Differentiation

CHAPTER 2

The marathon is one of the most famous running events, covering 26 miles and 385 yards. The race was invented for the 1896 Olympics in Greece to commemorate a famous Greek legend. Following a decisive victory over the Persian army at the Battle of Marathon, an army runner was dispatched to carry the news from Marathon to Athens. According to the legend, the runner reached Athens, shouted, "Rejoice! We conquer!" and then died.

The historic route from Marathon to Athens was used for the 2004 Olympic marathon, won by Stefano Baldini of Italy in a time of 2:10:55. It is interesting to compute his running speed. Using the physics formula "rate equals distance divided by time," we can compute Baldini's average speed of

$$\frac{26 + \dfrac{385}{1760}}{2 + \dfrac{10}{60} + \dfrac{55}{3600}} \approx 12.0 \text{ mph.}$$

This says that Baldini averaged less than 5 minutes per mile for over 26 miles! However, the 100-meter sprint was won by Justin Gatlin of the United States in 9.85 seconds, and the 200-meter sprint was won by Shawn Crawford of the United States in 19.79 seconds. Average speeds for these runners were

$$\frac{\dfrac{100}{1610}}{\dfrac{9.85}{3600}} \approx 22.7 \text{ mph} \qquad \text{and} \qquad \frac{\dfrac{200}{1610}}{\dfrac{19.79}{3600}} \approx 22.6 \text{ mph.}$$

Since these speeds are much faster than that of the marathon runner, the winners of these events are often called the "World's Fastest Human."

An interesting connection can be made with a thought experiment. If the same person ran 200 meters in 19.79 seconds with the first 100 meters covered in 9.85 seconds, compare the average speeds for the first and second 100 meters. In the second 100 meters, the distance run is $200 - 100 = 100$ meters and the time is $19.79 - 9.85 = 9.94$ seconds. The average speed is then

$$\frac{200 - 100}{19.79 - 9.85} = \frac{100}{9.94} \approx 10.06 \text{ m/s} \approx 22.5 \text{ mph.}$$

Notice that the speed calculation in m/s is the same calculation we would use for the slope between the points (9.85, 100) and (19.79, 200). The connection between slope and speed (and other quantities of interest) is explored in this chapter.

2.1 TANGENT LINES AND VELOCITY

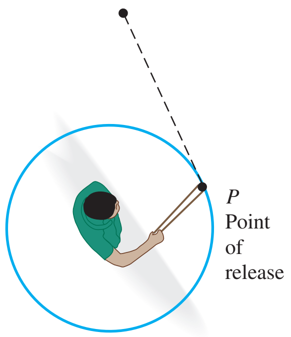


FIGURE 2.1
Path of rock

A traditional slingshot is essentially a rock on the end of a string, which you rotate around in a circular motion and then release. When you release the string, in which direction will the rock travel? An overhead view of this is illustrated in Figure 2.1. Many people mistakenly believe that the rock will follow a curved path, but Newton's first law of motion tells us that the path in the horizontal plane is straight. In fact, the rock follows a path along the tangent line to the circle at the point of release. Our aim in this section is to extend the notion of tangent line to more general curves.

To make our discussion more concrete, suppose that we want to find the tangent line to the curve $y = x^2 + 1$ at the point (1, 2) (see Figure 2.2). How could we define this? The tangent line hugs the curve near the point of tangency. In other words, like the tangent line to a circle, this tangent line has the same direction as the curve at the point of tangency. So, if you were standing on the curve at the point of tangency, took a small step and tried to stay on the curve, you would step in the direction of the tangent line. Another way to think of this is to observe that, if we zoom in sufficiently far, the graph appears to approximate that of a straight line. In Figure 2.3, we show the graph of $y = x^2 + 1$ zoomed in on the small rectangular box indicated in Figure 2.2. (Be aware that the "axes" indicated in Figure 2.3 do not intersect at the origin. We provide them only as a guide as to the scale used to produce the figure.) We now choose two points from the curve—for example, (1, 2) and (3, 10)—and compute the slope of the line joining these two points. Such a line is called a **secant** line and we denote its slope by m_{sec}:

$$m_{\text{sec}} = \frac{10 - 2}{3 - 1} = 4.$$

An equation of the secant line is then determined by

$$\frac{y - 2}{x - 1} = 4,$$

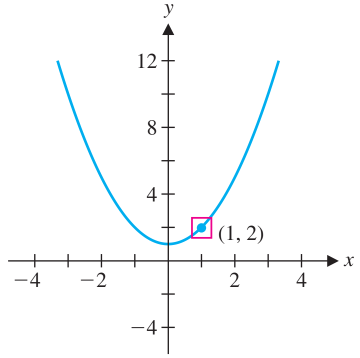


FIGURE 2.2
$y = x^2 + 1$

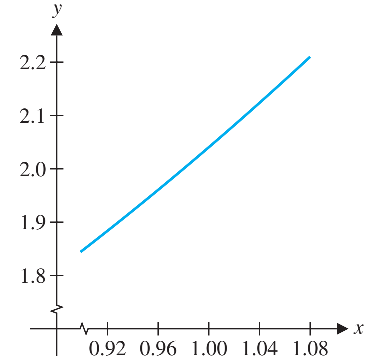


FIGURE 2.3
$y = x^2 + 1$

so that
$$y = 4(x - 1) + 2.$$

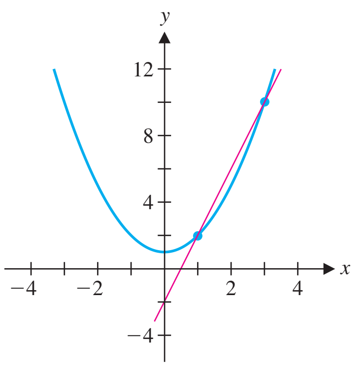


FIGURE 2.4a
Secant line joining (1, 2) and (3, 10)

As can be seen in Figure 2.4a, the secant line doesn't look very much like a tangent line.

Taking the second point a little closer to the point of tangency, say (2, 5), gives the slope of the secant line as

$$m_{\text{sec}} = \frac{5 - 2}{2 - 1} = 3$$

and an equation of the secant line as $y = 3(x - 1) + 2$. As seen in Figure 2.4b, this looks much more like a tangent line, but it's still not quite there. Choosing our second point much closer to the point of tangency, say (1.05, 2.1025), should give us an even better approximation to the tangent line. In this case, we have

$$m_{\text{sec}} = \frac{2.1025 - 2}{1.05 - 1} = 2.05$$

and an equation of the secant line is $y = 2.05(x - 1) + 2$. As can be seen in Figure 2.4c, the secant line looks very much like a tangent line, even when zoomed in quite far, as in Figure 2.4d. We continue this process by computing the slope of the secant line joining (1, 2) and the unspecified point $(1 + h, f(1 + h))$, for some value of h close to 0. The slope of this secant line is

$$\begin{aligned} m_{\text{sec}} &= \frac{f(1+h) - 2}{(1+h) - 1} = \frac{[(1+h)^2 + 1] - 2}{h} \\ &= \frac{(1 + 2h + h^2) - 1}{h} = \frac{2h + h^2}{h} && \text{Multiply out and cancel.} \\ &= \frac{h(2+h)}{h} = 2 + h. && \text{Factor out common } h \text{ and cancel.} \end{aligned}$$

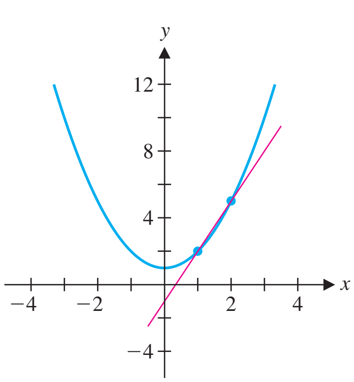


FIGURE 2.4b
Secant line joining (1, 2) and (2, 5)

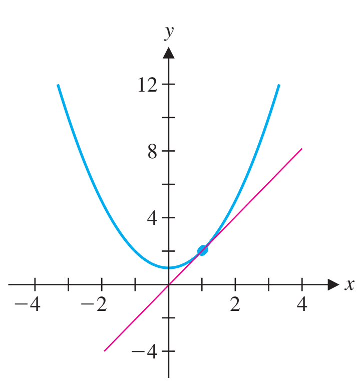


FIGURE 2.4c
Secant line joining (1, 2) and (1.05, 2.1025)

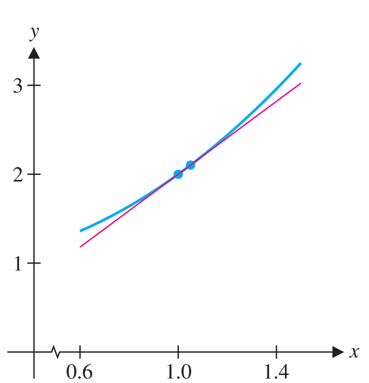


FIGURE 2.4d
Close-up of secant line

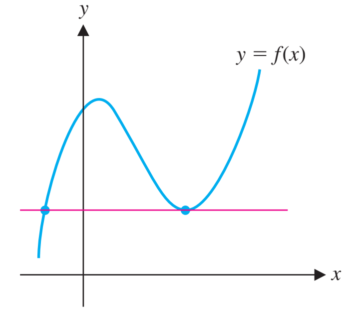


FIGURE 2.5
Tangent line intersecting a curve at more than one point

Notice that as h approaches 0, the slope of the secant line approaches 2, which we define to be the slope of the tangent line.

REMARK 1.1

We should make one more observation before moving on to the general case of tangent lines. Unlike the case for a circle, tangent lines may intersect a curve at more than one point, as indicated in Figure 2.5.

The General Case

To find the slope of the tangent line to $y = f(x)$ at $x = a$, first pick two points on the curve. One point is the point of tangency, $(a, f(a))$. Call the x-coordinate of the second point $x = a + h$, for some small number h; the corresponding y-coordinate is then $f(a + h)$. It is natural to think of h as being positive, as shown in Figure 2.6a, although h can also be negative, as shown in Figure 2.6b.

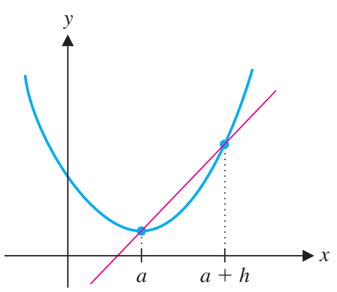


FIGURE 2.6a
Secant line ($h > 0$)

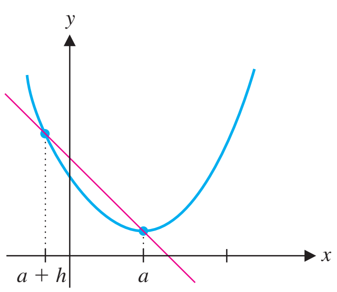


FIGURE 2.6b
Secant line ($h < 0$)

The slope of the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$ is given by

$$\boxed{m_{\text{sec}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.} \tag{1.1}$$

Notice that the expression in (1.1) (called a **difference quotient**) gives the slope of the secant line for any second point we might choose (i.e., for any $h \neq 0$). Recall that in order to obtain an improved approximation to the tangent line, we zoom in closer and closer toward the point of tangency. This makes the two points closer together, which in turn makes h closer to 0. Just how far should we zoom in? The farther, the better; this means that we want h to approach 0. We illustrate this process in Figure 2.7, where we have plotted a number of secant lines for $h > 0$. Notice that as the point Q approaches the point P (i.e., as $h \to 0$), the secant line approaches the tangent line at P.

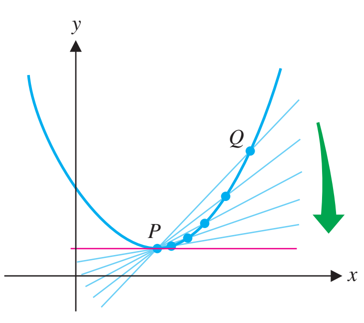


FIGURE 2.7
Secant lines approaching the tangent line at the point P

We define the slope of the tangent line to be the limit of the slopes of the secant lines in (1.1) as h tends to 0, whenever this limit exists.

DEFINITION 1.1

The **slope** $m_{\tan}$ of the tangent line to $y = f(x)$ at $x = a$ is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \qquad (1.2)$$

provided the limit exists.

The tangent line is then the line passing through the point $(a, f(a))$ with slope $m_{\tan}$ and so, the point-slope form of the equation of the tangent line is

Equation of tangent line

$$\boxed{y = m_{\tan}(x - a) + f(a).}$$

EXAMPLE 1.1 Finding the Equation of a Tangent Line

Find an equation of the tangent line to $y = x^2 + 1$ at $x = 1$.

Solution We compute the slope using (1.2):

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{[(1+h)^2 + 1] - (1+1)}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 - 2}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h \to 0} (2+h) = 2. \end{aligned}$$

Notice that the point corresponding to $x = 1$ is $(1, 2)$ and the line with slope 2 through the point $(1, 2)$ has equation

$$y = 2(x - 1) + 2 \quad \text{or} \quad y = 2x.$$

Note how closely this corresponds to the secant lines computed earlier. We show a graph of the function and this tangent line in Figure 2.8. ■

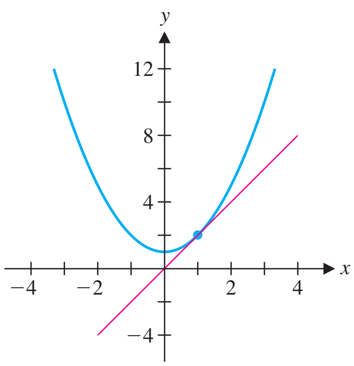


FIGURE 2.8
$y = x^2 + 1$ and the tangent line at $x = 1$

EXAMPLE 1.2 Tangent Line to the Graph of a Rational Function

Find an equation of the tangent line to $y = \dfrac{2}{x}$ at $x = 2$.

Solution From (1.2), we have

$$\begin{aligned} m_{\tan} &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{h} && \text{Since } f(2+h) = \frac{2}{2+h}. \\ &= \lim_{h \to 0} \frac{\left[\frac{2 - (2+h)}{(2+h)}\right]}{h} = \lim_{h \to 0} \frac{\left[\frac{2 - 2 - h}{(2+h)}\right]}{h} && \text{Add fractions and multiply out.} \\ &= \lim_{h \to 0} \frac{-h}{(2+h)h} = \lim_{h \to 0} \frac{-1}{2+h} = -\frac{1}{2}. && \text{Cancel } h\text{'s.} \end{aligned}$$

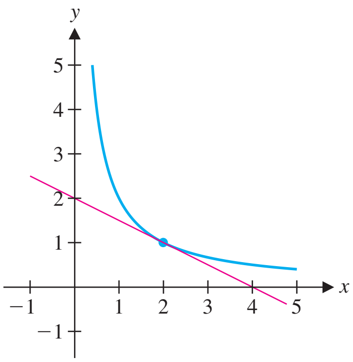


FIGURE 2.9
$y = \frac{2}{x}$ and tangent line at (2, 1)

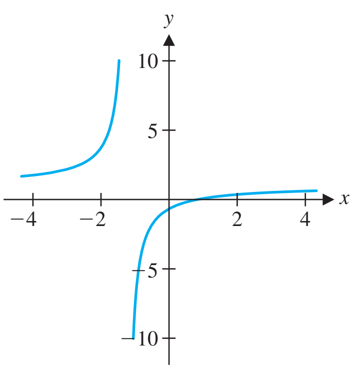


FIGURE 2.10a
$y = \frac{x-1}{x+1}$

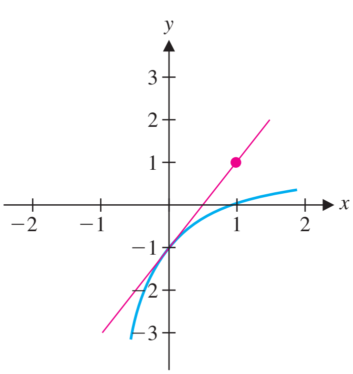


FIGURE 2.10b
Tangent line

The point corresponding to $x = 2$ is (2, 1), since $f(2) = 1$. An equation of the tangent line is then

$$y = -\frac{1}{2}(x - 2) + 1.$$

We show a graph of function and this tangent line in Figure 2.9. ■

In cases where we cannot (or cannot easily) evaluate the limit for the slope of the tangent line, we can approximate the limit numerically. We illustrate this in example 1.3.

EXAMPLE 1.3 Graphical and Numerical Approximation of Tangent Lines

Graphically and numerically approximate the slope of the tangent line to $y = \frac{x-1}{x+1}$ at $x = 0$.

Solution A graph of $y = \frac{x-1}{x+1}$ is shown in Figure 2.10a. We are interested in the tangent line at the point $(0, -1)$. We sketch a tangent line in Figure 2.10b, where we have zoomed in to provide better detail. To approximate the slope, we estimate the coordinates of one point on the tangent line other than $(0, -1)$. In Figure 2.10b, it appears that the tangent line passes through the point (1, 1). An estimate of the slope is then $m_{\tan} \approx \frac{1-(-1)}{1-0} = 2$. To approximate the slope numerically, we choose several points near $(0, -1)$ and compute the slopes of the secant lines. For example, rounding the y-values to four decimal places, we have

Second Point	m_{sec}	*Second Point*	m_{sec}
(1, 0)	$\frac{0-(-1)}{1-0} = 1$	$(-0.5, -3)$	$\frac{-3-(-1)}{-0.5-0} = 4.0$
$(0.1, -0.8182)$	$\frac{-0.8182-(-1)}{0.1-0} = 1.818$	$(-0.1, -1.2222)$	$\frac{-1.2222-(-1)}{-0.1-0} = 2.22$
$(0.01, -0.9802)$	$\frac{-0.9802-(-1)}{0.01-0} = 1.98$	$(-0.01, -1.0202)$	$\frac{-1.0202-(-1)}{-0.01-0} = 2.02$

In both columns, as the second point gets closer to $(0, -1)$, the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point $(0, -1)$ is then 2. ■

○ Velocity

The slopes of tangent lines have many important applications, of which one of the most important is in computing velocity. The term *velocity* is certainly familiar to you, but can you say precisely what it is? We often describe velocity as a quantity determining the speed *and direction* of an object, but what exactly is speed? If your car did not have a speedometer, you might determine your speed using the familiar formula

$$\text{distance} = \text{rate} \times \text{time}. \tag{1.3}$$

Using (1.3), you can find the rate (speed) by simply dividing the distance by the time. However, the rate in (1.3) refers to *average* speed over a period of time. We are interested in the speed at a specific instant. The following story should indicate the difference.

During traffic stops, police officers frequently ask drivers if they know how fast they were going. An overzealous student might answer that during the past, say, 3 years, 2 months, 7 days, 5 hours and 45 minutes, they've driven exactly 45,259.7 miles, so that their speed was

$$\text{rate} = \frac{\text{distance}}{\text{time}} = \frac{45{,}259.7 \text{ miles}}{27{,}917.75 \text{ hours}} \approx 1.62118 \text{ mph}.$$

Of course, most police officers would not be impressed with this analysis, but, *why* is it wrong? Certainly there's nothing wrong with formula (1.3) or the arithmetic. However, it's reasonable to argue that unless they were in their car during this entire 3-year period, the results are invalid.

Suppose that the driver substitutes the following argument instead: "I left home at 6:17 P.M. and by the time you pulled me over at 6:43 P.M., I had driven exactly 17 miles. Therefore, my speed was

$$\text{rate} = \frac{17 \text{ miles}}{26 \text{ minutes}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = 39.2 \text{ mph},$$

well under the posted 45-mph speed limit."

While this is a much better estimate of the velocity than the 1.6 mph computed previously, it's still an average velocity using too long of a time period.

More generally, suppose that the function $f(t)$ gives the position at time t of an object moving along a straight line. That is, $f(t)$ gives the displacement (**signed distance**) from a fixed reference point, so that $f(t) < 0$ means that the object is located $|f(t)|$ away from the reference point, but in the negative direction. Then, for two times a and b (where $a < b$), $f(b) - f(a)$ gives the signed distance between positions $f(a)$ and $f(b)$. The **average velocity** v_{avg} is then given by

$$\boxed{v_{avg} = \frac{\text{signed distance}}{\text{time}} = \frac{f(b) - f(a)}{b - a}.} \tag{1.4}$$

EXAMPLE 1.4 Finding Average Velocity

The position of a car after t minutes driving in a straight line is given by

$$s(t) = \frac{1}{2}t^2 - \frac{1}{12}t^3, \quad 0 \le t \le 4.$$

Approximate the velocity at time $t = 2$.

Solution Averaging over the 2 minutes from $t = 2$ to $t = 4$, we get from (1.4) that

$$\begin{aligned} v_{avg} &= \frac{s(4) - s(2)}{4 - 2} \approx \frac{2.666666667 - 1.333333333}{2} \\ &\approx 0.666666667 \text{ mile/minute} \\ &\approx 40 \text{ mph}. \end{aligned}$$

Of course, a 2-minute-long interval is rather long, given that cars can speed up and slow down a great deal in 2 minutes. We get an improved approximation by averaging over just one minute:

$$\begin{aligned} v_{avg} &= \frac{s(3) - s(2)}{3 - 2} \approx \frac{2.25 - 1.333333333}{1} \\ &\approx 0.9166666667 \text{ mile/minute} \\ &\approx 55 \text{ mph}. \end{aligned}$$

h	$\frac{s(2+h)-s(2)}{h}$
1.0	0.9166666667
0.1	0.9991666667
0.01	0.9999916667
0.001	0.999999917
0.0001	1.0
0.00001	1.0

While this latest estimate is certainly better than the first one, we can do better. As we make the time interval shorter and shorter, the average velocity should be getting closer and closer to the velocity at the instant $t = 2$. It stands to reason that, if we compute the average velocity over the time interval $[2, 2+h]$ and then let $h \to 0$, the resulting average velocities should be getting closer and closer to the velocity at the instant $t = 2$.

We have
$$v_{avg} = \frac{s(2+h)-s(2)}{(2+h)-2} = \frac{s(2+h)-s(2)}{h}.$$

A sequence of these average velocities is displayed in the accompanying table, for $h > 0$, with similar results if we allow h to be negative. It appears that the average velocity is approaching 1 mile/minute (60 mph), as $h \to 0$. We refer to this limiting value as the *instantaneous velocity*. ■

This leads us to make the following definition.

NOTE

Notice that if (for example) t is measured in seconds and $f(t)$ is measured in feet, then velocity (average or instantaneous) is measured in feet per second (ft/s). The term **velocity** is always used to refer to instantaneous velocity.

DEFINITION 1.2

If $f(t)$ represents the position of an object relative to some fixed location at time t as it moves along a straight line, then the **instantaneous velocity** at time $t = a$ is given by

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{(a+h)-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \tag{1.5}$$

provided the limit exists.

EXAMPLE 1.5 Finding Average and Instantaneous Velocity

Suppose that the height of a falling object t seconds after being dropped from a height of 64 feet is given by $f(t) = 64 - 16t^2$ feet. Find the average velocity between times $t = 1$ and $t = 2$; the average velocity between times $t = 1.5$ and $t = 2$; the average velocity between times $t = 1.9$ and $t = 2$ and the instantaneous velocity at time $t = 2$.

Solution The average velocity between times $t = 1$ and $t = 2$ is

$$v_{avg} = \frac{f(2)-f(1)}{2-1} = \frac{64-16(2)^2-[64-16(1)^2]}{1} = -48 \text{ ft/s}.$$

The average velocity between times $t = 1.5$ and $t = 2$ is

$$v_{avg} = \frac{f(2)-f(1.5)}{2-1.5} = \frac{64-16(2)^2-[64-16(1.5)^2]}{0.5} = -56 \text{ ft/s}.$$

The average velocity between times $t = 1.9$ and $t = 2$ is

$$v_{avg} = \frac{f(2)-f(1.9)}{2-1.9} = \frac{64-16(2)^2-[64-16(1.9)^2]}{0.1} = -62.4 \text{ ft/s}.$$

The instantaneous velocity is the limit of such average velocities. From (1.5), we have

$$\begin{aligned} v(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{(2+h) - 2} \\ &= \lim_{h\to 0} \frac{[64 - 16(2+h)^2] - [64 - 16(2)^2]}{h} \\ &= \lim_{h\to 0} \frac{[64 - 16(4 + 4h + h^2)] - [64 - 16(2)^2]}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h\to 0} \frac{-64h - 16h^2}{h} = \lim_{h\to 0} \frac{-16h(h+4)}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h\to 0} [-16(h+4)] = -64 \text{ ft/s.} \end{aligned}$$

Recall that velocity indicates both speed and direction. In this problem, $f(t)$ measures the height above the ground. So, the negative velocity indicates that the object is moving in the negative (or downward) direction. The **speed** of the object at the 2-second mark is then 64 ft/s. (Speed is simply the absolute value of velocity.) ■

Observe that the formulas for instantaneous velocity (1.5) and for the slope of a tangent line (1.2) are identical. We want to make this connection as strong as possible, by illustrating example 1.5 graphically. We graph the position function $f(t) = 64 - 16t^2$ for $0 \le t \le 2$. The average velocity between $t = 1$ and $t = 2$ corresponds to the slope of the secant line between the points at $t = 1$ and $t = 2$. (See Figure 2.11a.) Similarly, the average velocity between $t = 1.5$ and $t = 2$ gives the slope of the corresponding secant line. (See Figure 2.11b.) Finally, the instantaneous velocity at time $t = 2$ corresponds to the slope of the tangent line at $t = 2$. (See Figure 2.11c.)

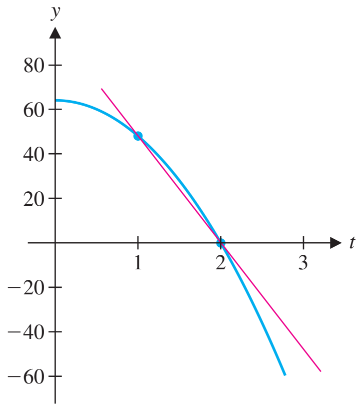


FIGURE 2.11a
Secant line between $t = 1$ and $t = 2$

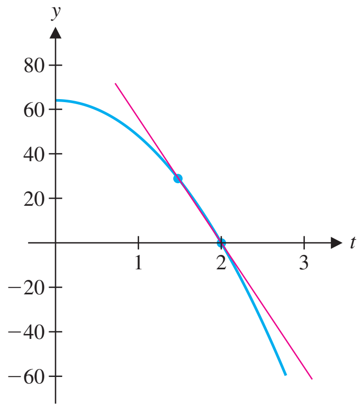


FIGURE 2.11b
Secant line between $t = 1.5$ and $t = 2$

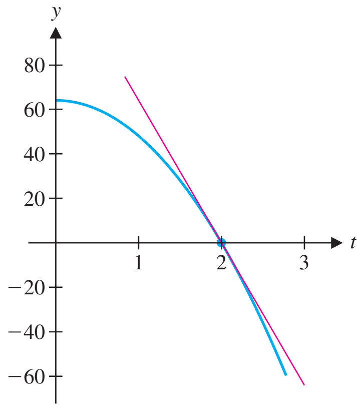


FIGURE 2.11c
Tangent line at $t = 2$

Velocity is a *rate* (more precisely, the instantaneous rate of change of position with respect to time). We now generalize this notion of instantaneous rate of change. In general, the **average rate of change** of a function $f(x)$ between $x = a$ and $x = b\,(a \neq b)$ is given by

$$\frac{f(b) - f(a)}{b - a}.$$

The **instantaneous rate of change** of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided the limit exists. The units of the instantaneous rate of change are the units of f divided by (or "per") the units of x.

EXAMPLE 1.6 Interpreting Rates of Change

If the function $f(t)$ gives the population of a city in millions of people t years after January 1, 2000, interpret each of the following quantities, assuming that they equal the given numbers. (a) $\frac{f(2) - f(0)}{2} = 0.34$, (b) $f(2) - f(1) = 0.31$ and (c) $\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = 0.3$.

Solution From the preceding, $\frac{f(b) - f(a)}{b - a}$ is the average rate of change of the function f between a and b. Expression (a) tells us that the average rate of change of f between $a = 0$ and $b = 2$ is 0.34. Stated in more common language, the city's population grew at an average rate of 0.34 million people per year between 2000 and 2002. Similarly, expression (b) is the average rate of change between $a = 1$ and $b = 2$. That is, the city's population grew at an average rate of 0.31 million people per year in 2001. Finally, expression (c) gives the instantaneous rate of change of the population at time $t = 2$. As of January 1, 2002, the city's population was growing at a rate of 0.3 million people per year. ■

Additional applications of the slope of a tangent line are innumerable. These include the rate of a chemical reaction, the inflation rate in economics and learning growth rates in psychology. Rates of change in nearly any discipline you can name can be thought of as slopes of tangent lines. We explore many of these applications as we progress through the text.

You hopefully noticed that we tacked the phrase "provided the limit exists" onto the end of the definitions of the slope of a tangent line, the instantaneous velocity and the instantaneous rate of change. This was important, since these defining limits do not always exist, as we see in example 1.7.

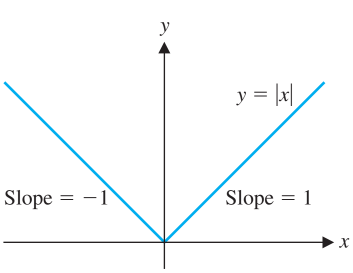


FIGURE 2.12
$y = |x|$

EXAMPLE 1.7 A Graph with No Tangent Line at a Point

Determine whether there is a tangent line to $y = |x|$ at $x = 0$.

Solution We can look at this problem graphically, numerically and symbolically. The graph is shown in Figure 2.12. Our graphical technique is to zoom in on the point of tangency until the graph appears straight. However, no matter how far we zoom in on (0, 0), the graph continues to look like Figure 2.12. (This is one reason why we left off the scale on Figure 2.12.) From this evidence alone, we would conjecture that the tangent line does not exist. Numerically, the slope of the tangent line is the limit of the slope of a secant line, as the second point approaches the point of tangency. Observe that the secant line through (0, 0) and (1, 1) has slope 1, as does the secant line through (0, 0) and (0.1, 0.1). In fact, if h is any positive number, the slope of the secant line through (0, 0) and $(h, |h|)$ is 1. However, the secant line through (0, 0) and $(-1, 1)$ has slope -1, as does the secant line through (0, 0) and $(h, |h|)$ for any negative number h. We therefore conjecture that the one-sided limits are different, so that the limit (and also the tangent line) does not exist. To prove this conjecture, we take our cue from the numerical work and look at one-sided limits: if $h > 0$, then $|h| = h$, so that

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

On the other hand, if $h < 0$, then $|h| = -h$ (remember that if $h < 0$, $-h > 0$), so that

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1.$$

Since the one-sided limits are different, we conclude that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \text{ does not exist}$$

and hence, the tangent line does not exist. ■

EXERCISES 2.1

WRITING EXERCISES

1. What does the phrase "off on a tangent" mean? Relate the common meaning of the phrase to the image of a tangent to a circle (use the slingshot example, if that helps). In what way does the zoomed image of the tangent promote the opposite view of the relationship between a curve and its tangent?

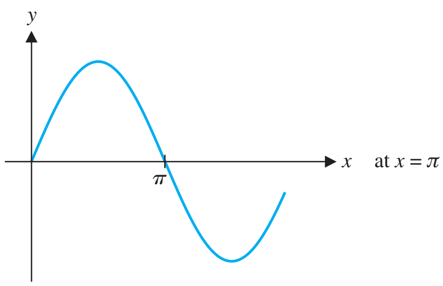
2. In general, the instantaneous velocity of an object cannot be computed directly; the limit process is the only way to compute velocity *at an instant*. Given this, how does a car's speedometer compute velocity? (Hint: Look this up in a reference book or on the Internet. An important aspect of the car's ability to do this seemingly difficult task is that it performs *analog* calculations. For example, the pitch of a fly's buzz gives us an analog device for computing the speed of a fly's wings, since pitch is proportional to speed.)

3. Look in the news media (TV, newspaper, Internet) and find references to at least five different *rates*. We have defined a rate of change as the limit of the difference quotient of a function. For your five examples, state as precisely as possible what the original function is. Is the rate given quantitatively or qualitatively? If it is given quantitatively, is the rate given as a percentage or a number? In calculus, we usually compute rates (quantitatively) as numbers; is this in line with the standard usage?

4. Sketch the graph of a function that is discontinuous at $x = 1$. Explain why there is no tangent line at $x = 1$.

In exercises 1–4, sketch in a plausible tangent line at the given point. (Hint: Mentally zoom in on the point and use the zoomed image of the tangent.)

1.



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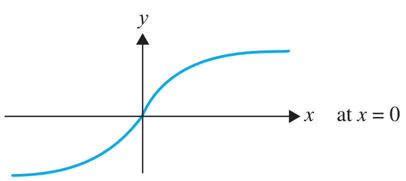

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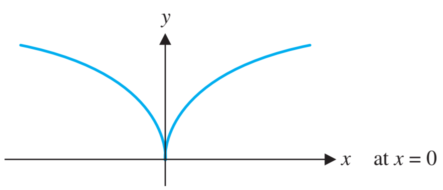
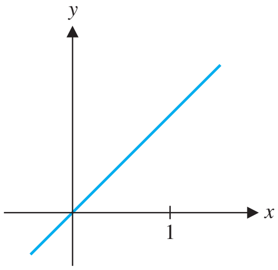


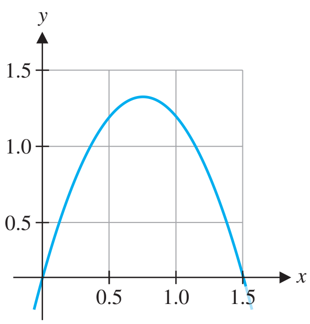
4.



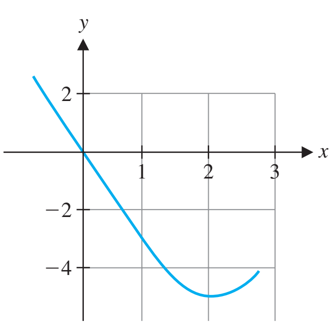
at $x = 1$

In exercises 5 and 6, estimate the slope of the tangent line to the curve at $x = 1$.

5.

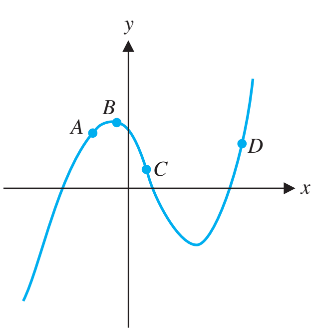


6.

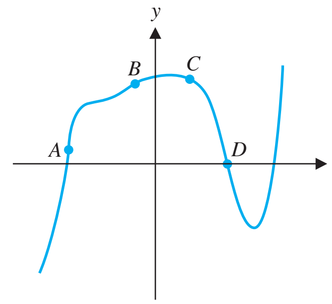


In exercises 7 and 8, list the points *A, B, C* and *D* in order of increasing slope of the tangent line.

7.



8.



In exercises 9–12, compute the slope of the secant line between the points at (a) $x = 1$ and $x = 2$, (b) $x = 2$ and $x = 3$, (c) $x = 1.5$ and $x = 2$, (d) $x = 2$ and $x = 2.5$, (e) $x = 1.9$ and $x = 2$, (f) $x = 2$ and $x = 2.1$, and (g) use parts (a)–(f) and other calculations as needed to estimate the slope of the tangent line at $x = 2$.

9. $f(x) = x^3 - x$

10. $f(x) = \sqrt{x^2 + 1}$

11. $f(x) = \cos x^2$

12. $f(x) = \tan(x/4)$

In exercises 13–16, use a CAS or graphing calculator.

13. On one graph, sketch the secant lines in exercise 9, parts (a)–(d) and the tangent line in part (g).

14. On one graph, sketch the secant lines in exercise 10, parts (a)–(d) and the tangent line in part (g).

15. Animate the secant lines in exercise 9, parts (a), (c) and (e), converging to the tangent line in part (g).

16. Animate the secant lines in exercise 9, parts (b), (d) and (f), converging to the tangent line in part (g).

In exercises 17–24, find the equation of the tangent line to $y = f(x)$ at $x = a$. Graph $y = f(x)$ and the tangent line to verify that you have the correct equation.

17. $f(x) = x^2 - 2, a = 1$

18. $f(x) = x^2 - 2, a = 0$

19. $f(x) = x^2 - 3x, a = -2$

20. $f(x) = x^3 + x, a = 1$

21. $f(x) = \dfrac{2}{x+1}, a = 1$

22. $f(x) = \dfrac{x}{x-1}, a = 0$

23. $f(x) = \sqrt{x+3}, a = -2$

24. $f(x) = \sqrt{x^2+1}, a = 1$

In exercises 25–30, use graphical and numerical evidence to determine whether the tangent line to $y = f(x)$ exists at $x = a$. If it does, estimate the slope of the tangent; if not, explain why not.

25. $f(x) = |x - 1|$ at $a = 1$

26. $f(x) = \dfrac{4x}{x-1}$ at $a = 1$

27. $f(x) = \begin{cases} -2x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$ at $a = 0$

28. $f(x) = \begin{cases} -2x & \text{if } x < 1 \\ x - 3 & \text{if } x \geq 1 \end{cases}$ at $a = 1$

29. $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ x^3 + 1 & \text{if } x \geq 0 \end{cases}$ at $a = 0$

30. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x^2 - 2x & \text{if } x > 0 \end{cases}$ at $a = 0$

In exercises 31–34, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

31. $f(t) = 16t^2 + 10$

32. $f(t) = 3t^3 + t$

33. $f(t) = \sqrt{t^2 + 8t}$

34. $f(t) = 100\sin(t/4)$

In exercises 35 and 36, use the position function $f(t)$ meters to find the velocity at time $t = a$ seconds.

35. $f(t) = -16t^2 + 5$, (a) $a = 1$; (b) $a = 2$

36. $f(t) = \sqrt{t + 16}$, (a) $a = 0$; (b) $a = 2$

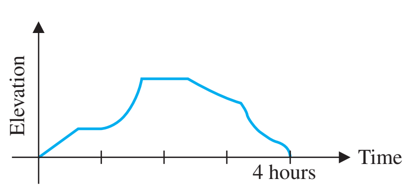
37. The table shows the freezing temperature of water in degrees Celsius at various pressures. Estimate the slope of the tangent line at $p = 1$ and interpret the result. Estimate the slope of the tangent line at $p = 3$ and interpret the result.

p (atm)	0	1	2	3	4
$^\circ C$	0	−7	−20	−16	−11

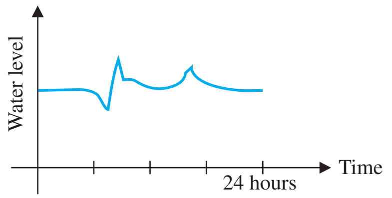
38. The table shows the range of a soccer kick launched at 30° above the horizontal at various initial speeds. Estimate the slope of the tangent line at $v = 50$ and interpret the result.

Distance (yd)	19	28	37	47	58
Speed (mph)	30	40	50	60	70

39. The graph shows the elevation of a person on a hike up a mountain as a function of time. When did the hiker reach the top? When was the hiker going the fastest on the way up? When was the hiker going the fastest on the way down? What do you think occurred at places where the graph is level?



40. The graph shows the amount of water in a city water tank as a function of time. When was the tank the fullest? the emptiest? When was the tank filling up at the fastest rate? When was the tank emptying at the fastest rate? What time of day do you think the level portion represents?



41. Suppose a hot cup of coffee is left in a room for 2 hours. Sketch a reasonable graph of what the temperature would look like as a function of time. Then sketch a graph of what the rate of change of the temperature would look like.

42. Sketch a graph representing the height of a bungee-jumper. Sketch the graph of the person's velocity (use + for upward velocity and − for downward velocity).

43. Suppose that $f(t)$ represents the balance in dollars of a bank account t years after January 1, 2000. Interpret each of the following. (a) $\dfrac{f(4) - f(2)}{2} = 21{,}034$, (b) $2[f(4) - f(3.5)] = 25{,}036$ and (c) $\lim_{h \to 0} \dfrac{f(4+h) - f(4)}{h} = 30{,}000$.

44. Suppose that $f(m)$ represents the value of a car that has been driven m thousand miles. Interpret each of the following. (a) $\dfrac{f(40) - f(38)}{2} = -2103$, (b) $f(40) - f(39) = -2040$ and (c) $\lim_{h \to 0} \dfrac{f(40+h) - f(40)}{h} = -2000$.

45. In using a slingshot, it is important to generate a large angular velocity. **Angular velocity** is defined by $\lim_{h \to 0} \dfrac{\theta(a+h) - \theta(a)}{h}$, where $\theta(t)$ is the angle of rotation at time t. If the angle of a slingshot is $\theta(t) = 0.4t^2$, what is the angular velocity after three rotations? [Hint: Which value of t (seconds) corresponds to three rotations?]

46. Find the angular velocity of the slingshot in exercise 45 after two rotations. Explain why the third rotation is helpful.

47. Sometimes an incorrect method accidentally produces a correct answer. For quadratic functions (but definitely *not* most other functions), the average velocity between $t = r$ and $t = s$ equals the average of the velocities at $t = r$ and $t = s$. To show this, assume that $f(t) = at^2 + bt + c$ is the distance function. Show that the average velocity between $t = r$ and $t = s$ equals $a(s + r) + b$. Show that the velocity at $t = r$ is $2ar + b$ and the velocity at $t = s$ is $2as + b$. Finally, show that $a(s + r) + b = \dfrac{(2ar + b) + (2as + b)}{2}$.

48. Find a cubic function [try $f(t) = t^3 + \cdots$] and numbers r and s such that the average velocity between $t = r$ and $t = s$ is different from the average of the velocities at $t = r$ and $t = s$.

49. Show that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. (Hint: Let $h = x - a$.)

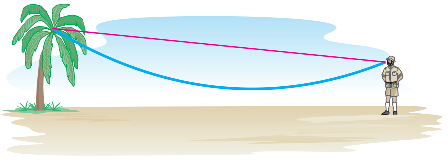
50. Use the second limit in exercise 49 to recompute the slope in exercises 17 and 19. Which limit do you prefer?

51. A car speeding around a curve in the shape of $y = x^2$ (moving from left to right) skids off at the point $\left(\frac{1}{2}, \frac{1}{4}\right)$. If the car continues in a straight path, will it hit a tree located at the point $\left(1, \frac{3}{4}\right)$?

52. For the car in exercise 51, show graphically that there is only one skid point on the curve $y = x^2$ such that the tangent line passes through the point $\left(1, \frac{3}{4}\right)$.

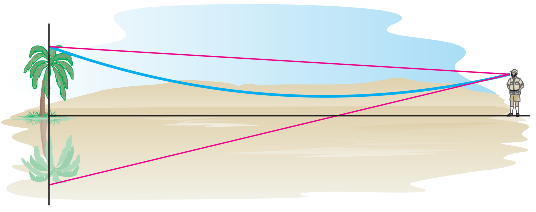
EXPLORATORY EXERCISES

1. Many optical illusions are caused by our brain's (unconscious) use of the tangent line in determining the positions of objects. Suppose you are in the desert 100 feet from a palm tree. You see a particular spot 10 feet up on the palm tree due to light reflecting from that spot to your eyes. Normally, it is a good approximation to say that the light follows a straight line (top path in the figure).



Two paths of light from tree to person.

However, when there is a large temperature difference in the air, light may follow nonlinear paths. If, as in the desert, the air near the ground is much hotter than the air higher up, light will bend as indicated by the bottom path in the figure. Our brains always interpret light coming in straight paths, so you would think the spot on the tree is at $y = 10$ because of the top path and also at some other y because of the bottom path. If the bottom curve is $y = 0.002x^2 - 0.24x + 10$, find an equation of the tangent line at $x = 100$ and show that it crosses the y-axis at $y = -10$. That is, you would "see" the spot at $y = 10$ and also at $y = -10$, a perfect reflection.



Two perceived locations of tree.

How do reflections normally occur in nature? From water! You would perceive a tree and its reflection in a pool of water. This is the desert mirage!

2. You can use a VCR to estimate speed. Most VCRs play at 30 frames per second. So, with a frame-by-frame advance, you can estimate time as the number of frames divided by 30. If you know the distance covered, you can compute the average velocity by dividing distance by time. Try this to estimate how fast you can throw a ball, run 50 yards, hit a tennis ball or whatever speed you find interesting. Some of the possible inaccuracies are explored in exercise 3.

3. What is the peak speed for a human being? It has been estimated that Carl Lewis reached a peak speed of 28 mph while winning a gold medal in the 1992 Olympics. Suppose that we have the following data for a sprinter.

Meters	*Seconds*
30	3.2
40	4.2
50	5.16666
56	5.76666
58	5.93333
60	6.1

Meters	*Seconds*
62	6.26666
64	6.46666
70	7.06666
80	8.0
90	9.0
100	10.0

We want to estimate peak speed. We could start by computing $\frac{\text{distance}}{\text{time}} = \frac{100 \text{ m}}{10 \text{ s}} = 10$ m/s, but this is the average speed over the entire race, not the peak speed. Argue that we want to compute average speeds only using adjacent measurements (e.g., 40 and 50 meters, or 50 and 56 meters). Do this for all 11 adjacent pairs and find the largest speed (if you want to convert to mph, divide by 0.447). We will then explore how accurate this estimate might be.

Notice that all times are essentially multiples of 1/30, since the data were obtained using the VCR technique in exercise 2. Given this, why is it suspicious that all the distances are whole numbers? To get an idea of how much this might affect your calculations, change some of the distances. For instance, if you change 60 (meters) to 59.8, how much do your average velocity calculations change? One possible way to identify where mistakes have been made is to look at the pattern of average velocities: does it seem reasonable? Would a sprinter speed up and slow down in such a pattern? In places where the pattern seems suspicious, try adjusting the distances and see if you can produce a more realistic pattern. Taking all this into account, try to quantify your error analysis: what is the highest (lowest) the peak speed could be?

2.2 THE DERIVATIVE

In section 2.1, we investigated two seemingly unrelated concepts: slopes of tangent lines and velocity, both of which are expressed in terms of the *same* limit. This is an indication of the power of mathematics, that otherwise unrelated notions are described by the *same* mathematical expression. This particular limit turns out to be so useful that we give it a special name.

DEFINITION 2.1

The **derivative** of the function $f(x)$ at $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{2.1}$$

provided the limit exists. If the limit exists, we say that f is **differentiable** at $x = a$.

An alternative form of (2.1) is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}. \tag{2.2}$$

(See exercise 49 in section 2.1.)

EXAMPLE 2.1 Finding the Derivative at a Point

Compute the derivative of $f(x) = 3x^3 + 2x - 1$ at $x = 1$.

Solution From (2.1), we have

$$\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h \to 0} \frac{\left[3(1+h)^3 + 2(1+h) - 1\right] - (3 + 2 - 1)}{h} \\
&= \lim_{h \to 0} \frac{3(1 + 3h + 3h^2 + h^3) + (2 + 2h) - 1 - 4}{h} && \text{Multiply out and cancel.} \\
&= \lim_{h \to 0} \frac{11h + 9h^2 + 3h^3}{h} && \text{Factor out common } h \text{ and cancel.} \\
&= \lim_{h \to 0} (11 + 9h + 3h^2) = 11. \quad \blacksquare
\end{aligned}$$

Suppose that in example 2.1 we had also needed to find $f'(2)$ and $f'(3)$. Must we now repeat the same long limit calculation to find each of $f'(2)$ and $f'(3)$? Instead, we compute the derivative without specifying a value for x, leaving us with a function from which we can calculate $f'(a)$ for any a, simply by substituting a for x.

EXAMPLE 2.2 Finding the Derivative at an Unspecified Point

Find the derivative of $f(x) = 3x^3 + 2x - 1$ at an unspecified value of x. Then, evaluate the derivative at $x = 1$, $x = 2$ and $x = 3$.

Solution Replacing a with x in the definition of the derivative (2.1), we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left[3(x+h)^3+2(x+h)-1\right]-(3x^3+2x-1)}{h} \\ &= \lim_{h\to 0} \frac{3(x^3+3x^2h+3xh^2+h^3)+(2x+2h)-1-3x^3-2x+1}{h} \\ &= \lim_{h\to 0} \frac{9x^2h+9xh^2+3h^3+2h}{h} \\ &= \lim_{h\to 0} (9x^2+9xh+3h^2+2) \\ &= 9x^2+0+0+2 = 9x^2+2. \end{aligned}$$

Multiply out and cancel.

Factor out common h and cancel.

Notice that in this case, we have derived a new *function,* $f'(x) = 9x^2 + 2$. Simply substituting in for x, we get $f'(1) = 9 + 2 = 11$ (the same as we got in example 2.1!), $f'(2) = 9(4) + 2 = 38$ and $f'(3) = 9(9) + 2 = 83$. ■

Example 2.2 leads us to the following definition.

DEFINITION 2.2

The **derivative** of $f(x)$ is the function $f'(x)$ given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \tag{2.3}$$

provided the limit exists. The process of computing a derivative is called **differentiation.**

Further, f is differentiable on an interval I if it is differentiable at every point in I.

In examples 2.3 and 2.4, observe that the name of the game is to write down the defining limit and then to find some way of evaluating that limit (which initially has the indeterminate form $\frac{0}{0}$).

EXAMPLE 2.3 Finding the Derivative of a Simple Rational Function

If $f(x) = \dfrac{1}{x}$ $(x \neq 0)$, find $f'(x)$.

Solution We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)}{h} \\ &= \lim_{h\to 0} \frac{\left[\dfrac{x-(x+h)}{x(x+h)}\right]}{h} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}, \end{aligned}$$

Since $f(x+h) = \dfrac{1}{x+h}$.

Add fractions and cancel.

Cancel h's.

or $f'(x) = -x^{-2}$. ■

EXAMPLE 2.4 The Derivative of the Square Root Function

If $f(x) = \sqrt{x}$ (for $x \geq 0$), find $f'(x)$.

Solution We have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) && \text{Multiply numerator and denominator by the conjugate: } \sqrt{x+h} + \sqrt{x}. \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h\left[\sqrt{x+h} + \sqrt{x}\right]} && \text{Multiply out and cancel.} \\
&= \lim_{h \to 0} \frac{h}{h\left[\sqrt{x+h} + \sqrt{x}\right]} && \text{Cancel common } h\text{'s.} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-1/2}.
\end{aligned}$$

Notice that $f'(x)$ is defined only for $x > 0$. ■

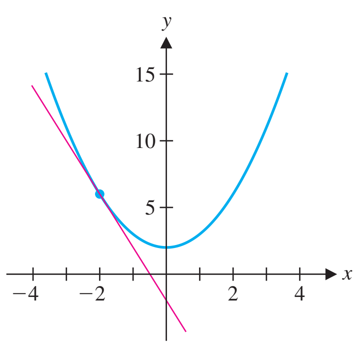


FIGURE 2.13a
$m_{\tan} < 0$

The benefits of having a derivative *function* go well beyond simplifying the computation of a derivative at multiple points. As we'll see, the derivative function tells us a great deal about the original function.

Keep in mind that the value of a derivative at a point is the slope of the tangent line at that point. In Figures 2.13a–2.13c, we have graphed a function along with its tangent lines at three different points. The slope of the tangent line in Figure 2.13a is negative; the slope of the tangent line in Figure 2.13c is positive and the slope of the tangent line in Figure 2.13b is zero. These three tangent lines give us three points on the graph of the derivative function (see Figure 2.13d), by estimating the value of $f'(x)$ at the three points. Thus, as x changes, the slope of the tangent line changes and hence $f'(x)$ changes.

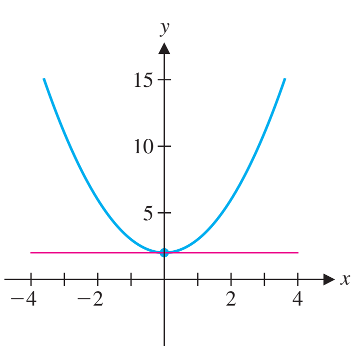


FIGURE 2.13b
$m_{\tan} = 0$

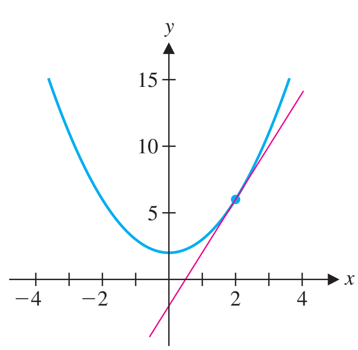


FIGURE 2.13c
$m_{\tan} > 0$

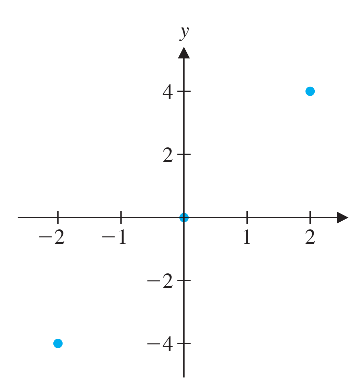


FIGURE 2.13d
$y = f'(x)$ (three points)

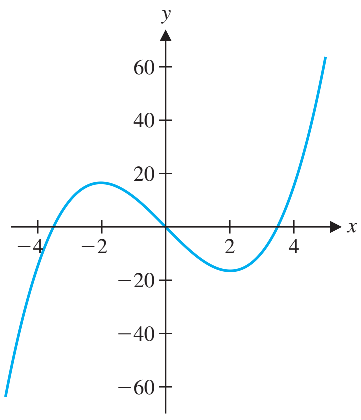


FIGURE 2.14
$y = f(x)$

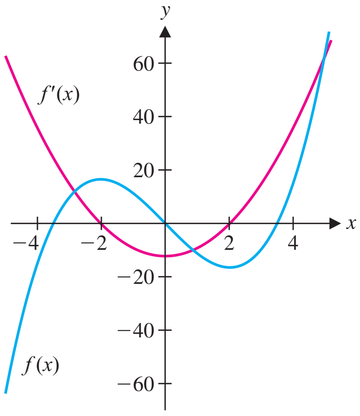


FIGURE 2.15
$y = f(x)$ and $y = f'(x)$

EXAMPLE 2.5 Sketching the Graph of $f'(x)$ Given the Graph of $f(x)$

Given the graph of $f(x)$ in Figure 2.14, sketch a plausible graph of $f'(x)$.

Solution Rather than worrying about exact values of the slope, we only wish to get the general shape right. As in Figures 2.13a–2.13d, pick a few important points to analyze carefully. You should focus on any discontinuities and any places where the graph of f turns around.

The graph levels out at approximately $x = -2$ and $x = 2$. At these points, the derivative is 0. As we move from left to right, the graph rises for $x < -2$, drops for $-2 < x < 2$ and rises again for $x > 2$. This means that $f'(x) > 0$ for $x < -2$, $f'(x) < 0$ for $-2 < x < 2$ and finally $f'(x) > 0$ for $x > 2$. We can say even more. As x approaches -2 from the left, observe that the tangent lines get less steep. Therefore, $f'(x)$ becomes less positive as x approaches -2 from the left. Moving to the right from $x = -2$, the graph gets steeper until about $x = 0$, then gets less steep until it levels out at $x = 2$. Thus, $f'(x)$ gets more negative until $x = 0$, then less negative until $x = 2$. Finally, the graph gets steeper as we move to the right from $x = 2$. Putting this all together, we have the possible graph of $f'(x)$ shown in red in Figure 2.15, superimposed on the graph of $f(x)$. ■

The opposite question to that asked in example 2.5 is even more interesting. That is, given the graph of a derivative, what might the graph of the original function look like? We explore this in example 2.6.

EXAMPLE 2.6 Sketching the Graph of $f(x)$ Given the Graph of $f'(x)$

Given the graph of $f'(x)$ in Figure 2.16, sketch a plausible graph of $f(x)$.

Solution Again, do not worry about getting exact values of the function, but rather only the general shape of the graph. Notice from the graph of $y = f'(x)$ that $f'(x) < 0$ for $x < -2$, so that on this interval, the slopes of the tangent lines to $y = f(x)$ are negative and the function is decreasing. On the interval $(-2, 1)$, $f'(x) > 0$, indicating that the tangent lines to the graph of $y = f(x)$ have positive slope and the function is increasing. Further, this says that the graph turns around (i.e., goes from decreasing to increasing) at $x = -2$. We have drawn a graph exhibiting this behavior in Figure 2.17

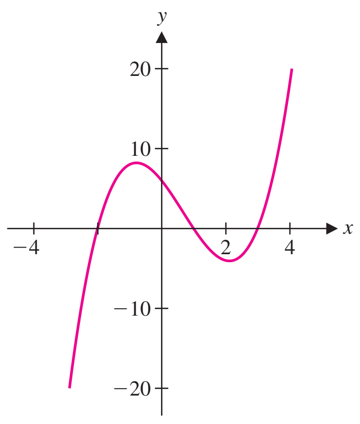


FIGURE 2.16
$y = f'(x)$

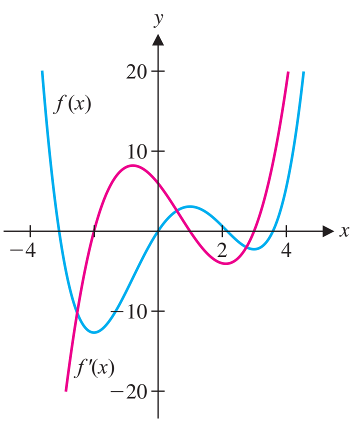


FIGURE 2.17
$y = f'(x)$ and a plausible graph of $y = f(x)$

superimposed on the graph of $y = f'(x)$. Further, $f'(x) < 0$ on the interval $(1, 3)$, so that the function decreases here. Finally, for $x > 3$, we have that $f'(x) > 0$, so that the function is increasing here. We show a graph exhibiting all of this behavior in Figure 2.17. We drew the graph of f so that the small "valley" on the right side of the y-axis was not as deep as the one on the left side of the y-axis for a reason. Look carefully at the graph of $f'(x)$ and notice that $|f'(x)|$ gets much larger on $(-2, 1)$ than on $(1, 3)$. This says that the tangent lines and hence, the graph will be much steeper on the interval $(-2, 1)$ than on $(1, 3)$. ■

Alternative Derivative Notations

We have denoted the derivative function by $f'(x)$. There are other commonly used notations, each with advantages and disadvantages. One of the coinventors of the calculus, Gottfried Leibniz, used the notation $\dfrac{df}{dx}$ (*Leibniz notation*) for the derivative. If we write $y = f(x)$, the following are all alternatives for denoting the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x).$$

The expression $\dfrac{d}{dx}$ is called a **differential operator** and tells you to take the derivative of whatever expression follows.

In section 2.1, we observed that $f(x) = |x|$, does not have a tangent line at $x = 0$ (i.e., it is not differentiable at $x = 0$), although it is continuous everywhere. Thus, there are continuous functions that are not differentiable. You might have already wondered whether the reverse is true. That is, are there differentiable functions that are not continuous? The answer (no) is provided by Theorem 2.1.

THEOREM 2.1

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

PROOF

For f to be continuous at $x = a$, we need only show that $\lim_{x\to a} f(x) = f(a)$. We consider

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}(x - a)\right] \qquad \text{Multiply and divide by } (x - a).$$

$$= \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}\right]\lim_{x\to a}(x - a) \qquad \text{By Theorem 3.1 (iii) from section 1.3.}$$

$$= f'(a)(0) = 0, \qquad \text{Since } f \text{ is differentiable at } x = a.$$

where we have used the alternative definition of derivative (2.2) discussed earlier. By Theorem 3.1 in section 1.3, it now follows that

$$0 = \lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a} f(x) - \lim_{x\to a} f(a)$$

$$= \lim_{x\to a} f(x) - f(a),$$

which gives us the result. ■

Note that Theorem 2.1 says that if a function is *not* continuous at a point then it *cannot* have a derivative at that point. It also turns out that functions are not differentiable at any point where their graph has a "sharp" corner, as is the case for $f(x) = |x|$ at $x = 0$. (See example 1.7.)

HISTORICAL NOTES

Gottfried Leibniz (1646–1716)
A German mathematician and philosopher who introduced much of the notation and terminology in calculus and who is credited (together with Sir Isaac Newton) with inventing the calculus. Leibniz was a prodigy who had already received his law degree and published papers on logic and jurisprudence by age 20. A true Renaissance man, Leibniz made important contributions to politics, philosophy, theology, engineering, linguistics, geology, architecture and physics, while earning a reputation as the greatest librarian of his time. Mathematically, he derived many fundamental rules for computing derivatives and helped promote the development of calculus through his extensive communications. The simple and logical notation he invented made calculus accessible to a wide audience and has only been marginally improved upon in the intervening 300 years. He wrote, "In symbols one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly . . . then indeed the labor of thought is wonderfully diminished."

EXAMPLE 2.7 Showing That a Function Is Not Differentiable at a Point

Show that $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ is not differentiable at $x = 2$.

Solution The graph (see Figure 2.18) indicates a sharp corner at $x = 2$, so you might expect that the derivative does not exist. To verify this, we investigate the derivative by evaluating one-sided limits. For $h > 0$, note that $(2 + h) > 2$ and so, $f(2 + h) = 2(2 + h)$. This gives us

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0^+} \frac{2(2+h) - 4}{h} \\ &= \lim_{h \to 0^+} \frac{4 + 2h - 4}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h \to 0^+} \frac{2h}{h} = 2. && \text{Cancel common } h\text{'s.} \end{aligned}$$

Likewise, if $h < 0$, $(2 + h) < 2$ and so, $f(2 + h) = 4$. Thus, we have

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{4 - 4}{h} = 0.$$

Since the one-sided limits do not agree $(0 \neq 2)$, $f'(2)$ does not exist (i.e., f is not differentiable at $x = 2$). ■

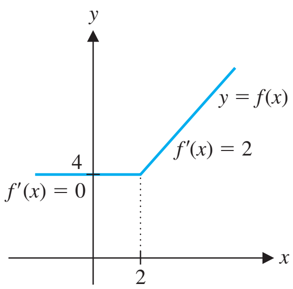


FIGURE 2.18
A sharp corner

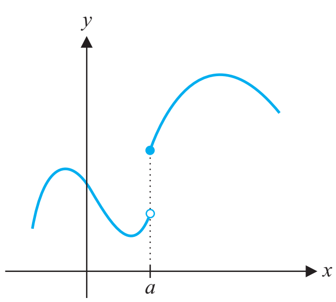


FIGURE 2.19a
A jump discontinuity

Figures 2.19a–2.19d show a variety of functions for which $f'(a)$ does not exist. In each case, convince yourself that the derivative does not exist.

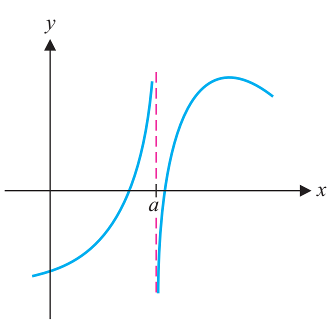


FIGURE 2.19b
A vertical asymptote

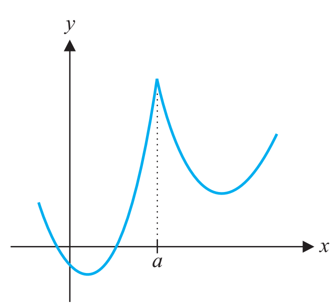


FIGURE 2.19c
A cusp

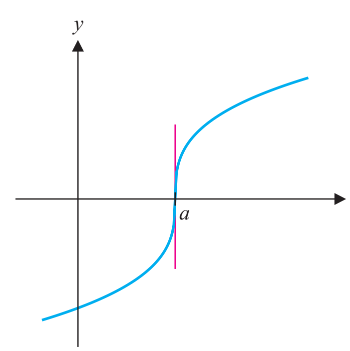


FIGURE 2.19d
A vertical tangent line

○ Numerical Differentiation

There are many times in applications when it is not possible or practical to compute derivatives symbolically. This is frequently the case in applications where we have only some data (i.e., a table of values) representing an otherwise *unknown* function. You will need an understanding of the limit definition to compute reasonable estimates of the derivative.

EXAMPLE 2.8 Approximating a Derivative Numerically

Numerically estimate the derivative of $f(x) = x^2\sqrt{x^3+2}$ at $x = 1$.

Solution We are not anxious to struggle through the limit definition for this function. The definition tells us, however, that the derivative at $x = 1$ is the limit of slopes of secant lines. We compute some of these below:

h	$\frac{f(1+h) - f(1)}{h}$
0.1	4.7632
0.01	4.3715
0.001	4.3342

h	$\frac{f(1+h) - f(1)}{h}$
−0.1	3.9396
−0.01	4.2892
−0.001	4.3260

Notice that the slopes seem to be converging to approximately 4.33 as h approaches 0. Thus, we make the approximation $f'(1) \approx 4.33$. ■

EXAMPLE 2.9 Estimating Velocity Numerically

Suppose that a sprinter reaches the following distances in the given times. Estimate the velocity of the sprinter at the 6-second mark.

$t(s)$	5.0	5.5	5.8	5.9	6.0	6.1	6.2	6.5	7.0
$f(t)$ (ft)	123.7	141.01	151.41	154.90	158.40	161.92	165.42	175.85	193.1

Solution The instantaneous velocity is the limit of the average velocity as the time interval shrinks. We first compute the average velocities over the shortest intervals given, from 5.9 to 6.0 and from 6.0 to 6.1.

Time Interval	*Average Velocity*
(5.9, 6.0)	35.0 ft/s
(6.0, 6.1)	35.2 ft/s

Time Interval	*Average Velocity*
(5.5, 6.0)	34.78 ft/s
(5.8, 6.0)	34.95 ft/s
(5.9, 6.0)	35.00 ft/s
(6.0, 6.1)	35.20 ft/s
(6.0, 6.2)	35.10 ft/s
(6.0, 6.5)	34.90 ft/s

Since these are the best individual estimates available from the data, we could just split the difference and estimate a velocity of 35.1 ft/s. However, there is useful information in the rest of the data. Based on the accompanying table, we can conjecture that the sprinter was reaching a peak speed at about the 6-second mark. Thus, we might accept the higher estimate of 35.2 ft/s. We should emphasize that there is not a single correct answer to this question, since the data are incomplete (i.e., we know the distance only at fixed times, rather than over a continuum of times). ■

BEYOND FORMULAS

In sections 2.3–2.8, we derive numerous formulas for computing derivatives. As you learn these formulas, keep in mind the reasons that we are interested in the derivative. Careful studies of the slope of the tangent line to a curve and the velocity of a moving object led us to the same limit, which we named the *derivative*. In general, the derivative represents the rate of change or the ratio of the change of one quantity to the change in another quantity. The study of change in a quantifiable way led directly to modern science and engineering. If we were limited to studying phenomena with only constant change, how much of the science that you have learned would still exist?

EXERCISES 2.2

WRITING EXERCISES

1. The derivative is important because of its many different uses and interpretations. Describe four aspects of the derivative: graphical (think of tangent lines), symbolic (the derivative function), numerical (approximations) and applications (velocity and others).
2. Mathematicians often use the word "smooth" to describe functions with certain (desirable) properties. Graphically, how are differentiable functions smoother than functions that are continuous but not differentiable, or functions that are not continuous?
3. Briefly describe what the derivative tells you about the original function. In particular, if the derivative is positive at a point, what do you know about the trend of the function at that point? What is different if the derivative is negative at the point?
4. Show that the derivative of $f(x) = 3x - 5$ is $f'(x) = 3$. Explain in terms of slope why this is true.

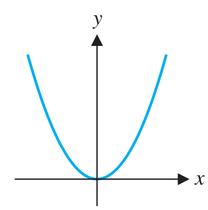
In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

1. $f(x) = 3x + 1, a = 1$
2. $f(x) = 3x^2 + 1, a = 1$
3. $f(x) = \sqrt{3x + 1}, a = 1$
4. $f(x) = \dfrac{3}{x + 1}, a = 2$

In exercises 5–12, compute the derivative function $f'(x)$ using (2.1) or (2.2).

5. $f(x) = 3x^2 + 1$
6. $f(x) = x^2 - 2x + 1$
7. $f(x) = \dfrac{3}{x + 1}$
8. $f(x) = \dfrac{2}{2x - 1}$
9. $f(x) = \sqrt{3x + 1}$
10. $f(x) = 2x + 3$
11. $f(x) = x^3 + 2x - 1$
12. $f(x) = x^4 - 2x^2 + 1$

In exercises 13–18, match the graphs of the functions on the left with the graphs of their derivatives on the right.

13.


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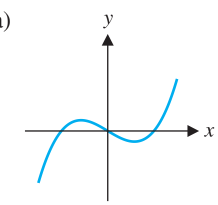

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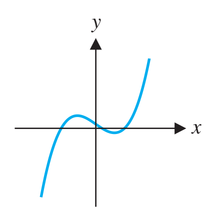

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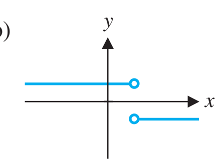

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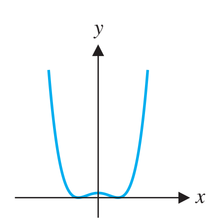

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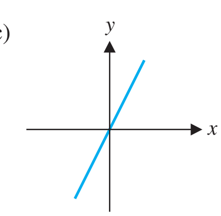

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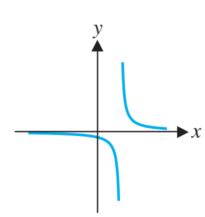

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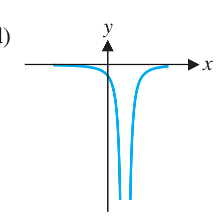

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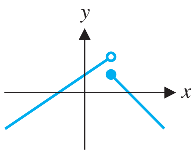

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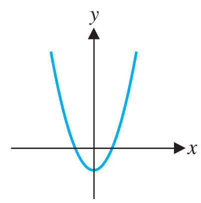

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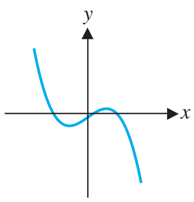

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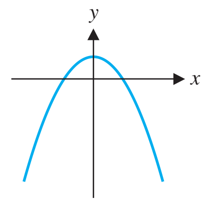
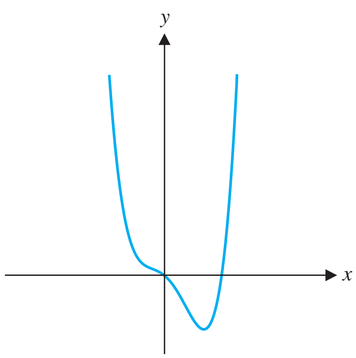


In exercises 19–22, use the given graph of $f(x)$ to sketch a graph of $f'(x)$.

19.



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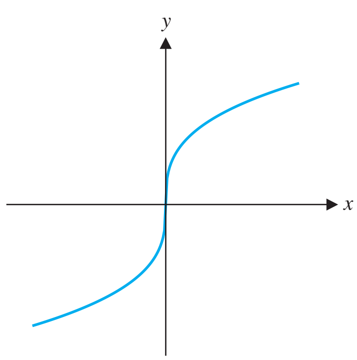

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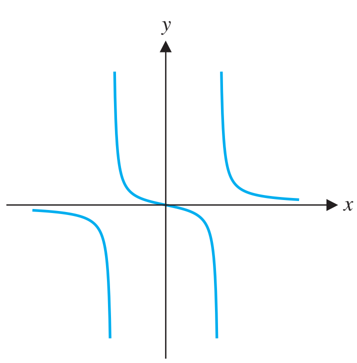

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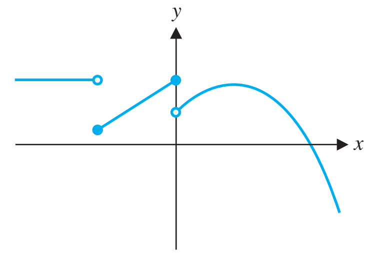
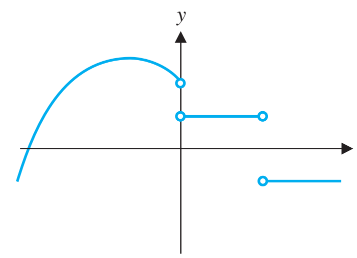


In exercises 23 and 24, use the given graph of $f'(x)$ to sketch a plausible graph of a continuous function $f(x)$.

23.



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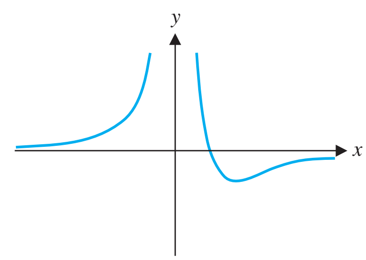


25. Graph $f(x) = |x| + |x - 2|$ and identify all x-values at which $f(x)$ is not differentiable.

26. Graph $f(x) = e^{-2/(x^3-x)}$ and identify all x-values at which $f(x)$ is not differentiable.

27. Find all real numbers p such that $f'(0)$ exists for $f(x) = x^p$.

28. Prove that if $f(x)$ is differentiable at $x = a$, then $\lim_{h \to 0} \frac{f(a + ch) - f(a)}{h} = cf'(a)$.

29. If $f(x)$ is differentiable at $x = a \neq 0$, evaluate $\lim_{x \to a} \frac{f(x^2) - f(a^2)}{x^2 - a^2}$.

30. Prove that if $f(x)$ is differentiable at $x = 0$, $f(x) \leq 0$ for all x and $f(0) = 0$, then $f'(0) = 0$.

31. The table shows the margin of error in degrees for tennis serves hit at 100 mph with various amounts of topspin (in units of revolutions per second). Estimate the slope of the derivative at $x = 60$, and interpret it in terms of the benefit of extra spin. (Data adapted from *The Physics and Technology of Tennis* by Brody, Cross and Lindsey.)

Topspin (rps)	20	40	60	80	100
Margin of error	1.8	2.4	3.1	3.9	4.6

32. The table shows the margin of error in degrees for tennis serves hit at 120 mph from various heights. Estimate the slope of the derivative at $x = 8.5$ and interpret it in terms of hitting a serve from a higher point. (Data adapted from *The Physics and Technology of Tennis* by Brody, Cross and Lindsey.)

Height (ft)	7.5	8.0	8.5	9.0	9.5
Margin of error	0.3	0.58	0.80	1.04	1.32

In exercises 33 and 34, use the distances $f(t)$ to estimate the velocity at $t = 2$.

33.

t	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(t)$	3.1	3.9	4.8	5.8	6.8	7.7	8.5

34.

t	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(t)$	4.6	5.3	6.1	7.0	7.8	8.6	9.3

35. The Environmental Protection Agency uses the measurement of ton-MPG to evaluate the power-train efficiency of vehicles. The ton-MPG rating of a vehicle is given by the weight of the vehicle (in tons) multiplied by a rating of the vehicle's fuel efficiency in miles per gallon. Several years of data for new cars are given in the table. Estimate the rate of change of ton-MPG in (a) 1994 and (b) 2000. Do your estimates imply that cars are becoming more or less efficient? Is the rate of change constant or changing?

Year	1992	1994	1996	1998	2000
Ton-MPG	44.9	45.7	46.5	47.3	47.7

36. The fuel efficiencies in miles per gallon of cars from 1992 to 2000 are shown in the following table. Estimate the rate of change in MPG in (a) 1994 and (b) 2000. Do your estimates imply that cars are becoming more or less fuel efficient? Comparing your answers to exercise 35, what must be happening to the average weight of cars? If weight had remained constant, what do you expect would have happened to MPG?

Year	1992	1994	1996	1998	2000
MPG	28.0	28.1	28.3	28.5	28.1

In exercises 37 and 38, use a CAS or graphing calculator.

37. Numerically estimate $f'(1)$ for $f(x) = x^x$ and verify your answer using a CAS.

38. Numerically estimate $f'(\pi)$ for $f(x) = x^{\sin x}$ and verify your answer using a CAS.

In exercises 39 and 40, compute the right-hand derivative $D_+ f(0) = \lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h}$ and the left-hand derivative $D_- f(0) = \lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h}$.

39. $f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ 3x + 1 & \text{if } x \ge 0 \end{cases}$

40. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \ge 0 \end{cases}$

41. Assume that $f(x) = \begin{cases} g(x) & \text{if } x < 0 \\ k(x) & \text{if } x \ge 0 \end{cases}$. If f is continuous at $x = 0$ and g and k are differentiable at $x = 0$, prove that $D_+ f(0) = k'(0)$ and $D_- f(0) = g'(0)$. Which statement is not true if f has a jump discontinuity at $x = 0$?

42. Explain why the derivative $f'(0)$ exists if and only if the one-sided derivatives exist and are equal.

43. If $f'(x) > 0$ for all x, use the tangent line interpretation to argue that f is an **increasing function;** that is, if $a < b$, then $f(a) < f(b)$.

44. If $f'(x) < 0$ for all x, use the tangent line interpretation to argue that f is a **decreasing function;** that is, if $a < b$, then $f(a) > f(b)$.

45. If $f(x) = x^{2/3}$, show graphically and numerically that f is continuous at $x = 0$ but $f'(0)$ does not exist.

46. If $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \ge 0 \end{cases}$, show graphically and numerically that f is continuous at $x = 0$ but $f'(0)$ does not exist.

47. Give an example showing that the following is not true for all functions f: if $f(x) \le x$, then $f'(x) \le 1$.

48. Determine whether the following is true for all functions f: if $f(0) = 0$, $f'(x)$ exists for all x and $f(x) \le x$, then $f'(x) \le 1$.

In exercises 49 and 50, give the units for the derivative function.

49. (a) $f(t)$ represents position, measured in meters, at time t seconds.
(b) $f(x)$ represents the demand, in number of items, of a product when the price is x dollars.

50. (a) $c(t)$ represents the amount of a chemical present, in grams, at time t minutes.
(b) $p(x)$ represents the mass, in kg, of the first x meters of a pipe.

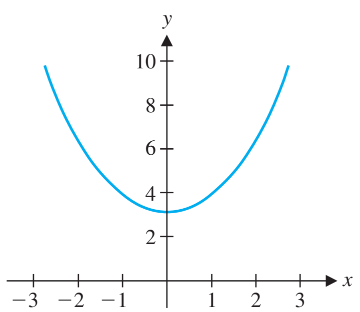
51. Let $f(t)$ represent the trading value of a stock at time t days. If $f'(t) < 0$, what does that mean about the stock? If you held some shares of this stock, should you sell what you have or buy more?

52. Suppose that there are two stocks with trading values $f(t)$ and $g(t)$, where $f(t) > g(t)$ and $0 < f'(t) < g'(t)$. Based on this information, which stock should you buy? Briefly explain.

53. One model for the spread of a disease assumes that at first the disease spreads very slowly, gradually the infection rate increases to a maximum and then the infection rate decreases back to zero, marking the end of the epidemic. If $I(t)$ represents the number of people infected at time t, sketch a graph of both $I(t)$ and $I'(t)$, assuming that those who get infected do not recover.

54. One model for urban population growth assumes that at first, the population is growing very rapidly, then the growth rate decreases until the population starts decreasing. If $P(t)$ is the population at time t, sketch a graph of both $P(t)$ and $P'(t)$.

55. Use the graph to list the following in increasing order: $f(1)$, $f(2) - f(1)$, $\dfrac{f(1.5) - f(1)}{0.5}$, $f'(1)$.



Exercises 55 and 56

56. Use the graph to list the following in increasing order: $f(0)$, $f(0) - f(-1)$, $\dfrac{f(0) - f(-0.5)}{0.5}$, $f'(0)$.

In exercises 57–60, the limit equals $f'(a)$ for some function $f(x)$ and some constant a. Determine $f(x)$ and a.

57. $\displaystyle\lim_{h\to 0} \frac{(1+h)^2 - (1+h)}{h}$

58. $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$

59. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$

60. $\displaystyle\lim_{h\to 0} \frac{(h-1)^2 - 1}{h}$

61. Sketch the graph of a function with the following properties: $f(0) = 1$, $f(1) = 0$, $f(3) = 6$, $f'(0) = 0$, $f'(1) = -1$ and $f'(3) = 4$.

62. Sketch the graph of a function with the following properties: $f(-2) = 4$, $f(0) = -2$, $f(2) = 1$, $f'(-2) = -2$, $f'(0) = 2$ and $f'(2) = 1$.

63. A phone company charges one dollar for the first 20 minutes of a call, then 10 cents per minute for the next 60 minutes and 8 cents per minute for each additional minute (or partial minute). Let $f(t)$ be the price in cents of a t-minute phone call, $t > 0$. Determine $f'(t)$ as completely as possible.

64. The table shows the percentage of English Premier League soccer players by birth month, where $x = 0$ represents November, $x = 1$ represents December and so on. (The data are adapted from John Wesson's *The Science of Soccer*.) If these data come from a differentiable function $f(x)$, estimate $f'(1)$. Interpret the derivative in terms of the effect of being a month older but in the same grade of school.

Month	0	1	2	3	4
Percent	13	11	9	7	7

EXPLORATORY EXERCISES

1. Compute the derivative function for x^2, x^3 and x^4. Based on your results, identify the pattern and conjecture a general formula for the derivative of x^n. Test your conjecture on the functions $\sqrt{x} = x^{1/2}$ and $1/x = x^{-1}$.

2. In Theorem 2.1, it is stated that a differentiable function is guaranteed to be continuous. The converse is not true: continuous functions are not necessarily differentiable (see example 2.7). This fact is carried to an extreme in **Weierstrass' function,** to be explored here. First, graph the function $f_4(x) = \cos x + \frac{1}{2}\cos 3x + \frac{1}{4}\cos 9x + \frac{1}{8}\cos 27x + \frac{1}{16}\cos 81x$ in the graphing window $0 \le x \le 2\pi$ and $-2 \le y \le 2$. Note that the graph appears to have several sharp corners, where a derivative would not exist. Next, graph the function $f_6(x) = f_4(x) + \frac{1}{32}\cos 243x + \frac{1}{64}\cos 729x$. Note that there are even more places where the graph appears to have sharp corners. Explore graphs of $f_{10}(x)$, $f_{14}(x)$ and so on, with more terms added. Try to give graphical support to the fact that the Weierstrass function $f_\infty(x)$ is continuous for all x but is not differentiable for any x. More graphical evidence comes from the fractal nature of the Weierstrass function: compare the graphs of $f_4(x)$ with $0 \le x \le 2\pi$ and $-2 \le y \le 2$ and $f_6(x) - \cos x - \frac{1}{2}\cos 3x$ with $0 \le x \le \frac{2\pi}{9}$ and $-\frac{1}{2} \le y \le \frac{1}{2}$. Explain why the graphs are identical. Then explain why this indicates that no matter how much you zoom in on a graph of the Weierstrass function, you will continue to see wiggles and corners. That is, you cannot zoom in to find a tangent line.

3. Suppose there is a function $F(x)$ such that $F(1) = 1$ and $F(0) = f_0$, where $0 < f_0 < 1$. If $F'(1) > 1$, show graphically that the equation $F(x) = x$ has a solution q where $0 < q < 1$. (Hint: Graph $y = x$ and a plausible $F(x)$ and look for intersections.) Sketch a graph where $F'(1) < 1$ and there are no solutions to the equation $F(x) = x$ between 0 and 1 (although $x = 1$ is a solution). Solutions have a connection with the probability of the extinction of animals or family names. Suppose you and your descendants have children according to the following probabilities: $f_0 = 0.2$ is the probability of having no children, $f_1 = 0.3$ is the probability of having exactly one child, and $f_2 = 0.5$ is the probability of having two children. Define $F(x) = 0.2 + 0.3x + 0.5x^2$ and show that $F'(1) > 1$. Find the solution of $F(x) = x$ between $x = 0$ and $x = 1$; this number is the probability that your "line" will go extinct some time into the future. Find nonzero values of f_0, f_1 and f_2 such that the corresponding $F(x)$ satisfies $F'(1) < 1$ and hence the probability of your line going extinct is 1.

4. The **symmetric difference quotient** of a function f centered at $x = a$ has the form $\dfrac{f(a+h) - f(a-h)}{2h}$. If $f(x) = x^2 + 1$ and $a = 1$, illustrate the symmetric difference quotient as a slope of a secant line for $h = 1$ and $h = 0.5$. Based on your picture, conjecture the limit of the symmetric difference quotient as h approaches 0. Then compute the limit and compare to the derivative $f'(1)$ found in example 1.1. For $h = 1$, $h = 0.5$ and $h = 0.1$, compare the actual values of the symmetric difference quotient and the usual difference quotient $\dfrac{f(a+h) - f(a)}{h}$. In general, which difference quotient provides a better estimate of the derivative? Next, compare the values of the difference quotients with $h = 0.5$ and $h = -0.5$ to the derivative $f'(1)$. Explain graphically why one is smaller and one is larger. Compare the average of these two difference quotients to the symmetric difference quotient with $h = 0.5$. Use this result to explain why the symmetric difference quotient might provide a better estimate of the derivative. Next, compute several symmetric difference quotients of $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ centered at $a = 2$. Recall that in example 2.7 we showed that the derivative $f'(2)$ does not exist. Given this, discuss one major problem with using the symmetric difference quotient to approximate derivatives. Finally, show that if $f'(a)$ exists, then $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h} = f'(a)$.

2.3 COMPUTATION OF DERIVATIVES: THE POWER RULE

You have now computed numerous derivatives using the limit definition. In fact, you may have computed enough that you have started taking some shortcuts. In exploratory exercise 1 in section 2.2, we asked you to compile a list of derivatives of basic functions and to generalize. We continue that process in this section.

The Power Rule

We first revisit the limit definition of derivative to compute two very simple derivatives.

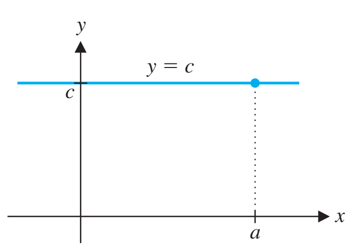


FIGURE 2.20
A horizontal line

For any constant c,
$$\frac{d}{dx} c = 0. \tag{3.1}$$

Notice that (3.1) says that for any constant c, the horizontal line $y = c$ has a tangent line with zero slope. That is, the tangent line to a horizontal line is the same horizontal line (see Figure 2.20).

Let $f(x) = c$, for all x. From the definition in equation (2.3), we have

$$\begin{aligned} \frac{d}{dx} c = f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0. \end{aligned}$$

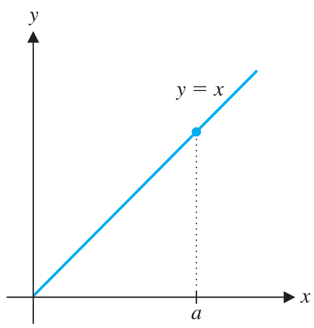


FIGURE 2.21
Tangent line to $y = x$

Similarly, we have

$$\frac{d}{dx}x = 1. \tag{3.2}$$

Notice that (3.2) says that the tangent line to the line $y = x$ is a line of slope one (i.e., $y = x$; see Figure 2.21). This is unsurprising, since intuitively, it should be clear that the tangent line to any line is that same line. We verify this result as follows.

Let $f(x) = x$. From equation (2.3), we have

$$\begin{aligned}\frac{d}{dx}x = f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)-x}{h}\\ &= \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1.\end{aligned}$$

$f(x)$	$f'(x)$
1	0
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

The table shown in the margin presents a short list of derivatives calculated previously either as examples or in the exercises using the limit definition. Can you identify any pattern to the derivatives shown in the table? There are two features to note. First, the power of x in the derivative is always one less than the power of x in the original function. Second, the coefficient of x in the derivative is the same as the power of x in the original function. Putting these ideas into symbolic form, we make and then prove a reasonable conjecture.

THEOREM 3.1 (Power Rule)

For any integer $n > 0$, $$\frac{d}{dx}x^n = nx^{n-1}.$$

PROOF

From the limit definition of derivative given in equation (2.3), if $f(x) = x^n$, then

$$\frac{d}{dx}x^n = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}. \tag{3.3}$$

To evaluate the limit, we will need to simplify the expression in the numerator. If n is a positive integer, we can multiply out $(x+h)^n$. Recall that $(x+h)^2 = x^2 + 2xh + h^2$ and $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. More generally, you may recall from the binomial theorem that

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n. \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$f'(x) = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h}$$

Cancel x^n terms.

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0} \frac{h\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h^1 + \cdots + nxh^{n-2} + h^{n-1}\right]}{h}$$

Factor out common h and cancel.

$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h^1 + \cdots + nxh^{n-2} + h^{n-1}\right] = nx^{n-1},$$

since every term but the first has a factor of h. ■

The power rule is very easy to apply, as we see in example 3.1.

EXAMPLE 3.1 Using the Power Rule

Find the derivative of $f(x) = x^8$ and $g(t) = t^{107}$.

Solution We have

$$f'(x) = \frac{d}{dx}x^8 = 8x^{8-1} = 8x^7.$$

Similarly, $$g'(t) = \frac{d}{dt}t^{107} = 107t^{107-1} = 107t^{106}.$$ ■

Recall that in section 2.2, we showed that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}. \tag{3.5}$$

Notice that we can rewrite (3.5) as

$$\frac{d}{dx}x^{-1} = (-1)x^{-2}.$$

That is, the derivative of x^{-1} follows the same pattern as the power rule that we just stated and proved for *positive* integer exponents.

Likewise, in section 2.2, we used the limit definition to show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}. \tag{3.6}$$

We can also rewrite (3.6) as $$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}.$$

Here, notice that the derivative of a rational power of x also follows the same pattern as the power rule that we proved for positive *integer* exponents.

REMARK 3.1

As we will see, the power rule holds for *any* power of x. We will not be able to prove this fact for some time now, as the proof of Theorem 3.1 does *not* generalize, since the expansion in equation (3.4) holds only for positive integer exponents. Even so, we will use the rule freely for any power of x. We state this in Theorem 3.2.

THEOREM 3.2 (General Power Rule)

For any real number r,
$$\frac{d}{dx}x^r = rx^{r-1}. \tag{3.7}$$

The power rule is simple to use, as we see in example 3.2.

CAUTION

Be careful here to avoid a common error:
$$\frac{d}{dx}x^{-19} \neq -19x^{-18}.$$
The power rule says to *subtract* 1 from the exponent (even if the exponent is negative).

EXAMPLE 3.2 Using the General Power Rule

Find the derivative of $\frac{1}{x^{19}}$, $\sqrt[3]{x^2}$ and x^π.

Solution From (3.7), we have
$$\frac{d}{dx}\left(\frac{1}{x^{19}}\right) = \frac{d}{dx}x^{-19} = -19x^{-19-1} = -19x^{-20}.$$

If we rewrite $\sqrt[3]{x^2}$ as a fractional power of x, we can use (3.7) to compute the derivative, as follows.
$$\frac{d}{dx}\sqrt[3]{x^2} = \frac{d}{dx}x^{2/3} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}.$$

Finally, we have
$$\frac{d}{dx}x^\pi = \pi x^{\pi-1}.$$

Notice that there is the additional conceptual problem in example 3.2 (which we resolve in Chapter 4) of deciding what x^π means. Since the exponent isn't rational, what exactly do we mean when we raise a number to the irrational power π?

General Derivative Rules

The power rule gives us a large class of functions whose derivatives we can quickly compute without using the limit definition. The following rules for combining derivatives further expand the number of derivatives we can compute without resorting to the definition. Keep in mind that a derivative is a limit; the differentiation rules in Theorem 3.3 then follow immediately from the corresponding rules for limits (found in Theorem 3.1 in Chapter 1).

THEOREM 3.3

If $f(x)$ and $g(x)$ are differentiable at x and c is any constant, then

(i) $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$,

(ii) $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$ and

(iii) $\frac{d}{dx}[cf(x)] = cf'(x)$.

PROOF

We prove only part (i). The proofs of parts (ii) and (iii) are left as exercises. Let $k(x) = f(x) + g(x)$. Then, from the limit definition of the derivative (2.3), we get

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] = k'(x) &= \lim_{h\to 0}\frac{k(x+h)-k(x)}{h} \\
&= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} && \text{By definition of } k(x). \\
&= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} && \text{Grouping the } f \text{ terms together and the } g \text{ terms together.} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{By Theorem 3.1 in Chapter 1.} \\
&= f'(x)+g'(x). && \text{Recognizing the derivatives of } f \text{ and of } g. \ \blacksquare
\end{aligned}$$

We illustrate Theorem 3.3 by working through the calculation of a derivative step by step, showing all of the details.

EXAMPLE 3.3 Finding the Derivative of a Sum

Find the derivative of $f(x) = 2x^6 + 3\sqrt{x}$.

Solution We have

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(2x^6) + \frac{d}{dx}\left(3\sqrt{x}\right) && \text{By Theorem 3.3 (i).} \\
&= 2\frac{d}{dx}(x^6) + 3\frac{d}{dx}(x^{1/2}) && \text{By Theorem 3.3 (iii).} \\
&= 2(6x^5) + 3\left(\frac{1}{2}x^{-1/2}\right) && \text{By the power rule.} \\
&= 12x^5 + \frac{3}{2\sqrt{x}}. && \text{Simplifying.}
\end{aligned}$$

EXAMPLE 3.4 Rewriting a Function before Computing the Derivative

Find the derivative of $f(x) = \dfrac{4x^2 - 3x + 2\sqrt{x}}{x}$.

Solution Note that we don't yet have any rule for computing the derivative of a quotient. So, we must first rewrite $f(x)$ by dividing out the x in the denominator. We have

$$f(x) = \frac{4x^2}{x} - \frac{3x}{x} + \frac{2\sqrt{x}}{x} = 4x - 3 + 2x^{-1/2}.$$

From Theorem 3.3 and the power rule (3.7), we get

$$f'(x) = 4\frac{d}{dx}(x) - 3\frac{d}{dx}(1) + 2\frac{d}{dx}(x^{-1/2}) = 4 - 0 + 2\left(-\frac{1}{2}x^{-3/2}\right) = 4 - x^{-3/2}.$$

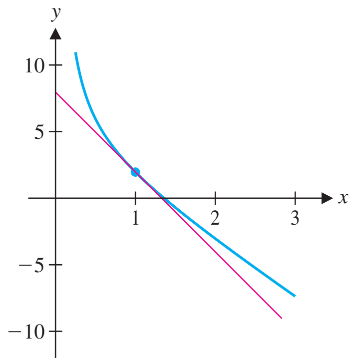


FIGURE 2.22
$y = f(x)$ and the tangent line at $x = 1$

EXAMPLE 3.5 Finding the Equation of the Tangent Line

Find an equation of the tangent line to $f(x) = 4 - 4x + \dfrac{2}{x}$ at $x = 1$.

Solution First, notice that $f(x) = 4 - 4x + 2x^{-1}$. From Theorem 3.3 and the power rule, we have

$$f'(x) = 0 - 4 - 2x^{-2} = -4 - 2x^{-2}.$$

At $x = 1$, the slope of the tangent line is then $f'(1) = -4 - 2 = -6$. The line with slope -6 through the point $(1, 2)$ has equation

$$y - 2 = -6(x - 1).$$

We show a graph of $y = f(x)$ and the tangent line at $x = 1$ in Figure 2.22. ■

Higher Order Derivatives

One consequence of having the derivative function is that we can compute the derivative of a derivative. It turns out that such **higher order** derivatives have important applications.

Suppose we start with a function $f(x)$ and compute its derivative $f'(x)$. We can then compute the derivative of $f'(x)$, called the **second derivative** of f and written $f''(x)$. We can then compute the derivative of $f''(x)$, called the **third derivative** of f, written $f'''(x)$. We can continue to take derivatives indefinitely. Below, we show common notations for the first five derivatives of f [where we assume that $y = f(x)$].

Order	*Prime Notation*	*Leibniz Notation*
1	$y' = f'(x)$	$\dfrac{df}{dx}$
2	$y'' = f''(x)$	$\dfrac{d^2 f}{dx^2}$
3	$y''' = f'''(x)$	$\dfrac{d^3 f}{dx^3}$
4	$y^{(4)} = f^{(4)}(x)$	$\dfrac{d^4 f}{dx^4}$
5	$y^{(5)} = f^{(5)}(x)$	$\dfrac{d^5 f}{dx^5}$

Computing higher order derivatives is done by simply computing several first derivatives, as we see in example 3.6.

EXAMPLE 3.6 Computing Higher Order Derivatives

If $f(x) = 3x^4 - 2x^2 + 1$, compute as many derivatives as possible.

Solution We have

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(3x^4 - 2x^2 + 1) = 12x^3 - 4x.$$

Then,
$$f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}(12x^3 - 4x) = 36x^2 - 4,$$
$$f'''(x) = \frac{d^3 f}{dx^3} = \frac{d}{dx}(36x^2 - 4) = 72x,$$
$$f^{(4)}(x) = \frac{d^4 f}{dx^4} = \frac{d}{dx}(72x) = 72,$$
$$f^{(5)}(x) = \frac{d^5 f}{dx^5} = \frac{d}{dx}(72) = 0$$
and so on. It follows that
$$f^{(n)}(x) = \frac{d^n f}{dx^n} = 0, \text{ for } n \geq 5.$$

Acceleration

What information does the second derivative of a function give us? Graphically, we get a property called *concavity,* which we develop in Chapter 3. One important application of the second derivative is acceleration, which we briefly discuss now.

You are probably familiar with the term **acceleration,** which is **the instantaneous rate of change of velocity.** Consequently, if the velocity of an object at time t is given by $v(t)$, then the acceleration is
$$a(t) = v'(t) = \frac{dv}{dt}.$$

EXAMPLE 3.7 Computing the Acceleration of a Skydiver

Suppose that the height of a skydiver t seconds after jumping from an airplane is given by $f(t) = 640 - 20t - 16t^2$ feet. Find the person's acceleration at time t.

Solution Since acceleration is the derivative of velocity, we first compute velocity:
$$v(t) = f'(t) = 0 - 20 - 32t = -20 - 32t \text{ ft/s}.$$
Computing the derivative of this function gives us
$$a(t) = v'(t) = -32.$$
To finish the problem, we need to determine the units of acceleration. Since the distance here is measured in feet and time is measured in seconds, the units of the velocity are feet per second, so that the units of acceleration are feet per second per second, written ft/s/s, or more commonly ft/s^2 (feet per second squared). Our answer says that the velocity changes by -32 ft/s every second. In this case, the speed in the downward (negative) direction increases by 32 ft/s every second due to gravity. ■

BEYOND FORMULAS

The power rule gives us a much-needed shortcut for computing many derivatives. Mathematicians always seek the shortest, most efficient computations. By skipping unnecessary lengthy steps and saving brain power, mathematicians free themselves to tackle complex problems with creativity. It's important to remember, however, that shortcuts such as the power rule must always be carefully proved. What shortcuts do you use when solving an equation such as $3x^2 - 4 = 5$ and what are the intermediate steps that you are skipping?

EXERCISES 2.3

WRITING EXERCISES

1. Explain to a non-calculus-speaking friend how to (mechanically) use the power rule. Decide whether it is better to give separate explanations for positive and negative exponents; integer and noninteger exponents; other special cases.

2. In the 1700s, mathematical "proofs" were, by modern standards, a bit fuzzy and lacked rigor. In 1734, the Irish metaphysician Bishop Berkeley wrote *The Analyst* to an "infidel mathematician" (thought to be Edmund Halley of Halley's comet fame). The accepted proof at the time of the power rule may be described as follows.

   If x is incremented to $x + h$, then x^n is incremented to $(x + h)^n$. It follows that $\dfrac{(x+h)^n - x^n}{(x+h) - x} = nx^{n-1} + \dfrac{n^2 - n}{2}hx^{n-2} + \cdots$. Now, let the increment h vanish, and the derivative is nx^{n-1}.

   Bishop Berkeley objected to this argument.

   "But it should seem that the reasoning is not fair or conclusive. For when it is said, 'let the increments vanish,' the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition is retained. Which . . . is a false way of reasoning. Certainly, when we suppose the increments to vanish, we must suppose . . . everything derived from the supposition of their existence to vanish with them."

   Do you think Berkeley's objection is fair? Is it logically acceptable to assume that something exists to draw one conclusion, and then assume that the same thing does not exist to avoid having to accept other consequences? Mathematically speaking, how does the limit avoid Berkeley's objection of the increment h both existing and not existing?

3. The historical episode in exercise 2 is just one part of an ongoing conflict between people who blindly use mathematical techniques without proof and those who insist on a full proof before permitting anyone to use the technique. To which side are you sympathetic? Defend your position in an essay. Try to anticipate and rebut the other side's arguments.

4. Explain the first two terms in the expansion $(x + h)^n = x^n + nhx^{n-1} + \cdots$, where n is a positive integer. Think of multiplying out $(x + h)(x + h)(x + h) \cdots (x + h)$; how many terms would include x^n? x^{n-1}?

In exercises 1–16, find the derivative of each function.

1. $f(x) = x^3 - 2x + 1$
2. $f(x) = x^9 - 3x^5 + 4x^2 - 4x$
3. $f(t) = 3t^3 - 2\sqrt{t}$
4. $f(s) = 5\sqrt{s} - 4s^2 + 3$
5. $f(x) = \dfrac{3}{x} - 8x + 1$
6. $f(x) = \dfrac{2}{x^4} - x^3 + 2$
7. $h(x) = \dfrac{10}{\sqrt{x}} - 2x$
8. $h(x) = 12x - x^2 - \dfrac{3}{\sqrt{x}}$
9. $f(s) = 2s^{3/2} - 3s^{-1/3}$
10. $f(t) = 3t^{\pi} - 2t^{1.3}$
11. $f(x) = 2\sqrt[3]{x} + 3$
12. $f(x) = 4x - 3\sqrt[3]{x^2}$
13. $f(x) = x\left(3x^2 - \sqrt{x}\right)$
14. $f(x) = (x + 1)(3x^2 - 4)$
15. $f(x) = \dfrac{3x^2 - 3x + 1}{2x}$
16. $f(x) = \dfrac{4x^2 - x + 3}{\sqrt{x}}$

In exercises 17–24, compute the indicated derivative.

17. $f''(x)$ for $f(x) = x^4 + 3x^2 - 2$
18. $\dfrac{d^2 f}{dx^2}$ for $f(x) = x^6 - \sqrt{x}$
19. $\dfrac{d^2 f}{dx^2}$ for $f(x) = 2x^4 - \dfrac{3}{\sqrt{x}}$
20. $f'''(t)$ for $f(t) = 4t^2 - 12 + \dfrac{4}{t^2}$
21. $f^{(4)}(x)$ for $f(x) = x^4 + 3x^2 - 2$
22. $f^{(5)}(x)$ for $f(x) = x^{10} - 3x^4 + 2x - 1$
23. $f'''(x)$ for $f(x) = \dfrac{x^2 - x + 1}{\sqrt{x}}$
24. $f^{(4)}(t)$ for $f(t) = (t^2 - 1)(\sqrt{t} + t)$

In exercises 25–28, use the given position function to find the velocity and acceleration functions.

25. $s(t) = -16t^2 + 40t + 10$
26. $s(t) = 12t^3 - 6t - 1$
27. $s(t) = \sqrt{t} + 2t^2$
28. $s(t) = 10 - \dfrac{10}{t}$

In exercises 29–32, the given function represents the height of an object. Compute the velocity and acceleration at time $t = t_0$. Is the object going up or down? Is the speed of the object increasing or decreasing?

29. $h(t) = -16t^2 + 40t + 5,\ t_0 = 1$

30. $h(t) = -16t^2 + 40t + 5,\ t_0 = 2$

31. $h(t) = 10t^2 - 24t,\ t_0 = 2$

32. $h(t) = 10t^2 - 24t,\ t_0 = 1$

In exercises 33–36, find an equation of the tangent line to $y = f(x)$ at $x = a$.

33. $f(x) = 4\sqrt{x} - 2x,\ a = 4$

34. $f(x) = x^2 - 2x + 1,\ a = 2$

35. $f(x) = x^2 - 2,\ a = 2$

36. $f(x) = 3x + 4,\ a = 2$

In exercises 37 and 38, determine the value(s) of x for which the tangent line to $y = f(x)$ is horizontal. Graph the function and determine the graphical significance of each point.

37. $f(x) = x^3 - 3x + 1$

38. $f(x) = x^4 - 2x^2 + 2$

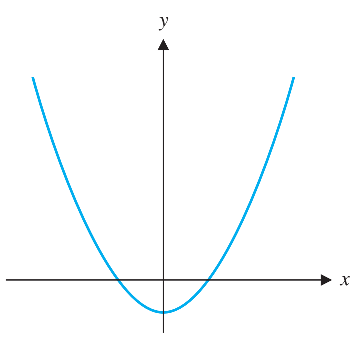
In exercises 39 and 40, determine the value(s) of x for which the slope of the tangent line to $y = f(x)$ does not exist. Graph the function and determine the graphical significance of each point.

39. $f(x) = x^{2/3}$

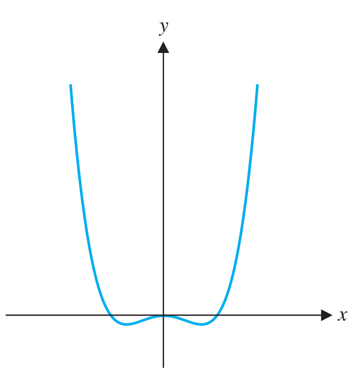
40. $f(x) = x^{1/3}$

In exercises 41 and 42, one curve represents a function $f(x)$ and the other two represent $f'(x)$ and $f''(x)$. Determine which is which.

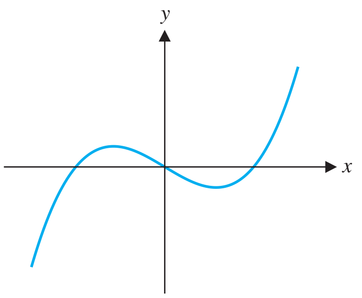
41. (a)



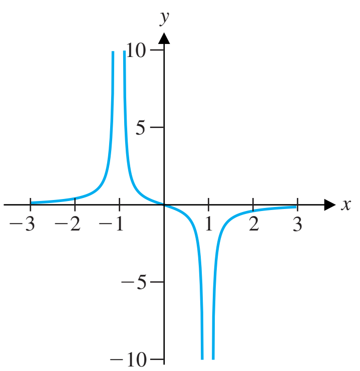
(b)



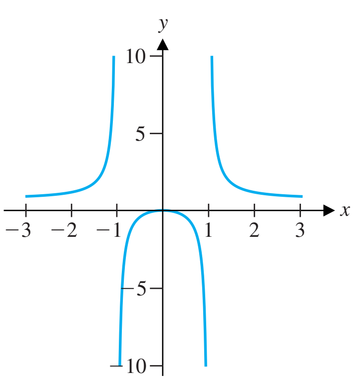
(c)



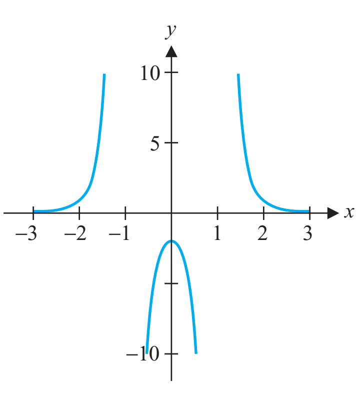
42. (a)



(b)



(c)



In exercises 43 and 44, find a general formula for the nth derivative $f^{(n)}(x)$.

43. $f(x) = \sqrt{x}$ **44.** $f(x) = \dfrac{2}{x}$

45. Find a second-degree polynomial (of the form $ax^2 + bx + c$) such that $f(0) = -2$, $f'(0) = 2$ and $f''(0) = 3$.

46. Find a second-degree polynomial (of the form $ax^2 + bx + c$) such that $f(0) = 0$, $f'(0) = 5$ and $f''(0) = 1$.

47. Find the area of the triangle bounded by $x = 0$, $y = 0$ and the tangent line to $y = \frac{1}{x}$ at $x = 1$. Repeat with the triangle bounded by $x = 0$, $y = 0$ and the tangent line to $y = \frac{1}{x}$ at $x = 2$. Show that you get the same area using the tangent line to $y = \frac{1}{x}$ at any $x = a > 0$.

48. Show that the result of exercise 47 does not hold for $y = \frac{1}{x^2}$. That is, the area of the triangle bounded by $x = 0$, $y = 0$ and the tangent line to $y = \frac{1}{x^2}$ at $x = a > 0$ does depend on the value of a.

49. Assume that a is a real number, $f(x)$ is differentiable for all $x \geq a$ and $g(x) = \max\limits_{a \leq t \leq x} f(t)$ for $x \geq a$. Find $g'(x)$ in the cases (a) $f'(x) > 0$ and (b) $f'(x) < 0$.

50. Assume that a is a real number, $f(x)$ is differentiable for all $x \geq a$ and $g(x) = \min\limits_{a \leq t \leq x} f(t)$ for $x \geq a$. Find $g'(x)$ in the cases (a) $f'(x) > 0$ and (b) $f'(x) < 0$.

51. A public official solemnly proclaims, "We have achieved a reduction in the rate at which the national debt is increasing." If $d(t)$ represents the national debt at time t years, which derivative of $d(t)$ is being reduced? What can you conclude about the size of $d(t)$ itself?

52. A rod made of an inhomogeneous material extends from $x = 0$ to $x = 4$ meters. The mass of the portion of the rod from $x = 0$ to $x = t$ is given by $m(t) = 3t^2$ kg. Compute $m'(t)$ and explain why it represents the density of the rod.

53. For most land animals, the relationship between leg width w and body length b follows an equation of the form $w = cb^{3/2}$ for some constant $c > 0$. Show that if b is large enough, $w'(b) > 1$. Conclude that for larger animals, leg width (necessary for support) increases faster than body length. Why does this put a limitation on the size of land animals?

54. Suppose the function $v(d)$ represents the average speed in m/s of the world record running time for d meters. For example, if the fastest 200-meter time ever is 19.32 s, then $v(200) = 200/19.32 \approx 10.35$. Compare the function $f(d) = 26.7d^{-0.177}$ to the values of $v(d)$, which you will have to research and compute, for distances ranging from $d = 400$ to $d = 2000$. Explain what $v'(d)$ would represent.

55. Let $f(t)$ equal the gross domestic product (GDP) in billions of dollars for the United States in year t. Several values are given in the table. Estimate and interpret $f'(2000)$ and $f''(2000)$. [Hint: to estimate the second derivative, estimate $f'(1998)$ and $f'(1999)$ and look for a trend.]

t	1996	1997	1998	1999	2000	2001
$f(t)$	7664.8	8004.5	8347.3	8690.7	9016.8	9039.5

56. Let $f(t)$ equal the average weight of a domestic SUV in year t. Several values are given in the table below. Estimate and interpret $f'(2000)$ and $f''(2000)$.

t	1985	1990	1995	2000
$f(t)$	4055	4189	4353	4619

57. If the position of an object is at time t given by $f(t)$, then $f'(t)$ represents velocity and $f''(t)$ gives acceleration. By Newton's second law, acceleration is proportional to the net force on the object (causing it to accelerate). Interpret the third derivative $f'''(t)$ in terms of force. The term **jerk** is sometimes applied to $f'''(t)$. Explain why this is an appropriate term.

58. Suppose that the daily output of a manufacturing plant is modeled by $Q = 1000K^{1/2}L^{1/3}$, where K is the capital investment in thousands of dollars and L is the size of the labor force in worker-hours. Assume that L stays constant and think of output as a function of capital investment, $Q(x) = 1000L^{1/3}x^{1/2}$. Find and interpret $Q'(40)$.

In exercises 59–62, find a function with the given derivative.

59. $f'(x) = 4x^3$ **60.** $f'(x) = 5x^4$

61. $f'(x) = \sqrt{x}$ **62.** $f'(x) = \dfrac{1}{x^2}$

63. Assume that a is a real number and $f''(a)$ exists. Then $\lim\limits_{h \to 0} \dfrac{f(a+h) - 2f(a) + f(a-h)}{h^2}$ also exists. Find its value.

64. For $f(x) = x|x|$, show that $\lim\limits_{h \to 0} \dfrac{f(h) - 2f(0) + f(-h)}{h^2}$ exists but $f''(0)$ does not exist. (That is, the converse of exercise 63 is not true.)

EXPLORATORY EXERCISES

1. A plane is cruising at an altitude of 2 miles at a distance of 10 miles from an airport. Choosing the airport to be at the point (0, 0), the plane starts its descent at the point (10, 2) and lands at the airport. Sketch a graph of a reasonable flight path $y = f(x)$, where y represents altitude and x gives the ground distance from the airport. (Think about it as you draw!) Explain what the derivative $f'(x)$ represents. (Hint: It's not velocity.) Explain why it is important and/or necessary to have $f(0) = 0$, $f(10) = 2$, $f'(0) = 0$ and $f'(10) = 0$. The simplest polynomial that can meet these requirements is a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ (Note: four requirements, four constants). Find values of the constants a, b, c and d to fit the flight path. [Hint: Start by setting $f(0) = 0$ and then set $f'(0) = 0$. You may want to use your CAS to solve the equations.] Graph the resulting function; does it look right? Suppose that airline regulations prohibit a derivative of $\frac{2}{10}$ or larger. Why might such a regulation exist? Show that the flight path you found is illegal. Argue that in fact all flight paths meeting the four requirements are illegal. Therefore, the descent needs to start farther away than 10 miles. Find a flight path with descent starting at 20 miles away that meets all requirements.

2. We discuss a graphical interpretation of the second derivative in Chapter 3. You can discover the most important aspects of that here. For $f(x) = x^4 - 2x^2 - 1$, solve the equations $f'(x) = 0$ and $f''(x) = 0$. What do the solutions of the equation $f'(x) = 0$ represent graphically? The solutions of the equation $f''(x) = 0$ are a little harder to interpret. Looking at the graph of $f(x)$ near $x = 0$, would you say that the graph is curving up or curving down? Compute $f''(0)$. Looking at the graph near $x = 2$ and $x = -2$, is the graph curving up or down? Compute $f''(2)$ and $f''(-2)$. Where does the graph change from curving up to curving down and vice versa? Hypothesize a relationship between $f''(x)$ and the curving of the graph of $y = f(x)$. Test your hypothesis on a variety of functions. (Try $y = x^4 - 4x^3$.)

3. In the enjoyable book *Surely You're Joking Mr. Feynman,* physicist Richard Feynman tells the story of a contest he had pitting his brain against the technology of the day (an abacus). The contest was to compute the cube root of 1729.03. Feynman came up with 12.002 before the abacus expert gave up. Feynman admits to some luck in the choice of the number 1729.03: he knew that a cubic foot contains 1728 cubic inches. Explain why this told Feynman that the answer is slightly greater than 12. How did he get three digits of accuracy? "I had learned in calculus that for small fractions, the cube root's excess is one-third of the number's excess. The excess, 1.03, is only one part in nearly 2000. So all I had to do is find the fraction 1/1728, divide by 3 and multiply by 12." To see what he did, find an equation of the tangent line to $y = x^{1/3}$ at $x = 1728$ and find the y-coordinate of the tangent line at $x = 1729.03$.

4. Suppose that you want to find solutions of the equation $x^3 - 4x^2 + 2 = 0$. Show graphically that there is a solution between $x = 0$ and $x = 1$. We will approximate this solution in stages. First, find an equation of the tangent line to $y = x^3 - 4x^2 + 2$ at $x = 1$. Then, determine where this tangent line crosses the x-axis. Show graphically that the x-intercept is considerably closer to the solution than is $x = 1$. Now, repeat the process: for the new x-value, find the equation of the tangent line, determine where it crosses the x-axis and show that this is closer still to the desired solution. This process of using tangent lines to produce continually improved approximations is referred to as **Newton's method.** We discuss this in some detail in section 3.1.

2.4 THE PRODUCT AND QUOTIENT RULES

We have now developed rules for computing the derivatives of a variety of functions, including general formulas for the derivative of a sum or difference of two functions. Given this, you might wonder whether the derivative of a product of two functions is the same as the product of the derivatives. We test this conjecture with a simple example.

Product Rule

Consider $\frac{d}{dx}[(x^2)(x^5)]$. We can compute this derivative by first combining the two terms:

$$\frac{d}{dx}[(x^2)(x^5)] = \frac{d}{dx}x^7 = 7x^6.$$

Is this derivative the same as the product of the two individual derivatives? Notice that

$$\begin{aligned}\left(\frac{d}{dx}x^2\right)\left(\frac{d}{dx}x^5\right) &= (2x)(5x^4) \\ &= 10x^5 \neq 7x^6 = \frac{d}{dx}[(x^2)(x^5)]. \qquad (4.1)\end{aligned}$$

You can now plainly see from (4.1) that the derivative of a product is *not* generally the product of the corresponding derivatives. In Theorem 4.1, we state a general rule for computing the derivative of a product of two differentiable functions.

THEOREM 4.1 (Product Rule)

Suppose that f and g are differentiable at x. Then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x). \tag{4.2}$$

PROOF

Since we are proving a general rule, we have only the limit definition of derivative to use. For $p(x) = f(x)g(x)$, we have

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] = p'(x) &= \lim_{h\to 0}\frac{p(x+h)-p(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}.\end{aligned} \tag{4.3}$$

Notice that the elements of the derivatives of f and g are present [limit, $f(x+h)$, $f(x)$ etc.], but we need to get them into the right form. The trick is to add and subtract $f(x)g(x+h)$ in the numerator.

From (4.3), we have

$$\begin{aligned} p'(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \quad \text{Subtract and add } f(x)g(x+h).\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h} \quad \text{Break into two pieces.}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x+h)\right] + \lim_{h\to 0}\left[f(x)\frac{g(x+h)-g(x)}{h}\right]\\ &= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0}g(x+h)\right] + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)g(x)+f(x)g'(x). \quad \text{Recognize the derivative of } f \text{ and the derivative of } g.\end{aligned}$$

Here, we identified the limits of the difference quotients as derivatives. There is also a subtle technical detail in the last step: since g is differentiable at x, recall that it must also be continuous at x, so that $g(x+h) \to g(x)$ as $h \to 0$. ■

In example 4.1, notice that the product rule saves us from multiplying out a messy product.

EXAMPLE 4.1 Using the Product Rule

Find $f'(x)$ if $f(x) = (2x^4 - 3x + 5)\left(x^2 - \sqrt{x} + \dfrac{2}{x}\right)$.

Solution Although we could first multiply out the expression, the product rule will simplify our work:

$$f'(x) = \left[\frac{d}{dx}(2x^4 - 3x + 5)\right]\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + (2x^4 - 3x + 5)\frac{d}{dx}\left(x^2 - \sqrt{x} + \frac{2}{x}\right)$$

$$= (8x^3 - 3)\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + (2x^4 - 3x + 5)\left(2x - \frac{1}{2\sqrt{x}} - \frac{2}{x^2}\right).$$

The product rule usually leaves derivatives in an "unsimplified" form as in example 4.1. Unless you have some particular reason to simplify (or some particularly obvious simplification to make), just leave the results of the product rule alone.

EXAMPLE 4.2 Finding the Equation of the Tangent Line

Find an equation of the tangent line to

$$y = (x^4 - 3x^2 + 2x)(x^3 - 2x + 3)$$

at $x = 0$.

Solution From the product rule, we have

$$y' = (4x^3 - 6x + 2)(x^3 - 2x + 3) + (x^4 - 3x^2 + 2x)(3x^2 - 2).$$

Evaluating at $x = 0$, we have $y'(0) = (2)(3) + (0)(-2) = 6$. The line with slope 6 and passing through the point (0, 0) [why (0, 0)?] has equation $y = 6x$.

Quotient Rule

Given our experience with the product rule, you probably have no expectation that the derivative of a quotient will turn out to be the quotient of the derivatives. Just to be sure, let's try a simple experiment. Note that

$$\frac{d}{dx}\left(\frac{x^5}{x^2}\right) = \frac{d}{dx}(x^3) = 3x^2,$$

while

$$\frac{\frac{d}{dx}(x^5)}{\frac{d}{dx}(x^2)} = \frac{5x^4}{2x^1} = \frac{5}{2}x^3 \neq 3x^2 = \frac{d}{dx}\left(\frac{x^5}{x^2}\right).$$

Since these are obviously not the same, we know that the derivative of a quotient is generally not the quotient of the corresponding derivatives.

Theorem 4.2 provides us with a general rule for computing the derivative of a quotient of two differentiable functions.

THEOREM 4.2 (Quotient Rule)

Suppose that f and g are differentiable at x and $g(x) \neq 0$. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \tag{4.4}$$

PROOF

You should be able to guess the structure of the proof. For $Q(x) = \frac{f(x)}{g(x)}$, we have from the limit definition of derivative that

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = Q'(x) &= \lim_{h\to 0}\frac{Q(x+h)-Q(x)}{h} \\
&= \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)}-\dfrac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{\left[\dfrac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\right]}{h} && \text{Add the fractions.} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)}. && \text{Simplify.}
\end{aligned}$$

As in the proof of the product rule, we look for the right term to add and subtract in the numerator, so that we can isolate the limit definitions of $f'(x)$ and $g'(x)$. In this case, we add and subtract $f(x)g(x)$, to get

$$\begin{aligned}
Q'(x) &= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x+h)g(x)} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{hg(x+h)g(x)} && \text{Subtract and add } f(x)g(x). \\
&= \lim_{h\to 0}\frac{\dfrac{f(x+h)-f(x)}{h}g(x)-f(x)\dfrac{g(x+h)-g(x)}{h}}{g(x+h)g(x)} && \text{Group first two and last two terms together and factor out common terms.} \\
&= \frac{\lim\limits_{h\to 0}\left[\dfrac{f(x+h)-f(x)}{h}\right]g(x)-f(x)\lim\limits_{h\to 0}\left[\dfrac{g(x+h)-g(x)}{h}\right]}{\lim\limits_{h\to 0}g(x+h)g(x)} \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2},
\end{aligned}$$

where we have again recognized the derivatives of f and g and used the fact that g is differentiable to imply that g is continuous, so that

$$\lim_{h\to 0} g(x+h) = g(x). \blacksquare$$

Notice that the numerator in the quotient rule looks very much like the product rule, but with a minus sign between the two terms. For this reason, you need to be very careful with the order.

EXAMPLE 4.3 Using the Quotient Rule

Compute the derivative of $f(x) = \frac{x^2-2}{x^2+1}$.

Solution Using the quotient rule, we have

$$f'(x) = \frac{\left[\frac{d}{dx}(x^2-2)\right](x^2+1) - (x^2-2)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{2x(x^2+1) - (x^2-2)(2x)}{(x^2+1)^2}$$

$$= \frac{6x}{(x^2+1)^2}.$$

In this case, we rewrote the numerator because it simplified significantly. This often occurs with the quotient rule. ■

Now that we have the quotient rule, we can justify the use of the power rule for negative integer exponents. (Recall that we have been using this rule without proof since section 2.3.)

THEOREM 4.3 (Power Rule)

For any integer exponent n, $\frac{d}{dx}x^n = nx^{n-1}$.

PROOF

We have already proved this for *positive* integer exponents. So, suppose that $n < 0$ and let $M = -n > 0$. Then, using the quotient rule, we get

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{-M} = \frac{d}{dx}\left(\frac{1}{x^M}\right) \qquad \text{Since } x^{-M} = \frac{1}{x^M}.$$

$$= \frac{\left[\frac{d}{dx}(1)\right]x^M - (1)\frac{d}{dx}(x^M)}{(x^M)^2} \qquad \text{By the quotient rule.}$$

$$= \frac{(0)x^M - (1)Mx^{M-1}}{x^{2M}} \qquad \text{By the power rule, since } M > 0.$$

$$= \frac{-Mx^{M-1}}{x^{2M}} = -Mx^{M-1-2M} \qquad \text{By the usual rules of exponents.}$$

$$= (-M)x^{-M-1} = nx^{n-1}, \qquad \text{Since } n = -M.$$

where we have used the fact that $\frac{d}{dx}x^M = Mx^{M-1}$, since $M > 0$. ■

As we see in example 4.4, it is sometimes preferable to rewrite a function, instead of automatically using the product or quotient rule.

EXAMPLE 4.4 A Case Where the Product and Quotient Rules Are Not Needed

Compute the derivative of $f(x) = x\sqrt{x} + \frac{2}{x^2}$.

Solution Although it may be tempting to use the product rule for the first term and the quotient rule for the second term, notice that we can rewrite the function and considerably simplify our work. We can combine the two powers of x in the first term and since the second term is a fraction with a constant numerator, we can more simply write it using a negative exponent. We have

$$f(x) = x\sqrt{x} + \frac{2}{x^2} = x^{3/2} + 2x^{-2}.$$

Using the power rule, we have simply

$$f'(x) = \frac{3}{2}x^{1/2} - 4x^{-3}.$$

Applications

You will see important uses of the product and quotient rules throughout your mathematical and scientific studies. We start you off with a couple of simple applications now.

EXAMPLE 4.5 Investigating the Rate of Change of Revenue

Suppose that a product currently sells for \$25, with the price increasing at the rate of \$2 per year. At this price, consumers will buy 150 thousand items, but the number sold is decreasing at the rate of 8 thousand per year. At what rate is the total revenue changing? Is the total revenue increasing or decreasing?

Solution To answer these questions, we need the basic relationship

$$\text{revenue} = \text{quantity} \times \text{price}$$

(e.g., if you sell 10 items at \$4 each, you earn \$40). Since these quantities are changing in time, we write $R(t) = Q(t)P(t)$, where $R(t)$ is revenue, $Q(t)$ is quantity sold and $P(t)$ is the price, all at time t. We don't have formulas for any of these functions, but from the product rule, we have

$$R'(t) = Q'(t)P(t) + Q(t)P'(t).$$

We have information about each of these terms: the initial price, $P(0)$, is 25 (dollars); the rate of change of the price is $P'(0) = 2$ (dollars per year); the initial quantity, $Q(0)$, is 150 (thousand items) and the rate of change of quantity is $Q'(0) = -8$ (thousand items per year). Note that the negative sign of $Q'(0)$ denotes a decrease in Q. Thus,

$$R'(0) = (-8)(25) + (150)(2) = 100 \text{ thousand dollars per year.}$$

Since the rate of change is positive, the revenue is increasing. This may be a surprise since one of the two factors in the equation is decreasing and the rate of decrease of the quantity is more than the rate of increase in the price.

EXAMPLE 4.6 Using the Derivative to Analyze Sports

A golf ball of mass 0.05 kg struck by a golf club of mass m kg with speed 50 m/s will have an initial speed of $u(m) = \dfrac{83m}{m + 0.05}$ m/s. Show that $u'(m) > 0$ and interpret this result in golf terms. Compare $u'(0.15)$ and $u'(0.20)$.

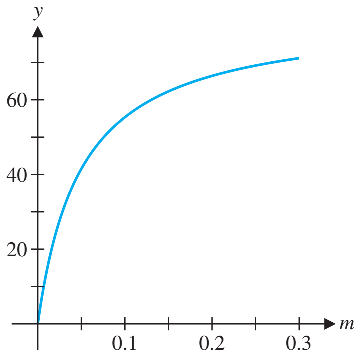


FIGURE 2.23
$$u(m) = \frac{83m}{m + 0.05}$$

Solution From the quotient rule, we have

$$u'(m) = \frac{83(m + 0.05) - 83m}{(m + 0.05)^2} = \frac{4.15}{(m + 0.05)^2}.$$

Both the numerator and denominator are positive, so $u'(m) > 0$. A positive slope for all tangent lines indicates that the graph of $u(m)$ should rise from left to right (see Figure 2.23). Said a different way, $u(m)$ increases as m increases. In golf terms, this says that (all other things being equal) the greater the mass of the club, the greater the velocity of the ball will be. Finally, we compute $u'(0.15) = 103.75$ and $u'(0.20) = 66.4$. This says that the rate of increase in ball speed is much less for the heavier club than for the lighter one. Since heavier clubs can be harder to control, the relatively small increase in ball speed obtained by making the heavy club even heavier may not compensate for the decrease in control. ■

EXERCISES 2.4

WRITING EXERCISES

1. The product and quotient rules give you the ability to symbolically calculate the derivative of a wide range of functions. However, many calculators and almost every computer algebra system (CAS) can do this work for you. Discuss why you should learn these basic rules anyway. (Keep example 4.5 in mind.)

2. Gottfried Leibniz is recognized (along with Sir Isaac Newton) as a coinventor of calculus. Many of the fundamental methods and (perhaps more importantly) much of the notation of calculus are due to Leibniz. The product rule was worked out by Leibniz in 1675, in the form $d(xy) = (dx)y + x(dy)$. His "proof," as given in a letter written in 1699, follows. "If we are to differentiate xy we write:

$$(x + dx)(y + dy) - xy = x\,dy + y\,dx + dx\,dy.$$

But here $dx\,dy$ is to be rejected as incomparably less than $x\,dy + y\,dx$. Thus, in any particular case the error is less than any finite quantity." Answer Leibniz' letter with one describing your own "discovery" of the product rule for $d(xyz)$. Use Leibniz' notation.

3. In example 4.1, we cautioned you against always multiplying out the terms of the derivative. To see one reason for this warning, suppose that you want to find solutions of the equation $f'(x) = 0$. (In fact, we do this routinely in Chapter 3.) Explain why having a factored form of $f'(x)$ is very helpful. Discuss the extent to which the product rule gives you a factored form.

4. Many students prefer the product rule to the quotient rule. Many computer algebra systems actually use the product rule to compute the derivative of $f(x)[g(x)]^{-1}$ instead of using the quotient rule on $\dfrac{f(x)}{g(x)}$ (see exercise 18 on the next page). Given the simplifications in problems like example 4.3, explain why the quotient rule can be preferable.

In exercises 1–16, find the derivative of each function.

1. $f(x) = (x^2 + 3)(x^3 - 3x + 1)$

2. $f(x) = (x^3 - 2x^2 + 5)(x^4 - 3x^2 + 2)$

3. $f(x) = (\sqrt{x} + 3x)\left(5x^2 - \dfrac{3}{x}\right)$

4. $f(x) = (x^{3/2} - 4x)\left(x^4 - \dfrac{3}{x^2} + 2\right)$

5. $f(x) = \dfrac{3x - 2}{5x + 1}$

6. $f(x) = \dfrac{x^2 + 2x + 5}{x^2 - 5x + 1}$

7. $f(x) = \dfrac{3x - 6\sqrt{x}}{5x^2 - 2}$

8. $f(x) = \dfrac{6x - 2/x}{x^2 + \sqrt{x}}$

9. $f(x) = \dfrac{(x + 1)(x - 2)}{x^2 - 5x + 1}$

10. $f(x) = \dfrac{x^2 - 2x}{x^2 + 5x}$

11. $f(x) = \dfrac{x^2 + 3x - 2}{\sqrt{x}}$

12. $f(x) = \dfrac{2x}{x^2 + 1}$

13. $f(x) = x(\sqrt[3]{x} + 3)$

14. $f(x) = \dfrac{x^2}{3} + \dfrac{5}{x^2}$

15. $f(x) = (x^2 - 1)\dfrac{x^3 + 3x^2}{x^2 + 2}$

16. $f(x) = (x + 2)\dfrac{x^2 - 1}{x^2 + x}$

17. Write out the product rule for the function $f(x)g(x)h(x)$. (Hint: Group the first two terms together.) Describe the general product rule: for n functions, what is the derivative of the product $f_1(x)\, f_2(x)\, f_3(x) \cdots f_n(x)$? How many terms are there? What does each term look like?

18. Use the quotient rule to show that the derivative of $[g(x)]^{-1}$ is $-g'(x)[g(x)]^{-2}$. Then use the product rule to compute the derivative of $f(x)[g(x)]^{-1}$.

In exercises 19 and 20, find the derivative of each function using the general product rule developed in exercise 17.

19. $f(x) = x^{2/3}(x^2 - 2)(x^3 - x + 1)$

20. $f(x) = (x + 4)(x^3 - 2x^2 + 1)(3 - 2/x)$

In exercises 21–24, assume that f and g are differentiable with $f(0) = -1$, $f(1) = -2$, $f'(0) = -1$, $f'(1) = 3$, $g(0) = 3$, $g(1) = 1$, $g'(0) = -1$ and $g'(1) = -2$.

21. Find an equation of the tangent line to $h(x) = f(x)g(x)$ at (a) $x = 1$, (b) $x = 0$.

22. Find an equation of the tangent line to $h(x) = \dfrac{f(x)}{g(x)}$ at (a) $x = 1$, (b) $x = 0$.

23. Find an equation of the tangent line to $h(x) = x^2 f(x)$ at (a) $x = 1$, (b) $x = 0$.

24. Find an equation of the tangent line to $h(x) = \dfrac{x^2}{g(x)}$ at (a) $x = 1$, (b) $x = 0$.

25. Suppose that for some toy, the quantity sold $Q(t)$ at time t years decreases at a rate of 4%; explain why this translates to $Q'(t) = -0.04Q(t)$. Suppose also that the price increases at a rate of 3%; write out a similar equation for $P'(t)$ in terms of $P(t)$. The revenue for the toy is $R(t) = Q(t)P(t)$. Substituting the expressions for $Q'(t)$ and $P'(t)$ into the product rule $R'(t) = Q'(t)P(t) + Q(t)P'(t)$, show that the revenue decreases at a rate of 1%. Explain why this is "obvious."

26. As in exercise 25, suppose that the quantity sold decreases at a rate of 4%. By what rate must the price be increased to keep the revenue constant?

27. Suppose the price of an object is \$20 and 20,000 units are sold. If the price increases at a rate of \$1.25 per year and the quantity sold increases at a rate of 2000 per year, at what rate will revenue increase?

28. Suppose the price of an object is \$14 and 12,000 units are sold. The company wants to increase the quantity sold by 1200 units per year, while increasing the revenue by \$20,000 per year. At what rate would the price have to be increased to reach these goals?

29. A baseball with mass 0.15 kg and speed 45 m/s is struck by a baseball bat of mass m and speed 40 m/s (in the opposite direction of the ball's motion). After the collision, the ball has initial speed $u(m) = \dfrac{82.5m - 6.75}{m + 0.15}$ m/s. Show that $u'(m) > 0$ and interpret this in baseball terms. Compare $u'(1)$ and $u'(1.2)$.

30. In exercise 29, if the baseball has mass M kg at speed 45 m/s and the bat has mass 1.05 kg at speed 40 m/s, the ball's initial speed is $u(M) = \dfrac{86.625 - 45M}{M + 1.05}$ m/s. Compute $u'(M)$ and interpret its sign (positive or negative) in baseball terms.

31. In example 4.6, it is reasonable to assume that the speed of the golf club at impact decreases as the mass of the club increases. If, for example, the speed of a club of mass m is $v = 8.5/m$ m/s at impact, then the initial speed of the golf ball is $u(m) = \dfrac{14.11}{m + 0.05}$ m/s. Show that $u'(m) < 0$ and interpret this in golf terms.

32. In example 4.6, if the golf club has mass 0.17 kg and strikes the ball with speed v m/s, the ball has initial speed $u(v) = \dfrac{0.2822v}{0.217}$ m/s. Compute and interpret the derivative $u'(v)$.

33. Assume that $g(x)$ is continuous at $x = 0$ and define $f(x) = xg(x)$. Show that $f(x)$ is differentiable at $x = 0$. Illustrate the result with $g(x) = |x|$.

34. Determine whether the result of exercise 33 still holds if $x = 0$ is replaced with $x = a \neq 0$.

In exercises 35–40, use the symbolic differentiation feature on your CAS or calculator.

35. Repeat example 4.4 with your CAS. If its answer is not in the same form as ours in the text, explain how the CAS computed its answer.

36. Repeat exercise 15 with your CAS. If its answer is not in the same form as ours in the back of the book, explain how the CAS computed its answer.

37. Use your CAS to sketch the derivative of $\sin x$. What function does this look like? Repeat with $\sin 2x$ and $\sin 3x$. Generalize to conjecture the derivative of $\sin kx$ for any constant k.

38. Repeat exercise 37 with $\sin x^2$. To identify the derivative, sketch a curve outlining the tops of the curves of the derivative graph and try to identify the amplitude of the derivative.

39. Find the derivative of $f(x) = \dfrac{\sqrt{3x^3 + x^2}}{x}$ on your CAS. Compare its answer to $\dfrac{3}{2\sqrt{3x + 1}}$ for $x > 0$ and $\dfrac{-3}{2\sqrt{3x + 1}}$ for $x < 0$. Explain how to get this answer and your CAS's answer, if it differs.

40. Find the derivative of $f(x) = \dfrac{x^2 - x - 2}{x - 2}\left(2x - \dfrac{2x^2}{x + 1}\right)$ on your CAS. Compare its answer to 2. Explain how to get this answer and your CAS's answer, if it differs.

41. Suppose that $F(x) = f(x)g(x)$ for infinitely differentiable functions $f(x)$ and $g(x)$ (that is, $f'(x)$, $f''(x)$, etc. exist for all x). Show that $F''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$. Compute $F'''(x)$. Compare $F''(x)$ to the binomial formula for $(a + b)^2$ and compare $F'''(x)$ to the formula for $(a + b)^3$.

42. Given that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, write out a formula for $F^{(4)}(x)$. (See exercise 41.)

43. Use the product rule to show that if $g(x) = [f(x)]^2$ and $f(x)$ is differentiable, then $g'(x) = 2f(x)f'(x)$. This is an example of the *chain rule,* to be discussed in section 2.5.

44. Use the result from exercise 43 and the product rule to show that if $g(x) = [f(x)]^3$ and $f(x)$ is differentiable, then $g'(x) = 3[f(x)]^2 f'(x)$. Hypothesize the correct chain rule for the derivative of $[f(x)]^n$.

45. The relationship among the pressure P, volume V and temperature T of a gas or liquid is given by **van der Waals' equation** $\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$ for positive constants a, b, n and R. Solve the equation for P. Treating T as a constant and V as the variable, find the critical point (T_c, P_c, V_c) such that $P'(V) = P''(V) = 0$. (Hint: Don't solve either equation separately, but substitute the result from one equation into the other.) For temperatures above T_c, the substance can exist only in its gaseous form; below T_c, the substance is a gas or liquid (depending on the pressure and volume). For water, take $R = 0.08206$ l-atm/mo-K, $a = 5.464$ 1^2-atm/mo^2 and $b = 0.03049$ l/mo. Find the highest temperature at which $n = 1$ mo of water may exist as a liquid. (Note: your answer will be in degrees Kelvin; subtract 273.15 to get degrees Celsius.)

46. The amount of an **allosteric enzyme** is affected by the presence of an activator. If x is the amount of activator and f is the amount of enzyme, then one model of an allosteric activation is $f(x) = \frac{x^{2.7}}{1 + x^{2.7}}$. Find and interpret $\lim_{x \to 0} f(x)$ and $\lim_{x \to \infty} f(x)$.

47. For the allosteric enzyme of exercise 46, compute and interpret $f'(x)$.

48. Enzyme production can also be inhibited. In this situation, the amount of enzyme as a function of the amount of inhibitor is modeled by $f(x) = \frac{1}{1 + x^{2.7}}$. Find and interpret $\lim_{x \to 0} f(x)$, $\lim_{x \to \infty} f(x)$ and $f'(x)$.

In exercises 49–52, find the derivative of the expression for an unspecified differentiable function f.

49. $x^3 f(x)$ **50.** $\frac{f(x)}{x^2}$ **51.** $\frac{\sqrt{x}}{f(x)}$ **52.** $\sqrt{x}\, f(x)$

EXPLORATORY EXERCISES

1. Most cars are rated for fuel efficiency by estimating miles per gallon in city driving (c) and miles per gallon in highway driving (h). The Environmental Protection Agency uses the formula $r = \frac{1}{0.55/c + 0.45/h}$ as its overall rating of gas usage (this is called the 55/45 combined mpg). You may want to rewrite the function (find a common denominator in the denominator and simplify) to work this exercise.

(a) Think of c as the variable and h as a constant, and show that $\frac{dr}{dc} > 0$. Interpret this result in terms of gas mileage.

(b) Think of h as the variable and c as a constant, and show that $\frac{dr}{dh} > 0$.

(c) Show that if $c = h$, then $r = c$.

(d) Show that if $c < h$, then $c < r < h$. To do this, assume that c is a constant and $c < h$. Explain why the results of parts (b) and (c) imply that $r > c$. Next, show that $\frac{dr}{dh} < 0.45$. Explain why this result along with the result of part (b) implies that $r < h$.

Explain why the results of parts (a)–(d) must be true if the EPA's combined formula is a reasonable way to average the ratings c and h. To get some sense of how the formula works, take $c = 20$ and graph r as a function of h. Comment on why the EPA might want to use a function whose graph flattens out as this one does.

2. In many sports, the collision between a ball and a striking implement is central to the game. Suppose the ball has weight w and velocity v before the collision and the striker (bat, tennis racket, golf club, etc.) has weight W and velocity $-V$ before the collision (the negative indicates the striker is moving in the opposite direction from the ball). The velocity of the ball after the collision will be $u = \frac{WV(1+c) + v(cW - w)}{W + w}$, where the parameter c, called the **coefficient of restitution,** represents the "bounciness" of the ball in the collision. Treating W as the independent variable (like x) and the other parameters as constants, compute the derivative and verify that $\frac{du}{dW} = \frac{V(1+c)w + cvw + vw}{(W+w)^2} \geq 0$ since all parameters are nonnegative. Explain why this implies that if the athlete uses a bigger striker (bigger W) with all other things equal, the speed of the ball increases. Does this match your intuition? What is doubtful about the assumption of all other things being equal? Similarly compute and interpret $\frac{du}{dw}$, $\frac{du}{dv}$, $\frac{du}{dV}$ and $\frac{du}{dc}$. (Hint: c is between 0 and 1 with 0 representing a dead ball and 1 the liveliest ball possible.)

3. Suppose that a soccer player strikes the ball with enough energy that a stationary ball would have initial speed 80 mph. Show that the same energy kick on a ball moving directly to the player at 40 mph will launch the ball at approximately 100 mph. (Use the general collision formula in exploratory exercise 2 with $c = 0.5$ and assume that the ball's weight is much less than the soccer player's weight.) In general, what proportion of the ball's incoming speed is converted by the kick into extra speed in the opposite direction?

2.5 THE CHAIN RULE

Suppose that the function $P(t) = \sqrt{100 + 8t}$ models the population of a city after t years. Then the rate of growth of that population after 2 years is given by $P'(2)$. At present, we need the limit definition to compute this derivative. However, observe that $P(t)$ is the composition of the two functions $f(t) = \sqrt{t}$ and $g(t) = 100 + 8t$, so that $P(t) = f(g(t))$. Also, notice that both $f'(t)$ and $g'(t)$ are easily computed using existing derivative rules. We now develop a general rule for the derivative of a composition of two functions.

The following simple examples will help us to identify the form of the chain rule. Notice that from the product rule

$$\begin{aligned}\frac{d}{dx}[(x^2+1)^2] &= \frac{d}{dx}[(x^2+1)(x^2+1)]\\ &= 2x(x^2+1) + (x^2+1)2x\\ &= 2(x^2+1)2x.\end{aligned}$$

Of course, we can write this as $4x(x^2 + 1)$, but the unsimplified form helps us to understand the form of the chain rule. Using this result and the product rule, notice that

$$\begin{aligned}\frac{d}{dx}[(x^2+1)^3] &= \frac{d}{dx}[(x^2+1)(x^2+1)^2]\\ &= 2x(x^2+1)^2 + (x^2+1)2(x^2+1)2x\\ &= 3(x^2+1)^2 2x.\end{aligned}$$

We leave it as a straightforward exercise to extend this result to

$$\frac{d}{dx}[(x^2+1)^4] = 4(x^2+1)^3 2x.$$

You should observe that, in each case, we have brought the exponent down, lowered the power by one and then multiplied by $2x$, the derivative of $x^2 + 1$. Notice that we can write $(x^2 + 1)^4$ as the composite function $f(g(x)) = (x^2 + 1)^4$, where $g(x) = x^2 + 1$ and $f(x) = x^4$. Finally, observe that the derivative of the composite function is

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[(x^2+1)^4] = 4(x^2+1)^3 2x = f'(g(x))g'(x).$$

This is an example of the *chain rule,* which has the following general form.

THEOREM 5.1 (Chain Rule)

If g is differentiable at x and f is differentiable at $g(x)$, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x).$$

PROOF

At this point, we can prove only the special case where $g'(x) \neq 0$. Let $F(x) = f(g(x))$. Then,

$$\begin{aligned}
\frac{d}{dx}[f(g(x))] &= F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{h} && \text{Since } F(x) = f(g(x)). \\
&= \lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{h} \frac{g(x+h)-g(x)}{g(x+h)-g(x)} && \text{Multiply numerator and denominator by } g(x+h)-g(x). \\
&= \lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} && \text{Regroup terms.} \\
&= \lim_{g(x+h)\to g(x)} \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(g(x))g'(x),
\end{aligned}$$

where the next to the last line is valid since as $h \to 0$, $g(x+h) \to g(x)$, by the continuity of g. (Recall that since g is differentiable, it is also continuous.) You will be asked in the exercises to fill in some of the gaps in this argument. In particular, you should identify why we need $g'(x) \neq 0$ in this proof. ■

It is often helpful to think of the chain rule in Leibniz notation. If $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$ and the chain rule says that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \tag{5.1}$$

REMARK 5.1

The chain rule should make sense intuitively as follows. We think of $\frac{dy}{dx}$ as the (instantaneous) rate of change of y with respect to x, $\frac{dy}{du}$ as the (instantaneous) rate of change of y with respect to u and $\frac{du}{dx}$ as the (instantaneous) rate of change of u with respect to x. So, if $\frac{dy}{du} = 2$ (i.e., y is changing at twice the rate of u) and $\frac{du}{dx} = 5$ (i.e., u is changing at five times the rate of x), it should make sense that y is changing at $2 \times 5 = 10$ times the rate of x. That is, $\frac{dy}{dx} = 10$, which is precisely what equation (5.1) says.

EXAMPLE 5.1 Using the Chain Rule

Differentiate $y = (x^3 + x - 1)^5$.

Solution For $u = x^3 + x - 1$, note that $y = u^5$. From (5.1), we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(u^5)\frac{du}{dx} && \text{Since } y = u^5. \\
&= 5u^4 \frac{d}{dx}(x^3 + x - 1) \\
&= 5(x^3 + x - 1)^4(3x^2 + 1).
\end{aligned}$$ ■

It is helpful to think of the chain rule in terms of *inside* functions and *outside* functions. For the composition $f(g(x))$, f is referred to as the **outside** function and g is referred to as the **inside** function. The chain rule derivative $f'(g(x))g'(x)$ can then be viewed as the derivative of the outside function times the derivative of the inside function. In example 5.1, the inside function is $x^3 + x - 1$ (the expression inside the parentheses) and the outside function is u^5. In example 5.2, we think of $\sqrt{100 + 8t}$ as composed of the inside function $100 + 8t$ and the outside function $\sqrt{u}$. Some careful thought about the pieces of a composition of functions will help you use the chain rule effectively.

EXAMPLE 5.2 Using the Chain Rule on a Radical Function

Find $\frac{d}{dt}(\sqrt{100+8t})$.

Solution Let $u = 100 + 8t$ and note that $\sqrt{100+8t} = u^{1/2}$. Then, from (5.1),

$$\frac{d}{dt}(\sqrt{100+8t}) = \frac{d}{dt}(u^{1/2}) = \frac{1}{2}u^{-1/2}\frac{du}{dt}$$
$$= \frac{1}{2\sqrt{100+8t}}\frac{d}{dt}(100+8t) = \frac{4}{\sqrt{100+8t}}.$$

Notice that here, the derivative of the *inside* is the derivative of the expression under the square root sign. ■

You are now in a position to calculate the derivative of a very large number of functions, by using the chain rule in combination with other differentiation rules.

EXAMPLE 5.3 Derivatives Involving Chain Rules and Other Rules

Compute the derivative of $f(x) = x^3\sqrt{4x+1}$, $g(x) = \frac{8x}{(x^3+1)^2}$ and $h(x) = \frac{8}{(x^3+1)^2}$.

Solution Notice the differences in these three functions. The first function $f(x)$ is a product of two functions, $g(x)$ is a quotient of two functions and $h(x)$ is a constant divided by a function. This tells us to use the product rule for $f(x)$, the quotient rule for $g(x)$ and simply the chain rule for $h(x)$. For the first function, we have

$$f'(x) = \frac{d}{dx}\left(x^3\sqrt{4x+1}\right) = 3x^2\sqrt{4x+1} + x^3\frac{d}{dx}\sqrt{4x+1} \quad \text{By the product rule.}$$
$$= 3x^2\sqrt{4x+1} + x^3\frac{1}{2}(4x+1)^{-1/2}\underbrace{\frac{d}{dx}(4x+1)}_{\text{derivative of the inside}} \quad \text{By the chain rule.}$$
$$= 3x^2\sqrt{4x+1} + 2x^3(4x+1)^{-1/2}. \quad \text{Simplifying.}$$

Next, we have

$$g'(x) = \frac{d}{dx}\left[\frac{8x}{(x^3+1)^2}\right] = \frac{8(x^3+1)^2 - 8x\frac{d}{dx}[(x^3+1)^2]}{(x^3+1)^4} \quad \text{By the quotient rule.}$$
$$= \frac{8(x^3+1)^2 - 8x\left[2(x^3+1)\underbrace{\frac{d}{dx}(x^3+1)}_{\text{derivative of the inside}}\right]}{(x^3+1)^4} \quad \text{By the chain rule.}$$
$$= \frac{8(x^3+1)^2 - 16x(x^3+1)3x^2}{(x^3+1)^4}$$
$$= \frac{8(x^3+1) - 48x^3}{(x^3+1)^3} = \frac{8-40x^3}{(x^3+1)^3}. \quad \text{Simplification.}$$

For $h(x)$, notice that instead of using the quotient rule, it is simpler to rewrite the function as $h(x) = 8(x^3+1)^{-2}$. Then

$$h'(x) = \frac{d}{dx}[8(x^3+1)^{-2}] = -16(x^3+1)^{-3}\underbrace{\frac{d}{dx}(x^3+1)}_{\text{derivative of the inside}} = -16(x^3+1)^{-3}(3x^2)$$
$$= -48x^2(x^3+1)^{-3}.$$

In example 5.4, we apply the chain rule to a composition of a function with a composition of functions.

EXAMPLE 5.4 A Derivative Involving Multiple Chain Rules

Find the derivative of $f(x) = (\sqrt{x^2+4} - 3x^2)^{3/2}$.

Solution We have

$$f'(x) = \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2} \frac{d}{dx}\left(\sqrt{x^2+4} - 3x^2\right)$$
$$= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2} \left[\frac{1}{2}(x^2+4)^{-1/2}\frac{d}{dx}(x^2+4) - 6x\right] \quad \text{By the chain rule.}$$
$$= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2} \left[\frac{1}{2}(x^2+4)^{-1/2}(2x) - 6x\right] \quad \text{By the chain rule.}$$
$$= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2} \left[x(x^2+4)^{-1/2} - 6x\right]. \quad \text{Simplification.}$$

We now use the chain rule to compute the derivative of an inverse function in terms of the original function. Recall that we write $g(x) = f^{-1}(x)$ if $g(f(x)) = x$ for all x in the domain of f and $f(g(x)) = x$ for all x in the domain of g. From this last equation, assuming that f and g are differentiable, it follows that

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}(x).$$

From the chain rule, we now have

$$f'(g(x))g'(x) = 1.$$

Solving this for $g'(x)$, we get $$g'(x) = \frac{1}{f'(g(x))},$$

assuming we don't divide by zero. We now state the result as Theorem 5.2.

THEOREM 5.2

If f is differentiable at all x and has an inverse function $g(x) = f^{-1}(x)$, then

$$g'(x) = \frac{1}{f'(g(x))},$$

provided $f'(g(x)) \neq 0$.

As we see in example 5.5, in order to use Theorem 5.2, we must be able to compute values of the inverse function.

EXAMPLE 5.5 The Derivative of an Inverse Function

Given that the function $f(x) = x^5 + 3x^3 + 2x + 1$ has an inverse function $g(x)$, compute $g'(7)$.

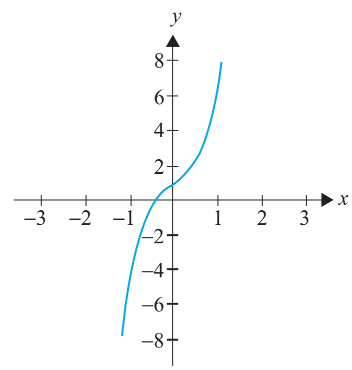


FIGURE 2.24
$y = x^5 + 3x^3 + 2x + 1$

Solution First, notice from Figure 2.24 that f appears to be one-to-one and so, will have an inverse. From Theorem 5.2, we have

$$g'(7) = \frac{1}{f'(g(7))}. \tag{5.2}$$

It's easy to compute $f'(x) = 5x^4 + 9x^2 + 2$, but to use Theorem 5.2 we also need the value of $g(7)$. If we write $x = g(7)$, then $x = f^{-1}(7)$, so that $f(x) = 7$. In general, solving the equation $f(x) = 7$ may be beyond our algebraic abilities. (Try solving $x^5 + 3x^3 + 2x + 1 = 7$ to see what we mean.) By trial and error, however, it is not hard to see that $f(1) = 7$, so that $g(7) = 1$. [Keep in mind that for inverse functions, $f(x) = y$ and $g(y) = x$ are equivalent statements.] Returning to equation (5.2), we now have

$$g'(7) = \frac{1}{f'(1)} = \frac{1}{16}.$$

Notice that our solution in example 5.5 is dependent on finding an x for which $f(x) = 7$. This particular example was workable by trial and error, but finding most values other than $g(7)$ would have been quite difficult or impossible to solve exactly.

BEYOND FORMULAS

If you think that the method used in example 5.5 is roundabout, then you have the right idea. The chain rule in particular and calculus in general give us methods for determining quantities that are not directly computable. In the case of example 5.5, we use the properties of one function to determine properties of another function. The key to our ability to do this is understanding the theory behind the chain rule. Can you think of situations in your life where your understanding of your friends' personalities helped you convince them, in a roundabout way, to do something that at first they didn't want to do?

TODAY IN MATHEMATICS

Fan Chung (1949–)
A Taiwanese mathematician with a highly successful career in American industry. She says, "As an undergraduate in Taiwan, I was surrounded by good friends and many women mathematicians. . . . A large part of education is learning from your peers, not just the professors." Collaboration has been a hallmark of her career. "Finding the right problem is often the main part of the work in establishing the connection. Frequently a good problem from someone else will give you a push in the right direction and the next thing you know, you have another good problem."

EXERCISES 2.5

WRITING EXERCISES

1. If Fred can run 10 mph and Greg can run twice as fast as Fred, how fast can Greg run? The answer is obvious for most people. Formulate this simple problem as a chain rule calculation and conclude that the chain rule (in this context) is obvious.

2. The biggest challenge in computing the derivatives of $\sqrt{(x^2+4)(x^3-x+1)}$, $(x^2+4)\sqrt{x^3-x+1}$ and $x^2+4\sqrt{x^3-x+1}$ is knowing which rule (product, chain etc.) to use when. Discuss how you know which rule to use when. (Hint: Think of the order in which you would perform operations to compute the value of each function for a specific choice of x.)

3. One simple implication of the chain rule is: if $g(x) = f(x-a)$, then $g'(x) = f'(x-a)$. Explain this derivative graphically: how does $g(x)$ compare to $f(x)$ graphically and why do the slopes of the tangent lines relate as the formula indicates?

4. Another simple implication of the chain rule is: if $h(x) = f(2x)$, then $h'(x) = 2f'(2x)$. Explain this derivative graphically: how does $h(x)$ compare to $f(x)$ graphically and why do the slopes of the tangent lines relate as the formula indicates?

In exercises 1–4, find the derivative with and without using the chain rule.

1. $f(x) = (x^3-1)^2$
2. $f(x) = (x^2+2x+1)^2$
3. $f(x) = (x^2+1)^3$
4. $f(x) = (2x+1)^4$

In exercises 5–22, find the derivative of each function.

5. $f(x) = \sqrt{x^2+4}$
6. $f(x) = (x^3+x-1)^3$
7. $f(x) = x^5\sqrt{x^3+2}$
8. $f(x) = (x^3+2)\sqrt{x^5}$
9. $f(x) = \dfrac{x^3}{(x^2+4)^2}$
10. $f(x) = \dfrac{x^2+4}{(x^3)^2}$
11. $f(x) = \dfrac{6}{\sqrt{x^2+4}}$
12. $f(x) = \dfrac{(x^3+4)^5}{8}$
13. $f(x) = (\sqrt{x}+3)^{4/3}$
14. $f(x) = \sqrt{x}(x^{4/3}+3)$
15. $f(x) = (\sqrt{x^3+2}+2x)^{-2}$
16. $f(x) = \sqrt{4x^2+(8-x^2)^2}$
17. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$
18. $f(x) = \dfrac{(x^2-1)^2}{x^2+1}$
19. $f(x) = \sqrt{\dfrac{x}{x^2+1}}$
20. $f(x) = \sqrt{(x^2+1)(\sqrt{x}+1)^3}$
21. $f(x) = \sqrt[3]{x\sqrt{x^4+2x\sqrt[4]{\dfrac{8}{x+2}}}}$
22. $f(x) = \dfrac{3x^2+2\sqrt{x^3+4/x^4}}{(x^3-4)\sqrt{x^2+2}}$

In exercises 23 and 24, find an equation of the tangent line to $y = f(x)$ at $x = a$.

23. $f(x) = \sqrt{x^2+16}$, $a = 3$
24. $f(x) = \dfrac{6}{x^2+4}$, $a = -2$

In exercises 25 and 26, use the position function to find the velocity at time $t = 2$. (Assume units of meters and seconds.)

25. $s(t) = \sqrt{t^2+8}$
26. $s(t) = \dfrac{60t}{\sqrt{t^2+1}}$

In exercises 27 and 28, compute $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$, and identify a pattern for the nth derivative $f^{(n)}(x)$.

27. $f(x) = \sqrt{2x+1}$
28. $f(x) = \dfrac{2}{x+1}$

In exercises 29–32, use the table of values to estimate the derivative of $h(x) = f(g(x))$ or $k(x) = g(f(x))$.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-2	-1	0	-1	-2	-3	-2	0	2
$g(x)$	6	4	2	2	4	6	4	2	1

29. $h'(1)$
30. $k'(1)$
31. $k'(3)$
32. $h'(3)$

In exercises 33 and 34, use the relevant information to compute the derivative for $h(x) = f(g(x))$.

33. $h'(1)$, where $f(1) = 3$, $g(1) = 2$, $f'(1) = 4$, $f'(2) = 3$, $g'(1) = -2$ and $g'(3) = 5$

34. $h'(2)$, where $f(2) = 1$, $g(2) = 3$, $f'(2) = -1$, $f'(3) = -3$, $g'(1) = 2$ and $g'(2) = 4$

In exercises 35–40, $f(x)$ has an inverse $g(x)$. Use Theorem 5.2 to find $g'(a)$.

35. $f(x) = x^3+4x-1$, $a = -1$
36. $f(x) = x^3+2x+1$, $a = -2$
37. $f(x) = x^5+3x^3+x$, $a = 5$
38. $f(x) = x^5+4x-2$, $a = -2$
39. $f(x) = \sqrt{x^3+2x+4}$, $a = 2$
40. $f(x) = \sqrt{x^5+4x^3+3x+1}$, $a = 3$

In exercises 41–44, find a function $g(x)$ such that $g'(x) = f(x)$.

41. $f(x) = (x^2+3)^2\,(2x)$

42. $f(x) = x^2(x^3+4)^{2/3}$

43. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

44. $f(x) = \dfrac{x}{(x^2+1)^2}$

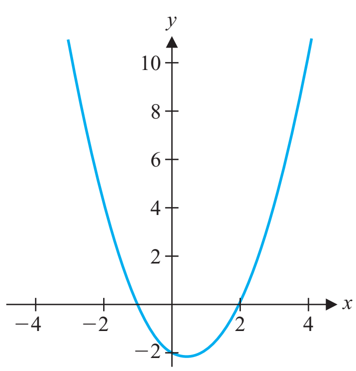
45. A function $f(x)$ is an **even function** if $f(-x) = f(x)$ for all x and is an **odd function** if $f(-x) = -f(x)$ for all x. Prove that the derivative of an even function is odd and the derivative of an odd function is even.

In exercises 46–49, find the derivative of the expression for an unspecified differentiable function $f(x)$.

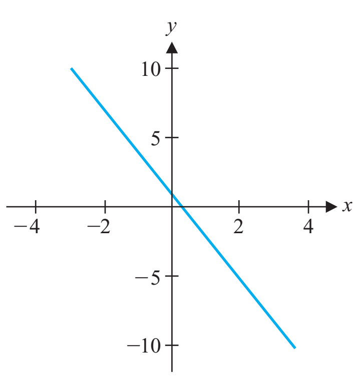
46. $f(x^2)$ **47.** $f(\sqrt{x})$ **48.** $\sqrt{4f(x)+1}$ **49.** $\dfrac{1}{1+[f(x)]^2}$

50. If the graph of a differentiable function $f(x)$ is symmetric about the line $x = a$, what can you say about the symmetry of the graph of $f'(x)$?

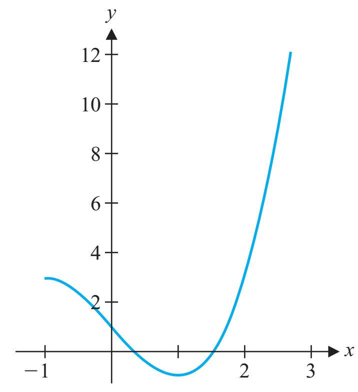
In exercises 51–54, use the given graphs to estimate the derivative.



$y = a(x)$



$y = b(x)$



$y = c(x)$

51. $f'(2)$, where $f(x) = b(a(x))$

52. $f'(0)$, where $f(x) = a(b(x))$

53. $f'(-1)$, where $f(x) = c(a(x))$

54. $f'(1)$, where $f(x) = b(c(x))$

55. Determine all values of x such that $f(x) = \sqrt[3]{x^3 - 3x^2 + 2x}$ is not differentiable. Describe the graphical property that prevents the derivative from existing.

56. Determine all values of x for which $f(x) = |2x| + |x-4| + |x+4|$ is not differentiable. Describe the graphical property that prevents the derivative from existing.

57. Which steps in our outline of the proof of the chain rule are not well documented? Where do we use the assumption that $g'(x) \neq 0$?

EXPLORATORY EXERCISES

1. A guitar string of length L, density p and tension T will vibrate at the frequency $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{p}}$. Compute the derivative $\dfrac{df}{dT}$, where we think of T as the independent variable and treat p and L as constants. Interpret this derivative in terms of a guitarist tightening or loosening the string to "tune" it. Compute the derivative $\dfrac{df}{dL}$ and interpret it in terms of a guitarist playing notes by pressing the string against a fret.

2. Newton's second law of motion is $F = ma$, where m is the mass of the object that undergoes an acceleration a due to an applied force F. This law is accurate at low speeds. At high speeds, we use the corresponding formula from Einstein's theory of relativity, $F = m\dfrac{d}{dt}\left(\dfrac{v(t)}{\sqrt{1-v^2(t)/c^2}}\right)$, where $v(t)$ is the velocity function and c is the speed of light. Compute $\dfrac{d}{dt}\left(\dfrac{v(t)}{\sqrt{1-v^2(t)/c^2}}\right)$. What has to be "ignored" to simplify this expression to the acceleration $a = v'(t)$ in Newton's second law?

2.6 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

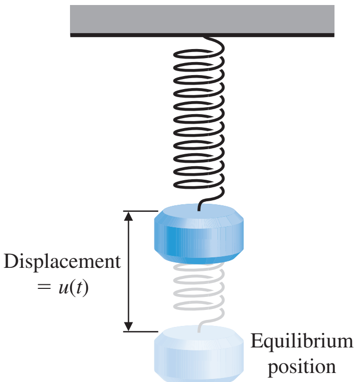


FIGURE 2.25
Spring-mass system

Springs are essential components of many mechanical systems, including shock absorbing systems for cars, stereo equipment and other sensitive devices. Imagine a weight hanging from a spring suspended from the ceiling (see Figure 2.25). If we set the weight in motion (e.g., by tapping down on it), the weight will bounce up and down in ever-shortening strokes until it eventually is again at rest (equilibrium). For short periods of time, this motion is nearly periodic. Suppose we measure the vertical displacement of the weight from its natural resting (equilibrium) position (see Figure 2.25).

When we pull the weight down, its vertical displacement is negative. The weight then swings up to where the displacement is positive, swings down to a negative displacement and so on. The only functions we've experienced that exhibit this kind of behavior are the sine and cosine functions. We calculate the derivatives of these and the other trigonometric functions in this section.

We can learn a lot about the derivatives of $\sin x$ and $\cos x$ from their graphs. From the graph of $y = \sin x$ in Figure 2.26a, notice the horizontal tangents at $x = -3\pi/2, -\pi/2, \pi/2$ and $3\pi/2$. At these x-values, the derivative must equal 0. The tangent lines have positive slope for $-2\pi < x < -3\pi/2$, negative slope for $-3\pi/2 < x < -\pi/2$ and so on. For each interval on which the derivative is positive (or negative), the graph appears to be steepest in the middle of the interval: for example, from $x = -\pi/2$, the graph gets steeper until about $x = 0$ and then gets less steep until leveling out at $x = \pi/2$. A sketch of the derivative graph should then look like the graph in Figure 2.26b, which looks like the graph of $y = \cos x$. We show here that this conjecture is, in fact, correct. In the exercises, you are asked to perform a similar graphical analysis to conjecture that the derivative of $f(x) = \cos x$ equals $-\sin x$.

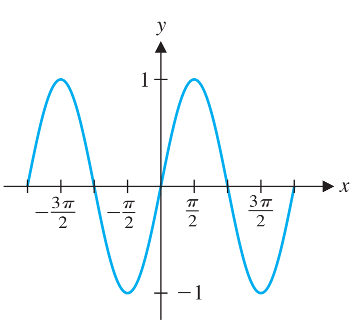


FIGURE 2.26a
$y = \sin x$

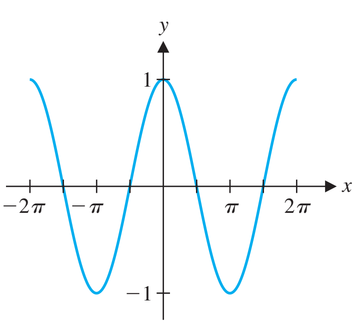


FIGURE 2.26b
The derivative of $f(x) = \sin x$

Before we move to the calculation of the derivatives of the six trigonometric functions, we first consider a few limits involving trigonometric functions. (We refer to these results as *lemmas*—minor theorems that lead up to some more significant result.) You will see shortly why we must consider these first.

LEMMA 6.1

$$\lim_{\theta \to 0} \sin \theta = 0.$$

This result certainly seems reasonable, especially when we consider the graph of $y = \sin x$. In fact, we have been using this for some time now, having stated this (without proof) as part of Theorem 3.4 in section 1.3. We now prove the result.

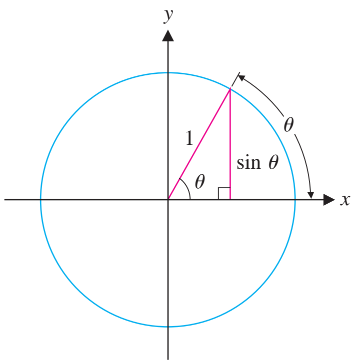


FIGURE 2.27
Definition of $\sin\theta$

PROOF

For $0 < \theta < \dfrac{\pi}{2}$, consider Figure 2.27. From the figure, observe that

$$0 \le \sin\theta \le \theta. \tag{6.1}$$

It is a simple matter to see that

$$\lim_{\theta\to 0^+} 0 = 0 = \lim_{\theta\to 0^+} \theta. \tag{6.2}$$

From the Squeeze Theorem (see section 1.3), it now follows from (6.1) and (6.2) that

$$\lim_{\theta\to 0^+} \sin\theta = 0,$$

also. Similarly, you can show that

$$\lim_{\theta\to 0^-} \sin\theta = 0.$$

This is left as an exercise. Since both one-sided limits are the same, it follows that

$$\lim_{\theta\to 0} \sin\theta = 0. \quad \blacksquare$$

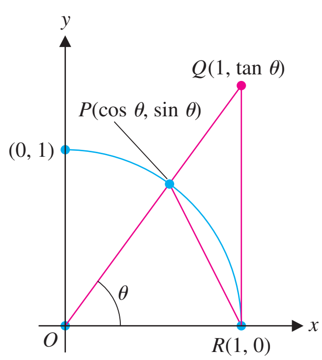


FIGURE 2.28a

LEMMA 6.2

$$\lim_{\theta\to 0} \cos\theta = 1.$$

The proof of this result is straightforward and follows from Lemma 6.1 and the Pythagorean Theorem. We leave this as an exercise.

The following result was conjectured to be true (based on a graph and some computations) when we first examined limits in section 1.2. We can now prove the result.

LEMMA 6.3

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

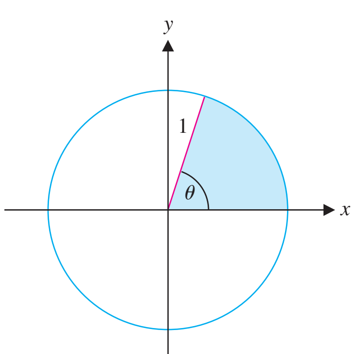


FIGURE 2.28b
A circular sector

PROOF

Assume $0 < \theta < \dfrac{\pi}{2}$.

Referring to Figure 2.28a, observe that the area of the circular sector OPR is larger than the area of the triangle OPR, but smaller than the area of the triangle OQR. That is,

$$0 < \text{Area } \Delta OPR < \text{Area sector } OPR < \text{Area } \Delta OQR. \tag{6.3}$$

You can see from Figure 2.28b that

$$\begin{aligned}\text{Area sector } OPR &= \pi(\text{radius})^2 \text{ (fraction of circle included)}\\ &= \pi(1^2)\frac{\theta}{2\pi} = \frac{\theta}{2}.\end{aligned} \tag{6.4}$$

Also, $$\text{Area } \Delta OPR = \frac{1}{2}(\text{base})\,(\text{height}) = \frac{1}{2}(1)\sin\theta \tag{6.5}$$

and $$\text{Area } \Delta OQR = \frac{1}{2}(1)\tan\theta. \tag{6.6}$$

Thus, from (6.3), (6.4), (6.5) and (6.6), we have

$$0 < \frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta. \tag{6.7}$$

If we divide (6.7) by $\frac{1}{2}\sin\theta$ (note that this is positive, so that the inequalities are not affected), we get

$$1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}.$$

Taking reciprocals (again, everything here is positive), we find

$$1 > \frac{\sin\theta}{\theta} > \cos\theta. \tag{6.8}$$

The inequality (6.8) also holds if $-\frac{\pi}{2} < \theta < 0$. (This is left as an exercise.) Finally, note that

$$\lim_{\theta\to 0}\cos\theta = 1 = \lim_{\theta\to 0} 1.$$

Thus, it follows from (6.8) and the Squeeze Theorem that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

also. ■

We need one additional limit result before we tackle the derivatives of the trigonometric functions.

LEMMA 6.4

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.$$

Before we prove this, we want to make this conjecture seem reasonable. We draw a graph of $y = \dfrac{1-\cos x}{x}$ in Figure 2.29. The tables of function values that follow should prove equally convincing.

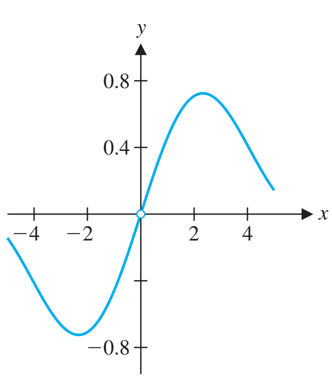


FIGURE 2.29
$y = \dfrac{1-\cos x}{x}$

x	$\dfrac{1-\cos x}{x}$
0.1	0.04996
0.01	0.00499996
0.001	0.0005
0.0001	0.00005

x	$\dfrac{1-\cos x}{x}$
−0.1	−0.04996
−0.01	−0.00499996
−0.001	−0.0005
−0.0001	−0.00005

Now that we have strong evidence for the conjecture, we prove the lemma.

PROOF

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right)$$ Multiply numerator and denominator by $1+\cos\theta$.

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$ Multiply out numerator and denominator.

$$= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}$$ Since $\sin^2\theta+\cos^2\theta = 1$.

$$= \lim_{\theta\to 0}\left[\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin\theta}{1+\cos\theta}\right)\right]$$

$$= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right)$$ Split up terms, since both limits exist.

$$= (1)\left(\frac{0}{1+1}\right) = 0,$$

as conjectured. ■

We are finally in a position to compute the derivatives of the sine and cosine functions. The derivatives of the other trigonometric functions will then follow by the quotient rule.

THEOREM 6.1

$$\frac{d}{dx}\sin x = \cos x.$$

PROOF

From the limit definition of derivative, for $f(x) = \sin x$, we have

$$\frac{d}{dx}\sin x = f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$ Trig identity: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$.

$$= \lim_{h\to 0}\frac{\sin x\cos h - \sin x}{h} + \lim_{h\to 0}\frac{\sin h\cos x}{h}$$ Grouping terms with $\sin x$ and terms with $\sin h$ separately.

$$= (\sin x)\lim_{h\to 0}\frac{\cos h - 1}{h} + (\cos x)\lim_{h\to 0}\frac{\sin h}{h}$$ Factoring $\sin x$ from the first term and $\cos x$ from the second term.

$$= (\sin x)(0) + (\cos x)(1) = \cos x,$$

from Lemmas 6.3 and 6.4. ■

THEOREM 6.2

$$\frac{d}{dx}\cos x = -\sin x.$$

The proof of Theorem 6.2 is left as an exercise.

For the remaining four trigonometric functions, we can use the quotient rule in conjunction with the derivatives of $\sin x$ and $\cos x$.

THEOREM 6.3

$$\frac{d}{dx}\tan x = \sec^2 x.$$

PROOF

We have from the quotient rule that

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}\\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2}\\ &= \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \sec^2 x,\end{aligned}$$

where we have used the quotient rule and the preceding results on the derivatives of $\sin x$ and $\cos x$ (Theorems 6.1 and 6.2). ■

The derivatives of the remaining trigonometric functions are left as exercises. The derivatives of all six trigonometric functions are summarized below.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Example 6.1 shows where the product rule is necessary.

EXAMPLE 6.1 A Derivative That Requires the Product Rule

Find the derivative of $f(x) = x^5 \cos x$.

Solution From the product rule, we have

$$\begin{aligned}\frac{d}{dx}(x^5\cos x) &= \left[\frac{d}{dx}(x^5)\right]\cos x + x^5\frac{d}{dx}(\cos x)\\ &= 5x^4\cos x - x^5 \sin x.\end{aligned}$$ ■

EXAMPLE 6.2 Computing Some Routine Derivatives

Compute the derivatives of (a) $f(x) = \sin^2 x$ and (b) $g(x) = 4\tan x - 5\csc x$.

Solution For (a), we first rewrite the function as $f(x) = (\sin x)^2$ and use the chain rule. We have

$$f'(x) = (2\sin x)\underbrace{\frac{d}{dx}(\sin x)}_{\text{derivative of the inside}} = 2\sin x\cos x.$$

For (b), we have

$$g'(x) = 4\sec^2 x + 5\csc x\cot x. \blacksquare$$

You must be very careful to distinguish between similar notations with very different meanings, as you see in example 6.3.

EXAMPLE 6.3 The Derivatives of Some Similar Trigonometric Functions

Compute the derivative of $f(x) = \cos x^3$, $g(x) = \cos^3 x$ and $h(x) = \cos 3x$.

Solution Note the differences in these three functions. Using the implied parentheses we normally do not bother to include, we have $f(x) = \cos(x^3)$, $g(x) = (\cos x)^3$ and $h(x) = \cos(3x)$. For the first function, we have

$$f'(x) = \frac{d}{dx}\cos(x^3) = -\sin(x^3)\underbrace{\frac{d}{dx}(x^3)}_{\text{derivative of the inside}} = -\sin(x^3)(3x^2) = -3x^2\sin x^3.$$

Next, we have

$$\begin{aligned} g'(x) &= \frac{d}{dx}(\cos x)^3 = 3(\cos x)^2\underbrace{\frac{d}{dx}(\cos x)}_{\text{derivative of the inside}} \\ &= 3(\cos x)^2(-\sin x) = -3\sin x\cos^2 x. \end{aligned}$$

Finally, we have

$$h'(x) = \frac{d}{dx}(\cos 3x) = -\sin(3x)\underbrace{\frac{d}{dx}(3x)}_{\text{derivative of the inside}} = -\sin(3x)(3) = -3\sin 3x. \blacksquare$$

By combining our trigonometric rules with the product, quotient and chain rules, we can now differentiate many complicated functions.

EXAMPLE 6.4 A Derivative Involving the Chain Rule and the Quotient Rule

Find the derivative of $f(x) = \sin\left(\frac{2x}{x+1}\right)$.

Solution We have

$$\begin{aligned} f'(x) &= \cos\left(\frac{2x}{x+1}\right)\underbrace{\frac{d}{dx}\left(\frac{2x}{x+1}\right)}_{\text{derivative of the inside}} && \text{By the chain rule.} \\ &= \cos\left(\frac{2x}{x+1}\right)\frac{2(x+1) - 2x(1)}{(x+1)^2} && \text{By the quotient rule.} \\ &= \cos\left(\frac{2x}{x+1}\right)\frac{2}{(x+1)^2}. \blacksquare \end{aligned}$$

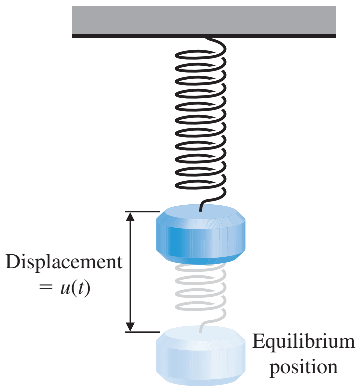


FIGURE 2.30
Spring-mass system

Applications

The trigonometric functions arise quite naturally in the solution of numerous physical problems of interest. For instance, it can be shown that the vertical displacement of a weight suspended from a spring, in the absence of damping (i.e., when resistance to the motion, such as air resistance, is negligible), is given by

$$u(t) = a\cos(\omega t) + b\sin(\omega t),$$

where ω is the frequency, t is time and a and b are constants. (See Figure 2.30 for a depiction of such a spring-mass system.)

EXAMPLE 6.5 Analysis of a Spring-Mass System

Suppose that $u(t)$ measures the displacement (measured in inches) of a weight suspended from a spring t seconds after it is released and that

$$u(t) = 4\cos t.$$

Find the velocity at any time t and determine the maximum velocity.

Solution Since $u(t)$ represents position (displacement), the velocity is given by $u'(t)$. We have

$$u'(t) = 4(-\sin t) = -4\sin t,$$

where $u'(t)$ is measured in inches per second. Of course, $\sin t$ oscillates between -1 and 1 and hence, the largest that $u'(t)$ can be is $-4(-1) = 4$ inches per second. This occurs when $\sin t = -1$, that is, at $t = 3\pi/2$, $t = 7\pi/2$ and so on. Notice that at these times $u(t) = 0$, so that the weight is moving fastest when it is passing through its resting position. ■

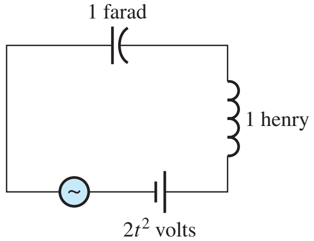


FIGURE 2.31
A simple circuit

EXAMPLE 6.6 Analysis of a Simple Electrical Circuit

The diagram in Figure 2.31 shows a simple electrical circuit. If the capacitance is 1 (farad), the inductance is 1 (henry) and the impressed voltage is $2t^2$ (volts) at time t, then a model for the total charge $Q(t)$ in the circuit at time t is

$$Q(t) = 2\sin t + 2t^2 - 4 \text{ (coulombs)}.$$

The current is defined to be the rate of change of the charge with respect to time and so is given by

$$I(t) = \frac{dQ}{dt} \text{ (amperes)}.$$

Compare the current at times $t = 0$ and $t = 1$.

Solution In general, the current is given by

$$I(t) = \frac{dQ}{dt} = 2\cos t + 4t \text{ (amperes)}.$$

Notice that at time $t = 0$, $I(0) = 2$ (amperes). At time $t = 1$,

$$I(1) = 2\cos 1 + 4 \approx 5.08 \text{ (amperes)}.$$

This represents an increase of

$$\frac{I(1) - I(0)}{I(0)}(100)\% \approx \frac{3.08}{2}(100)\% = 154\%. \quad ■$$

EXAMPLE 6.7 Finding the Equation of the Tangent Line

Find an equation of the tangent line to

$$y = 3\tan x - 2\csc x$$

at $x = \dfrac{\pi}{3}$.

Solution The derivative is

$$y' = 3\sec^2 x - 2(-\csc x \cot x) = 3\sec^2 x + 2\csc x \cot x.$$

At $x = \dfrac{\pi}{3}$, we have

$$y'\left(\frac{\pi}{3}\right) = 3(2)^2 + 2\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = 12 + \frac{4}{3} = \frac{40}{3} \approx 13.33333.$$

The tangent line with slope $\dfrac{40}{3}$ and point of tangency $\left(\dfrac{\pi}{3}, 3\sqrt{3} - \dfrac{4}{\sqrt{3}}\right)$ has equation

$$y = \frac{40}{3}\left(x - \frac{\pi}{3}\right) + 3\sqrt{3} - \frac{4}{\sqrt{3}}.$$

We show a graph of the function and the tangent line in Figure 2.32. ■

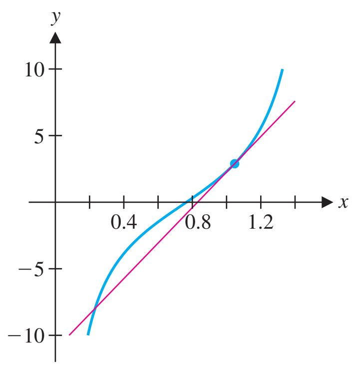


FIGURE 2.32
$y = 3\tan x - 2\csc x$ and the tangent line at $x = \dfrac{\pi}{3}$

EXERCISES 2.6

WRITING EXERCISES

1. Most people draw sine curves that are very steep and rounded. Given the results of this section, discuss the actual shape of the sine curve. Starting at (0, 0), how steep should the graph be drawn? What is the steepest the graph should be drawn anywhere? In which regions is the graph almost straight and where does it curve a lot?

2. In many common physics and engineering applications, the term $\sin x$ makes calculations difficult. A common simplification is to replace $\sin x$ with x, accompanied by the justification "$\sin x$ approximately equals x for small angles." Discuss this approximation in terms of the tangent line to $y = \sin x$ at $x = 0$. How small is the "small angle" for which the approximation is good? The tangent line to $y = \cos x$ at $x = 0$ is simply $y = 1$, but the simplification "$\cos x$ approximately equals 1 for small angles" is almost never used. Why would this approximation be less useful than $\sin x \approx x$?

---

1. Use a graphical analysis as in the text to argue that the derivative of $\cos x$ is $-\sin x$.

2. In the proof of $\dfrac{d}{dx}(\sin x) = \cos x$, where do we use the assumption that x is in radians? If you have access to a CAS, find out what the derivative of $\sin x^\circ$ is; explain where the $\pi/180$ came from.

In exercises 3–20, find the derivative of each function.

3. $f(x) = 4\sin x - x$

4. $f(x) = x^2 + 2\cos^2 x$

5. $f(x) = \tan^3 x - \csc^4 x$

6. $f(x) = 4\sec x^2 - 3\cot x$

7. $f(x) = x\cos 5x^2$

8. $f(x) = 4x^2 - 3\tan x$

9. $f(x) = \sin(\tan x^2)$

10. $f(x) = \sqrt{\sin^2 x + 2}$

11. $f(x) = \dfrac{\sin x^2}{x^2}$

12. $f(x) = \dfrac{x^2}{\csc^4 x}$

13. $f(t) = \sin t \sec t$

14. $f(t) = \sqrt{\cos t \sec t}$

15. $f(x) = \dfrac{1}{\sin 4x}$

16. $f(x) = x^2 \sec^2 3x$

17. $f(x) = 2\sin x \cos x$

18. $f(x) = 4\sin^2 x + 4\cos^2 x$

19. $f(x) = \tan\sqrt{x^2 + 1}$

20. $f(x) = 4x^2 \sin x \sec 3x$

In exercises 21–24, use your CAS or graphing calculator.

21. Repeat exercise 17 with your CAS. If its answer is not in the same form as ours in the back of the book, explain how the CAS computed its answer.

22. Repeat exercise 18 with your CAS. If its answer is not 0, explain how the CAS computed its answer.

23. Find the derivative of $f(x) = 2\sin^2 x + \cos 2x$ on your CAS. Compare its answer to 0. Explain how to get this answer and your CAS's answer, if it differs.

24. Find the derivative of $f(x) = \dfrac{\tan x}{\sin x}$ on your CAS. Compare its answer to $\sec x \tan x$. Explain how to get this answer and your CAS's answer, if it differs.

In exercises 25–28, find an equation of the tangent line to $y = f(x)$ at $x = a$.

25. $f(x) = \sin 4x$, $a = \dfrac{\pi}{8}$

26. $f(x) = \tan 3x$, $a = 0$

27. $f(x) = \cos x$, $a = \dfrac{\pi}{2}$

28. $f(x) = x \sin x$, $a = \dfrac{\pi}{2}$

In exercises 29–32, use the position function to find the velocity at time $t = t_0$. Assume units of feet and seconds.

29. $s(t) = t^2 - \sin 2t$, $t_0 = 0$

30. $s(t) = t\cos(t^2 + \pi)$, $t_0 = 0$

31. $s(t) = \dfrac{\cos t}{t}$, $t_0 = \pi$

32. $s(t) = 4 + 3\sin t$, $t_0 = \pi$

33. A spring hanging from the ceiling vibrates up and down. Its vertical position at time t is given by $f(t) = 4\sin 3t$. Find the velocity of the spring at time t. What is the spring's maximum speed? What is its location when it reaches its maximum speed?

34. In exercise 33, for what time values is the velocity 0? What is the location of the spring when its velocity is 0? When does the spring change directions?

In exercises 35 and 36, refer to example 6.6.

35. If the total charge in an electrical circuit at time t is given by $Q(t) = 3\sin 2t + t + 4$ coulombs, compare the current at times $t = 0$ and $t = 1$.

36. If the total charge in an electrical circuit at time t is given by $Q(t) = 4\cos 4t - 3t + 1$ coulombs, compare the current at times $t = 0$ and $t = 1$.

37. For $f(x) = \sin x$, find $f^{(75)}(x)$ and $f^{(150)}(x)$.

38. For $f(x) = \cos x$, find $f^{(77)}(x)$ and $f^{(120)}(x)$.

39. For Lemma 6.1, show that $\lim\limits_{\theta \to 0^-} \sin\theta = 0$.

40. Use Lemma 6.1 and the identity $\cos^2\theta + \sin^2\theta = 1$ to prove Lemma 6.2.

41. Use the identity $\cos(x + h) = \cos x \cos h - \sin x \sin h$ to prove Theorem 6.2.

42. Use the quotient rule to derive formulas for the derivatives of $\cot x$, $\sec x$ and $\csc x$.

43. Use the basic limits $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ and $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$ to find the following limits:

(a) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$ (b) $\lim\limits_{t \to 0} \dfrac{\sin t}{4t}$

(c) $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{5x}$ (d) $\lim\limits_{x \to 0} \dfrac{\sin x^2}{x^2}$

44. Use the basic limits $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ and $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$ to find the following limits:

(a) $\lim\limits_{t \to 0} \dfrac{2t}{\sin t}$ (b) $\lim\limits_{x \to 0} \dfrac{\cos x^2 - 1}{x^2}$

(c) $\lim\limits_{x \to 0} \dfrac{\sin 6x}{\sin 5x}$ (d) $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$

45. For $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ show that f is continuous and differentiable for all x. (Hint: Focus on $x = 0$.)

46. For the function of exercise 45, show that the derivative $f'(x)$ is continuous. (In this case, we say that the function f is C^1.)

47. Show that the function of exercise 45 is C^2. (That is, $f''(x)$ exists and is continuous for all x.)

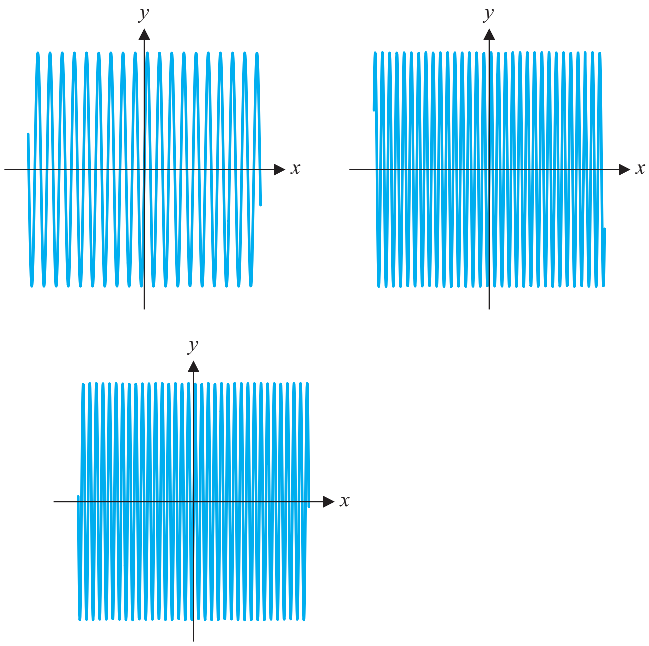
48. As in exercise 46, show that

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{is } C^1.$$

49. Sketch a graph of $y = \sin x$ and its tangent line at $x = 0$. Try to determine how many times they intersect by zooming in on the graph (but don't spend too much time on this). Show that for $f(x) = \sin x$, $f'(x) < 1$ for $0 < x < 1$. Explain why this implies that $\sin x < x$ for $0 < x < 1$. Use a similar argument to show that $\sin x > x$ for $-1 < x < 0$. Explain why $y = \sin x$ intersects $y = x$ at only one point.

50. For different positive values of k, determine how many times $y = \sin kx$ intersects $y = x$. In particular, what is the largest value of k for which there is only one intersection? Try to determine the largest value of k for which there are three intersections.

51. On a graphing calculator, graph $y = \sin x$ using the following range of x-values: $[-50, 50]$, $[-60, 60]$ and so on. You should find graphs similar to the following.



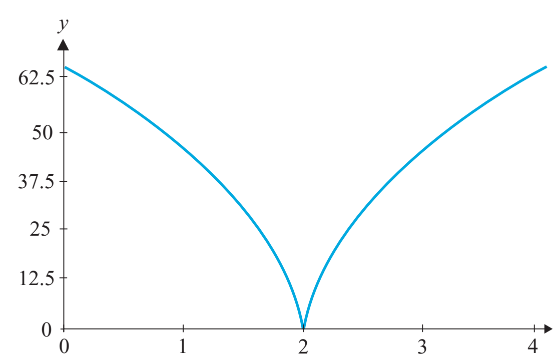
Briefly explain why the graph seems to change so much. In particular, is the calculator showing all of the graph or are there large portions missing?

EXPLORATORY EXERCISES

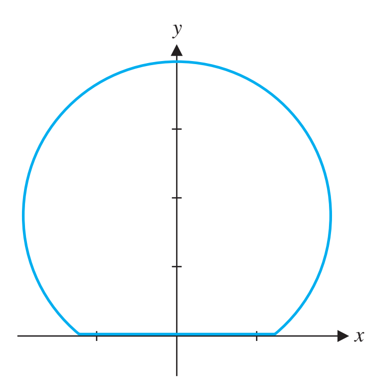
1. The function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ has several unusual properties. Show that f is continuous and differentiable at $x = 0$. However, $f'(x)$ is discontinuous at $x = 0$. To see this, show that $f'(x) = -1$ for $x = \dfrac{1}{2\pi}$, $x = \dfrac{1}{4\pi}$ and so on. Then show that $f'(x) = 1$ for $x = \dfrac{1}{\pi}$, $x = \dfrac{1}{3\pi}$ and so on. Explain why this proves that $f'(x)$ cannot be continuous at $x = 0$.

2. We have seen that the approximation $\sin x \approx x$ for small x derives from the tangent line to $y = \sin x$ at $x = 0$. We can also think of it as arising from the result $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. Explain why the limit implies $\sin x \approx x$ for small x. How can we get a better approximation? Instead of using the tangent line, we can try a quadratic function. To see what to multiply x^2 by, numerically compute $a = \lim\limits_{x\to 0} \dfrac{\sin x - x}{x^2}$. Then $\sin x - x \approx ax^2$ or $\sin x \approx x + ax^2$ for small x. How about a cubic approximation? Compute $b = \lim\limits_{x\to 0} \dfrac{\sin x - (x + ax^2)}{x^3}$. Then $\sin x - (x + ax^2) \approx bx^3$ or $\sin x \approx x + ax^2 + bx^3$ for small x. Starting with $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$, find a cubic approximation of the cosine function. If you have a CAS, take the approximations for sine and cosine out to a seventh-order polynomial and identify the pattern. (Hint: Write the coefficients—a, b, etc.—in the form $1/n!$ for some integer n.) We will take a longer look at these approximations, called **Taylor polynomials,** in Chapter 8.

3. When a ball bounces, we often think of the bounce occurring instantaneously, as in the accompanying figure. The sharp corner in the graph at the point of impact does not take into account that the ball actually compresses and maintains contact with the ground for a brief period of time. As shown in John Wesson's *The Science of Soccer*, the amount s that the ball is compressed satisfies the equation $s''(t) = -\frac{cp}{m}s(t)$, where c is the circumference of the ball, p is the pressure of air in the ball and m is the mass of the ball. Assume that the ball hits the ground at time 0 with vertical speed v m/s. Then $s(0) = 0$ and $s'(0) = v$. Show that $s(t) = \frac{v}{k}\sin kt$ satisfies the three conditions $s''(t) = -\frac{cp}{m}s(t)$, $s(0) = 0$ and $s'(0) = v$ with $k = \sqrt{\frac{cp}{m}}$. Use the properties of the sine function to show that the duration of the bounce is $\frac{\pi}{k}$ seconds and find the maximum compression. For a soccer ball with $c = 0.7$ m, $p = 0.86 \times 10^5$ N/m^2, $v = 15$ m/s, radius $R = 0.112$ m and $m = 0.43$ kg, compute the duration of the bounce and the maximum compression.



An idealized bounce



A ball being compressed

Putting together the physics for before, during and after the bounce, we obtain the height of the center of mass of a ball of radius R:

$$h(t) = \begin{cases} -4.9t^2 - vt + R & \text{if } t < 0 \\ R - \frac{v}{k}\sin kt & \text{if } 0 \le t \le \frac{\pi}{k} \\ -4.9(t - \frac{\pi}{k})^2 + v(t - \frac{\pi}{k}) + R & \text{if } t > \frac{\pi}{k}. \end{cases}$$

Determine whether $h(t)$ is continuous for all t and sketch a reasonable graph of this function to replace the figure shown here.

2.7 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

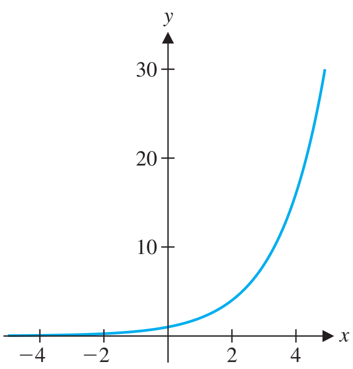


FIGURE 2.33a
$y = 2^x$

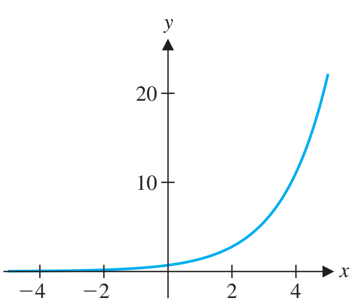


FIGURE 2.33b
The derivative of $f(x) = 2^x$

Exponential and logarithmic functions are among the most common functions encountered in applications. We begin with a simple application in business.

Suppose that you have an investment that doubles in value every year. If you start with \$100, then the value of the investment after 1 year is \$100(2) or \$200. After 2 years, its value is \$100(2)(2) = \$400 and after 3 years, its value is \$100(2^3) = \$800. In general, after t years, the value of your investment is \$100(2^t). Since the value doubles every year, you might describe the *rate* of return on your investment as 100% (usually called the **annual percentage yield** or **APY**). To a calculus student, the term *rate* should suggest the derivative.

We first consider $f(x) = a^x$ for some constant $a > 1$ (called the **base**). Recall from our discussion in Chapter 0 that the graph will look something like the graph of $f(x) = 2^x$, shown in Figure 2.33a.

Observe that as you look from left to right, the graph rises. Thus, the slopes of the tangent lines and so, too, the values of the derivative, are always positive. Also, the farther to the right you look, the steeper the graph is and so, the more positive the derivative is. Further, to the left of the origin, the tangent lines are nearly horizontal and hence, the derivative is positive but close to zero. The sketch of $y = f'(x)$ shown in Figure 2.33b is consistent with all of the above information. (Use your computer algebra system to generate a series of graphs of $y = a^x$ for various values of $a > 0$, along with the graphs of the corresponding derivatives and you may detect a pattern.) In particular, notice that the sketch of the derivative closely resembles the graph of the function itself.

Derivatives of the Exponential Functions

Let's start by using the usual limit definition to try to compute the derivative of $f(x) = a^x$, for $a > 1$. We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} && \text{From the usual rules of exponents.} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h}. && \text{Factor out the common term of } a^x. \end{aligned} \tag{7.1}$$

Unfortunately, we have, at present, no means of computing the limit in (7.1). Nonetheless, *assuming* the limit exists, (7.1) says that

$$\frac{d}{dx} a^x = (\text{constant})\, a^x. \tag{7.2}$$

Further, (7.2) is consistent with what we observed graphically in Figures 2.33a and 2.33b. The question that we now face is: does

$$\lim_{h \to 0} \frac{a^h - 1}{h}$$

exist for all (or any) values of $a > 1$? We explore this limit numerically in the following table for $a = 2$.

h	$\frac{2^h - 1}{h}$
0.01	0.6955550
0.0001	0.6931712
0.000001	0.6931474
0.0000001	0.6931470

h	$\frac{2^h - 1}{h}$
−0.01	0.6907505
−0.0001	0.6931232
−0.000001	0.6931469
−0.0000001	0.6931472

The numerical evidence suggests that the limit in question exists and that

$$\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693147.$$

In the same way, the numerical evidence suggests that

$$\lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.098612.$$

The approximate values of these limits are largely unremarkable until you observe that

$$\ln 2 \approx 0.693147 \quad \text{and} \quad \ln 3 \approx 1.098612.$$

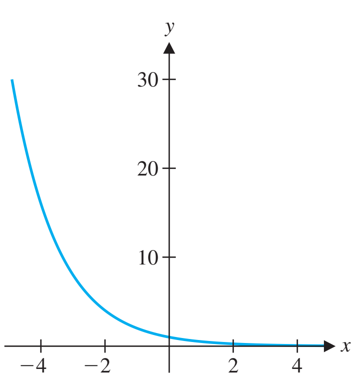


FIGURE 2.34a
$y = 2^{-x}$

You'll find similar results if you consider the limit in (7.1) for other values of $a > 1$. (Try estimating several of these for yourself.) This suggests that for $a > 1$,

$$\frac{d}{dx} a^x = a^x \ln a.$$

Next, consider the exponential function $f(x) = a^x$ for $0 < a < 1$, for instance, $f(x) = \left(\frac{1}{2}\right)^x$. Notice that we may write such a function as $f(x) = b^{-x}$, where $b = \frac{1}{a} > 1$. For example, for $b = \frac{1}{2}$, we have

$$f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}.$$

The graph of this type of exponential looks something like the graph of $y = 2^{-x}$ shown in Figure 2.34a.

Assuming that $$\frac{d}{dx} 2^x = 2^x \ln 2,$$

the chain rule gives us

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}\right)^x &= \frac{d}{dx} 2^{-x} \\ &= 2^{-x} \ln 2 \frac{d}{dx}(-x) \\ &= 2^{-x}(-\ln 2) \\ &= \left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right), \end{aligned}$$

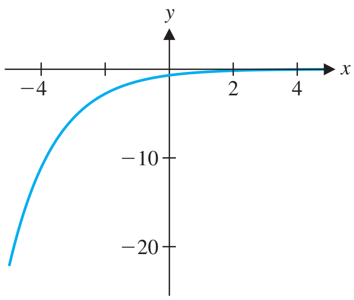


FIGURE 2.34b
The derivative of $f(x) = 2^{-x}$

where we have used the fact that

$$\ln\left(\frac{1}{2}\right) = \ln(2^{-1}) = -\ln 2.$$

A sketch of the derivative is shown in Figure 2.34b. This suggests the following general result.

THEOREM 7.1

For any constant $a > 0$,
$$\frac{d}{dx}a^x = a^x \ln a. \tag{7.3}$$

The proof of this result is complete except for the precise evaluation of the limit in (7.1). Unfortunately, we will not be in a position to complete this work until Chapter 4. For the moment, you should be content with the strong numerical, graphical and (nearly complete) algebraic arguments supporting this conjecture.

You should notice that (7.3) is consistent with what we observed in Figures 2.33a, 2.33b, 2.34a and 2.34b. In particular, recall that for $0 < a < 1$, $\ln a < 0$. This accounts for why the graph of the derivative for $0 < a < 1$ resembles the reflection of the graph of the original function through the x-axis.

EXAMPLE 7.1 Finding the Rate of Change of an Investment

If the value of a 100-dollar investment doubles every year, its value after t years is given by $v(t) = 100\ 2^t$. Find the instantaneous percentage rate of change of the worth.

Solution The instantaneous rate of change is the derivative

$$v'(t) = 100\ 2^t \ln 2.$$

The relative rate of change is then

$$\frac{v'(t)}{v(t)} = \frac{100\ 2^t \ln 2}{100\ 2^t} = \ln 2 \approx 0.693.$$

The percentage change is then about 69.3% per year. This is surprising to most people. A percentage rate of 69.3% will double your investment each year if it is compounded "continuously." We explore the notion of continuous compounding of interest further in exercise 37. ■

The most commonly used base (by far) is the (naturally occurring) irrational number e. You should immediately notice the significance of this choice. Since $\ln e = 1$, the derivative of $f(x) = e^x$ is simply

$$\frac{d}{dx}e^x = e^x \ln e = e^x.$$

We now have the following result.

THEOREM 7.2

$$\frac{d}{dx}e^x = e^x.$$

PROOF

As we have just seen, the proof of this formula follows immediately from Theorem 7.1. ■

You will probably agree that this is the easiest derivative formula to remember. In section 2.6, we looked at a simple model of the oscillations of a weight hanging from a spring. Using the cosine function, we had a model that produced the oscillations of the spring, but did not reproduce the physical reality of the spring eventually coming to rest. Coupling the trigonometric functions with the exponential functions produces a more realistic model.

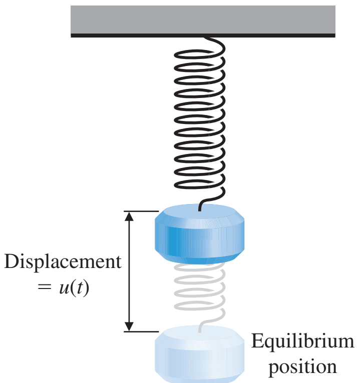


FIGURE 2.35
Spring-mass system

EXAMPLE 7.2 Finding the Velocity of a Hanging Weight

If we build damping (i.e., resistance to the motion due to friction, for instance) into our model spring-mass system (see Figure 2.35), the vertical displacement at time t of a weight hanging from a spring can be described by

$$u(t) = Ae^{\alpha t}\cos(\omega t) + Be^{\alpha t}\sin(\omega t),$$

where A, B, α and ω are constants. For each of

$$\text{(a) } u(t) = e^{-t}\cos t \quad \text{and} \quad \text{(b) } v(t) = e^{-t/6}\cos 4t,$$

sketch a graph of the motion of the weight and find its velocity at any time t.

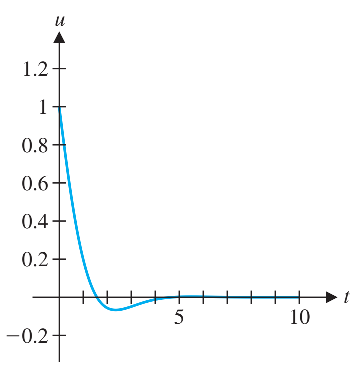


FIGURE 2.36a
$u(t) = e^{-t}\cos t$

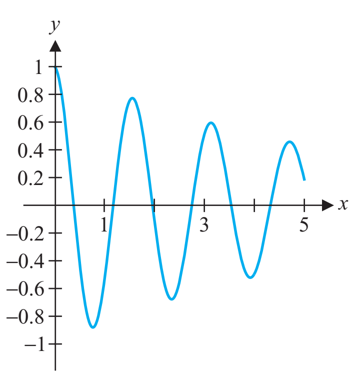


FIGURE 2.36b
$y = e^{-t/6}\cos(4t)$

Solution Figure 2.36a displays a graph of $u(t) = e^{-t}\cos t$. Notice that it seems to briefly oscillate and then quickly come to rest at $u = 0$. You should observe that this is precisely the kind of behavior you expect from your car's suspension system (the spring-mass system most familiar to you) when you hit a bump in the road. If your car's suspension system needs repair, you might get the behavior shown in Figure 2.36b, which is a graph of $v(t) = e^{-t/6}\cos(4t)$.

The velocity of the weight is given by the derivative. From the product rule, we get

$$\begin{aligned} u'(t) &= \frac{d}{dt}(e^{-t})\cos t + e^{-t}\frac{d}{dt}(\cos t) \\ &= e^{-t}\frac{d}{dt}(-t)\cos t - e^{-t}\sin t \\ &= -e^{-t}(\cos t + \sin t) \end{aligned}$$

and

$$\begin{aligned} v'(t) &= \frac{d}{dt}(e^{-t/6})\cos(4t) + e^{-t/6}\frac{d}{dt}[\cos(4t)] \\ &= e^{-t/6}\frac{d}{dt}\left(-\frac{t}{6}\right)\cos(4t) + e^{-t/6}[-\sin(4t)]\frac{d}{dt}(4t) \\ &= -\frac{1}{6}e^{-t/6}\cos(4t) - 4e^{-t/6}\sin(4t). \end{aligned}$$

■

At first glance, the chain rule for exponential functions may look a bit different, in part because the "inside" of the exponential function $e^{f(x)}$ is the exponent $f(x)$. Be careful not to change the exponent when computing the derivative of an exponential function.

EXAMPLE 7.3 The Chain Rule with Exponential Functions

Find the derivative of $f(x) = 3e^{x^2}$, $g(x) = xe^{2/x}$ and $h(x) = 3^{2x^2}$.

Solution From the chain rule, we have

$$f'(x) = 3e^{x^2}\frac{d}{dx}(x^2) = 3e^{x^2}(2x) = 6xe^{x^2}.$$

Using the product rule and the chain rule, we get

$$\begin{aligned} g'(x) &= (1)e^{2/x} + xe^{2/x}\frac{d}{dx}\left(\frac{2}{x}\right) \\ &= e^{2/x} + xe^{2/x}\left(-\frac{2}{x^2}\right) \\ &= e^{2/x} - 2\frac{e^{2/x}}{x} \\ &= e^{2/x}(1 - 2/x). \end{aligned}$$

Finally, we have

$$\begin{aligned} h'(x) &= 3^{2x^2}\ln 3\frac{d}{dx}(2x^2) \\ &= 3^{2x^2}\ln 3\,(4x) \\ &= 4(\ln 3)\,x3^{2x^2}. \end{aligned}$$

Derivative of the Natural Logarithm

The natural logarithm function $\ln x$ is closely connected with exponential functions. We've already seen it arise as a part of the general exponential derivative formula (7.3). Recall from Chapter 0 that the graph of the natural logarithm looks like the one shown in Figure 2.37a.

The function is defined only for $x > 0$ and as you look to the right, the graph always rises. Thus, the slopes of the tangent lines and hence, also the values of the derivative are always positive. Further, as $x \to \infty$, the slopes of the tangent lines become less positive and seem to approach 0. On the other hand, as x approaches 0 from the right, the graph gets steeper and steeper and hence, the derivative gets larger and larger, without bound. The graph shown in Figure 2.37b is consistent with all of these observations. Do you recognize it?

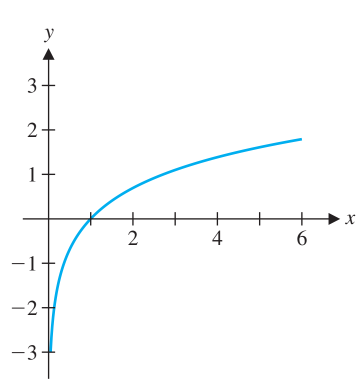


FIGURE 2.37a
$y = \ln x$

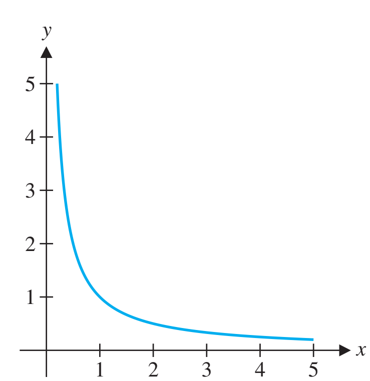


FIGURE 2.37b
The derivative of $f(x) = \ln x$

Using the definition of derivative, we get the following for $f(x) = \ln x$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}.$$

Unfortunately, we don't yet know how to evaluate this limit or even whether it exists. (We'll develop all of this in Chapter 4.) Our only option at present is to investigate this limit numerically. In the following tables, we compute some values for $x = 2$.

h	$\frac{\ln(2+h) - \ln 2}{h}$
0.1	0.4879016
0.001	0.4998750
0.00001	0.4999988
0.0000001	0.5000002

h	$\frac{\ln(2+h) - \ln 2}{h}$
−0.1	0.5129329
−0.001	0.5001250
−0.00001	0.5000013
−0.0000001	0.5000000

This leads us to the approximation

$$f'(2) = \lim_{h\to 0} \frac{\ln(2+h) - \ln 2}{h} \approx \frac{1}{2}.$$

Similarly, we compute some values for $x = 3$.

h	$\frac{\ln(3+h) - \ln 3}{h}$
0.1	0.3278982
0.001	0.3332778
0.00001	0.3333328
0.0000001	0.3333333

h	$\frac{\ln(3+h) - \ln 3}{h}$
−0.1	0.3390155
−0.001	0.3333889
−0.00001	0.3333339
−0.0000001	0.3333333

This leads us to the approximation

$$f'(3) = \lim_{h\to 0} \frac{\ln(3+h) - \ln 3}{h} \approx \frac{1}{3}.$$

You should verify the approximations $f'(4) \approx \frac{1}{4}$, $f'(5) \approx \frac{1}{5}$ and so on. Look back one more time at Figure 2.37b. Do you recognize the function now? You should convince yourself that the result stated in Theorem 7.3 makes sense.

THEOREM 7.3

For $x > 0$,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}. \tag{7.4}$$

PROOF

The proof of this follows directly from Theorems 5.2 and 7.2. Recall that $y = \ln x$ if and only if $e^y = x$. From Theorem 5.2, we have for $f(x) = \ln x$ and $g(x) = e^x$

$$f'(x) = \frac{1}{g'(y)} = \frac{1}{e^y} = \frac{1}{x},$$

as desired. ■

EXAMPLE 7.4 Derivatives of Logarithms

Find the derivative of $f(x) = x \ln x$, $g(x) = \ln x^3$ and $h(x) = \ln(x^2 + 1)$.

Solution Using the product rule, we get

$$f'(x) = (1)\ln x + x\left(\frac{1}{x}\right) = \ln x + 1.$$

We could certainly use the chain rule to differentiate $g(x)$. However, using the properties of logarithms, recall that we can rewrite $g(x) = \ln x^3 = 3\ln x$ and using (7.4), we get

$$g'(x) = 3\frac{d}{dx}(\ln x) = 3\left(\frac{1}{x}\right) = \frac{3}{x}.$$

Using the chain rule for $h(x)$, we get

$$h'(x) = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1) = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1}.$$ ■

EXAMPLE 7.5 Finding the Maximum Concentration of a Chemical

The concentration x of a certain chemical after t seconds of an autocatalytic reaction is given by $x(t) = \dfrac{10}{9e^{-20t}+1}$. Show that $x'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 10.

Solution Before computing the derivative, look carefully at the function $x(t)$. The independent variable is t and the only term involving t is in the denominator. So, we don't need to use the quotient rule. Instead, first rewrite the function as $x(t) = 10(9e^{-20t} + 1)^{-1}$ and use the chain rule. We get

$$\begin{aligned} x'(t) &= -10(9e^{-20t}+1)^{-2}\frac{d}{dt}(9e^{-20t}+1) \\ &= -10(9e^{-20t}+1)^{-2}(-180e^{-20t}) \\ &= 1800e^{-20t}(9e^{-20t}+1)^{-2} \\ &= \frac{1800e^{-20t}}{(9e^{-20t}+1)^2}. \end{aligned}$$

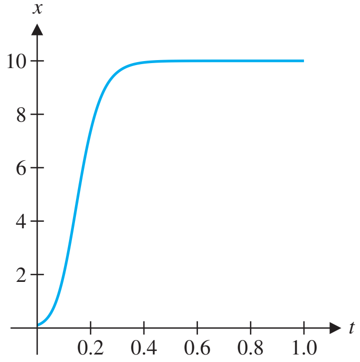


FIGURE 2.38
Chemical concentration

Notice that since $e^{-20t} > 0$ for all t, both the numerator and denominator are positive, so that $x'(t) > 0$. Since all of the tangent lines have positive slope, the graph of $y = x(t)$ rises from left to right, as shown in Figure 2.38.

Since the concentration increases for all time, the concentration is always less than the limiting value $\lim_{t\to\infty} x(t)$, which is easily computed to be

$$\lim_{t\to\infty} \frac{10}{9e^{-20t}+1} = \frac{10}{0+1} = 10. \blacksquare$$

Logarithmic Differentiation

A clever technique called **logarithmic differentiation** uses the rules of logarithms to help find derivatives of certain functions for which we don't presently have derivative formulas. For instance, note that the function $f(x) = x^x$ is not a power function because the exponent is not a constant and is not an exponential function because the base is not constant. In example 7.6, we show how to take advantage of the properties of logarithms to find the derivative of such a function.

EXAMPLE 7.6 Logarithmic Differentiation

Find the derivative of $f(x) = x^x$, for $x > 0$.

Solution As already noted, none of our existing derivative rules apply. We begin by taking the natural logarithm of both sides of the equation $f(x) = x^x$. We have

$$\begin{aligned} \ln[f(x)] &= \ln(x^x) \\ &= x \ln x, \end{aligned}$$

from the usual properties of logarithms. We now differentiate both sides of this last equation. Using the chain rule on the left side and the product rule on the right side, we get

$$\frac{1}{f(x)} f'(x) = (1)\ln x + x\frac{1}{x}$$

or

$$\frac{f'(x)}{f(x)} = \ln x + 1.$$

Solving for $f'(x)$, we get $$f'(x) = (\ln x + 1) f(x).$$

Substituting $f(x) = x^x$ gives us $f'(x) = (\ln x + 1)x^x$. $\blacksquare$

EXERCISES 2.7

WRITING EXERCISES

1. The graph of $f(x) = e^x$ curves upward in the interval from $x = -1$ to $x = 1$. Interpreting $f'(x) = e^x$ as the slopes of tangent lines and noting that the larger x is, the larger e^x is, explain why the graph curves upward. For larger values of x, the graph of $f(x) = e^x$ appears to shoot straight up with no curve. Using the tangent line, determine whether this is correct or just an optical illusion.

2. The graph of $f(x) = \ln x$ appears to get flatter as x gets larger. Interpret the derivative $f'(x) = \frac{1}{x}$ as the slopes of tangent lines to determine whether this is correct or just an optical illusion.

3. Graphically compare and contrast the functions x^2, x^3, x^4 and e^x for $x > 0$. Sketch the graphs for large x (and *very* large

y's) and compare the relative growth rates of the functions. In general, how does the exponential function compare to polynomials?

4. Graphically compare and contrast the functions $x^{1/2}$, $x^{1/3}$, $x^{1/4}$ and $\ln x$ for $x > 1$. Sketch the graphs for large x and compare the relative growth rates of the functions. In general, how does $\ln x$ compare to $\sqrt[n]{x}$?

In exercises 1–24, find the derivative of the function.

1. $f(x) = x^3 e^x$
2. $f(x) = e^{2x} \cos 4x$
3. $f(x) = x + 2^x$
4. $f(x) = x4^{3x}$
5. $f(x) = 2e^{4x+1}$
6. $f(x) = (1/e)^x$
7. $f(x) = (1/3)^{x^2}$
8. $f(x) = 4^{-x^2}$
9. $f(x) = 4^{-3x+1}$
10. $f(x) = (1/2)^{1-x}$
11. $f(x) = \dfrac{e^{4x}}{x}$
12. $f(x) = \dfrac{x}{e^{6x}}$
13. $f(x) = \ln 2x$
14. $f(x) = \ln \sqrt{8x}$
15. $f(x) = \ln (x^3 + 3x)$
16. $f(x) = x^3 \ln x$
17. $f(x) = \ln (\cos x)$
18. $f(x) = e^{\sin 2x}$
19. $f(x) = \sin [\ln (\cos x^3)]$
20. $f(x) = \ln (\sin x^2)$
21. $f(x) = \dfrac{\sqrt{\ln x^2}}{x}$
22. $f(x) = \dfrac{e^x}{2^x}$
23. $f(x) = \ln (\sec x + \tan x)$
24. $f(x) = \sqrt[3]{e^{2x} x^3}$

In exercises 25–30, find an equation of the tangent line to $y = f(x)$ at $x = 1$.

25. $f(x) = 3e^x$
26. $f(x) = 2e^{x-1}$
27. $f(x) = 3^x$
28. $f(x) = 2^x$
29. $f(x) = x^2 \ln x$
30. $f(x) = 2 \ln x^3$

In exercises 31–34, the value of an investment at time t is given by $v(t)$. Find the instantaneous percentage rate of change.

31. $v(t) = 100\ 3^t$
32. $v(t) = 100\ 4^t$
33. $v(t) = 100\ e^t$
34. $v(t) = 100\ e^{-t}$

35. A bacterial population starts at 200 and triples every day. Find a formula for the population after t days and find the percentage rate of change in population.

36. A bacterial population starts at 500 and doubles every four days. Find a formula for the population after t days and find the percentage rate of change in population.

37. An investment of A dollars receiving $100r$ percent (per year) interest compounded continuously will be worth $f(t) = Ae^{rt}$ dollars after t years. APY can be defined as $[f(1) - A]/A$, the relative increase of worth in one year. Find the APY for the following interest rates:

(a) 5% (b) 10% (c) 20%
(d) $100 \ln 2\%$ (e) 100%

38. Determine the interest rate needed to obtain an APY of

(a) 100% (b) 10%

In exercises 39–44, use logarithmic differentiation to find the derivative.

39. $f(x) = x^{\sin x}$
40. $f(x) = x^{4-x^2}$
41. $f(x) = (\sin x)^x$
42. $f(x) = (x^2)^{4x}$
43. $f(x) = x^{\ln x}$
44. $f(x) = x^{\sqrt{x}}$

45. The motion of a spring is described by $f(t) = e^{-t} \cos t$. Compute the velocity at time t. Graph the velocity function. When is the velocity zero? What is the position of the spring when the velocity is zero?

46. The motion of a spring is described by $f(t) = e^{-2t} \sin 3t$. Compute the velocity at time t. Graph the velocity function. When is the velocity zero? What is the position of the spring when the velocity is zero?

47. In exercise 45, graphically estimate the value of $t > 0$ at which the maximum velocity is reached.

48. In exercise 46, graphically estimate the value of $t > 0$ at which the maximum velocity is reached.

In exercises 49–52, involve the hyperbolic sine and hyperbolic cosine functions: $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

49. Show that $\dfrac{d}{dx}(\sinh x) = \cosh x$ and $\dfrac{d}{dx}(\cosh x) = \sinh x$.

50. Find the derivative of the hyperbolic tangent function: $\tanh x = \dfrac{\sinh x}{\cosh x}$.

51. Show that both $\sinh x$ and $\cosh x$ have the property that $f''(x) = f(x)$.

52. Find the derivative of (a) $f(x) = \sinh (\cos x)$ and (b) $f(x) = \cosh (x^2) - \sinh (x^2)$.

53. Find the value of a such that the tangent to $\ln x$ at $x = a$ is a line through the origin.

54. Find the value of a such that the tangent to e^x at $x = a$ is a line through the origin. Compare the slopes of the lines in exercises 53 and 54.

In exercises 55–58, use a CAS or graphing calculator.

55. Find the derivative of $f(x) = e^{\ln x^2}$ on your CAS. Compare its answer to $2x$. Explain how to get this answer and your CAS's answer, if it differs.

56. Find the derivative of $f(x) = e^{\ln(-x^2)}$ on your CAS. The correct answer is that it does not exist. Explain how to get this answer and your CAS's answer, if it differs.

57. Find the derivative of $f(x) = \ln\sqrt{4e^{3x}}$ on your CAS. Compare its answer to $\frac{3}{2}$. Explain how to get this answer and your CAS's answer, if it differs.

58. Find the derivative of $f(x) = \ln\left(\frac{e^{4x}}{x^2}\right)$ on your CAS. Compare its answer to $4 - 2/x$. Explain how to get this answer and your CAS's answer, if it differs.

59. Numerically estimate the limit in (7.1) for $a = 3$ and compare your answer to $\ln 3$.

60. Numerically estimate the limit in (7.1) for $a = \frac{1}{3}$ and compare your answer to $\ln \frac{1}{3}$.

61. The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $x(t) = \dfrac{6}{2e^{-8t} + 1}$. Show that $x'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 6.

62. The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $x(t) = \dfrac{10}{9e^{-10t} + 2}$. Show that $x'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 5.

63. The **Padé approximation** of e^x is the function of the form $f(x) = \dfrac{a + bx}{1 + cx}$ for which the values of $f(0)$, $f'(0)$ and $f''(0)$ match the corresponding values of e^x. Show that these values all equal 1 and find the values of a, b and c that make $f(0) = 1$, $f'(0) = 1$ and $f''(0) = 1$. Compare the graphs of $f(x)$ and e^x.

64. In a *World Almanac* or Internet resource, look up the population of the United States by decade for as many years as are available. If there are not columns indicating growth by decade, both numerically and by percentage, compute these yourself. (A spreadsheet is helpful here.) The United States has had periods of both linear and exponential growth. Explain why linear growth corresponds to a constant numerical increase; during which decades has the numerical growth been (approximately) constant? Explain why exponential growth corresponds to a constant percentage growth. During which decades has the percentage growth been (approximately) constant?

65. In statistics, the function $f(x) = e^{-x^2/2}$ is used to analyze random quantities that have a bell-shaped distribution. Solutions of the equation $f''(x) = 0$ give statisticians a measure of the variability of the random variable. Find all solutions.

66. Repeat exercise 65 for the function $f(x) = e^{-x^2/8}$. Comparing the graphs of the two functions, explain why you would say that this distribution is more spread out than that of exercise 65.

67. Repeat exercise 65 for the general function $f(x) = e^{-(x-m)^2/2c^2}$, where m and c are constants.

68. In exercise 67, find the solution to the equation $f'(x) = 0$. This value is known as the mode (or average) of the distribution.

EXPLORATORY EXERCISES

1. The **Hill functions** $f(x) = \dfrac{Ax^n}{\theta^n + x^n}$ for positive constants A, n and θ are used to model a variety of chemical and biological processes. In this exercise, we explore a technique for determining the values of the constants. First, show that $f'(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} f(x) = A$. Second, if $u = \ln\left(\dfrac{f(x)/A}{1 - f(x)/A}\right)$ and $v = \ln x$, show that u is a linear function of v. To see why these facts are important, consider the following data collected in a study of the binding of oxygen to hemoglobin. Here, x is the percentage of oxygen in the air and y is the percentage of hemoglobin saturated with oxygen.

x	1	2	3	4	5	6	7	8	9
y	2	13	32	52	67	77	84	88	91

Plot these data points. As you already showed, Hill functions are increasing and level off at a horizontal asymptote. Explain why it would be reasonable to try to find a Hill function to match these data. Use the limiting value of $f(x)$ to explain why in this data set, $A = 100$. Next, for each x-y pair, compute u and v as defined above. Plot the u-v points and show that they are (almost exactly) linear. Find the slope and use this to determine the values of n and θ.

2. In Chapter 0, we defined e as the limit $e = \lim_{n\to\infty}(1 + 1/n)^n$. In this section, e is given as the value of the base a such that $\dfrac{d}{dx}a^x = a^x$. There are a variety of other interesting properties of this important number. We discover one here. Graph the functions x^2 and 2^x. For $x > 0$, how many times do they intersect? Graph the functions x^3 and 3^x. For $x > 0$, how many times do they intersect? Try the pair $x^{2.5}$ and 2.5^x and the pair x^4 and 4^x. What would be a reasonable conjecture for the number of intersections ($x > 0$) of the functions x^a and a^x? Explain why $x = a$ is always a solution. Under what circumstances is there another solution less than a? greater than a? By trial and error, verify that e is the value of a at which the "other" solution changes.

3. For $n = 1$ and $n = 2$, investigate $\lim_{x\to 0}\dfrac{e^{-1/x}}{x^n}$ numerically and graphically. Conjecture the value of $\lim_{x\to 0}\dfrac{e^{-1/x}}{x^n}$ for any positive integer n and use your conjecture for the remainder of the exercise. For $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$, show that f is differentiable at each x and that $f'(x)$ is continuous for all x. Then show that $f''(0)$ exists and compare the work needed to show that $f'(x)$ is continuous at $x = 0$ and to show that $f''(0)$ exists.

2.8 IMPLICIT DIFFERENTIATION AND INVERSE TRIGONOMETRIC FUNCTIONS

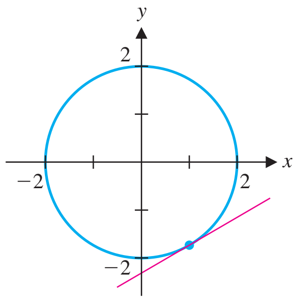


FIGURE 2.39
The tangent line at the point $(1, -\sqrt{3})$

Compare the following two equations describing familiar curves:

$$y = x^2 + 3 \text{ (parabola)}$$

and

$$x^2 + y^2 = 4 \text{ (circle)}.$$

The first equation defines y as a function of x *explicitly,* since for each x, the equation gives an explicit formula $y = f(x)$ for finding the corresponding value of y. On the other hand, the second equation does not define a function, since the circle in Figure 2.39 doesn't pass the vertical line test. However, you can solve for y and find at least two functions $\left(y = \sqrt{4 - x^2} \text{ and } y = -\sqrt{4 - x^2}\right)$ that are defined *implicitly* by the equation $x^2 + y^2 = 4$.

Suppose that we want to find the slope of the tangent line to the circle $x^2 + y^2 = 4$ at the point $\left(1, -\sqrt{3}\right)$ (see Figure 2.39). We can think of the circle as the graph of two semicircles, $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Since we are interested in the point $(1, -\sqrt{3})$, we use the equation describing the bottom semicircle, $y = -\sqrt{4 - x^2}$ to compute the derivative

$$y'(x) = -\frac{1}{2\sqrt{4 - x^2}}(-2x) = \frac{x}{\sqrt{4 - x^2}}.$$

So, the slope of the tangent line at the point $\left(1, -\sqrt{3}\right)$ is then $y'(1) = \dfrac{1}{\sqrt{3}}$.

This calculation was not especially challenging, although we will soon see an easier way to do it. However, it's not always possible to explicitly solve for a function defined implicitly by a given equation. For example, van der Waals' equation relating the pressure P, volume V and temperature T of a gas has the form

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT, \tag{8.1}$$

where a, n, b and R are constants. Notice the difficulty in solving this for V as a function of P. If we want the derivative $\frac{dV}{dP}$, we will need a method for computing the derivative directly from the implicit representation given in (8.1).

Consider each of the following calculations:

$$\frac{d}{dx}(x^3 + 4)^2 = 2(x^3 + 4)(3x^2),$$

$$\frac{d}{dx}(\sin x - 3x)^2 = 2(\sin x - 3x)(\cos x - 3)$$

and

$$\frac{d}{dx}(\tan x + 2)^2 = 2(\tan x + 2)\sec^2 x.$$

Notice that each of these calculations has the form

$$\frac{d}{dx}[y(x)]^2 = 2[y(x)]y'(x),$$

for some choice of the function $y(x)$. This last equation is simply an expression of the chain rule. We can use this notion to find the derivatives of functions defined implicitly by equations.

We first return to the simple case of the circle $x^2 + y^2 = 4$. Assuming this equation defines one or more differentiable functions of x: $y = y(x)$, the equation is

$$x^2 + [y(x)]^2 = 4. \tag{8.2}$$

Differentiating both sides of equation (8.2) with respect to x, we obtain

$$\frac{d}{dx}\left\{x^2 + [y(x)]^2\right\} = \frac{d}{dx}(4).$$

From the chain rule, $\frac{d}{dx}[y(x)]^2 = 2y(x)y'(x)$, as above and so, we have

$$2x + 2y(x)y'(x) = 0. \tag{8.3}$$

Subtracting $2x$ from both sides of (8.3) gives us

$$2y(x)y'(x) = -2x$$

and dividing by $2y(x)$ (assuming this is not zero), we have solved for $y'(x)$:

$$y'(x) = \frac{-2x}{2y(x)} = \frac{-x}{y(x)}.$$

Notice that here $y'(x)$ is expressed in terms of both x and y. To get the slope at the point $(1, -\sqrt{3})$, substitute $x = 1$ and $y = -\sqrt{3}$. We have

$$y'(1) = \left.\frac{-x}{y(x)}\right|_{x=1} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Notice that this is the same as we had found earlier by first solving for y explicitly and then differentiating. This process of differentiating both sides of an equation with respect to x and then solving for $y'(x)$ is called **implicit differentiation.**

When faced with an equation implicitly defining one or more differentiable functions $y = y(x)$, differentiate both sides with respect to x, being careful to recognize that differentiating any function of y will require the chain rule:

$$\boxed{\frac{d}{dx}g(y) = g'(y)y'(x).}$$

Then, gather any terms with a factor of $y'(x)$ on one side of the equation, with the remaining terms on the other side of the equation and solve for $y'(x)$. We illustrate this process in the examples that follow.

EXAMPLE 8.1 Finding a Slope Implicitly

Find $y'(x)$ for $x^2 + y^3 - 2y = 3$. Then, find the slope of the tangent line at the point $(2, 1)$.

Solution Since we can't (easily) solve for y in terms of x explicitly, we compute the derivative implicitly. Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^2 + y^3 - 2y) = \frac{d}{dx}(3)$$

and so,

$$2x + 3y^2y'(x) - 2y'(x) = 0.$$

To solve for $y'(x)$, simply write all terms involving $y'(x)$ on one side of the equation and all other terms on the other side. We have

$$3y^2y'(x) - 2y'(x) = -2x \quad \text{Subtracting } 2x \text{ from both sides.}$$

and hence, after factoring, we have

$$(3y^2 - 2)y'(x) = -2x. \quad \text{Factoring } y'(x) \text{ from both terms on the left side.}$$

Solving for $y'(x)$, we get

$$y'(x) = \frac{-2x}{3y^2 - 2}. \quad \text{Dividing by } (3y^2 - 2).$$

Substituting $x = 2$ and $y = 1$, we find that the slope of the tangent line at the point (2, 1) is

$$y'(2) = \frac{-4}{3 - 2} = -4.$$

The equation of the tangent line is then

$$y - 1 = -4(x - 2).$$

We have plotted a graph of the equation and the tangent line in Figure 2.40 using the implicit plot mode of our computer algebra system. ■

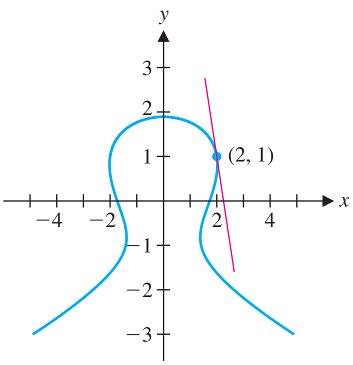


FIGURE 2.40
Tangent line at (2, 1)

EXAMPLE 8.2 Finding a Tangent Line by Implicit Differentiation

Find $y'(x)$ for $x^2y^2 - 2x = 4 - 4y$. Then, find an equation of the tangent line at the point $(2, -2)$.

Solution Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^2y^2 - 2x) = \frac{d}{dx}(4 - 4y).$$

Since the first term is the product of x^2 and y^2, we must use the product rule. We get

$$2xy^2 + x^2(2y)y'(x) - 2 = 0 - 4y'(x).$$

Grouping the terms with $y'(x)$ on one side, we get

$$(2x^2y + 4)y'(x) = 2 - 2xy^2,$$

so that

$$y'(x) = \frac{2 - 2xy^2}{2x^2y + 4}.$$

Substituting $x = 2$ and $y = -2$, we get the slope of the tangent line,

$$y'(2) = \frac{2 - 16}{-16 + 4} = \frac{7}{6}.$$

Finally, an equation of the tangent line is given by

$$y + 2 = \frac{7}{6}(x - 2).$$

We have plotted the curve and the tangent line at $(2, -2)$ in Figure 2.41 using the implicit plot mode of our computer algebra system. ■

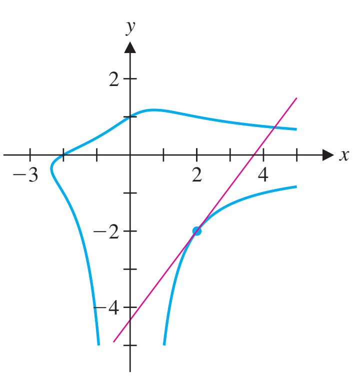


FIGURE 2.41
Tangent line at (2, −2)

You can use implicit differentiation to find a needed derivative from virtually any equation you can write down. We illustrate this next for an application.

EXAMPLE 8.3 Rate of Change of Volume with Respect to Pressure

Suppose that van der Waals' equation for a specific gas is

$$\left(P + \frac{5}{V^2}\right)(V - 0.03) = 9.7. \tag{8.4}$$

Thinking of the volume V as a function of pressure P, use implicit differentiation to find the derivative $\frac{dV}{dP}$ at the point (5, 1).

Solution Differentiating both sides of (8.4) with respect to P, we have

$$\frac{d}{dP}[(P + 5V^{-2})(V - 0.03)] = \frac{d}{dP}(9.7).$$

From the product rule and the chain rule, we get

$$\left(1 - 10V^{-3}\frac{dV}{dP}\right)(V - 0.03) + (P + 5V^{-2})\frac{dV}{dP} = 0.$$

Grouping the terms containing $\frac{dV}{dP}$, we get

$$[-10V^{-3}(V - 0.03) + P + 5V^{-2}]\frac{dV}{dP} = 0.03 - V,$$

so that

$$\frac{dV}{dP} = \frac{0.03 - V}{-10V^{-3}(V - 0.03) + P + 5V^{-2}}.$$

We now have

$$V'(5) = \frac{0.03 - 1}{-10(1)(0.97) + 5 + 5(1)} = \frac{-0.97}{0.3} = -\frac{97}{30}.$$

(The units are in terms of volume per unit pressure.) We show a graph of van der Waals' equation, along with the tangent line to the graph at the point (5, 1) in Figure 2.42. ■

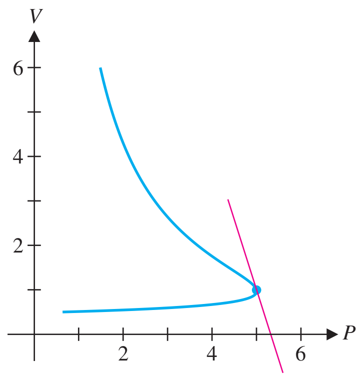


FIGURE 2.42
Graph of van der Waals' equation and the tangent line at the point (5, 1)

Of course, since we can find one derivative implicitly, we can also find second and higher order derivatives implicitly. In example 8.4, notice that you can choose which equation to use for the second derivative. A smart choice can save you time and effort.

EXAMPLE 8.4 Finding a Second Derivative Implicitly

Find $y''(x)$ implicitly for $y^2 + 2e^{-xy} = 6$. Then find the value of y'' at the point (0, 2).

Solution As always, start by differentiating both sides of the equation with respect to x. We have

$$\frac{d}{dx}(y^2 + 2e^{-xy}) = \frac{d}{dx}(6).$$

By the chain rule, we have

$$2yy'(x) + 2e^{-xy}[-y - xy'(x)] = 0. \tag{8.5}$$

Notice that we don't need to solve this for $y'(x)$. Dividing out the common factor of 2 and differentiating again, we get

$$y'(x)y'(x) + yy''(x) - e^{-xy}[-y - xy'(x)][y + xy'(x)] \\ - e^{-xy}[y'(x) + y'(x) + xy''(x)] = 0.$$

Grouping all the terms involving $y''(x)$ on one side of the equation gives us

$$yy''(x) - xe^{-xy}y''(x) = -[y'(x)]^2 - e^{-xy}[y + xy'(x)]^2 + 2e^{-xy}y'(x).$$

Factoring out the $y''(x)$ on the left, we get

$$(y - xe^{-xy})y''(x) = -[y'(x)]^2 - e^{-xy}[y + xy'(x)]^2 + 2e^{-xy}y'(x),$$

so that

$$y''(x) = \frac{-[y'(x)]^2 - e^{-xy}[y + xy'(x)]^2 + 2e^{-xy}y'(x)}{y - xe^{-xy}}. \tag{8.6}$$

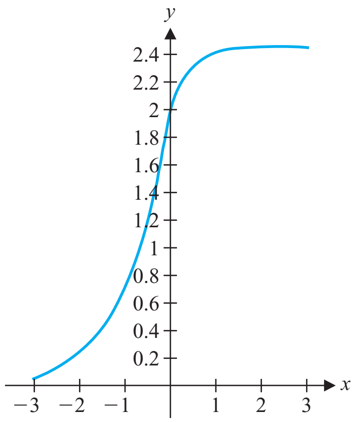


FIGURE 2.43
$y^2 + 2e^{-xy} = 6$

Notice that (8.6) gives us a (rather messy) formula for $y''(x)$ in terms of x, y and $y'(x)$. If we need to have $y''(x)$ in terms of x and y only, we can solve (8.5) for $y'(x)$ and substitute into (8.6). However, we don't need to do this to find $y''(0)$. Instead, first substitute $x = 0$ and $y = 2$ into (8.5) to get

$$4y'(0) + 2(-2) = 0,$$

from which we conclude that $y'(0) = 1$. Then substitute $x = 0$, $y = 2$ and $y'(0) = 1$ into (8.6) to get

$$y''(0) = \frac{-1 - (2)^2 + 2}{2} = -\frac{3}{2}.$$

See Figure 2.43 for a graph of $y^2 + 2e^{-xy} = 6$ near the point (0, 2). ■

Recall that, up to this point, we have proved the power rule

$$\frac{d}{dx}x^r = rx^{r-1}$$

only for *integer* exponents (see Theorems 3.1 and 4.3), although we have been freely using this result for any real exponent, r. Now that we have developed implicit differentiation, however, we have the tools we need to prove the power rule for the case of any *rational* exponent.

THEOREM 8.1

For any rational exponent, r, $\dfrac{d}{dx}x^r = rx^{r-1}$.

PROOF

Suppose that r is any rational number. Then $r = \dfrac{p}{q}$, for some integers p and q. Let

$$y = x^r = x^{p/q}. \tag{8.7}$$

Then, raising both sides of equation (8.7) to the qth power, we get

$$y^q = x^p. \tag{8.8}$$

TODAY IN MATHEMATICS

Dusa McDuff (1945–)
A British mathematician who has won prestigious awards for her research in multidimensional geometry. McDuff was inspired by Russian mathematician Israel Gelfand. "Gelfand amazed me by talking of mathematics as if it were poetry. . . . I had always thought of mathematics as being much more straightforward: a formula is a formula, and an algebra is an algebra, but Gelfand found hedgehogs lurking in the rows of his spectral sequences!" McDuff has made important contributions to undergraduate teaching and Women in Science and Engineering, lectures around the world and has coauthored several research monographs.

Differentiating both sides of equation (8.8) with respect to x, we get

$$\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p).$$

From the chain rule, we have $$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

Solving for $\frac{dy}{dx}$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} = \frac{px^{p-1}}{q(x^{p/q})^{q-1}} && \text{Since } y = x^{p/q}. \\ &= \frac{px^{p-1}}{qx^{p-p/q}} = \frac{p}{q}x^{p-1-p+p/q} && \text{Using the usual rules of exponents.} \\ &= \frac{p}{q}x^{p/q-1} = rx^{r-1}, && \text{Since } \frac{p}{q} = r. \end{aligned}$$

as desired. ■

Derivatives of the Inverse Trigonometric Functions

The inverse trigonometric functions are useful in any number of applications and are essential for solving equations. We now develop derivative rules for these functions. Recall from our discussion in Chapter 0 that you must pay very close attention to the domains and ranges for these functions. In particular, the inverse sine (or arcsine) function is defined by restricting the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Specifically, we had

$$y = \sin^{-1} x \text{ if and only if } \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

Differentiating the equation $\sin y = x$ implicitly, we have

$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$

and so, $$\cos y\frac{dy}{dx} = 1.$$

Solving this for $\frac{dy}{dx}$, we find (for $\cos y \neq 0$) that

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

This is not entirely satisfactory, though, since this gives us the derivative in terms of y. Notice that for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$ and hence,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

This leaves us with $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}},$$

for $-1 < x < 1$. That is,

$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1.}$$

Alternatively, we can derive this formula using Theorem 5.2 in section 2.5.

Similarly, we can show that

$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1.}$$

To find $\frac{d}{dx}\tan^{-1}x$, recall that we have

$$y = \tan^{-1}x \text{ if and only if } \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Using implicit differentiation, we then have

$$\frac{d}{dx}\tan y = \frac{d}{dx}x$$

and so,

$$(\sec^2 y)\frac{dy}{dx} = 1.$$

We solve this for $\frac{dy}{dx}$, to obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2}.\end{aligned}$$

That is,

$$\boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.}$$

The derivatives of the remaining inverse trigonometric functions are left as exercises. The derivatives of all six inverse trigonometric functions are summarized here.

$$\boxed{\begin{aligned}
\frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}}, && \text{for } -1 < x < 1 \\
\frac{d}{dx}\cos^{-1}x &= \frac{-1}{\sqrt{1-x^2}}, && \text{for } -1 < x < 1 \\
\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2} && \\
\frac{d}{dx}\cot^{-1}x &= \frac{-1}{1+x^2} && \\
\frac{d}{dx}\sec^{-1}x &= \frac{1}{|x|\sqrt{x^2-1}}, && \text{for } |x| > 1 \\
\frac{d}{dx}\csc^{-1}x &= \frac{-1}{|x|\sqrt{x^2-1}}, && \text{for } |x| > 1
\end{aligned}}$$

EXAMPLE 8.5 Finding the Derivative of an Inverse Trigonometric Function

Compute the derivative of (a) $\cos^{-1}(3x^2)$, (b) $(\sec^{-1} x)^2$ and (c) $\tan^{-1}(x^3)$.

Solution From the chain rule, we have

(a)
$$\frac{d}{dx}\cos^{-1}(3x^2) = \frac{-1}{\sqrt{1-(3x^2)^2}}\frac{d}{dx}(3x^2) = \frac{-6x}{\sqrt{1-9x^4}}.$$

(b)
$$\frac{d}{dx}(\sec^{-1} x)^2 = 2(\sec^{-1} x)\frac{d}{dx}(\sec^{-1} x) = 2(\sec^{-1} x)\frac{1}{|x|\sqrt{x^2-1}}$$

and (c)
$$\frac{d}{dx}[\tan^{-1}(x^3)] = \frac{1}{1+(x^3)^2}\frac{d}{dx}(x^3) = \frac{3x^2}{1+x^6}. \blacksquare$$

EXAMPLE 8.6 Modeling the Rate of Change of a Ballplayer's Gaze

One of the guiding principles of most sports is to "keep your eye on the ball." In baseball, a batter stands 2 feet from home plate as a pitch is thrown with a velocity of 130 ft/s (about 90 mph). At what rate does the batter's angle of gaze need to change to follow the ball as it crosses home plate?

Solution First, look at the triangle shown in Figure 2.44. We denote the distance from the ball to home plate by d and the angle of gaze by θ. Since the distance is changing with time, we write $d = d(t)$. The velocity of 130 ft/s means that $d'(t) = -130$. [Why

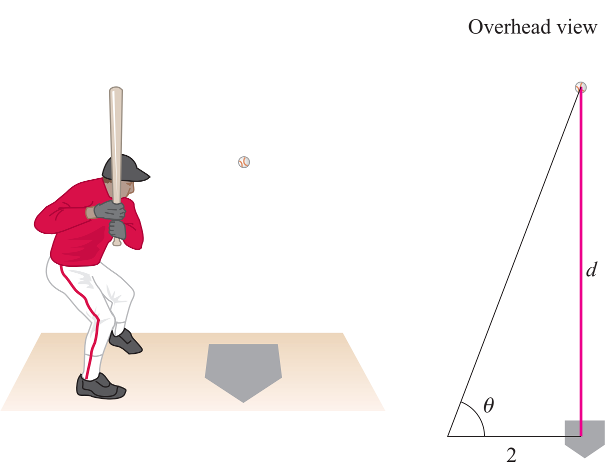


FIGURE 2.44
A ballplayer's gaze

would $d'(t)$ be negative?] From Figure 2.44, notice that

$$\theta(t) = \tan^{-1}\left[\frac{d(t)}{2}\right].$$

The rate of change of the angle is then

$$\theta'(t) = \frac{1}{1+\left[\frac{d(t)}{2}\right]^2}\frac{d'(t)}{2}$$

$$= \frac{2d'(t)}{4+[d(t)]^2} \text{ radians/second.}$$

When $d(t) = 0$ (i.e., when the ball is crossing home plate), the rate of change is then

$$\theta'(t) = \frac{2(-130)}{4} = -65 \text{ radians/second.}$$

One problem with this is that most humans can accurately track objects only at the rate of about 3 radians/second. Keeping your eye on the ball in this case is thus physically impossible. (See Watts and Bahill, *Keep Your Eye on the Ball.*) ■

BEYOND FORMULAS

Implicit differentiation allows us to find the derivative of functions even when we don't have a formula for the function. This remarkable result means that if we have any equation for the relationship between two quantities, we can find the rate of change of one with respect to the other. Here is a case where mathematics requires creative thinking beyond formula memorization. In what other situations have you seen the need for creativity in mathematics?

EXERCISES 2.8

WRITING EXERCISES

1. For implicit differentiation, we assume that y is a function of x: we write $y(x)$ to remind ourselves of this. However, for the circle $x^2 + y^2 = 1$, it is not true that y is a function of x. Since $y = \pm\sqrt{1-x^2}$, there are actually (at least) two functions of x defined implicitly. Explain why this is not really a contradiction; that is, explain exactly what we are assuming when we do implicit differentiation.

2. To perform implicit differentiation on an equation such as $x^2y^2 + 3 = x$, we start by differentiating all terms. We get $2xy^2 + x^2(2y)y' = 1$. Many students learn the rules this way: take "regular" derivatives of all terms, and tack on a y' every time you take a y-derivative. Explain why this works, and rephrase the rule in a more accurate and understandable form.

3. In implicit differentiation, the derivative is typically a function of both x and y; for example, on the circle $x^2 + y^2 = r^2$, we have $y' = -x/y$. If we take the derivative $-x/y$ and plug in any x and y, will it always be the slope of a tangent line? That is, are there any requirements on which x's and y's we can plug in?

4. In each example in this section, after we differentiated the given equation, we were able to rewrite the resulting equation in the form $f(x, y)y'(x) = g(x, y)$ for some functions $f(x, y)$ and $g(x, y)$. Explain why this can always be done; that is, why doesn't the chain rule ever produce a term like $[y'(x)]^2$ or $\frac{1}{y'(x)}$?

In exercises 1–4, compute the slope of the tangent line at the given point both explicitly (first solve for y as a function of x) and implicitly.

1. $x^2 + 4y^2 = 8$ at $(2, 1)$

2. $x^3y - 4\sqrt{x} = x^2y$ at $(2, \sqrt{2})$

3. $y - 3x^2y = \cos x$ at $(0, 1)$

4. $y^2 + 2xy + 4 = 0$ at $(-2, 2)$

In exercises 5–16, find the derivative $y'(x)$ implicitly.

5. $x^2y^2 + 3y = 4x$

6. $3xy^3 - 4x = 10y^2$

7. $\sqrt{xy} - 4y^2 = 12$

8. $\sin xy = x^2 - 3$

9. $\dfrac{x+3}{y} = 4x + y^2$

10. $3x + y^3 - 4y = 10x^2$

11. $e^{x^2y} - e^y = x$

12. $xe^y - 3y\sin x = 1$

13. $\sqrt{x+y} - 4x^2 = y$

14. $\cos y - y^2 = 8$

15. $e^{4y} - \ln y = 2x$

16. $e^{x^2}y - 3y = x^2 + 1$

In exercises 17–22, find an equation of the tangent line at the given point. If you have a CAS that will graph implicit curves, sketch the curve and the tangent line.

17. $x^2 - 4y^3 = 0$ at $(2, 1)$

18. $x^2y^2 = 4x$ at $(1, 2)$

19. $x^2y^2 = 4y$ at $(2, 1)$

20. $x^3y^2 = -3xy$ at $(-1, -3)$

21. $x^4 = 4(x^2 - y^2)$ at $(1, \frac{\sqrt{3}}{2})$

22. $x^4 = 8(x^2 - y^2)$ at $(2, -\sqrt{2})$

In exercises 23 and 24, find the locations of all horizontal and vertical tangents.

23. $x^2 + y^3 - 3y = 4$

24. $xy^2 - 2y = 2$

In exercises 25–28, find the second derivative $y''(x)$.

25. $x^2y^2 + 3x - 4y = 5$

26. $x^{2/3} + y^{2/3} = 4$

27. $y^2 = x^3 - 6x + 4\cos y$

28. $e^{xy} + 2y - 3x = \sin y$

In exercises 29–38, find the derivative of the given function.

29. $f(x) = \tan^{-1}\sqrt{x}$

30. $f(x) = \sin^{-1}(x^3 + 1)$

31. $f(x) = \tan^{-1}(\cos x)$

32. $f(x) = 4\sec^{-1}(x^4)$

33. $f(x) = 4\sec(x^4)$

34. $f(x) = \sqrt{2 + \tan^{-1} x}$

35. $f(x) = e^{\tan^{-1} x}$

36. $f(x) = \dfrac{x^2}{\cot^{-1} x}$

37. $f(x) = \dfrac{\tan^{-1} x}{x^2 + 1}$

38. $f(x) = \sin^{-1}(\sin x)$

39. In example 8.1, it is easy to find a y-value for $x = 2$, but other y-values are not so easy to find. Try solving for y if $x = 1.9$. Use the tangent line found in example 8.1 to estimate a y-value. Repeat for $x = 2.1$.

40. Use the tangent line found in example 8.2 to estimate a y-value corresponding to $x = 1.9$; $x = 2.1$.

41. For **elliptic curves,** there are nice ways of finding points with rational coordinates (see Ezra Brown's article "Three Fermat Trails to Elliptic Curves" in the May 2000 *College Mathematics Journal* for more information). If you have access to an implicit plotter, graph the elliptic curve defined by $y^2 = x^3 - 6x + 9$. Show that the points $(-3, 0)$ and $(0, 3)$ are on the curve. Find the line through these two points and show that the line intersects the curve in another point with rational (in this case, integer) coordinates.

42. For the elliptic curve $y^2 = x^3 - 6x + 4$, show that the point $(-1, 3)$ is on the curve. Find the tangent line to the curve at this point and show that it intersects the curve at another point with rational coordinates.

43. The functions $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ and $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ both have inverses on the appropriate domains and ranges. Show that $\dfrac{d}{dx}\tanh^{-1}x = \dfrac{1}{1-x^2}$ and $\dfrac{d}{dx}\coth^{-1}x = \dfrac{1}{1-x^2}$. Identify the set of x-values for which each formula is valid. Is it fair to say that the derivatives are equal?

44. Use a CAS to plot the set of points for which $(\cos x)^2 + (\sin y)^2 = 1$. Determine whether the segments plotted are straight or not.

45. Find and simplify the derivative of $\sin^{-1}x + \cos^{-1}x$. Use the result to write out an equation relating $\sin^{-1}x$ and $\cos^{-1}x$.

46. Find and simplify the derivative of $\sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right)$. Use the result to write out an equation relating $\sin^{-1}\left(\dfrac{x}{\sqrt{x^2+1}}\right)$ and $\tan^{-1}x$.

47. Use implicit differentiation to find $y'(x)$ for $x^2y - 2y = 4$. Based on this equation, why would you expect to find vertical tangents at $x = \pm\sqrt{2}$ and horizontal tangents at $y = 0$? Show that there are no points for these values. To see what's going on, solve the original equation for y and sketch the graph. Describe what's happening at $x = \pm\sqrt{2}$ and $y = 0$.

48. Show that any curve of the form $xy = c$ for some constant c intersects any curve of the form $x^2 - y^2 = k$ for some constant k at right angles (that is, the tangent lines to the curves at the intersection points are perpendicular). In this case, we say that the families of curves are **orthogonal.**

In exercises 49–52, show that the families of curves are orthogonal (see exercise 48).

49. $y = \dfrac{c}{x}$ and $y^2 = x^2 + k$

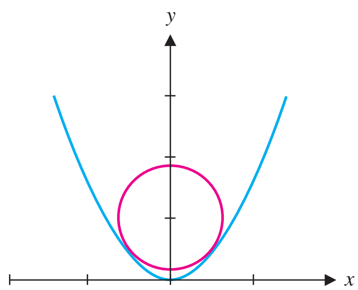
50. $x^2 + y^2 = cx$ and $x^2 + y^2 = ky$

51. $y = cx^3$ and $x^2 + 3y^2 = k$

52. $y = cx^4$ and $x^2 + 4y^2 = k$

53. Based on exercises 51 and 52, make a conjecture for a family of functions that is orthogonal to $y = cx^n$. Show that your conjecture is correct. Are there any values of n that must be excluded?

54. Suppose that a circle of radius r and center $(0, c)$ is inscribed in the parabola $y = x^2$. At the point of tangency, the slopes must be the same. Find the slope of the circle implicitly and show that at the point of tangency, $y = c - \frac{1}{2}$. Then use the equations of the circle and parabola to show that $c = r^2 + \frac{1}{4}$.



55. In example 8.6, it was shown that by the time the baseball reached home plate, the rate of rotation of the player's gaze (θ') was too fast for humans to track. Given a maximum rotational rate of $\theta' = -3$ radians per second, find d such that $\theta' = -3$. That is, find how close to the plate a player *can* track the ball. In a major league setting, the player must start swinging by the time the pitch is halfway $(30')$ to home plate. How does this correspond to the distance at which the player loses track of the ball?

56. Suppose the pitching speed d' in example 8.6 is different. Then θ' will be different and the value of d for which $\theta' = -3$ will change. Find d as a function of d' for d' ranging from 30 ft/s (slowpitch softball) to 140 ft/s (major league fastball), and sketch the graph.

57. Suppose a painting hangs on a wall. The frame extends from 6 feet to 8 feet above the floor. A person stands x feet from the wall and views the painting, with a viewing angle A formed by the ray from the person's eye (5 feet above the floor) to the top of the frame and the ray from the person's eye to the bottom of the frame. Find the value of x that maximizes the viewing angle A.

58. What changes in exercise 57 if the person's eyes are 6 feet above the floor?

EXPLORATORY EXERCISES

1. Suppose a slingshot (see section 2.1) rotates counterclockwise along the circle $x^2 + y^2 = 9$ and the rock is released at the point (2.9, 0.77). If the rock travels 300 feet, where does it land? [Hint: Find the tangent line at (2.9, 0.77), and find the point (x, y) on that line such that the distance is $\sqrt{(x - 2.9)^2 + (y - 0.77)^2} = 300$.]

2. A landowner's property line runs along the path $y = 6 - x$. The landowner wants to run an irrigation ditch from a reservoir bounded by the ellipse $4x^2 + 9y^2 = 36$. The landowner wants to build the shortest ditch possible from the reservoir to the closest point on the property line. We explore how to find the best path. Sketch the line and ellipse, and draw in a tangent line to the ellipse that is parallel to the property line. Argue that the ditch should start at the point of tangency and run perpendicular to the two lines. We start by identifying the point on the right side of the ellipse with tangent line parallel to $y = 6 - x$. Find the slope of the tangent line to the ellipse at (x, y) and set it equal to -1. Solve for x and substitute into the equation of the ellipse. Solve for y and you have the point on the ellipse at which to start the ditch. Find an equation of the (normal) line through this point perpendicular to $y = 6 - x$ and find the intersection of the normal line and $y = 6 - x$. This point is where the ditch ends.

3. In this exercise, you will design a movie theater with all seats having an equal view of the screen. Suppose the screen extends vertically from 10 feet to 30 feet above the floor. The first row of seats is 15 feet from the screen. Your task is to determine a function $h(x)$ such that if seats x feet from the screen are raised $h(x)$ feet above floor level, then the angle from the bottom of the screen to the viewer to the top of the screen will be the same as for a viewer sitting in the first row. You will be able to accomplish this only for a limited range of x-values. Beyond the maximum such x, find the height that maximizes the viewing angle. [Hint: Write the angle as a difference of inverse tangents and use the formula $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$.]

2.9 THE MEAN VALUE THEOREM

In this section, we present the Mean Value Theorem, which is so significant that we will be deriving new ideas from it for many chapters to come. Before considering the main result, we look at a special case, called Rolle's Theorem.

The idea behind Rolle's Theorem is really quite simple. For any function f that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and where $f(a) = f(b)$, there must be at least one point between $x = a$ and $x = b$ where the tangent line to $y = f(x)$ is horizontal. In Figures 2.45a to 2.45c, we draw a number of graphs satisfying the above criteria. Notice that each one has at least one point where there is a horizontal tangent line. Draw your own graphs, to convince yourself that, under these circumstances, it's not possible to connect the two points $(a, f(a))$ and $(b, f(b))$ without having at least one horizontal tangent line.

Note that since $f'(x) = 0$ at a horizontal tangent, this says that there is at least one point c in (a, b), for which $f'(c) = 0$ (see Figures 2.45a to 2.45c).

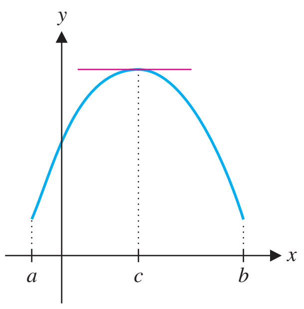


FIGURE 2.45a
Graph initially rising

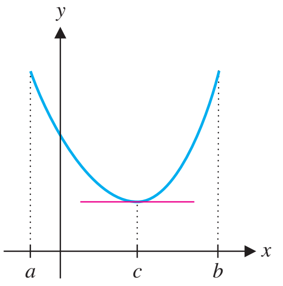


FIGURE 2.45b
Graph initially falling

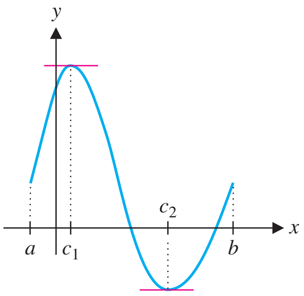


FIGURE 2.45c
Graph with two horizontal tangents

HISTORICAL NOTES

Michel Rolle (1652–1719)
A French mathematician who proved Rolle's Theorem for polynomials. Rolle came from a poor background, being largely self-taught and struggling through a variety of jobs including assistant attorney, scribe and elementary school teacher. He was a vigorous member of the French Academy of Sciences, arguing against such luminaries as Descartes that if $a < b$ then $-b < -a$ (so, for instance, $-2 < -1$). Oddly, Rolle was known as an opponent of the newly developed calculus, calling it a "collection of ingenious fallacies."

THEOREM 9.1 (Rolle's Theorem)

Suppose that f is continuous on the interval $[a, b]$, differentiable on the interval (a, b) and $f(a) = f(b)$. Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

SKETCH OF PROOF

We present the main ideas of the proof from a graphical perspective. First, note that if $f(x)$ is constant on $[a, b]$, then $f'(x) = 0$ for *all* x's between a and b. On the other hand, if $f(x)$ is not constant on $[a, b]$, then, as you look from left to right, the graph must at some point start to either rise or fall (see Figures 2.46a and 2.46b). For the case where the graph starts to rise, notice that in order to return to the level at which it started, it will need to turn around at some point and start to fall. (Think about it this way: if you start to climb up a mountain—so that your altitude rises—if you are to get back down to where you started, you will need to turn around at some point—where your altitude starts to fall.)

So, there is at least one point where the graph turns around, changing from rising to falling (see Figure 2.46a). Likewise, in the case where the graph first starts to fall, the graph

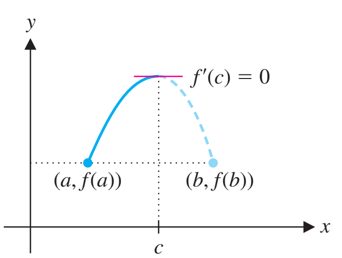


FIGURE 2.46a
Graph rises and turns around to fall back to where it started.

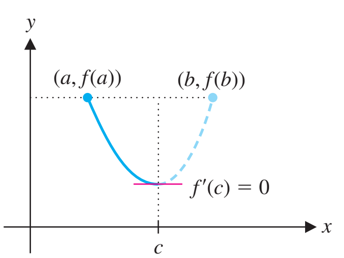


FIGURE 2.46b
Graph falls and then turns around to rise back to where it started.

must turn around from falling to rising (see Figure 2.46b). We name this point $x = c$. Since we know that $f'(c)$ exists, we have that either $f'(c) > 0$, $f'(c) < 0$ or $f'(c) = 0$. We want to argue that $f'(c) = 0$, as Figures 2.46a and 2.46b suggest. To establish this, it is easier to show that it is *not* true that $f'(c) > 0$ or $f'(c) < 0$. If it were true that $f'(c) > 0$, then from the alternative definition of the derivative given in equation (2.2) in section 2.2, we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0.$$

This says that for every x sufficiently close to c,

$$\frac{f(x) - f(c)}{x - c} > 0. \tag{9.1}$$

In particular, for the case where the graph first rises, if $x - c > 0$ (i.e., $x > c$), this says that $f(x) - f(c) > 0$ or $f(x) > f(c)$, which can't happen for every $x > c$ (with x sufficiently close to c) if the graph has turned around at c and started to fall. From this, we conclude that it can't be true that $f'(c) > 0$. Similarly, we can show that it is not true that $f'(c) < 0$. Therefore, $f'(c) = 0$, as desired. The proof for the case where the graph first falls is nearly identical and is left to the reader. ■

We now give an illustration of the conclusion of Rolle's Theorem.

EXAMPLE 9.1 An Illustration of Rolle's Theorem

Find a value of c satisfying the conclusion of Rolle's Theorem for

$$f(x) = x^3 - 3x^2 + 2x + 2$$

on the interval [0, 1].

Solution First, we verify that the hypotheses of the theorem are satisfied: f is differentiable and continuous for all x [since $f(x)$ is a polynomial and all polynomials are continuous and differentiable everywhere]. Also, $f(0) = f(1) = 2$. We have

$$f'(x) = 3x^2 - 6x + 2.$$

We now look for values of c such that

$$f'(c) = 3c^2 - 6c + 2 = 0.$$

By the quadratic formula, we get $c = 1 + \frac{1}{3}\sqrt{3} \approx 1.5774$ [not in the interval (0, 1)] and $c = 1 - \frac{1}{3}\sqrt{3} \approx 0.42265 \in (0, 1)$. ■

REMARK 9.1

We want to emphasize that example 9.1 is merely an *illustration* of Rolle's Theorem. Finding the number(s) c satisfying the conclusion of Rolle's Theorem is *not* the point of our discussion. Rather, Rolle's Theorem is of interest to us primarily because we use it to prove one of the fundamental results of elementary calculus, the Mean Value Theorem.

Although Rolle's Theorem is a simple result, we can use it to derive numerous properties of functions. For example, we are often interested in finding the zeros of a function f (that is, solutions of the equation $f(x) = 0$). In practice, it is often difficult to determine how many zeros a given function has. Rolle's Theorem can be of help here.

THEOREM 9.2

If f is continuous on the interval $[a, b]$, differentiable on the interval (a, b) and $f(x) = 0$ has two solutions in $[a, b]$, then $f'(x) = 0$ has at least one solution in (a, b).

PROOF

This is just a special case of Rolle's Theorem. Identify the two zeros of $f(x)$ as $x = s$ and $x = t$, where $s < t$. Since $f(s) = f(t)$, Rolle's Theorem guarantees that there is a number c such that $s < c < t$ (and hence $a < c < b$) where $f'(c) = 0$. ■

We can easily generalize the result of Theorem 9.2, as in the following theorem.

THEOREM 9.3

For any integer $n > 0$, if f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) and $f(x) = 0$ has n solutions in $[a, b]$, then $f'(x) = 0$ has at least $(n - 1)$ solutions in (a, b).

PROOF

From Theorem 9.2, between every pair of solutions of $f(x) = 0$ is at least one solution of $f'(x) = 0$. In this case, there are $(n - 1)$ consecutive pairs of solutions of $f(x) = 0$ and so, the result follows. ■

We can use Theorems 9.2 and 9.3 to investigate the number of zeros a given function has. (Here, we consider only real zeros of a function and not complex zeros.)

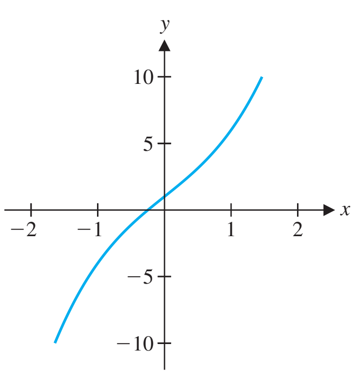


FIGURE 2.47
$y = x^3 + 4x + 1$

EXAMPLE 9.2 Determining the Number of Zeros of a Function

Prove that $x^3 + 4x + 1 = 0$ has exactly one solution.

Solution The graph shown in Figure 2.47 makes the result seem reasonable: we can see one solution, but how can we be sure there are no others outside of the displayed window? Notice that if $f(x) = x^3 + 4x + 1$, then

$$f'(x) = 3x^2 + 4 > 0$$

for all x. By Theorem 9.2, if $f(x) = 0$ had two solutions, then $f'(x) = 0$ would have at least one solution. However, since $f'(x) \neq 0$ for all x, it can't be true that $f(x) = 0$ has two (or more) solutions. Therefore, $f(x) = 0$ has exactly one solution. ■

We now generalize Rolle's Theorem to one of the most significant results of elementary calculus.

THEOREM 9.4 (Mean Value Theorem)

Suppose that f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{9.2}$$

PROOF

Note that the hypotheses are identical to those of Rolle's Theorem, except that there is no assumption about the values of f at the endpoints. The expression $\dfrac{f(b)-f(a)}{b-a}$ is the slope of the secant line connecting the endpoints, $(a, f(a))$ and $(b, f(b))$.

The theorem states that there is a line tangent to the curve at some point $x = c$ in (a, b) that has the same slope as (and hence, is parallel to) the secant line (see Figures 2.48 and 2.49). If you tilt your head so that the line segment looks horizontal, Figure 2.49 will look like a figure for Rolle's Theorem (Figures 2.46a and 2.46b). The proof is to "tilt" the function and then apply Rolle's Theorem.

NOTE

Note that in the special case where $f(a) = f(b)$, (9.2) simplifies to the conclusion of Rolle's Theorem, that $f'(c) = 0$.

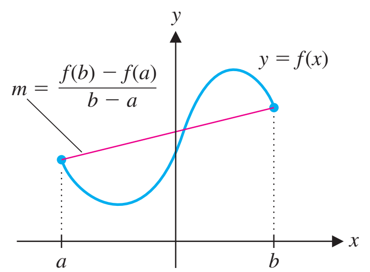


FIGURE 2.48
Secant line

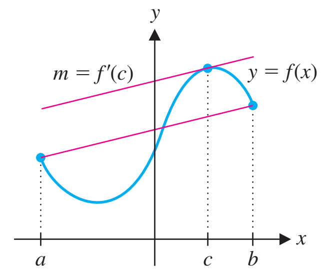


FIGURE 2.49
Mean Value Theorem

The equation of the secant line through the endpoints is

$$y - f(a) = m(x - a),$$

where

$$m = \frac{f(b)-f(a)}{b-a}.$$

Define the "tilted" function g to be the difference between f and the function whose graph is the secant line:

$$g(x) = f(x) - [m(x-a) + f(a)]. \tag{9.3}$$

Note that g is continuous on $[a, b]$ and differentiable on (a, b), since f is. Further,

$$g(a) = f(a) - [0 + f(a)] = 0$$

and

$$\begin{aligned} g(b) &= f(b) - [m(b-a) + f(a)] \\ &= f(b) - [f(b) - f(a) + f(a)] = 0. \quad \text{Since } m = \frac{f(b)-f(a)}{b-a}. \end{aligned}$$

Since $g(a) = g(b)$, we have by Rolle's Theorem that there exists a number c in the interval (a, b) such that $g'(c) = 0$. Differentiating (9.3), we get

$$0 = g'(c) = f'(c) - m. \tag{9.4}$$

Finally, solving (9.4) for $f'(c)$ gives us

$$f'(c) = m = \frac{f(b)-f(a)}{b-a},$$

as desired. ■

Before we demonstrate some of the power of the Mean Value Theorem, we first briefly illustrate its conclusion.

EXAMPLE 9.3 An Illustration of the Mean Value Theorem

Find a value of c satisfying the conclusion of the Mean Value Theorem for

$$f(x) = x^3 - x^2 - x + 1$$

on the interval [0, 2].

Solution Notice that f is continuous on [0, 2] and differentiable on (0, 2). The Mean Value Theorem then says that there is a number c in (0, 2) for which

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2 - 0} = 1.$$

To find this number c, we set

$$f'(c) = 3c^2 - 2c - 1 = 1$$

or

$$3c^2 - 2c - 2 = 0.$$

From the quadratic formula, we get $c = \frac{1 \pm \sqrt{7}}{3}$. In this case, only one of these, $c = \frac{1 + \sqrt{7}}{3}$, is in the interval (0, 2). In Figure 2.50, we show the graphs of $y = f(x)$, the secant line joining the endpoints of the portion of the curve on the interval [0, 2] and the tangent line at $x = \frac{1 + \sqrt{7}}{3}$. ■

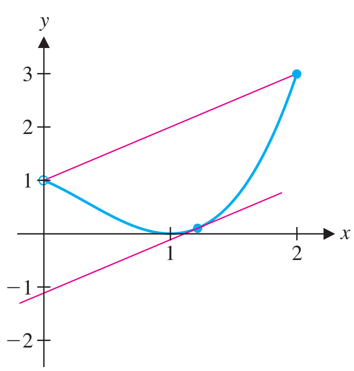


FIGURE 2.50
Mean Value Theorem

The illustration in example 9.3, where we found the number c whose existence is guaranteed by the Mean Value Theorem, is not the point of the theorem. In fact, these c's usually remain unknown. The significance of the Mean Value Theorem is that it relates a difference of function values to the difference of the corresponding x-values, as in equation (9.5) below.

Note that if we take the conclusion of the Mean Value Theorem (9.2) and multiply both sides by the quantity $(b - a)$, we get

$$f(b) - f(a) = f'(c)(b - a). \tag{9.5}$$

As it turns out, many of the most important results in the calculus (including one known as the *Fundamental Theorem of Calculus*) follow from the Mean Value Theorem. For now, we derive a result essential to our work in Chapter 4. The question concerns how many functions share the same derivative.

Recall that for any constant c,

$$\frac{d}{dx}(c) = 0.$$

A question that you probably haven't thought to ask is: Are there any other functions whose derivative is zero? The answer is no, as we see in Theorem 9.5.

THEOREM 9.5

Suppose that $f'(x) = 0$ for all x in some open interval I. Then, $f(x)$ is constant on I.

PROOF

Pick any two numbers, say a and b, in I, with $a < b$. Since f is differentiable in I and $(a, b) \subset I$, f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, we know that

$$\frac{f(b) - f(a)}{b - a} = f'(c), \tag{9.6}$$

for some number $c \in (a, b) \subset I$. Since, $f'(x) = 0$ for all $x \in I$, $f'(c) = 0$ and it follows from (9.6) that

$$f(b) - f(a) = 0 \quad \text{or} \quad f(b) = f(a).$$

Since a and b were arbitrary points in I, this says that f is constant on I, as desired. ■

A question closely related to Theorem 9.5 is the following. We know, for example, that

$$\frac{d}{dx}(x^2 + 2) = 2x.$$

But are there any other functions with the same derivative? You should quickly come up with several. For instance, $x^2 + 3$ and $x^2 - 4$ also have the derivative $2x$. In fact,

$$\frac{d}{dx}(x^2 + c) = 2x,$$

for any constant c. Are there any other functions, though, with the derivative $2x$? Corollary 9.1 says that there are no such functions.

COROLLARY 9.1

Suppose that $g'(x) = f'(x)$ for all x in some open interval I. Then, for some constant c,

$$g(x) = f(x) + c, \text{ for all } x \in I.$$

PROOF

Define $h(x) = g(x) - f(x)$. Then

$$h'(x) = g'(x) - f'(x) = 0$$

for all x in I. From Theorem 9.5, $h(x) = c$, for some constant c. The result then follows immediately from the definition of $h(x)$. ■

We see in Chapter 4 that Corollary 9.1 has significant implications when we try to reverse the process of differentiation (called *antidifferentiation*). We take a look ahead to this in example 9.4.

EXAMPLE 9.4 Finding Every Function with a Given Derivative

Find all functions that have a derivative equal to $3x^2 + 1$.

Solution We first write down (from our experience with derivatives) one function with the correct derivative: $x^3 + x$. Then, Corollary 9.1 tells us that any other function with the same derivative differs by at most a constant. So, *every* function whose derivative equals $3x^2 + 1$ has the form $x^3 + x + c$, for some constant c. ■

As our final example, we demonstrate how the Mean Value Theorem can be used to establish a useful inequality.

EXAMPLE 9.5 Proving an Inequality for sin x

Prove that
$$|\sin a| \le |a| \text{ for all } a.$$

Solution First, note that $f(x) = \sin x$ is continuous and differentiable on any interval and that for any a,
$$|\sin a| = |\sin a - \sin 0|,$$
since $\sin 0 = 0$. From the Mean Value Theorem, we have that (for $a \neq 0$)
$$\frac{\sin a - \sin 0}{a - 0} = f'(c) = \cos c, \tag{9.7}$$
for some number c between a and 0. Notice that if we multiply both sides of (9.7) by a and take absolute values, we get
$$|\sin a| = |\sin a - \sin 0| = |\cos c|\,|a - 0| = |\cos c|\,|a|. \tag{9.8}$$
But, $|\cos c| \le 1$, for all real numbers c and so, from (9.8), we have
$$|\sin a| = |\cos c|\,|a| \le (1)\,|a| = |a|,$$
as desired. ■

BEYOND FORMULAS

The Mean Value Theorem is subtle, but its implications are far-reaching. Although the illustration in Figure 2.49 makes the result seem obvious, the consequences of the Mean Value Theorem, such as example 9.4, are powerful and not at all obvious. For example, most of the rest of the calculus developed in this book depends on the Mean Value Theorem either directly or indirectly. A thorough understanding of the theory of calculus can lead you to important conclusions, particularly when the problems are beyond what your intuition alone can handle. What other theorems have you learned that continue to provide insight beyond their original context?

EXERCISES 2.9

WRITING EXERCISES

1. Notice that for both Rolle's Theorem and the Mean Value Theorem, we have assumed that the function is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Recall that if a function is differentiable at $x = a$, then it is also continuous at $x = a$. Therefore, if we had assumed that the function was differentiable on the closed interval, we would not have had to mention continuity. We do not do so because the assumption of being differentiable at the endpoints is not necessary. The "ethics" of the statement of a theorem is to include only assumptions that are absolutely necessary. Discuss the virtues of this tradition. Is this common practice in our social dealings, such as financial obligations or personal gossip?

2. One of the results in this section is that if $f'(x) = g'(x)$, then $g(x) = f(x) + c$ for some constant c. Explain this result graphically.

3. Explain the result of Corollary 9.1 in terms of position and velocity functions. That is, if two objects have the same velocity functions, what can you say about the relative positions of the two objects?

4. As we mentioned, you can derive Rolle's Theorem from the Mean Value Theorem simply by setting $f(a) = f(b)$. Given this, it may seem odd that Rolle's Theorem rates its own name and portion of the book. To explain why we do this, discuss ways in which Rolle's Theorem is easier to understand than the Mean Value Theorem.

In exercises 1–6, check the hypotheses of Rolle's Theorem and the Mean Value Theorem and find a value of c that makes the appropriate conclusion true. Illustrate the conclusion with a graph.

1. $f(x) = x^2 + 1, [-2, 2]$
2. $f(x) = x^2 + 1, [0, 2]$
3. $f(x) = x^3 + x^2, [0, 1]$
4. $f(x) = x^3 + x^2, [-1, 1]$
5. $f(x) = \sin x, [0, \pi/2]$
6. $f(x) = \sin x, [-\pi, 0]$
7. If $f'(x) > 0$ for all x, prove that $f(x)$ is an *increasing* function: that is, if $a < b$, then $f(a) < f(b)$.
8. If $f'(x) < 0$ for all x, prove that $f(x)$ is a *decreasing* function: that is, if $a < b$, then $f(a) > f(b)$.

In exercises 9–16, determine whether the function is increasing, decreasing or neither.

9. $f(x) = x^3 + 5x + 1$
10. $f(x) = x^5 + 3x^3 - 1$
11. $f(x) = -x^3 - 3x + 1$
12. $f(x) = x^4 + 2x^2 + 1$
13. $f(x) = e^x$
14. $f(x) = e^{-x}$
15. $f(x) = \ln x$
16. $f(x) = \ln x^2$
17. Prove that $x^3 + 5x + 1 = 0$ has exactly one solution.
18. Prove that $x^3 + 4x - 3 = 0$ has exactly one solution.
19. Prove that $x^4 + 3x^2 - 2 = 0$ has exactly two solutions.
20. Prove that $x^4 + 6x^2 - 1 = 0$ has exactly two solutions.
21. Prove that $x^3 + ax + b = 0$ has exactly one solution for $a > 0$.
22. Prove that $x^4 + ax^2 - b = 0$ $(a > 0, b > 0)$ has exactly two solutions.
23. Prove that $x^5 + ax^3 + bx + c = 0$ has exactly one solution for $a > 0, b > 0$.
24. Prove that a third-degree (cubic) polynomial has at most three zeros. (You may use the quadratic formula.)
25. Suppose that $s(t)$ gives the position of an object at time t. If s is differentiable on the interval $[a, b]$, prove that at some time $t = c$, the instantaneous velocity at $t = c$ equals the average velocity between times $t = a$ and $t = b$.
26. Two runners start a race at time 0. At some time $t = a$, one runner has pulled ahead, but the other runner has taken the lead by time $t = b$. Prove that at some time $t = c > 0$, the runners were going exactly the same speed.
27. If f and g are differentiable functions on the interval $[a, b]$ with $f(a) = g(a)$ and $f(b) = g(b)$, prove that at some point in the interval $[a, b]$, f and g have parallel tangent lines.
28. Prove that the result of exercise 27 still holds if the assumptions $f(a) = g(a)$ and $f(b) = g(b)$ are relaxed to requiring $f(b) - f(a) = g(b) - g(a)$.

In exercises 29–34, find all functions g such that $g'(x) = f(x)$.

29. $f(x) = x^2$
30. $f(x) = 9x^4$
31. $f(x) = 1/x^2$
32. $f(x) = \sqrt{x}$
33. $f(x) = \sin x$
34. $f(x) = \cos x$

In exercises 35–38, explain why it is not valid to use the Mean Value Theorem. When the hypotheses are not true, the theorem does not tell you anything about the truth of the conclusion. In three of the four cases, show that there is no value of c that makes the conclusion of the theorem true. In the fourth case, find the value of c.

35. $f(x) = \dfrac{1}{x}, [-1, 1]$
36. $f(x) = \dfrac{1}{x^2}, [-1, 2]$
37. $f(x) = \tan x, [0, \pi]$
38. $f(x) = x^{1/3}, [-1, 1]$
39. Assume that f is a differentiable function such that $f(0) = f'(0) = 0$ and $f''(0) > 0$. Argue that there exists a positive constant $a > 0$ such that $f(x) > 0$ for all x in the interval $(0, a)$. Can anything be concluded about $f(x)$ for negative x's?
40. Show that for any real numbers u and v, $|\cos u - \cos v| \le |u - v|$. (Hint: Use the Mean Value Theorem.)
41. Prove that $|\sin a| < |a|$ for all $a \ne 0$ and use the result to show that the only solution to the equation $\sin x = x$ is $x = 0$. What happens if you try to find all intersections with a graphing calculator?
42. Use the Mean Value Theorem to show that $|\tan^{-1} a| < |a|$ for all $a \ne 0$ and use this inequality to find all solutions of the equation $\tan^{-1} x = x$.
43. Prove that $|x| < |\sin^{-1} x|$ for $0 < |x| < 1$.
44. Prove that $|x| \le |\tan x|$ for $|x| < \dfrac{\pi}{2}$.
45. For $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 2x - 4 & \text{if } x > 0 \end{cases}$ show that f is continuous on the interval $(0, 2)$, differentiable on the interval $(0, 2)$ and has $f(0) = f(2)$. Show that there *does not* exist a value of c such that $f'(c) = 0$. Which hypothesis of Rolle's Theorem is not satisfied?

46. Assume that f is a differentiable function such that $f(0) = f'(0) = 0$. Show by example that it is not necessarily true that $f(x) = 0$ for all x. Find the flaw in the following bogus "proof." Using the Mean Value Theorem with $a = x$ and $b = 0$, we have $f'(c) = \frac{f(x) - f(0)}{x - 0}$. Since $f(0) = 0$ and $f'(c) = 0$, we have $0 = \frac{f(x)}{x}$ so that $f(x) = 0$.

EXPLORATORY EXERCISES

1. In section 2.1, we gave an example of computing the velocity of a moving car. The point of the story was that computing instantaneous velocity requires us to compute a limit. However, we left an interesting question unanswered. If you have an average velocity of 60 mph over 1 hour and the speed limit is 65 mph, you are unable to prove that you never exceeded the speed limit. What is the longest time interval over which you can average 60 mph and still guarantee no speeding? We can use the Mean Value Theorem to answer the question after clearing up a couple of preliminary questions. First, argue that we need to know the maximum acceleration of a car and the maximum positive acceleration may differ from the maximum negative acceleration. Based on your experience, what is the fastest your car could accelerate (speed up)? What is the fastest your car could decelerate (slow down)? Back up your estimates with some real data (e.g., my car goes from 0 to 60 in 15 seconds). Call the larger number A (use units of mph per second). Next, argue that if acceleration (the derivative of velocity) is constant, then the velocity function is linear. Therefore, if the velocity varies from 55 mph to 65 mph at constant acceleration, the average velocity will be 60 mph. Now, apply the Mean Value Theorem to the velocity function $v(t)$ on a time interval $[0, T]$, where the velocity changes from 55 mph to 65 mph at constant acceleration A: $v'(c) = \frac{v(T) - v(0)}{T - 0}$ becomes $A = \frac{65 - 55}{T - 0}$ and so $T = 10/A$. For how long is the guarantee good?

2. Suppose that a pollutant is dumped into a lake at the rate of $p'(t) = t^2 - t + 4$ tons per month. The amount of pollutant dumped into the lake in the first two months is $A = p(2) - p(0)$. Using $c = 1$ (the midpoint of the interval), estimate A by applying the Mean Value Theorem to $p(t)$ on the interval $[0, 2]$. To get a better estimate, apply the Mean Value Theorem to the intervals $[0, 1/2]$, $[1/2, 1]$, $[1, 3/2]$ and $[3/2, 2]$. [Hint: $A = p(1/2) - p(0) + p(1) - p(1/2) + p(3/2) - p(1) + p(2) - p(3/2)$.] If you have access to a CAS, get better estimates by dividing the interval $[0, 2]$ into more and more pieces and try to conjecture the limit of the estimates. You will be well on your way to understanding Chapter 4 on integration.

3. A result known as the Cauchy Mean Value Theorem states that if f and g are differentiable on the interval (a, b) and continuous on $[a, b]$, then there exists a number c with $a < c < b$ and $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$. Find all flaws in the following invalid attempt to prove the result, and then find a correct proof. *Invalid attempt:* The hypotheses of the Mean Value Theorem are satisfied by both functions, so there exists a number c with $a < c < b$ and $f'(c) = \frac{f(b) - f(a)}{b - a}$ and $g'(c) = \frac{g(b) - g(a)}{b - a}$. Then $b - a = \frac{f(b) - f(a)}{f'(c)} = \frac{g(b) - g(a)}{g'(c)}$ and thus $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Tangent line	Velocity	Average velocity
Derivative	Power rule	Acceleration
Product rule	Quotient rule	Chain rule
Implicit differentiation	Mean Value Theorem	Rolle's Theorem

State the derivative of each function:
$\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$, e^x, b^x, $\ln x$, $\log_b x$

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. If a function is continuous at $x = a$, then it has a tangent line at $x = a$.

2. The average velocity between $t = a$ and $t = b$ is the average of the velocities at $t = a$ and $t = b$.

3. The derivative of a function gives its slope.

4. Given the graph of $f'(x)$, you can construct the graph of $f(x)$.

Review Exercises

5. The power rule gives the rule for computing the derivative of any polynomial.
6. If a function is written as a quotient, use the quotient rule to find its derivative.
7. The chain rule gives the derivative of the composition of two functions. The order does not matter.
8. The derivative of an inverse function is the inverse of the derivative of the function.
9. The slope of $f(x) = \sin 4x$ is never larger than 1.
10. The derivative of any exponential function is itself.
11. The derivative of $f(x) = \ln ax$ is $\frac{1}{x}$ for any $a > 0$.
12. In implicit differentiation, you do not have to solve for y as a function of x to find $y'(x)$.
13. The Mean Value Theorem and Rolle's Theorem are special cases of each other.
14. The Mean Value Theorem can be used to show that for a fifth-degree polynomial, $f'(x) = 0$ for at most four values of x.

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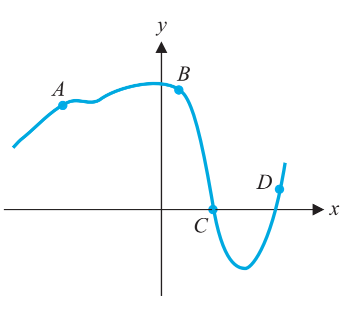
1. Estimate the value of $f'(1)$ from the given data.

x	0	0.5	1	1.5	2
$f(x)$	2.0	2.6	3.0	3.4	4.0

2. List the points A, B, C and D in order of increasing slope of the tangent line.



In exercises 3–8, use the limit definition to find the indicated derivative.

3. $f'(2)$ for $f(x) = x^2 - 2x$
4. $f'(1)$ for $f(x) = 1 + \frac{1}{x}$
5. $f'(1)$ for $f(x) = \sqrt{x}$
6. $f'(0)$ for $f(x) = x^3 - 2x$
7. $f'(x)$ for $f(x) = x^3 + x$
8. $f'(x)$ for $f(x) = \frac{3}{x}$

In exercises 9–14, find an equation of the tangent line.

9. $y = x^4 - 2x + 1$ at $x = 1$
10. $y = \sin 2x$ at $x = 0$
11. $y = 3e^{2x}$ at $x = 0$
12. $y = \sqrt{x^2 + 1}$ at $x = 0$
13. $y - x^2y^2 = x - 1$ at $(1, 1)$
14. $y^2 + xe^y = 4 - x$ at $(2, 0)$

In exercises 15–18, use the given position function to find velocity and acceleration.

15. $s(t) = -16t^2 + 40t + 10$
16. $s(t) = -9.8t^2 - 22t + 6$
17. $s(t) = 10e^{-2t} \sin 4t$
18. $s(t) = \sqrt{4t + 16} - 4$
19. In exercise 15, $s(t)$ gives the height of a ball at time t. Find the ball's velocity at $t = 1$; is the ball going up or down? Find the ball's velocity at $t = 2$; is the ball going up or down?
20. In exercise 17, $s(t)$ gives the position of a spring at time t. Compare the velocities at $t = 0$ and $t = \pi$. Is the spring moving in the same direction or opposite directions? At which time is the spring moving faster?

In exercises 21 and 22, compute the slopes of the secant lines between (a) $x = 1$ and $x = 2$, (b) $x = 1$ and $x = 1.5$, (c) $x = 1$ and $x = 1.1$ and (d) estimate the slope of the tangent line at $x = 1$.

21. $f(x) = \sqrt{x + 1}$
22. $f(x) = e^{2x}$

In exercises 23–50, find the derivative of the given function.

23. $f(x) = x^4 - 3x^3 + 2x - 1$
24. $f(x) = x^{2/3} - 4x^2 + 5$
25. $f(x) = \frac{3}{\sqrt{x}} + \frac{5}{x^2}$
26. $f(x) = \frac{2 - 3x + x^2}{\sqrt{x}}$
27. $f(t) = t^2(t + 2)^3$
28. $f(t) = (t^2 + 1)(t^3 - 3t + 2)$
29. $g(x) = \frac{x}{3x^2 - 1}$
30. $g(x) = \frac{3x^2 - 1}{x}$
31. $f(x) = x^2 \sin x$
32. $f(x) = \sin x^2$
33. $f(x) = \tan \sqrt{x}$
34. $f(x) = \sqrt{\tan x}$
35. $f(t) = t \csc t$
36. $f(t) = \sin 3t \cos 4t$
37. $u(x) = 2e^{-x^2}$
38. $u(x) = (2e^{-x})^2$
39. $f(x) = x \ln x^2$
40. $f(x) = \sqrt{\ln x + 1}$
41. $f(x) = \sqrt{\sin 4x}$
42. $f(x) = \cos^2 3x$

Review Exercises

43. $f(x) = \left(\dfrac{x+1}{x-1}\right)^2$

44. $f(x) = e^{\sqrt{3x}}$

45. $f(t) = te^{4t}$

46. $f(x) = \dfrac{6x}{(x-1)^2}$

47. $\sin^{-1} 2x$

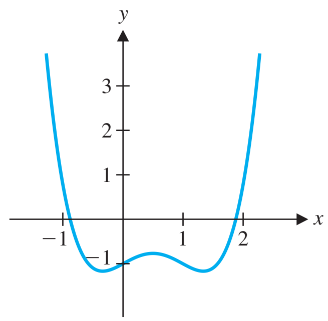
48. $\cos^{-1} x^2$

49. $\tan^{-1}(\cos 2x)$

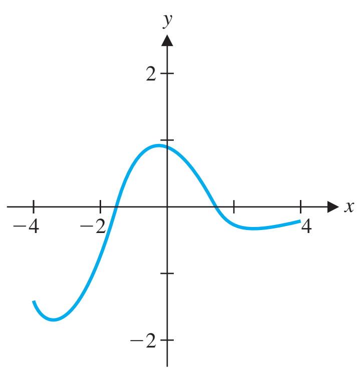
50. $\sec^{-1}(3x^2)$

In exercises 51 and 52, use the graph of $y = f(x)$ to sketch the graph of $y = f'(x)$.

51.



52.



In exercises 53–60, find the indicated derivative.

53. $f''(x)$ for $f(x) = x^4 - 3x^3 + 2x^2 - x - 1$

54. $f'''(x)$ for $f(x) = \sqrt{x+1}$

55. $f'''(x)$ for $f(x) = xe^{2x}$

56. $f''(x)$ for $f(x) = \dfrac{4}{x+1}$

57. $f''(x)$ for $f(x) = \tan x$

58. $f^{(4)}(x)$ for $f(x) = x^6 - 3x^4 + 2x^3 - 7x + 1$

59. $f^{(26)}(x)$ for $f(x) = \sin 3x$

60. $f^{(31)}(x)$ for $f(x) = e^{-2x}$

61. Revenue equals price times quantity. Suppose that the current price is \$2.40 and 12,000 items are sold at that price. If the price is increasing at the rate of 10 cents per year and the quantity sold decreases at the rate of 1500 items per year, at what rate is the revenue changing?

62. The value (in dollars) of an investment as a function of time (years) is given by $v(t) = 200\left(\frac{3}{2}\right)^t$. Find the instantaneous percentage rate of change of the value of the investment.

63. The position at time t of a spring moving vertically is given by $f(t) = 4\cos 2t$. Find the position of the spring when it has (a) zero velocity, (b) maximum velocity and (c) minimum velocity.

64. The position at time t of a spring moving vertically is given by $f(t) = e^{-2t}\sin 3t$. Find the velocity of the spring at any time t.

In exercises 65–68, find the derivative $y'(x)$.

65. $x^2y - 3y^3 = x^2 + 1$

66. $\sin(xy) + x^2 = x - y$

67. $\dfrac{y}{x+1} - 3y = \tan x$

68. $x - 2y^2 = 3e^{x/y}$

69. If you have access to a CAS, sketch the graph in exercise 65. Find the y-value corresponding to $x = 0$. Find the slope of the tangent line to the curve at this point. Also, find $y''(0)$.

70. If you have access to a CAS, sketch the graph in exercise 67. Find the y-value corresponding to $x = 0$. Find the slope of the tangent line to the curve at this point. Also, find $y''(0)$.

In exercises 71–74, find all points at which the tangent line to the curve is (a) horizontal and (b) vertical.

71. $y = x^3 - 6x^2 + 1$

72. $y = x^{2/3}$

73. $x^2y - 4y = x^2$

74. $y = x^4 - 2x^2 + 3$

75. Prove that the equation $x^3 + 7x - 1 = 0$ has exactly one solution.

76. Prove that the equation $x^4 + 2x^2 - 3 = 0$ has exactly two solutions.

Review Exercises

In exercises 77 and 78, do both parts without solving for the inverse: (a) find the derivative of the inverse at $x = a$ and (b) graph the inverse.

77. $x^5 + 2x^3 - 1, a = 2$

78. $e^{x^3+2x}, a = 1$

79. Prove that $|\cos x - 1| < |x|$ for all x.

80. Prove that $x + x^3/3 + 2x^5/15 < \tan x < x + x^3/3 + 2x^5/5$ for $0 < x < 1$.

81. If $f(x)$ is differentiable at $x = a$, show that $g(x)$ is continuous at $x = a$ where $g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases}$.

82. If f is differentiable at $x = a$ and $T(x) = f(a) + f'(a)(x - a)$ is the tangent line to $f(x)$ at $x = a$, prove that $f(x) - T(x) = e(x)(x - a)$ for some error function $e(x)$ with $\lim_{x \to a} e(x) = 0$.

In exercises 83 and 84, find a value of c as guaranteed by the Mean Value Theorem.

83. $f(x) = x^2 - 2x$ on the interval [0, 2]

84. $f(x) = x^3 - x$ on the interval [0, 2]

In exercises 85 and 86, find all functions g such that $g'(x) = f(x)$.

85. $f(x) = 3x^2 - \cos x$

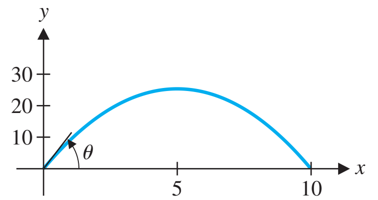
86. $f(x) = x^3 - e^{2x}$

87. A polynomial $f(x)$ has a **double root** at $x = a$ if $(x - a)^2$ is a factor of $f(x)$ but $(x - a)^3$ is not. The line through the point (1, 2) with slope m has equation $y = m(x - 1) + 2$. Find m such that $f(x) = x^3 + 1 - [m(x - 1) + 2]$ has a double root at $x = 1$. Show that $y = m(x - 1) + 2$ is the tangent line to $y = x^3 + 1$ at the point (1, 2).

88. Repeat exercise 87 for $f(x) = x^3 + 2x$ and the point (2, 12).

EXPLORATORY EXERCISES

1. Knowing where to aim a ball is an important skill in many sports. If the ball doesn't follow a straight path (because of gravity or other factors), aiming can be a difficult task. When throwing a baseball, for example, the player must take gravity into account and aim higher than the target. Ignoring air resistance and any lateral movement, the motion of a thrown ball may be approximated by $y = -\dfrac{16}{v^2 \cos^2 \theta}x^2 + (\tan \theta)x$. Here, the ball is thrown from the position (0, 0) with initial speed v ft/s at angle θ from the horizontal.



Given such a curve, we can compute the slope of the tangent line at $x = 0$, but how can we compute the proper angle θ? Show that if m is the slope of the tangent line at $x = 0$, then $\tan \theta = m$. (Hint: Draw a triangle using the tangent line and x-axis and recall that slope is rise over run.) Tangent is a good name, isn't it? Now, for some baseball problems. We will look at how high players need to aim to make throws that are easy to catch. Throwing height is also a good catching height. If L (ft) is the length of the throw and we want the ball to arrive at the same height as it is released (as shown in the figure), the parabola can be determined from the following relationship between angle and velocity: $\sin 2\theta = 32L/v^2$. A third baseman throwing at 130 ft/s (about 90 mph) must throw 120 ft to reach first base. Find the angle of release (substitute L and v and, by trial and error, find a value of θ that works), the slope of the tangent line and the height at which the third baseman must aim (that is, the height at which the ball would arrive if there were no gravity). How much does this change for a soft throw at 100 ft/s? How about for an outfielder's throw of 300 feet at 130 ft/s? Most baseball players would deny that they are aiming this high; what in their experience would make it difficult for them to believe the calculations?

2. We continue our explorations in the theory of mathematical chaos here. (Related exploratory exercises can be found in the review exercises of Chapter 1.) We have seen how to iterate functions and that such iterations may result in *attractors* consisting of a single number, two numbers alternating (a two-cycle), a four-cycle, other size cycles or chaos (bounded but not repeating). We now have the tools to determine which cycles attract and which do not. First, try iterating $\cos x$: start at any x_0 and compute $x_1 = \cos x_0$, $x_2 = \cos x_1$ and so on. Without knowing what your choice of x_0 is, we can predict that the iterations will converge to the mysterious-looking number 0.739085133215. From our previous work, you should be able to explain where this number comes from: it is the only intersection of $y = \cos x$ and $y = x$. Similarly, iterates of $\sin x$ will converge to 0, the only solution of the equation $\sin x = x$. You might hypothesize that iterates of $f(x)$ always converge to a solution of the equation $f(x) = x$. However, iterates for

Review Exercises

most starting points of $f(x) = x(3.2 - x)$ converge to a two-cycle (approximately 2.5582 and 1.6417) and not to either solution of the equation $x(3.2 - x) = x$. Those **fixed points,** as they're called, are $x = 0$ and $x = 2.2$. They are called **repelling fixed points:** pick x_0 very close to one of them and the iterates will get "repelled" away from the fixed points. For example, if $x_0 = 2.22$, then $x_1 = 2.1756$, $x_2 = 2.22868$, $x_3 = 2.1648$, $x_4 = 2.24105$ and so on, with succeeding iterates getting farther and farther from 2.2 (and closer to the two-cycle 2.5582 and 1.6417). A simple rule tells whether a fixed point will be attracting or repelling. If a is a *fixed point* of $f(x)$ (that is, a is a solution of the equation $f(x) = x$) and $|f'(a)| < 1$, then a is attracting. If $|f'(a)| > 1$, then a is repelling. Verify that 0.739085133215 is an attracting fixed point of $\cos x$, that 0 is an attracting fixed point of $\sin x$, that 0 is a repelling fixed point of $x(3.2 - x)$ and that 2.2 is a repelling fixed point of $x(3.2 - x)$.

3. Based on exercise 2, we can find a **bifurcation point** for the family of functions $f_c(x) = x(c - x)$. Show that $x = 0$ and $x = c - 1$ are the fixed points of $f_c(x)$. If $-1 < c < 1$, show that 0 attracts and $c - 1$ repels. Illustrate this by computing iterates of $f_0(x)$ and $f_{1/2}(x)$ starting at $x_0 = 0.2$. If $1 < c < 3$, show that 0 repels and $c - 1$ attracts. Illustrate this by computing iterates of $f_2(x)$ and $f_{2.8}(x)$ starting at $x_0 = 0.5$. If $c > 3$, show that both 0 and $c - 1$ repel. Illustrate this by computing iterates of $f_{3.1}(x)$ and $f_{3.2}(x)$ starting at $x_0 = 0.5$. We call $c = 3$ a bifurcation point because of the abrupt change of attractor from a one-cycle for $c < 3$ to a two-cycle for $c > 3$. Many diseases, such as some forms of heart disease, epilepsy and Parkinson's disease, are now viewed by some researchers as bifurcations from healthy behavior to unhealthy behavior.

4. In exercise 2, we learned a simple rule for determining whether a fixed point is attracting or repelling. If there are no attracting fixed points, the iterates may converge to a two-cycle (two numbers alternating) or some other pattern. We can do analysis on the two-cycles similar to our work on the fixed points. Numbers a and b are a two-cycle if $f(a) = b$ and $f(b) = a$; both numbers are solutions of the equation $f(f(x)) = x$. For $f(x) = x^2 - 1$, set up and find all four solutions of the equation $f(f(x)) = x$. Show that 0 and -1 form a two-cycle; what do the other two solutions represent? Compute iterates starting at $x_0 = 0.2$. Does it appear that the two-cycle attracts or repels? The test for attracting or repelling two-cycles is similar to the test for fixed points, except applied to the function $g(x) = f(f(x))$. If a is part of a two-cycle and $|g'(a)| < 1$, then the two-cycle is attracting; if $|g'(a)| > 1$, the two-cycle is repelling. Show that if a and b are a two-cycle, then $g'(a) = f'(f(a))f'(a) = f'(b)f'(a)$. Using this rule, show that 0 and -1 form an attracting two-cycle. Explore the two-cycle for $x^2 - 3$.