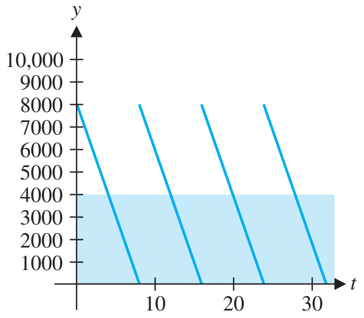
Integration

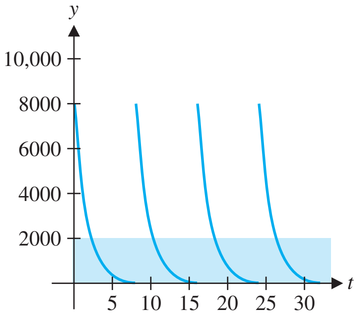
CHAPTER 4

In the modern business world, companies must find the most cost-efficient method of handling their inventory. One method is **just-in-time inventory,** where new inventory arrives just as existing stock is running out. As a simplified example of this, suppose that a heating oil company's terminal receives shipments of 8000 gallons of oil at a time and orders are shipped out to customers at a constant rate of 1000 gallons per day, where each shipment of oil arrives just as the last gallon on hand is shipped out. Inventory costs are determined based on the average number of gallons held at the terminal. So, how would we calculate this average?

To translate this into a calculus problem, let $f(t)$ represent the number of gallons of oil at the terminal at time t (days), where a shipment arrives at time $t = 0$. In this case, $f(0) = 8000$. Further, for $0 < t < 8$, there is no oil coming in, but oil is leaving at the rate of 1000 gallons per day. Since "rate" means derivative, we have $f'(t) = -1000$, for $0 < t < 8$. This tells us that the graph of $y = f(t)$ has slope -1000 until time $t = 8$, at which point another shipment arrives to refill the terminal, so that $f(8) = 8000$. Continuing in this way, we generate the graph of $f(t)$ shown here at the left.



$y = f(t)$



$y = g(t)$

Since the inventory ranges from 0 gallons to 8000 gallons, you might guess that the average inventory of oil is 4000 gallons. However, look at the graph at the right, showing a different inventory function $g(t)$, where the oil is not shipped out at a constant rate. Although the inventory again ranges from 0 to 8000, the drop in inventory is so rapid immediately following each delivery that the average number of gallons on hand is well below 4000.

As we will see in this chapter, our usual way of averaging a set of numbers is analogous to an area problem. Specifically, the average value of a function is the height of the rectangle that has the same area as the area between the graph of the function and the x-axis. For our original $f(t)$, notice that 4000 appears to work well, while for $g(t)$, an average of 2000 appears to be better, as you can see in the graphs.

Actually, several problems were just introduced: finding a function from its derivative, finding the average value of a function and finding the area under a curve. In this chapter, you will explore the relationships among these problems and learn a variety of techniques for solving them.

4.1 ANTIDERIVATIVES

NOTES

For a realistic model of a system as complex as a space shuttle, we must consider much more than the simple concepts discussed here. For a very interesting presentation of this problem, see the article by Long and Weiss in the February 1999 issue of *The American Mathematical Monthly.*

Space shuttle *Endeavor*

Calculus provides us with a powerful set of tools for understanding the world around us. When engineers originally designed the space shuttle for NASA, it was equipped with aircraft engines to power its flight through the atmosphere after reentry. In order to cut costs, the aircraft engines were scrapped and the space shuttle became a huge glider. As a result, once the shuttle has begun its reentry, there is only *one* choice of landing site. NASA engineers use the calculus to provide precise answers to flight control problems. While we are not in a position to deal with the vast complexities of a space shuttle flight, we can consider an idealized model.

As we often do with real-world problems, we begin with a physical principle(s) and use this to produce a *mathematical model* of the physical system. We then solve the mathematical problem and interpret the solution in terms of the physical problem.

If we consider only the vertical motion of an object falling toward the ground, the physical principle governing the motion is Newton's second law of motion:

$$\text{Force} = \text{mass} \times \text{acceleration} \quad \text{or} \quad F = ma.$$

This says that the sum of all the forces acting on an object equals the product of its mass and acceleration. Two forces that you might identify here are gravity pulling downward and air drag pushing in the direction opposite the motion. From experimental evidence, we know that the force due to air drag, F_d, is proportional to the square of the speed of the object and acts in the direction opposite the motion. So, for the case of a falling object,

$$F_d = kv^2,$$

for some constant $k > 0$.

The force due to gravity is simply the weight of the object, $W = -mg$, where the gravitational constant g is approximately 32 ft/s^2. (The minus sign indicates that the force of gravity acts downward.) Putting this together, Newton's second law of motion gives us

$$F = ma = -mg + kv^2.$$

Recognizing that $a = v'(t)$, we have

$$mv'(t) = -mg + kv^2(t). \tag{1.1}$$

Notice that equation (1.1) involves both the unknown function $v(t)$ and its derivative $v'(t)$. Such an equation is called a **differential equation.** We discuss differential equations in detail in Chapter 7. To get started now, we simplify the problem by assuming that gravity is the only force acting on the object. Taking $k = 0$ in (1.1) gives us

$$mv'(t) = -mg \quad \text{or} \quad v'(t) = -g.$$

Now, let $y(t)$ be the position function, giving the altitude of the object in feet t seconds after the start of reentry. Since $v(t) = y'(t)$ and $a(t) = v'(t)$, we have

$$y''(t) = -32.$$

From this, we'd like to determine $y(t)$. More generally, we need to find a way to *undo* differentiation. That is, given a function, $f(x)$, we'd like to find another function $F(x)$ such that $F'(x) = f(x)$. We call such a function F an **antiderivative** of f.

EXAMPLE 1.1 Finding Several Antiderivatives of a Given Function

Find an antiderivative of $f(x) = x^2$.

Solution Notice that $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x)$, since

$$F'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2.$$

Further, observe that
$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 5\right) = x^2,$$

so that $G(x) = \frac{1}{3}x^3 + 5$ is also an antiderivative of f. In fact, for any constant c, we have

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2.$$

Thus, $H(x) = \frac{1}{3}x^3 + c$ is also an antiderivative of $f(x)$, for any choice of the constant c. Graphically, this gives us a family of antiderivative curves, as illustrated in Figure 4.1. Note that each curve is a vertical translation of every other curve in the family. ■

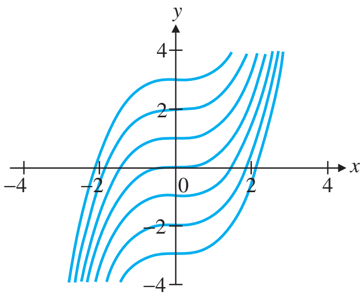


FIGURE 4.1
A family of antiderivative curves

In general, observe that if F is any antiderivative of f and c is any constant, then

$$\frac{d}{dx}[F(x) + c] = F'(x) + 0 = f(x).$$

Thus, $F(x) + c$ is also an antiderivative of $f(x)$, for any constant c. On the other hand, are there any other antiderivatives of $f(x)$ besides $F(x) + c$? The answer, as provided in Theorem 1.1, is no.

THEOREM 1.1

Suppose that F and G are both antiderivatives of f on an interval I. Then,

$$G(x) = F(x) + c,$$

for some constant c.

PROOF

Since F and G are both antiderivatives for f, we have that $G'(x) = F'(x)$. It now follows, from Corollary 9.1 in section 2.9, that $G(x) = F(x) + c$, for some constant c, as desired. ■

NOTES

Theorem 1.1 says that given *any* antiderivative F of f, *every* possible antiderivative of f can be written in the form $F(x) + c$, for some constant, c. We give this most general antiderivative a name in Definition 1.1.

DEFINITION 1.1

Let F be any antiderivative of f. The **indefinite integral** of $f(x)$ (with respect to x), is defined by

$$\int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant (the **constant of integration**).

The process of computing an integral is called **integration.** Here, $f(x)$ is called the **integrand** and the term dx identifies x as the **variable of integration.**

EXAMPLE 1.2 An Indefinite Integral

Evaluate $\int 3x^2 dx$.

Solution You should recognize $3x^2$ as the derivative of x^3 and so,

$$\int 3x^2 dx = x^3 + c.$$

EXAMPLE 1.3 Determining the Coefficient in an Indefinite Integral

Evaluate $\int t^5 dt$.

Solution We know that $\frac{d}{dt}t^6 = 6t^5$ and so, $\frac{d}{dt}\left(\frac{1}{6}t^6\right) = t^5$. Therefore,

$$\int t^5 dt = \frac{1}{6}t^6 + c.$$

We should point out that every differentiation rule gives rise to a corresponding integration rule. For instance, recall that for every rational power, r, $\frac{d}{dx}x^r = rx^{r-1}$. Likewise, we have

$$\frac{d}{dx}x^{r+1} = (r+1)x^r.$$

This proves the following result.

REMARK 1.1

Theorem 1.2 says that to integrate a power of x (other than x^{-1}), you simply raise the power by 1 and divide by the new power. Notice that this rule obviously doesn't work for $r = -1$, since this would produce a division by 0. Later in this section, we develop a rule to cover this case.

THEOREM 1.2 (Power Rule)

For any rational power $r \neq -1$,

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c. \qquad (1.2)$$

EXAMPLE 1.4 Using the Power Rule

Evaluate $\int x^{17} dx$.

Solution From the power rule, we have

$$\int x^{17} dx = \frac{x^{17+1}}{17+1} + c = \frac{x^{18}}{18} + c.$$

EXAMPLE 1.5 The Power Rule with a Negative Exponent

Evaluate $\int \frac{1}{x^3}\,dx$.

Solution We can use the power rule if we first rewrite the integrand. We have

$$\int \frac{1}{x^3}\,dx = \int x^{-3}dx = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{2}x^{-2} + c.$$

EXAMPLE 1.6 The Power Rule with a Fractional Exponent

Evaluate (a) $\int \sqrt{x}\,dx$ and (b) $\int \frac{1}{\sqrt[3]{x}}\,dx$.

Solution (a) As in example 1.5, we first rewrite the integrand and then apply the power rule. We have

$$\int \sqrt{x}\,dx = \int x^{1/2}dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c.$$

Notice that the fraction $\frac{2}{3}$ in the last expression is exactly what it takes to cancel the new exponent $3/2$. (This is what happens if you differentiate.)

(b) Similarly,

$$\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-1/3}dx = \frac{x^{-1/3+1}}{-1/3+1} + c$$
$$= \frac{x^{2/3}}{2/3} + c = \frac{3}{2}x^{2/3} + c.$$

Notice that since $\frac{d}{dx}\sin x = \cos x$, we have

$$\boxed{\int \cos x\,dx = \sin x + c.}$$

Again, by reversing any derivative formula, we get a corresponding integration formula. The following table contains a number of important formulas. The proofs of these are left as straightforward, yet important, exercises. Notice that we do not yet have integration formulas for several familiar functions: $\frac{1}{x}$, $\ln x$, $\tan x$, $\cot x$ and others.

$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$, for $r \neq -1$ (power rule)	$\int \sec x \tan x\,dx = \sec x + c$
$\int \sin x\,dx = -\cos x + c$	$\int \csc x \cot x\,dx = -\csc x + c$
$\int \cos x\,dx = \sin x + c$	$\int e^x\,dx = e^x + c$
$\int \sec^2 x\,dx = \tan x + c$	$\int e^{-x}\,dx = -e^{-x} + c$
$\int \csc^2 x\,dx = -\cot x + c$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

At this point, we are simply reversing the most basic derivative rules we know. We will develop more sophisticated techniques later. For now, we need a general rule to allow us to combine our basic integration formulas.

THEOREM 1.3

Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants, a and b,

$$\int [af(x) + bg(x)]\, dx = a\int f(x)\, dx + b\int g(x)\, dx. \tag{1.3}$$

PROOF

We have that $\frac{d}{dx}\int f(x)\, dx = f(x)$ and $\frac{d}{dx}\int g(x)\, dx = g(x)$. It then follows that

$$\frac{d}{dx}\left[a\int f(x)\, dx + b\int g(x)\, dx\right] = af(x) + bg(x),$$

as desired. ■

Note that Theorem 1.3 says that we can easily compute integrals of sums, differences and constant multiples of functions. However, it turns out that the integral of a product (or a quotient) is not generally the product (or quotient) of the integrals.

EXAMPLE 1.7 An Indefinite Integral of a Sum

Evaluate $\int(3\cos x + 4x^8)\, dx$.

Solution

$$\begin{aligned}\int(3\cos x + 4x^8)\, dx &= 3\int \cos x\, dx + 4\int x^8\, dx \quad \text{From (1.3).}\\ &= 3\sin x + 4\,\frac{x^2}{9} + c\\ &= 3\sin x + \frac{4}{9}x^9 + c.\end{aligned}$$

EXAMPLE 1.8 An Indefinite Integral of a Difference

Evaluate $\int\left(3e^x - \frac{2}{1+x^2}\right) dx$.

Solution

$$\int\left(3e^x - \frac{2}{1+x^2}\right) dx = 3\int e^x\, dx - 2\int \frac{1}{1+x^2}\, dx = 3e^x - 2\tan^{-1} x + c.$$

From the power rule, we know how to evaluate $\int x^r\, dx$ for any rational exponent *except* $r = -1$. We can deal with this exceptional case if we make the following observation. First, recall from our discussion in section 2.7 that for $x > 0$,

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Now, note that $\ln|x|$ is defined for $x \neq 0$. For $x > 0$, we have $\ln|x| = \ln x$ and hence,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}.$$

Similarly, for $x < 0$, $\ln|x| = \ln(-x)$, and hence,

$$\begin{aligned}\frac{d}{dx}\ln|x| &= \frac{d}{dx}\ln(-x)\\ &= \frac{1}{-x}\frac{d}{dx}(-x) \qquad \text{By the chain rule.}\\ &= \frac{1}{-x}(-1) = \frac{1}{x}.\end{aligned}$$

Notice that we got the same derivative in either case. This proves the following result.

THEOREM 1.4

For $x \neq 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$.

EXAMPLE 1.9 The Derivative of the Log of an Absolute Value

For any x for which $\tan x \neq 0$, evaluate $\dfrac{d}{dx}\ln|\tan x|$.

Solution From Theorem 1.4 and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\ln|\tan x| &= \frac{1}{\tan x}\frac{d}{dx}\tan x\\ &= \frac{1}{\tan x}\sec^2 x = \frac{1}{\sin x\cos x}.\end{aligned}$$

Of course, with the new differentiation rule in Theorem 1.4, we get a new integration rule.

COROLLARY 1.1

For $x \neq 0$,

$$\int \frac{1}{x}\,dx = \ln|x| + c.$$

More generally, notice that if $f(x) \neq 0$ and f is differentiable, we have by the chain rule that

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)}.$$

This proves the following integration rule.

COROLLARY 1.2

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c, \qquad (1.4)$$

provided $f(x) \neq 0$.

EXAMPLE 1.10 The Indefinite Integral of a Fraction of the Form $\dfrac{f'(x)}{f(x)}$

Evaluate $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$.

Solution Notice that the numerator ($\sec^2 x$) is the derivative of the denominator ($\tan x$). From (1.4) we then have

$$\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + c.$$

Before concluding the section by examining another falling object, we should emphasize that we have developed only a small number of integration rules. Further, unlike with derivatives, we will never have rules to cover all of the functions with which we are familiar. Thus, it is important to recognize when you *cannot* find an antiderivative.

EXAMPLE 1.11 Identifying Integrals That We Cannot Yet Evaluate

Which of the following integrals can you evaluate given the rules developed in this section? (a) $\displaystyle\int \frac{1}{\sqrt[3]{x^2}}\,dx$, (b) $\displaystyle\int \sec x\,dx$, (c) $\displaystyle\int \frac{2x}{x^2+1}\,dx$, (d) $\displaystyle\int \frac{x^3+1}{x}\,dx$, (e) $\displaystyle\int (x+1)(x-1)\,dx$ and (f) $\displaystyle\int x\sin 2x\,dx$.

Solution First, notice that we can rewrite problems (a), (c), (d) and (e) into forms where we can recognize an antiderivative, as follows. For (a),

$$\int \frac{1}{\sqrt[3]{x^2}}\,dx = \int x^{-2/3}\,dx = \frac{x^{-2/3+1}}{-\frac{2}{3}+1} + c = 3x^{1/3} + c.$$

In part (c), notice that $\dfrac{d}{dx}(x^2+1) = 2x$ (the numerator). From (1.4), we then have

$$\int \frac{2x}{x^2+1}\,dx = \ln|x^2+1| + c = \ln(x^2+1) + c,$$

where we can remove the absolute value signs since $x^2+1>0$ for all x.

In part (d), if we divide out the integrand, we find

$$\int \frac{x^3+1}{x}\,dx = \int (x^2 + x^{-1})\,dx = \frac{1}{3}x^3 + \ln|x| + c.$$

Finally, in part (e), if we multiply out the integrand, we get

$$\int (x+1)(x-1)\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + c.$$

Parts (b) and (f) require us to find functions whose derivatives equal $\sec x$ and $x\sin 2x$. As yet, we cannot evaluate these integrals.

Now that we know how to find antiderivatives for a number of functions, we return to the problem of the falling object that opened the section.

EXAMPLE 1.12 Finding the Position of a Falling Object Given Its Acceleration

If an object's downward acceleration is given by $y''(t) = -32$ ft/s^2, find the position function $y(t)$. Assume that the initial velocity is $y'(0) = -100$ ft/s and the initial position is $y(0) = 100{,}000$ feet.

Solution We have to undo two derivatives, so we compute two antiderivatives. First, we have

$$y'(t) = \int y''(t)\,dt = \int (-32)\,dt = -32t + c.$$

Recall that $y'(t)$ is the velocity of the object, given in units of feet per second. We can evaluate the constant c using the given initial velocity. Since

$$v(t) = y'(t) = -32t + c$$

and $v(0) = y'(0) = -100$, we must have

$$-100 = v(0) = -32(0) + c = c,$$

so that $c = -100$. Thus, the velocity is $y'(t) = -32t - 100$. Next, we have

$$y(t) = \int y'(t)\,dt = \int (-32t - 100)\,dt = -16t^2 - 100t + c.$$

Recall that $y(t)$ gives the height of the object, measured in feet. Using the initial position, we have

$$100{,}000 = y(0) = -16(0) - 100(0) + c = c.$$

Thus, $c = 100{,}000$ and $\quad y(t) = -16t^2 - 100t + 100{,}000.$

Keep in mind that this models the object's height assuming that the only force acting on the object is gravity (i.e., there is no air drag or lift). ■

EXERCISES 4.1

WRITING EXERCISES

1. In the text, we emphasized that the indefinite integral represents *all* antiderivatives of a given function. To understand why this is important, consider a situation where you know the net force, $F(t)$, acting on an object. By Newton's second law, $F = ma$. For the position function $s(t)$, this translates to $a(t) = s''(t) = F(t)/m$. To compute $s(t)$, you need to compute an antiderivative of the force function $F(t)/m$ followed by an antiderivative of the first antiderivative. However, suppose you were unable to find *all* antiderivatives. How would you know whether you had computed the antiderivative that corresponds to the position function? In physical terms, explain why it is reasonable to expect that there is only one antiderivative corresponding to a given set of initial conditions.

2. In the text, we presented a one-dimensional model of the motion of a falling object. We ignored some of the forces on the object so that the resulting mathematical equation would be one that we could solve. You may wonder what the benefit of doing this is. Weigh the relative worth of having an unsolvable but realistic model versus having a solution of a model that is only partially accurate. Keep in mind that when you toss trash into a wastebasket you do not take the curvature of the earth into account.

3. Verify that $\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c$ and $\int xe^x dx = xe^x - e^x + c$ by computing derivatives of the proposed antiderivatives. Which derivative rules did you use? Why does this make it unlikely that we will find a general product (antiderivative) rule for $\int f(x)g(x)\,dx$?

4. We stated in the text that we do not yet have a formula for the antiderivative of several elementary functions, including $\ln x$, $\sec x$ and $\csc x$. Given a function $f(x)$, explain what determines whether or not we have a simple formula for $\int f(x)\,dx$. For example, why is there a simple formula for $\int \sec x \tan x\,dx$ but not $\int \sec x\,dx$?

In exercises 1–4, sketch several members of the family of functions defined by the antiderivative.

1. $\int x^3\,dx$

2. $\int (x^3 - x)\,dx$

3. $\int e^x\,dx$

4. $\int \cos x\,dx$

In exercises 5–30, find the general antiderivative.

5. $\int (3x^4 - 3x)\,dx$

6. $\int (x^3 - 2)\,dx$

7. $\int \left(3\sqrt{x} - \frac{1}{x^4}\right)dx$

8. $\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right)dx$

9. $\int \frac{x^{1/3} - 3}{x^{2/3}}\,dx$

10. $\int \frac{x + 2x^{3/4}}{x^{5/4}}\,dx$

11. $\int (2\sin x + \cos x)\,dx$

12. $\int (3\cos x - \sin x)\,dx$

13. $\int 2\sec x \tan x\,dx$

14. $\int \frac{4}{\sqrt{1 - x^2}}\,dx$

15. $\int 5\sec^2 x\,dx$

16. $\int 4\frac{\cos x}{\sin^2 x}\,dx$

17. $\int (3e^x - 2)\,dx$

18. $\int (4x - 2e^x)\,dx$

19. $\int (3\cos x - 1/x)\,dx$

20. $\int (2x^{-1} + \sin x)\,dx$

21. $\int \frac{4x}{x^2 + 4}\,dx$

22. $\int \frac{3}{4x^2 + 4}\,dx$

23. $\int \left(5x - \frac{3}{e^x}\right)dx$

24. $\int \left(2\cos x - \sqrt{e^{2x}}\right)dx$

25. $\int \frac{e^x}{e^x + 3}\,dx$

26. $\int \frac{\cos x}{\sin x}\,dx$

27. $\int \frac{e^x + 3}{e^x}\,dx$

28. $\int \frac{(e^x)^2 - 2}{e^x}\,dx$

29. $\int x^{1/4}(x^{5/4} - 4)\,dx$

30. $\int x^{2/3}(x^{-4/3} - 3)\,dx$

In exercises 31–34, one of the two antiderivatives can be determined using basic algebra and the derivative formulas we have presented. Find the antiderivative of this one and label the other "N/A."

31. (a) $\int \sqrt{x^3 + 4}\,dx$ (b) $\int \left(\sqrt{x^3} + 4\right)dx$

32. (a) $\int \frac{3x^2 - 4}{x^2}\,dx$ (b) $\int \frac{x^2}{3x^2 - 4}\,dx$

33. (a) $\int 2\sec x\,dx$ (b) $\int \sec^2 x\,dx$

34. (a) $\int \left(\frac{1}{x^2} - 1\right)dx$ (b) $\int \frac{1}{x^2 - 1}\,dx$

35. In example 1.11, use your CAS to evaluate the antiderivatives in parts (b) and (f). Verify that these are correct by computing the derivatives.

36. For each of the problems in exercises 31–34 that you labeled N/A, try to find an antiderivative on your CAS. Where possible, verify that the antiderivative is correct by computing the derivatives.

37. Use a CAS to find an antiderivative, then verify the answer by computing a derivative.

(a) $\int x^2 e^{-x^3}\,dx$ (b) $\int \frac{1}{x^2 - x}\,dx$ (c) $\int \csc x\,dx$

38. Use a CAS to find an antiderivative, then verify the answer by computing a derivative.

(a) $\int \frac{x}{x^4 + 1}\,dx$ (b) $\int 3x\sin 2x\,dx$ (c) $\int \ln x\,dx$

In exercises 39–42, find the function $f(x)$ satisfying the given conditions.

39. $f'(x) = 3e^x + x,\ f(0) = 4$

40. $f'(x) = 4\cos x,\ f(0) = 3$

41. $f''(x) = 12,\ f'(0) = 2,\ f(0) = 3$

42. $f''(x) = 2x,\ f'(0) = -3,\ f(0) = 2$

In exercises 43–46, find all functions satisfying the given conditions.

43. $f''(x) = 3\sin x + 4x^2$

44. $f''(x) = \sqrt{x} - 2\cos x$

45. $f'''(x) = 4 - 2/x^3$

46. $f'''(x) = \sin x - e^x$

47. Determine the position function if the velocity function is $v(t) = 3 - 12t$ and the initial position is $s(0) = 3$.

48. Determine the position function if the velocity function is $v(t) = 3e^{-t} - 2$ and the initial position is $s(0) = 0$.

49. Determine the position function if the acceleration function is $a(t) = 3\sin t + 1$, the initial velocity is $v(0) = 0$ and the initial position is $s(0) = 4$.

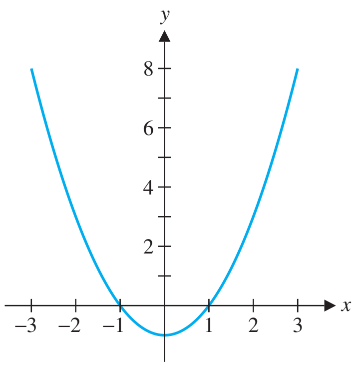
50. Determine the position function if the acceleration function is $a(t) = t^2 + 1$, the initial velocity is $v(0) = 4$ and the initial position is $s(0) = 0$.

51. Suppose that a car can accelerate from 30 mph to 50 mph in 4 seconds. Assuming a constant acceleration, find the acceleration (in miles per second squared) of the car and find the distance traveled by the car during the 4 seconds.

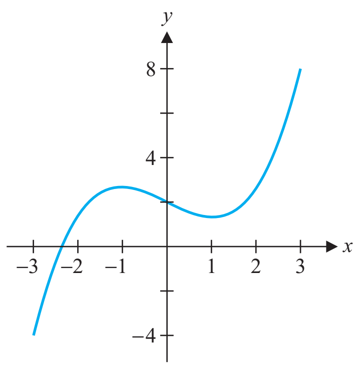
52. Suppose that a car can come to rest from 60 mph in 3 seconds. Assuming a constant (negative) acceleration, find the acceleration (in miles per second squared) of the car and find the distance traveled by the car during the 3 seconds (i.e., the stopping distance).

In exercises 53 and 54, sketch the graph of a function $f(x)$ corresponding to the given graph of $y = f'(x)$.

53.



54.



55. Sketch the graphs of three functions, each of which has the derivative sketched in exercise 53.

56. Repeat exercise 53 if the given graph is of $f''(x)$.

57. The following table shows the velocity of a falling object at different times. For each time interval, estimate the distance fallen and the acceleration.

t (s)	0	0.5	1.0	1.5	2.0
$v(t)$ (ft/s)	−4.0	−19.8	−31.9	−37.7	−39.5

58. The following table shows the velocity of a falling object at different times. For each time interval, estimate the distance fallen and the acceleration.

t (s)	0	1.0	2.0	3.0	4.0
$v(t)$ (m/s)	0.0	−9.8	−18.6	−24.9	−28.5

59. The following table shows the acceleration of a car moving in a straight line. If the car is traveling 70 ft/s at time $t = 0$, estimate the speed and distance traveled at each time.

t (s)	0	0.5	1.0	1.5	2.0
$a(t)$ (ft/s^2)	−4.2	2.4	0.6	−0.4	1.6

60. The following table shows the acceleration of a car moving in a straight line. If the car is traveling 20 m/s at time $t = 0$, estimate the speed and distance traveled at each time.

t (s)	0	0.5	1.0	1.5	2.0
$a(t)$ (m/s^2)	0.6	−2.2	−4.5	−1.2	−0.3

61. Find a function $f(x)$ such that the point $(1, 2)$ is on the graph of $y = f(x)$, the slope of the tangent line at $(1, 2)$ is 3 and $f''(x) = x - 1$.

62. Find a function $f(x)$ such that the point $(-1, 1)$ is on the graph of $y = f(x)$, the slope of the tangent line at $(-1, 1)$ is 2 and $f''(x) = 6x + 4$.

In exercises 63–68, find an antiderivative by reversing the chain rule, product rule or quotient rule.

63. $\int 2x \cos x^2 dx$

64. $\int x^2\sqrt{x^3+2}\,dx$

65. $\int (x \sin 2x + x^2 \cos 2x)\,dx$

66. $\int \frac{2xe^{3x} - 3x^2e^{3x}}{e^{6x}}\,dx$

67. $\int \frac{x \cos x^2}{\sqrt{\sin x^2}}\,dx$

68. $\int \left(\sqrt{x^2+1}\cos x + \frac{x}{\sqrt{x^2+1}}\sin x\right) dx$

69. Show that $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c$ and $\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\sin^{-1} x + c$. Explain why this does not imply that $\cos^{-1} x = -\sin^{-1} x$. Find an equation relating $\cos^{-1} x$ and $\sin^{-1} x$.

70. Derive the formulas $\int \sec^2 x\,dx = \tan x + c$ and $\int \sec x \tan x\,dx = \sec x + c$.

71. Derive the formulas $\int e^x\,dx = e^x + c$ and $\int e^{-x} dx = -e^{-x} + c$.

72. For the antiderivative $\int \frac{1}{kx}\,dx$, (a) factor out the k and then use a basic formula and (b) rewrite the problem as $\frac{1}{k}\int \frac{k}{kx}\,dx$ and use formula (1.4). Discuss the difference between the antiderivatives (a) and (b) and explain why they are both correct.

EXPLORATORY EXERCISES

1. Compute the derivatives of $e^{\sin x}$ and e^{x^2}. Given these derivatives, evaluate the indefinite integrals $\int \cos x\, e^{\sin x}\,dx$

and $\int 2x\, e^{x^2} dx$. Next, evaluate $\int x\, e^{x^2} dx$. (Hint: $\int x\, e^{x^2} dx = \frac{1}{2} \int 2x\, e^{x^2} dx$.) Similarly, evaluate $\int x^2 e^{x^3} dx$. In general, evaluate

$$\int f'(x)\, e^{f(x)}\, dx.$$

Next, evaluate $\int e^x \cos(e^x)\, dx$, $\int 2x \cos(x^2)\, dx$ and the more general

$$\int f'(x) \cos(f(x))\, dx.$$

As we have stated, there is no general rule for the antiderivative of a product, $\int f(x)g(x)\, dx$. Instead, there are many special cases that you evaluate case by case.

2. A **differential equation** is an equation involving an unknown function and one or more of its derivatives, for instance, $v'(t) = 2t + 3$. To solve this differential equation, you simply find the antiderivative $v(t) = \int (2t + 3)\, dt = t^2 + 3t + c$. Notice that solutions of a differential equation are functions. In general, differential equations can be challenging to solve. For example, we introduced the differential equation $mv'(t) = -mg + kv^2(t)$ for the vertical motion of an object subject to gravity and air drag. Taking specific values of m and k gives the equation $v'(t) = -32 + 0.0003v^2(t)$. To solve this, we would need to find a function whose derivative equals -32 plus 0.0003 times the square of the function. It is difficult to find a function whose derivative is written in terms of $[v(t)]^2$ when $v(t)$ is precisely what is unknown. We can nonetheless construct a graphical representation of the solution using what is called a **direction field.** Suppose we want to construct a solution passing through the point $(0, -100)$, corresponding to an initial velocity of $v(0) = -100$ ft/s. At $t = 0$, with $v = -100$, we know that the slope of the solution is $v' = -32 + 0.0003(-100)^2 = -29$. Starting at $(0, -100)$, sketch in a short line segment with slope -29. Such a line segment would connect to the point $(1, -129)$ if you extended it that far (but make yours much shorter). At $t = 1$ and $v = -129$, the slope of the solution is $v' = -32 + 0.0003(-129)^2 \approx -27$. Sketch in a short line segment with slope -27 starting at the point $(1, -129)$. This line segment points to $(2, -156)$. At this point, $v' = -32 + 0.0003(-156)^2 \approx -24.7$. Sketch in a short line segment with slope -24.7 at $(2, -156)$. Do you see a graphical solution starting to emerge? Is the solution increasing or decreasing? Concave up or concave down? If your CAS has a direction field capability, sketch the direction field and try to visualize the solutions starting at point $(0, -100)$, $(0, 0)$ and $(0, -300)$.

4.2 SUMS AND SIGMA NOTATION

In section 4.1, we discussed how to calculate backward from the velocity function for an object to arrive at the position function for the object. We next investigate the same process graphically. In this section, we develop an important skill necessary for this new interpretation.

Driving at a constant 60 mph, in 2 hours, you travel 120 miles; in 4 hours, you travel 240 miles. There's no surprise here, but notice that you can see this graphically by looking at several graphs of the (constant) velocity function $v(t) = 60$. In Figure 4.2a, the area under the graph from $t = 0$ to $t = 2$ (shaded) equals 120, the distance traveled in this time interval. In Figure 4.2b, the shaded region from $t = 0$ to $t = 4$ has area equal to the distance of 240 miles.

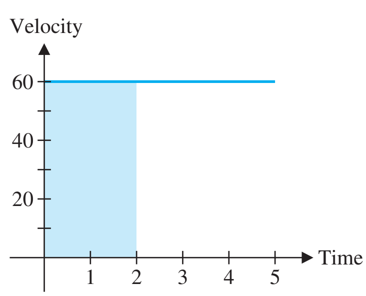


FIGURE 4.2a
$y = v(t)$ on $[0, 2]$

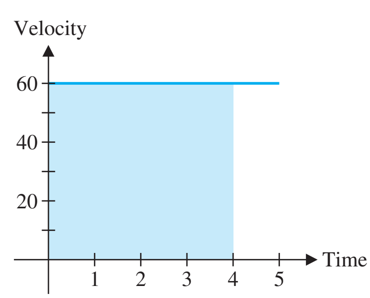


FIGURE 4.2b
$y = v(t)$ on $[0, 4]$

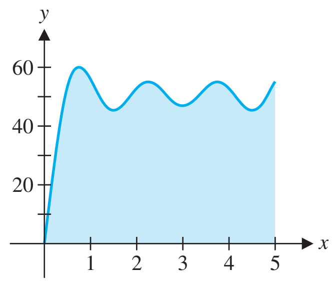


FIGURE 4.3
Nonconstant velocity

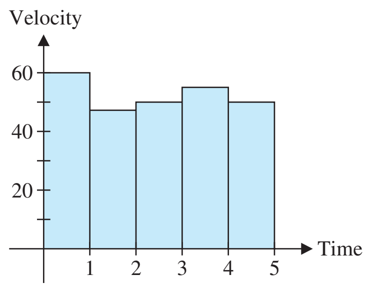


FIGURE 4.4
Approximate area

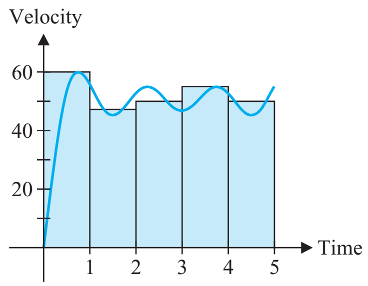


FIGURE 4.5
Approximate area

So, it appears that the distance traveled over a particular time interval equals the area of the region bounded by $y = v(t)$ and the t-axis on that interval. For the case of constant velocity, this is no surprise, as we have that

$$d = r \times t = \text{velocity} \times \text{time}.$$

We would also like to compute the area under the curve (equal to the distance traveled) for a nonconstant velocity function, such as the one shown in Figure 4.3 for the time interval [0, 5]. Our work in this section provides the first step toward a powerful technique for computing such areas. To indicate the direction we will take, suppose that the velocity curve in Figure 4.3 is replaced by the approximation in Figure 4.4, where the velocity is assumed to be constant over each of five 1-hour time intervals.

The area on the interval from $t = 0$ to $t = 5$ is then approximately the sum of the areas of the five rectangles:

$$A \approx 60 + 45 + 50 + 55 + 50 = 260 \text{ miles}.$$

Of course, this is a fairly crude estimate of the area in Figure 4.3 (see Figure 4.5 to see how good this approximation is), but you should observe that we could get a better estimate by approximating the area using more (and smaller) rectangles. Certainly, we had no problem adding up the areas of five rectangles, but for 5000 rectangles, you will want some means for simplifying and automating the process. Dealing with such sums is the topic of this section.

We begin by introducing some notation. Suppose that you want to sum the squares of the first 20 positive integers. Notice that

$$1 + 4 + 9 + \cdots + 400 = 1^2 + 2^2 + 3^2 + \cdots + 20^2.$$

The pattern is obvious; each term in the sum has the form i^2, for $i = 1, 2, 3, \ldots, 20$. To reduce the amount of writing, we use the Greek capital letter sigma, $\sum$, as a symbol for *sum* and write the sum in **summation notation** as

$$\sum_{i=1}^{20} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 20^2,$$

to indicate that we add together terms of the form i^2, starting with $i = 1$ and ending with $i = 20$. The variable i is called the **index of summation.**

In general, for any real numbers $a_1, a_2, \ldots, a_n$, we have

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

EXAMPLE 2.1 Using Summation Notation

Write in summation notation: $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{10}$ and $3^3 + 4^3 + 5^3 + \cdots + 45^3$.

Solution We have the sum of the square roots of the integers from 1 to 10:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{10} = \sum_{i=1}^{10} \sqrt{i}$$

and the sum of the cubes of the integers from 3 to 45:

$$3^3 + 4^3 + 5^3 + \cdots + 45^3 = \sum_{i=3}^{45} i^3.$$

REMARK 2.1

The index of summation is a **dummy variable,** since it is used only as a counter to keep track of terms. The value of the summation does not depend on the letter used as the index. For this reason, you may use any letter you like as an index. By tradition, we most frequently use i, j, k, m and n, but any index will do. For instance,

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

EXAMPLE 2.2 Summation Notation for a Sum Involving Odd Integers

Write in summation notation: the sum of the first 200 odd positive integers.

Solution First, notice that $(2i)$ is even for every integer i and hence, both $(2i - 1)$ and $(2i + 1)$ are odd. So, we have

$$1 + 3 + 5 + \cdots + 399 = \sum_{i=1}^{200} (2i - 1).$$

Alternatively, we can write this as the equivalent expression $\sum_{i=0}^{199} (2i + 1)$. (Write out the terms to see why these are equivalent.) ■

EXAMPLE 2.3 Computing Sums Given in Summation Notation

Write out all terms and compute the sums $\sum_{i=1}^{8} (2i + 1)$, $\sum_{i=2}^{6} \sin(2\pi i)$ and $\sum_{i=4}^{10} 5$.

Solution We have

$$\sum_{i=1}^{8} (2i + 1) = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 80$$

and

$$\sum_{i=2}^{6} \sin(2\pi i) = \sin 4\pi + \sin 6\pi + \sin 8\pi + \sin 10\pi + \sin 12\pi = 0.$$

(Note that the sum started at $i = 2$.) Finally,

$$\sum_{i=4}^{10} 5 = 5 + 5 + 5 + 5 + 5 + 5 + 5 = 35.$$ ■

As example 2.3 suggests, there are sometimes shortcuts for computing sums. For instance, the easy way of evaluating the third sum above is to notice that 5 appears 7 times, and 7 times 5 is 35. We now state this and two other useful formulas.

THEOREM 2.1

If n is any positive integer and c is any constant, then

(i) $\sum_{i=1}^{n} c = cn$ **(sum of constants),**

(ii) $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$ **(sum of the first n positive integers)** and

(iii) $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$ **(sum of the squares of the first n positive integers).**

HISTORICAL NOTES

Karl Friedrich Gauss (1777–1855)
A German mathematician widely considered to be the greatest mathematician of all time. A prodigy who had proved important theorems by age 14, Gauss was the acknowledged master of almost all areas of mathematics. He proved the Fundamental Theorem of Algebra and numerous results in number theory and mathematical physics. Gauss was instrumental in starting new fields of research including the analysis of complex variables, statistics, vector calculus and non-Euclidean geometry. Gauss was truly the "Prince of Mathematicians."

PROOF

(i) $\sum_{i=1}^{n} c$ indicates to add the same constant c to itself n times and hence, the sum is simply c times n.

(ii) The following clever proof has been credited to then 10-year-old Karl Friedrich Gauss. (For more on Gauss, see the historical note in the margin.) First notice that

$$\sum_{i=1}^{n} i = \underbrace{1 + 2 + 3 + \cdots + (n-2) + (n-1) + n}_{n \text{ terms}}. \tag{2.1}$$

Since the order in which we add the terms does not matter, we add the terms in (2.1) in reverse order, to get

$$\sum_{i=1}^{n} i = \underbrace{n + (n-1) + (n-2) + \cdots + 3 + 2 + 1}_{\text{same } n \text{ terms (backward)}}. \tag{2.2}$$

Adding equations (2.1) and (2.2) term by term, we get

$$\begin{aligned} 2\sum_{i=1}^{n} i &= (1+n) + (2+n-1) + (3+n-2) + \cdots + (n-1+2) + (n+1) \\ &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}_{n \text{ terms}} \\ &= n(n+1), \qquad \text{Adding each term in parentheses.} \end{aligned}$$

since $(n+1)$ appears n times in the sum. Dividing both sides by 2 gives us

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

as desired. The proof of (iii) requires a more sophisticated proof using mathematical induction and we defer it to the end of this section. ■

We also have the following general rule for expanding sums. The proof is straightforward and is left as an exercise.

THEOREM 2.2

For any constants c and d,

$$\sum_{i=1}^{n} (ca_i + db_i) = c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i.$$

Using Theorems 2.1 and 2.2, we can now compute several simple sums with ease. Note that we have no more difficulty summing 800 terms than we do summing 8.

EXAMPLE 2.4 Computing Sums Using Theorems 2.1 and 2.2

Compute $\sum_{i=1}^{8}(2i+1)$ and $\sum_{i=1}^{800}(2i+1)$.

Solution From Theorems 2.1 and 2.2, we have

$$\sum_{i=1}^{8}(2i+1) = 2\sum_{i=1}^{8} i + \sum_{i=1}^{8} 1 = 2\frac{8(9)}{2} + (1)(8) = 72 + 8 = 80.$$

Similarly, $$\sum_{i=1}^{800}(2i+1) = 2\sum_{i=1}^{800} i + \sum_{i=1}^{800} 1 = 2\frac{800(801)}{2} + (1)(800)$$
$$= 640{,}800 + 800 = 641{,}600.$$

EXAMPLE 2.5 Computing Sums Using Theorems 2.1 and 2.2

Compute $\sum_{i=1}^{20} i^2$ and $\sum_{i=1}^{20}\left(\frac{i}{20}\right)^2$.

Solution From Theorems 2.1 and 2.2, we have

$$\sum_{i=1}^{20} i^2 = \frac{20(21)(41)}{6} = 2870$$

and $$\sum_{i=1}^{20}\left(\frac{i}{20}\right)^2 = \frac{1}{20^2}\sum_{i=1}^{20} i^2 = \frac{1}{400}\frac{20(21)(41)}{6} = \frac{1}{400}2870 = 7.175.$$

We return to the study of general sums in Chapter 8. Recall that our initial motivation for studying sums was to calculate distance from velocity. In the beginning of this section, we approximated distance by summing several values of the velocity function. In section 4.3, we will further develop these sums to allow us to compute areas exactly.

EXAMPLE 2.6 Computing a Sum of Function Values

Sum the values of $f(x) = x^2 + 3$ evaluated at $x = 0.1, x = 0.2, \ldots, x = 1.0$.

Solution We first formulate this in summation notation, so that we can use the rules we have developed in this section. The terms to be summed are $a_1 = f(0.1) = 0.1^2 + 3$, $a_2 = f(0.2) = 0.2^2 + 3$ and so on. Note that since each of the x-values is a multiple of 0.1, we can write the x's in the form $0.1i$, for $i = 1, 2, \ldots, 10$. In general, we have

$$a_i = f(0.1i) = (0.1i)^2 + 3, \quad \text{for } i = 1, 2, \ldots, 10.$$

From Theorem 2.1 (i) and (iii), we then have

$$\sum_{i=1}^{10} a_i = \sum_{i=1}^{10} f(0.1i) = \sum_{i=1}^{10}[(0.1i)^2 + 3] = 0.1^2\sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} 3$$
$$= 0.01\frac{10(11)(21)}{6} + (3)(10) = 3.85 + 30 = 33.85.$$

EXAMPLE 2.7 A Sum of Function Values at Equally Spaced x's

Sum the values of $f(x) = 3x^2 - 4x + 2$ evaluated at $x = 1.05$, $x = 1.15$, $x = 1.25, \ldots, x = 2.95$.

Solution You will need to think carefully about the x's. The distance between successive x-values is 0.1, and there are 20 such values. (Be sure to count these for yourself.) Notice that we can write the x's in the form $0.95 + 0.1i$, for $i = 1, 2, \ldots, 20$.

We now have

$$\begin{aligned}\sum_{i=1}^{20} f(0.95+0.1i) &= \sum_{i=1}^{20}[3(0.95+0.1i)^2 - 4(0.95+0.1i)+2]\\ &= \sum_{i=1}^{20}(0.03i^2 + 0.17i + 0.9075) && \text{Multiply out terms.}\\ &= 0.03\sum_{i=1}^{20} i^2 + 0.17\sum_{i=1}^{20} i + \sum_{i=1}^{20} 0.9075 && \text{From Theorem 2.2.}\\ &= 0.03\frac{20(21)(41)}{6} + 0.17\frac{20(21)}{2} + 0.9075(20) && \text{From Theorem 2.1 (i), (ii) and (iii).}\\ &= 139.95.\end{aligned}$$

Over the next several sections, we will see how sums such as those found in examples 2.6 and 2.7 play a very significant role. We end this section by looking at a powerful mathematical principle.

Principle of Mathematical Induction

For any proposition that depends on a positive integer, n, we first show that the result is true for a specific value $n = n_0$. We then *assume* that the result is true for an *unspecified* $n = k \geq n_0$. (This is called the **induction assumption.**) If we can show that it follows that the proposition is true for $n = k + 1$, we have proved that the result is true for any positive integer $n \geq n_0$. Think about why this must be true. (Hint: If P_1 is true and P_k true implies P_{k+1} is true, then P_1 true implies P_2 is true, which in turn implies P_3 is true and so on.)

We can now use mathematical induction to prove the last part of Theorem 2.1, which states that for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

PROOF OF THEOREM 2.1 (iii)

For $n = 1$, we have

$$1 = \sum_{i=1}^{1} i^2 = \frac{1(2)(3)}{6},$$

as desired. So, the proposition is true for $n = 1$. Next, **assume** that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}, \qquad \text{Induction assumption.} \tag{2.3}$$

for some integer $k \geq 1$.

In this case, we have by the induction assumption that for $n = k + 1$,

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + \sum_{i=k+1}^{k+1} i^2 \quad \text{Split off the last term.}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{From (2.3).}$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \quad \text{Add the fractions.}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \quad \text{Factor out } (k+1).$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6} \quad \text{Combine terms.}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{Factor the quadratic.}$$

$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \quad \text{Rewrite the terms.}$$

$$= \frac{n(n+1)(2n+1)}{6}, \quad \text{Since } n = k+1.$$

as desired.

EXERCISES 4.2

WRITING EXERCISES

1. In the text, we mentioned that one of the benefits of using the summation notation is the simplification of calculations. To help understand this, write out in words what is meant by $\sum_{i=1}^{40}(2i^2 - 4i + 11)$.

2. Following up on exercise 1, calculate the sum $\sum_{i=1}^{40}(2i^2 - 4i + 11)$ and then describe in words how you did so. Be sure to describe any formulas and your use of them in words.

In exercises 1–4, a calculation is described in words. Translate each into summation notation and then compute the sum.

1. The sum of the squares of the first 50 positive integers.
2. The square of the sum of the first 50 positive integers.
3. The sum of the square roots of the first 10 positive integers.
4. The square root of the sum of the first 10 positive integers.

In exercises 5–8, write out all terms and compute the sums.

5. $\sum_{i=1}^{6} 3i^2$
6. $\sum_{i=3}^{7} (i^2 + i)$
7. $\sum_{i=6}^{10} (4i + 2)$
8. $\sum_{i=6}^{8} (i^2 + 2)$

In exercises 9–18, use summation rules to compute the sum.

9. $\sum_{i=1}^{70} (3i - 1)$
10. $\sum_{i=1}^{45} (3i - 4)$
11. $\sum_{i=1}^{40} (4 - i^2)$
12. $\sum_{i=1}^{50} (8 - i)$
13. $\sum_{i=1}^{100} (i^2 - 3i + 2)$
14. $\sum_{i=1}^{140} (i^2 + 2i - 4)$
15. $\sum_{i=1}^{200} (4 - 3i - i^2)$
16. $\sum_{i=1}^{250} (i^2 + 8)$
17. $\sum_{i=3}^{n} (i^2 - 3)$
18. $\sum_{i=0}^{n} (i^2 + 5)$

In exercises 19–22, compute the sum and the limit of the sum as $n \to \infty$.

19. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$
20. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$

21. $\sum_{i=1}^{n} \frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$

22. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$

In exercises 23–26, compute sums of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ for the given values.

23. $f(x) = x^2 + 4x$; $x = 0.2, 0.4, 0.6, 0.8, 1.0$; $\Delta x = 0.2$; $n = 5$

24. $f(x) = 3x + 5$; $x = 0.4, 0.8, 1.2, 1.6, 2.0$; $\Delta x = 0.4$; $n = 5$

25. $f(x) = 4x^2 - 2$; $x = 2.1, 2.2, 2.3, 2.4, \ldots, 3.0$; $\Delta x = 0.1$; $n = 10$

26. $f(x) = x^3 + 4$; $x = 2.05, 2.15, 2.25, 2.35, \ldots, 2.95$; $\Delta x = 0.1$; $n = 10$

27. Suppose that a car has velocity 50 mph for 2 hours, velocity 60 mph for 1 hour, velocity 70 mph for 30 minutes and velocity 60 mph for 3 hours. Find the distance traveled.

28. Suppose that a car has velocity 50 mph for 1 hour, velocity 40 mph for 1 hour, velocity 60 mph for 30 minutes and velocity 55 mph for 3 hours. Find the distance traveled.

29. Suppose that a runner has velocity 15 mph for 20 minutes, velocity 18 mph for 30 minutes, velocity 16 mph for 10 minutes and velocity 12 mph for 40 minutes. Find the distance run.

30. Suppose that a runner has velocity 12 mph for 20 minutes, velocity 14 mph for 30 minutes, velocity 18 mph for 10 minutes and velocity 15 mph for 40 minutes. Find the distance run.

31. The table shows the velocity of a projectile at various times. Estimate the distance traveled.

time (s)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0
velocity (ft/s)	120	116	113	110	108	106	104	103	102

32. The table shows the (downward) velocity of a falling object. Estimate the distance fallen.

time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
velocity (m/s)	10	14.9	19.8	24.7	29.6	34.5	39.4	44.3	49.2

33. Use mathematical induction to prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$.

34. Use mathematical induction to prove that $\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$ for all integers $n \geq 1$.

In exercises 35–38, use the formulas in exercises 33 and 34 to compute the sums.

35. $\sum_{i=1}^{10}(i^3 - 3i + 1)$

36. $\sum_{i=1}^{20}(i^3 + 2i)$

37. $\sum_{i=1}^{100}(i^5 - 2i^2)$

38. $\sum_{i=1}^{100}(2i^5 + 2i + 1)$

39. Prove Theorem 2.2.

40. Use induction to derive the geometric series formula $a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1 - r}$ for constants a and $r \neq 1$.

In exercises 41 and 42, use the result of exercise 40 to evaluate the sum and the limit of the sum as $n \to \infty$.

41. $\sum_{i=1}^{n} e^{(6i)/n}\frac{6}{n}$

42. $\sum_{i=1}^{n} e^{(2i)/n}\frac{2}{n}$

EXPLORATORY EXERCISES

1. Suppose that the velocity of a car is given by $v(t) = 3\sqrt{t} + 30$ mph at time t hours $(0 \leq t \leq 4)$. We will try to determine the distance traveled in the 4 hours. To start, we can note that the velocity at $t = 0$ is $v(0) = 3\sqrt{0} + 30 = 30$ mph and the velocity at time $t = 1$ is $v(1) = 3\sqrt{1} + 30 = 33$ mph. Since the average of these velocities is 31.5 mph, we could estimate that the car traveled 31.5 miles in the first hour. Carefully explain why this is not necessarily correct. Even so, it will serve as a first approximation. Since $v(1) = 33$ mph and $v(2) = 3\sqrt{2} + 30 \approx 34$ mph, we can estimate that the car traveled 33.5 mph in the second hour. Using $v(3) \approx 35$ mph and $v(4) = 36$ mph, find similar estimates for the distance traveled in the third and fourth hours and then estimate the total distance. To improve this estimate, we can find an estimate for the distance covered each half hour. The first estimate would take $v(0) = 30$ mph and $v(0.5) \approx 32.1$ mph and estimate an average velocity of 31.05 mph and a distance of 15.525 miles. Estimate the average velocity and then the distance for the remaining 7 half hours and estimate the total distance. We can improve this estimate, too. By estimating the average velocity every quarter hour, find a third estimate of the total distance. Based on these three estimates, conjecture the limit of these approximations as the time interval considered goes to zero.

2. In this exercise, we investigate a generalization of a finite sum called an **infinite series.** Suppose a bouncing ball has **coefficient of restitution** equal to 0.6. This means that if the ball hits the ground with velocity v ft/s, it rebounds with velocity $0.6v$. Ignoring air resistance, a ball launched with velocity v ft/s will stay in the air $v/16$ seconds before hitting the ground. Suppose a ball with coefficient of restitution 0.6 is launched with initial velocity 60 ft/s. Explain why the total time in the air is given by $60/16 + (0.6)(60)/16 + (0.6)(0.6)(60)/16 + \cdots$. It might seem like the ball would continue to bounce forever. To see otherwise, use the result of exercise 40 to find the limit that these sums approach. The limit is the number of seconds that the ball continues to bounce.

3. The following statement is obviously false: Given any set of n numbers, the numbers are all equal. Find the flaw in the attempted use of mathematical induction. Let $n = 1$. One number is equal to itself. Assume that for $n = k$, any k numbers are equal. Let S be any set of $k+1$ numbers $a_1, a_2, \ldots, a_{k+1}$. By the induction hypothesis, the first k numbers are equal: $a_1 = a_2 = \cdots = a_k$ and the last k numbers are equal: $a_2 = a_3 = \cdots = a_{k+1}$. Combining these results, all $k+1$ numbers are equal: $a_1 = a_2 = \cdots = a_k = a_{k+1}$, as desired.

4.3 AREA

Several times now, we have considered how to compute the distance traveled from a given velocity function. We examined this in terms of antiderivatives in section 4.1 and reworked this as an area problem in section 4.2. In this section, we will develop the general problem of calculation of areas in some detail.

You are familiar with the formulas for computing the area of a rectangle, a circle and a triangle. From endless use of these formulas over the years, you should have a clear idea of what area is: one measure of the size of a two-dimensional region. However, how would you compute the area of a region that's not a rectangle, circle or triangle?

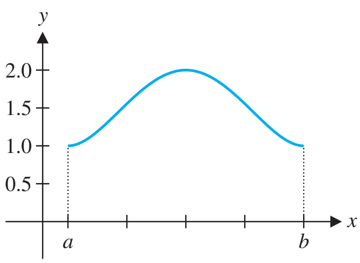


FIGURE 4.6
Area under $y = f(x)$

We need a more general description of area, one that can be used to find the area of almost any two-dimensional region imaginable. In this section, we develop a general process for computing area. It turns out that this process (which we generalize to the notion of the *definite integral* in section 4.4) has significance far beyond the calculation of area. In fact, this powerful and flexible tool is one of the central ideas of calculus, with applications in a wide variety of fields.

The general problem is to estimate the area below the graph of $y = f(x)$ and above the x-axis for $a \le x \le b$. For now, we assume that $f(x) \ge 0$ and f is continuous on the interval $[a, b]$, as in Figure 4.6.

We start by dividing the interval $[a, b]$ into n equal pieces. This is called a **regular partition** of $[a, b]$. The width of each subinterval in the partition is then $\dfrac{b-a}{n}$, which we denote by Δx (meaning a small change in x). The points in the partition are denoted by $x_0 = a$, $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$ and so on. In general,

$$x_i = x_0 + i\,\Delta x, \quad \text{for } i = 1, 2, \ldots, n.$$

See Figure 4.7 for an illustration of a regular partition for the case where $n = 6$. On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, \ldots, n$), construct a rectangle of height $f(x_i)$ (the value

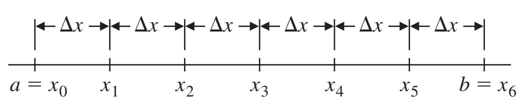


FIGURE 4.7
Regular partition of $[a, b]$

of the function at the right endpoint of the subinterval), as illustrated in Figure 4.8 for the case where $n = 4$. It should be clear from Figure 4.8 that the area under the curve A is roughly the same as the sum of the areas of the four rectangles,

$$A \approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x = A_4.$$

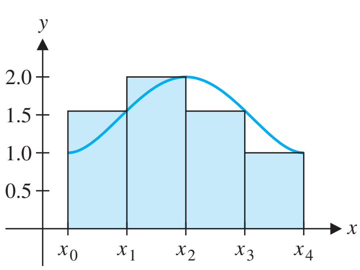


FIGURE 4.8
$A \approx A_4$

In particular, notice that although two of these rectangles enclose more area than that under the curve and two enclose less area, on the whole the sum of the areas of the four rectangles

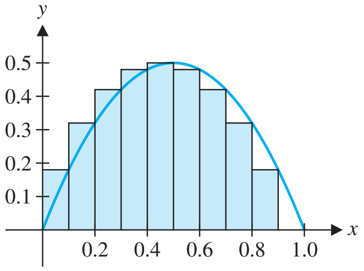


FIGURE 4.9
$A \approx A_{10}$

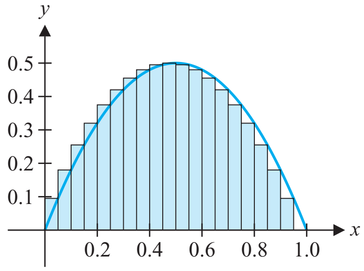


FIGURE 4.10
$A \approx A_{20}$

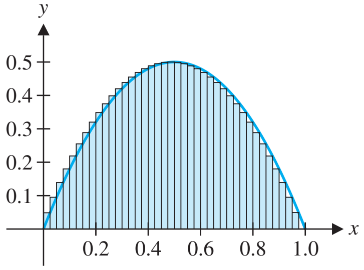


FIGURE 4.11
$A \approx A_{40}$

n	A_n
10	0.33
20	0.3325
30	0.332963
40	0.333125
50	0.3332
60	0.333241
70	0.333265
80	0.333281
90	0.333292
100	0.3333

provides an approximation to the total area under the curve. More generally, if we construct n rectangles of equal width on the interval $[a, b]$, we have

$$\begin{aligned} A &\approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\,\Delta x = A_n. \end{aligned} \tag{3.1}$$

EXAMPLE 3.1 Approximating an Area with Rectangles

Approximate the area under the curve $y = f(x) = 2x - 2x^2$ on the interval [0, 1], using 10 rectangles.

Solution The partition divides the interval into 10 subintervals, each of length $\Delta x = 0.1$, namely [0, 0.1], [0.1, 0.2], . . . , [0.9, 1.0]. In Figure 4.9, we have drawn in rectangles of height $f(x_i)$ on each subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, 10$. Notice that the sum of the areas of the 10 rectangles indicated provides an approximation to the area under the curve. That is,

$$\begin{aligned} A \approx A_{10} &= \sum_{i=1}^{10} f(x_i)\,\Delta x \\ &= [f(0.1) + f(0.2) + \cdots + f(1.0)](0.1) \\ &= (0.18 + 0.32 + 0.42 + 0.48 + 0.5 + 0.48 + 0.42 + 0.32 + 0.18 + 0)(0.1) \\ &= 0.33. \; ■ \end{aligned}$$

EXAMPLE 3.2 A Better Approximation Using More Rectangles

Repeat example 3.1, with $n = 20$.

Solution Here, we partition the interval [0, 1] into 20 subintervals, each of width

$$\Delta x = \frac{1 - 0}{20} = \frac{1}{20} = 0.05.$$

We then have $x_0 = 0$, $x_1 = 0 + \Delta x = 0.05$, $x_2 = x_1 + \Delta x = 2(0.05)$ and so on, so that $x_i = (0.05)i$, for $i = 0, 1, 2, \ldots, 20$. From (3.1), the area is then approximately

$$\begin{aligned} A \approx A_{20} &= \sum_{i=1}^{20} f(x_i)\,\Delta x = \sum_{i=1}^{20} \left(2x_i - 2x_i^2\right)\Delta x \\ &= \sum_{i=1}^{20} 2[0.05i - (0.05i)^2](0.05) = 0.3325, \end{aligned}$$

where the details of the calculation are left for the reader. Figure 4.10 shows an approximation using 20 rectangles and in Figure 4.11, we see 40 rectangles.

Based on Figures 4.9–4.11, you should expect that the larger we make n, the better A_n will approximate the actual area, A. The obvious drawback to this idea is the length of time it would take to compute A_n, for n large. However, your CAS or programmable calculator can compute these sums for you, with ease. The table shown in the margin indicates approximate values of A_n for various values of n.

Notice that as n gets larger and larger, A_n seems to be approaching $\frac{1}{3}$. ■

Examples 3.1 and 3.2 give strong evidence that the larger the number of rectangles we use, the better our approximation of the area becomes. Thinking this through, we arrive at the following definition of the area under a curve.

DEFINITION 3.1

For a function f defined on the interval $[a, b]$, if f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, the **area A under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n \to \infty} A_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x. \tag{3.2}$$

In example 3.3, we use the limit defined in (3.2) to find the exact area under the curve from examples 3.1 and 3.2.

EXAMPLE 3.3 Computing the Area Exactly

Find the area under the curve $y = f(x) = 2x - 2x^2$ on the interval $[0, 1]$.

Solution Here, using n subintervals, we have

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

and so, $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = x_1 + \Delta x = \frac{2}{n}$ and so on. Then, $x_i = \frac{i}{n}$, for $i = 0, 1, 2, \ldots, n$. From (3.1), the area is approximately

$$\begin{aligned}
A \approx A_n &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n} \left[2\frac{i}{n} - 2\left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n} \left[2\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)\right] - \sum_{i=1}^{n} \left[2\left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\right] \\
&= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{2}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{2}{n^2}\frac{n(n+1)}{2} - \frac{2}{n^3}\frac{n(n+1)(2n+1)}{6} && \text{From Theorem 2.1 (ii) and (iii).} \\
&= \frac{n+1}{n} - \frac{(n+1)(2n+1)}{3n^2} \\
&= \frac{(n+1)(n-1)}{3n^2}.
\end{aligned}$$

Since we have a formula for A_n, for any n, we can compute various values with ease. We have

$$A_{200} = \frac{(201)(199)}{3(40{,}000)} = 0.333325,$$

$$A_{500} = \frac{(501)(499)}{3(250{,}000)} = 0.333332$$

and so on. Finally, we can compute the limiting value of A_n explicitly. We have

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{n^2 - 1}{3n^2} = \lim_{n \to \infty} \frac{1 - 1/n^2}{3} = \frac{1}{3}.$$

Therefore, the exact area in Figure 4.9 is $1/3$, as we had suspected. ■

EXAMPLE 3.4 Estimating the Area Under a Curve

Estimate the area under the curve $y = f(x) = \sqrt{x+1}$ on the interval $[1, 3]$.

Solution Here, we have

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

and $x_0 = 1$, so that

$$x_1 = x_0 + \Delta x = 1 + \frac{2}{n},$$

$$x_2 = 1 + 2\left(\frac{2}{n}\right)$$

and so on, so that

$$x_i = 1 + \frac{2i}{n}, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

n	A_n
10	3.50595
50	3.45942
100	3.45357
500	3.44889
1000	3.44830
5000	3.44783

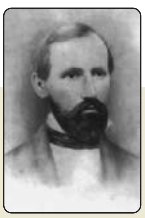
Thus, we have from (3.1) that

$$A \approx A_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \sqrt{x_i + 1}\,\Delta x$$

$$= \sum_{i=1}^{n} \sqrt{\left(1 + \frac{2i}{n}\right) + 1}\left(\frac{2}{n}\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n} \sqrt{2 + \frac{2i}{n}}.$$

We have no formulas like those in Theorem 2.1 for simplifying this last sum (unlike the sum in example 3.3). Our only choice, then, is to compute A_n for a number of values of n using a CAS or programmable calculator. The accompanying table lists approximate values of A_n. Although we can't compute the area exactly (as yet), you should get the sense that the area is approximately 3.4478. ■



HISTORICAL NOTES

Bernhard Riemann (1826–1866)
A German mathematician who made important generalizations to the definition of the integral. Riemann died at a young age without publishing many papers, but each of his papers was highly influential. His work on integration was a small portion of a paper on Fourier series. Pressured by Gauss to deliver a talk on geometry, Riemann developed his own geometry, which provided a generalization of both Euclidean and non-Euclidean geometry. Riemann's work often formed unexpected and insightful connections between analysis and geometry.

We pause now to define some of the mathematical objects we have been examining.

DEFINITION 3.2

Let $\{x_0, x_1, \ldots, x_n\}$ be a regular partition of the interval $[a, b]$, with $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for all i. Pick points $c_1, c_2, \ldots, c_n$, where c_i is any point in the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. (These are called **evaluation points.**) The **Riemann sum** for this partition and set of evaluation points is

$$\sum_{i=1}^{n} f(c_i)\,\Delta x.$$

So far, we have shown that for a continuous, nonnegative function f, the area under the curve $y = f(x)$ is the limit of the Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \tag{3.3}$$

where $c_i = x_i$, for $i = 1, 2, \ldots, n$. Surprisingly, for any continuous function f, the limit in (3.3) is the same for *any* choice of the evaluation points $c_i \in [x_{i-1}, x_i]$ (although the proof is beyond the level of this course). In examples 3.3 and 3.4, we used the evaluation points $c_i = x_i$, for each i (the right endpoint of each subinterval). This is usually the most convenient choice when working by hand, but does *not* generally produce the most accurate approximation for a given value of n.

REMARK 3.1

Most often, we *cannot* compute the limit of Riemann sums indicated in (3.3) exactly (at least not directly). However, we can *always* obtain an approximation to the area by calculating Riemann sums for some large values of n. The most common (and obvious) choices for the evaluation points c_i are x_i (the right endpoint), x_{i-1} (the left endpoint) and $\frac{1}{2}(x_{i-1} + x_i)$ (the midpoint). As it turns out, the midpoint usually provides the best approximation, for a given n. See Figures 4.12a, 4.12b and 4.12c for the right endpoint, left endpoint and midpoint approximations, respectively, for $f(x) = 9x^2 + 2$, on the interval [0, 1], using $n = 10$. You should note that in this case (as with any increasing function), the rectangles corresponding to the right endpoint evaluation (Figure 4.12a) give too much area on each subinterval, while the rectangles corresponding to left endpoint evaluation (Figure 4.12b) give too little area. We leave it to you to observe that the reverse is true for a decreasing function.

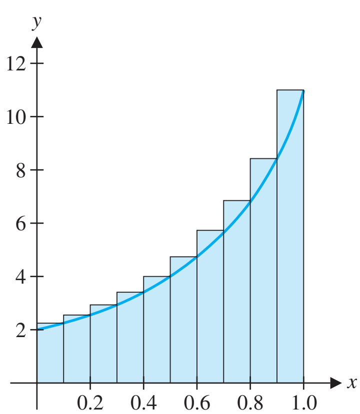


FIGURE 4.12a
$c_i = x_i$

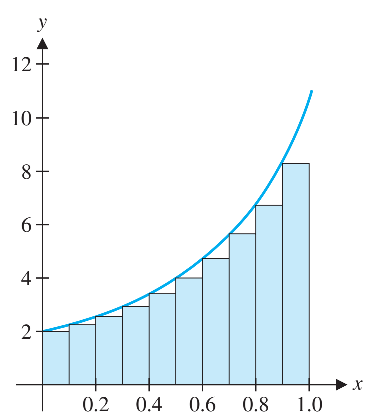


FIGURE 4.12b
$c_i = x_{i-1}$

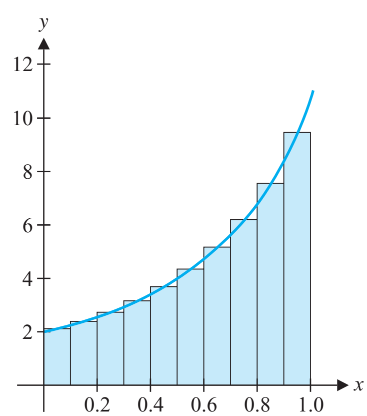


FIGURE 4.12c
$c_i = \frac{1}{2}(x_{i-1} + x_i)$

EXAMPLE 3.5 Computing Riemann Sums with Different Evaluation Points

Compute Riemann sums for $f(x) = \sqrt{x+1}$ on the interval [1, 3], for $n = 10$, 50, 100, 500, 1000 and 5000, using the left endpoint, right endpoint and midpoint of each subinterval as the evaluation points.

Solution The numbers given in the following table are from a program written for a programmable calculator. We suggest that you test your own program or one built into your CAS against these values (rounded off to six digits).

TODAY IN MATHEMATICS

Louis de Branges (1932–)
A French mathematician who proved the Bieberbach conjecture in 1985. To solve this famous 70-year-old problem, de Branges actually proved a related but much stronger result. In 2004, de Branges posted on the Internet what he believes is a proof of another famous problem, the Riemann hypothesis. To qualify for the $1 million prize offered for the first proof of the Riemann hypothesis, the result will have to be verified by expert mathematicians. However, de Branges has said, "I am enjoying the happiness of having a theory which is in my own hands and not in that of eventual readers. I would not want to end that situation for a million dollars."

n	*Left Endpoint*	*Midpoint*	*Right Endpoint*
10	3.38879	3.44789	3.50595
50	3.43599	3.44772	3.45942
100	3.44185	3.44772	3.45357
500	3.44654	3.44772	3.44889
1000	3.44713	3.44772	3.44830
5000	3.44760	3.44772	3.44783

There are several conclusions to be drawn from these numbers. First, there is good evidence that all three sets of numbers are converging to a common limit of approximately 3.4477. You should notice that the limit is independent of the particular evaluation point used. Second, even though the limits are the same, the different rules approach the limit at different rates. You should try computing left and right endpoint sums for larger values of n, to see that these eventually approach 3.44772, also. ■

Riemann sums using midpoint evaluation usually approach the limit far faster than left or right endpoint rules. If you think about the rectangles being drawn, you may be able to explain why. Finally, notice that the left and right endpoint sums in example 3.5 approach the limit from opposite directions and at about the same rate. We take advantage of this in an approximation technique called the Trapezoidal Rule, to be discussed in section 4.7. If your CAS or graphics calculator does not have a command for calculating Riemann sums, we suggest that you write a program for computing them yourself.

BEYOND FORMULAS

We have now developed a technique for using limits to compute certain areas exactly. This parallels the derivation of the slope of the tangent line as the limit of the slopes of secant lines. Recall that this limit became known as the derivative and turned out to have applications far beyond the slope of a tangent line. Similarly, Riemann sums lead us to a second major area of calculus, called integration. Based on your experience with the derivative, do you expect this new limit to solve problems beyond the area of a region? Do you expect that there will be rules developed to simplify the calculations?

EXERCISES 4.3

WRITING EXERCISES

1. For many functions, the limit of the Riemann sums is independent of the choice of evaluation points. Discuss why this is a somewhat surprising result. To make the result more believable, consider a continuous function $f(x)$. As the number of partition points gets larger, the distance between the endpoints gets smaller. For a continuous function $f(x)$, explain why the difference between the function values at any two points in a given subinterval will have to get smaller.

2. Rectangles are not the only basic geometric shapes for which we have an area formula. Discuss how you might approximate the area under a parabola using circles or triangles. Which geometric shape do you think is the easiest to use?

In exercises 1–4, list the evaluation points corresponding to the midpoint of each subinterval, sketch the function and approximating rectangles and evaluate the Riemann sum.

1. $f(x) = x^2 + 1$, (a) $[0, 1]$, $n = 4$; (b) $[0, 2]$, $n = 4$
2. $f(x) = x^3 - 1$, (a) $[1, 2]$, $n = 4$; (b) $[1, 3]$, $n = 4$

3. $f(x) = \sin x$, (a) $[0, \pi]$, $n = 4$; (b) $[0, \pi]$, $n = 8$

4. $f(x) = 4 - x^2$, (a) $[-1, 1]$, $n = 4$; (b) $[-3, -1]$, $n = 4$

In exercises 5–10, approximate the area under the curve on the given interval using *n* rectangles and the evaluation rules (a) left endpoint (b) midpoint (c) right endpoint.

5. $y = x^2 + 1$ on $[0, 1]$, $n = 16$

6. $y = x^2 + 1$ on $[0, 2]$, $n = 16$

7. $y = \sqrt{x + 2}$ on $[1, 4]$, $n = 16$

8. $y = e^{-2x}$ on $[-1, 1]$, $n = 16$

9. $y = \cos x$ on $[0, \pi/2]$, $n = 50$

10. $y = x^3 - 1$ on $[-1, 1]$, $n = 100$

In exercises 11–14, construct a table of Riemann sums as in example 3.5 to show that sums with right-endpoint, midpoint and left-endpoint evaluation all converge to the same value as $n \to \infty$.

11. $f(x) = 4 - x^2$, $[-2, 2]$

12. $f(x) = \sin x$, $[0, \pi/2]$

13. $f(x) = x^3 - 1$, $[1, 3]$

14. $f(x) = x^3 - 1$, $[-1, 1]$

In exercises 15–18, use Riemann sums and a limit to compute the exact area under the curve.

15. $y = x^2 + 1$ on $[0, 1]$

16. $y = x^2 + 3x$ on $[0, 1]$

17. $y = 2x^2 + 1$ on $[1, 3]$

18. $y = 4x + 2$ on $[1, 3]$

In exercises 19–22, use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation.

19.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	2.0	2.4	2.6	2.7	2.6	2.4	2.0	1.4	0.6

20.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	2.0	2.2	1.6	1.4	1.6	2.0	2.2	2.4	2.0

21.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.8	1.4	1.1	0.7	1.2	1.4	1.8	2.4	2.6

22.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
$f(x)$	0.0	0.4	0.6	0.8	1.2	1.4	1.2	1.4	1.0

In exercises 23–26, graphically determine whether a Riemann sum with (a) left-endpoint, (b) midpoint and (c) right-endpoint evaluation points will be greater than or less than the area under the curve $y = f(x)$ on $[a, b]$.

23. $f(x)$ is increasing and concave up on $[a, b]$.

24. $f(x)$ is increasing and concave down on $[a, b]$.

25. $f(x)$ is decreasing and concave up on $[a, b]$.

26. $f(x)$ is decreasing and concave down on $[a, b]$.

27. For the function $f(x) = x^2$ on the interval [0, 1], by trial and errror find evaluation points for $n = 2$ such that the Riemann sum equals the exact area of $1/3$.

28. For the function $f(x) = \sqrt{x}$ on the interval [0, 1], by trial and error find evaluation points for $n = 2$ such that the Riemann sum equals the exact area of $2/3$.

29. Show that for right-endpoint evaluation on the interval $[a, b]$ with each subinterval of length $\Delta x = (b - a)/n$, the evaluation points are $c_i = a + i\Delta x$, for $i = 1, 2, \ldots, n$.

30. Show that for left-endpoint evaluation on the interval $[a, b]$ with each subinterval of length $\Delta x = (b - a)/n$, the evaluation points are $c_i = a + (i - 1)\Delta x$, for $i = 1, 2, \ldots, n$.

31. As in exercises 29 and 30, find a formula for the evaluation points for midpoint evaluation.

32. As in exercises 29 and 30, find a formula for evaluation points that are one-third of the way from the left endpoint to the right endpoint.

33. Economists use a graph called the **Lorentz curve** to describe how equally a given quantity is distributed in a given population. For example, the **gross domestic product** (GDP) varies considerably from country to country. The accompanying data from the Energy Information Administration show percentages for the 100 top-GDP countries in the world in 2001, arranged in order of increasing GDP. The data indicate that the first 10 (lowest 10%) countries account for only 0.2% of the world's total GDP; the first 20 countries account for 0.4% and so on. The first 99 countries account for 73.6% of the total GDP. What percentage does country #100 (the United States) produce? The Lorentz curve is a plot of y versus x. Graph the Lorentz curve for these data. Estimate the area between the curve and the x-axis. (Hint: Notice that the x-values are not equally spaced. You will need to decide how to handle this. Depending on your choice, your answer may not exactly match the back of the book; this is OK!)

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	0.002	0.004	0.008	0.014	0.026	0.048	0.085

x	0.8	0.9	0.95	0.98	0.99	1.0
y	0.144	0.265	0.398	0.568	0.736	1.0

34. The Lorentz curve (see exercise 33) can be used to compute the **Gini index,** a numerical measure of how inequitable a given distribution is. Let A_1 equal the area between the Lorentz curve and the x-axis. Construct the Lorentz curve for the situation of all countries being exactly equal in GDP and let A_2 be the area

between this new Lorentz curve and the x-axis. The Gini index G equals A_1 divided by A_2. Explain why $0 \le G \le 1$ and show that $G = 2A_1$. Estimate G for the data in exercise 33.

In exercises 35–40, use the following definitions. The upper sum of f on P is given by $U(P, f) = \sum_{i=1}^{n} f(c_i)\,\Delta x$, where $f(c_i)$ is the maximum of f on the subinterval $[x_{i-1}, x_i]$. Similarly, the lower sum of f on P is given by $L(P, f) = \sum_{i=1}^{n} f(d_i)\,\Delta x$, where $f(d_i)$ is the minimum of f on the subinterval $[x_{i-1}, x_i]$.

35. Compute the upper sum and lower sum of $f(x) = x^2$ on $[0, 2]$ for the regular partition with $n = 4$.

36. Compute the upper sum and lower sum of $f(x) = x^2$ on $[-2, 2]$ for the regular partition with $n = 8$.

37. Find (a) the general upper sum and (b) the general lower sum for $f(x) = x^2$ on $[0, 2]$ and show that both sums approach the same number as $n \to \infty$.

38. Repeat exercise 37 for $f(x) = x^2$ on the interval $[-1, 0]$.

39. Repeat exercise 37 for $f(x) = x^3 + 1$ on the interval $[0, 2]$.

40. Repeat exercise 37 for $f(x) = x^2 - 2x$ on the interval $[0, 1]$.

EXPLORATORY EXERCISES

1. Riemann sums can also be defined on **irregular partitions,** for which subintervals are not of equal size. An example of an irregular partition of the interval $[0, 1]$ is $x_0 = 0, x_1 = 0.2, x_2 = 0.6, x_3 = 0.9, x_4 = 1$. Explain why the corresponding Riemann sum would be

$$f(c_1)(0.2) + f(c_2)(0.4) + f(c_3)(0.3) + f(c_4)(0.1),$$

for evaluation points c_1, c_2, c_3 and c_4. Identify the interval from which each c_i must be chosen and give examples of evaluation points. To see why irregular partitions might be useful, consider the function $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \ge 1 \end{cases}$ on the interval $[0, 2]$. One way to approximate the area under the graph of this function is to compute Riemann sums using midpoint evaluation for $n = 10$, $n = 50$, $n = 100$ and so on. Show graphically and numerically that with midpoint evaluation, the Riemann sum with $n = 2$ gives the correct area on the subinterval $[0, 1]$. Then explain why it would be wasteful to compute Riemann sums on this subinterval for larger and larger values of n. A more efficient strategy would be to compute the areas on $[0, 1]$ and $[1, 2]$ separately and add them together. The area on $[0, 1]$ can be computed exactly using a small value of n, while the area on $[1, 2]$ must be approximated using larger and larger values of n. Use this technique to estimate the area for $f(x)$ on the interval $[0, 2]$. Try to determine the area to within an error of 0.01 and discuss why you believe your answer is this accurate.

2. Graph the function $f(x) = e^{-x^2}$. You may recognize this curve as the so-called "bell curve," which is of fundamental importance in statistics. We define the **area function** $g(t)$ to be the area between this graph and the x-axis between $x = 0$ and $x = t$ (for now, assume that $t > 0$). Sketch the area that defines $g(1)$ and $g(2)$ and argue that $g(2) > g(1)$. Explain why the function $g(x)$ is increasing and hence $g'(x) > 0$ for $x > 0$. Further, argue that $g'(2) < g'(1)$. Explain why $g'(x)$ is a decreasing function. Thus, $g'(x)$ has the same general properties (positive, decreasing) that $f(x)$ does. In fact, we will discover in section 4.5 that $g'(x) = f(x)$. To collect some evidence for this result, use Riemann sums to estimate $g(2)$, $g(1.1)$, $g(1.01)$ and $g(1)$. Use these values to estimate $g'(1)$ and compare to $f(1)$.

3. The following result has been credited to Archimedes. (See the historical note on page 442). For the general parabola $y = a^2 - x^2$ with $-a \le x \le a$, show that the area under the parabola is $\frac{2}{3}$ of the base times the height [that is, $\frac{2}{3}(2a)(a^2)$]. Generalize the result to any parabola and its circumscribing rectangle.

4.4 THE DEFINITE INTEGRAL

A sky diver who steps out of an airplane (starting with zero downward velocity) gradually picks up speed until reaching *terminal velocity,* the speed at which the force due to air resistance cancels out the force due to gravity. A function that models the velocity x seconds into the jump is $f(x) = 30(1 - e^{-x/3})$ (see Figure 4.13).

We saw in section 4.2 that the area A under this curve on the interval $0 \le x \le t$ corresponds to the distance fallen in the first t seconds. For any given value of t, we approximate A by partitioning the interval into n subintervals of equal width, Δx. On each subinterval, $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we construct a rectangle of height $f(c_i)$, for any choice of

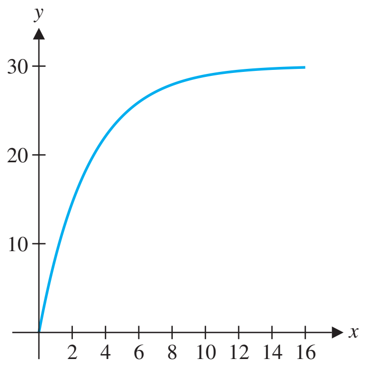


FIGURE 4.13
$y = f(x)$

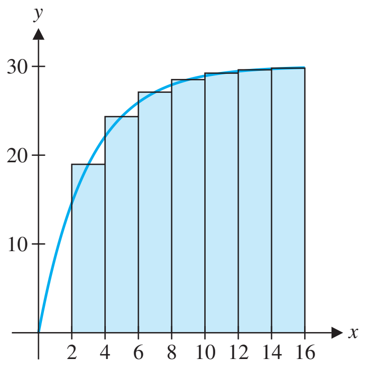


FIGURE 4.14
Approximate area

$c_i \in [x_{i-1}, x_i]$ (see Figure 4.14). Finally, summing the areas of the rectangles gives us the approximation

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

The exact area is then given by the limit of these Riemann sums,

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x.} \tag{4.1}$$

Notice that the sum in (4.1) makes sense even when some (or all) of the function values $f(c_i)$ are negative. The general definition follows.

REMARK 4.1

Definition 4.1 is adequate for most functions (those that are continuous except for at most a finite number of discontinuities). For more general functions, we broaden the definition to include partitions with subintervals of different lengths. You can find a suitably generalized definition in Chapter 13.

DEFINITION 4.1

For any function f defined on $[a, b]$, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

whenever the limit exists and is the same for any choice of evaluation points, $c_1, c_2, \ldots, c_n$. When the limit exists, we say that f is **integrable** on $[a, b]$.

We should observe that in the Riemann sum, the Greek letter $\sum$ indicates a sum; so does the elongated "S", $\int$ used as the integral sign. The **lower** and **upper limits of integration,** a and b, respectively, indicate the endpoints of the interval over which you are integrating. The dx in the integral corresponds to the increment Δx in the Riemann sum and also indicates the variable of integration. The letter used for the variable of integration (called a dummy variable) is irrelevant since the value of the integral is a constant and not a function of x. Here, $f(x)$ is called the **integrand.**

To calculate a definite integral, we have two options: if the function is simple enough (say, a polynomial of degree 2 or less) we can symbolically compute the limit of the Riemann sums. Otherwise, we can numerically compute a number of Riemann sums and approximate the value of the limit. We frequently use the **Midpoint Rule,** which uses the midpoints as the evaluation points for the Riemann sum.

NOTES

If f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{Area under the curve } \geq 0.$$

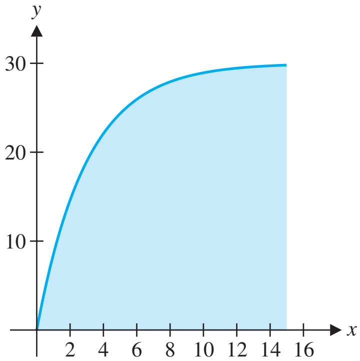


FIGURE 4.15
$y = 30(1 - e^{-x/3})$

n	R_n
10	361.5
20	360.8
50	360.6
100	360.6

EXAMPLE 4.1 A Midpoint Rule Approximation of a Definite Integral

Use the Midpoint Rule to estimate $\int_0^{15} 30\,(1 - e^{-x/3})\,dx$.

Solution The integral gives the area under the curve indicated in Figure 4.15. (Note that this corresponds to the distance fallen by the sky diver in this section's introduction.) From the Midpoint Rule we have

$$\int_0^{15} 30(1 - e^{-x/3})\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = 30 \sum_{i=1}^{n} (1 - e^{-c_i/3}) \left(\frac{15 - 0}{n} \right),$$

where $c_i = \dfrac{x_i + x_{i-1}}{2}$. Using a CAS or a calculator program, you can get the sequence of approximations found in the accompanying table.

One remaining question is when to stop increasing n. In this case, we continued to increase n until it seemed clear that 361 feet was a reasonable approximation. ■

Now, think carefully about the limit in Definition 4.1. How can we interpret this limit when f is both positive and negative on the interval $[a, b]$? Notice that if $f(c_i) < 0$, for some i, then the height of the rectangle shown in Figure 4.16 is $-f(c_i)$ and so,

$$f(c_i)\Delta x = -\text{Area of the } i\text{th rectangle}.$$

To see the effect this has on the sum, consider example 4.2.

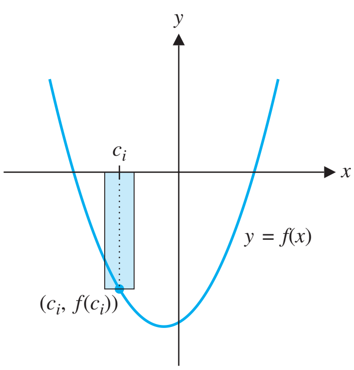


FIGURE 4.16
$f(c_i) < 0$

EXAMPLE 4.2 A Riemann Sum for a Function with Positive and Negative Values

For $f(x) = \sin x$ on $[0, 2\pi]$, give an area interpretation of $\lim\limits_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$.

Solution For this illustration, we take c_i to be the midpoint of $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. In Figure 4.17a, we see 10 rectangles constructed between the x-axis and the curve $y = f(x)$.

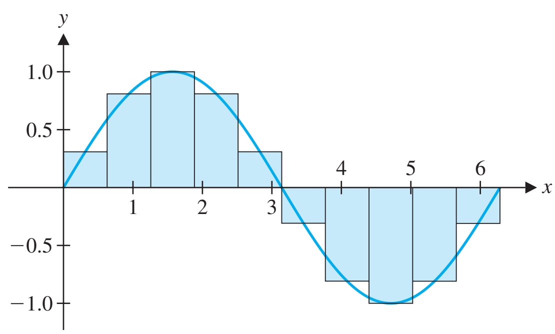


FIGURE 4.17a
Ten rectangles

The first five rectangles [where $f(c_i) > 0$] lie above the x-axis and have height $f(c_i)$. The remaining five rectangles [where $f(c_i) < 0$] lie below the x-axis and have height $-f(c_i)$. So, here

$$\sum_{i=1}^{10} f(c_i)\,\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis}).$$

In Figures 4.17b and 4.17c, we show 20 and 40 rectangles, respectively, constructed in the same way. From this, observe that

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x = (\text{Area above the } x\text{-axis}) - (\text{Area below the } x\text{-axis}),$$

which turns out to be zero.

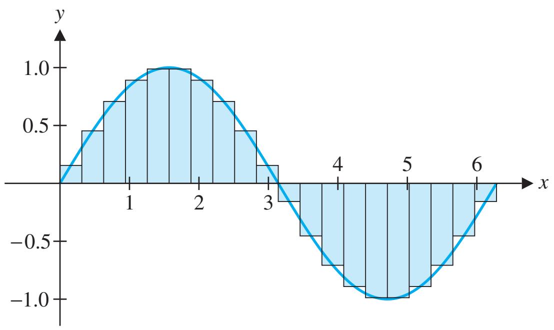


FIGURE 4.17b
Twenty rectangles

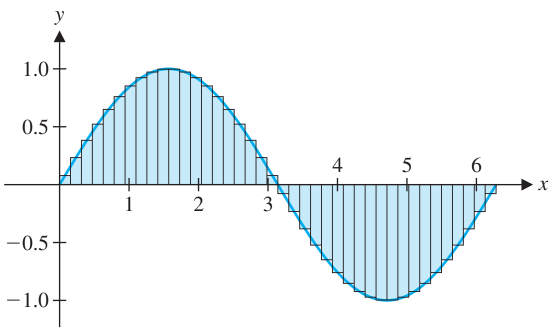


FIGURE 4.17c
Forty rectangles

We illustrate the area interpretation of the integral in example 4.3.

EXAMPLE 4.3 Using Riemann Sums to Compute a Definite Integral

Compute $\int_0^2 (x^2 - 2x)\,dx$ exactly.

Solution The definite integral is the limit of a sequence of Riemann sums, where we can choose any evaluation points we wish. It is usually easiest to write out the formula using right endpoints, as we do here. In this case,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

We then have $x_0 = 0$, $x_1 = x_0 + \Delta x = \frac{2}{n}$,

$$x_2 = x_1 + \Delta x = \frac{2}{n} + \frac{2}{n} = \frac{2(2)}{n}$$

and so on. We then have $c_i = x_i = \frac{2i}{n}$. The nth Riemann sum R_n is then

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} (x_i^2 - 2x_i)\,\Delta x \\
&= \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 - 2\left(\frac{2i}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{4i}{n}\right)\left(\frac{2}{n}\right) \\
&= \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i \\
&= \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6} - \left(\frac{8}{n^2}\right)\frac{n(n+1)}{2} \qquad \text{From Theorem 2.1 (ii) and (iii).} \\
&= \frac{4(n+1)(2n+1)}{3n^2} - \frac{4(n+1)}{n} = \frac{8n^2+12n+4}{3n^2} - \frac{4n+4}{n}.
\end{aligned}$$

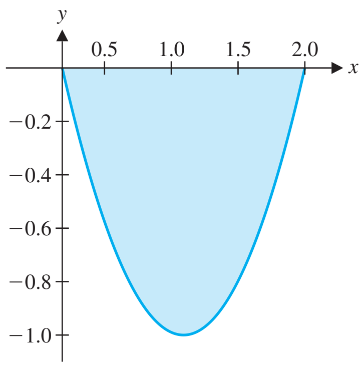


FIGURE 4.18
$y = x^2 - 2x$ on [0, 2]

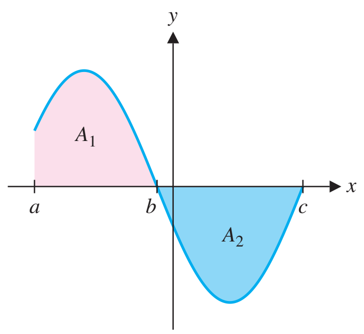


FIGURE 4.19
Signed area

Taking the limit of R_n as $n \to \infty$ gives us the exact value of the integral:

$$\int_0^2 (x^2 - 2x)\,dx = \lim_{n\to\infty}\left(\frac{8n^2+12n+4}{3n^2} - \frac{4n+4}{n}\right) = \frac{8}{3} - 4 = -\frac{4}{3}.$$

To interpret the negative value of the integral in terms of area, look at a graph of $y = x^2 - 2x$ (see Figure 4.18). On the interval [0, 2], the integrand is always negative. Notice that the *absolute value* of the integral, $\frac{4}{3}$, corresponds to the area between the curve and the x-axis. The negative value of the integral indicates that the area lies *below* the x-axis. We refer to this as *signed area,* which we now define. ■

DEFINITION 4.2

Suppose that $f(x) \geq 0$ on the interval $[a, b]$ and A_1 is the area bounded between the curve $y = f(x)$ and the x-axis for $a \leq x \leq b$. Further, suppose that $f(x) \leq 0$ on the interval $[b, c]$ and A_2 is the area bounded between the curve $y = f(x)$ and the x-axis for $b \leq x \leq c$. The **signed area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 - A_2$, and the **total area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 + A_2$ (see Figure 4.19).

Definition 4.2 says that signed area is the difference between any areas lying above the x-axis and any areas lying below the x-axis, while the total area is the sum total of the area bounded between the curve $y = f(x)$ and the x-axis.

Example 4.4 examines the general case where the integrand may be both positive and negative on the interval of integration.

EXAMPLE 4.4 Relating Definite Integrals to Signed Area

Compute three related integrals: $\int_0^2 (x^2 - 2x)\,dx$, $\int_2^3 (x^2 - 2x)\,dx$ and $\int_0^3 (x^2 - 2x)\,dx$, and interpret each in terms of area.

Solution From example 4.3, we already know that $\int_0^2 (x^2 - 2x)\,dx = -\frac{4}{3}$. (See Figure 4.18 to interpret this result graphically.)

On the interval [2, 3], we have $\Delta x = \frac{1}{n}$, $x_0 = 2$, $x_1 = x_0 + \Delta x = 2 + \frac{1}{n}$,

$$x_2 = x_1 + \Delta x = \left(2 + \frac{1}{n}\right) + \frac{1}{n} = 2 + \frac{2}{n}$$

and so on. Using right-endpoint evaluation, we have $c_i = x_i = 2 + \frac{i}{n}$. This gives us the Riemann sum

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(x_i^2 - 2x_i\right)\Delta x \\
&= \sum_{i=1}^{n} \left[\left(2 + \frac{i}{n}\right)^2 - 2\left(2 + \frac{i}{n}\right)\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n} \left(4 + 4\frac{i}{n} + \frac{i^2}{n^2} - 4 - \frac{2i}{n}\right)\left(\frac{1}{n}\right) \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n^2}\sum_{i=1}^{n} i \\
&= \left(\frac{1}{n^3}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{2}{n^2}\right)\frac{n(n+1)}{2} && \text{From Theorem 2.1 (ii) and (iii).} \\
&= \frac{(n+1)(2n+1)}{6n^2} + \frac{n+1}{n}.
\end{aligned}$$

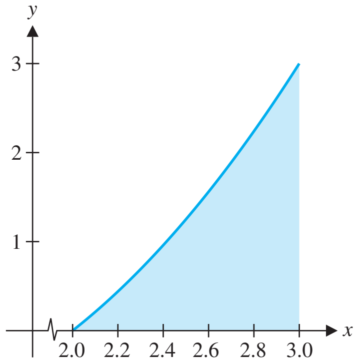


FIGURE 4.20a
$y = x^2 - 2x$ on [2, 3]

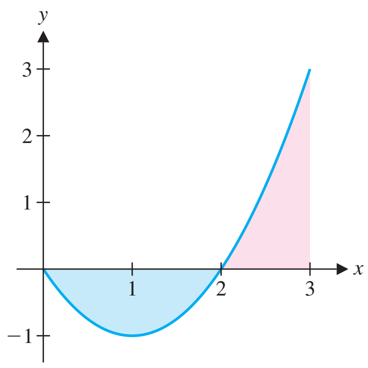


FIGURE 4.20b
$y = x^2 - 2x$ on [0, 3]

Taking the limit of this Riemann sum as $n \to \infty$, we have

$$\int_2^3 (x^2 - 2x)\,dx = \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{6n^2} + \frac{n+1}{n}\right] = \frac{2}{6} + 1 = \frac{4}{3}.$$

A graph of $y = x^2 - 2x$ on the interval [2, 3] is shown in Figure 4.20a. Notice that since the function is always positive on the interval [2, 3], the integral corresponds to the area under the curve.

Finally, on the interval [0, 3], we have $\Delta x = \frac{3}{n}$ and $x_0 = 0$, $x_1 = x_0 + \Delta x = \frac{3}{n}$,

$$x_2 = x_1 + \Delta x = \frac{3}{n} + \frac{3}{n} = \frac{3(2)}{n}$$

and so on. Using right-endpoint evaluation, we have $c_i = x_i = \frac{3i}{n}$. This gives us the Riemann sum

$$\begin{aligned} R_n &= \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(\frac{9i^2}{n^2} - \frac{6i}{n}\right)\left(\frac{3}{n}\right) \\ &= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{18}{n^2}\sum_{i=1}^{n} i \\ &= \left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6} - \left(\frac{18}{n^2}\right)\frac{n(n+1)}{2} \qquad \text{From Theorem 2.1 (ii) and (iii).} \\ &= \frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n}. \end{aligned}$$

Taking the limit as $n \to \infty$ gives us

$$\int_0^3 (x^2 - 2x)\,dx = \lim_{n\to\infty}\left[\frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n}\right] = \frac{18}{2} - 9 = 0.$$

On the interval [0, 2], notice that the curve $y = x^2 - 2x$ lies below the x-axis and the area bounded between the curve and the x-axis is $\frac{4}{3}$. On the interval [2, 3], the curve lies above the x-axis and the area bounded between the curve and the x-axis is also $\frac{4}{3}$. Notice that the integral of 0 on the interval [0, 3] indicates that the signed areas have canceled out one another. (See Figure 4.20b for a graph of $y = x^2 - 2x$ on the interval [0, 3].) You should also observe that the total area A bounded between $y = x^2 - 2x$ and the x-axis is the sum of the two areas indicated above, $A = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$. ■

We can also interpret signed area in terms of velocity and position. Suppose that $v(t)$ is the velocity function for an object moving back and forth along a straight line. Notice that the velocity may be both positive and negative. If the velocity is positive on the interval $[t_1, t_2]$, then $\int_{t_1}^{t_2} v(t)\,dt$ gives the distance traveled (here, in the positive direction). If the velocity is negative on the interval $[t_3, t_4]$, then the object is moving in the negative direction and the distance traveled (here, in the negative direction) is given by $-\int_{t_3}^{t_4} v(t)\,dt$. Notice that if the object starts moving at time 0 and stops at time T, then $\int_0^T v(t)\,dt$ gives the distance traveled in the positive direction minus the distance traveled in the negative direction. That is, $\int_0^T v(t)\,dt$ corresponds to the *overall change* in position from start to finish.

EXAMPLE 4.5 Estimating Overall Change in Position

An object moving along a straight line has velocity function $v(t) = \sin t$. If the object starts at position 0, determine the total distance traveled and the object's position at time $t = 3\pi/2$.

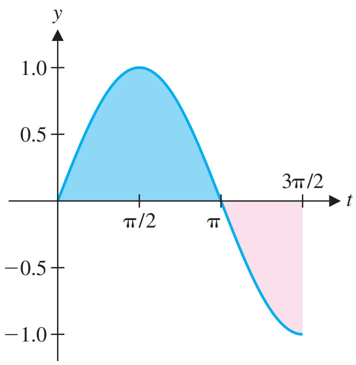


FIGURE 4.21
$y = \sin t$ on $\left[0, \frac{3\pi}{2}\right]$

Solution From the graph (see Figure 4.21), notice that $\sin t \geq 0$ for $0 \leq t \leq \pi$ and $\sin t \leq 0$ for $\pi \leq t \leq 3\pi/2$. The total distance traveled corresponds to the area of the shaded regions in Figure 4.21, given by

$$A = \int_0^{\pi} \sin t\, dt - \int_{\pi}^{3\pi/2} \sin t\, dt.$$

You can use the Midpoint Rule to get the following Riemann sums:

n	$R_n \approx \int_0^{\pi} \sin t\, dt$
10	2.0082
20	2.0020
50	2.0003
100	2.0001

n	$R_n \approx \int_{\pi}^{3\pi/2} \sin t\, dt$
10	−1.0010
20	−1.0003
50	−1.0000
100	−1.0000

Observe that the sums appear to be converging to 2 and −1, respectively, which we will soon be able to show are indeed correct. The total area bounded between $y = \sin t$ and the t-axis on $\left[0, \frac{3\pi}{2}\right]$ is then

$$\int_0^{\pi} \sin t\, dt - \int_{\pi}^{3\pi/2} \sin t\, dt = 2 + 1 = 3,$$

so that the total distance traveled is 3 units. The overall change in position of the object is given by

$$\int_0^{3\pi/2} \sin t\, dt = \int_0^{\pi} \sin t\, dt + \int_{\pi}^{3\pi/2} \sin t\, dt = 2 + (-1) = 1.$$

So, if the object starts at position 0, it ends up at position $0 + 1 = 1$. ■

We have defined the definite integral of a function in terms of a limit, but we have not yet discussed the circumstances under which the limit actually exists. Theorem 4.1 indicates that many of the functions with which you are familiar are indeed integrable.

THEOREM 4.1

If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

The proof of Theorem 4.1 is too technical to include here. However, if you think about the area interpretation of the definite integral, the result should seem plausible.

Next, we give some general rules for integrals.

THEOREM 4.2

If f and g are integrable on $[a, b]$, then the following are true.

(i) For any constants c and d, $\int_a^b [cf(x) + dg(x)]\, dx = c\int_a^b f(x)\, dx + d\int_a^b g(x)\, dx$ and

(ii) For any c in $[a, b]$, $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

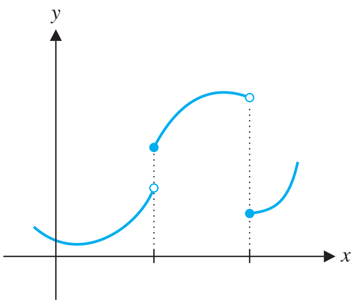


FIGURE 4.22
Piecewise continuous function

PROOF

By definition, for any constants c and d, we have

$$\begin{aligned}\int_a^b [cf(x) + dg(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [cf(c_i) + dg(c_i)]\,\Delta x \\ &= \lim_{n\to\infty} \left[c\sum_{i=1}^{n} f(c_i)\,\Delta x + d\sum_{i=1}^{n} g(c_i)\,\Delta x \right] \quad \text{From Theorem 2.2.} \\ &= c\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x + d\lim_{n\to\infty} \sum_{i=1}^{n} g(c_i)\,\Delta x \\ &= c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx,\end{aligned}$$

where we have used our usual rules for summations plus the fact that f and g are integrable. We leave the proof of part (ii) to the exercises, but note that we have already illustrated the idea in example 4.5. ■

We now make a pair of reasonable definitions. First, for any integrable function f, if $a < b$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{4.2}$$

This should appear reasonable in that if we integrate "backward" along an interval, the width of the rectangles corresponding to a Riemann sum (Δx) would seem to be negative. Second, if $f(a)$ is defined, we define

$$\int_a^a f(x)\,dx = 0.$$

If you think of the definite integral as area, this says that the area from a up to a is zero.

It turns out that a function is integrable even when it has a finite number of jump discontinuities, but is otherwise continuous. (Such a function is called **piecewise continuous;** see Figure 4.22 for the graph of such a function.)

In example 4.6, we evaluate the integral of a discontinuous function.

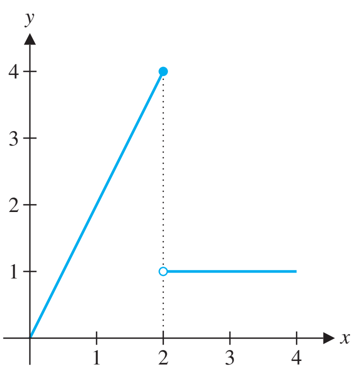


FIGURE 4.23a
$y = f(x)$

EXAMPLE 4.6 An Integral with a Discontinuous Integrand

Evaluate $\int_0^3 f(x)\,dx$, where $f(x)$ is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \le 2 \\ 1, & \text{if } x > 2 \end{cases}.$$

Solution We start by looking at a graph of $y = f(x)$ in Figure 4.23a. Notice that although f is discontinuous at $x = 2$, it has only a single jump discontinuity and so, is piecewise continuous on [0, 3]. By Theorem 4.2 (ii), we have that

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx.$$

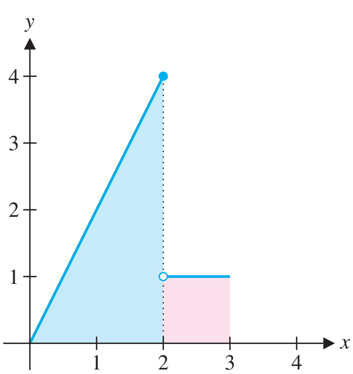


FIGURE 4.23b
The area under the curve $y = f(x)$ on [0, 3]

Referring to Figure 4.23b, observe that $\int_0^2 f(x)\,dx$ corresponds to the area of the triangle of base 2 and altitude 4 shaded in the figure, so that

$$\int_0^2 f(x)\,dx = \frac{1}{2}(\text{base})\,(\text{height}) = \frac{1}{2}(2)(4) = 4.$$

Next, also notice from Figure 4.23b that $\int_2^3 f(x)\,dx$ corresponds to the area of the square of side 1, so that

$$\int_2^3 f(x)\,dx = 1.$$

We now have that

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = 4 + 1 = 5.$$

Notice that in this case, the areas corresponding to the two integrals could be computed using simple geometric formulas and so, there was no need to compute Riemann sums here. ■

Another simple property of definite integrals is the following.

THEOREM 4.3

Suppose that $g(x) \leq f(x)$ for all $x \in [a, b]$ and that f and g are integrable on $[a, b]$. Then,

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx.$$

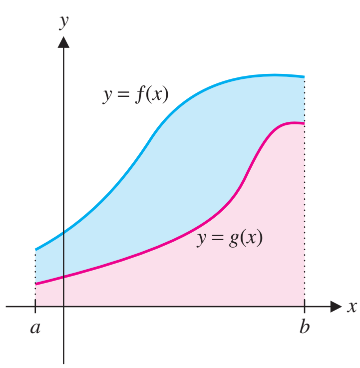


FIGURE 4.24
Larger functions have larger integrals

PROOF

Since $g(x) \leq f(x)$, we must also have that $0 \leq [f(x) - g(x)]$ on $[a, b]$ and in view of this, $\int_a^b [f(x) - g(x)]\,dx$ represents the area under the curve $y = f(x) - g(x)$, which can't be negative. Using Theorem 4.2 (i), we now have

$$0 \leq \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx,$$

from which the result follows. ■

Notice that Theorem 4.3 simply says that larger functions have larger integrals. We illustrate this for the case of two positive functions in Figure 4.24.

Average Value of a Function

To compute the average age of the students in your calculus class, note that you need only add up each student's age and divide the total by the number of students in your class. By contrast, how would you find the average depth of a cross section of a lake? In this case, there are an infinite number of depths to average. You would get a reasonable idea of the average depth by sampling the depth of the lake at a number of points spread out along the length of the lake and then averaging these depths, as indicated in Figure 4.25.

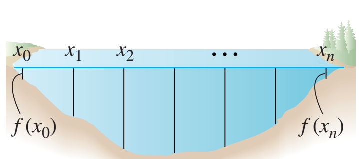


FIGURE 4.25
Average depth of a cross section of a lake

More generally, we often want to calculate the average value of a function f on some interval $[a, b]$. To do this, we form a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

where the difference between successive points is $\Delta x = \frac{b-a}{n}$. The **average value,** f_{ave}, is then given approximately by the average of the function values at $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} f_{\text{ave}} &\approx \frac{1}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)] \\ &= \frac{1}{n}\sum_{i=1}^{n} f(x_i) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) && \text{Multiply and divide by } (b-a). \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\Delta x. && \text{Since } \Delta x = \frac{b-a}{n}. \end{aligned}$$

Notice that the last summation is a Riemann sum. Further, observe that the more points we sample, the better our approximation should be. So, letting $n \to \infty$, we arrive at an integral representing average value:

$$\boxed{f_{\text{ave}} = \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x\right] = \frac{1}{b-a}\int_a^b f(x)\,dx.} \tag{4.3}$$

EXAMPLE 4.7 Computing the Average Value of a Function

Compute the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution From (4.3), we have

$$f_{\text{ave}} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx.$$

We can approximate the value of this integral by calculating some Riemann sums, to obtain the approximate average, $f_{\text{ave}} \approx 0.6366198$. (This is left as an exercise.) In Figure 4.26, we show a graph of $y = \sin x$ and its average value on the interval $[0, \pi]$. You should note that the two shaded areas are the same. ■

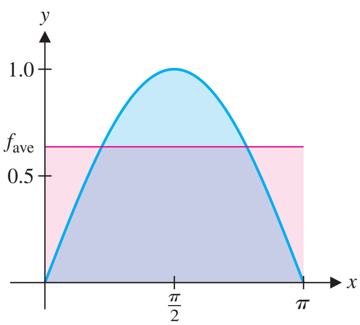


FIGURE 4.26
$y = \sin x$ and its average

Returning to the problem of finding the average depth of a lake, imagine the dirt at the bottom of the lake settling out to form a flat bottom. The depth of the lake would then be constant and equal to the average value of the depth of the original lake. In the settling out process, the depth at one (and possibly more) points would not change (see Figure 4.27). That is, the average depth of the lake exactly equals the depth of the lake at one or more points. A precise statement of this result is given on the next page as the Integral Mean Value Theorem.

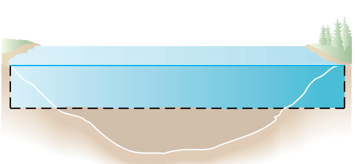


FIGURE 4.27
A lake and its average depth

Notice that for any constant, c,

$$\int_a^b c\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} c\,\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x = c(b-a),$$

since $\sum_{i=1}^{n}\Delta x$ is simply the sum of the lengths of the subintervals in the partition.

Let f be any continuous function defined on $[a, b]$. Recall that by the Extreme Value Theorem, since f is continuous, it has a minimum, m, and a maximum, M, on $[a, b]$. It follows that

$$m \le f(x) \le M, \quad \text{for all } x \in [a, b]$$

and consequently, from Theorem 4.3,

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

Since m and M are constants, we get

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{4.4}$$

Finally, dividing by $(b-a) > 0$, we obtain

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

That is, $\frac{1}{b-a}\int_a^b f(x)\,dx$ (the average value of f on $[a, b]$) lies between the minimum and the maximum values of f on $[a, b]$. Since f is a continuous function, we have by the Intermediate Value Theorem that there must be some $c \in (a, b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We have just proved a theorem:

THEOREM 4.4 (Integral Mean Value Theorem)

If f is continuous on $[a, b]$, then there is a number $c \in (a, b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The Integral Mean Value Theorem is a fairly simple idea (that a continuous function will take on its average value at some point), but it has some significant applications. The first of these will be found in section 4.5, in the proof of one of the most significant results in the calculus, the Fundamental Theorem of Calculus.

Referring back to our derivation of the Integral Mean Value Theorem, observe that along the way we proved that for any integrable function f, if $m \le f(x) \le M$, for all $x \in [a, b]$, then inequality (4.4) holds:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

This enables us to estimate the value of a definite integral. Although the estimate is generally only a rough one, it still has importance in that it gives us an interval in which the value must lie. We illustrate this in example 4.8.

EXAMPLE 4.8 Estimating the Value of an Integral

Use inequality (4.4) to estimate the value of $\int_0^1 \sqrt{x^2+1}\,dx$.

Solution First, notice that it's beyond your present abilities to compute the value of this integral exactly. However, notice that

$$1 \le \sqrt{x^2+1} \le \sqrt{2}, \quad \text{for all } x \in [0, 1].$$

From inequality (4.4), we now have

$$1 \le \int_0^1 \sqrt{x^2+1}\,dx \le \sqrt{2} \approx 1.414214.$$

In other words, although we still do not know the exact value of the integral, we know that it must be between 1 and $\sqrt{2} \approx 1.414214$. ■

EXERCISES 4.4

WRITING EXERCISES

1. Sketch a graph of a function f that is both positive and negative on an interval $[a, b]$. Explain in terms of area what it means to have $\int_a^b f(x)\,dx = 0$. Also, explain what it means to have $\int_a^b f(x)\,dx > 0$ and $\int_a^b f(x)\,dx < 0$.
2. To get a physical interpretation of the result in Theorem 4.3, suppose that $f(x)$ and $g(x)$ are velocity functions for two different objects starting at the same position. If $f(x) \ge g(x) \ge 0$, explain why it follows that $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
3. The Integral Mean Value Theorem says that if $f(x)$ is continuous on the interval $[a, b]$, then there exists a number c between a and b such that $f(c)(b-a) = \int_a^b f(x)\,dx$. By thinking of the left-hand side of this equation as the area of a rectangle, sketch a picture that illustrates this result, and explain why the result follows.
4. Write out the Integral Mean Value Theorem as applied to the derivative $f'(x)$. Then write out the Mean Value Theorem for derivatives (see section 2.9). If the c-values identified by each theorem are the same, what does $\int_a^b f'(x)\,dx$ have to equal? Explain why, at this point, we don't know whether or not the c-values are the same.

In exercises 1–4, use the Midpoint Rule to estimate the value of the integral (obtain two digits of accuracy).

1. $\int_0^3 (x^3+x)\,dx$
2. $\int_0^3 \sqrt{x^2+1}\,dx$
3. $\int_0^{\pi} \sin x^2\,dx$
4. $\int_{-2}^2 e^{-x^2}\,dx$

In exercises 5–10, evaluate the integral by computing the limit of Riemann sums.

5. $\int_0^1 2x\,dx$
6. $\int_1^2 2x\,dx$
7. $\int_0^2 x^2\,dx$
8. $\int_0^3 (x^2+1)\,dx$
9. $\int_1^3 (x^2-3)\,dx$
10. $\int_{-2}^2 (x^2-1)\,dx$

In exercises 11–18, write the given (total) area as an integral or sum of integrals.

11. The area above the x-axis and below $y = 4 - x^2$
12. The area above the x-axis and below $y = 4x - x^2$
13. The area below the x-axis and above $y = x^2 - 4$
14. The area below the x-axis and above $y = x^2 - 4x$
15. The area between $y = \sin x$ and the x-axis for $0 \le x \le \pi$
16. The area between $y = \sin x$ and the x-axis for $-\pi/2 \le x \le \pi/4$
17. The area between $y = x^3 - 3x^2 + 2x$ and the x-axis for $0 \le x \le 2$
18. The area between $y = x^3 - 4x$ and the x-axis for $-2 \le x \le 3$

In exercises 19–20, use the given velocity function and initial position to estimate the final position $s(b)$.

19. $v(t) = 40(1 - e^{-2t})$, $s(0) = 0$, $b = 4$
20. $v(t) = 30e^{-t/4}$, $s(0) = -1$, $b = 4$

In exercises 21–24, use Theorem 4.2 to write the expression as a single integral.

21. $\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$
22. $\int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx$
23. $\int_0^2 f(x)\,dx + \int_2^1 f(x)\,dx$
24. $\int_{-1}^2 f(x)\,dx + \int_2^3 f(x)\,dx$

In exercises 25–28, sketch the area corresponding to the integral.

25. $\int_1^2 (x^2-x)\,dx$
26. $\int_2^4 (x^2-x)\,dx$
27. $\int_0^{\pi/2} \cos x\,dx$
28. $\int_{-2}^2 e^{-x}\,dx$

In exercises 29–32, use the Integral Mean Value Theorem to estimate the value of the integral.

29. $\int_{\pi/3}^{\pi/2} 3\cos x^2\,dx$

30. $\int_0^{1/2} e^{-x^2}\,dx$

31. $\int_0^2 \sqrt{2x^2+1}\,dx$

32. $\int_{-1}^{1} \frac{3}{x^3+2}\,dx$

In exercises 33 and 34, find a value of c that satisfies the conclusion of the Integral Mean Value Theorem.

33. $\int_0^2 3x^2 dx\ (= 8)$

34. $\int_{-1}^{1} (x^2 - 2x)\,dx\ (= \frac{2}{3})$

In exercises 35–38, compute the average value of the function on the given interval.

35. $f(x) = 2x + 1$, $[0, 4]$

36. $f(x) = x^2 + 2x$, $[0, 1]$

37. $f(x) = x^2 - 1$, $[1, 3]$

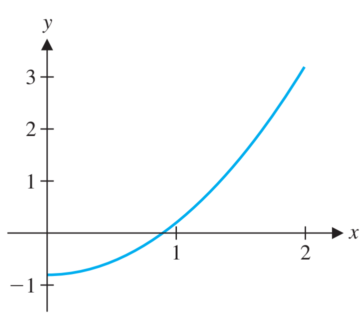
38. $f(x) = 2x - 2x^2$, $[0, 1]$

39. Prove that if f is continuous on the interval $[a, b]$, then there exists a number c in (a, b) such that $f(c)$ equals the average value of f on the interval $[a, b]$.

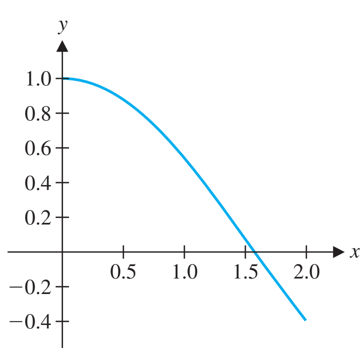
40. Prove part (ii) of Theorem 4.2 for the special case where $c = \frac{1}{2}(a + b)$.

In exercises 41–44, use the graph to determine whether $\int_0^2 f(x)\,dx$ is positive or negative.

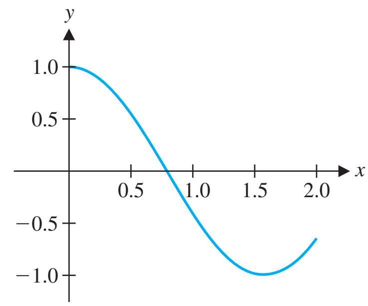
41.



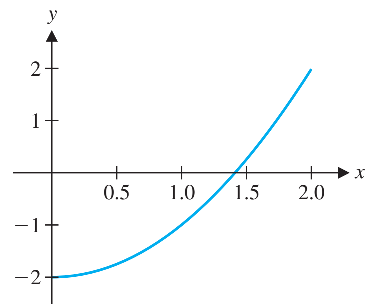
42.



43.



44.



45. For the functions $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2+2 & \text{if } x \geq 1 \end{cases}$ and $g(x) = \begin{cases} 2x & \text{if } x \leq 1 \\ x^2+2 & \text{if } x > 1 \end{cases}$, assume that $\int_0^2 f(x)\,dx$ and $\int_0^2 g(x)\,dx$ exist. Explain why the approximating Riemann sums with midpoint evaluations are equal for any even value of n. Argue that this result implies that the two integrals are both equal to the sum $\int_0^1 2x\,dx + \int_1^2 (x^2+2)\,dx$.

46. Prove that $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$. (Hint: Use Theorem 4.3.)

In exercises 47 and 48, compute $\int_0^4 f(x)\,dx$.

47. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 & \text{if } x \geq 1 \end{cases}$

48. $f(x) = \begin{cases} 2 & \text{if } x \leq 2 \\ 3x & \text{if } x > 2 \end{cases}$

49. Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 410 - 0.3t$ organisms per month and the death rate is given by $a(t) = 390 + 0.2t$ organisms per month. Explain why $\int_0^{12} [b(t) - a(t)]\,dt$ represents the net change in population in the first 12 months. Determine for which values of t it is true that $b(t) > a(t)$. At which times is the population increasing? Decreasing? Determine the time at which the population reaches a maximum.

50. Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 400 - 3\sin t$ organisms per month and the death rate is given by $a(t) = 390 + t$ organisms per month. Explain why $\int_0^{12} [b(t) - a(t)]\,dt$ represents the net change in population in the first 12 months. Graphically determine for which values of t it is true that $b(t) > a(t)$. At which times is the population increasing? Decreasing? Estimate the time at which the population reaches a maximum.

51. For a particular ideal gas at constant temperature, pressure P and volume V are related by $PV = 10$. The work required to increase the volume from $V = 2$ to $V = 4$ is given by the integral $\int_2^4 P(V)\,dV$. Estimate the value of this integral.

52. Suppose that the temperature t months into the year is given by $T(t) = 64 - 24\cos\frac{\pi}{6}t$ (degrees Fahrenheit). Estimate the average temperature over an entire year. Explain why this answer is obvious from the graph of $T(t)$.

53. Suppose that the average value of a function $f(x)$ over the interval $[0, 2]$ is 5 and the average value of $f(x)$ over the interval $[2, 6]$ is 11. Find the average value of $f(x)$ over the interval $[0, 6]$.

54. Suppose that the average value of a function $f(x)$ over an interval $[a, b]$ is v and the average value of $f(x)$ over the interval $[b, c]$ is w. Find the average value of $f(x)$ over the interval $[a, c]$.

In exercises 55–58, use a geometric formula to compute the integral.

55. $\int_0^2 3x\,dx$ **56.** $\int_1^4 2x\,dx$

57. $\int_0^2 \sqrt{4 - x^2}\,dx$ **58.** $\int_{-3}^0 \sqrt{9 - x^2}\,dx$

59. The table shows the temperature at different times of the day. Estimate the average temperature using (a) right-endpoint evaluation and (b) left-endpoint evaluation. Explain why the estimates are different.

time	12:00	3:00	6:00	9:00	12:00	3:00	6:00	9:00	12:00
temperature	46	44	52	70	82	86	80	72	56

60. In exercise 59, describe the average of the estimates in parts (a) and (b) in terms of the "usual" way of averaging, that is, adding up some numbers and dividing by how many numbers were added.

Exercises 61–64 involve the just-in-time inventory discussed in the chapter introduction.

61. For a business using just-in-time inventory, a delivery of Q items arrives just as the last item is shipped out. Suppose that items are shipped out at the constant rate of r items per day. If a delivery arrives at time 0, show that $f(t) = Q - rt$ gives the number of items in inventory for $0 \le t \le \frac{Q}{r}$. Find the average value of f on the interval $\left[0, \frac{Q}{r}\right]$.

62. The **Economic Order Quantity (EOQ)** model uses the assumptions in exercise 61 to determine the optimal quantity Q to order at any given time. Assume that D items are ordered annually, so that the number of shipments equals $\frac{D}{Q}$. If C_o is the cost of placing an order and C_c is the annual cost for storing an item in inventory, then the total annual cost is given by $f(Q) = C_o\frac{D}{Q} + C_c\frac{Q}{2}$. Find the value of Q that minimizes the total cost. For the optimal order size, show that the total ordering cost $C_o\frac{D}{Q}$ equals the total carrying cost (for storage) $C_c\frac{Q}{2}$.

63. The EOQ model of exercise 62 can be modified to take into account noninstantaneous receipt. In this case, instead of a full delivery arriving at one instant, the delivery arrives at a rate of p items per day. Assume that a delivery of size Q starts at time 0, with shipments out continuing at the rate of r items per day (assume that $p > r$). Show that when the delivery is completed, the inventory equals $Q(1 - r/p)$. From there, inventory drops at a steady rate of r items per day until no items are left. Show that the average inventory equals $\frac{1}{2}Q(1 - r/p)$ and find the order size Q that minimizes the total cost.

64. A further refinement we can make to the EOQ model of exercises 62–63 is to allow discounts for ordering large quantities. To make the calculations easier, take specific values of $D = 4000$, $C_o = \$50{,}000$ and $C_c = \$3800$. If 1–99 items are ordered, the price is \$2800 per item. If 100–179 items are ordered, the price is \$2200 per item. If 180 or more items are ordered, the price is \$1800 per item. The total cost is now $C_o\frac{D}{Q} + C_c\frac{Q}{2} + PD$, where P is the price per item. Find the order size Q that minimizes the total cost.

65. The *impulse-momentum equation* states the relationship between a force $F(t)$ applied to an object of mass m and the resulting change in velocity Δv of the object. The equation is $m\Delta v = \int_a^b F(t)\,dt$, where $\Delta v = v(b) - v(a)$. Suppose that the force of a baseball bat on a ball is approximately $F(t) = 9 - 10^8(t - 0.0003)^2$ thousand pounds, for t between 0 and 0.0006 second. What is the maximum force on the ball? Using $m = 0.01$ for the mass of a baseball, estimate the change in velocity Δv (in ft/s).

66. Measurements taken of the feet of badminton players lunging for a shot indicate a vertical force of approximately $F(t) = 1000 - 25{,}000(t - 0.2)^2$ Newtons, for t between 0 and 0.4 second (see *The Science of Racquet Sports*). For a player of mass $m = 5$, use the impulse-momentum equation in exercise 65 to estimate the change in vertical velocity of the player.

67. Use a graph to explain why $\int_{-1}^1 x^3 dx = 0$. Use your knowledge of e^{-x} to determine whether $\int_{-1}^1 x^3 e^{-x} dx$ is positive or negative.

68. Use the Integral Mean Value Theorem to prove the following fact for a continuous function. For any positive integer n, there exists a set of evaluation points for which the Riemann sum approximation of $\int_a^b f(x)\,dx$ is exact.

EXPLORATORY EXERCISES

1. Many of the basic quantities used by epidemiologists to study the spread of disease are described by integrals. In the case of AIDS, a person becomes infected with the HIV virus and, after an incubation period, develops AIDS. Our goal is to derive a formula for the number of AIDS cases given the HIV infection rate $g(t)$ and the incubation distribution $F(t)$. To take a simple case, suppose that the infection rate the first month is 20 people per month, the infection rate the second month is 30 people per month and the infection rate the third month is 25 people per month. Then $g(1) = 20$, $g(2) = 30$ and $g(3) = 25$. Also, suppose that 20% of those infected develop AIDS after 1 month, 50% develop AIDS after 2 months and 30% develop AIDS after 3 months (fortunately, these figures are not at all realistic). Then $F(1) = 0.2$, $F(2) = 0.5$ and $F(3) = 0.3$. Explain why the number of people developing AIDS in the fourth month would be $g(1)F(3) + g(2)F(2) + g(3)F(1)$. Compute this number. Next, suppose that $g(0.5) = 16$, $g(1) = 20$, $g(1.5) = 26$, $g(2) = 30$, $g(2.5) = 28$, $g(3) = 25$ and $g(3.5) = 22$. Further, suppose that $F(0.5) = 0.1$, $F(1) = 0.1$, $F(1.5) = 0.2$, $F(2) = 0.3$, $F(2.5) = 0.1$, $F(3) = 0.1$ and $F(3.5) = 0.1$. Compute the number of people developing AIDS in the fourth month. If we have $g(t)$ and $F(t)$ defined at all real numbers t, explain why the number of people developing AIDS in the fourth month equals $\int_0^4 g(t)F(4-t)\,dt$.

2. **Riemann's condition** states that $\int_a^b f(x)\,dx$ exists if and only if for every $\epsilon > 0$ there exists a partition P such that the upper sum U and lower sum L (see exercises 35–40 in section 4.3) satisfy $|U - L| < \epsilon$. Use this condition to prove that $f(x) = \begin{cases} -1 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is not integrable on the interval $[0, 1]$. A function f is called a **Lipschitz** function on the interval $[a, b]$ if $|f(x) - f(y)| \le |x - y|$ for all x and y in $[a, b]$. Use Riemann's condition to prove that every Lipschitz function on $[a, b]$ is integrable on $[a, b]$.

4.5 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section, we present a pair of results known collectively as the Fundamental Theorem of Calculus. On a practical level, the Fundamental Theorem provides us with a much-needed shortcut for computing definite integrals without struggling to find limits of Riemann sums. On a conceptual level, the Fundamental Theorem unifies the seemingly disconnected studies of derivatives and definite integrals, showing us that differentiation and integration are, in fact, inverse processes. In this sense, the theorem is truly *fundamental* to calculus as a coherent discipline.

One hint as to the nature of the first part of the Fundamental Theorem is that we used suspiciously similar notations for indefinite and definite integrals. We have also used both antidifferentiation and area calculations to compute distance from velocity. However, the Fundamental Theorem makes much stronger statements about the relationship between differentiation and integration.

NOTES

The Fundamental Theorem, Part 1, says that to compute a definite integral, we need only find an antiderivative and then evaluate it at the two limits of integration. Observe that this is a vast improvement over computing limits of Riemann sums, which we could compute exactly for only a few simple cases.

THEOREM 5.1 (Fundamental Theorem of Calculus, Part I)

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.1}$$

PROOF

First, we partition $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

HISTORICAL NOTES

The Fundamental Theorem of Calculus marks the beginning of calculus as a unified discipline and is credited to both Isaac Newton and Gottfried Leibniz. Newton developed his calculus in the late 1660s but did not publish his results until 1687. Leibniz rediscovered the same results in the mid-1670s but published before Newton in 1684 and 1686. Leibniz' original notation and terminology, much of which is in use today, is superior to Newton's (Newton referred to derivatives and integrals as *fluxions* and *fluents*), but Newton developed the central ideas earlier than Leibniz. A bitter controversy, centering on some letters from Newton to Leibniz in the 1670s, developed over which man would receive credit for inventing the calculus. The dispute evolved into a battle between England and the rest of the European mathematical community. Communication between the two groups ceased for over 100 years and greatly influenced the development of mathematics in the 1700s.

where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for $i = 1, 2, \ldots, n$. Working backward, note that by virtue of all the cancellations, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \cdots + [F(x_n) - F(x_{n-1})] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned} \tag{5.2}$$

Since F is an antiderivative of f, F is differentiable on (a, b) and continuous on $[a, b]$. By the Mean Value Theorem, we then have for each $i = 1, 2, \ldots, n$, that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x, \tag{5.3}$$

for some $c_i \in (x_{i-1}, x_i)$. Thus, from (5.2) and (5.3), we have

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\,\Delta x. \tag{5.4}$$

You should recognize this last expression as a Riemann sum for f on $[a, b]$. Taking the limit of both sides of (5.4) as $n \to \infty$, we find that

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} [F(b) - F(a)] \\ &= F(b) - F(a), \end{aligned}$$

as desired, since this last quantity is a constant. ■

REMARK 5.1

We will often use the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

This enables us to write down the antiderivative before evaluating it at the endpoints.

EXAMPLE 5.1 Using the Fundamental Theorem

Compute $\int_0^2 (x^2 - 2x)\,dx$.

Solution Notice that $f(x) = x^2 - 2x$ is continuous on the interval $[0, 2]$ and so, we can apply the Fundamental Theorem. We find an antiderivative from the power rule and simply evaluate:

$$\int_0^2 (x^2 - 2x)\,dx = \left(\frac{1}{3}x^3 - x^2\right)\Bigg|_0^2 = \left(\frac{8}{3} - 4\right) - (0) = -\frac{4}{3}.$$ ■

Recall that we had already evaluated the integral in example 5.1 by computing the limit of Riemann sums (see example 4.3). Given a choice, which method would you prefer?

While you had a choice in example 5.1, you *cannot* evaluate the integrals in examples 5.2–5.5 by computing the limit of a Riemann sum directly, as we have no formulas for the summations involved.

EXAMPLE 5.2 Computing a Definite Integral Exactly

Compute $\displaystyle\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx.$

Solution Observe that since $f(x) = x^{1/2} - x^{-2}$ is continuous on $[1, 4]$, we can apply the Fundamental Theorem. Since an antiderivative of $f(x)$ is $F(x) = \frac{2}{3}x^{3/2} + x^{-1}$, we have

$$\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx = \frac{2}{3}x^{3/2} + x^{-1}\bigg|_1^4 = \left[\frac{2}{3}(4)^{3/2} + 4^{-1}\right] - \left(\frac{2}{3} + 1\right) = \frac{47}{12}.$$ ■

TODAY IN MATHEMATICS

Benoit Mandelbrot (1924–)
A French mathematician who invented and developed fractal geometry (see the Mandelbrot set in the exercises for section 9.1). Mandelbrot has always been guided by a strong geometric intuition. He explains, "Faced with some complicated integral, I instantly related it to a familiar shape. . . . I knew an army of shapes I'd encountered once in some book and remembered forever, with their properties and their peculiarities." The fractal geometry that Mandelbrot developed has greatly extended our ability to accurately describe the peculiarities of such phenomena as the structure of the lungs and heart, or mountains and clouds, as well as the stock market and weather.

EXAMPLE 5.3 Using the Fundamental Theorem to Compute Areas

Find the area under the curve $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution Since $\sin x \geq 0$ and $\sin x$ is continuous on $[0, \pi]$, we have that

$$\text{Area} = \int_0^\pi \sin x \, dx.$$

Notice that an antiderivative of $\sin x$ is $F(x) = -\cos x$. By the Fundamental Theorem, then, we have

$$\int_0^\pi \sin x \, dx = F(\pi) - F(0) = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2.$$ ■

EXAMPLE 5.4 A Definite Integral Involving an Exponential Function

Compute $\displaystyle\int_0^4 e^{-2x} dx.$

Solution Since $f(x) = e^{-2x}$ is continuous, we can apply the Fundamental Theorem. Notice that an antiderivative of e^{-2x} is $-\frac{1}{2}e^{-2x}$, so that

$$\int_0^4 e^{-2x} dx = -\frac{1}{2}e^{-2x}\bigg|_0^4 = -\frac{1}{2}e^{-8} - \left(-\frac{1}{2}e^0\right) \approx 0.49983.$$

Note that the exact value of the integral is $-\frac{1}{2}e^{-8} + \frac{1}{2}$, while 0.49983 is simply a decimal approximation. ■

EXAMPLE 5.5 A Definite Integral Involving a Logarithm

Evaluate $\displaystyle\int_{-3}^{-1} \frac{2}{x} dx.$

Solution Since $f(x) = \dfrac{2}{x}$ is continuous on $[-3, -1]$, we can apply the Fundamental Theorem. First, recall that an antiderivative for $f(x)$ is $2\ln|x|$. (It's a common error to leave off the absolute values. In this case, the error is fatal! Look carefully at the following to see why.)

$$\begin{aligned}\int_{-3}^{-1} \frac{2}{x} dx &= 2\ln|x|\bigg|_{-3}^{-1} = 2(\ln|-1| - \ln|-3|) \\ &= 2(\ln 1 - \ln 3) = -2\ln 3.\end{aligned}$$ ■

EXAMPLE 5.6 A Definite Integral with a Variable Upper Limit

Evaluate $\int_1^x 12t^5\,dt$.

Solution Even though the upper limit of integration is a variable, we can use the Fundamental Theorem to evaluate this, since $f(t) = 12t^5$ is continuous on any interval. We have

$$\int_1^x 12t^5\,dt = 12\frac{t^6}{6}\bigg|_1^x = 2(x^6 - 1).$$

It's not surprising that the definite integral in example 5.6 is a function of x, since one of the limits of integration involves x. The following observation may be surprising, though. Note that

$$\frac{d}{dx}[2(x^6 - 1)] = 12x^5,$$

which is the same as the original integrand, except that the (dummy) variable of integration, t, has been replaced by the variable in the upper limit of integration, x.

The seemingly odd coincidence observed here is, in fact, not an isolated occurrence, as we see in Theorem 5.2. First, you need to be clear about what a function such as $F(x) = \int_1^x 12t^5\,dt$ means. Notice that the function value at $x = 2$ is found by replacing x by 2:

$$F(2) = \int_1^2 12t^5\,dt,$$

which corresponds to the area under the curve $y = 12t^5$ from $t = 1$ to $t = 2$ (see Figure 4.28a). Similarly, the function value at $x = 3$ is

$$F(3) = \int_1^3 12t^5\,dt,$$

which is the area under the curve $y = 12t^5$ from $t = 1$ to $t = 3$ (see Figure 4.28b). More generally, for any $x > 1$, $F(x)$ gives the area under the curve $y = 12t^5$ from $t = 1$ up to $t = x$ (see Figure 4.28c). For this reason, the function F is sometimes called an **area function.** Notice that for $x > 1$, as x increases, $F(x)$ gives more and more of the area under the curve to the right of $t = 1$.

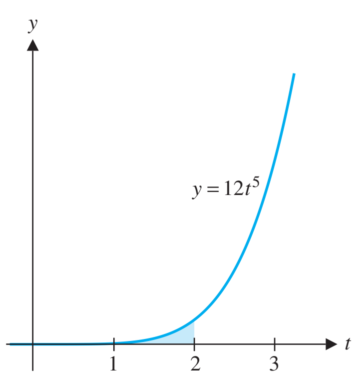


FIGURE 4.28a
Area from $t = 1$ to $t = 2$

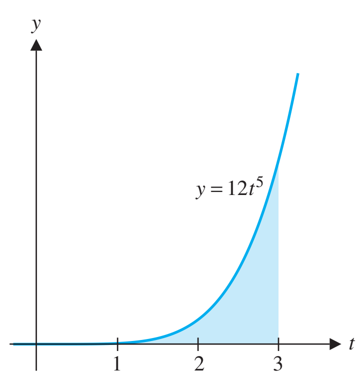


FIGURE 4.28b
Area from $t = 1$ to $t = 3$

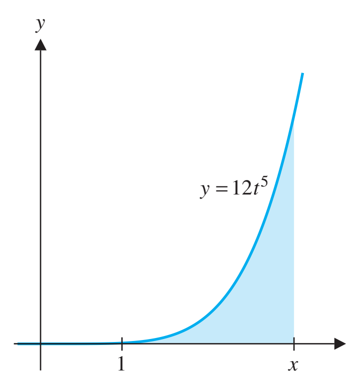


FIGURE 4.28c
Area from $t = 1$ to $t = x$

THEOREM 5.2 (Fundamental Theorem of Calculus, Part II)

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$, on $[a, b]$.

PROOF

Using the definition of derivative, we have

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt, \tag{5.5}$$

where we switched the limits of integration according to equation (4.2) and combined the integrals according to Theorem 4.2 (ii).

Look very carefully at the last term in (5.5). You may recognize it as the limit of the average value of $f(t)$ on the interval $[x, x+h]$ (if $h > 0$). By the Integral Mean Value Theorem (Theorem 4.4), we have

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c), \tag{5.6}$$

for some number c between x and $x + h$. Finally, since c is between x and $x + h$, we have that $c \to x$, as $h \to 0$. Since f is continuous, it follows from (5.5) and (5.6) that

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x),$$

as desired. ■

REMARK 5.2

Part II of the Fundamental Theorem says that *every* continuous function f has an antiderivative, namely, $\int_a^x f(t)\,dt$.

EXAMPLE 5.7 Using the Fundamental Theorem, Part II

For $F(x) = \int_1^x (t^2 - 2t + 3)\,dt$, compute $F'(x)$.

Solution Here, the integrand is $f(t) = t^2 - 2t + 3$. By Theorem 5.2, the derivative is

$$F'(x) = f(x) = x^2 - 2x + 3.$$

That is, $F'(x)$ is the function in the integrand with t replaced by x. It really is that easy! ■

Before moving on to more complicated examples, let's look at example 5.7 in more detail, just to get more comfortable with the meaning of Part II of the Fundamental Theorem. First, we can use Part I of the Fundamental Theorem to find

$$F(x) = \int_1^x (t^2 - 2t + 3)\,dt = \frac{1}{3}t^3 - t^2 + 3t\Big|_1^x = \left(\frac{1}{3}x^3 - x^2 + 3x\right) - \left(\frac{1}{3} - 1 + 3\right).$$

It's easy to differentiate this directly, to get

$$F'(x) = \frac{1}{3}\cdot 3x^2 - 2x + 3 - 0 = x^2 - 2x + 3.$$

Notice that the lower limit of integration (in this case, 1) has no effect on the value of $F'(x)$. In the definition of $F(x)$, the lower limit of integration merely determines the value of the constant that is subtracted at the end of the calculation of $F(x)$. Since the derivative of any constant is 0, this value does not affect $F'(x)$.

REMARK 5.3

The general form of the chain rule used in example 5.8 is: if $g(x) = \int_a^{u(x)} f(t)\,dt$, then $g'(x) = f(u(x))u'(x)$ or

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))u'(x).$$

EXAMPLE 5.8 Using the Chain Rule and the Fundamental Theorem, Part II

If $F(x) = \int_2^{x^2} \cos t\,dt$, compute $F'(x)$.

Solution Let $u(x) = x^2$, so that

$$F(x) = \int_2^{u(x)} \cos t\,dt.$$

From the chain rule,

$$F'(x) = \cos u(x)\frac{du}{dx} = \cos u(x)(2x) = 2x\cos x^2. \blacksquare$$

EXAMPLE 5.9 An Integral with Variable Upper and Lower Limits

If $F(x) = \int_{2x}^{x^2} \sqrt{t^2+1}\,dt$, compute $F'(x)$.

Solution The Fundamental Theorem applies only to definite integrals with variables in the upper limit, so we will first rewrite the integral by Theorem 4.2 (ii) as

$$F(x) = \int_{2x}^{0} \sqrt{t^2+1}\,dt + \int_0^{x^2} \sqrt{t^2+1}\,dt = -\int_0^{2x} \sqrt{t^2+1}\,dt + \int_0^{x^2} \sqrt{t^2+1}\,dt,$$

where we have also used equation (4.2) to switch the limits of integration in the first integral. Using the chain rule as in example 5.8, we get

$$\begin{aligned} F'(x) &= -\sqrt{(2x)^2+1}\,\frac{d}{dx}(2x) + \sqrt{(x^2)^2+1}\,\frac{d}{dx}(x^2) \\ &= -2\sqrt{4x^2+1} + 2x\sqrt{x^4+1}. \blacksquare \end{aligned}$$

Before discussing the theoretical significance of the two parts of the Fundamental Theorem, we present two examples that remind you of why you might want to compute integrals and derivatives.

EXAMPLE 5.10 Computing the Distance Fallen by an Object

Suppose the (downward) velocity of a sky diver is given by $v(t) = 30(1 - e^{-t})$ ft/s for the first 5 seconds of a jump. Compute the distance fallen.

Solution Recall that the distance d is given by the definite integral

$$\begin{aligned} d &= \int_0^5 (30 - 30e^{-t})\,dt = 30t + 30e^{-t}\Big|_0^5 \\ &= (150 + 30e^{-5}) - (0 + 30e^0) = 120 + 30e^{-5} \approx 120.2 \text{ feet}. \blacksquare \end{aligned}$$

Recall that velocity is the *instantaneous rate of change* of the distance function with respect to time. We see in example 5.10 that the definite integral of velocity gives the *total change* of the distance function over the given time interval. A similar interpretation of derivative and the definite integral holds for many quantities of interest. In example 5.11, we look at the rate of change and total change of water in a tank.

EXAMPLE 5.11 Rate of Change and Total Change of Volume of a Tank

Suppose that water flows in and out of a storage tank. The net rate of change (that is, the rate in minus the rate out) of water is $f(t) = 20(t^2 - 1)$ gallons per minute. (a) For $0 \le t \le 3$, determine when the water level is increasing and when the water level is decreasing. (b) If the tank has 200 gallons of water at time $t = 0$, determine how many gallons are in the tank at time $t = 3$.

Solution Let $w(t)$ be the number of gallons in the tank at time t. (a) Notice that the water level decreases if $w'(t) = f(t) < 0$. We have

$$f(t) = 20(t^2 - 1) < 0, \quad \text{if } 0 \le t < 1.$$

Alternatively, the water level increases if $w'(t) = f(t) > 0$. In this case, we have

$$f(t) = 20(t^2 - 1) > 0, \quad \text{if } 1 < t \le 3.$$

(b) We start with $w'(t) = 20(t^2 - 1)$. Integrating from $t = 0$ to $t = 3$, we have

$$\int_0^3 w'(t)\,dt = \int_0^3 20(t^2 - 1)\,dt$$

Evaluating the integrals on both sides yields

$$w(3) - w(0) = 20\left(\frac{t^3}{3} - t\right)\Bigg|_{t=0}^{t=3}.$$

Since $w(0) = 200$, we have

$$w(3) - 200 = 20(9 - 3) = 120$$

and hence,

$$w(3) = 200 + 120 = 320,$$

so that the tank will have 320 gallons at time 3. ■

In example 5.12, we use Part II of the Fundamental Theorem to determine information about a seemingly complicated function. Notice that although we don't know how to evaluate the integral, we can use the Fundamental Theorem to obtain some important information about the function.

EXAMPLE 5.12 Finding a Tangent Line for a Function Defined as an Integral

For the function $F(x) = \int_4^{x^2} \ln(t^3 + 4)\,dt$, find an equation of the tangent line at $x = 2$.

Solution Notice that there are almost no function values that we can compute exactly, yet we can easily find an equation of a tangent line! From Part II of the Fundamental Theorem and the chain rule, we get the derivative

$$F'(x) = \ln[(x^2)^3 + 4]\frac{d}{dx}(x^2) = \ln[(x^2)^3 + 4](2x) = 2x\ln(x^6 + 4).$$

So, the slope at $x = 2$ is $F'(2) = 4\ln(68) \approx 16.878$. The tangent passes through the point with $x = 2$ and $y = F(2) = \int_4^4 \ln(t^3 + 4)\,dt = 0$ (since the upper limit equals the lower limit). An equation of the tangent line is then

$$y = 4\ln 68(x - 2). \quad ■$$

BEYOND FORMULAS

The two parts of the Fundamental Theorem are different sides of the same theoretical coin. Recall the conclusions of Parts I and II of the Fundamental Theorem:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{and} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

In both cases, we are saying that differentiation and integration are in some sense *inverse operations:* their effects (with appropriate hypotheses) cancel each other out. This fundamental connection is what unifies seemingly unrelated calculation techniques into *the calculus*. What are some results in algebra and trigonometry that similarly tie together different areas of study and are thus fundamental results?

EXERCISES 4.5

WRITING EXERCISES

1. To explore Part I of the Fundamental Theorem graphically, first suppose that $F(x)$ is increasing on the interval $[a, b]$. Explain why both of the expressions $F(b) - F(a)$ and $\int_a^b F'(x)\,dx$ will be positive. Further, explain why the faster $F(x)$ increases, the larger each expression will be. Similarly, explain why if $F(x)$ is decreasing, both expressions will be negative.

2. You can think of Part I of the Fundamental Theorem in terms of position $s(t)$ and velocity $v(t) = s'(t)$. Start by assuming that $v(t) \geq 0$. Explain why $\int_a^b v(t)\,dt$ gives the total distance traveled and explain why this equals $s(b) - s(a)$. Discuss what changes if $v(t) < 0$.

3. To explore Part II of the Fundamental Theorem graphically, consider the function $g(x) = \int_a^x f(t)\,dt$. If $f(t)$ is positive on the interval $[a, b]$, explain why $g'(x)$ will also be positive. Further, the larger $f(t)$ is, the larger $g'(x)$ will be. Similarly, explain why if $f(t)$ is negative then $g'(x)$ will also be negative.

4. In Part I of the Fundamental Theorem, F can be *any* antiderivative of f. Recall that any two antiderivatives of f differ by a constant. Explain why $F(b) - F(a)$ is **well defined;** that is, if F_1 and F_2 are different antiderivatives, explain why $F_1(b) - F_1(a) = F_2(b) - F_2(a)$. When evaluating a definite integral, explain why you do not need to include "$+\,c$" with the antiderivative.

In exercises 1–20, use Part I of the Fundamental Theorem to compute each integral exactly.

1. $\int_0^2 (2x - 3)\,dx$
2. $\int_0^3 (x^2 - 2)\,dx$
3. $\int_{-1}^1 (x^3 + 2x)\,dx$
4. $\int_0^2 (x^3 + 3x - 1)\,dx$
5. $\int_0^4 (\sqrt{x} + 3x)\,dx$
6. $\int_1^2 (4x - 2/x^2)\,dx$
7. $\int_0^1 (x\sqrt{x} + x^{1/3})\,dx$
8. $\int_0^8 (\sqrt[3]{x} - x^{2/3})\,dx$
9. $\int_0^{\pi/4} \sec x \tan x\,dx$
10. $\int_0^{\pi/4} \sec^2 x\,dx$
11. $\int_{\pi/2}^{\pi} (2\sin x - \cos x)\,dx$
12. $\int_0^1 (e^x - e^{-x})\,dx$
13. $\int_0^{1/2} \frac{3}{\sqrt{1 - x^2}}\,dx$
14. $\int_{-1}^1 \frac{4}{1 + x^2}\,dx$
15. $\int_1^4 \frac{x - 3}{x}\,dx$
16. $\int_1^2 \frac{x^2 - 3x + 4}{x^2}\,dx$
17. $\int_0^4 x(x - 2)\,dx$
18. $\int_0^{\pi/3} \frac{3}{\cos^2 x}\,dx$
19. $\int_0^{\ln 2} (e^{x/2})^2\,dx$
20. $\int_0^{\pi} (\sin^2 x + \cos^2 x)\,dx$

In exercises 21–26, use the Fundamental Theorem if possible or estimate the integral using Riemann sums. (Hint: Three problems can be worked using antiderivative formulas we have covered so far.)

21. $\int_0^2 \sqrt{x^2 + 1}\,dx$
22. $\int_0^2 (\sqrt{x} + 1)^2 dx$
23. $\int_1^4 \frac{x^2}{x^2 + 4}\,dx$
24. $\int_1^4 \frac{x^2 + 4}{x^2}\,dx$
25. $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$
26. $\int_0^{\pi/4} \frac{\tan x}{\sec^2 x}\,dx$

In exercises 27–32, find the derivative $f'(x)$.

27. $f(x) = \int_0^x (t^2 - 3t + 2)\,dt$
28. $f(x) = \int_2^x (t^2 - 3t - 4)\,dt$

29. $f(x) = \int_0^{x^2} (e^{-t^2} + 1)\, dt$

30. $f(x) = \int_2^{x^2+1} \sin t\, dt$

31. $f(x) = \int_x^{-1} \ln(t^2 + 1)\, dt$

32. $f(x) = \int_x^2 \sec t\, dt$

In exercises 33–36, find an equation of the tangent line at the given value of x.

33. $y = \int_0^x \sin\sqrt{t^2 + \pi^2}\, dt,\ x = 0$

34. $y = \int_{-1}^x \ln(t^2 + 2t + 2)\, dt,\ x = -1$

35. $y = \int_2^x \cos(\pi t^3)\, dt,\ x = 2$

36. $y = \int_0^x e^{-t^2+1}\, dt,\ x = 0$

37. Identify all local extrema of $f(x) = \int_0^x (t^2 - 3t + 2)\, dt$.

38. Katie drives a car at speed $f(t) = 55 + 10\cos t$ mph, and Michael drives a car at speed $g(t) = 50 + 2t$ mph at time t minutes. Suppose that Katie and Michael are at the same location at time $t = 0$. Compute $\int_0^x [f(t) - g(t)]\, dt$, and interpret the integral in terms of a race between Katie and Michael.

In exercises 39–44, find the given area.

39. The area above the x-axis and below $y = 4 - x^2$

40. The area below the x-axis and above $y = x^2 - 4x$

41. The area of the region bounded by $y = x^2$, $x = 2$ and the x-axis

42. The area of the region bounded by $y = x^3$, $x = 3$ and the x-axis

43. The area between $y = \sin x$ and the x-axis for $0 \le x \le \pi$

44. The area between $y = \sin x$ and the x-axis for $-\pi/2 \le x \le \pi/4$

In exercises 45 and 46, (a) explain how you know the proposed integral value is wrong and (b) find all mistakes.

45. $\int_{-1}^1 \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{x=-1}^{x=1} = -1 - (1) = -2$

46. $\int_0^\pi \sec^2 x\, dx = \tan x\Big|_{x=0}^{x=\pi} = \tan\pi - \tan 0 = 0$

In exercises 47–50, find the position function $s(t)$ from the given velocity or acceleration function and initial value(s). Assume that units are feet and seconds.

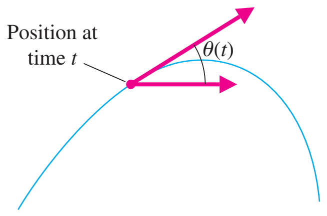
47. $v(t) = 40 - \sin t,\ s(0) = 2$

48. $v(t) = 10e^{-t},\ s(0) = 2$

49. $a(t) = 4 - t,\ v(0) = 8,\ s(0) = 0$

50. $a(t) = 16 - t^2,\ v(0) = 0,\ s(0) = 30$

51. If $\theta(t)$ is the angle between the path of a moving object and a fixed ray (see the figure), the **angular velocity** of the object is $\omega(t) = \theta'(t)$ and the **angular acceleration** of the object is $\alpha(t) = \omega'(t)$.



Suppose a baseball batter swings with a constant angular acceleration of $\alpha(t) = 10$ rad/s^2. If the batter hits the ball 0.8 s later, what is the angular velocity? The (linear) speed of the part of the bat located 3 feet from the pivot point (the batter's body) is $v = 3\omega$. How fast is this part of the bat moving at the moment of contact? Through what angle was the bat rotated during the swing?

52. Suppose a golfer rotates a golf club through an angle of $3\pi/2$ with a constant angular acceleration of α rad/s^2. If the clubhead is located 4 feet from the pivot point (the golfer's body), the (linear) speed of the clubhead is $v = 4\omega$. Find the value of α that will produce a clubhead speed of 100 mph at impact.

In exercises 53–58, find the average value of the function on the given interval.

53. $f(x) = x^2 - 1,\ [1, 3]$

54. $f(x) = x^2 + 2x,\ [0, 1]$

55. $f(x) = 2x - 2x^2,\ [0, 1]$

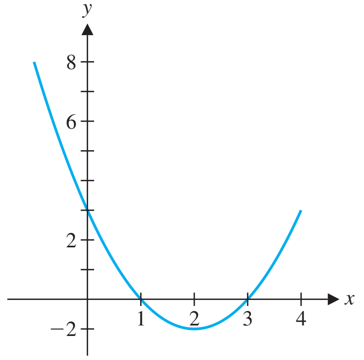
56. $f(x) = x^3 - 3x^2 + 2x,\ [1, 2]$

57. $f(x) = \cos x,\ [0, \pi/2]$

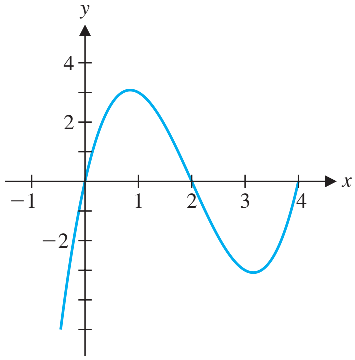
58. $f(x) = \sin x,\ [0, \pi/2]$

In exercises 59 and 60, use the graph to list $\int_0^1 f(x)\,dx$, $\int_0^2 f(x)\,dx$ and $\int_0^3 f(x)\,dx$ in order, from smallest to largest.

59.



60.



61. Use the Fundamental Theorem of Calculus to find an antiderivative of e^{-x^2}.

62. Use the Fundamental Theorem of Calculus to find an antiderivative of $\sin\sqrt{x^2+1}$.

63. The number of items that consumers are willing to buy depends on the price of the item. Let $p = D(q)$ represent the price (in dollars) at which q items can be sold. The integral $\int_0^Q D(q)\,dq$ is interpreted as the total number of dollars that consumers would be willing to spend on Q items. If the price is fixed at $P = D(Q)$ dollars, then the actual amount of money spent is PQ. The **consumer surplus** is defined by $CS = \int_0^Q D(q)\,dq - PQ$. Compute the consumer surplus for $D(q) = 150 - 2q - 3q^2$ at $Q = 4$ and at $Q = 6$. What does the difference in CS values tell you about how many items to produce?

64. Repeat exercise 63 for $D(q) = 40e^{-0.05q}$ at $Q = 10$ and $Q = 20$.

65. For a business using just-in-time inventory, a delivery of Q items arrives just as the last item is shipped out. Suppose that items are shipped out at a nonconstant rate such that $f(t) = Q - r\sqrt{t}$ gives the number of items in inventory. Find the time T at which the next shipment must arrive. Find the average value of f on the interval $[0, T]$.

66. The Economic Order Quantity (EOQ) model uses the assumptions in exercise 65 to determine the optimal quantity Q to order at any given time. If C_o is the cost of placing an order, C_c is the annual cost for storing an item in inventory and A is the average value from exercise 65, then the total annual cost is given by $f(Q) = C_0\frac{D}{Q} + C_cA$. Find the value of Q that minimizes the total cost. Show that for this order size, the total ordering cost $C_0\frac{D}{Q}$ equals the total carrying cost (for storage) C_cA.

67. Let $f(x) = \begin{cases} x & \text{if } x < 2 \\ x+1 & \text{if } x \geq 2 \end{cases}$ and define $F(x) = \int_0^x f(t)\,dt$. Show that $F(x)$ is continuous but that it is not true that $F'(x) = f(x)$ for all x. Explain why this does not contradict the Fundamental Theorem of Calculus.

68. Find the derivative of $f(x) = \frac{1}{k}\int_x^{x+k} g(t)\,dt$, where g is a continuous function.

69. Find the first and second derivatives of $g(x) = \int_0^x \left(\int_0^u f(t)\,dt\right) du$, where f is a continuous function. Identify the graphical feature of $y = g(x)$ that corresponds to a zero of $f(x)$.

70. Let f be a continuous function on the interval [0, 1], and define $g_n(x) = f(x^n)$ for $n = 1, 2$ and so on. For a given x with $0 \leq x \leq 1$, find $\lim_{n\to\infty} g_n(x)$. Then, find $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx$.

In exercises 71 and 72, identify the integrals to which the Fundamental Theorem of Calculus applies; the other integrals are called improper integrals.

71. (a) $\displaystyle\int_0^4 \frac{1}{x-4}\,dx$ (b) $\displaystyle\int_0^1 \sqrt{x}\,dx$ (c) $\displaystyle\int_0^1 \ln x\,dx$

72. (a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x+2}}\,dx$ (b) $\displaystyle\int_0^2 \frac{1}{(x-3)^2}\,dx$ (c) $\displaystyle\int_0^2 \sec x\,dx$

In exercises 73 and 74, identify each sum as a Riemann sum and evaluate the limit.

73. (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\pi\right]$

(b) $\displaystyle\lim_{n\to\infty} \frac{2}{n}\left[\frac{1}{1+2/n} + \frac{1}{1+4/n} + \cdots + \frac{1}{3}\right]$

74. (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left[e^{4/n} + e^{8/n} + \cdots + e^4\right]$

(b) $\displaystyle\lim_{n\to\infty} \frac{4}{n}\left[\frac{2}{\sqrt{n}} + \frac{2\sqrt{2}}{\sqrt{n}} + \cdots + 2\right]$

75. Derive **Leibniz' Rule:**

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\,b'(x) - f(a(x))\,a'(x).$$

EXPLORATORY EXERCISES

1. Suppose that a communicable disease has an infection stage and an incubation stage (like HIV and AIDS). Assume that the infection rate is a constant $f(t) = 100$ people per month and the incubation distribution is $b(t) = \frac{1}{900}te^{-t/30}$ month^{-1}. The rate at which people develop the disease at time $t = T$ is given by $r(T) = \int_0^T f(t)b(T-t)\,dt$ people per month. Use your CAS to find expressions for both the rate $r(T)$ and the number of people $p(x) = \int_0^x r(T)\,dT$ who develop the disease between times $t = 0$ and $t = x$. Explain why the graph $y = r(T)$ has a horizontal asymptote. For small x's, the graph of $y = p(x)$ is concave up; explain what happens for large x's. Repeat this for $f(t) = 100 + 10\sin t$, where the infection rate oscillates up and down.

2. When solving differential equations of the form $\frac{dy}{dt} = f(y)$ for the unknown function $y(t)$, it is often convenient to make use of a **potential function** $V(y)$. This is a function such that $-\frac{dV}{dy} = f(y)$. For the function $f(y) = y - y^3$, find a potential function $V(y)$. Find the locations of the local minima of $V(y)$ and use a graph of $V(y)$ to explain why this is called a "double-well" potential. Explain each step in the calculation

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = -f(y)f(y) \le 0.$$

Since $\frac{dV}{dt} \le 0$, does the function V increase or decrease as time goes on? Use your graph of V to predict the possible values of $\lim_{t\to\infty} y(t)$. Thus, you can predict the limiting value of the solution of the differential equation without ever solving the equation itself. Use this technique to predict $\lim_{t\to\infty} y(t)$ if $y' = 2 - 2y$.

3. Let $f_n(x) = \begin{cases} 2n + 4n^2x & -\frac{1}{2n} \le x \le 0 \\ 2n - 4n^2x & 0 \le x \le \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$

for $n = 1, 2, 3, \ldots$. For an arbitrary n, sketch $y = f_n(x)$ and show that $\int_{-1}^{1} f_n(x)\,dx = 1$. Compute $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\,dx$. For an arbitrary $x \ne 0$ in $[-2, 2]$, compute $\lim_{n\to\infty} f_n(x)$ and compute $\int_{-1}^{1} \lim_{n\to\infty} f_n(x)\,dx$. Is it always true that $\lim_{n\to\infty} \int_{-1}^{1} f_n(x)\,dx = \int_{-1}^{1} \lim_{n\to\infty} f_n(x)\,dx$?

4.6 INTEGRATION BY SUBSTITUTION

In this section, we expand our ability to compute antiderivatives by developing a useful technique called **integration by substitution.** This method gives us a process for helping to recognize a whole range of new antiderivatives.

EXAMPLE 6.1 Finding an Antiderivative by Trial and Error

Evaluate $\int 2xe^{x^2}\,dx$.

Solution We need to find a function $F(x)$ for which $F'(x) = 2xe^{x^2}$. You might be tempted to guess that since x^2 is an antiderivative of $2x$,

$$F(x) = x^2e^{x^2}$$

is an antiderivative of $2xe^{x^2}$. To see that this is incorrect, observe that, from the product rule,

$$\frac{d}{dx}\left(x^2e^{x^2}\right) = 2xe^{x^2} + x^2e^{x^2}(2x) \ne 2xe^{x^2}.$$

So much for our guess. Before making another guess, look closely at the integrand. Notice that $2x$ is the derivative of x^2 and x^2 already appears in the integrand, as the exponent of e^{x^2}. Further, by the chain rule, for $F(x) = e^{x^2}$,

$$F'(x) = e^{x^2}\frac{d}{dx}(x^2) = 2xe^{x^2},$$

which is the integrand. To finish this example, recall that we need to add an arbitrary constant, to get

$$\int 2xe^{x^2}\,dx = e^{x^2} + c. \blacksquare$$

More generally, recognize that when one factor in an integrand is the derivative of another part of the integrand, you may be looking at a chain rule derivative.

Note that, in general, if F is any antiderivative of f, then from the chain rule, we have

$$\frac{d}{dx}[F(u)] = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}.$$

From this, we have that

$$\int f(u)\frac{du}{dx}\,dx = \int \frac{d}{dx}[F(u)]\,dx = F(u) + c = \int f(u)\,du, \tag{6.1}$$

since F is an antiderivative of f. If you read the expressions on the far left and the far right sides of (6.1), this suggests that

$$du = \frac{du}{dx}\,dx.$$

So, if we cannot compute the integral $\int h(x)\,dx$ directly, we often look for a new variable u and function $f(u)$ for which

$$\int h(x)\,dx = \int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du,$$

where the second integral is easier to evaluate than the first.

NOTES

In deciding how to choose a new variable, there are several things to look for:

- terms that are derivatives of other terms (or pieces thereof) and
- terms that are particularly troublesome. (You can often substitute your troubles away.)

EXAMPLE 6.2 Using Substitution to Evaluate an Integral

Evaluate $\int (x^3 + 5)^{100}(3x^2)\,dx$.

Solution You probably cannot evaluate this as it stands. However, observe that

$$\frac{d}{dx}(x^3 + 5) = 3x^2,$$

which is part of the integrand. This leads us to make the substitution $u = x^3 + 5$, so that $du = \frac{d}{dx}(x^3 + 5)\,dx = 3x^2\,dx$. This gives us

$$\int \underbrace{(x^3 + 5)^{100}}_{u^{100}}\underbrace{(3x^2)\,dx}_{du} = \int u^{100}\,du = \frac{u^{101}}{101} + c.$$

We are not done quite yet. Since we invented the new variable u, we need to convert back to the original variable x, to obtain

$$\int (x^3 + 5)^{100}(3x^2)\,dx = \frac{u^{101}}{101} + c = \frac{(x^3 + 5)^{101}}{101} + c.$$

It's always a good idea to perform a quick check on the antiderivative. (Remember that integration and differentiation are inverse processes!) Here, we compute

$$\frac{d}{dx}\left[\frac{(x^3 + 5)^{101}}{101}\right] = \frac{101(x^3 + 5)^{100}(3x^2)}{101} = (x^3 + 5)^{100}(3x^2),$$

which is the original integrand. This confirms that we have indeed found an antiderivative. ■

INTEGRATION BY SUBSTITUTION

Integration by substitution consists of the following general steps, as illustrated in example 6.2.

- **Choose a new variable u:** a common choice is the innermost expression or "inside" term of a composition of functions. (In example 6.2, note that $x^3 + 5$ is the inside term of $(x^3 + 5)^{100}$.)
- **Compute** $du = \dfrac{du}{dx}\,dx$.
- **Replace all terms** in the original integrand with expressions involving u and du.
- **Evaluate** the resulting (u) integral. If you still can't evaluate the integral, you may need to try a different choice of u.
- **Replace each occurrence of u** in the antiderivative with the corresponding expression in x.

Always keep in mind that finding antiderivatives is the reverse process of finding derivatives. In example 6.3, we are not so fortunate as to have the exact derivative we want in the integrand.

EXAMPLE 6.3 Using Substitution: A Power Function Inside a Cosine

Evaluate $\int x \cos x^2\, dx$.

Solution Notice that

$$\frac{d}{dx}x^2 = 2x.$$

Unfortunately, we don't quite have a factor of $2x$ in the integrand. This does not present a problem, though, as you can always push constants back and forth past an integral sign. Notice that we can rewrite the integral as

$$\int x \cos x^2 dx = \frac{1}{2}\int 2x \cos x^2 dx.$$

We now substitute $u = x^2$, so that $du = 2x\ dx$ and we have

$$\int x \cos x^2 dx = \frac{1}{2}\int \underbrace{\cos x^2}_{\cos u}\underbrace{(2x)\,dx}_{du}$$

$$= \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + c = \frac{1}{2}\sin x^2 + c.$$

Again, as a check, observe that

$$\frac{d}{dx}\left(\frac{1}{2}\sin x^2\right) = \frac{1}{2}\cos x^2(2x) = x\cos x^2,$$

which is the original integrand. ■

EXAMPLE 6.4 Using Substitution: A Trigonometric Function Inside a Power

Evaluate $\int (3\sin x + 4)^5 \cos x\ dx$.

Solution As with most integrals, you probably can't evaluate this one as it stands. So, what do you notice about the integrand? Observe that there's a $\sin x$ term and a factor of

$\cos x$ in the integrand and that $\frac{d}{dx}\sin x = \cos x$. Thus, we let $u = 3\sin x + 4$, so that $du = 3\cos x\,dx$. We then have

$$\begin{aligned}\int (3\sin x + 4)^5 \cos x\,dx &= \frac{1}{3}\int \underbrace{(3\sin + 4)^5}_{u^5}\underbrace{(3\cos x)\,dx}_{du} \\ &= \frac{1}{3}\int u^5 du = \left(\frac{1}{3}\right)\frac{u^6}{6} + c \\ &= \frac{1}{18}(3\sin x + 4)^6 + c.\end{aligned}$$

Sometimes you will need to look a bit deeper into an integral to see terms that are derivatives of other terms, as in example 6.5.

EXAMPLE 6.5 Using Substitution: A Root Function Inside a Sine

Evaluate $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

Solution This integral is not especially obvious. It never hurts to try something, though. If you had to substitute for something, what would you choose? You might notice that $\sin\sqrt{x} = \sin x^{1/2}$ and letting $u = \sqrt{x} = x^{1/2}$ (the "inside"), we get $du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$. Since there is a factor of $\frac{1}{\sqrt{x}}\,dx$ in the integrand, we can proceed. We have

$$\begin{aligned}\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx &= 2\int \underbrace{\sin\sqrt{x}}_{\sin u}\underbrace{\left(\frac{1}{2\sqrt{x}}\right)dx}_{du} \\ &= 2\int \sin u\,du = -2\cos u + c = -2\cos\sqrt{x} + c.\end{aligned}$$

EXAMPLE 6.6 Substitution: Where the Numerator Is the Derivative of the Denominator

Evaluate $\displaystyle\int \frac{x^2}{x^3+5}\,dx$.

Solution Since $\frac{d}{dx}(x^3+5) = 3x^2$, we let $u = x^3 + 5$, so that $du = 3x^2dx$. We now have

$$\begin{aligned}\int \frac{x^2}{x^3+5}\,dx &= \frac{1}{3}\int \underbrace{\frac{1}{x^3+5}}_{u}\underbrace{(3x^2)\,dx}_{du} = \frac{1}{3}\int \frac{1}{u}\,du \\ &= \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|x^3+5| + c.\end{aligned}$$

Example 6.6 is an illustration of a very common type of integral, one where the numerator is the derivative of the denominator. More generally, we have the result in Theorem 6.1.

THEOREM 6.1

For any continuous function, f

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c,$$

provided $f(x) \neq 0$.

PROOF

Let $u = f(x)$. Then $du = f'(x)\,dx$ and

$$\int \frac{f'(x)}{f(x)}\,dx = \int \underbrace{\frac{1}{f(x)}}_{u}\,\underbrace{f'(x)\,dx}_{du}$$

$$= \int \frac{1}{u}\,du = \ln|u| + c = \ln|f(x)| + c,$$

as desired. As an alternative to this proof, you might simply compute $\frac{d}{dx}\ln|f(x)|$ directly, to obtain the integrand. ■

You should recall that we already stated this result in section 4.1 (as Corollary 1.2). It is important enough to repeat here in the context of substitution.

EXAMPLE 6.7 An Antiderivative for the Tangent Function

Evaluate $\int \tan x\,dx$.

Solution Note that this is *not* one of our basic integration formulas. However, you might notice that

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \underbrace{\frac{1}{\cos x}}_{u}\,\underbrace{(-\sin x)\,dx}_{du}$$

$$= -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln|\cos x| + c,$$

where we have used the fact that $\frac{d}{dx}(\cos x) = -\sin x$. ■

EXAMPLE 6.8 A Substitution for an Inverse Tangent

Evaluate $\displaystyle\int \frac{(\tan^{-1} x)^2}{1+x^2}\,dx$.

Solution Again, the key is to look for a substitution. Since

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2},$$

we let $u = \tan^{-1} x$, so that $du = \frac{1}{1+x^2}\,dx$. We now have

$$\int \frac{(\tan^{-1} x)^2}{1+x^2}\,dx = \int \underbrace{(\tan^{-1} x)^2}_{u^2} \underbrace{\frac{1}{1+x^2}\,dx}_{du}$$

$$= \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}(\tan^{-1} x)^3 + c.$$

So far, every one of our examples has been solved by spotting a term in the integrand that was the derivative of another term. We present an integral now where this is not the case, but where a substitution is made to deal with a particularly troublesome term in the integrand.

EXAMPLE 6.9 A Substitution That Lets You Expand the Integrand

Evaluate $\int x\sqrt{2-x}\,dx$.

Solution You certainly cannot evaluate this as it stands. If you look for terms that are derivatives of other terms, you will come up empty-handed. The real problem here is that there is a square root of a sum (or difference) in the integrand. A reasonable step would be to substitute for the expression under the square root. We let $u = 2 - x$, so that $du = -dx$. That doesn't seem so bad, but what are we to do with the extra x in the integrand? Well, since $u = 2 - x$, it follows that $x = 2 - u$. Making these substitutions in the integral, we get

$$\int x\sqrt{2-x}\,dx = (-1)\int \underbrace{x}_{2-u}\,\underbrace{\sqrt{2-x}}_{\sqrt{u}}\,\underbrace{(-1)\,dx}_{du}$$

$$= -\int (2-u)\sqrt{u}\,du.$$

While we can't evaluate this integral directly, if we multiply out the terms, we get

$$\int x\sqrt{2-x}\,dx = -\int (2-u)\sqrt{u}\,du$$

$$= -\int (2u^{1/2} - u^{3/2})\,du$$

$$= -2\frac{u^{3/2}}{\left(\frac{3}{2}\right)} + \frac{u^{5/2}}{\left(\frac{5}{2}\right)} + c$$

$$= -\frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c$$

$$= -\frac{4}{3}(2-x)^{3/2} + \frac{2}{5}(2-x)^{5/2} + c.$$

You should check the validity of this antiderivative via differentiation. ■

○ Substitution in Definite Integrals

There is only one slight difference in using substitution for evaluating a definite integral: you must also change the limits of integration to correspond to the new variable. The procedure here is then precisely the same as that used for examples 6.2 through 6.9, except that when you introduce the new variable u, the limits of integration change from $x = a$ and $x = b$ to the corresponding limits for $u : u = u(a)$ and $u = u(b)$. We have

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

EXAMPLE 6.10 Using Substitution in a Definite Integral

Evaluate $\int_1^2 x^3\sqrt{x^4+5}\,dx$.

Solution Of course, you probably can't evaluate this as it stands. However, since $\frac{d}{dx}(x^4+5) = 4x^3$, we make the substitution $u = x^4 + 5$, so that $du = 4x^3 dx$. For the limits of integration, note that when $x = 1$,

$$u = x^4 + 5 = 1^4 + 5 = 6$$

and when $x = 2$,

$$u = x^4 + 5 = 2^4 + 5 = 21.$$

We now have

$$\int_1^2 x^3\sqrt{x^4+5}\,dx = \frac{1}{4}\int_1^2 \underbrace{\sqrt{x^4+5}}_{\sqrt{u}}\,\underbrace{(4x^3)\,dx}_{du} = \frac{1}{4}\int_6^{21}\sqrt{u}\,du$$

$$= \frac{1}{4}\frac{u^{3/2}}{\left(\frac{3}{2}\right)}\bigg|_6^{21} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)(21^{3/2} - 6^{3/2}).$$

Notice that because we changed the limits of integration to match the new variable, we did *not* need to convert back to the original variable, as we do when we make a substitution in an indefinite integral. (Note that, if we had switched the variables back, we would also have needed to switch the limits of integration back to their original values before evaluating!) ■

CAUTION

You must change the limits of integration as soon as you change variables!

It may have occurred to you that you could use a substitution in a definite integral only to find an antiderivative and then switch back to the original variable to do the evaluation. Although this method will work for many problems, we recommend that you avoid it, for several reasons. First, changing the limits of integration is not very difficult and results in a much more readable mathematical expression. Second, in many applications requiring substitution, you will *need* to change the limits of integration, so you might as well get used to doing so now.

EXAMPLE 6.11 Substitution in a Definite Integral Involving an Exponential

Compute $\int_0^{15} te^{-t^2/2}dt$.

Solution As always, we look for terms that are derivatives of other terms. Here, you should notice that $\frac{d}{dt}\left(\frac{-t^2}{2}\right) = -t$. So, we set $u = -\frac{t^2}{2}$ and compute $du = -t\,dt$. For the upper limit of integration, we have that $t = 15$ corresponds to $u = -\frac{(15)^2}{2} = -\frac{225}{2}$. For the lower limit, we have that $t = 0$ corresponds to $u = 0$. This gives us

$$\int_0^{15} te^{-t^2/2}dt = -\int_0^{15}\underbrace{e^{-t^2/2}}_{e^u}\,\underbrace{(-t)\,dt}_{du}$$

$$= -\int_0^{-225/2} e^u\,du = -e^u\bigg|_0^{-112.5} = -e^{-112.5} + 1.$$ ■

EXERCISES 4.6

WRITING EXERCISES

1. It is never *wrong* to make a substitution in an integral, but sometimes it is not very helpful. For example, using the substitution $u = x^2$, you can correctly conclude that
$$\int x^3\sqrt{x^2+1}\,dx = \int \frac{1}{2}u\sqrt{u+1}\,du,$$
but the new integral is no easier than the original integral. In this case, a better substitution makes this workable. (Can you find it?) However, the general problem remains of how you can tell whether or not to give up on a substitution. Give some guidelines for answering this question, using the integrals $\int x\sin x^2 dx$ and $\int x\sin x^3 dx$ as illustrative examples.

2. It is not uncommon for students learning substitution to use incorrect notation in the intermediate steps. Be aware of this—it can be harmful to your grade! Carefully examine the following string of equalities and find each mistake. Using $u = x^2$,
$$\int_0^2 x\sin x^2 dx = \int_0^2 (\sin u)x\,dx = \int_0^2 (\sin u)\frac{1}{2}\,du$$
$$= -\frac{1}{2}\cos u\Big|_0^2 = -\frac{1}{2}\cos x^2\Big|_0^2$$
$$= -\frac{1}{2}\cos 4 + \frac{1}{2}.$$
The final answer is correct, but because of several errors, this work would not earn full credit. Discuss each error and write this in a way that would earn full credit.

3. Suppose that an integrand has a term of the form $e^{f(x)}$. For example, suppose you are trying to evaluate $\int x^2 e^{x^3} dx$. Discuss why you should immediately try the substitution $u = f(x)$. If this substitution does not work, what could you try next? (Hint: Think about $\int x^2 e^{\ln x} dx$.)

4. Suppose that an integrand has a composite function of the form $f(g(x))$. Explain why you should look to see if the integrand also has the term $g'(x)$. Discuss possible substitutions.

In exercises 1–4, use the given substitution to evaluate the indicated integral.

1. $\int x^2\sqrt{x^3+2}\,dx,\ u = x^3+2$

2. $\int x^3(x^4+1)^{-2/3}dx,\ u = x^4+1$

3. $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}\,dx,\ u = \sqrt{x}+2$

4. $\int \sin x\cos x\,dx,\ u = \sin x$

In exercises 5–30, evaluate the indicated integral.

5. $\int x^3\sqrt{x^4+3}\,dx$
6. $\int \sec^2 x\sqrt{\tan x}\,dx$
7. $\int \frac{\sin x}{\sqrt{\cos x}}\,dx$
8. $\int \sin^3 x\cos x\,dx$
9. $\int x^2\cos x^3 dx$
10. $\int \sin x(\cos x+3)^{3/4}dx$
11. $\int x e^{x^2+1}dx$
12. $\int e^x\sqrt{e^x+4}\,dx$
13. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
14. $\int \frac{x+1}{(x^2+2x-1)^2}\,dx$
15. $\int \frac{\sqrt{\ln x}}{x}\,dx$
16. $\int \frac{\cos(1/x)}{x^2}\,dx$
17. $\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}\,dx$
18. $\int \frac{x}{x^2+4}\,dx$
19. $\int \frac{4}{x(\ln x+1)^2}\,dx$
20. $\int \tan 2x\,dx$
21. $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\,dx$
22. $\int x^2\sec^2 x^3 dx$
23. $\int \frac{x}{\sqrt{1-x^4}}\,dx$
24. $\int \frac{x^3}{\sqrt{1-x^4}}\,dx$
25. $\int \frac{x^2}{1+x^6}\,dx$
26. $\int \frac{x^5}{1+x^6}\,dx$
27. $\int \frac{2x+3}{x+7}\,dx$
28. $\int \frac{x^2}{\sqrt[3]{x+3}}\,dx$
29. $\int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$
30. $\int \frac{1}{x\sqrt{x^4-1}}\,dx$

In exercises 31–40, evaluate the definite integral.

31. $\int_0^2 x\sqrt{x^2+1}\,dx$
32. $\int_1^3 x\sin(\pi x^2)\,dx$
33. $\int_{-1}^1 \frac{x}{(x^2+1)^2}\,dx$
34. $\int_0^2 x^2 e^{x^3}dx$
35. $\int_0^2 \frac{e^x}{1+e^{2x}}\,dx$
36. $\int_0^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$
37. $\int_{\pi/4}^{\pi/2} \cot x\,dx$
38. $\int_1^e \frac{\ln x}{x}\,dx$
39. $\int_1^4 \frac{x-1}{\sqrt{x}}\,dx$
40. $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$

In exercises 41–44, evaluate the integral exactly, if possible. Otherwise, estimate it numerically.

41. (a) $\int_0^\pi \sin x^2\,dx$ (b) $\int_0^\pi x\sin x^2\,dx$

42. (a) $\int_{-1}^1 xe^{-x^2}dx$ (b) $\int_{-1}^1 e^{-x^2}dx$

43. (a) $\int_0^2 \frac{4x^2}{(x^2+1)^2}\,dx$ (b) $\int_0^2 \frac{4x^3}{(x^2+1)^2}\,dx$

44. (a) $\int_0^{\pi/4} \sec x\,dx$ (b) $\int_0^{\pi/4} \sec^2 x\,dx$

In exercises 45–48, make the indicated substitution for an unspecified function $f(x)$.

45. $u = x^2$ for $\int_0^2 x f(x^2)\,dx$

46. $u = x^3$ for $\int_1^2 x^2 f(x^3)\,dx$

47. $u = \sin x$ for $\int_0^{\pi/2} (\cos x) f(\sin x)\,dx$

48. $u = \sqrt{x}$ for $\int_0^4 \frac{f(\sqrt{x})}{\sqrt{x}}\,dx$

49. A function f is said to be **even** if $f(-x) = f(x)$ for all x. A function f is said to be **odd** if $f(-x) = -f(x)$. Suppose that f is continuous for all x. Show that if f is even, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$. Also, if f is odd, show that $\int_{-a}^a f(x)\,dx = 0$.

50. Assume that f is periodic with period T; that is, $f(x+T) = f(x)$ for all x. Show that $\int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx$ for any real number a. (Hint: First, work with $0 \le a \le T$.)

51. For the integral $I = \int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}\,dx$, use a substitution to show that $I = \int_0^{10} \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$. Use these two representations of I to evaluate I.

52. Generalize exercise 51 to $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ for any positive, continuous function f and then quickly evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$.

53. For $I = \int_2^4 \frac{\sin^2(9-x)}{\sin^2(9-x)+\sin^2(x+3)}\,dx$, use the substitution $u = 6 - x$ to show that $I = \int_2^4 \frac{\sin^2(x+3)}{\sin^2(9-x)+\sin^2(x+3)}\,dx$ and evaluate I.

54. Generalize the result of exercise 53 to $\int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}\,dx$, for any positive, continuous function f on [2, 4].

55. As in exercise 54, evaluate $\int_0^2 \frac{f(x+4)}{f(x+4)+f(6-x)}\,dx$ for any positive, continuous function f on [0, 2].

56. Use the substitution $u = x^{1/6}$ to evaluate $\int \frac{1}{x^{5/6}+x^{2/3}}\,dx$.

57. Evaluate $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$. (See exercise 56.)

58. Generalize exercises 56 and 57 to $\int \frac{1}{x^{(p+1)/q}+x^{p/q}}\,dx$ for positive integers p and q.

59. There are often multiple ways of computing an antiderivative. For $\int \frac{1}{x \ln \sqrt{x}}\,dx$, first use the substitution $u = \ln \sqrt{x}$ to find the indefinite integral $2\ln|\ln\sqrt{x}| + c$. Then rewrite $\ln\sqrt{x}$ and use the substitution $u = \ln x$ to find the indefinite integral $2\ln|\ln x| + c$. Show that these two answers are equivalent.

60. As in exercise 59, use different substitutions to find two forms for $\int \frac{1}{x \ln x^2}\,dx$ and then show that they are equivalent.

61. Find each mistake in the following calculations and then show how to correctly do the substitution. Start with $\int_{-2}^1 4x^4 dx = \int_{-2}^1 x(4x^3)\,dx$ and then use the substitution $u = x^4$ with $du = 4x^3 dx$. Then

$$\int_{-2}^1 x(4x^3)\,dx = \int_{16}^1 u^{1/4} du = \frac{4}{5}u^{5/4}\Big|_{u=16}^{u=1} = \frac{4}{5} - \frac{32}{5} = -\frac{18}{5}$$

62. Find each mistake in the following calculations and then show how to correctly do the substitution. Start with $\int_0^\pi \cos^2 x\,dx = \int_0^\pi \cos x(\cos x)\,dx$ and then use the substitution $u = \sin x$ with $du = \cos x\,dx$. Then

$$\int_0^\pi \cos x(\cos x)\,dx = \int_0^0 \sqrt{1-u^2}du = 0$$

63. For $a > 0$, show that $\int_a^1 \frac{1}{x^2+1}\,dx = \int_1^{1/a} \frac{1}{x^2+1}\,dx$. Use this equality to derive an identity involving $\tan^{-1} x$.

64. Evaluate $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx$ by rewriting the integrand as $\frac{1}{x^2\sqrt{1-1/x^2}}$ and then making the substitution $u = 1/x$. Use your answer to derive an identity involving $\sin^{-1}(1/x)$ and $\sec^{-1} x$.

65. The location $(\bar{x}, \bar{y})$ of the center of gravity (balance point) of a flat plate bounded by $y = f(x) > 0$, $a \le x \le b$ and the x-axis is given by $\bar{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$ and $\bar{y} = \frac{\int_a^b [f(x)]^2\,dx}{2\int_a^b f(x)\,dx}$. For the semicircle $y = f(x) = \sqrt{4-x^2}$, use symmetry to argue that $\bar{x} = 0$ and $\bar{y} = \frac{1}{2\pi}\int_0^2 (4-x^2)\,dx$. Compute $\bar{y}$.

66. Suppose that the population density of a group of animals can be described by $f(x) = xe^{-x^2}$ thousand animals per mile for $0 \le x \le 2$, where x is the distance from a pond. Graph $y = f(x)$ and briefly describe where these animals are likely to be found. Find the total population $\int_0^2 f(x)\,dx$.

67. The voltage in an AC (alternating current) circuit is given by $V(t) = V_p \sin(2\pi f t)$, where f is the frequency. A voltmeter does not indicate the amplitude V_p. Instead, the voltmeter reads the **root-mean-square** (rms), the square root of the average value of the square of the voltage over one cycle. That is, rms $= \sqrt{f \int_0^{1/f} V^2(t)\,dt}$. Use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ to show that rms $= V_p/\sqrt{2}$.

68. Graph $y = f(t)$ and find the root-mean-square of

$$f(t) = \begin{cases} -1 & \text{if } -2 \le t < -1 \\ t & \text{if } -1 \le t \le 1 \\ 1 & \text{if } 1 < t \le 2 \end{cases},$$

where rms $= \sqrt{\dfrac{1}{4}\displaystyle\int_{-2}^{2} f^2(t)\,dt}$.

EXPLORATORY EXERCISES

1. A **predator-prey system** is a set of differential equations modeling the change in population of interacting species of organisms. A simple model of this type is

$$\begin{cases} x'(t) = x(t)[a - by(t)] \\ y'(t) = y(t)[dx(t) - c] \end{cases}$$

for positive constants *a, b, c* and *d*. Both equations include a term of the form $x(t)y(t)$, which is intended to represent the result of confrontations between the species. Noting that the contribution of this term is negative to $x'(t)$ but positive to $y'(t)$, explain why it must be that $x(t)$ represents the population of the prey and $y(t)$ the population of the predator. If $x(t) = y(t) = 0$, compute $x'(t)$ and $y'(t)$. In this case, will x and y increase, decrease or stay constant? Explain why this makes sense physically. Determine $x'(t)$ and $y'(t)$ and the subsequent change in x and y at the so-called **equilibrium point** $x = c/d$, $y = a/b$. If the population is periodic, we can show that the equilibrium point gives the average population (even if the population does not remain constant). To do so, note that $\dfrac{x'(t)}{x(t)} = a - by(t)$. Integrating both sides of this equation from $t = 0$ to $t = T$ [the period of $x(t)$ and $y(t)$], we get $\displaystyle\int_0^T \frac{x'(t)}{x(t)}\,dt = \int_0^T a\,dt - \int_0^T by(t)\,dt$. Evaluate each integral to show that $\ln x(T) - \ln x(0) = aT - \int_0^T by(t)\,dt$. Assuming that $x(t)$ has period *T*, we have $x(T) = x(0)$ and so, $0 = aT - \int_0^T by(t)\,dt$. Finally, rearrange terms to show that $1/T \int_0^T y(t)\,dt = a/b$; that is, the average value of the population $y(t)$ is the equilibrium value $y = a/b$. Similarly, show that the average value of the population $x(t)$ is the equilibrium value $x = c/d$.

2. Evaluate the integrals $\displaystyle\int \frac{5}{3+x}\,dx$ and $\displaystyle\int \frac{5}{3+x^2}\,dx$ by hand. Also use your CAS to evaluate $\displaystyle\int \frac{5}{3+x^3}\,dx$, $\displaystyle\int \frac{5}{3+x^4}\,dx$ and $\displaystyle\int \frac{5}{3+x^5}\,dx$. Describe any patterns you see. In particular, are there any constants with recognizable patterns? What types of functions appear? If the arguments of the logarithms are multiplied, what is the result? Conjecture as much as possible about the form of $\displaystyle\int \frac{5}{3+x^n}\,dx$ for positive integer *n*.

3. Physicists define something called the **Dirac delta** $\delta(x)$, for which a defining property is that $\int_{-a}^{b} \delta(x)\,dx = 1$ for any $a, b > 0$. Assuming that $\delta(x)$ acts like a continuous function (this is a significant issue!), use this property to evaluate (a) $\int_0^1 \delta(x-2)\,dx$, (b) $\int_0^1 \delta(2x-1)\,dx$ and (c) $\int_{-1}^{1} \delta(2x)\,dx$. Assuming that it applies, use the Fundamental Theorem of Calculus to prove that $\delta(x) = 0$ for all $x \neq 0$ and to prove that $\delta(x)$ is unbounded in $[-1, 1]$. What do you find troublesome about this? Do you think that $\delta(x)$ is really a continuous function, or even a function at all?

4. Suppose that f is a continuous function such that for all x, $f(2x) = 3f(x)$ and $f(x + \frac{1}{2}) = \frac{1}{3} + f(x)$. Compute $\displaystyle\int_0^1 f(x)\,dx$.

4.7 NUMERICAL INTEGRATION

Thus far, our development of the integral has paralleled our development of the derivative. In both cases, we began with a limit definition that was difficult to use for calculation and then, proceeded to develop simplified rules for calculation. At this point, you should be able to find the derivative of nearly any function you can write down. You might expect that with a few more rules you will be able to do the same for integrals. Unfortunately, this is not the case. There are many functions for which *no* elementary antiderivative is available. (By elementary antiderivative, we mean an antiderivative expressible in terms of the elementary functions with which you are familiar: the algebraic, trigonometric, exponential and logarithmic functions.) For instance,

$$\int_0^2 \cos(x^2)\,dx$$

cannot be calculated exactly, since $\cos(x^2)$ does not have an elementary antiderivative. (Try to find one, but don't spend much time on it.)

In fact, most definite integrals cannot be calculated exactly. When we can't compute the value of an integral exactly, we do the next best thing: we approximate its value numerically. In this section, we develop three methods of approximating definite integrals. None will replace the built-in integration routine on your calculator or computer. However, by exploring these methods, you will gain a basic understanding of some of the ideas behind more sophisticated numerical integration routines.

Since a definite integral is the limit of a sequence of Riemann sums, any Riemann sum serves as an approximation of the integral,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i is any point chosen from the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. Further, the larger n is, the better the approximation tends to be. The most common choice of the evaluation points $c_1, c_2, \ldots, c_n$ leads to a method called the **Midpoint Rule:**

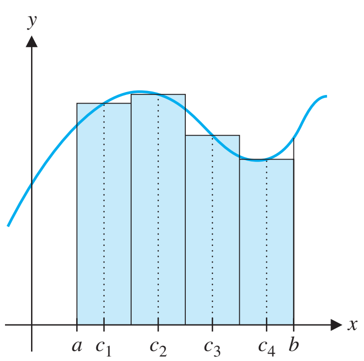


FIGURE 4.29
Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i is the midpoint of the subinterval $[x_{i-1}, x_i]$,

$$c_i = \frac{1}{2}(x_{i-1} + x_i), \quad \text{for } i = 1, 2, \ldots, n.$$

We illustrate this approximation for the case where $f(x) \geq 0$ on $[a, b]$, in Figure 4.29.

EXAMPLE 7.1 Using the Midpoint Rule

Write out the Midpoint Rule approximation of $\int_0^1 3x^2\,dx$ with $n = 4$.

Solution For $n = 4$, the regular partition of the interval [0, 1] is $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$ and $x_4 = 1$. The midpoints are then $c_1 = \frac{1}{8}$, $c_2 = \frac{3}{8}$, $c_3 = \frac{5}{8}$ and $c_4 = \frac{7}{8}$. With $\Delta x = \frac{1}{4}$, the Riemann sum is then

$$\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]\left(\frac{1}{4}\right) = \left(\frac{3}{64} + \frac{27}{64} + \frac{75}{64} + \frac{147}{64}\right)\left(\frac{1}{4}\right)$$
$$= \frac{252}{256} = 0.984375.$$

Of course, from the Fundamental Theorem, the exact value of the integral in example 7.1 is

$$\int_0^1 3x^2 dx = \left.\frac{3x^3}{3}\right|_0^1 = 1.$$

So, our approximation in example 7.1 is not especially accurate. To obtain greater accuracy, notice that you could always compute an approximation using more rectangles. You can simplify this process by writing a simple program for your calculator or computer to implement the Midpoint Rule. A suggested outline for such a program follows.

MIDPOINT RULE

1. Store $f(x)$, a, b and n.
2. Compute $\Delta x = \dfrac{b-a}{n}$.
3. Compute $c_1 = a + \dfrac{\Delta x}{2}$ and start the sum with $f(c_1)$.
4. Compute the next $c_i = c_{i-1} + \Delta x$ and add $f(c_i)$ to the sum.
5. Repeat step 4 until $i = n$ [i.e., perform step 4 a total of $(n-1)$ times].
6. Multiply the sum by Δx.

EXAMPLE 7.2 Using a Program for the Midpoint Rule

Repeat example 7.1 using a program to compute the Midpoint Rule approximations for $n = 8, 16, 32, 64$ and 128.

Solution You should confirm the values in the following table. We include a column displaying the error in the approximation for each n (i.e., the difference between the exact value of 1 and the approximate values).

n	*Midpoint Rule*	*Error*
4	0.984375	0.015625
8	0.99609375	0.00390625
16	0.99902344	0.00097656
32	0.99975586	0.00024414
64	0.99993896	0.00006104
128	0.99998474	0.00001526

You should note that each time the number of steps is doubled, the error is reduced approximately by a factor of 4. Although this precise reduction in error will not occur with all integrals, this rate of improvement in the accuracy of the approximation is typical of the Midpoint Rule. ■

Of course, we won't know the error in a Midpoint Rule approximation, except where we know the value of the integral exactly. We started with a simple integral, whose value we knew exactly, so that you could get a sense of how accurate the Midpoint Rule approximation is.

Note that in example 7.3, we can't compute an exact value of the integral, since we do not know an antiderivative for the integrand.

EXAMPLE 7.3 Finding an Approximation with a Given Accuracy

Use the Midpoint Rule to approximate $\int_0^2 \sqrt{x^2+1}\,dx$ accurate to three decimal places.

Solution Given the instructions, how do we know how large n should be? We continue increasing n until it appears unlikely the third decimal will change further. (The size of n will vary substantially from integral to integral.) You should test your program against the numbers in the accompanying table.

n	*Midpoint Rule*
10	2.95639
20	2.95751
30	2.95772
40	2.95779

From the table, we can make the reasonable approximation

$$\int_0^2 \sqrt{x^2+1}\,dx \approx 2.958.$$ ■

REMARK 7.1

Computer and calculator programs that estimate the values of integrals face the same challenge we did in example 7.3—that is, knowing when a given approximation is good enough. Such software generally includes sophisticated algorithms for estimating the accuracy of its approximations. You can find an introduction to such algorithms in most texts on numerical analysis.

Another important reason for pursuing numerical methods is for the case where we don't know the function that we're trying to integrate. That's right: we often know only some *values* of a function at a collection of points, while a symbolic representation of a function is unavailable. This is often the case in the physical and biological sciences and engineering, in situations where the only information available about a function comes from measurements made at a finite number of points.

x	$f(x)$
0.0	1.0
0.25	0.8
0.5	1.3
0.75	1.1
1.0	1.6

EXAMPLE 7.4 Estimating an Integral from a Table of Function Values

Estimate $\int_0^1 f(x)\,dx$, where we have values of the unknown function $f(x)$ as given in the table shown in the margin.

Solution Approaching the problem graphically, we have five data points (see Figure 4.30). How can we estimate the area under the curve from five points? Conceptually, we have two tasks. First, we need a reasonable way to connect the given points. Second, we need to compute the area of the resulting region. So, how should we connect the dots? The most obvious way is the same way any child would: connect the dots with straight-line segments as in Figure 4.31a.

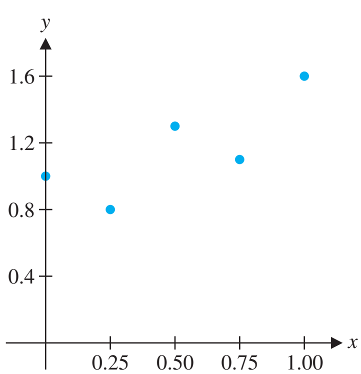


FIGURE 4.30
Data from an unknown function

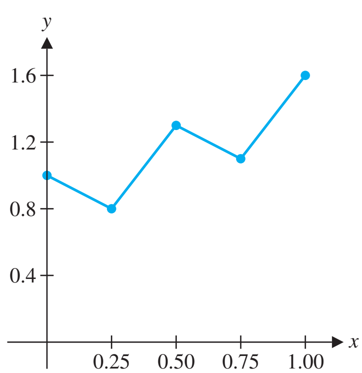


FIGURE 4.31a
Connecting the dots

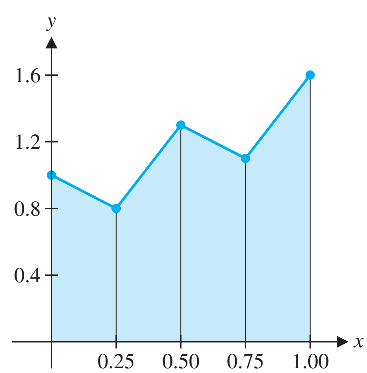


FIGURE 4.31b
Four trapezoids

Notice that the region bounded by the graph and the x-axis on the interval [0, 1] consists of four trapezoids (see Figure 4.31b).

Recall that the area of a trapezoid with sides h_1 and h_2 and base b is given by $\left(\frac{h_1 + h_2}{2}\right) b$. (It's an easy exercise to verify this.) You can think of this as the average of the areas of the rectangle whose height is the value of the function at the left endpoint and the rectangle whose height is the value of the function at the right endpoint.

The total area of the four trapezoids is then

$$\frac{f(0)+f(0.25)}{2}0.25+\frac{f(0.25)+f(0.5)}{2}0.25+\frac{f(0.5)+f(0.75)}{2}0.25$$
$$+\frac{f(0.75)+f(1)}{2}0.25$$
$$=[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)]\frac{0.25}{2}=1.125.$$

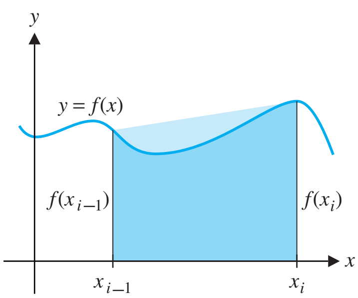


FIGURE 4.32
Trapezoidal Rule

More generally, for any continuous function f defined on the interval $[a, b]$, we partition $[a, b]$ as follows:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where the points in the partition are equally spaced, with spacing $\Delta x = \frac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$, approximate the area under the curve by the area of the trapezoid whose sides have length $f(x_{i-1})$ and $f(x_i)$, as indicated in Figure 4.32. The area under the curve on the interval $[x_{i-1}, x_i]$ is then approximately

$$A_i \approx \frac{1}{2}[f(x_{i-1})+f(x_i)]\,\Delta x,$$

for each $i = 1, 2, \ldots, n$. Adding together the approximations for the area under the curve on each subinterval, we get that

$$\int_a^b f(x)\,dx \approx \left[\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\cdots+\frac{f(x_{n-1})+f(x_n)}{2}\right]\Delta x$$
$$=\frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)].$$

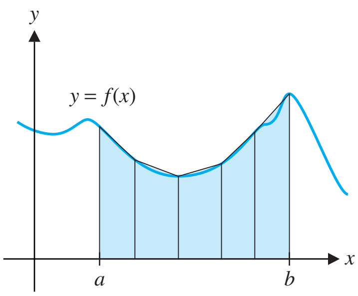


FIGURE 4.33
The $(n+1)$-point Trapezoidal Rule

We illustrate this in Figure 4.33. Notice that each of the middle terms is multiplied by 2, since each one is used in two trapezoids, once as the height of the trapezoid at the right endpoint and once as the height of the trapezoid at the left endpoint. We refer to this as the $(n+1)$-point **Trapezoidal Rule,** $T_n(f)$,

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n(f)=\frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)].$$

One way to write a program for the Trapezoidal Rule is to add together $[f(x_{i-1})+f(x_i)]$ for $i = 1, 2, \ldots, n$ and then multiply by $\Delta x/2$. As discussed in the exercises, an alternative is to add together the Riemann sums using left- and right-endpoint evaluations, and then divide by 2.

EXAMPLE 7.5 Using the Trapezoidal Rule

Compute the Trapezoidal Rule approximations with $n = 4$ (by hand) and $n = 8$, 16, 32, 64 and 128 (use a program) for $\int_0^1 3x^2 dx$.

Solution As we saw in examples 7.1 and 7.2, the exact value of this integral is 1. For the Trapezoidal Rule with $n = 4$, we have

$$T_4(f)=\frac{1-0}{(2)(4)}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)+f(1)\right]$$
$$=\frac{1}{8}\left(0+\frac{3}{8}+\frac{12}{8}+\frac{27}{8}+3\right)=\frac{66}{64}=1.03125.$$

NOTES

Since the Trapezoidal Rule formula is an average of two Riemann sums, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} T_n(f).$$

Using a program, you can easily get the values in the accompanying table.

n	$T_n(f)$	Error
4	1.03125	0.03125
8	1.0078125	0.0078125
16	1.00195313	0.00195313
32	1.00048828	0.00048828
64	1.00012207	0.00012207
128	1.00003052	0.00003052

We have included a column showing the error (the absolute value of the difference between the exact value of 1 and the approximate value). Notice that (as with the Midpoint Rule) as the number of steps doubles, the error is reduced by approximately a factor of 4. ■

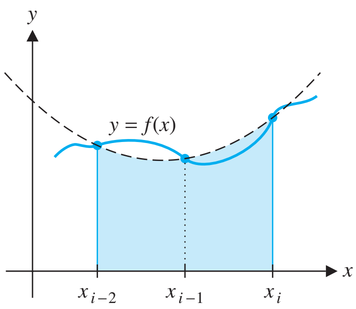


FIGURE 4.34
Simpson's Rule

Simpson's Rule

Consider the following alternative to the Trapezoidal Rule. First, construct a regular partition of the interval $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where
$$x_i - x_{i-1} = \frac{b-a}{n} = \Delta x,$$

for each $i = 1, 2, \ldots, n$ and where n is an *even* number. Instead of connecting each pair of points with a straight line segment (as we did with the Trapezoidal Rule), we connect each set of *three* consecutive points, $(x_{i-2}, f(x_{i-2}))$, $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ for $i = 2, 4, \ldots, n$, with a parabola (see Figure 4.34). That is, we look for the quadratic function $p(x)$ whose graph passes through these three points, so that

$$p(x_{i-2}) = f(x_{i-2}), \quad p(x_{i-1}) = f(x_{i-1}) \quad \text{and} \quad p(x_i) = f(x_i).$$

Using this to approximate the value of the integral of f on the interval $[x_{i-2}, x_i]$, we have

$$\int_{x_{i-2}}^{x_i} f(x)\,dx \approx \int_{x_{i-2}}^{x_i} p(x)\,dx.$$

Notice why we want to approximate f by a polynomial: polynomials are easy to integrate. A straightforward though tedious computation (try this; your CAS may help) gives

$$\int_{x_{i-2}}^{x_i} f(x)\,dx \approx \int_{x_{i-2}}^{x_i} p(x)\,dx = \frac{x_i - x_{i-2}}{6}[f(x_{i-2}) + 4f(x_{i-1}) + f(x_i)]$$
$$= \frac{b-a}{3n}[f(x_{i-2}) + 4f(x_{i-1}) + f(x_i)].$$

Adding together the integrals over each subinterval $[x_{i-2}, x_i]$, for $i = 2, 4, 6, \ldots, n$, we get

$$\int_a^b f(x)\,dx$$
$$\approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{b-a}{3n}[f(x_2) + 4f(x_3) + f(x_4)] + \cdots$$
$$+ \frac{b-a}{3n}[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$
$$= \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

HISTORICAL NOTES

Thomas Simpson (1710–1761) An English mathematician who popularized the numerical method now known as Simpson's Rule. Trained as a weaver, Simpson also earned a living as a fortune-teller, as the editor of the *Ladies' Diary* and as a textbook author. Simpson's calculus textbook (titled *A New Treatise on Fluxions*, using Newton's calculus terminology) introduced many mathematicians to Simpson's Rule, although the method had been developed years earlier.

Be sure to notice the pattern that the coefficients follow. We refer to this as the $(n+1)$-point **Simpson's Rule,** $S_n(f)$,

SIMPSON'S RULE

$$\int_a^b f(x)\,dx \approx S_n(f) = \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

Next, we illustrate the use of Simpson's Rule for a simple integral.

EXAMPLE 7.6 Using Simpson's Rule

Approximate the value of $\int_0^1 3x^2 dx$ using Simpson's Rule with $n = 4$.

Solution We have

$$S_4(f) = \frac{1-0}{(3)(4)}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] = 1,$$

which is in fact, the *exact* value. Notice that this is far more accurate than the Midpoint and Trapezoidal Rules and yet requires no more effort. ■

Recall that Simpson's Rule computes the area beneath approximating parabolas. Given this, it shouldn't surprise you that Simpson's Rule gives the exact area in example 7.6. As you will discover in the exercises, Simpson's Rule gives exact values of integrals for *any* polynomial of degree 3 or less.

In example 7.7, we illustrate Simpson's Rule for an integral that you do not know how to compute exactly.

EXAMPLE 7.7 Using a Program for Simpson's Rule

Compute Simpson's Rule approximations with $n = 4$ (by hand), $n = 8$, 16, 32, 64 and 128 (use a program) for $\int_0^2 \sqrt{x^2+1}\,dx$.

Solution For $n = 4$, we have

$$S_4(f) = \frac{2-0}{(3)(4)}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + f(2)\right]$$

$$= \left(\frac{1}{6}\right)\left[1 + 4\sqrt{\frac{5}{4}} + 2\sqrt{2} + 4\sqrt{\frac{13}{4}} + \sqrt{5}\right] \approx 2.95795560.$$

n	$S_n(f)$
4	2.9579556
8	2.9578835
16	2.95788557
32	2.95788571
64	2.95788571
128	2.95788572

Using a program, you can easily obtain the values in the accompanying table. Based on these calculations, we would expect 2.9578857 to be a very good approximation of $\int_0^2 \sqrt{x^2+1}\,dx$. ■

Since most graphs curve somewhat, you might expect the parabolas of Simpson's Rule to better track the curve than the line segments of the Trapezoidal Rule. As example 7.8 shows, Simpson's Rule can be much more accurate than either the Midpoint Rule or the Trapezoidal Rule.

EXAMPLE 7.8 Comparing the Midpoint, Trapezoidal and Simpson's Rules

Compute the Midpoint, Trapezoidal and Simpson's Rule approximations of $\int_0^1 \frac{4}{x^2+1}\,dx$ with $n = 10$, $n = 20$, $n = 50$ and $n = 100$. Compare to the exact value of π.

Solution

n	*Midpoint Rule*	*Trapezoidal Rule*	*Simpson's Rule*
10	3.142425985	3.139925989	3.141592614
20	3.141800987	3.141175987	3.141592653
50	3.141625987	3.141525987	3.141592654
100	3.141600987	3.141575987	3.141592654

Compare these values to the exact value of $\pi \approx 3.141592654$. Note that the Midpoint Rule tends to be slightly closer to π than the Trapezoidal Rule, but neither is as close with $n = 100$ as Simpson's Rule is with $n = 10$. ■

REMARK 7.2

Notice that for a given value of n, the number of computations (and hence the effort) required to produce the Midpoint, Trapezoidal and Simpson's Rule approximations are all roughly the same. So, example 7.8 gives an indication of how much more efficient Simpson's Rule is than the other two methods. This is particularly significant when the function $f(x)$ is difficult to evaluate. For instance, in the case of experimental data, each function value $f(x)$ could be the result of an expensive and time-consuming experiment.

In example 7.9, we revise our estimate of the area in Figure 4.30, first examined in example 7.4.

EXAMPLE 7.9 Using Simpson's Rule with Data

Use Simpson's Rule to estimate $\int_0^1 f(x)\,dx$, where the only information known about f is given in the table of values shown in the margin.

x	*f(x)*
0.0	1.0
0.25	0.8
0.5	1.3
0.75	1.1
1.0	1.6

Solution From Simpson's Rule with $n = 4$, we have

$$\int_0^1 f(x)\,dx \approx \frac{1-0}{(3)(4)}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)]$$
$$= \left(\frac{1}{12}\right)[1 + 4(0.8) + 2(1.3) + 4(1.1) + 1.6] \approx 1.066667.$$

Since Simpson's Rule is generally much more accurate than the Trapezoidal Rule (for the same number of points), we expect that this approximation is more accurate than the approximation of 1.125 arrived at in example 7.4 via the Trapezoidal Rule. ■

REMARK 7.3

Most graphing calculators and computer algebra systems have very fast and accurate programs for numerical approximation of definite integrals. Some ask you to specify an error tolerance and then calculate a value accurate to within that tolerance. Most calculators and CAS's use *adaptive quadrature* routines, which automatically calculate how many points are needed to obtain a desired accuracy. You should feel comfortable using these programs. However, if the integral you are approximating is a critical part of an important project, it's a good idea to check your result. You can do this by using Simpson's Rule, $S_n(f)$, for a sequence of values of n. Of course, if all you know about a function is its value at a fixed number of points, most calculator and CAS programs will not help you, but the three methods discussed here will, as we saw in examples 7.4 and 7.9. We will pursue this idea further in the exercises.

Error Bounds for Numerical Integration

We have used examples where we know the value of an integral exactly to compare the accuracy of our three numerical integration methods. However, in practice, where the value of an integral is not known exactly, how do we determine how accurate a given numerical estimate is? In Theorems 7.1 and 7.2, we give bounds on the error in our three numerical integration methods. First, we introduce some notation. Let ET_n represent the error in using the $(n+1)$-point Trapezoidal Rule to approximate $\int_a^b f(x)\,dx$. That is,

$$ET_n = \text{exact} - \text{approximate} = \int_a^b f(x)\,dx - T_n(f).$$

Similarly, we denote the error in the Midpoint Rule and Simpson's Rule by EM_n and ES_n, respectively. We now have:

THEOREM 7.1

Suppose that f'' is continuous on $[a, b]$ and that $|f''(x)| \le K$, for all x in $[a, b]$. Then,

$$|ET_n| \le K\frac{(b-a)^3}{12n^2}$$

and

$$|EM_n| \le K\frac{(b-a)^3}{24n^2}.$$

Notice that both of the estimates found in Theorem 7.1 say that the error in using the indicated numerical method is no larger (in absolute value) than the given bound. This says that if the bound is small, so too will be the error. In particular, observe that the error bound for the Midpoint Rule is *half* that for the Trapezoidal Rule. This doesn't say that the actual error in the Midpoint Rule will be half that of the Trapezoidal Rule, but it does explain why the Midpoint Rule tends to be somewhat more accurate than the Trapezoidal Rule for the same value of n. Also notice that the constant K is determined by the concavity $|f''(x)|$ of the function f. The larger $|f''(x)|$ is, the more the graph curves and consequently, the less accurate are the straight-line approximations of the Midpoint Rule and the Trapezoidal Rule. We have a corresponding result for Simpson's Rule.

THEOREM 7.2

Suppose that $f^{(4)}$ is continuous on $[a, b]$ and that $|f^{(4)}(x)| \leq L$, for all x in $[a, b]$. Then,

$$|ES_n| \leq L\frac{(b-a)^5}{180n^4}.$$

The proofs of Theorems 7.1 and 7.2 are beyond the level of this course and we refer the interested reader to a text on numerical analysis. In comparing Theorems 7.1 and 7.2, notice that the denominators of the error bounds for both the Trapezoidal Rule and the Midpoint Rule contain a factor of n^2, while the error bound for Simpson's Rule contains a factor of n^4. For $n = 10$, observe that $n^2 = 100$ and $n^4 = 10{,}000$. Since these powers of n are in the denominators of the error bounds, this says that the error bound for Simpson's Rule tends to be *much* smaller than that of either the Trapezoidal Rule or the Midpoint Rule for the same value of n. This accounts for the far greater accuracy we have seen with using Simpson's Rule over the other two methods. We illustrate the use of the error bounds in example 7.10.

EXAMPLE 7.10 Finding a Bound on the Error in Numerical Integration

Find bounds on the error in using each of the Midpoint Rule, the Trapezoidal Rule and Simpson's Rule to approximate the value of the integral $\int_1^3 \frac{1}{x}dx$, using $n = 10$.

Solution Your first inclination might be to observe that you already know the value of this integral exactly, since by the Fundamental Theorem of Calculus,

$$\int_1^3 \frac{1}{x}\,dx = \ln|x|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

However, you don't really know the value of ln 3, but must use your calculator to compute an approximate value of this. On the other hand, you can approximate this integral using Trapezoidal, Midpoint or Simpson's Rules. Here, $f(x) = 1/x = x^{-1}$, so that $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$ and $f^{(4)}(x) = 24x^{-5}$. This says that for $x \in [1, 3]$,

$$|f''(x)| = |2x^{-3}| = \frac{2}{x^3} \leq 2.$$

From Theorem 7.1, we now have

$$|EM_{10}| \leq K\frac{(b-a)^3}{24n^2} = 2\frac{(3-1)^3}{24(10^2)} \approx 0.006667.$$

Similarly, we have

$$|ET_{10}| \leq K\frac{(b-a)^3}{12n^2} = 2\frac{(3-1)^3}{12(10^2)} \approx 0.013333.$$

Turning to Simpson's Rule, for $x \in [1, 3]$, we have $S_{10}(f) \approx 1.09866$ and

$$|f^{(4)}(x)| = |24x^{-5}| = \frac{24}{x^5} \leq 24,$$

so that Theorem 7.2 now gives us

$$|ES_{10}| \leq L\frac{(b-a)^5}{180n^4} = 24\frac{(3-1)^5}{180(10^4)} \approx 0.000427.$$

From example 7.10, we now know that the Simpson's Rule approximation $S_{10}(f) \approx 1.09866$ is off by no more than about 0.000427. This is certainly very useful information, but a more interesting question is the following. In practice, we start out needing a certain accuracy and then must produce an approximation with at least that accuracy. We explore this in example 7.11.

EXAMPLE 7.11 Determining the Number of Steps That Guarantee a Given Accuracy

Determine the number of steps that will guarantee an accuracy of at least 10^{-7} for using each of Trapezoidal Rule and Simpson's Rule to approximate $\int_1^3 \frac{1}{x}\, dx$.

Solution From example 7.10, we know that $|f''(x)| \le 2$ and $|f^{(4)}(x)| \le 24$, for all $x \in [1, 3]$. So, from Theorem 7.1, we now have that

$$|ET_n| \le K\frac{(b-a)^3}{12n^2} = 2\frac{(3-1)^3}{12n^2} = \frac{4}{3n^2}.$$

If we require the above bound on the error to be no larger than the required accuracy of 10^{-7}, we have

$$|ET_n| \le \frac{4}{3n^2} \le 10^{-7}.$$

Solving this inequality for n^2 gives us

$$\frac{4}{3}10^7 \le n^2$$

and taking the square root of both sides yields

$$n \ge \sqrt{\frac{4}{3}10^7} \approx 3651.48.$$

So, any value of $n \ge 3652$ will give the required accuracy. Similarly, for Simpson's Rule, we have

$$|ES_n| \le L\frac{(b-a)^5}{180n^4} = 24\frac{(3-1)^5}{180n^4}.$$

Again, requiring that the error bound be no larger than 10^{-7} gives us

$$|ES_n| \le 24\frac{(3-1)^5}{180n^4} \le 10^{-7}$$

and solving for n^4, we have $\quad n^4 \ge 24\frac{(3-1)^5}{180}10^7.$

Upon taking fourth roots, we get

$$n \ge \sqrt[4]{24\frac{(3-1)^5}{180}10^7} \approx 80.8,$$

so that taking any value of $n \ge 82$ will guarantee the required accuracy. (If you expected us to say that $n \ge 81$, keep in mind that Simpson's Rule requires n to be even.) ■

In example 7.11, compare the number of steps required to guarantee 10^{-7} accuracy in Simpson's Rule (82) to the number required to guarantee the same accuracy in the Trapezoidal Rule (3652). Simpson's Rule typically requires far fewer steps than either the Trapezoidal Rule or the Midpoint Rule to get the same accuracy. Finally, from example 7.11, observe that we now know that

$$\ln 3 = \int_1^3 \frac{1}{x}\,dx \approx S_{82} \approx 1.0986123,$$

which is guaranteed (by Theorem 7.2) to be correct to within 10^{-7}. Compare this with the approximate value of ln 3 generated by your calculator.

EXERCISES 4.7

WRITING EXERCISES

1. Ideally, approximation techniques should be both simple and accurate. How do the numerical integration methods presented in this section compare in terms of simplicity and accuracy? Which criterion would be more important if you were working entirely by hand? Which method would you use? Which criterion would be more important if you were using a very fast computer? Which method would you use?

2. Suppose you were going to construct your own rule for approximate integration. (Name it after yourself!) In the text, new methods were obtained both by choosing evaluation points for Riemann sums (Midpoint Rule) and by geometric construction (Trapezoidal Rule and Simpson's Rule). Without working out the details, explain how you would develop a very accurate but simple rule.

3. Test your calculator or computer on $\int_0^1 \sin(1/x)\,dx$. Discuss what your options are when your technology does not immediately return an accurate approximation. Based on a quick sketch of $y = \sin(1/x)$, describe why a numerical integration routine would have difficulty with this integral.

4. Explain why we did not use the Midpoint Rule in example 7.4.

In exercises 1–4, compute Midpoint, Trapezoidal and Simpson's Rule approximations by hand (leave your answer as a fraction) for $n = 4$.

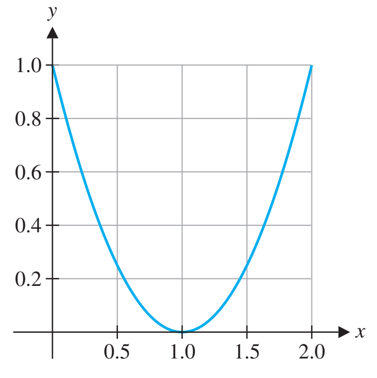
1. $\int_0^1 (x^2+1)\,dx$

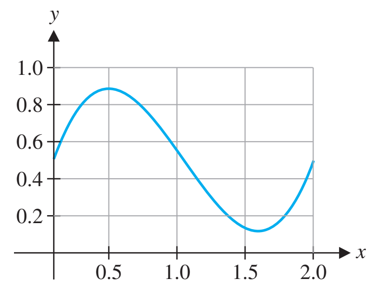
2. $\int_0^2 (x^2+1)\,dx$

3. $\int_1^3 \frac{1}{x}\,dx$

4. $\int_{-1}^1 (2x - x^2)\,dx$

In exercises 5 and 6, use the graph to estimate (a) Riemann sum with left-endpoint evaluation, (b) Midpoint Rule and (c) Trapezoidal Rule approximations with $n = 4$ of $\int_0^2 f(x)\,dx$.

5.


6.


In exercises 7–12, use a computer or calculator to compute the Midpoint, Trapezoidal and Simpson's Rule approximations with $n = 10$, $n = 20$ and $n = 50$. Compare these values to the approximation given by your calculator or computer.

7. $\int_0^\pi \cos x^2\,dx$

8. $\int_0^{\pi/4} \sin \pi x^2\,dx$

9. $\int_0^2 e^{-x^2}\,dx$

10. $\int_0^3 e^{-x^2}\,dx$

11. $\int_0^\pi e^{\cos x}\,dx$

12. $\int_0^1 \sqrt[3]{x^2+1}\,dx$

In exercises 13–16, compute the exact value and compute the error (the difference between the approximation and the exact value) in each of the Midpoint, Trapezoidal and Simpson's Rule approximations using $n = 10, n = 20, n = 40$ and $n = 80$.

13. $\int_0^1 5x^4\,dx$

14. $\int_1^2 \frac{1}{x}\,dx$

15. $\int_0^\pi \cos x\,dx$

16. $\int_0^{\pi/4} \cos x\,dx$

17. Fill in the blanks with the most appropriate power of 2 (2, 4, 8 etc.). If you double n, the error in the Midpoint Rule is divided by_____. If you double n, the error in the Trapezoidal Rule is divided by_____. If you double n, the error in Simpson's Rule is divided by_____.

18. Fill in the blanks with the most appropriate power of 2 (2, 4, 8 etc.). If you halve the interval length $b - a$, the error in the Midpoint Rule is divided by_____, the error in the Trapezoidal Rule is divided by_____ and the error in Simpson's Rule is divided by_____.

In exercises 19–22, approximate the given value using (a) Midpoint Rule, (b) Trapezoidal Rule and (c) Simpson's Rule with $n = 4$.

19. $\ln 4 = \int_1^4 \frac{1}{x}\,dx$

20. $\ln 8 = \int_1^8 \frac{1}{x}\,dx$

21. $\sin 1 = \int_0^1 \cos x\,dx$

22. $e^2 = \int_0^1 (2e^{2x} + 1)\,dx$

23. For exercise 19, find bounds on the errors made by each method.

24. For exercise 21, find bounds on the errors made by each method.

25. For exercise 19, find the number of steps needed to guarantee an accuracy of 10^{-7}.

26. For exercise 21, find the number of steps needed to guarantee an accuracy of 10^{-7}.

27. For each rule in exercise 13, compute the error bound and compare it to the actual error.

28. For each rule in exercise 15, compute the error bound and compare it to the actual error.

In exercises 29–30, use (a) Trapezoidal Rule and (b) Simpson's Rule to estimate $\int_0^2 f(x)\,dx$ from the given data.

29.

x	0.0	0.25	0.5	0.75	1.0
$f(x)$	4.0	4.6	5.2	4.8	5.0

x	1.25	1.5	1.75	2.0
$f(x)$	4.6	4.4	3.8	4.0

30.

x	0.0	0.25	0.5	0.75	1.0
$f(x)$	1.0	0.6	0.2	−0.2	−0.4

x	1.25	1.5	1.75	2.0
$f(x)$	0.4	0.8	1.2	2.0

In exercises 31 and 32, the table gives the measurements (in feet) of the width of a plot of land at 10-foot intervals. Estimate the area of the plot.

31.

x	0	10	20	30	40	50	60
$f(x)$	56	54	58	62	58	58	62

x	70	80	90	100	110	120
$f(x)$	56	52	48	40	32	22

32.

x	0	10	20	30	40	50	60
$f(x)$	26	30	28	22	28	32	30

x	70	80	90	100	110	120
$f(x)$	33	31	28	30	32	22

In exercises 33 and 34, the velocity of an object at various times is given. Use the data to estimate the distance traveled.

33.

t (s)	0	1	2	3	4	5	6
$v(t)$ (ft/s)	40	42	40	44	48	50	46

t (s)	7	8	9	10	11	12
$v(t)$ (ft/s)	46	42	44	40	42	42

34.

t (s)	0	2	4	6	8	10	12
$v(t)$ (ft/s)	26	30	28	30	28	32	30

t (s)	14	16	18	20	22	24
$v(t)$ (ft/s)	33	31	28	30	32	32

In exercises 35 and 36, the data come from a pneumotachograph, which measures air flow through the throat (in liters per second). The integral of the air flow equals the volume of air exhaled. Estimate this volume.

35.

t (s)	0	0.2	0.4	0.6	0.8	1.0	1.2
$f(t)$ (l/s)	0	0.2	0.4	1.0	1.6	2.0	2.2

t (s)	1.4	1.6	1.8	2.0	2.2	2.4
$f(t)$ (l/s)	2.0	1.6	1.2	0.6	0.2	0

36.

t (s)	0	0.2	0.4	0.6	0.8	1.0	1.2
$f(t)$ (l/s)	0	0.1	0.4	0.8	1.4	1.8	2.0

t (s)	1.4	1.6	1.8	2.0	2.2	2.4
$f(t)$ (l/s)	2.0	1.6	1.0	0.6	0.2	0

In exercises 37–42, use the given information about $f(x)$ and its derivatives to determine whether (a) the Midpoint Rule would be exact, underestimate or overestimate the integral (or there's not enough information to tell). Repeat for (b) Trapezoidal Rule and (c) Simpson's Rule.

37. $f''(x) > 0,\ f'(x) > 0$

38. $f''(x) > 0,\ f'(x) < 0$

39. $f''(x) < 0,\ f'(x) > 0$

40. $f''(x) < 0,\ f'(x) < 0$

41. $f''(x) = 4,\ f'(x) > 0$

42. $f''(x) = 0,\ f'(x) > 0$

43. Suppose that R_L and R_R are the Riemann sum approximations of $\int_a^b f(x)\,dx$ using left- and right-endpoint evaluation rules, respectively, for some $n > 0$. Show that the trapezoidal approximation T_n is equal to $(R_L + R_R)/2$.

44. For the data in Figure 4.30, sketch in the two approximating parabolas for Simpson's Rule. Compare the Simpson's Rule approximation to the Trapezoidal Rule approximation. Explain graphically why the Simpson's Rule approximation is smaller.

45. Show that both $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ and $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$ equal $\dfrac{\pi}{4}$. Use Simpson's Rule on each integral with $n = 4$ and $n = 8$ and compare to the exact value. Which integral provides a better algorithm for estimating π?

46. Prove the following formula, which is basic to Simpson's Rule. If $f(x) = Ax^2 + Bx + C$, then $\int_{-h}^{h} f(x)\,dx = \frac{h}{3}[f(-h) + 4f(0) + f(h)]$.

47. A commonly used type of numerical integration algorithms is called **Gaussian quadrature.** For an integral on the interval $[-1, 1]$, a simple Gaussian quadrature approximation is $\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$. Show that, like Simpson's Rule, this Gaussian quadrature gives the exact value of the integrals of the power functions x, x^2 and x^3.

48. Referring to exercise 47, compare the Simpson's Rule ($n = 2$) and Gaussian quadrature approximations of $\int_{-1}^{1} \pi\cos\left(\frac{\pi x}{2}\right) dx$ to the exact value.

49. Explain why Simpson's Rule can't be used to approximate $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx$. Find $L = \displaystyle\lim_{x\to 0} \frac{\sin x}{x}$ and argue that if $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ L & \text{if } x = 0 \end{cases}$ then $\displaystyle\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{\sin x}{x}\,dx$. Use an appropriate numerical method to conjecture that $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \approx 1.18\left(\frac{\pi}{2}\right)$.

50. As in exercise 49, approximate $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x}\,dx$.

51. In most of the calculations that you have done, it is true that the Trapezoidal Rule and Midpoint Rule are on opposite sides of the exact integral (i.e., one is too large, the other too small). Also, you may have noticed that the Trapezoidal Rule tends to be about twice as far from the exact value as the Midpoint Rule. Given this, explain why the linear combination $\frac{1}{3}T_n + \frac{2}{3}M_n$ should give a good estimate of the integral. (Here, T_n represents the Trapezoidal Rule approximation using n partitions and M_n the corresponding Midpoint Rule approximation.)

52. Show that the approximation rule $\frac{1}{3}T_n + \frac{2}{3}M_n$ in exercise 51 is identical to Simpson's Rule.

EXPLORATORY EXERCISES

1. Compute the Trapezoidal Rule approximations T_4, T_8 and T_{16} of $\int_0^1 3x^2\,dx$, and compute the error (the difference between the approximation and the exact value of 1). Verify that when the step size is cut in half, the error is divided by four. When such patterns emerge, they can be taken advantage of using **extrapolation.** The idea is simple: if the approximations continually get smaller, then the value of the integral is smaller and we should be able to predict (extrapolate) how much smaller the integral is. Given that $(T_4 - I) = 4(T_8 - I)$, where $I = 1$ is the exact integral, show that $I = T_8 + \dfrac{T_8 - T_4}{3}$. Also, show that $I = T_{16} + \dfrac{T_{16} - T_8}{3}$. In general, we have the *approximations* $(T_4 - I) \approx 4(T_8 - I)$ and $I \approx T_8 + \dfrac{T_8 - T_4}{3}$. Then the extrapolation $E_{2n} = T_{2n} + \dfrac{T_{2n} - T_n}{3}$ is closer to the exact integral than either of the individual Trapezoidal Rule approximations T_{2n} and T_n. Show that, in fact, E_{2n} equals the Simpson's Rule approximation for $2n$.

2. The geometric construction of Simpson's Rule makes it clear that Simpson's Rule will compute integrals such as $\int_0^1 3x^2\,dx$ exactly. Briefly explain why. Now, compute Simpson's Rule with $n = 2$ for $\int_0^1 4x^3\,dx$. It turns out that Simpson's Rule also computes integrals of cubics exactly. In this exercise, we want to understand why a method that uses parabolas can compute integrals of cubics exactly. But first, sketch out the Midpoint Rule approximation of $\int_0^1 2x\,dx$ with $n = 1$. On part of the interval, the midpoint rectangle is above the straight line and on part of the interval, the midpoint rectangle is below the line. Explain why the Midpoint Rule computes the area exactly. Now, back to Simpson's Rule. To see how Simpson's Rule works on $\int_0^1 4x^3\,dx$, we need to determine the actual parabola being used. The parabola must pass through the points $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(1, 4)$. Find the quadratic function $y = ax^2 + bx + c$ that accomplishes this. (Hint: Explain why $0 = 0 + 0 + c$, $\dfrac{1}{2} = \dfrac{a}{4} + \dfrac{b}{2} + c$ and $4 = a + b + c$, and then solve for a, b and c.) Graph this parabola and $y = 4x^3$ on the same axes, carefully choosing the graphing window so that you can see what is happening on the interval $[0, 1]$. Where is the vertex of the parabola? How do the integrals of the parabola and cubic compare on the subinterval $[0, \frac{1}{2}]$? $[\frac{1}{2}, 1]$? Why does Simpson's Rule compute the integral exactly?

4.8 THE NATURAL LOGARITHM AS AN INTEGRAL

In Chapter 0, we defined the natural logarithm as the ordinary logarithm with base e. That is,

$$\ln x = \log_e x,$$

where e is a (so far) mysterious transcendental number, $e \approx 2.718\ldots$. So, why would this be called *natural* and why would anyone be interested in such a seemingly unusual function? We will resolve both of these issues in this section.

First, recall the power rule for integrals,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad \text{for } n \neq -1.$$

Of course, this rule doesn't hold for $n = -1$, since this would result in division by zero. Assume, for the moment, that we have not yet defined $\ln x$. Then, what can we say about

$$\int \frac{1}{x}\,dx?$$

(Although we found this integral in section 4.1, our observation hinged on the conjecture that $\frac{d}{dx}\ln x = \frac{1}{x}$, which we have not yet proved.) From Theorem 4.1, we know that since $f(x) = \frac{1}{x}$ is continuous for $x \neq 0$, it must be integrable on any interval not including $x = 0$. The question is how to find an antiderivative. Notice that by Part II of the Fundamental Theorem of Calculus,

$$\int_1^x \frac{1}{t}\,dt$$

is an antiderivative of $\frac{1}{x}$ for $x > 0$. We give this new (and naturally arising) function a name in Definition 8.1.

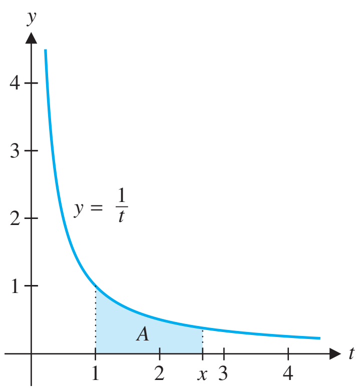


FIGURE 4.35a
$\ln x\ (x > 1)$

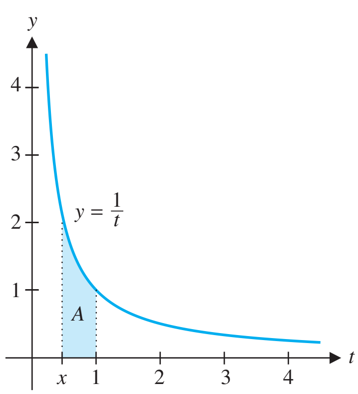


FIGURE 4.35b
$\ln x\ (0 < x < 1)$

DEFINITION 8.1

For $x > 0$, we define the **natural logarithm** function, written $\ln x$, by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

We'll see later that this definition is, in fact, consistent with how we defined the function in Chapter 0. First, let's interpret this function graphically. Notice that for $x > 1$, this definite integral corresponds to the area A under the curve $y = \frac{1}{t}$ from 1 to x, as indicated in Figure 4.35a. That is,

$$\ln x = \int_1^x \frac{1}{t}\,dt = A > 0.$$

Similarly, for $0 < x < 1$, notice from Figure 4.35b that for the area A under the curve $y = \frac{1}{t}$ from x to 1, we have

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt = -A < 0.$$

Using Definition 8.1, we get by Part II of the Fundamental Theorem of Calculus that

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}, \quad \text{for } x > 0, \tag{8.1}$$

which is the same derivative formula we had obtained in section 2.7.

Recall that we had shown in section 4.1 that for any $x \neq 0$, we can extend (8.1) to obtain $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This, in turn gives us the familiar integration rule

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c.}$$

EXAMPLE 8.1 Approximating Several Values of the Natural Logarithm

Approximate the value of ln 2 and ln 3.

Solution Notice that since $\ln x$ is defined by a definite integral, we can use any convenient numerical integration method to compute approximate values of the function. For instance, using Simpson's Rule, we obtain

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx 0.693147$$

and

$$\ln 3 = \int_1^3 \frac{1}{t}\,dt \approx 1.09861.$$

We leave the details of these approximations as exercises. (You should also check the values with the 'ln' key on your calculator.) ■

We now briefly sketch a graph of $y = \ln x$. As we've already observed, the domain of $f(x) = \ln x$ is $(0, \infty)$ and

$$\ln x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$$

Further, we've shown that

$$f'(x) = \frac{1}{x} > 0, \quad \text{for } x > 0,$$

so that f is increasing throughout its domain. Next,

$$f''(x) = -\frac{1}{x^2} < 0, \quad \text{for } x > 0$$

and hence, the graph is concave down everywhere. You can easily use Simpson's Rule or the Trapezoidal Rule (this is left as an exercise) to make the conjectures

$$\lim_{x\to\infty} \ln x = \infty \tag{8.2}$$

and

$$\lim_{x\to 0^+} \ln x = -\infty. \tag{8.3}$$

We postpone the proof of (8.2) until after Theorem 8.1. The proof of (8.3) is left as an exercise. We now obtain the graph pictured in Figure 4.36.

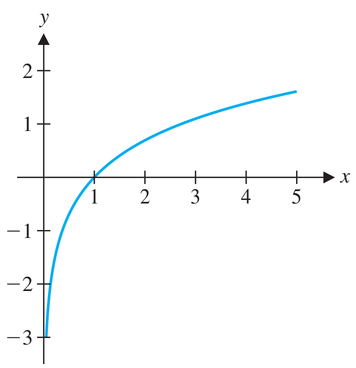


FIGURE 4.36
$y = \ln x$

Now, it remains for us to explain why this function should be called a *logarithm*. The answer is simple: it satisfies all of the properties satisfied by other logarithms. Since $\ln x$ behaves like any other logarithm, we call it (what else?) a logarithm. We summarize this in Theorem 8.1.

THEOREM 8.1

For any real numbers $a, b > 0$ and any rational number r,

(i) $\ln 1 = 0$
(ii) $\ln(ab) = \ln a + \ln b$
(iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ and
(iv) $\ln(a^r) = r \ln a$.

PROOF

(i) From Definition 8.1,

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

(ii) Also from the definition, we have

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt,$$

from part (ii) of Theorem 4.2 in section 4.4. Make the substitution $u = \frac{t}{a}$ in the last integral only. This gives us $du = \frac{1}{a}\,dt$. Finally, the limits of integration must be changed to reflect the new variable (when $t = a$, we have $u = \frac{a}{a} = 1$, and when $t = ab$, we have $u = \frac{ab}{a} = b$) to yield

$$\begin{aligned}\ln(ab) &= \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \underbrace{\frac{a}{t}}_{\frac{1}{u}} \underbrace{\left(\frac{1}{a}\right) dt}_{du} \\ &= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{u}\,du = \ln a + \ln b. \quad \text{From Definition 8.1.}\end{aligned}$$

(iv) Note that

$$\begin{aligned}\frac{d}{dx}\ln(x^r) &= \frac{1}{x^r}\frac{d}{dx}x^r && \text{From (8.1) and the chain rule.} \\ &= \frac{1}{x^r} r x^{r-1} = \frac{r}{x}. && \text{From the power rule.}\end{aligned}$$

Likewise,

$$\frac{d}{dx}[r \ln x] = r\frac{d}{dx}(\ln x) = \frac{r}{x}.$$

Now, since $\ln(x^r)$ and $r \ln x$ have the same derivative, it follows from Corollary 9.1 in section 2.9 that for all $x > 0$,

$$\ln(x^r) = r \ln x + k,$$

for some constant, k. In particular, taking $x = 1$, we find that

$$\ln(1^r) = r \ln 1 + k,$$

where since $1^r = 1$ and $\ln 1 = 0$, we have

$$0 = r(0) + k.$$

So, $k = 0$ and $\ln(x^r) = r \ln x$, for all $x > 0$.

Part (iii) follows from (ii) and (iv) and is left as an exercise. ■

Using the properties of logarithms will often simplify the calculation of derivatives. We illustrate this in example 8.2.

EXAMPLE 8.2 Using Properties of Logarithms to Simplify Differentiation

Find the derivative of $\ln \sqrt{\dfrac{(x-2)^3}{x^2+5}}$.

Solution Rather than directly differentiating this expression by applying the chain rule and the quotient rule, notice that we can considerably simplify our work by first using the properties of logarithms. We have

$$\begin{aligned}
\frac{d}{dx} \ln \sqrt{\frac{(x-2)^3}{x^2+5}} &= \frac{d}{dx} \ln \left[\frac{(x-2)^3}{x^2+5}\right]^{1/2} \\
&= \frac{1}{2} \frac{d}{dx} \ln \left[\frac{(x-2)^3}{x^2+5}\right] && \text{From Theorem 8.1 (iv).} \\
&= \frac{1}{2} \frac{d}{dx} [\ln(x-2)^3 - \ln(x^2+5)] && \text{From Theorem 8.1 (iii).} \\
&= \frac{1}{2} \frac{d}{dx} [3 \ln(x-2) - \ln(x^2+5)] && \text{From Theorem 8.1 (iv).} \\
&= \frac{1}{2} \left[3 \left(\frac{1}{x-2}\right) \frac{d}{dx}(x-2) - \left(\frac{1}{x^2+5}\right) \frac{d}{dx}(x^2+5)\right] && \text{From (8.1) and the chain rule.} \\
&= \frac{1}{2} \left(\frac{3}{x-2} - \frac{2x}{x^2+5}\right).
\end{aligned}$$

Compare our work here to computing the derivative directly using the original expression to see how the rules of logarithms have simplified our work. ■

EXAMPLE 8.3 Examining the Limiting Behavior of ln x

Use the properties of logarithms in Theorem 8.1 to prove that

$$\lim_{x \to \infty} \ln x = \infty.$$

Solution We can verify this as follows. First, recall that $\ln 3 \approx 1.0986 > 1$. Taking $x = 3^n$, we have by the rules of logarithms that for any integer n,

$$\ln 3^n = n \ln 3.$$

Since $x = 3^n \to \infty$, as $n \to \infty$, it now follows that

$$\lim_{x \to \infty} \ln x = \lim_{n \to \infty} \ln 3^n = \lim_{n \to \infty} (n \ln 3) = +\infty,$$

where the first equality depends on the fact that $\ln x$ is a strictly increasing function. ■

The Exponential Function as the Inverse of the Natural Logarithm

Next, we revisit the natural exponential function, e^x. As we did with the natural logarithm, we now carefully define this function and develop its properties. First, recall that in Chapter 0, we gave the (usual) mysterious description of e as an irrational number $e = 2.71828\ldots$, without attempting to explain why this number is significant. We then proceeded to define $\ln x$ as $\log_e x$, the logarithm with base e. Now that we have carefully defined $\ln x$ (independent of the definition of e), we can clearly define e, as well as calculate its approximate value.

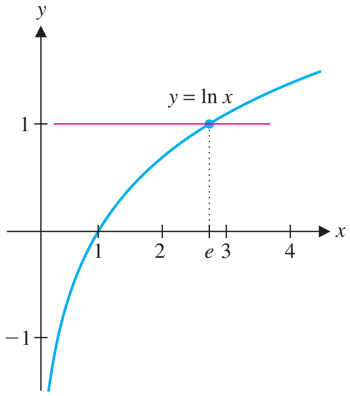


FIGURE 4.37
Definition of e

DEFINITION 8.2

We define e to be that number for which

$$\ln e = 1.$$

That is, e is the x-coordinate of the point of intersection of the graphs of $y = \ln x$ and $y = 1$ (see Figure 4.37). In other words, e is the solution of the equation

$$\ln x - 1 = 0.$$

You can solve this equation approximately (e.g., using Newton's method) to obtain

$$e \approx 2.71828182846.$$

In Chapter 0, we had defined e by $e = \lim_{n\to\infty}(1 + 1/n)^n$. We leave it as an exercise to show that these two definitions of e define the same number. So, having defined the irrational number e, you might wonder what the big deal is with defining the function e^x? Of course, there's no problem at all, when x is rational. For instance, we have

$$\begin{aligned} e^2 &= e \cdot e \\ e^3 &= e \cdot e \cdot e \\ e^{1/2} &= \sqrt{e} \\ e^{5/7} &= \sqrt[7]{e^5} \end{aligned}$$

and so on. In fact, for any rational power, $x = p/q$ (where p and q are integers), we have

$$e^x = e^{p/q} = \sqrt[q]{e^p}.$$

On the other hand, what does it mean to raise a number to an *irrational* power? For instance, what is e^π? In Chapter 0, we gave a necessarily vague answer to this important question. We are now in a position to give a complete answer.

First, observe that for $f(x) = \ln x$ $(x > 0)$, $f'(x) = 1/x > 0$. So, f is a strictly increasing and hence, one-to-one function, which therefore, has an inverse, f^{-1}. As is often the case, there is no algebraic method of solving for this inverse function. However, from Theorem 8.1 (iv), we have that for any rational power x,

$$\ln(e^x) = x \ln e = x,$$

since we had defined e so that $\ln e = 1$. Observe that this says that

$$f^{-1}(x) = e^x, \quad \text{for } x \text{ rational}.$$

That is, the (otherwise unknown) inverse function, $f^{-1}(x)$, agrees with e^x at every rational number x. Since e^x so far has no meaning when x is irrational, we now *define* it to be the value of $f^{-1}(x)$, as follows.

DEFINITION 8.3

For x irrational, we define $y = e^x$ to be that number for which

$$\ln y = \ln(e^x) = x.$$

This says that for any irrational exponent x, we define e^x to be that real number for which $\ln(e^x) = x$. According to this definition, notice that for any $x > 0$, $e^{\ln x}$ is that real number for which

$$\ln(e^{\ln x}) = \ln x. \tag{8.4}$$

Since $\ln x$ is a one-to-one function, (8.4) says that

$$\boxed{e^{\ln x} = x, \quad \text{for } x > 0.} \tag{8.5}$$

Notice that (8.5) says that

$$\ln x = \log_e x.$$

That is, the integral definition of $\ln x$ is consistent with our earlier definition of $\ln x$ as $\log_e x$. Observe also that with this definition of the exponential function, we have

$$\boxed{\ln(e^x) = x, \quad \text{for all } x \in (-\infty, \infty).}$$

Together with (8.5), this says that e^x and $\ln x$ are inverse functions. Keep in mind that for x irrational, e^x is defined *only* through the inverse function relationship given in Definition 8.3. We now state some familiar laws of exponents and prove that they hold even for the case of irrational exponents.

THEOREM 8.2

For r, s any real numbers and t any rational number,

(i) $e^r e^s = e^{r+s}$

(ii) $\dfrac{e^r}{e^s} = e^{r-s}$ and

(iii) $(e^r)^t = e^{rt}$.

PROOF

These laws are already known when the exponents are rational. If the exponent is irrational though, we only know the value of these exponentials indirectly, through the inverse function relationship with $\ln x$, given in Definition 8.3.

(i) Note that using the rules of logarithms, we have

$$\ln(e^r e^s) = \ln(e^r) + \ln(e^s) = r + s = \ln(e^{r+s}).$$

Since $\ln x$ is one-to-one, it must follow that

$$e^r e^s = e^{r+s}.$$

The proofs of (ii) and (iii) are similar and are left as exercises. ■

In Chapter 2, we found the derivative of e^x using the limit definition of derivative. You may recall that the derivation was complete, except for the evaluation of the limit

$$\lim_{h\to 0} \frac{e^h - 1}{h}.$$

At the time, we conjectured that the value of this limit is 1, but we were unable to prove it, with the tools at our disposal. We revisit this limit in exercises 41 and 42 at the end of this section. We now present an alternative derivation, based on our new, refined definition of the exponential function. Again, from Definition 8.3, we have that

$$y = e^x \quad \text{if and only if} \quad \ln y = x.$$

Differentiating this last equation with respect to x gives us

$$\frac{d}{dx} \ln y = \frac{d}{dx} x = 1.$$

From the chain rule, we now have

$$1 = \frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}. \tag{8.6}$$

Multiplying both sides of (8.6) by y, we have

$$\frac{dy}{dx} = y = e^x$$

or

$$\frac{d}{dx}(e^x) = e^x. \tag{8.7}$$

Note that (8.7) is the same derivative formula as that conjectured in Chapter 2, but we have now carefully verified it. Of course, this also verifies the corresponding integration rule

$$\int e^x \, dx = e^x + c.$$

We now have the tools to revisit the graph of $f(x) = e^x$. Since $e = 2.718\ldots > 1$, we have

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to -\infty} e^x = 0.$$

We also have that

$$f'(x) = e^x > 0,$$

so that f is increasing for all x and

$$f''(x) = e^x > 0,$$

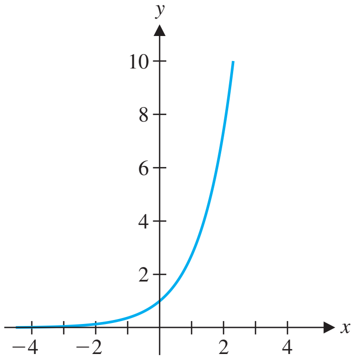


FIGURE 4.38
$y = e^x$

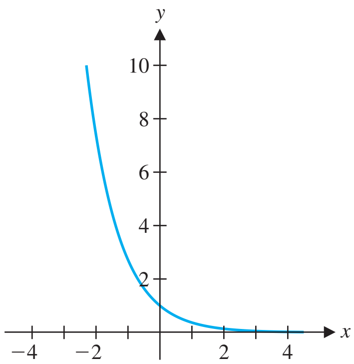


FIGURE 4.39
$y = e^{-x}$

NOTES

We will occasionally write $e^x = \exp(x)$. This is particularly helpful when the exponent is a complicated expression. For example,

$$\exp(x^3 - 5x^2 + 2x + 7) = e^{x^3 - 5x^2 + 2x + 7},$$

where the former is more easily read than the latter.

so that the graph is concave up everywhere. You should now readily obtain the graph in Figure 4.38.

Similarly, for $f(x) = e^{-x}$, we have

$$\lim_{x \to \infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-x} = \infty.$$

Further, from the chain rule, $\qquad f'(x) = -e^{-x} < 0,$

so that f is decreasing for all x. We also have

$$f''(x) = e^{-x} > 0,$$

so that the graph is concave up everywhere. You should easily obtain the graph in Figure 4.39.

More general exponential functions, such as $f(x) = b^x$, for any base $b > 0$, are easy to express in terms of the natural exponential function, as follows. Notice that by the usual rules of logs and exponentials, we have

$$\boxed{b^x = e^{\ln(b^x)} = e^{x \ln b}.}$$

It now follows that

$$\begin{aligned} \frac{d}{dx} b^x &= \frac{d}{dx} e^{x \ln b} = e^{x \ln b} \frac{d}{dx}(x \ln b) \\ &= e^{x \ln b}(\ln b) = b^x(\ln b), \end{aligned}$$

as we had conjectured in section 2.7. Similarly, for $b > 0\,(b \neq 1)$, we have

$$\begin{aligned} \int b^x \, dx &= \int e^{x \ln b} dx = \frac{1}{\ln b} \int e^{\overbrace{x \ln b}^{u}} \underbrace{(\ln b)\, dx}_{du} \\ &= \frac{1}{\ln b} e^{x \ln b} + c = \frac{1}{\ln b} b^x + c. \end{aligned}$$

You can now see that the general exponential functions are easily dealt with in terms of the natural exponential. In fact, you should not bother to memorize the formulas for the derivatives and integrals of general exponentials. Rather, each time you encounter the exponential function $f(x) = b^x$, simply rewrite it as $f(x) = e^{x \ln b}$ and then use the familiar rules for the derivative and integral of the natural exponential and the chain rule.

In a similar way, we can use our knowledge of the natural logarithm to discuss more general logarithms. First, recall that for any base $a > 0\,(a \neq 1)$ and any $x > 0$, $y = \log_a x$ if and only if $x = a^y$. Taking the natural logarithm of both sides of this equation, we have

$$\ln x = \ln(a^y) = y \ln a.$$

Solving for y gives us

$$y = \frac{\ln x}{\ln a},$$

which proves Theorem 8.3.

THEOREM 8.3

For any base $a > 0\,(a \neq 1)$ and any $x > 0$, $\log_a x = \dfrac{\ln x}{\ln a}$.

Calculators typically have built-in programs for evaluating $\ln x$ and $\log_{10} x$, but not for general logarithms. Theorem 8.3 enables us to easily evaluate logarithms with any base. For instance, we have

$$\log_7 3 = \frac{\ln 3}{\ln 7} \approx 0.564575.$$

More importantly, observe that we can use Theorem 8.3 to find derivatives of general logarithms in terms of the derivative of the natural logarithm. In particular, for any base $a > 0\,(a \neq 1)$, we have

$$\begin{aligned}\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln x)\\ &= \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{x\ln a}.\end{aligned}$$

As with the derivative formula for general exponentials, there is little point in learning this as a new differentiation rule. Rather, simply use Theorem 8.3.

BEYOND FORMULAS

You might wonder why we have returned to the natural logarithm and exponential functions to carefully define them. Part of the answer is versatility. The integral definition of the logarithm gives you a convenient formula for working with properties of the natural logarithm. A more basic reason for this section is to be sure that the logarithm function is not just a button on your calculator. Instead, $\ln x$ can be thought of in terms of area. You can visualize this easily and use this image to help understand properties of the function. What are some examples from your life (such as behavior rules or sports techniques) where you used a rule before understanding the reason for the rule? Did understanding the rule help?

EXERCISES 4.8

WRITING EXERCISES

1. Explain why it is mathematically legal to define $\ln x$ as $\int_1^x \frac{1}{t}\,dt$. For some, this type of definition is not very satisfying. Try the following comparison. Clearly, it is easier to compute function values for x^2 than for $\ln x$ and therefore x^2 is easier to understand. However, compare how you would compute (without a calculator) function values for $\sin x$ versus function values for $\ln x$. Describe which is more "natural" and easy to understand.

2. In this section, we discussed two different "definitions" of $\ln x$. Explain why it is logically invalid to give different definitions unless you can show that they define the same thing. If they define the same object, both definitions are equally valid and you should use whichever definition is clearer for the task at hand. Explain why, in this section, the integral definition is more convenient than the base e logarithm.

3. Use the integral definition of $\ln x$ (interpreted as area) to explain why it is reasonable that $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$.

4. Use the integral definition of $\ln x$ (interpreted as area) to explain why the graph of $\ln x$ is increasing and concave down for $x > 0$.

In exercises 1–4, express the number as an integral and sketch the corresponding area.

1. $\ln 4$
2. $\ln 5$
3. $\ln 8.2$
4. $\ln 24$
5. Use Simpson's Rule with $n = 4$ to estimate $\ln 4$.
6. Use Simpson's Rule with $n = 4$ to estimate $\ln 5$.

7. Use Simpson's Rule with (a) $n = 32$ and (b) $n = 64$ to estimate $\ln 4$.

8. Use Simpson's Rule with (a) $n = 32$ and (b) $n = 64$ to estimate $\ln 5$.

In exercises 9–12, use the properties of logarithms to rewrite the expression as a single term.

9. $\ln\sqrt{2} + 3\ln 2$

10. $\ln 8 - 2\ln 2$

11. $2\ln 3 - \ln 9 + \ln\sqrt{3}$

12. $2\ln\left(\frac{1}{3}\right) - \ln 3 + \ln\left(\frac{1}{9}\right)$

In exercises 13–16, evaluate the derivative using properties of logarithms where needed.

13. $\dfrac{d}{dx}(\ln\sqrt{x^2+1})$

14. $\dfrac{d}{dx}[\ln(x^5 \sin x \cos x)]$

15. $\dfrac{d}{dx}\left(\ln\dfrac{x^4}{x^5+1}\right)$

16. $\dfrac{d}{dx}\left(\ln\sqrt{\dfrac{x^3}{x^5+1}}\right)$

In exercises 17–30, evaluate the integral.

17. $\displaystyle\int \frac{3x^3}{x^4+5}\,dx$

18. $\displaystyle\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$

19. $\displaystyle\int \frac{1}{x\ln x}\,dx$

20. $\displaystyle\int \frac{1}{\sqrt{1-x^2}\sin^{-1}x}\,dx$

21. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$

22. $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

23. $\displaystyle\int \frac{e^{2/x}}{x^2}\,dx$

24. $\displaystyle\int \frac{\sin(\ln x^3)}{x}\,dx$

25. $\displaystyle\int_0^1 \frac{x^2}{x^3-4}\,dx$

26. $\displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

27. $\displaystyle\int_0^1 \tan x\,dx$

28. $\displaystyle\int_1^2 \frac{\ln x}{x}\,dx$

29. $\displaystyle\int_0^1 \frac{e^x - 1}{e^{2x}}\,dx$

30. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

In exercises 31–36, graph the function.

31. $y = \ln(x-2)$

32. $y = \ln(3x+5)$

33. $y = \ln(x^2+1)$

34. $y = \ln(x^3+1)$

35. $y = x\ln x$

36. $y = x^2\ln x$

37. Use property (ii) of Theorem 8.1 to prove property (iii) that $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$.

38. Starting with $e^x = \displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, show that $\ln x = \displaystyle\lim_{n\to\infty}[n(x^{1/n}-1)]$. Assume that if $\displaystyle\lim_{n\to\infty} x_n = x$, then $\displaystyle\lim_{n\to\infty}\left[n(x_n^{1/n}-1)\right] = \lim_{n\to\infty}[n(x^{1/n}-1)]$.

39. The **sigmoid function** $f(x) = \dfrac{1}{1+e^{-x}}$ is used to model situations with a threshold. For example, in the brain, each neuron receives inputs from numerous other neurons and fires only after its total input crosses some threshold. Graph $y = f(x)$ and find $\displaystyle\lim_{x\to-\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} f(x)$. Define the function $g(x)$ to be the value of $f(x)$ rounded off to the nearest integer. What value of x is the threshold for this function to switch from "off" (0) to "on" (1)? How could you modify the function to move the threshold to $x = 4$ instead?

40. Suppose you have a 1-in-10 chance of winning a prize with some purchase (like a lottery). If you make 10 purchases (i.e., you get 10 tries), the probability of winning at least one prize is $1-(9/10)^{10}$. If the prize had probability 1-in-20 and you tried 20 times, would the probability of winning at least once be higher or lower? Compare $1-(9/10)^{10}$ and $1-(19/20)^{20}$ to find out. To see what happens for larger and larger odds, compute $\displaystyle\lim_{n\to\infty}[1-((n-1)/n)^n]$.

41. In the text, we deferred the proof of $\displaystyle\lim_{h\to\infty}\frac{e^h-1}{h} = 1$ to the exercises. In this exercise, we guide you through one possible proof. (Another proof is given in exercise 42.) Starting with $h > 0$, write $h = \ln e^h = \displaystyle\int_1^{e^h}\frac{1}{x}\,dx$. Use the Integral Mean Value Theorem to write $\displaystyle\int_1^{e^h}\frac{1}{x}\,dx = \frac{e^h-1}{\bar{x}}$ for some number $\bar{x}$ between 1 and e^h. This gives you $\dfrac{e^h-1}{h} = \bar{x}$. Now, take the limit as $h \to 0^+$. For $h < 0$, repeat this argument, with h replaced with $-h$.

42. In this exercise, we guide you through a different proof of $\displaystyle\lim_{h\to 0}\frac{e^h-1}{h} = 1$. Start with $f(x) = \ln x$ and the fact that $f'(1) = 1$. Using the alternative definition of derivative, we write this as $f'(1) = \displaystyle\lim_{x\to 1}\frac{\ln x - \ln 1}{x-1} = 1$. Explain why this implies that $\displaystyle\lim_{x\to 1}\frac{x-1}{\ln x} = 1$. Finally, substitute $x = e^h$.

43. In this exercise, we show that if $e = \displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, then $\ln e = 1$. Define $x_n = \left(1+\dfrac{1}{n}\right)^n$. By the continuity of $\ln x$, we have $\ln e = \ln\left[\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right] = \lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n}\right)^n\right]$. Use l'Hôpital's Rule on $\displaystyle\lim_{n\to\infty}\left[\frac{\ln\left(1+\frac{1}{n}\right)}{1/n}\right]$ to evaluate this limit.

44. Apply Newton's method to the function $f(x) = \ln x - 1$ to find an iterative scheme for approximating e. Discover how many steps are needed to start at $x_0 = 3$ and obtain five digits of accuracy.

45. A telegraph cable is made of an outer winding around an inner core. If x is defined as the core radius divided by the outer radius, the transmission speed is proportional to $s(x) = x^2 \ln(1/x)$. Find an x that maximizes the transmission speed.

EXPLORATORY EXERCISES

1. A special function used in many applications is the **error function,** defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} f(x)$, where $f(x) = \int_0^x e^{-u^2}\,du$. Explore the function $f(x)$. For which x's is $f(x)$ positive? Negative? Increasing? Decreasing? Concave up? Concave down? Estimate some function values for large x. Conjecture $\lim_{x\to\infty} f(x)$. Sketch a graph of $f(x)$.

2. Verify that $\int \sec x\,dx = \ln|\sec x + \tan x| + c$. (Hint: Differentiate the suspected antiderivative and show that you get the integrand.) This integral appears in the construction of a special type of map called a Mercator map. On this map, the latitude lines are not equally spaced. Instead, they are placed so that straight lines on a Mercator map correspond to paths of constant heading. (If you travel due northeast, your path on a map with equally spaced latitude will appear to curve due to the curvature of the earth.) Let R be the (average) radius of the earth. Assuming the earth is a sphere, the actual distance from the equator to a place at latitude $b°$ is $\frac{\pi}{180}Rb$. On a Mercator map, this distance is scaled to $\frac{\pi}{180}R\int_0^b \sec x\,dx$. Tampa, Florida, has latitude $28°$ north. Moscow, Russia, is twice as far from the equator at $56°$ north. What is the spacing for Tampa and Moscow on a Mercator map?

3. Define the **log integral function** $Li(x) = \int_0^x \frac{1}{\ln t}\,dt$ for $x > 1$. For $x = 4$ and $n = 4$, explain why Simpson's Rule does not give an estimate of $Li(4)$. Sketch a picture of the area represented by $Li(4)$. It turns out that $Li(x) = 0$ for $x \approx 1.45$. Explain why $Li(4) \approx \int_{1.45}^4 \frac{1}{\ln t}\,dt$ and estimate this with Simpson's Rule using $n = 4$. This function is used to estimate $\pi(N)$, the number of prime numbers less than N. Another common estimate of $\pi(N)$ is $\frac{N}{\ln N}$. Estimate $\frac{N}{\ln N}$, $\pi(N)$ and $Li(N)$ for (a) $N = 20$; (b) $N = 40$ and (c) $N = 100{,}000{,}000$, where we'll give you $\pi(N) = 5{,}761{,}455$. Discuss any patterns that you find. (See *Prime Obsession* by John Derbyshire for more about this area of number theory.)

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Area	Average value	Indefinite integral
Signed area	Integration by substitution	Simpson's Rule
Midpoint Rule	Trapezoidal Rule	Natural logarithm
Integral Mean Value Theorem	Fundamental Theorem of Calculus	
Riemann sum	Definite integral	

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. The Midpoint Rule always gives better approximations than left-endpoint evaluation.
2. The larger n is, the better is the Riemann sum approximation.
3. All piecewise continuous functions are integrable.
4. The definite integral of velocity gives the total distance traveled.
5. There are some elementary functions that do not have an antiderivative.
6. To evaluate a definite integral, you can use *any* antiderivative.
7. A substitution is not correct unless the derivative term du is present in the original integrand.
8. With Simpson's Rule, if n is doubled, the error is reduced by a factor of 16.

In exercises 1–20, find the antiderivative.

1. $\int (4x^2 - 3)\,dx$ **2.** $\int (x - 3x^5)\,dx$

Review Exercises

3. $\int \frac{4}{x} dx$

4. $\int \frac{4}{x^2} dx$

5. $\int 2\sin 4x\, dx$

6. $\int 3\sec^2 x\, dx$

7. $\int (x - e^{4x})\, dx$

8. $\int 3\sqrt{x}\, dx$

9. $\int \frac{x^2+4}{x} dx$

10. $\int \frac{x}{x^2+4} dx$

11. $\int e^x(1 - e^{-x})\, dx$

12. $\int e^x(1 + e^x)^2 dx$

13. $\int x\sqrt{x^2+4}\, dx$

14. $\int x(x^2+4)\, dx$

15. $\int 6x^2 \cos x^3 dx$

16. $\int 4x \sec x^2 \tan x^2 dx$

17. $\int \frac{e^{1/x}}{x^2} dx$

18. $\int \frac{\ln x}{x} dx$

19. $\int \tan x\, dx$

20. $\int \sqrt{3x+1}\, dx$

21. Find a function $f(x)$ satisfying $f'(x) = 3x^2 + 1$ and $f(0) = 2$.

22. Find a function $f(x)$ satisfying $f'(x) = e^{-2x}$ and $f(0) = 3$.

23. Determine the position function if the velocity is $v(t) = -32t + 10$ and the initial position is $s(0) = 2$.

24. Determine the position function if the acceleration is $a(t) = 6$ with initial velocity $v(0) = 10$ and initial position $s(0) = 0$.

25. Write out all terms and compute $\sum_{i=1}^{6}(i^2 + 3i)$.

26. Translate into summation notation and compute: the sum of the squares of the first 12 positive integers.

In exercises 27 and 28, use summation rules to compute the sum.

27. $\sum_{i=1}^{100}(i^2 - 1)$

28. $\sum_{i=1}^{100}(i^2 + 2i)$

29. Compute the sum $\frac{1}{n^3}\sum_{i=1}^{n}(i^2 - i)$ and the limit of the sum as n approaches ∞.

30. For $f(x) = x^2 - 2x$ on the interval $[0, 2]$, list the evaluation points for the Midpoint Rule with $n = 4$, sketch the function and approximating rectangles and evaluate the Riemann sum.

In exercises 31–34, approximate the area under the curve using n rectangles and the given evaluation rule.

31. $y = x^2$ on $[0, 2]$, $n = 8$, midpoint evaluation

32. $y = x^2$ on $[-1, 1]$, $n = 8$, right-endpoint evaluation

33. $y = \sqrt{x+1}$ on $[0, 3]$, $n = 8$, midpoint evaluation

34. $y = e^{-x}$ on $[0, 1]$, $n = 8$, left-endpoint evaluation

In exercises 35 and 36, use the given function values to estimate the area under the curve using (a) left-endpoint evaluation, (b) right-endpoint evaluation, (c) Trapezoidal Rule and (d) Simpson's Rule.

35.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	1.0	1.4	1.6	2.0	2.2	2.4	2.0	1.6	1.4

36.

x	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8	4.2
$f(x)$	4.0	3.4	3.6	3.0	2.6	2.4	3.0	3.6	3.4

37. In exercises 35 and 36, which of the four area estimates would you expect to be the most accurate? Briefly explain.

38. If $f(x)$ is positive and concave up, will the Midpoint Rule give an overestimate or underestimate of the actual area? Will the Trapezoidal Rule give an overestimate or underestimate of the actual area?

In exercises 39 and 40, evaluate the integral by computing the limit of Riemann sums.

39. $\int_0^1 2x^2 dx$

40. $\int_0^2 (x^2 + 1)\, dx$

In exercises 41 and 42, write the total area as an integral or sum of integrals and then evaluate it.

41. The area above the x-axis and below $y = 3x - x^2$

42. The area between the x-axis and $y = x^3 - 3x^2 + 2x$, $0 \le x \le 2$

In exercises 43 and 44, use the velocity function to compute the distance traveled in the given time interval.

43. $v(t) = 40 - 10t$, $[1, 2]$

44. $v(t) = 20e^{-t/2}$, $[0, 2]$

In exercises 45 and 46, compute the average value of the function on the interval.

45. $f(x) = e^x$, $[0, 2]$

46. $f(x) = 4x - x^2$, $[0, 4]$

Review Exercises

In exercises 47–58, evaluate the integral.

47. $\int_0^2 (x^2 - 2)\,dx$

48. $\int_{-1}^1 (x^3 - 2x)\,dx$

49. $\int_0^{\pi/2} \sin 2x\,dx$

50. $\int_0^{\pi/4} \sec^2 x\,dx$

51. $\int_0^{10} (1 - e^{-t/4})\,dt$

52. $\int_0^1 te^{-t^2}\,dt$

53. $\int_0^2 \frac{x}{x^2+1}\,dx$

54. $\int_1^2 \frac{\ln x}{x}\,dx$

55. $\int_0^2 x\sqrt{x^2+4}\,dx$

56. $\int_0^2 x(x^2+1)\,dx$

57. $\int_0^1 (e^x - 2)^2 dx$

58. $\int_{-\pi}^{\pi} \cos(x/2)\,dx$

In exercises 59 and 60, find the derivative.

59. $f(x) = \int_2^x (\sin t^2 - 2)\,dt$

60. $f(x) = \int_0^{x^2} \sqrt{t^2+1}\,dt$

In exercises 61 and 62, compute the (a) Midpoint Rule, (b) Trapezoidal Rule and (c) Simpson's Rule approximations with $n = 4$ by hand.

61. $\int_0^1 \sqrt{x^2+4}\,dx$

62. $\int_0^2 e^{-x^2/4} dx$

63. Repeat exercise 61 using a computer or calculator and $n = 20$; $n = 40$.

64. Repeat exercise 62 using a computer or calculator and $n = 20$; $n = 40$.

EXPLORATORY EXERCISES

1. Suppose that $f(t)$ is the rate of occurrence of some event (e.g., the birth of an animal or the lighting of a firefly). Then the average rate of occurrence R over a time interval $[0, T]$ is $R = \frac{1}{T}\int_0^T f(t)\,dt$. We will assume that the function $f(t)$ is periodic with period T. [That is, $f(t+T) = f(t)$ for all t.] **Perfect asynchrony** means that the event is equally likely to occur at all times. Argue that this corresponds to a constant rate function $f(t) = c$ and find the value of c (in terms of R and T). **Perfect synchrony** means that the event occurs only once every period (e.g., the fireflies all light at the same time, or all babies are born simultaneously). We will see what the rate function $f(t)$ looks like in this case. First, define the **degree of synchrony** to be $\frac{\text{area under } f \text{ and above } R}{RT}$. Show that if $f(t)$ is constant, then the degree of synchrony is 0. Then graph and find the degree of synchrony for the following functions (assuming $T > 2$):

$$f_1(t) = \begin{cases} (RT)(t - \frac{T}{2}) + RT & \text{if } \frac{T}{2} - 1 \le t \le \frac{T}{2} \\ (-RT)(t - \frac{T}{2}) + RT & \text{if } \frac{T}{2} \le t \le \frac{T}{2} + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(t) = \begin{cases} (4RT)(t - \frac{T}{2}) + 2RT & \text{if } \frac{T}{2} - \frac{1}{2} \le t \le \frac{T}{2} \\ (-4RT)(t - \frac{T}{2}) + 2RT & \text{if } \frac{T}{2} \le t \le \frac{T}{2} + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

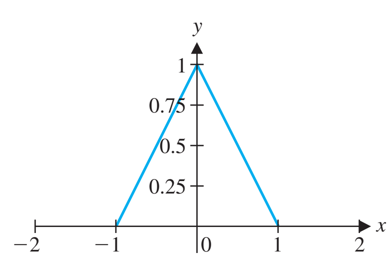
$$f_3(t) = \begin{cases} (9RT)(t - \frac{T}{2}) + 3RT & \text{if } \frac{T}{2} - \frac{1}{3} \le t \le \frac{T}{2} \\ (-9RT)(t - \frac{T}{2}) + 3RT & \text{if } \frac{T}{2} \le t \le \frac{T}{2} + \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

What would you conjecture as the limit of the degrees of synchrony of $f_n(t)$ as $n \to \infty$? The "function" that $f_n(t)$ approaches as $n \to \infty$ is called an **impulse function** of strength RT. Discuss the appropriateness of this name.

2. The **Omega function** is used for risk/reward analysis of financial investments. Suppose that $f(x)$ is a function defined on the interval (A, B) that gives the distribution of returns on an investment. (This means that $\int_a^b f(x)\,dx$ is the probability that the investment returns between \$$a$ and \$$b$.) Let $F(x) = \int_A^x f(t)\,dt$ be the **cumulative distribution function** for returns.

Then $\Omega(r) = \dfrac{\int_r^B [1 - F(x)]\,dx}{\int_A^r F(x)\,dx}$ is the Omega function for the investment.

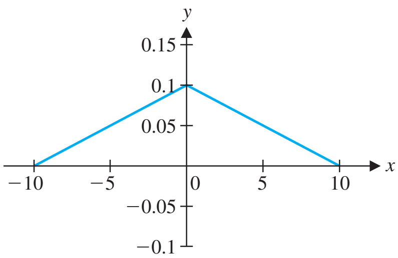
(a) For the distribution $f_1(x)$ shown, compute the cumulative distribution function $F_1(x)$.



$y = f_1(x)$

Review Exercises

(b) Repeat part (a) for the distribution $f_2(x)$ shown.



$y = f_2(x)$

(c) Compute $\Omega_1(r)$ for the distribution $f_1(x)$. Note that $\Omega_1(r)$ will be undefined (∞) for $r \leq -1$ and $\Omega_1(r) = 0$ for $r \geq 1$.

(d) Compute $\Omega_2(r)$ for the distribution $f_2(x)$. Note that $\Omega_2(r)$ will be undefined (∞) for $r \leq -10$ and $\Omega_2(r) = 0$ for $r \geq 10$.

(e) Even though the means (average values) are the same, investments with distributions $f_1(x)$ and $f_2(x)$ are not equivalent. Use the graphs of $f_1(x)$ and $f_2(x)$ to explain why $f_2(x)$ corresponds to a riskier investment than $f_2(x)$.

(f) Show that $\Omega_2(r) > \Omega_1(r)$ for $r > 0$ and $\Omega_2(r) < \Omega_1(r)$ for $r < 0$. In general, the larger $\Omega(r)$ is, the better the investment is. Explain this in terms of this example.