

In exercises 65–68, use a graphing calculator or computer to determine the number of solutions of each equation, and numerically estimate the solutions (x is in radians).

65. $2\cos x = 2 - x$

66. $3\sin x = x$

67. $\cos x = x^2 - 2$

68. $\sin x = x^2$

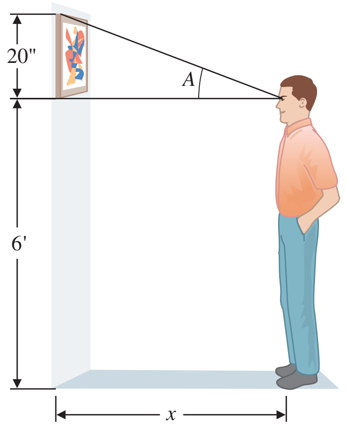
69. A person sitting 2 miles from a rocket launch site measures 20° up to the current location of the rocket. How high up is the rocket?

70. A person who is 6 feet tall stands 4 feet from the base of a light pole and casts a 2-foot-long shadow. How tall is the light pole?

71. A surveyor stands 80 feet from the base of a building and measures an angle of 50° to the top of the steeple on top of the building. The surveyor figures that the center of the steeple lies 20 feet inside the front of the structure. Find the distance from the ground to the top of the steeple.

72. Suppose that the surveyor of exercise 71 estimates that the center of the steeple lies between 20′ and 21′ inside the front of the structure. Determine how much the extra foot would change the calculation of the height of the building.

73. A picture hanging in an art gallery has a frame 20 inches high, and the bottom of the frame is 6 feet above the floor. A person whose eyes are 6 feet above the floor stands x feet from the wall. Let A be the angle formed by the ray from the person's eye to the bottom of the frame and the ray from the person's eye to the top of the frame. Write A as a function of x and graph $y = A(x)$.



74. In golf, the goal is to hit a ball into a hole of diameter 4.5 inches. Suppose a golfer stands x feet from the hole trying to putt the ball into the hole. A first approximation of the margin of error in a putt is to measure the angle A formed by the ray from the ball to the right edge of the hole and the ray from the ball to the left edge of the hole. Find A as a function of x.

75. In an AC circuit, the voltage is given by $v(t) = v_p \sin 2\pi f t$, where v_p is the peak voltage and f is the frequency in Hz. A voltmeter actually measures an average (called the **root-mean-square**) voltage, equal to $v_p/\sqrt{2}$. If the voltage has amplitude 170 and period $\pi/30$, find the frequency and meter voltage.

76. An old-style LP record player rotates records at $33\frac{1}{3}$ rpm (revolutions per minute). What is the period (in minutes) of the rotation? What is the period for a 45-rpm record?

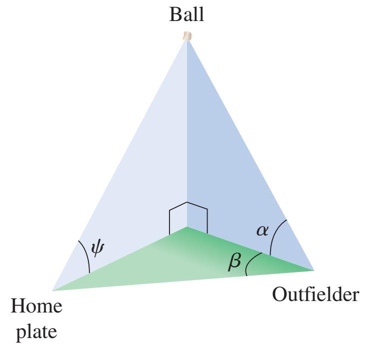
77. Suppose that the ticket sales of an airline (in thousands of dollars) is given by $s(t) = 110 + 2t + 15 \sin\left(\frac{1}{6}\pi t\right)$, where t is measured in months. What real-world phenomenon might cause the fluctuation in ticket sales modeled by the sine term? Based on your answer, what month corresponds to $t = 0$? Disregarding seasonal fluctuations, by what amount is the airline's sales increasing annually?

78. Piano tuners sometimes start by striking a tuning fork and then the corresponding piano key. If the tuning fork and piano note each have frequency 8, then the resulting sound is $\sin 8t + \sin 8t$. Graph this. If the piano is slightly out-of-tune at frequency 8.1, the resulting sound is $\sin 8t + \sin 8.1t$. Graph this and explain how the piano tuner can hear the small difference in frequency.

79. Give precise definitions of $\csc^{-1} x$ and $\cot^{-1} x$.

80. In baseball, outfielders are able to easily track down and catch fly balls that have very long and high trajectories. Physicists have argued for years about how this is done. A recent explanation involves the following geometry.

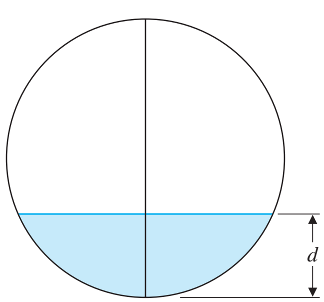
The player can catch the ball by running to keep the angle ψ constant (this makes it appear that the ball is moving in a straight line). Assuming that all triangles shown are right triangles, show that $\tan \psi = \dfrac{\tan \alpha}{\tan \beta}$ and then solve for ψ.



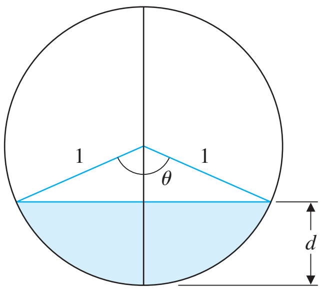
EXPLORATORY EXERCISES

1. In his book and video series *The Ring of Truth,* physicist Philip Morrison performed an experiment to estimate the circumference of the earth. In Nebraska, he measured the angle to a bright star in the sky, then drove 370 miles due south into Kansas and measured the new angle to the star. Some geometry shows that the difference in angles, about 5.02°, equals the angle from the center of the earth to the two locations in Nebraska and Kansas. If the earth is perfectly spherical (it's not) and the circumference of the portion of the circle measured out by 5.02° is 370 miles, estimate the circumference of the earth. This experiment was based on a similar experiment by the ancient Greek scientist Eratosthenes. The ancient Greeks and the Spaniards of Columbus' day knew that the earth was round, they just disagreed about the circumference. Columbus argued for a figure about half of the actual value, since a ship couldn't survive on the water long enough to navigate the true distance.

2. An oil tank with circular cross sections lies on its side. A stick is inserted in a hole at the top and used to measure the depth d of oil in the tank. Based on this measurement, the goal is to compute the percentage of oil left in the tank.



To simplify calculations, suppose the circle is a unit circle with center at (0, 0). Sketch radii extending from the origin to the top of the oil. The area of oil at the bottom equals the area of the portion of the circle bounded by the radii minus the area of the triangle formed above the oil in the figure.



Start with the triangle, which has area one-half base times height. Explain why the height is $1 - d$. Find a right triangle in the figure (there are two of them) with hypotenuse 1 (the radius of the circle) and one vertical side of length $1 - d$. The horizontal side has length equal to one-half the base of the larger triangle. Show that this equals $\sqrt{1-(1-d)^2}$. The area of the portion of the circle equals $\pi\theta/2\pi = \theta/2$, where θ is the angle at the top of the triangle. Find this angle as a function of d. (Hint: Go back to the right triangle used above with upper angle $\theta/2$.) Then find the area filled with oil and divide by π to get the portion of the tank filled with oil.

3. Computer graphics can be misleading. This exercise works best using a "disconnected" graph (individual dots, not connected). Graph $y = \sin x^2$ using a graphing window for which each pixel represents a step of 0.1 in the x- or y-direction. You should get the impression of a sine wave that oscillates more and more rapidly as you move to the left and right. Next, change the graphing window so that the middle of the original screen (probably $x = 0$) is at the far left of the new screen. You will likely see what appears to be a random jumble of dots. Continue to change the graphing window by increasing the x-values. Describe the patterns or lack of patterns that you see. You should find one pattern that looks like two rows of dots across the top and bottom of the screen; another pattern looks like the original sine wave. For each pattern that you find, pick adjacent points with x-coordinates a and b. Then change the graphing window so that $a \le x \le b$ and find the portion of the graph that is missing. Remember that, whether the points are connected or not, computer graphs always leave out part of the graph; it is part of your job to know whether or not the missing part is important.

0.5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Some bacteria reproduce very quickly, as you may have discovered if you have ever had an infected cut or strep throat. Under the right circumstances, the number of bacteria in certain cultures will double in as little as an hour. In this section, we discuss some functions that can be used to model such rapid growth.

Suppose that initially there are 100 bacteria at a given site and the population doubles every hour. Call the population function $P(t)$, where t represents time (in hours) and start the clock running at time $t = 0$. Since the initial population is 100, we have $P(0) = 100$. After 1 hour, the population will double to 200, so that $P(1) = 200$. After another hour, the population will have doubled again to 400, making $P(2) = 400$ and so on.

To compute the bacterial population after 10 hours, you could calculate the population at 4 hours, 5 hours and so on, or you could use the following shortcut. To find $P(1)$, double the initial population, so that $P(1) = 2 \cdot 100$. To find $P(2)$, double the population at time $t = 1$, so that $P(2) = 2 \cdot 2 \cdot 100 = 2^2 \cdot 100$. Similarly, $P(3) = 2^3 \cdot 100$. This pattern leads us to

$$P(10) = 2^{10} \cdot 100 = 102{,}400.$$

Observe that the population can be modeled by the function

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an **exponential** function because the variable t is in the exponent. There is a subtle question here: what is the domain of this function? We have so far used only integer values of t, but for what other values of t does $P(t)$ make sense? Certainly, rational powers make sense, as in $P(1/2) = 2^{1/2} \cdot 100$, where $2^{1/2} = \sqrt{2}$. This says that the number of bacteria in the culture after a half hour is approximately

$$P(1/2) = 2^{1/2} \cdot 100 = \sqrt{2} \cdot 100 \approx 141.$$

It's a simple matter to interpret fractional powers as roots. For instance,

$$\begin{aligned} x^{1/2} &= \sqrt{x}, \\ x^{1/3} &= \sqrt[3]{x}, \\ x^{1/4} &= \sqrt[4]{x}, \\ x^{2/3} &= \sqrt[3]{x^2} = (\sqrt[3]{x})^2, \\ x^{3.1} &= x^{31/10} = \sqrt[10]{x^{31}} \end{aligned}$$

and so on. But, what about irrational powers? They are harder to define, but they work exactly the way you would want them to. For instance, since π is between 3.14 and 3.15, 2^π is between $2^{3.14}$ and $2^{3.15}$. In this way, we define 2^x for x irrational to fill in the gaps in the graph of $y = 2^x$ for x rational. That is, if x is irrational and $a < x < b$, for rational numbers a and b, then $2^a < 2^x < 2^b$. This is the logic behind the definition of irrational powers.

If for some reason you wanted to find the bacterial population after π hours, you can use your calculator or computer to obtain the approximate population:

$$P(\pi) = 2^\pi \cdot 100 \approx 882.$$

For your convenience, we now summarize the usual rules of exponents.

RULES OF EXPONENTS

- For any integers m and n,

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m.$$

- For any real number p,

$$x^{-p} = \frac{1}{x^p}.$$

- For any real numbers p and q,

$$(x^p)^q = x^{p \cdot q}.$$

- For any real numbers p and q,

$$x^p \cdot x^q = x^{p+q}.$$

Throughout your calculus course, you will need to be able to quickly convert back and forth between exponential form and fractional or root form.

EXAMPLE 5.1 Converting Expressions to Exponential Form

Convert each to exponential form: (a) $3\sqrt{x^5}$, (b) $\dfrac{5}{\sqrt[3]{x}}$, (c) $\dfrac{3x^2}{2\sqrt{x}}$ and (d) $(2^x \cdot 2^{3+x})^2$.

Solution For (a), simply leave the 3 alone and convert the power:

$$3\sqrt{x^5} = 3x^{5/2}.$$

For (b), use a negative exponent to write x in the numerator:

$$\frac{5}{\sqrt[3]{x}} = 5x^{-1/3}.$$

For (c), first separate the constants from the variables and then simplify:

$$\frac{3x^2}{2\sqrt{x}} = \frac{3}{2}\frac{x^2}{x^{1/2}} = \frac{3}{2}x^{2-1/2} = \frac{3}{2}x^{3/2}.$$

For (d), first work inside the parentheses and then square:

$$(2^x \cdot 2^{3+x})^2 = (2^{x+3+x})^2 = (2^{2x+3})^2 = 2^{4x+6}.$$

The function in part (d) of example 5.1 is called an *exponential* function with a *base* of 2.

DEFINITION 5.1

For any constant $b > 0$, the function $f(x) = b^x$ is called an **exponential function.** Here, b is called the **base** and x is the **exponent.**

Be careful to distinguish between algebraic functions such as $f(x) = x^3$ and $g(x) = x^{2/3}$ and exponential functions. For exponential functions such as $h(x) = 2^x$, the variable is in the exponent (hence the name), instead of in the base. Also, notice that the domain of an exponential function is the entire real line, $(-\infty, \infty)$, while the range is the open interval $(0, \infty)$, since $b^x > 0$ for all x.

While any positive real number can be used as a base for an exponential function, three bases are the most commonly used in practice. Base 2 arises naturally when analyzing processes that double at regular intervals (such as the bacteria at the beginning of this section). Our standard counting system is base 10, so this base is commonly used. However, far and away the most useful base is the irrational number e. Like π, the number e has a surprising tendency to occur in important calculations. We define e by

$$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.} \tag{5.1}$$

Note that equation (5.1) has at least two serious shortcomings. First, we have not yet said what the notation $\lim\limits_{n\to\infty}$ means. (In fact, we won't define this until Chapter 1.) Second, it's unclear why anyone would ever define a number in such a strange way. We will not be in a position to answer the second question until Chapter 4 (but the answer is worth the wait).

It suffices for the moment to say that equation (5.1) means that e can be approximated by calculating values of $(1 + 1/n)^n$ for large values of n and that the larger the value of n, the closer the approximation will be to the actual value of e. In particular, if you look at the sequence of numbers $(1 + 1/2)^2, (1 + 1/3)^3, (1 + 1/4)^4$ and so on, they will get progressively closer and closer to (i.e., home in on) the irrational number e.

To get an idea of the value of e, compute several of these numbers:

$$\left(1+\frac{1}{10}\right)^{10} = 2.5937\ldots,$$

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.7169\ldots,$$

$$\left(1+\frac{1}{10{,}000}\right)^{10{,}000} = 2.7181\ldots$$

and so on. You should compute enough of these values to convince yourself that the first few digits of the decimal representation of e ($e \approx 2.718281828459\ldots$) are correct.

EXAMPLE 5.2 Computing Values of Exponentials

Approximate e^4, $e^{-1/5}$ and e^0.

Solution From a calculator, we find that

$$e^4 = e \cdot e \cdot e \cdot e \approx 54.598.$$

From the usual rules of exponents,

$$e^{-1/5} = \frac{1}{e^{1/5}} = \frac{1}{\sqrt[5]{e}} \approx 0.81873.$$

(On a calculator, it is convenient to replace $-1/5$ with -0.2.) Finally, $e^0 = 1$. ■

The graphs of the exponential functions summarize many of their important properties.

EXAMPLE 5.3 Sketching Graphs of Exponentials

Sketch the graphs of the exponential functions $y = 2^x$, $y = e^x$, $y = e^{2x}$, $y = e^{x/2}$, $y = (1/2)^x$ and $y = e^{-x}$.

Solution Using a calculator or computer, you should get graphs similar to those that follow.

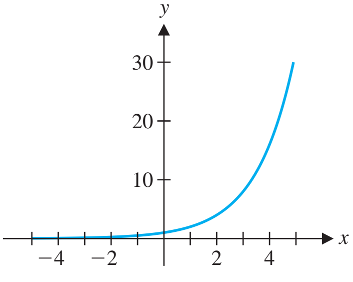


FIGURE 0.69a
$y = 2^x$

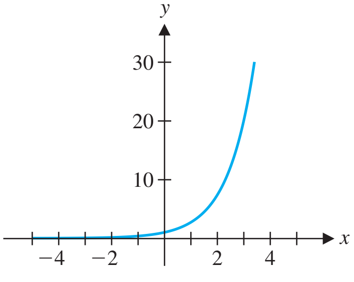


FIGURE 0.69b
$y = e^x$

Notice that each of the graphs in Figures 0.69a, 0.69b, 0.70a and 0.70b starts very near the x-axis (reading left to right), passes through the point (0, 1) and then rises steeply. This is true for all exponentials with base greater than 1 and with a positive coefficient in the exponent. Note that the larger the base ($e > 2$) or the larger the coefficient in the

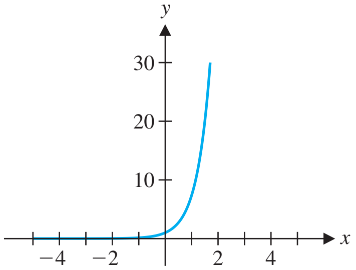


FIGURE 0.70a
$y = e^{2x}$

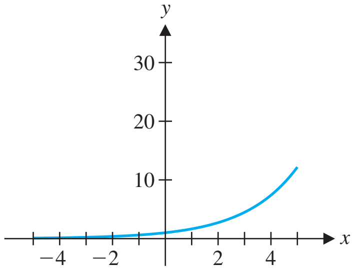


FIGURE 0.70b
$y = e^{x/2}$

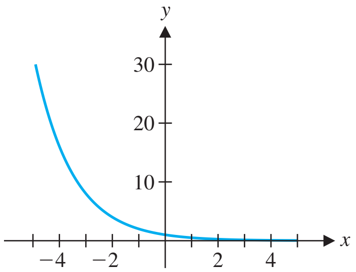


FIGURE 0.71a
$y = (1/2)^x$

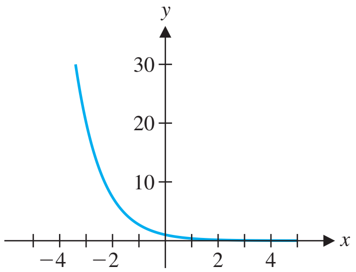


FIGURE 0.71b
$y = e^{-x}$

exponent ($2 > 1 > 1/2$), the more quickly the graph rises to the right (and drops to the left). Note that the graphs in Figures 0.71a and 0.71b are the mirror images in the y-axis of Figures 0.69a and 0.69b, respectively. The graphs rise as you move to the left and drop toward the x-axis as you move to the right. It's worth noting that by the rules of exponents, $(1/2)^x = 2^{-x}$ and $(1/e)^x = e^{-x}$. ■

In Figures 0.69–0.71, each exponential function is one-to-one and, hence, has an inverse function. We define the logarithmic functions to be inverses of the exponential functions.

DEFINITION 5.2

For any positive number $b \neq 1$, the **logarithm** function with base b, written $\log_b x$, is defined by

$$y = \log_b x \quad \text{if and only if} \quad x = b^y.$$

That is, the logarithm $\log_b x$ gives the exponent to which you must raise the base b to get the given number x. For example,

$$\begin{aligned} \log_{10} 10 &= 1 \quad (\text{since } 10^1 = 10), \\ \log_{10} 100 &= 2 \quad (\text{since } 10^2 = 100), \\ \log_{10} 1000 &= 3 \quad (\text{since } 10^3 = 1000) \end{aligned}$$

and so on. The value of $\log_{10} 45$ is less clear than the preceding three values, but the idea is the same: you need to find the number y such that $10^y = 45$. The answer is between

1 and 2, but to be more precise, you will need to employ trial and error. You should get $\log_{10} 45 \approx 1.6532$.

Observe from Definition 5.2 that for any base $b > 0\,(b \neq 1)$, if $y = \log_b x$, then $x = b^y > 0$. That is, the domain of $f(x) = \log_b x$ is the interval $(0, \infty)$. Likewise, the range of f is the entire real line, $(-\infty, \infty)$.

As with exponential functions, the most useful bases turn out to be 2, 10, and e. We usually abbreviate $\log_{10} x$ by $\log x$. Similarly, $\log_e x$ is usually abbreviated $\ln x$ (for **natural logarithm**).

EXAMPLE 5.4 Evaluating Logarithms

Without using your calculator, determine $\log(1/10)$, $\log(0.001)$, $\ln e$ and $\ln e^3$.

Solution Since $1/10 = 10^{-1}$, $\log(1/10) = -1$. Similarly, since $0.001 = 10^{-3}$, we have that $\log(0.001) = -3$. Since $\ln e = \log_e e^1$, $\ln e = 1$. Similarly, $\ln e^3 = 3$. ■

We want to emphasize the inverse relationship defined by Definition 5.2. That is, b^x and $\log_b x$ are inverse functions for any $b > 0\,(b \neq 1)$.

In particular, for the base e, we have

$$\boxed{e^{\ln x} = x \quad \text{for any } x > 0 \qquad \text{and} \qquad \ln(e^x) = x \quad \text{for any } x.} \tag{5.2}$$

We demonstrate this as follows. Let

$$y = \ln x = \log_e x.$$

By Definition 5.2, we have that

$$x = e^y = e^{\ln x}.$$

We can use this relationship between natural logarithms and exponentials to solve equations involving logarithms and exponentials, as in examples 5.5 and 5.6.

EXAMPLE 5.5 Solving a Logarithmic Equation

Solve the equation $\ln(x + 5) = 3$ for x.

Solution Taking the exponential of both sides of the equation and writing things backward (for convenience), we have

$$e^3 = e^{\ln(x+5)} = x + 5,$$

from (5.2). Subtracting 5 from both sides gives us

$$e^3 - 5 = x. \quad ■$$

EXAMPLE 5.6 Solving an Exponential Equation

Solve the equation $e^{x+4} = 7$ for x.

Solution Taking the natural logarithm of both sides and writing things backward (for simplicity), we have from (5.2) that

$$\ln 7 = \ln(e^{x+4}) = x + 4.$$

Subtracting 4 from both sides yields

$$\ln 7 - 4 = x. \quad ■$$

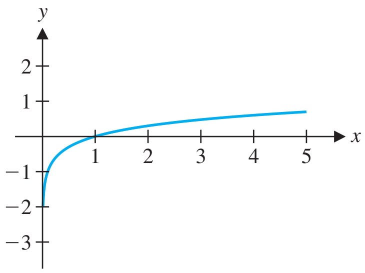


FIGURE 0.72a
$y = \log x$

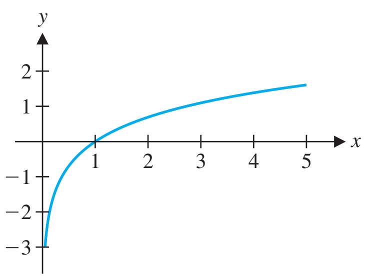


FIGURE 0.72b
$y = \ln x$

As always, graphs provide excellent visual summaries of the important properties of a function.

EXAMPLE 5.7 Sketching Graphs of Logarithms

Sketch graphs of $y = \log x$ and $y = \ln x$, and briefly discuss the properties of each.

Solution From a calculator or computer, you should obtain graphs resembling those in Figures 0.72a and 0.72b. Notice that both graphs appear to have a vertical asymptote at $x = 0$ (why would that be?), cross the x-axis at $x = 1$ and very gradually increase as x increases. Neither graph has any points to the left of the y-axis, since $\log x$ and $\ln x$ are defined only for $x > 0$. The two graphs are very similar, although not identical. ■

The properties just described graphically are summarized in Theorem 5.1.

THEOREM 5.1

For any positive base $b \neq 1$,

(i) $\log_b x$ is defined only for $x > 0$,
(ii) $\log_b 1 = 0$ and
(iii) if $b > 1$, then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

PROOF

(i) Note that since $b > 0$, $b^y > 0$ for any y. So, if $\log_b x = y$, then $x = b^y > 0$.
(ii) Since $b^0 = 1$ for any number $b \neq 0$, $\log_b 1 = 0$ (i.e., the exponent to which you raise the base b to get the number 1 is 0).
(iii) We leave this as an exercise. ■

All logarithms share a set of defining properties, as stated in Theorem 5.2.

THEOREM 5.2

For any positive base $b \neq 1$ and positive numbers x and y, we have

(i) $\log_b(xy) = \log_b x + \log_b y$,
(ii) $\log_b(x/y) = \log_b x - \log_b y$ and
(iii) $\log_b(x^y) = y \log_b x$.

As with most algebraic rules, each one of these properties can dramatically simplify calculations when it applies.

EXAMPLE 5.8 Simplifying Logarithmic Expressions

Write each as a single logarithm: (a) $\log_2 27^x - \log_2 3^x$ and (b) $\ln 8 - 3\ln(1/2)$.

Solution First, note that there is more than one order in which to work each problem. For part (a), we have $27 = 3^3$ and so, $27^x = (3^3)^x = 3^{3x}$. This gives us

$$\begin{aligned}\log_2 27^x - \log_2 3^x &= \log_2 3^{3x} - \log_2 3^x \\ &= 3x\log_2 3 - x\log_2 3 = 2x\log_2 3 = \log_2 3^{2x}.\end{aligned}$$

For part (b), note that $8 = 2^3$ and $1/2 = 2^{-1}$. Then,

$$\begin{aligned}\ln 8 - 3\ln(1/2) &= 3\ln 2 - 3(-\ln 2)\\ &= 3\ln 2 + 3\ln 2 = 6\ln 2 = \ln 2^6 = \ln 64.\end{aligned}$$

In some circumstances, it is beneficial to use the rules of logarithms to expand a given expression, as in example 5.9.

EXAMPLE 5.9 Expanding a Logarithmic Expression

Use the rules of logarithms to expand the expression $\ln\left(\frac{x^3y^4}{z^5}\right)$.

Solution From Theorem 5.2, we have that

$$\begin{aligned}\ln\left(\frac{x^3y^4}{z^5}\right) &= \ln(x^3y^4) - \ln(z^5) = \ln(x^3) + \ln(y^4) - \ln(z^5)\\ &= 3\ln x + 4\ln y - 5\ln z.\end{aligned}$$

Using the rules of exponents and logarithms, we can rewrite any exponential as an exponential with base e, as follows. For any base $a > 0$, we have

$$\boxed{a^x = e^{\ln(a^x)} = e^{x\ln a}.} \tag{5.3}$$

This follows from Theorem 5.2 (iii) and the fact that $e^{\ln y} = y$, for all $y > 0$.

EXAMPLE 5.10 Rewriting Exponentials as Exponentials with Base e

Rewrite the exponentials 2^x, 5^x and $(2/5)^x$ as exponentials with base e.

Solution From (5.3), we have

$$\begin{aligned}2^x &= e^{\ln(2^x)} = e^{x\ln 2},\\ 5^x &= e^{\ln(5^x)} = e^{x\ln 5}\end{aligned}$$

and

$$\left(\frac{2}{5}\right)^x = e^{\ln[(2/5)^x]} = e^{x\ln(2/5)}.$$

Just as we can rewrite an exponential with any positive base in terms of an exponential with base e, we can rewrite any logarithm in terms of natural logarithms, as follows. For any positive base $b\,(b \neq 1)$, we will show that

$$\boxed{\log_b x = \frac{\ln x}{\ln b}.} \tag{5.4}$$

Let $y = \log_b x$. Then by Definition 5.2, we have that $x = b^y$. Taking the natural logarithm of both sides of this equation, we get by Theorem 5.2 (iii) that

$$\ln x = \ln(b^y) = y\ln b.$$

Dividing both sides by $\ln b$ (since $b \neq 1$, $\ln b \neq 0$) gives us

$$y = \frac{\ln x}{\ln b},$$

establishing (5.4).

Equation (5.4) is useful for computing logarithms with bases other than e or 10. This is important since, more than likely, your calculator has keys only for $\ln x$ and $\log x$. We illustrate this idea in example 5.11.

EXAMPLE 5.11 Approximating the Value of Logarithms

Approximate the value of $\log_7 12$.

Solution From (5.4), we have

$$\log_7 12 = \frac{\ln 12}{\ln 7} \approx 1.2769894.$$

Saint Louis Gateway Arch

Hyperbolic Functions

There are two special combinations of exponential functions, called the **hyperbolic sine** and **hyperbolic cosine** functions, that have important applications. For instance, the Gateway Arch in Saint Louis was built in the shape of a hyperbolic cosine graph. (See the photograph in the margin.) The hyperbolic sine function [denoted by $\sinh(x)$] and the hyperbolic cosine function [denoted by $\cosh(x)$] are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Graphs of these functions are shown in Figures 0.73a and 0.73b. The hyperbolic functions (including the hyperbolic tangent, $\tanh x$, defined in the expected way) are often convenient to use when solving equations. For now, we verify several basic properties that the hyperbolic functions satisfy in parallel with their trigonometric counterparts.

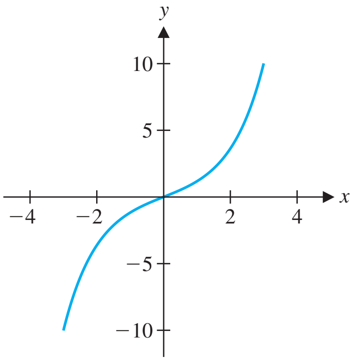


FIGURE 0.73a
$y = \sinh x$

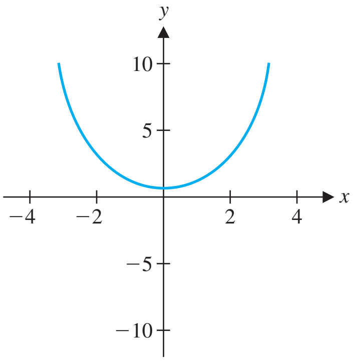


FIGURE 0.73b
$y = \cosh x$

EXAMPLE 5.12 Computing Values of Hyperbolic Functions

Compute $f(0)$, $f(1)$ and $f(-1)$, and determine how $f(x)$ and $f(-x)$ compare for each function: (a) $f(x) = \sinh x$ and (b) $f(x) = \cosh x$.

Solution For part (a), we have $\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$. Note that this means that $\sinh 0 = \sin 0 = 0$. Also, we have $\sinh 1 = \frac{e^1 - e^{-1}}{2} \approx 1.18$, while $\sinh(-1) = \frac{e^{-1} - e^1}{2} \approx -1.18$. Notice that $\sinh(-1) = -\sinh 1$. In fact, for any x,

$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh x.$$

[The same rule holds for the sine function: $\sin(-x) = -\sin x$.] For part (b), we have $\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$. Note that this means that $\cosh 0 = \cos 0 = 1$. Also, we have $\cosh 1 = \frac{e^1 + e^{-1}}{2} \approx 1.54$, while $\cosh(-1) = \frac{e^{-1} + e^1}{2} \approx 1.54$. Notice that $\cosh(-1) = \cosh 1$. In fact, for any x,

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

[The same rule holds for the cosine function: $\cos(-x) = \cos x$.] ■

Fitting a Curve to Data

You are familiar with the idea that two points determine a straight line. As we see in example 5.13, two points will also determine an exponential function.

EXAMPLE 5.13 Matching Data to an Exponential Curve

Find the exponential function of the form $f(x) = ae^{bx}$ that passes through the points (0, 5) and (3, 9).

Solution We must solve for a and b, using the properties of logarithms and exponentials. First, for the graph to pass through the point (0, 5), this means that

$$5 = f(0) = ae^{b \cdot 0} = a,$$

so that $a = 5$. Next, for the graph to pass through the point (3, 9), we must have

$$9 = f(3) = ae^{3b} = 5e^{3b}.$$

To solve for b, we divide both sides of the equation by 5 and take the natural logarithm of both sides, which yields

$$\ln\left(\frac{9}{5}\right) = \ln e^{3b} = 3b,$$

from (5.2). Finally, dividing by 3 gives us the value for b:

$$b = \frac{1}{3}\ln\left(\frac{9}{5}\right).$$

Thus, $f(x) = 5e^{\frac{1}{3}\ln(9/5)x}$. ■

Year	*U.S. Population*
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,866,020
1840	17,069,453
1850	23,191,876
1860	31,443,321

Consider the population of the United States from 1790 to 1860, found in the accompanying table. A plot of these data points can be seen in Figure 0.74 (where the vertical scale represents the population in millions). This shows that the population was increasing, with larger and larger increases each decade. If you sketch an imaginary curve through these points, you will probably get the impression of a parabola or perhaps the right half of a

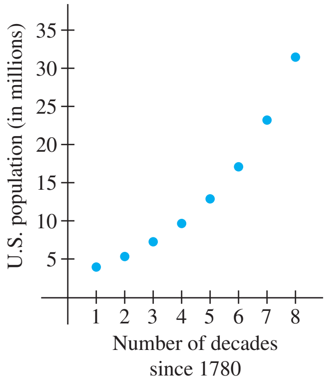


FIGURE 0.74
U.S. Population 1790–1860

cubic or exponential. And that's the question: are these data best modeled by a quadratic function, a cubic function, an exponential function or what?

We can use the properties of logarithms from Theorem 5.2 to help determine whether a given set of data is modeled better by a polynomial or an exponential function, as follows. Suppose that the data actually come from an exponential, say, $y = ae^{bx}$ (i.e., the data points lie on the graph of this exponential). Then,

$$\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx} = \ln a + bx.$$

If you draw a new graph, where the horizontal axis shows values of x and the vertical axis corresponds to values of $\ln y$, then the graph will be the line $\ln y = bx + c$ (where the constant $c = \ln a$). On the other hand, suppose the data actually came from a polynomial. If $y = bx^n$ (for any n), then observe that

$$\ln y = \ln(bx^n) = \ln b + \ln x^n = \ln b + n \ln x.$$

In this case, a graph with horizontal and vertical axes corresponding to x and $\ln y$, respectively, will look like the graph of a logarithm, $\ln y = n \ln x + c$. Such **semi-log graphs** (i.e., graphs of $\ln y$ versus x) let us distinguish the graph of an exponential from that of a polynomial: graphs of exponentials become straight lines, while graphs of polynomials (of degree ≥ 1) become logarithmic curves. Scientists and engineers frequently use semi-log graphs to help them gain an understanding of physical phenomena represented by some collection of data.

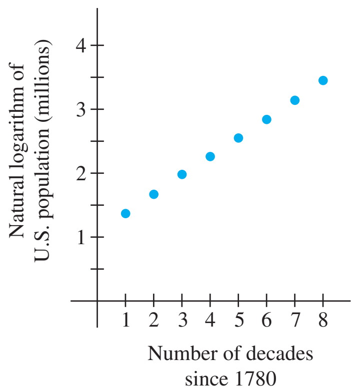


FIGURE 0.75
Semi-log plot of U.S. population

EXAMPLE 5.14 Using a Semi-Log Graph to Identify a Type of Function

Determine whether the population of the United States from 1790 to 1860 was increasing exponentially or as a polynomial.

Solution As already indicated, the trick is to draw a semi-log graph. That is, instead of plotting (1, 3.9) as the first data point, plot (1, ln 3.9) and so on. A semi-log plot of this data set is seen in Figure 0.75. Although the points are not exactly colinear (how would you prove this?), the plot is very close to a straight line with $\ln y$-intercept of 1 and slope 0.3. You should conclude that the population is well modeled by an exponential function. The exponential model would be $y = P(t) = ae^{bt}$, where t represents the number of decades since 1780. Here, b is the slope and $\ln a$ is the $\ln y$-intercept of the line in the semi-log graph. That is, $b \approx 0.3$ and $\ln a \approx 1$ (why?), so that $a \approx e$. The population is then modeled by

$$P(t) = e \cdot e^{0.3t} \text{ million.} \blacksquare$$

EXERCISES 0.5

WRITING EXERCISES

1. Starting from a single cell, a human being is formed by 50 generations of cell division. Explain why after n divisions there are 2^n cells. Guess how many cells will be present after 50 divisions, then compute 2^{50}. Briefly discuss how rapidly exponential functions increase.

2. Explain why the graphs of $f(x) = 2^{-x}$ and $g(x) = \left(\frac{1}{2}\right)^x$ are the same.

3. Compare $f(x) = x^2$ and $g(x) = 2^x$ for $x = \frac{1}{2}, x = 1, x = 2, x = 3$ and $x = 4$. In general, which function is bigger for large values of x? For small values of x?

4. Compare $f(x) = 2^x$ and $g(x) = 3^x$ for $x = -2, x = -\frac{1}{2}$, $x = \frac{1}{2}$ and $x = 2$. In general, which function is bigger for negative values of x? For positive values of x?

In exercises 1–6, convert each exponential expression into fractional or root form.

1. 2^{-3}
2. 4^{-2}
3. $3^{1/2}$
4. $6^{2/5}$
5. $5^{2/3}$
6. $4^{-2/3}$

In exercises 7–12, convert each expression into exponential form.

7. $\dfrac{1}{x^2}$
8. $\sqrt[3]{x^2}$
9. $\dfrac{2}{x^3}$
10. $\dfrac{4}{x^2}$
11. $\dfrac{1}{2\sqrt{x}}$
12. $\dfrac{3}{2\sqrt{x^3}}$

In exercises 13–16, find the integer value of the given expression without using a calculator.

13. $4^{3/2}$
14. $8^{2/3}$
15. $\dfrac{\sqrt{8}}{2^{1/2}}$
16. $\dfrac{2}{(1/3)^2}$

In exercises 17–20, use a calculator or computer to estimate each value.

17. $2e^{-1/2}$
18. $4e^{-2/3}$
19. $\dfrac{12}{e}$
20. $\dfrac{14}{\sqrt{e}}$

In exercises 21–30, sketch a graph of the given function.

21. $f(x) = e^{2x}$
22. $f(x) = e^{3x}$
23. $f(x) = 2e^{x/4}$
24. $f(x) = e^{-x^2}$
25. $f(x) = 3e^{-2x}$
26. $f(x) = 10e^{-x/3}$
27. $f(x) = \ln 2x$
28. $f(x) = \ln x^2$
29. $f(x) = e^{2\ln x}$
30. $f(x) = e^{-x/4}\sin x$

In exercises 31–40, solve the given equation for x.

31. $e^{2x} = 2$
32. $e^{4x} = 3$
33. $e^x(x^2 - 1) = 0$
34. $xe^{-2x} + 2e^{-2x} = 0$
35. $\ln 2x = 4$
36. $2\ln 3x = 1$
37. $4\ln x = -8$
38. $x^2\ln x - 9\ln x = 0$
39. $e^{2\ln x} = 4$
40. $\ln(e^{2x}) = 6$

In exercises 41 and 42, use the definition of logarithm to determine the value.

41. (a) $\log_3 9$ (b) $\log_4 64$ (c) $\log_3 \frac{1}{27}$
42. (a) $\log_4 \frac{1}{16}$ (b) $\log_4 2$ (c) $\log_9 3$

In exercises 43 and 44, use equation (5.4) to approximate the value.

43. (a) $\log_3 7$ (b) $\log_4 60$ (c) $\log_3 \frac{1}{24}$
44. (a) $\log_4 \frac{1}{10}$ (b) $\log_4 3$ (c) $\log_9 8$

In exercises 45–50, rewrite the expression as a single logarithm.

45. $\ln 3 - \ln 4$
46. $2\ln 4 - \ln 3$
47. $\frac{1}{2}\ln 4 - \ln 2$
48. $3\ln 2 - \ln \frac{1}{2}$
49. $\ln \frac{3}{4} + 4\ln 2$
50. $\ln 9 - 2\ln 3$

In exercises 51–54, find a function of the form $f(x) = ae^{bx}$ with the given function values.

51. $f(0) = 2,\ f(2) = 6$
52. $f(0) = 3,\ f(3) = 4$
53. $f(0) = 4,\ f(2) = 2$
54. $f(0) = 5,\ f(1) = 2$

55. A fast-food restaurant gives every customer a game ticket. With each ticket, the customer has a 1-in-10 chance of winning a free meal. If you go 10 times, estimate your chances of winning at least one free meal. The exact probability is $1 - \left(\frac{9}{10}\right)^{10}$. Compute this number and compare it to your guess.

56. In exercise 55, if you had 20 tickets with a 1-in-20 chance of winning, would you expect your probability of winning at least once to increase or decrease? Compute the probability $1 - \left(\frac{19}{20}\right)^{20}$ to find out.

57. In general, if you have n chances of winning with a 1-in-n chance on each try, the probability of winning at least once is $1 - \left(1 - \frac{1}{n}\right)^n$. As n gets larger, what number does this probability approach? (Hint: There is a very good reason that this question is in this section!)

58. If $y = a \cdot x^m$, show that $\ln y = \ln a + m\ln x$. If $v = \ln y$, $u = \ln x$ and $b = \ln a$, show that $v = mu + b$. Explain why the graph of v as a function of u would be a straight line. This graph is called the **log-log plot** of y and x.

59. For the given data, compute $v = \ln y$ and $u = \ln x$, and plot points (u, v). Find constants m and b such that $v = mu + b$ and use the results of exercise 58 to find a constant a such that $y = a \cdot x^m$.

x	2.2	2.4	2.6	2.8	3.0	3.2
y	14.52	17.28	20.28	23.52	27.0	30.72

60. Repeat exercise 59 for the given data.

x	2.8	3.0	3.2	3.4	3.6	3.8
y	9.37	10.39	11.45	12.54	13.66	14.81

61. Construct a log-log plot (see exercise 58) of the U.S. population data in example 5.14. Compared to the semi-log plot of the data in Figure 0.75, does the log-log plot look linear? Based on this, are the population data modeled better by an exponential function or a polynomial (power) function?

62. Construct a semi-log plot of the data in exercise 59. Compared to the log-log plot already constructed, does this plot look linear? Based on this, are these data better modeled by an exponential or power function?

63. The concentration $[H^+]$ of free hydrogen ions in a chemical solution determines the solution's pH, as defined by $\text{pH} = -\log[H^+]$. Find $[H^+]$ if the pH equals (a) 7, (b) 8 and (c) 9. For each increase in pH of 1, by what factor does $[H^+]$ change?

64. Gastric juice is considered an acid, with a pH of about 2.5. Blood is considered alkaline, with a pH of about 7.5. Compare the concentrations of hydrogen ions in the two substances (see exercise 63).

65. The Richter magnitude M of an earthquake is defined in terms of the energy E in joules released by the earthquake, with $\log_{10} E = 4.4 + 1.5M$. Find the energy for earthquakes with magnitudes (a) 4, (b) 5 and (c) 6. For each increase in M of 1, by what factor does E change?

66. It puzzles some people who have not grown up around earthquakes that a magnitude 6 quake is considered much more severe than a magnitude 3 quake. Compare the amount of energy released in the two quakes. (See exercise 65.)

67. The decibel level of a noise is defined in terms of the intensity I of the noise, with $\text{dB} = 10\log(I/I_0)$. Here, $I_0 = 10^{-12}\ \text{W/m}^2$ is the intensity of a barely audible sound. Compute the intensity levels of sounds with (a) dB = 80, (b) dB = 90 and (c) dB = 100. For each increase of 10 decibels, by what factor does I change?

68. At a basketball game, a courtside decibel meter shows crowd noises ranging from 60 dB to 110 dB. Compare the intensity level of the 110-dB crowd noise with that of the 60-dB noise. (See exercise 67.)

69. Use a graphing calculator to graph $y = xe^{-x}$, $y = xe^{-2x}$, $y = xe^{-3x}$ and so on. Estimate the location of the maximum for each. In general, state a rule for the location of the maximum of $y = xe^{-kx}$.

70. In golf, the task is to hit a golf ball into a small hole. If the ground near the hole is not flat, the golfer must judge how much the ball's path will curve. Suppose the golfer is at the point $(-13, 0)$, the hole is at the point $(0, 0)$ and the path of the ball is, for $-13 \le x \le 0$, $y = -1.672x + 72\ln(1 + 0.02x)$. Show that the ball goes in the hole and estimate the point on the y-axis at which the golfer aimed.

Exercises 71–76 refer to the hyperbolic functions.

71. Show that the range of the hyperbolic cosine is $\cosh x \ge 1$ and the range of the hyperbolic sine is the entire real line.

72. Show that $\cosh^2 x - \sinh^2 x = 1$ for all x.

73. The Saint Louis Gateway Arch is both 630 feet wide and 630 feet tall. (Most people think that it looks taller than it is wide.) One model for the outline of the arch is $y = 757.7 - 127.7\cosh\left(\frac{x}{127.7}\right)$ for $y \ge 0$. Use a graphing calculator to approximate the x- and y-intercepts and determine if the model has the correct horizontal and vertical measurements.

74. To model the outline of the Gateway Arch with a parabola, you can start with $y = -c(x + 315)(x - 315)$ for some constant c. Explain why this gives the correct x-intercepts. Determine the constant c that gives a y-intercept of 630. Graph this parabola and the hyperbolic cosine in exercise 73 on the same axes. Are the graphs nearly identical or very different?

75. Find all solutions of $\sinh(x^2 - 1) = 0$.

76. Find all solutions of $\cosh(3x + 2) = 0$.

77. On a standard piano, the A below middle C produces a sound wave with frequency 220 Hz (cycles per second). The frequency of the A one octave higher is 440 Hz. In general, doubling the frequency produces the same note an octave higher. Find an exponential formula for the frequency f as a function of the number of octaves x above the A below middle C.

78. There are 12 notes in an octave on a standard piano. Middle C is 3 notes above A (see exercise 77). If the notes are tuned equally, this means that middle C is a quarter-octave above A. Use $x = \frac{1}{4}$ in your formula from exercise 77 to estimate the frequency of middle C.

EXPLORATORY EXERCISES

1. Graph $y = x^2$ and $y = 2^x$ and approximate the two positive solutions of the equation $x^2 = 2^x$. Graph $y = x^3$ and $y = 3^x$, and approximate the two positive solutions of the equation $x^3 = 3^x$. Explain why $x = a$ will always be a solution of $x^a = a^x$, $a > 0$. What is different about the role of $x = 2$ as a solution of $x^2 = 2^x$ compared to the role of $x = 3$ as a solution of $x^3 = 3^x$? To determine the a-value at which the change occurs, graphically solve $x^a = a^x$ for $a = 2.1, 2.2, \ldots, 2.9$, and note that $a = 2.7$ and $a = 2.8$ behave differently. Continue to narrow down the interval of change by testing $a = 2.71, 2.72, \ldots, 2.79$. Then guess the exact value of a.

2. Graph $y = \ln x$ and describe the behavior near $x = 0$. Then graph $y = x\ln x$ and describe the behavior near $x = 0$. Repeat this for $y = x^2\ln x$, $y = x^{1/2}\ln x$ and $y = x^a\ln x$ for a variety of positive constants a. Because the function "blows up" at $x = 0$, we say that $y = \ln x$ has a **singularity** at $x = 0$. The **order** of the singularity at $x = 0$ of a function $f(x)$ is the smallest value of a such that $y = x^a f(x)$ doesn't have a singularity at $x = 0$. Determine the order of the singularity at $x = 0$ for (a) $f(x) = \frac{1}{x}$, (b) $f(x) = \frac{1}{x^2}$ and (c) $f(x) = \frac{1}{x^3}$. The higher the order of the singularity, the "worse" the singularity is. Based on your work, how bad is the singularity of $y = \ln x$ at $x = 0$?

0.6 TRANSFORMATIONS OF FUNCTIONS

You are now familiar with a long list of functions: polynomials, rational functions, trigonometric functions, exponentials and logarithms. One important goal of this course is to more fully understand the properties of these functions. To a large extent, you will build your understanding by examining a few key properties of functions.

We expand on our list of functions by combining them. We begin in a straight-forward fashion with Definition 6.1.

DEFINITION 6.1

Suppose that $f(x)$ and $g(x)$ are functions with domains D_1 and D_2, respectively. The functions $f+g$, $f-g$ and $f \cdot g$ are defined by

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x)$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x),$$

for all x in $D_1 \cap D_2$ (i.e., $x \in D_1$, *and* $x \in D_2$). The function $\dfrac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

for all x in $D_1 \cap D_2$ such that $g(x) \neq 0$.

In example 6.1, we examine various combinations of several simple functions.

EXAMPLE 6.1 Combinations of Functions

If $f(x) = x - 3$ and $g(x) = \sqrt{x-1}$, determine the functions $f+g$, $3f-g$ and $\dfrac{f}{g}$, stating the domains of each.

Solution First, note that the domain of f is the entire real line and the domain of g is the set of all $x \geq 1$. Now,

$$(f+g)(x) = x - 3 + \sqrt{x-1}$$

and

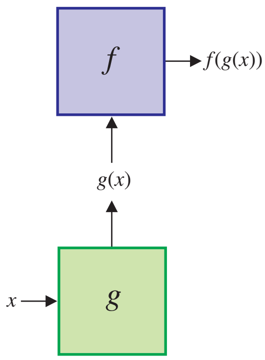
$$(3f-g)(x) = 3(x-3) - \sqrt{x-1} = 3x - 9 - \sqrt{x-1}.$$

Notice that the domain of both $(f+g)$ and $(3f-g)$ is $\{x | x \geq 1\}$. For

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-3}{\sqrt{x-1}},$$

the domain is $\{x | x > 1\}$, where we have added the restriction $x \neq 1$ to avoid dividing by 0. ■

Definition 6.1 and example 6.1 show us how to do arithmetic with functions. An operation on functions that does not directly correspond to arithmetic is the *composition* of two functions.



$(f \circ g)(x) = f(g(x))$

DEFINITION 6.2

The **composition** of functions f and g, written $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)),$$

for all x such that x is in the domain of g and $g(x)$ is in the domain of f.

The composition of two functions is a two-step process, as indicated in the margin schematic. Be careful to notice what this definition is saying. In particular, for $f(g(x))$ to be defined, you first need $g(x)$ to be defined, so x must be in the domain of g. Next, f must be defined at the point $g(x)$, so that the number $g(x)$ will need to be in the domain of f.

EXAMPLE 6.2 Finding the Composition of Two Functions

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution First, we have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = f(\sqrt{x-2}) \\ &= (\sqrt{x-2})^2 + 1 = x - 2 + 1 = x - 1.\end{aligned}$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g, we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x, the domain of $(f \circ g)$ is $\{x | x \geq 2\}$.

For the second composition,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(x^2 + 1) \\ &= \sqrt{(x^2+1) - 2} = \sqrt{x^2 - 1}.\end{aligned}$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the "inside" function f is defined for all x, the domain of $g \circ f$ is $\{x \in \mathbb{R} \big| |x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$. ■

As you progress through the calculus, you will often find yourself needing to recognize that a given function is a composition of simpler functions. For now, it is an important skill to practice.

EXAMPLE 6.3 Identifying Compositions of Functions

Identify functions f and g such that the given function can be written as $(f \circ g)(x)$ for each of (a) $\sqrt{x^2+1}$, (b) $(\sqrt{x}+1)^2$, (c) $\sin x^2$ and (d) $\cos^2 x$. Note that more than one answer is possible for each function.

Solution (a) Notice that $x^2 + 1$ is *inside* the square root. So, one choice is to have $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$.

(b) Here, $\sqrt{x}+1$ is *inside* the square. So, one choice is $g(x)=\sqrt{x}+1$ and $f(x)=x^2$.

(c) The function can be rewritten as $\sin(x^2)$, with x^2 clearly *inside* the sine function. Then, $g(x)=x^2$ and $f(x)=\sin x$ is one choice.

(d) The function as written is shorthand for $(\cos x)^2$. So, one choice is $g(x)=\cos x$ and $f(x)=x^2$. ■

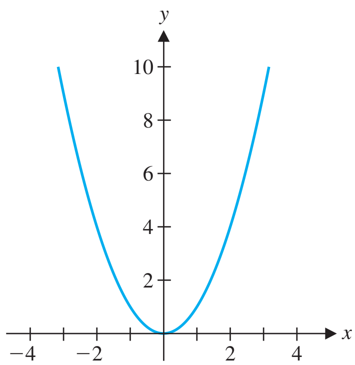


FIGURE 0.76a
$y=x^2$

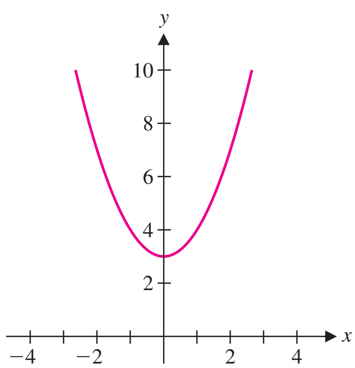


FIGURE 0.76b
$y=x^2+3$

In general, it is quite difficult to take the graphs of $f(x)$ and $g(x)$ and produce the graph of $f(g(x))$. If one of the functions f and g is linear, however, there is a simple graphical procedure for graphing the composition. Such **linear transformations** are explored in the remainder of this section.

The first case is to take the graph of $f(x)$ and produce the graph of $f(x)+c$ for some constant c. You should be able to deduce the general result from example 6.4.

EXAMPLE 6.4 Vertical Translation of a Graph

Graph $y=x^2$ and $y=x^2+3$; compare and contrast the graphs.

Solution You can probably sketch these by hand. You should get graphs like those in Figures 0.76a and 0.76b. Both figures show parabolas opening upward. The main obvious difference is that x^2 has a y-intercept of 0 and x^2+3 has a y-intercept of 3. In fact, for *any* given value of x, the point on the graph of $y=x^2+3$ will be plotted exactly 3 units higher than the corresponding point on the graph of $y=x^2$. This is shown in Figure 0.77a.

In Figure 0.77b, the two graphs are shown on the same set of axes. To many people, it does not look like the top graph is the same as the bottom graph moved up 3 units.

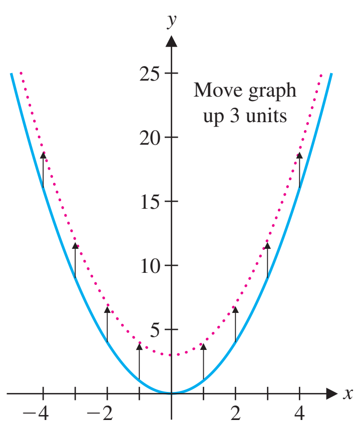


FIGURE 0.77a
Translate graph up

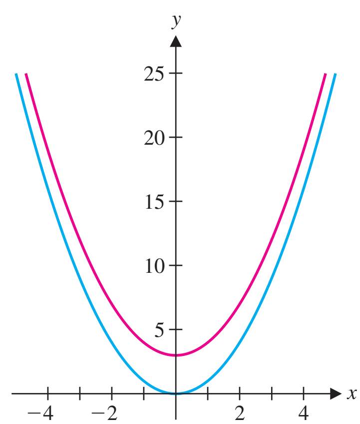


FIGURE 0.77b
$y=x^2$ and $y=x^2+3$

This is an unfortunate optical illusion. Humans usually mentally judge distance between curves as the shortest distance between the curves. For these parabolas, the shortest distance is vertical at $x=0$ but becomes increasingly horizontal as you move away from the y-axis. The distance of 3 between the parabolas is measured *vertically*. ■

In general, the graph of $y=f(x)+c$ is the same as the graph of $f(x)$ shifted up (if $c>0$) or down (if $c<0$) by $|c|$ units. We usually refer to $f(x)+c$ as a **vertical translation** (up or down, by $|c|$ units).

In example 6.5, we explore what happens if a constant is added to x.

EXAMPLE 6.5 A Horizontal Translation

Compare and contrast the graphs of $y = x^2$ and $y = (x - 1)^2$.

Solution The graphs are shown in Figures 0.78a and 0.78b, respectively.

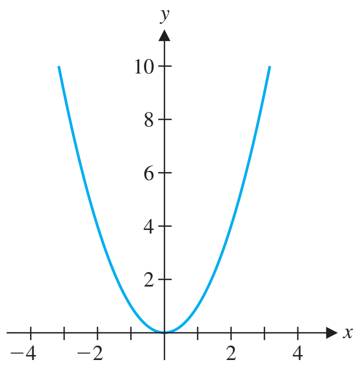


FIGURE 0.78a
$y = x^2$

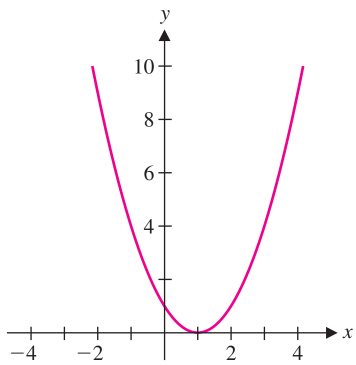


FIGURE 0.78b
$y = (x - 1)^2$

Notice that the graph of $y = (x - 1)^2$ appears to be the same as the graph of $y = x^2$, except that it is shifted 1 unit to the right. This should make sense for the following reason. Pick a value of x, say, $x = 13$. The value of $(x - 1)^2$ at $x = 13$ is 12^2, the same as the value of x^2 at $x = 12$, 1 unit to the left. Observe that this same pattern holds for any x you choose. A simultaneous plot of the two functions (see Figure 0.79) shows this. ■

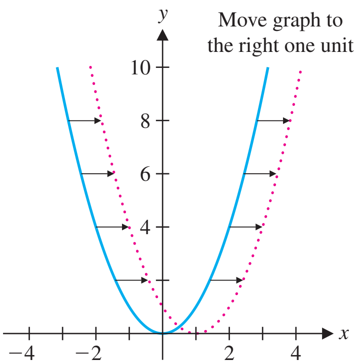


FIGURE 0.79
Translation to the right

In general, for $c > 0$, the graph of $y = f(x - c)$ is the same as the graph of $y = f(x)$ shifted c units to the right. Likewise (again, for $c > 0$), you get the graph of $f(x + c)$ by moving the graph of $y = f(x)$ to the left c units. We usually refer to $f(x - c)$ and $f(x + c)$ as **horizontal translations** (to the right and left, respectively, by c units).

To avoid confusion on which way to translate the graph of $y = f(x)$, focus on what makes the argument (the quantity inside the parentheses) zero. For $f(x)$, this is $x = 0$, but for $f(x - c)$ you must have $x = c$ to get $f(0)$ [i.e., the same y-value as $f(x)$ when $x = 0$]. This says that the point on the graph of $y = f(x)$ at $x = 0$ corresponds to the point on the graph of $y = f(x - c)$ at $x = c$.

EXAMPLE 6.6 Comparing Vertical and Horizontal Translations

Given the graph of $y = f(x)$ shown in Figure 0.80a, sketch the graphs of $y = f(x) - 2$ and $y = f(x - 2)$.

Solution To graph $y = f(x) - 2$, simply translate the original graph down 2 units, as shown in Figure 0.80b. To graph $y = f(x - 2)$, simply translate the original graph to the right 2 units (so that the x-intercept at $x = 0$ in the original graph corresponds to an x-intercept at $x = 2$ in the translated graph), as seen in Figure 0.80c.

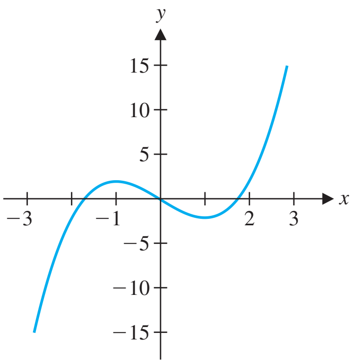


FIGURE 0.80a
$y = f(x)$

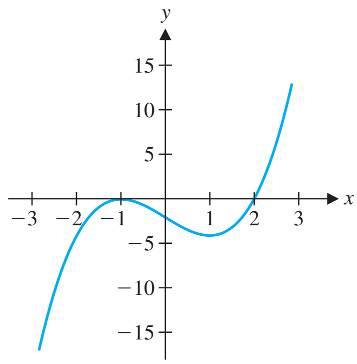


FIGURE 0.80b
$y = f(x) - 2$

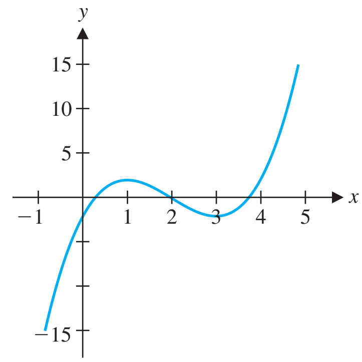


FIGURE 0.80c
$y = f(x - 2)$

Example 6.7 explores the effect of multiplying or dividing x or y by a constant.

EXAMPLE 6.7 Comparing Some Related Graphs

Compare and contrast the graphs of $y = x^2 - 1$, $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$.

Solution The first two graphs are shown in Figures 0.81a and 0.81b, respectively.

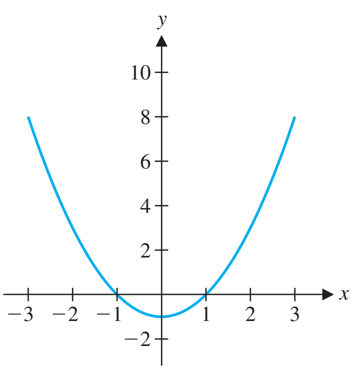


FIGURE 0.81a
$y = x^2 - 1$

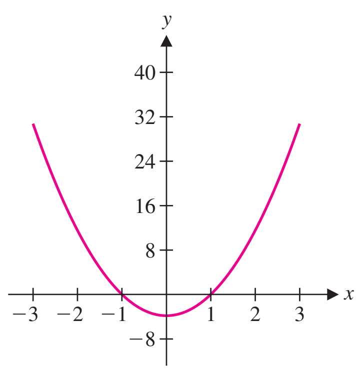


FIGURE 0.81b
$y = 4(x^2 - 1)$

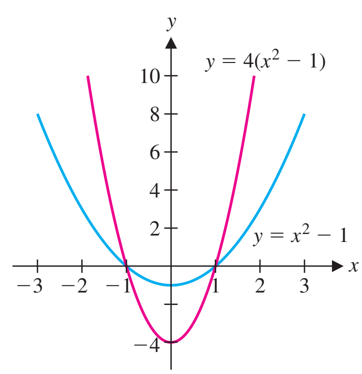


FIGURE 0.81c
$y = x^2 - 1$ and $y = 4(x^2 - 1)$

These graphs look identical until you compare the scales on the y-axes. The scale in Figure 0.81b is four times as large, reflecting the multiplication of the original function by 4. The effect looks different when the functions are plotted on the same scale, as in Figure 0.81c. Here, the parabola $y = 4(x^2 - 1)$ looks thinner and has a different y-intercept. Note that the x-intercepts remain the same. (Why would that be?)

The graphs of $y = x^2 - 1$ and $y = (4x)^2 - 1$ are shown in Figures 0.82a and 0.82b, respectively.

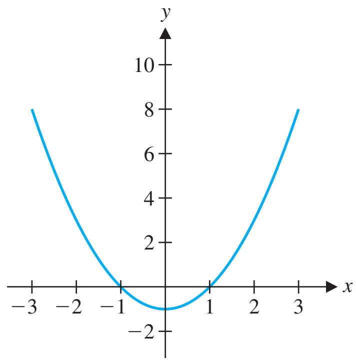


FIGURE 0.82a
$y = x^2 - 1$

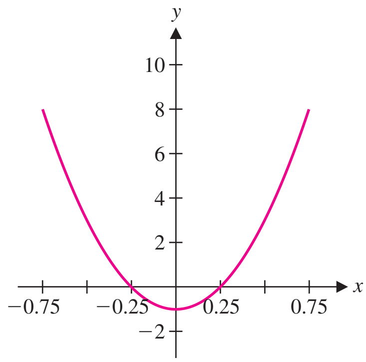


FIGURE 0.82b
$y = (4x)^2 - 1$

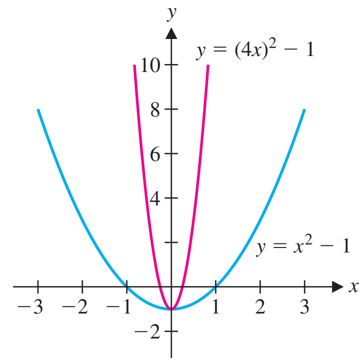


FIGURE 0.82c
$y = x^2 - 1$ and $y = (4x)^2 - 1$

Can you spot the difference here? In this case, the x-scale has now changed, by the same factor of 4 as in the function. To see this, note that substituting $x = 1/4$ into $(4x)^2 - 1$ produces $(1)^2 - 1$, exactly the same as substituting $x = 1$ into the original function. When plotted on the same set of axes (as in Figure 0.82c), the parabola $y = (4x)^2 - 1$ looks thinner. Here, the x-intercepts are different, but the y-intercepts are the same. ■

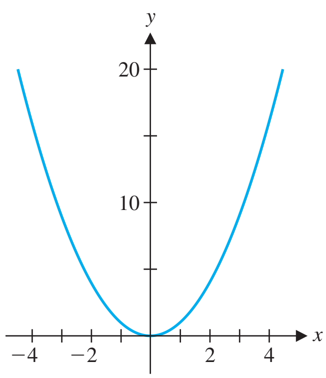


FIGURE 0.83a
$y = x^2$

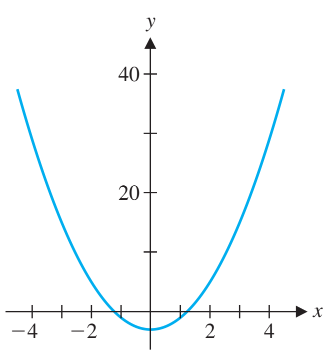


FIGURE 0.83b
$y = 2x^2 - 3$

We can generalize the observations made in example 6.7. Before reading our explanation, try to state a general rule for yourself. How are the graphs of the functions $cf(x)$ and $f(cx)$ related to the graph of $y = f(x)$?

Based on example 6.7, notice that to obtain a graph of $y = cf(x)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the y-axis by c. To obtain a graph of $y = f(cx)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the x-axis by $1/c$.

These basic rules can be combined to understand more complicated graphs.

EXAMPLE 6.8 A Translation and a Stretching

Describe how to get the graph of $y = 2x^2 - 3$ from the graph of $y = x^2$.

Solution You can get from x^2 to $2x^2 - 3$ by multiplying by 2 and then subtracting 3. In terms of the graph, this has the effect of multiplying the y-scale by 2 and then shifting the graph down by 3 units (see the graphs in Figures 0.83a and 0.83b). ■

EXAMPLE 6.9 A Translation in Both x- and y-Directions

Describe how to get the graph of $y = x^2 + 4x + 3$ from the graph of $y = x^2$.

Solution We can again relate this (and the graph of *every* quadratic) to the graph of $y = x^2$. We must first **complete the square.** Recall that in this process, you take the coefficient of x (4), divide by 2 $(4/2 = 2)$ and square the result $(2^2 = 4)$. Add and subtract this number and then, rewrite the x-terms as a perfect square. We have

$$y = x^2 + 4x + 3 = (x^2 + 4x + 4) - 4 + 3 = (x + 2)^2 - 1.$$

To graph this function, take the parabola $y = x^2$ (see Figure 0.84a) and translate the graph 2 units to the left and 1 unit down (see Figure 0.84b).

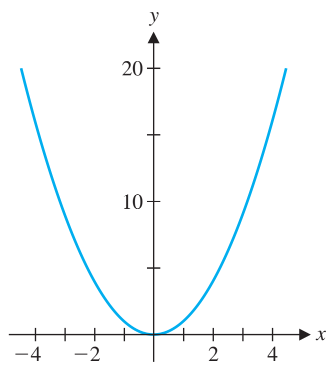


FIGURE 0.84a
$y = x^2$

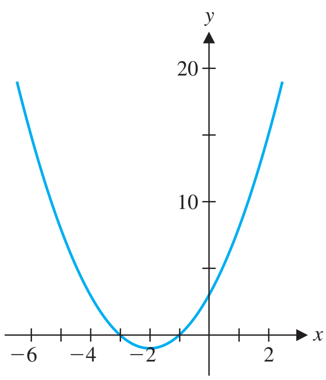


FIGURE 0.84b
$y = (x + 2)^2 - 1$

The following table summarizes our discoveries in this section.

Transformations of $f(x)$

Transformation	*Form*	*Effect on Graph*
Vertical translation	$f(x) + c$	$\lvert c\rvert$ units up ($c > 0$) or down ($c < 0$)
Horizontal translation	$f(x + c)$	$\lvert c\rvert$ units left ($c > 0$) or right ($c < 0$)
Vertical scale	$cf(x)\,(c > 0)$	multiply vertical scale by c
Horizontal scale	$f(cx)\,(c > 0)$	divide horizontal scale by c

You will explore additional transformations in the exercises.

EXERCISES 0.6

WRITING EXERCISES

1. The restricted domain of example 6.2 may be puzzling. Consider the following analogy. Suppose you have an airplane flight from New York to Los Angeles with a stop for refueling in Minneapolis. If bad weather has closed the airport in Minneapolis, explain why your flight will be canceled (or at least rerouted) even if the weather is great in New York and Los Angeles.

2. Explain why the graphs of $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$ in Figures 0.81c and 0.82c appear "thinner" than the graph of $y = x^2 - 1$.

3. As illustrated in example 6.9, completing the square can be used to rewrite any quadratic function in the form $a(x - d)^2 + e$. Using the transformation rules in this section, explain why this means that all parabolas (with $a > 0$) will look essentially the same.

4. Explain why the graph of $y = f(x + 4)$ is obtained by moving the graph of $y = f(x)$ four units to the left, instead of to the right.

In exercises 1–6, find the compositions $f \circ g$ and $g \circ f$, and identify their respective domains.

1. $f(x) = x + 1, \quad g(x) = \sqrt{x - 3}$
2. $f(x) = x - 2, \quad g(x) = \sqrt{x + 1}$
3. $f(x) = e^x, \quad g(x) = \ln x$
4. $f(x) = \sqrt{1 - x}, \quad g(x) = \ln x$
5. $f(x) = x^2 + 1, \quad g(x) = \sin x$
6. $f(x) = \dfrac{1}{x^2 - 1}, \quad g(x) = x^2 - 2$

In exercises 7–16, identify functions $f(x)$ and $g(x)$ such that the given function equals $(f \circ g)(x)$.

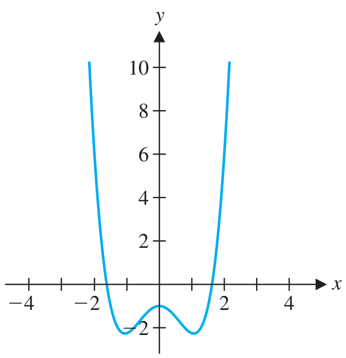
7. $\sqrt{x^4 + 1}$
8. $\sqrt[3]{x + 3}$
9. $\dfrac{1}{x^2 + 1}$
10. $\dfrac{1}{x^2} + 1$
11. $(4x + 1)^2 + 3$
12. $4(x + 1)^2 + 3$
13. $\sin^3 x$
14. $\sin x^3$
15. e^{x^2+1}
16. e^{4x-2}

In exercises 17–22, identify functions $f(x)$, $g(x)$ and $h(x)$ such that the given function equals $[f \circ (g \circ h)](x)$.

17. $\dfrac{3}{\sqrt{\sin x + 2}}$
18. $\sqrt{e^{4x} + 1}$
19. $\cos^3(4x - 2)$
20. $\ln \sqrt{x^2 + 1}$
21. $4e^{x^2} - 5$
22. $\left[\tan^{-1}(3x + 1)\right]^2$

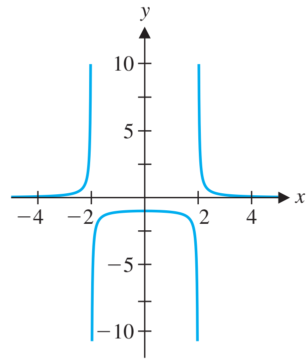
In exercises 23–30, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

23. $f(x) - 3$
24. $f(x + 2)$
25. $f(x - 3)$
26. $f(x) + 2$
27. $f(2x)$
28. $3f(x)$
29. $4f(x) - 1$
30. $3f(x + 2)$



In exercises 31–38, use the graph of $y = f(x)$ given in the figure to graph the indicated function.

31. $f(x - 4)$
32. $f(x + 3)$
33. $f(2x)$
34. $f(2x - 4)$
35. $f(3x + 3)$
36. $3f(x)$
37. $2f(x) - 4$
38. $3f(x) + 3$



In exercises 39–44, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

39. $f(x) = x^2 + 2x + 1$
40. $f(x) = x^2 - 4x + 4$
41. $f(x) = x^2 + 2x + 4$
42. $f(x) = x^2 - 4x + 2$
43. $f(x) = 2x^2 + 4x + 4$
44. $f(x) = 3x^2 - 6x + 2$

In exercises 45–48, graph the given function and compare to the graph of $y = x^2 - 1$.

45. $f(x) = -2(x^2 - 1)$
46. $f(x) = -3(x^2 - 1)$
47. $f(x) = -3(x^2 - 1) + 2$
48. $f(x) = -2(x^2 - 1) - 1$

In exercises 49–52, graph the given function and compare to the graph of $y = (x - 1)^2 - 1 = x^2 - 2x$.

49. $f(x) = (-x)^2 - 2(-x)$
50. $f(x) = (-2x)^2 - 2(-2x)$
51. $f(x) = (-x)^2 - 2(-x) + 1$
52. $f(x) = (-3x)^2 - 2(-3x) - 3$
53. Based on exercises 45–48, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = cf(x)$ for $c < 0$.
54. Based on exercises 49–52, state a rule for transforming the graph of $y = f(x)$ into the graph of $y = f(cx)$ for $c < 0$.
55. Sketch the graph of $y = |x|^3$. Explain why the graph of $y = |x|^3$ is identical to that of $y = x^3$ to the right of the y-axis. For $y = |x|^3$, describe how the graph to the left of the y-axis compares to the graph to the right of the y-axis. In general, describe how to draw the graph of $y = f(|x|)$ given the graph of $y = f(x)$.
56. For $y = x^3$, describe how the graph to the left of the y-axis compares to the graph to the right of the y-axis. Show that for $f(x) = x^3$, we have $f(-x) = -f(x)$. In general, if you have the graph of $y = f(x)$ to the right of the y-axis and $f(-x) = -f(x)$ for all x, describe how to graph $y = f(x)$ to the left of the y-axis.

57. **Iterations** of functions are important in a variety of applications. To iterate $f(x)$, start with an initial value x_0 and compute $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$ and so on. For example, with $f(x) = \cos x$ and $x_0 = 1$, the **iterates** are $x_1 = \cos 1 \approx 0.54$, $x_2 = \cos x_1 \approx \cos 0.54 \approx 0.86$, $x_3 \approx \cos 0.86 \approx 0.65$ and so on. Keep computing iterates and show that they get closer and closer to 0.739085. Then pick your own x_0 (any number you like) and show that the iterates with this new x_0 also converge to 0.739085.

58. Referring to exercise 57, show that the iterates of a function can be written as $x_1 = f(x_0)$, $x_2 = f(f(x_0))$, $x_3 = f(f(f(x_0)))$ and so on. Graph $y = \cos(\cos x)$, $y = \cos(\cos(\cos x))$ and $y = \cos(\cos(\cos(\cos x)))$. The graphs should look more and more like a horizontal line. Use the result of exercise 57 to identify the limiting line.

59. Compute several iterates of $f(x) = \sin x$ (see exercise 57) with a variety of starting values. What happens to the iterates in the long run?

60. Repeat exercise 59 for $f(x) = x^2$.

61. In cases where the iterates of a function (see exercise 57) repeat a single number, that number is called a **fixed point.** Explain why any fixed point must be a solution of the equation $f(x) = x$. Find all fixed points of $f(x) = \cos x$ by solving the equation $\cos x = x$. Compare your results to that of exercise 57.

62. Find all fixed points of $f(x) = \sin x$ (see exercise 61). Compare your results to those of exercise 59.

EXPLORATORY EXERCISES

1. You have explored how completing the square can transform any quadratic function into the form $y = a(x-d)^2 + e$. We concluded that all parabolas with $a > 0$ look alike. To see that the same statement is not true of cubic polynomials, graph $y = x^3$ and $y = x^3 - 3x$. In this exercise, you will use completing the cube to determine how many different cubic graphs there are. To see what "completing the cube" would look like, first show that $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. Use this result to transform the graph of $y = x^3$ into the graphs of (a) $y = x^3 - 3x^2 + 3x - 1$ and (b) $y = x^3 - 3x^2 + 3x + 2$. Show that you can't get a simple transformation to $y = x^3 - 3x^2 + 4x - 2$. However, show that $y = x^3 - 3x^2 + 4x - 2$ can be obtained from $y = x^3 + x$ by basic transformations. Show that the following statement is true: any cubic ($y = ax^3 + bx^2 + cx + d$) can be obtained with basic transformations from $y = ax^3 + kx$ for some constant k.

2. In many applications, it is important to take a section of a graph (e.g., some data) and extend it for predictions or other analysis. For example, suppose you have an electronic signal equal to $f(x) = 2x$ for $0 \le x \le 2$. To predict the value of the signal at $x = -1$, you would want to know whether the signal was periodic. If the signal is periodic, explain why $f(-1) = 2$ would be a good prediction. In some applications, you would assume that the function is *even.* That is, $f(x) = f(-x)$ for all x. In this case, you want $f(x) = 2(-x) = -2x$ for $-2 \le x \le 0$. Graph the *even extension* $f(x) = \begin{cases} -2x & \text{if } -2 \le x \le 0 \\ 2x & \text{if } 0 \le x \le 2 \end{cases}$. Find the even extension for (a) $f(x) = x^2 + 2x + 1, 0 \le x \le 2$ and (b) $f(x) = e^{-x}$, $0 \le x \le 2$.

3. Similar to the even extension discussed in exploratory exercise 2, applications sometimes require a function to be *odd;* that is, $f(-x) = -f(x)$. For $f(x) = x^2$, $0 \le x \le 2$, the odd extension requires that for $-2 \le x \le 0$, $f(x) = -f(-x) = -(-x)^2 = -x^2$ so that $f(x) = \begin{cases} -x^2 & \text{if } -2 \le x \le 0 \\ x^2 & \text{if } 0 \le x \le 2 \end{cases}$. Graph $y = f(x)$ and discuss how to graphically rotate the right half of the graph to get the left half of the graph. Find the odd extension for (a) $f(x) = x^2 + 2x$, $0 \le x \le 2$ and (b) $f(x) = e^{-x} - 1$, $0 \le x \le 2$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Slope of a line	Parallel lines	Perpendicular lines
Domain	Intercepts	Zeros of a function
Graphing window	Local maximum	Vertical asymptote
Inverse function	One-to-one function	Periodic function
Sine function	Cosine function	Arcsine function
e	Exponential function	Logarithm
Composition		

Review Exercises

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to “fix it” by modifying the given statement to a new statement that is true.

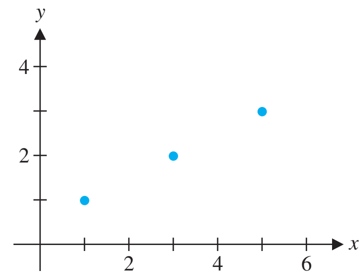
1. For a graph, you can compute the slope using any two points and get the same value.
2. All graphs must pass the vertical line test.
3. A cubic function has a graph with one local maximum and one local minimum.
4. If a function has no local maximum or minimum, then it is one-to-one.
5. The graph of the inverse of f can be obtained by reflecting the graph of f across the diagonal $y = x$.
6. If f is a trigonometric function, then the solution of the equation $f(x) = 1$ is $f^{-1}(1)$.
7. Exponential and logarithmic functions are inverses of each other.
8. All quadratic functions have graphs that look like the parabola $y = x^2$.

In exercises 1 and 2, find the slope of the line through the given points.

1. (2, 3), (0, 7)
2. (1, 4), (3, 1)

In exercises 3 and 4, determine whether the lines are parallel, perpendicular or neither.

3. $y = 3x + 1$ and $y = 3(x - 2) + 4$
4. $y = -2(x + 1) - 1$ and $y = \frac{1}{2}x + 2$
5. Determine whether the points (1, 2), (2, 4) and (0, 6) form the vertices of a right triangle.
6. The data represent populations at various times. Plot the points, discuss any patterns and predict the population at the next time: (0, 2100), (1, 3050), (2, 4100) and (3, 5050).
7. Find an equation of the line through the points indicated in the graph that follows and compute the y-coordinate corresponding to $x = 4$.



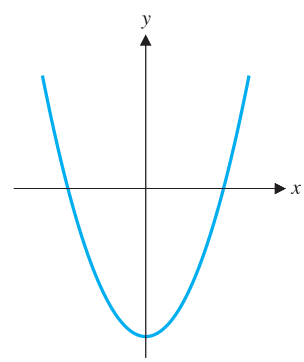
8. For $f(x) = x^2 - 3x - 4$, compute $f(0)$, $f(2)$ and $f(4)$.

In exercises 9 and 10, find an equation of the line with given slope and point.

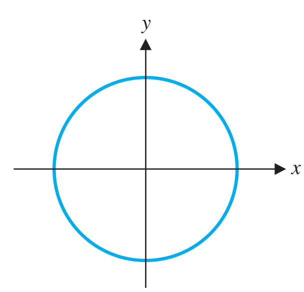
9. $m = -\frac{1}{3}$, $(-1, -1)$ 10. $m = \frac{1}{4}$, $(0, 2)$

In exercises 11 and 12, use the vertical line test to determine whether the curve is the graph of a function.

11.



12.



In exercises 13 and 14, find the domain of the given function.

13. $f(x) = \sqrt{4 - x^2}$ 14. $f(x) = \dfrac{x - 2}{x^2 - 2}$

Review Exercises

In exercises 15–28, sketch a graph of the function showing extrema, intercepts and asymptotes.

15. $f(x) = x^2 + 2x - 8$

16. $f(x) = x^3 - 6x + 1$

17. $f(x) = x^4 - 2x^2 + 1$

18. $f(x) = x^5 - 4x^3 + x - 1$

19. $f(x) = \dfrac{4x}{x+2}$

20. $f(x) = \dfrac{x-2}{x^2 - x - 2}$

21. $f(x) = \sin 3x$

22. $f(x) = \tan 4x$

23. $f(x) = \sin x + 2\cos x$

24. $f(x) = \sec 2x$

25. $f(x) = 4e^{2x}$

26. $f(x) = 3e^{-4x}$

27. $f(x) = \ln 3x$

28. $f(x) = e^{\ln 2x}$

29. Determine all intercepts of $y = x^2 + 2x - 8$ (see exercise 15).

30. Determine all intercepts of $y = x^4 - 2x^2 + 1$ (see exercise 17).

31. Find all vertical asymptotes of $y = \dfrac{4x}{x+2}$.

32. Find all vertical asymptotes of $y = \dfrac{x-2}{x^2 - x - 2}$.

In exercises 33–36, find or estimate all zeros of the given function.

33. $f(x) = x^2 - 3x - 10$

34. $f(x) = x^3 + 4x^2 + 3x$

35. $f(x) = x^3 - 3x^2 + 2$

36. $f(x) = x^4 - 3x - 2$

In exercises 37 and 38, determine the number of solutions.

37. $\sin x = x^3$

38. $\sqrt{x^2+1} = x^2 - 1$

39. A surveyor stands 50 feet from a telephone pole and measures an angle of 34° to the top. How tall is the pole?

40. Find $\sin\theta$ given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{1}{5}$.

41. Convert to fractional or root form: (a) $5^{-1/2}$ (b) 3^{-2}.

42. Convert to exponential form: (a) $\dfrac{2}{\sqrt{x}}$ (b) $\dfrac{3}{x^2}$.

43. Rewrite $\ln 8 - 2\ln 2$ as a single logarithm.

44. Solve the equation for x: $e^{\ln 4x} = 8$.

In exercises 45 and 46, solve the equation for x.

45. $3e^{2x} = 8$

46. $2\ln 3x = 5$

In exercises 47 and 48, find $f \circ g$ and $g \circ f$, and identify their respective domains.

47. $f(x) = x^2$, $g(x) = \sqrt{x-1}$

48. $f(x) = x^2$, $g(x) = \dfrac{1}{x^2 - 1}$

In exercises 49 and 50, identify functions $f(x)$ and $g(x)$ such that $(f \circ g)(x)$ equals the given function.

49. e^{3x^2+2}

50. $\sqrt{\sin x + 2}$

In exercises 51 and 52, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

51. $f(x) = x^2 - 4x + 1$

52. $f(x) = x^2 + 4x + 6$

In exercises 53–56, determine whether the function is one-to-one. If so, find its inverse.

53. $x^3 - 1$

54. e^{-4x}

55. e^{2x^2}

56. $x^3 - 2x + 1$

In exercises 57–60, graph the inverse without solving for the inverse.

57. $x^5 + 2x^3 - 1$

58. $x^3 + 5x + 2$

59. $\sqrt{x^3 + 4x}$

60. e^{x^3+2x}

In exercises 61–64, evaluate the quantity using the unit circle.

61. $\sin^{-1} 1$

62. $\cos^{-1}\left(-\frac{1}{2}\right)$

63. $\tan^{-1}(-1)$

64. $\csc^{-1}(-2)$

In exercises 65–68, simplify the expression.

65. $\sin(\sec^{-1} 2)$

66. $\tan(\cos^{-1}(4/5))$

67. $\sin^{-1}(\sin(3\pi/4))$

68. $\cos^{-1}(\sin(-\pi/4))$

In exercises 69 and 70, find all solutions of the equation.

69. $\sin 2x = 1$

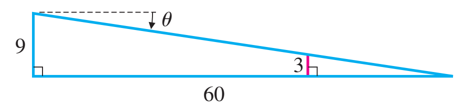
70. $\cos 3x = \frac{1}{2}$

EXPLORATORY EXERCISES

1. Sketch a graph of any function $y = f(x)$ that has an inverse. (Your choice.) Sketch the graph of the inverse function $y = f^{-1}(x)$. Then sketch the graph of $y = g(x) = f(x+2)$. Sketch the graph of $y = g^{-1}(x)$, and use the graph to determine a formula for $g^{-1}(x)$ in terms of $f^{-1}(x)$. Repeat this for $h(x) = f(x) + 3$ and $k(x) = f(x-4) + 5$.

Review Exercises

2. In tennis, a serve must clear the net and then land inside of a box drawn on the other side of the net. In this exercise, you will explore the margin of error for successfully serving. First, consider a straight serve (this essentially means a serve hit infinitely hard) struck 9 feet above the ground. Call the starting point (0, 9). The back of the service box is 60 feet away, at (60, 0). The top of the net is 3 feet above the ground and 39 feet from the server, at (39, 3). Find the service angle θ (i.e., the angle as measured from the horizontal) for the triangle formed by the points (0, 9), (0, 0) and (60, 0). Of course, most serves curve down due to gravity. Ignoring air resistance, the path of the ball struck at angle θ and initial speed v ft/s is $y = -\frac{16}{(v\cos\theta)^2}x^2 - (\tan\theta)x + 9$. To hit the back of the service line, you need $y = 0$ when $x = 60$. Substitute in these values along with $v = 120$. Multiply by $\cos^2\theta$ and replace $\sin\theta$ with $\sqrt{1-\cos^2\theta}$. Replacing $\cos\theta$ with z gives you an algebraic equation in z. Numerically estimate z. Similarly, substitute $x = 39$ and $y = 3$ and find an equation for $w = \cos\theta$. Numerically estimate w. The margin of error for the serve is given by $\cos^{-1} z < \theta < \cos^{-1} w$.



3. Baseball players often say that an unusually fast pitch rises or even hops up as it reaches the plate. One explanation of this illusion involves the players' inability to track the ball all the way to the plate. The player must compensate by predicting where the ball will be when it reaches the plate. Suppose the height of a pitch when it reaches home plate is $h = -(240/v)^2 + 6$ feet for a pitch with velocity v ft/s. (This equation takes into consideration gravity but not air resistance.) Halfway to the plate, the height would be $h = -(120/v)^2 + 6$ feet. Compare the halfway heights for pitches with $v = 132$ and $v = 139$ (about 90 and 95 mph, respectively). Would a batter be able to tell much difference between them? Now compare the heights at the plate. Why might the batter think that the faster pitch hopped up right at the plate? How many inches did the faster pitch hop?