Second-Order Differential Equations

CHAPTER 15

In a classic television commercial from years ago, the great jazz vocalist Ella Fitzgerald broke a wineglass by singing a particular high-pitched note. The phenomenon that makes this possible is called *resonance,* which is one of the topics in this chapter. Resonance results from the fact that the crystalline structures of certain solids have natural frequencies of vibration. An external force of the same frequency will "resonate" with the object and create a huge increase in energy. For instance, if the frequency of a musical note matches the natural vibration of a crystal wineglass, the glass will vibrate with increasing amplitude until it shatters.

Resonance causes such a dramatic increase in energy that engineers pay special attention to its presence. Buildings are designed to eliminate the chance of destructive resonance. Electronic devices are built to limit some forms of resonance (for example, vibrations in a CD player) and utilize others (for example, stochastic resonance to amplify the desirable portions of a signal).

One commonly known instance of resonance is that caused by soldiers marching in step across a footbridge. Should the frequency of their steps match the natural frequency of the bridge, the resulting resonance can cause the bridge to start moving violently up and down. To avoid this, soldiers are instructed to march out of step when crossing a bridge.

A related incident is the Tacoma Narrows Bridge disaster of 1940. You have likely seen video clips of this large suspension bridge twisting and undulating more and more until it finally tears itself apart. This disaster was initially thought to be the result of resonance, but the cause has come under renewed scrutiny in recent years. While engineers are still not in complete agreement as to its cause, it has been shown that the bridge was not a victim of resonance, but failed due to some related design flaw. Some of the stability issues that had a role in this disaster will be explored in this chapter.

In this chapter, we extend our study of differential equations to those of second order. We develop the basic theory and explore a small number of important applications.

15.1 SECOND-ORDER EQUATIONS WITH CONSTANT COEFFICIENTS

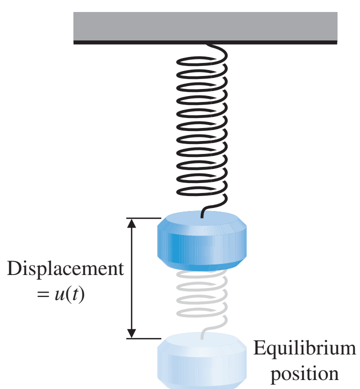


FIGURE 15.1
Spring-mass system

Today's sophisticated technology often requires very precise motion control to maintain acceptable performance. For instance, portable CD players must absorb bumps and twists without skipping. Similarly, a video camera should record a steady image even when the hand holding it is shaking. In this section and section 15.2, we begin to explore the mathematics behind such mechanical vibrations.

A simple version of this problem is easy to visualize. In Figure 15.1, we show a mass hanging from a spring that is suspended from the ceiling. We call the natural length of the spring l. Observe that hanging the mass from the spring will stretch the spring a distance Δl from its natural length. We measure the displacement $u(t)$ of the mass from this **equilibrium position.** Further, we consider downward (where the spring is stretched beyond its equilibrium position) to be a positive displacement and consider upward (where the spring is compressed from its equilibrium position) to be the negative direction. So, the mass in Figure 15.1 has been displaced from its natural length by a total of $u(t) + \Delta l$.

To describe the motion of a spring-mass system, we begin with Newton's second law of motion: $F = ma$. Note that there are three primary forces acting on the mass. First, gravity pulls the mass downward, with force mg. Next, the spring exerts a restoring force when it is stretched or compressed. If the spring is compressed to less than its natural length, the spring exerts a downward force. If the spring is stretched beyond its natural length, the spring exerts an upward force. So, the spring force has the opposite sign from the total displacement from its natural length. Hooke's law states that this force is proportional to the displacement from the spring's natural length. (That is, the more you stretch or compress the spring, the harder the spring resists.) Putting this together, the spring force is given by

$$\text{Spring force} = -k(u + \Delta l),$$

for some positive constant k (called the **spring constant**), determined by the stiffness of the spring. The third force acting on the mass is the **damping force** that resists the motion, due to friction such as air resistance. (A familiar device for adding damping to a mechanical system is the shock absorber in your car.) The damping force depends on velocity: the faster an object moves, the more damping there is. A simple model of the damping force is then

$$\text{Damping force} = -cv,$$

where $v = u'$ is the velocity of the mass and c is a positive constant.

Combining these three forces, Newton's second law gives the following:

$$mu''(t) = ma = F = mg - k[u(t) + \Delta l] - cu'(t)$$

or

$$mu''(t) + cu'(t) + ku(t) = mg - k\Delta l. \tag{1.1}$$

We can simplify this equation with a simple observation. If the mass is not in motion, then $u(t) = 0$, for all t. In this case, $u'(t) = u''(t) = 0$ for all t and equation (1.1) reduces to

$$0 = mg - k\Delta l.$$

REMARK 1.1

The spring constant k is given by

$$k = \frac{mg}{\Delta l}$$

(weight divided by displacement from natural length).

While we can use this to solve for the spring constant k in terms of the mass and Δl, this also simplifies (1.1) to

$$mu''(t) + cu'(t) + ku(t) = 0. \tag{1.2}$$

Equation (1.2) is a **second-order** differential equation, since it includes a second derivative. Solving equations such as (1.2) gives us insight into spring motion as well as many other diverse phenomena.

Before trying to solve this general equation, we first solve a few simple examples of second-order equations. The simplest second-order equation is $y''(t) = 0$. Integrating this once gives us

$$y'(t) = c_1,$$

for some constant c_1. Integrating again yields

$$y(t) = c_1 t + c_2,$$

where c_2 is another arbitrary constant. We refer to this as the **general solution** of the differential equation, meaning that *every* solution of the equation can be written in this form. It should not be surprising that the general solution of a second-order differential equation should involve two arbitrary constants, since it requires two integrations to undo the two derivatives. A slightly more complicated equation is

$$y'' - y = 0. \tag{1.3}$$

We can discover the solution of this, if we first rewrite the equation as

$$y'' = y.$$

Think about it this way: we are looking for a function whose second derivative is itself. One such function is $y = e^t$. It's not hard to see that a second solution is $y = e^{-t}$. It turns out that every possible solution of the equation can be written as a combination of these two solutions, so that the general solution of (1.3) is

General solution of $y'' = y$

$$\boxed{y = c_1 e^t + c_2 e^{-t},}$$

for constants c_1 and c_2.

More generally, we want to solve

$$ay''(t) + by'(t) + cy(t) = 0, \tag{1.4}$$

where a, b and c are constants. Notice that equation (1.4) is the same as equation (1.2), except for the name of the dependent variable. In a full course on differential equations, you will see that if you can find two solutions $y_1(t)$ and $y_2(t)$, neither of which is a constant multiple of the other, then *all solutions* can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

for some constants c_1 and c_2. The question remains as to how to find these two solutions. As we've already seen, the answer starts with making an educated guess.

Notice that equation (1.4) asks us to find a function whose first and second derivatives are similar enough that the combination $ay''(t) + by'(t) + cy(t)$ adds up to zero. As we already saw with equation (1.3), one candidate for such a function is the exponential function e^{rt}. So, we look for some (constant) value(s) of r for which $y = e^{rt}$ is a solution of (1.4).

Observe that if $y(t) = e^{rt}$, then $y'(t) = re^{rt}$ and $y''(t) = r^2 e^{rt}$. Substituting into (1.4), we get

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

and factoring out the common e^{rt}, we have

$$(ar^2 + br + c)e^{rt} = 0.$$

Since $e^{rt} > 0$, this can happen only if

$$ar^2 + br + c = 0. \tag{1.5}$$

Equation (1.5) is called the **characteristic equation,** whose solution(s) can be found by the quadratic formula to be

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

So, there are three possibilities for solutions of (1.5) : (1) r_1 and r_2 are distinct real solutions (if $b^2 - 4ac > 0$), (2) $r_1 = r_2$ is a (repeated) real solution (if $b^2 - 4ac = 0$) or (3) r_1 and r_2 are complex solutions (if $b^2 - 4ac < 0$). All three of these cases lead to different solutions of the differential equation (1.4), which we must consider separately.

Case 1: Distinct Real Roots

If r_1 and r_2 are distinct real solutions of (1.5), then $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are two solutions of (1.4) and $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the general solution of (1.4). We illustrate this in example 1.1.

EXAMPLE 1.1 Finding General Solutions

Find the general solution of (a) $y'' - y' - 6y = 0$ and (b) $y'' + 4y' - 2y = 0$.

Solution In each case, we solve the characteristic equation and interpret the solution(s).

For part (a), the characteristic equation is

$$0 = r^2 - r - 6 = (r - 3)(r + 2).$$

So, there are two distinct real solutions of the characteristic equation: $r_1 = 3$ and $r_2 = -2$. The general solution is then

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}.$$

For part (b), the characteristic equation is

$$0 = r^2 + 4r - 2.$$

Since the polynomial does not easily factor, we use the quadratic formula to get

$$r = \frac{-4 \pm \sqrt{16 + 8}}{2} = -2 \pm \sqrt{6}.$$

We again have two distinct real solutions and so, the general solution of the differential equation is

$$y(t) = c_1 e^{(-2+\sqrt{6})t} + c_2 e^{(-2-\sqrt{6})t}.$$

Case 2: Repeated Root

If $r_1 = r_2$ (repeated root of the characteristic equation), then we have found only one solution of (1.4): $y_1 = e^{r_1 t}$. We leave it as an exercise to show that a second solution in this case is $y_2 = te^{r_1 t}$. The general solution of (1.4) is then $y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$. We illustrate this case in example 1.2.

EXAMPLE 1.2 Finding General Solutions (Repeated Root)

Find the general solution of $y'' - 6y' + 9y = 0$.

Solution Here, the characteristic equation is

$$0 = r^2 - 6r + 9 = (r - 3)^2.$$

So, here we have the repeated root $r = 3$. The general solution is then

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}. \blacksquare$$

Case 3: Complex Roots

If r_1 and r_2 are complex roots of the characteristic equation, we can write these as $r_1 = u + vi$ and $r_2 = u - vi$, where i is the imaginary number $i = \sqrt{-1}$. The question is how to interpret a complex exponential like $e^{(u+vi)t}$. The answer lies with Euler's formula, which says that $e^{i\theta} = \cos\theta + i\sin\theta$. The solution corresponding to $r = u + vi$ is then

$$e^{(u+vi)t} = e^{ut+vti} = e^{ut}e^{vti} = e^{ut}(\cos vt + i\sin vt).$$

It can be shown that both the real and the imaginary parts of this solution (that is, both $y_1 = e^{ut}\cos vt$ and $y_2 = e^{ut}\sin vt$) are solutions of the differential equation. So, in this case, the general solution of (1.4) is

$$y(t) = c_1 e^{ut}\cos vt + c_2 e^{ut}\sin vt. \tag{1.6}$$

In example 1.3, we see how to use this solution.

EXAMPLE 1.3 Finding General Solutions (Complex Roots)

Find the general solution of the equations (a) $y'' + 2y' + 5y = 0$ and (b) $y'' + 4y = 0$.

Solution For part (a), the characteristic equation is

$$0 = r^2 + 2r + 5.$$

Since this does not factor, we use the quadratic formula to obtain

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

From (1.6) the general solution is

$$y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t.$$

For part (b), there is no y'-term and so, the characteristic equation is simply

$$r^2 + 4 = 0.$$

This gives us $r^2 = -4$, so that $r = \pm\sqrt{-4} = \pm 2i$. From (1.6) the general solution is then

$$y(t) = c_1\cos 2t + c_2\sin 2t. \blacksquare$$

We can now find the general solution of any equation of the form (1.4). Notice that the general solution of a second-order differential equation always involves two arbitrary constants. In order to determine the value of these constants, we specify two initial conditions, most often $y(0)$ and $y'(0)$ (corresponding to the initial position and initial velocity of the mass, in the case of a spring-mass system). A second-order differential equation plus two initial conditions is called an **initial value problem.** Example 1.4 illustrates how to apply these conditions to the general solution of a differential equation.

EXAMPLE 1.4 Solving an Initial Value Problem

Find the solution of the initial value problem $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = 0$.

Solution Here, the characteristic equation is

$$0 = r^2 + 4r + 3 = (r + 3)(r + 1),$$

so that $r = -3$ and $r = -1$. The general solution is then

$$y(t) = c_1 e^{-3t} + c_2 e^{-t},$$

so that

$$y'(t) = -3c_1 e^{-3t} - c_2 e^{-t}.$$

The two initial conditions then give us

$$2 = y(0) = c_1 + c_2 \tag{1.7}$$

and

$$0 = y'(0) = -3c_1 - c_2. \tag{1.8}$$

We solve the two equations (1.7) and (1.8) for c_1 and c_2, as follows. From (1.8), $c_2 = -3c_1$. Substituting this into (1.7) gives us

$$2 = c_1 + c_2 = c_1 - 3c_1 = -2c_1,$$

so that $c_1 = -1$. Then $c_2 = -3c_1 = 3$. The solution of the initial value problem is then

$$y(t) = -e^{-3t} + 3e^{-t}.$$

A graph of this solution is shown in Figure 15.2. ■

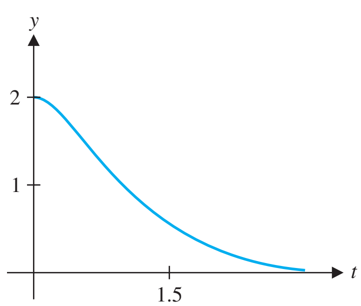


FIGURE 15.2
$y = -e^{-3t} + 3e^{-t}$

In example 1.5, the differential equation has no y'-term. Physically, this corresponds to the case of a spring-mass system with no damping.

EXAMPLE 1.5 Solving an Initial Value Problem

Find the solution of the initial value problem $y'' + 9y = 0$, $y(0) = 4$, $y'(0) = -6$.

Solution Here, the characteristic equation is $r^2 + 9 = 0$, so that $r^2 = -9$ and $r = \pm\sqrt{-9} = \pm 3i$. The general solution is then

$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$

and

$$y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t.$$

From the initial conditions, we now have

$$4 = y(0) = c_1$$

and

$$-6 = y'(0) = 3c_2.$$

So, $c_2 = -2$ and the solution of the initial value problem is

$$y(t) = 4\cos 3t - 2\sin 3t.$$

A graph is shown in Figure 15.3. ■

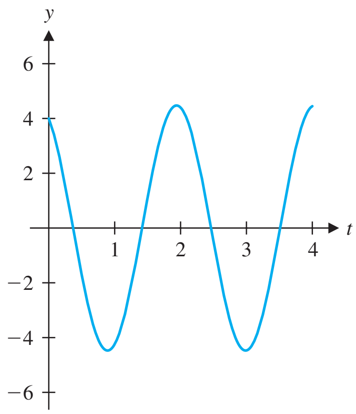


FIGURE 15.3
$y = 4\cos 3t - 2\sin 3t$

We now have the mathematical machinery needed to analyze a simple spring-mass system. In example 1.6, pay careful attention to the amount of work we do in setting up the differential equation. Remember: you can't get the right solution if you don't have the right equation!

EXAMPLE 1.6 Spring-Mass System with No Damping

A spring is stretched 6 inches by an 8-pound weight. The mass is then pulled down an additional 4 inches and released. Neglect damping. Find an equation for the position of the mass at any time t and graph the position function.

Solution The general equation describing the spring-mass system is $mu'' + cu' + ku = 0$. We need to identify the mass m, damping constant c and spring constant k. Since we are neglecting damping, we have $c = 0$. The mass m is related to the weight W by $W = mg$, where g is the gravitational constant. Since the weight is 8 pounds, we have $8 = m(32)$ or $m = \frac{8}{32} = \frac{1}{4}$ (the units of mass here are slugs). The spring constant k is determined from the equation $mg = k\Delta l$. Here, the mass stretches the spring 6 inches, which we must convert to $\frac{1}{2}$ foot. So, $\Delta l = \frac{1}{2}$ and $8 = k\left(\frac{1}{2}\right)$, leaving us with $k = 16$. The equation of motion is then

$$\frac{1}{4}u'' + 0u' + 16u = 0$$

or

$$u'' + 64u = 0.$$

Here, the characteristic equation is $r^2 + 64 = 0$, so that $r = \pm\sqrt{-64} = \pm 8i$ and the general solution is

$$u(t) = c_1\cos 8t + c_2\sin 8t. \tag{1.9}$$

To determine the values of c_1 and c_2, we need the initial values $u(0)$ and $u'(0)$. Read the problem carefully and notice that the spring is released after it is pulled down 4 inches. This says that the initial position is 4 inches or $\frac{1}{3}$ foot down (the positive direction) and so, $u(0) = \frac{1}{3}$. Further, since the weight is pulled down and simply released, its initial velocity is zero, $u'(0) = 0$. We then have the initial conditions

$$u(0) = \frac{1}{3} \quad \text{and} \quad u'(0) = 0.$$

From (1.9), we have $u'(t) = -8c_1\sin 8t + 8c_2\cos 8t$. The initial conditions now give us

$$\frac{1}{3} = u(0) = c_1(1) + c_2(0) = c_1$$

and

$$0 = u'(0) = -8c_1(0) + 8c_2(1) = 8c_2,$$

so that $c_1 = \frac{1}{3}$ and $c_2 = 0$. The solution of the initial value problem is now

$$u(t) = \frac{1}{3}\cos 8t.$$

The graph of this function in Figure 15.4 shows the smooth up and down motion of an undamped spring (called **simple harmonic motion**). ■

REMARK 1.2

In the English system of units, with pounds (weight), feet and seconds,

$$g \approx 32\ \text{ft/s}^2.$$

In the metric system with kg (mass), meters and seconds,

$$g \approx 9.8\ \text{m/s}^2.$$

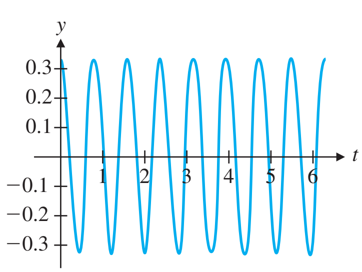


FIGURE 15.4
$u(t) = \frac{1}{3}\cos 8t$

For a real spring-mass system, there is always some damping, so that the idealized perpetual motion of example 1.6 must be modified somewhat. In example 1.7, again take note of the steps required to obtain the equation of motion.

EXAMPLE 1.7 Spring-Mass System with Damping

A spring is stretched 5 cm by a 2-kg mass. The mass is set in motion from its equilibrium position with an upward velocity of 2 m/s. The damping constant equals $c = 4$. Find an equation for the position of the mass at any time t and graph the position function.

Solution The equation of motion is $mu'' + cu' + ku = 0$, where the damping constant is $c = 4$ and the mass is $m = 2$ kg. With these units, $g \approx 9.8$ m/s^2, so that the weight is $W = mg = 2(9.8) = 19.6$. The displacement of the mass is 5 cm or 0.05 meter. The spring constant is then $k = \frac{W}{\Delta l} = \frac{19.6}{0.05} = 392$ and the equation of motion is

$$2u'' + 4u' + 392u = 0$$

or

$$u'' + 2u' + 196u = 0.$$

Here, the characteristic equation is $r^2 + 2r + 196 = 0$. From the quadratic formula, we have $r = \dfrac{-2 \pm \sqrt{4 - 784}}{2} = -1 \pm \sqrt{195}i$. The general solution is then

$$u(t) = c_1 e^{-t} \cos \sqrt{195}t + c_2 e^{-t} \sin \sqrt{195}t,$$

so that

$$u'(t) = -c_1 e^{-t} \cos \sqrt{195}t - c_1\sqrt{195}e^{-t} \sin \sqrt{195}t - c_2 e^{-t} \sin \sqrt{195}t$$
$$+ c_2\sqrt{195}e^{-t} \cos \sqrt{195}t.$$

Since the mass is set in motion from its equilibrium position, we have $u(0) = 0$ and since it's set in motion with an upward velocity of 2 m/s, we have $u'(0) = -2$. (Keep in mind that upward motion corresponds to negative displacement.) These initial conditions now give us

$$0 = u(0) = c_1$$

and

$$-2 = u'(0) = -c_1 + c_2\sqrt{195} = c_2\sqrt{195},$$

since $c_1 = 0$. So, $c_2 = \frac{-2}{\sqrt{195}}$ and the displacement of the mass at any given time is given by

$$u(t) = -\frac{2}{\sqrt{195}} e^{-t} \sin \sqrt{195}t.$$

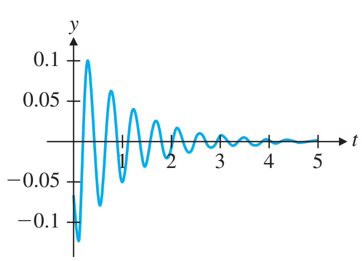


FIGURE 15.5
$u(t) = -\frac{2}{\sqrt{195}} e^{-t} \sin \sqrt{195}t$

The graph of this solution in Figure 15.5 shows a spring whose oscillations rapidly die out. ■

BEYOND FORMULAS

A major difference between solving second-order equations and solving first-order equations is that for second-order equations of the form $ay''(t) + by'(t) + cy(t) = 0$, you need to find two different solutions y_1 and y_2 (neither one of which is a constant multiple of the other). The form of these solutions depends on the type of the solutions to the characteristic equation, but in all cases the general solution is given by $c_1 y_1(t) + c_2 y_2(t)$, where the values of c_1 and c_2 are determined from two initial conditions.

EXERCISES 15.1

WRITING EXERCISES

1. Briefly discuss the role that theory plays in this section. In particular, if we didn't know that two different solutions were enough, would our method of guessing exponential solutions lead to solutions?

2. Briefly describe why our method of guessing exponential solutions would not work on equations with nonconstant coefficients. (You may want to work with a specific example like $y'' + 2ty' + 3y = 0$.)

3. It can be shown that e^{2t} and $2e^{2t}$ are both solutions of $y'' - 3y' + 2y = 0$. Explain why these can't be used as the two functions in the general solution. That is, you can't write all solutions in the form $c_1 e^{2t} + c_2 2e^{2t}$.

4. Discuss Figures 15.4 and 15.5 in physical terms. In particular, discuss the significance of the y-intercept and the increasing/decreasing properties of the graph in terms of the motion of the spring. Further, relate the motion of the spring to the forces acting on the spring.

In exercises 1–12, find the general solution of the differential equation.

1. $y'' - 2y' - 8y = 0$
2. $y'' - 2y' - 6y = 0$
3. $y'' - 4y' + 4y = 0$
4. $y'' + 2y' + 6y = 0$
5. $y'' - 2y' + 5y = 0$
6. $y'' + 6y' + 9y = 0$
7. $y'' - 2y' = 0$
8. $y'' - 6y = 0$
9. $y'' - 2y' - 6y = 0$
10. $y'' + y' + 3y = 0$
11. $y'' - \sqrt{5}y' + y = 0$
12. $y'' - \sqrt{3}y' + y = 0$

In exercises 13–20, solve the initial value problem.

13. $y'' + 4y = 0$, $y(0) = 2$, $y'(0) = -3$
14. $y'' + 2y' + 10y = 0$, $y(0) = 1$, $y'(0) = 0$
15. $y'' - 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$
16. $y'' + y' - 2y = 0$, $y(0) = 3$, $y'(0) = 0$
17. $y'' - 2y' + 5y = 0$, $y(0) = 2$, $y'(0) = 0$
18. $y'' - 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$
19. $y'' - 2y' + y = 0$, $y(0) = -1$, $y'(0) = 2$
20. $y'' + 3y' = 0$, $y(0) = 4$, $y'(0) = 0$

21. Show that $c_1 \cos kt + c_2 \sin kt = A \sin(kt + \delta)$, where $A = \sqrt{c_1^2 + c_2^2}$ and $\tan \delta = \dfrac{c_1}{c_2}$. We call A the **amplitude** and δ the **phase shift.** Use this identity to find the amplitude and phase shift of the solution of $y'' + 9y = 0$, $y(0) = 3$ and $y'(0) = -6$.

In exercises 22–24, solve the initial value problem and use the result of exercise 21 to find the amplitude and phase shift of the solution.

22. $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = -2$
23. $y'' + 20y = 0$, $y(0) = -2$, $y'(0) = 2$
24. $y'' + 12y = 0$, $y(0) = -1$, $y'(0) = -2$

25. A spring is stretched 6 inches by a 12-pound weight. The weight is then pulled down an additional 8 inches and released. Neglect damping. Find an equation for the position of the spring at any time t and graph the position function.

26. A spring is stretched 20 cm by a 4-kg mass. The weight is released with a downward velocity of 2 m/s. Neglect damping. Find an equation for the position of the spring at any time t and graph the position function.

27. A spring is stretched 10 cm by a 4-kg mass. The weight is pulled down an additional 20 cm and released with an upward velocity of 4 m/s. Neglect damping. Find an equation for the position of the spring at any time t and graph the position function. Find the amplitude and phase shift of the motion.

28. A spring is stretched 2 inches by a 6-pound weight. The weight is then pulled down an additional 4 inches and released with a downward velocity of 4 ft/s. Neglect damping. Find an equation for the position of the spring at any time t and graph the position function. Find the amplitude and phase shift of the motion.

29. A spring is stretched 4 inches by a 16-pound weight. The damping constant equals 10. The weight is then pushed up 6 inches and released. Find an equation for the position of the spring at any time t and graph the position function.

30. A spring is stretched 8 inches by a 32-pound weight. The damping constant equals 0.4. The weight is released with a downward velocity of 3 ft/s. Find an equation for the position of the spring at any time t and graph the position function.

31. A spring is stretched 25 cm by a 4-kg mass. The weight is pushed up $\frac{1}{2}$ meter and released. The damping constant equals $c = 2$. Find an equation for the position of the spring at any time t and graph the position function.

32. A spring is stretched 10 cm by a 5-kg mass. The weight is released with a downward velocity of 2 m/s. The damping constant equals $c = 5$. Find an equation for the position of the spring at any time t and graph the position function.

33. Show that in the case of a repeated root $r = r_1$ to the characteristic equation, the function $y = te^{r_1 t}$ is a second solution of the differential equation $ay'' + by' + cy = 0$.

34. Show that in the case of complex roots $r = u \pm vi$ to the characteristic equation, the functions $y = e^{ut} \cos vt$ and $y = e^{ut} \sin vt$ are solutions of the differential equation $ay'' + by' + cy = 0$.

35. For the equation $u'' + cu' + 16u = 0$, compare solutions with $c = 7$, $c = 8$ and $c = 9$. The first case is called **underdamped,** the second case is called **critically damped** and the last case is called **overdamped.** Briefly explain why these terms are appropriate.

36. For the general equation $mu'' + cu' + ku = 0$, show that critical damping occurs with $c = 2\sqrt{km}$. Without solving any equations, briefly describe what the graph of solutions look like with $c < 2\sqrt{km}$, compared to $c > 2\sqrt{km}$.

37. A spring is stretched 3 inches by a 16-pound weight. Use exercise 36 to find the critical damping value.

38. Show that for both the critically damped case and the overdamped case, the mass can pass through its equilibrium position at most once. (Hint: Show that $u(t) = 0$ has at most one solution.)

39. If you are designing a screen door, you can control the damping by changing the viscosity of the fluid in the cylinder in which the closure rod is embedded. Discuss whether overdamping or underdamping would be more appropriate.

40. Show that e^t and e^{-t} are solutions of the equation $y'' - y = 0$, and conclude that a general solution is given by $y = c_1 e^t + c_2 e^{-t}$. Then show that $\sinh t$ and $\cosh t$ are solutions of $y'' - y = 0$, and conclude that a general solution is given by $y = c_1 \sinh t + c_2 \cosh t$. Discuss whether or not these two general solutions are equivalent.

41. As in exercise 40, show that $y = c_1 \sinh at + c_2 \cosh at$ is a general solution of $y'' - a^2 y = 0$, for any constant $a > 0$. Compare this to the general solution of $y'' + a^2 y = 0$.

42. For the general equation $ay'' + by' + cy = 0$, if the roots of the characteristic equation are complex and $b > 0$, show that the solution $y(t) \to 0$ as $t \to \infty$.

43. For the general equation $ay'' + by' + cy = 0$, if $ac > 0$, $b > 0$ and the roots of the characteristic equation are real numbers $r_1 < r_2$, show that both roots are negative and thus, the solution $y(t) \to 0$ as $t \to \infty$.

44. For the general equation $ay'' + by' + cy = 0$, suppose that there is a repeated root $r_1 < 0$ of the characteristic equation. Show that $\lim_{t \to \infty} te^{r_1 t} = 0$ and thus, the solution $y(t) \to 0$ as $t \to \infty$.

45. Use the results of exercises 42–44 to show that if a, b and c are all positive, then the solution $y(t)$ of $ay'' + by' + cy = 0$ goes to 0 as $t \to \infty$.

46. Interpret the result of exercise 45 in terms of the spring equation $mu'' + cu' + ku = 0$. In particular, if there is nonzero damping, then what is the eventual motion of the spring?

EXPLORATORY EXERCISES

1. In this exercise, you will explore solutions of a different type of second-order equation. An **Euler equation** has the form $x^2 y'' + axy' + by = 0$ for constants a and b. Notice that this equation requires that x times the first derivative and x^2 times the second derivative be similar to the original function. Explain why a reasonable guess is $y = x^r$. Substitute this into the equation and (similar to our derivation of the characteristic equation) show that r must satisfy the equation

$$r^2 + (a - 1)r + b = 0.$$

Use this to find the general solution of (a) $x^2 y'' + 4xy' + 2y = 0$ and (b) $x^2 y'' - 3xy' + 3y = 0$. Discuss the main difference in the graphs of solutions to (a) and (b). Can you say anything definite about the graph of a solution of (c) $x^2 y'' + 2xy' - 6y = 0$ near $x = 0$? There remains the issue of what to do with complex and repeated roots. Show that if you get complex roots $r = u \pm vi$, then $y = x^u \cos(v \ln x)$ and $y = x^u \sin(v \ln x)$ are solutions for $x > 0$. Use this information to find the general solution of (d) $x^2 y'' + xy' + y = 0$. Use the form of the solutions corresponding to complex roots to guess the second solution in the repeated roots case. Find the general solution of (e) $x^2 y'' + 5xy' + 4y = 0$.

2. In this exercise, you will explore solutions of **higher-order** differential equations. For a third-order equation with constant coefficients such as (a) $y''' - 3y'' - y' + 3y = 0$, make a reasonable guess of the form of the solution, write down the characteristic equation and solve the equation (which factors). Use this idea to find the general solution of (b) $y''' + y'' + 3y' - 5y = 0$ and (c) $y''' - y'' - y' + y = 0$. Oddly enough, an equation like (d) $y''' - y = 0$ causes more problems than (a)–(c). How many solutions of the characteristic equation do you find? Show that $y = te^t$ is *not* a solution. Show that $y = e^{t/2} \cos \frac{\sqrt{3}}{2} t$ and $y = e^{t/2} \sin \frac{\sqrt{3}}{2} t$ are two additional solutions. Identify the two r-values to which these solutions correspond. Show that these r-values are in fact solutions of the characteristic equation. Conclude that a more thorough understanding of solutions of complex equations is necessary to fully master third-order equations. To end on a more positive note, find the general solutions of the fourth-order equation (e) $y^{(4)} - y = 0$ and the fifth-order equation (f) $y^{(5)} - 3y^{(4)} - 5y''' + 15y'' + 4y' - 12y = 0$.

15.2 NONHOMOGENEOUS EQUATIONS: UNDETERMINED COEFFICIENTS

Imagine yourself trying to videotape an important event. You might be more concerned with keeping a steady hand than with understanding the mathematics of motion control, but mathematics plays a vital role in helping you produce a professional-looking tape. In section 15.1, we modeled mechanical vibrations when the motion is started by an initial displacement or velocity. In this section, we extend that model to cases where an external force such as a shaky hand continues to affect the system.

The starting place for our model again is Newton's second law of motion: $F = ma$. We now add an external force to the spring force and damping force considered in section 15.1. If the external force is $F(t)$ and $u(t)$ gives the displacement from equilibrium, as defined before, we have

$$mu''(t) = -ku(t) - cu'(t) + F(t)$$

or

$$mu''(t) + cu'(t) + ku(t) = F(t). \tag{2.1}$$

The only change from the spring model in section 15.1 is that the right-hand side of the equation is no longer zero. Equations of the form (2.1) with zero on the right-hand side are called **homogeneous.** In the case where $F(t) \neq 0$, we call the equation **nonhomogeneous.**

Our goal is to find the general solution of such equations (that is, the form of all solutions). We can do this by first finding one **particular solution** $u_p(t)$ of the nonhomogeneous equation (2.1). Notice that if $u(t)$ is any other solution of (2.1), then we have that

$$\begin{aligned} m(u-u_p)'' + c(u-u_p)' + k(u-u_p) &= (mu'' + cu' + ku) - (mu_p'' + cu_p' + ku_p) \\ &= F(t) - F(t) = 0. \end{aligned}$$

That is, the function $u - u_p$ is a solution of the homogeneous equation $mu'' + cu' + ku = 0$ solved in section 15.1. So, if the general solution of the homogeneous equation is $c_1u_1 + c_2u_2$, then $u - u_p = c_1u_1 + c_2u_2$, for some constants c_1 and c_2 and

$$u = c_1u_1 + c_2u_2 + u_p.$$

We summarize this in Theorem 2.1.

> **TODAY IN MATHEMATICS**
>
> **Shigefumi Mori (1951–)**
> A Japanese mathematician who earned the Fields Medal in 1990. A colleague wrote, "The most profound and exciting development in algebraic geometry during the last decade or so was the Minimal Model Program or Mori's Program. . . . Shigefumi Mori initiated the program with a decisively new and powerful technique, guided the general research direction with some good collaborators along the way and finally finished up the program by himself overcoming the last difficulty. . . . Mori's theorems were stunning and beautiful by the totally new features unimaginable by those who had been working, probably very hard too, only in the traditional world of algebraic . . . surfaces."

THEOREM 2.1

Let $u = c_1u_1 + c_2u_2$ be the general solution of $mu'' + cu' + ku = 0$ and let u_p be any solution of $mu'' + cu' + ku = F(t)$. Then the general solution of $mu'' + cu' + ku = F(t)$ is given by

$$u = c_1u_1 + c_2u_2 + u_p.$$

We illustrate this result with example 2.1.

EXAMPLE 2.1 Solving a Nonhomogeneous Equation

Find the general solution of $u'' + 4u' + 3u = 30e^{2t}$, given that $u_p = 2e^{2t}$ is a solution.

Solution One of the two pieces of the general solution is given to us: we have $u_p = 2e^{2t}$. The other piece is the solution of the homogeneous equation

$u'' + 4u' + 3u = 0$. Here, the characteristic equation is

$$0 = r^2 + 4r + 3 = (r + 3)(r + 1),$$

so that $r = -3$ or $r = -1$. The general solution of the homogeneous equation is then $c_1 e^{-3t} + c_2 e^{-t}$, so that the general solution of the nonhomogeneous equation is

$$u(t) = c_1 e^{-3t} + c_2 e^{-t} + 2e^{2t}.$$

While Theorem 2.1 shows us how to piece together the solution of a nonhomogeneous equation from a particular solution and the general solution of the corresponding homogeneous equation, we still do not know how to find a particular solution. The method presented here, called the **method of undetermined coefficients,** works for equations with constant coefficients and where the nonhomogeneous term is not too complicated. The method relies on our ability to make an educated guess about the form of a particular solution. We begin by illustrateing this technique for example 2.1.

If $u'' + 4u' + 3u = 30e^{2t}$, then the most likely candidate for the form of $u(t)$ is a constant multiple of e^{2t}. (How else would u'', $4u'$ and $3u$ all add up to $30e^{2t}$?) Be sure that you understand the logic here, because we will be using it in the examples to come. So, an educated guess is $u_p(t) = Ae^{2t}$, for some constant A. Substituting this into the differential equation, we try to solve for A. (If it turns out to be impossible to solve for A, then we have simply made a bad guess.) Here, we have $u_p' = 2Ae^{2t}$ and $u_p'' = 4Ae^{2t}$ and so, requiring u_p to be a solution of the nonhomogeneous equation gives as

$$\begin{aligned} 30e^{2t} &= u_p'' + 4u_p' + 3u_p \\ &= 4Ae^{2t} + 4(2Ae^{2t}) + 3(Ae^{2t}) \\ &= 15Ae^{2t}. \end{aligned}$$

So, $15A = 30$ or $A = 2$. A particular solution is then $u_p(t) = 2e^{2t}$, as desired.

We learn more about making good guesses in examples 2.2 through 2.4.

EXAMPLE 2.2 Solving a Nonhomogeneous Equation

Find the general solution of $u'' + 2u' - 3u = -30 \sin 3t$.

Solution First, we solve the corresponding homogeneous equation: $u'' + 2u' - 3u = 0$. The characteristic equation here is

$$0 = r^2 + 2r - 3 = (r + 3)(r - 1),$$

so that $r = -3$ or $r = 1$. This gives us $u = c_1 e^{-3t} + c_2 e^t$ as the general solution of the homogeneous equation. Next, we guess the form of a particular solution. Since the right-hand side is a constant multiple of $\sin 3t$, a reasonable guess might seem to be $u_p = A \sin 3t$. However, it turns out that this is too specific a guess, since when we compute derivatives to substitute into the equation, we will also get $\cos 3t$ terms. This suggests the slightly more general guess $u_p = A \sin 3t + B \cos 3t$. Substituting this into the equation, we get

$$\begin{aligned} -30 \sin 3t &= -9A \sin 3t - 9B \cos 3t + 2(3A \cos 3t - 3B \sin 3t) \\ &\quad - 3(A \sin 3t + B \cos 3t) \\ &= (-12A - 6B) \sin 3t + (6A - 12B) \cos 3t. \end{aligned}$$

$u_p = A \sin 3t + B \cos 3t$
$u_p' = 3A \cos 3t - 3B \sin 3t$
$u_p'' = -9A \sin 3t - 9B \cos 3t$

Equating the corresponding coefficients of the sine and cosine terms (imagine the additional term $0 \cos 3t$ on the left-hand side), we have

$$-12A - 6B = -30$$

or

$$2A + B = 5$$

and

$$6A - 12B = 0$$

or

$$A = 2B.$$

Substituting into the first equation, we get $2(2B) + B = 5$ or $B = 1$. Then, $A = 2B = 2$ and the particular solution is $u_p = 2\sin 3t + \cos 3t$. We now put together all of the pieces to obtain the general solution of the original equation:

$$u(t) = c_1 e^{-3t} + c_2 e^t + 2\sin 3t + \cos 3t. \blacksquare$$

Observe that while the calculations get a bit messy, the process of making a guess is not exceptionally challenging. To keep the details of calculation from getting in the way of the ideas, we next focus on making good guesses. In general, you start with the function $F(t)$ and then add terms corresponding to each derivative. For instance, in example 2.2, we started with $A \sin 3t$ and then added a term corresponding to its derivative: $B \cos 3t$. We do not need to add other terms, because all other derivatives are simply constant multiples of either $\sin 3t$ or $\cos 3t$. However, suppose that $F(t) = 7t^5$. The initial guess would include At^5 and the derivatives Bt^4, Ct^3 and so on. To save letters, you can use subscripts and write the initial guess as

$$A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0.$$

There is one exception to the preceding rule. If any term in the initial guess is also a solution of the homogeneous equation, you must multiply the initial guess by a sufficiently high power of t so that nothing in the modified guess is a solution of the homogeneous equation. (In the case of second-order equations, this means multiplying the initial guess by either t or t^2.) To see why, consider $u'' + 2u' - 3u = 4e^{-3t}$. The initial guess Ae^{-3t} won't work, as seen in example 2.3.

EXAMPLE 2.3 Modifying an Initial Guess

Show that $u(t) = Ae^{-3t}$ is not a solution of $u'' + 2u' - 3u = 4e^{-3t}$, for any value of A, but that there is a solution of the form $u(t) = Ate^{-3t}$.

Solution For $u(t) = Ae^{-3t}$, we have $u'(t) = -3Ae^{-3t}$ and $u''(t) = 9Ae^{-3t}$ and so,

$$u'' + 2u' - 3u = 9Ae^{-3t} + 2(-3Ae^{-3t}) - 3Ae^{-3t} = 0.$$

That is, $u(t) = Ae^{-3t}$ is a solution of the homogeneous equation for every choice of A. As a result, Ae^{-3t} is not a solution of the nonhomogeneous equation for any choice of A. However, if we multiply our initial guess by t, we have $u(t) = Ate^{-3t}$, $u'(t) = Ae^{-3t} + At(-3e^{-3t})$ and $u''(t) = -3Ae^{-3t} - 3Ae^{-3t} + At(9e^{-3t}) = -6Ae^{-3t} + 9Ate^{-3t}$. Substituting into the equation, we have

$$\begin{aligned} 4e^{-3t} &= u'' + 2u' - 3u = -6Ae^{-3t} + 9Ate^{-3t} + 2(Ae^{-3t} - 3Ate^{-3t}) - 3Ate^{-3t} \\ &= -4Ae^{-3t}. \end{aligned}$$

So, $-4A = 4$ and $A = -1$. A particular solution of the nonhomogeneous equation is then $u_p(t) = -te^{-3t}$. $\blacksquare$

A summary of rules for making good guesses is given in the accompanying table. Notice that sine and cosine terms always go together, and all polynomials are complete with terms from t^n all the way down to t and a constant.

The form of u_p for $au'' + bu' + cu = F(t)$

$F(t)$	*Initial Guess*	*Modify Initial Guess if* $ar^2 + br + c = 0$ *for*
$e^{r_1 t}$	$Ae^{r_1 t}$	$r = r_1$
$\cos kt$ or $\sin kt$	$A\cos kt + B\sin kt$	$r = \pm ki$
t^n	$C_n t^n + C_{n-1}t^{n-1} + \cdots + C_1 t + C_0$	$r = 0$
$e^{ut}\cos vt$ or $e^{ut}\sin vt$	$e^{ut}(A\cos vt + B\sin vt)$	$r = u \pm vi$
$t^n e^{r_1 t}$	$(C_n t^n + C_{n-1}t^{n-1} + \cdots + C_1 t + C_0)e^{r_1 t}$	$r = r_1$

We illustrate the process of making good guesses in example 2.4.

EXAMPLE 2.4 Finding the Form of Particular Solutions

Determine the form for a particular solution of the following equations:
(a) $y'' + 4y' = t^4 + 3t^2 + 2e^{-4t}\sin t + 3e^{-4t}$ and (b) $y'' + 4y = 3t^2 \sin 2t + 3te^{2t}$.

Solution For part (a), the characteristic equation is $0 = r^2 + 4r = r(r+4)$, so that $r = 0$ and $r = -4$. The solution of the homogeneous equation is then $y = c_1 + c_2 e^{-4t}$. Looking at the right-hand side, we have a sum of three types of terms: a polynomial, an exponential/sine combination and an exponential. From the table, our initial guess is

$$y_p = (C_4t^4 + C_3t^3 + C_2t^2 + C_1t + C_0) + e^{-4t}(A\cos t + B\sin t) + De^{-4t}.$$

However, referring back to the solution of the homogeneous equation, note that the constant C_0 and the exponential De^{-4t} are solutions of the homogeneous equation. (Also note that the exponential/sine term is not a solution of the homogeneous equation.) Multiplying the first and third terms by t, the correct form of a solution to the nonhomogeneous equation is

$$y_p = t(C_4t^4 + C_3t^3 + C_2t^2 + C_1t + C_0) + e^{-4t}(A\cos t + B\sin t) + Dte^{-4t}.$$

Now that we have the correct form of a solution, it's a straightforward (though tedious) matter to find the value of all the constants.

For part (b), the characteristic equation is $r^2 + 4 = 0$, so that $r = \pm 2i$ and the solution of the homogeneous equation is then $y = c_1\cos 2t + c_2\sin 2t$. Here, the right-hand side consists of two terms: the product of a polynomial and a sine function and the product of a polynomial and an exponential function. Multiplying guesses from the table, we make our initial guess

$$y_p = (A_2t^2 + A_1t + A_0)\sin 2t + (B_2t^2 + B_1t + B_0)\cos 2t + (C_1t + C_0)e^{2t}.$$

Observe that both $A_0\sin 2t$ and $B_0\cos 2t$ are solutions of the homogeneous equation. So, we must multiply the first two terms by t to obtain the modified guess:

$$y_p = t(A_2t^2 + A_1t + A_0)\sin 2t + t(B_2t^2 + B_1t + B_0)\cos 2t + (C_1t + C_0)e^{2t}.$$

NOTES

The letters used in writing the forms of the solutions are completely arbitrary.

We now return to the study of mechanical vibrations. Recall that the movement of a spring-mass system with an external force $F(t)$ is modeled by $mu'' + cu' + ku = F(t)$.

EXAMPLE 2.5 The Motion of a Spring Subject to an External Force

A mass of 0.2 kg stretches a spring by 10 cm. The damping constant is $c = 0.4$. External vibrations create a force of $F(t) = 0.2 \sin 4t$ newtons, setting the spring in motion from its equilibrium position. Find an equation for the position of the spring at any time t.

Solution We are given $m = 0.2$ and $c = 0.4$. Recall that the spring constant k satisfies the equation $mg = k\Delta l$, where $\Delta l = 10\,\text{cm} = 0.1$ m. (Notice that since the mass is given in kg, $g = 9.8\ \text{m/s}^2$ and we need the value of Δl in meters.) This enables us to solve for k, as follows:

$$k = \frac{mg}{\Delta l} = \frac{(0.2)(9.8)}{0.1} = 19.6.$$

The equation of motion is then

$$0.2u'' + 0.4u' + 19.6u = 0.2 \sin 4t$$

or

$$u'' + 2u' + 98u = \sin 4t.$$

This gives us the characteristic equation $r^2 + 2r + 98 = 0$, which has solutions $r = \dfrac{-2 \pm \sqrt{4 - 392}}{2} = -1 \pm \sqrt{97}i$. The solution of the homogeneous equation is then $u_0(t)\, c_1 e^{-t} \cos \sqrt{97}t + c_2 e^{-t} \sin \sqrt{97}t$. A particular solution has the form $u_p = A \sin 4t + B \cos 4t$. Substituting this into the equation, we get

$$\begin{aligned}
\sin 4t &= u'' + 2u' + 98u \\
&= -16A \sin 4t - 16B \cos 4t + 2(4A \cos 4t - 4B \sin 4t) \\
&\quad + 98(A \sin 4t + B \cos 4t) \\
&= (82A - 8B) \sin 4t + (8A + 82B) \cos 4t.
\end{aligned}$$

Then, $82A - 8B = 1$ and $8A + 82B = 0$. The solution is $A = \frac{41}{3394}$ and $B = \frac{-2}{1697}$ and so,

$$u(t) = c_1 e^{-t} \cos \sqrt{97}t + c_2 e^{-t} \sin \sqrt{97}t + \frac{41}{3394} \sin 4t - \frac{2}{1697} \cos 4t.$$

The initial conditions are $u(0) = 0$ and $u'(0) = 0$. With $t = 0$ and $u = 0$, we get $0 = c_1 - \frac{2}{1697}$ or $c_1 = \frac{2}{1697}$. Computing the derivative $u'(t)$ and substituting in $t = 0$ and $u' = 0$, we get $0 = -c_1 + \sqrt{97}c_2 + \frac{82}{1697}$ or $c_2 = \frac{-80}{1697\sqrt{97}}$. The general solution of the nonhomogeneous equation is then

$$\begin{aligned}
u(t) = {} & \frac{2}{1697} e^{-t} \cos \sqrt{97}t - \frac{80}{1697\sqrt{97}} e^{-t} \sin \sqrt{97}t \\
& + \frac{41}{3394} \sin 4t - \frac{2}{1697} \cos 4t.
\end{aligned}$$

A graph is shown in Figure 15.6. ■

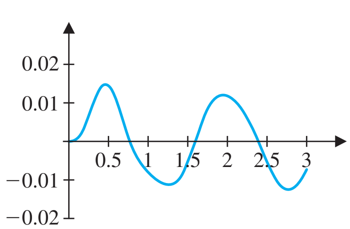


FIGURE 15.6
Spring motion with an external force

Notice in Figure 15.6 that after a very brief time, the motion appears to be simple harmonic motion. We can verify this by a quick analysis of our solution in example 2.5.

Recall that the solution comes in two pieces, the particular solution

$$u_p(t) = \frac{41}{3394}\sin 4t - \frac{2}{1697}\cos 4t$$

and the solution of the homogeneous equation

$$c_1 e^{-t}\cos\sqrt{97}t + c_2 e^{-t}\sin\sqrt{97}t.$$

As t increases, the presence of the exponential e^{-t} causes the homogeneous solution to approach 0, regardless of the value of the constants c_1 and c_2. So, for any initial conditions, the solution will eventually be dominated by the particular solution, which is a simple oscillation. For this reason, the solution of the homogeneous equation is called the **transient solution** and the particular solution is called the **steady-state solution.** This is true of many, but not all equations. (Can you think of cases where the homogeneous solution does not tend to 0 as t increases?) If we are interested only in the steady-state solution, we can avoid much of the work in example 2.5 and simply solve for the particular solution.

EXAMPLE 2.6 Finding a Steady-State Solution

For $u'' + 3u' + 2u = 20\cos 2t$, find the steady-state solution.

Solution For the homogeneous solution, the characteristic equation is $0 = r^2 + 3r + 2 = (r+1)(r+2)$ and so, the solutions are $r = -2$ and $r = -1$. The solution of the homogeneous equation is then $u(t) = c_1 e^{-2t} + c_2 e^{-t}$. Since this tends to 0 as $t \to \infty$, we ignore it. A particular solution has the form $u_p(t) = A\cos 2t + B\sin 2t$. Substituting into the equation, we have

$$\begin{aligned} 20\cos 2t &= u'' + 3u' + 2u \\ &= -4A\cos 2t - 4B\sin 2t + 3(-2A\sin 2t + 2B\cos 2t) \\ &\quad + 2(A\cos 2t + B\sin 2t) \\ &= (-2A + 6B)\cos 2t + (-6A - 2B)\sin 2t. \end{aligned}$$

So, we must have

$$-2A + 6B = 20$$

and

$$-6A - 2B = 0.$$

From the second equation, $B = -3A$. Substituting into the first equation, we have $-2A - 18A = 20$ or $A = -1$. Then, $B = -3A = 3$ and the steady-state solution is

$$u_p(t) = -\cos 2t + 3\sin 2t.$$

A graph is shown in Figure 15.7. ■

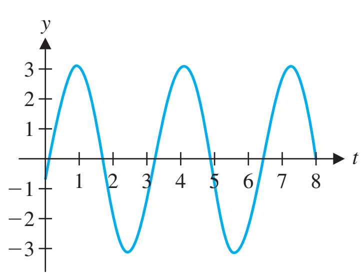


FIGURE 15.7
Steady-state solution

There are several possibilities for the steady-state motion of a mechanical system. Two interesting cases, called **resonance** and **beats,** are introduced here. In their pure forms, both occur only when there is no damping and the external force is a sine or cosine. In these cases, then, the equation of motion is

$$mu'' + ku = F(t).$$

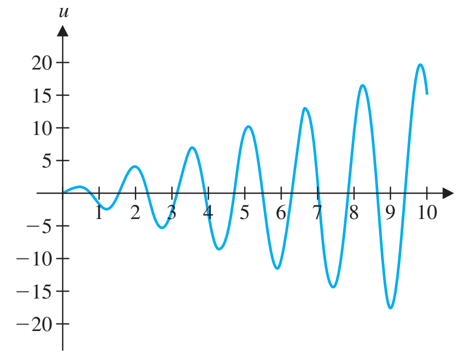


FIGURE 15.8
Resonance: $u = 2t \sin 4t$

The characteristic equation for the homogeneous equation is $mr^2 + k = 0$, which has solutions $r = \pm\sqrt{\frac{k}{m}}i$ and the solution of the homogeneous equation is

$$u(t) = c_1 \cos \omega t + c_2 \sin \omega t,$$

where $\omega = \sqrt{\frac{k}{m}}$ is called the **natural frequency** of the system.

Resonance occurs in a mechanical system when the external force is a sine or cosine whose frequency exactly matches the natural frequency of the system. For example, suppose that $F(t) = \sin \omega t$. Then our initial guess $u_p(t) = A \sin \omega t + B \cos \omega t$ matches the homogeneous solution and must be modified to the guess $u_p(t) = t(A \sin \omega t + B \cos \omega t)$. The graph of such a function would oscillate, but the presence of the factor t would cause the oscillations to grow larger and larger without bound. The graph of $u = 2t \sin 4t$ in Figure 15.8 illustrates this behavior.

Physically, resonance can cause impressive disasters. A singer hitting a note (thus producing an external force) at exactly the natural frequency of a wineglass can shatter it. Soldiers marching in step across a bridge at exactly the natural frequency of the bridge can create large oscillations in the bridge that can cause it to collapse.

The phenomenon of beats occurs when the forcing frequency is close to (but not equal to) the natural frequency. For example, for $u'' + 4u = 2 \sin(2.1t)$ with $u(0) = u'(0) = 0$, the homogeneous solution is $u(t) = c_1 \sin 2t + c_2 \cos 2t$ and so, the forcing frequency of 2.1 is close to the natural frequency of 2. We leave it as an exercise to show that the solution is

$$u(t) = 5.1219 \sin(2t) - 4.878 \sin(2.1t)$$

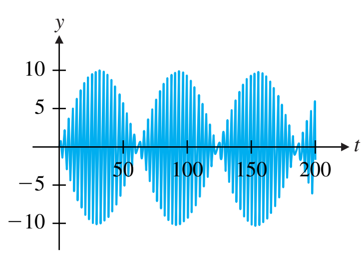


FIGURE 15.9
Beats

The graph in Figure 15.9 illustrates the beats phenomenon of periodically increasing and decreasing amplitudes.

This can be heard when tuning a piano. If a note is slightly off, its frequency is close to the frequency of the external tuning fork and you will hear the amplitude variation illustrated in Figure 15.9.

EXAMPLE 2.7 Resonance and Beats

For the system $u'' + 5u = 3 \sin \omega t$, find the natural frequency, the value of ω that produces resonance and a value of ω that produces beats.

Solution The characteristic equation for the homogeneous equation is $r^2 + 5 = 0$, with solutions $r = \pm\sqrt{5}i$. The natural frequency is then $\sqrt{5}$ and this is the value of ω that produces resonance. Values close to $\sqrt{5}$ (such as $\omega = 2$) produce beats. ■

BEYOND FORMULAS

Be sure that you understand the difference between the equations solved in section 15.2 and those solved in section 15.1. For the nonhomogeneous equations explored in this section, you must first find the homogeneous solution $c_1 y_1(t) + c_2 y_2(t)$ as in section 15.1 and then find a particular solution $y_p(t)$. Always keep in mind that the overall structure of the general solution of a nonhomogeneous equation is $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$. Your task is then to fill in the details one at a time.

EXERCISES 15.2

WRITING EXERCISES

1. In many cases, a guess for the form of a particular solution may seem logical but turn out to be a bad guess. Identify the criterion for whether a guess is ultimately good or bad. (See example 2.3.)
2. In example 2.4 part (a), the initial guess e^{-4t} is multiplied by t but $e^{-4t}\cos t$ is not. Explain why these terms are treated differently by comparing the r-values in a characteristic equation with solution e^{-4t} to one with solution $e^{-4t}\cos t$.
3. Soldiers are taught to break step when marching across a bridge. Briefly explain why this is a good idea.
4. Is there any danger to a party of people dancing on a strong balcony? Would it help if some of the people had a bad sense of rhythm?

In exercises 1–4, find the general solution of the equation given the particular solution.

1. $u'' + 2u' + 5u = 15e^{-2t}$, $u_p(t) = 3e^{-2t}$
2. $u'' + 2u' - 8u = 14e^{3t}$, $u_p(t) = 2e^{3t}$
3. $u'' + 4u' + 4u = 4t^2$, $u_p(t) = t^2 - 2t + \frac{3}{2}$
4. $u'' + 4u = 6\sin t$, $u_p(t) = 2\sin t$

In exercises 5–10, find the general solution of the equation.

5. $u'' + 2u' + 10u = 26e^{-3t}$
6. $u'' - 2u' + 5u = 10e^{2t}$
7. $u'' + 2u' + u = 25\sin t$
8. $u'' + 4u = 24\cos 4t$
9. $u'' - 4u = 2t^3$
10. $u'' + u' - 6u = 18t^2$

In exercises 11–18, determine the form of a particular solution of the equation.

11. $u'' + 2u' + 10u = 2e^{-t} + 3e^{-t}\cos 3t + 2\sin 3t$
12. $u'' - 2u' + 5u = e^t \sin 2t - t^2 e^t$
13. $u'' + 2u' = 5t^3 - 2t + 4e^{2t}$
14. $u'' + 4u = 2t\cos 2t - t^2 \sin t$
15. $u'' + 9u = e^t\cos 3t - 2t\sin 3t$
16. $u'' - 4u = t^3 e^{2t} + t^2 e^{-2t}$
17. $u'' + 4u' + 4u = t^2 e^{-2t} + 2te^{-2t}\sin t$
18. $u'' + 2u' + u = t^2 - 4 + 2e^{-t}$
19. A mass of 0.1 kg stretches a spring by 2 mm. The damping constant is $c = 0.2$. External vibrations create a force of $F(t) = 0.1\cos 4t$ newtons, setting the spring in motion from its equilibrium position with zero initial velocity. Find an equation for the position of the spring at any time t.
20. A mass of 0.4 kg stretches a spring by 2 mm. The damping constant is $c = 0.4$. External vibrations create a force of $F(t) = 0.8\sin 3t$ newtons, setting the spring in motion from its equilibrium position with zero initial velocity. Find an equation for the position of the spring at any time t.
21. A mass weighing 0.4 lb stretches a spring by 3 inches. The damping constant is $c = 0.4$. External vibrations create a force of $F(t) = 0.2e^{-t/2}$ lb. The spring is set in motion from its equilibrium position with a downward velocity of 1 ft/s. Find an equation for the position of the spring at any time t.
22. A mass weighing 0.1 lb stretches a spring by 2 inches. The damping constant is $c = 0.2$. External vibrations create a force of $F(t) = 0.2e^{-t/4}$ lb. The spring is set in motion by pulling it down 4 inches and releasing it. Find an equation for the position of the spring at any time t.

Exercises 23–28 refer to amplitude and phase shift. (See exercise 21 in section 15.1.)

23. For $u'' + 2u' + 6u = 15\cos 3t$, find the steady-state solution and identify its amplitude and phase shift.

24. For $u'' + 3u' + u = 5\sin 2t$, find the steady-state solution and identify its amplitude and phase shift.

25. For $u'' + 4u' + 8u = 15\cos t + 10\sin t$, find the steady-state solution and identify its amplitude and phase shift.

26. For $u'' + u' + 6u = 12\cos t + 8\sin t$, find the steady-state solution and identify its amplitude and phase shift.

27. A mass weighing 2 lb stretches a spring by 6 inches. The damping constant is $c = 0.4$. External vibrations create a force of $F(t) = 2\sin 2t$ lb. Find the steady-state solution and identify its amplitude and phase shift.

28. A mass of 0.5 kg stretches a spring by 20 cm. The damping constant is $c = 1$. External vibrations create a force of $F(t) = 3\cos 2t$ N. Find the steady-state solution and identify its amplitude and phase shift.

29. For the system $u'' + 3u = 4\sin \omega t$, find the natural frequency, the value of ω that produces resonance and a value of ω that produces beats.

30. For the system $u'' + 10u = 2\cos \omega t$, find the natural frequency, the value of ω that produces resonance and a value of ω that produces beats.

31. A mass weighing 0.4 lb stretches a spring by 3 inches. Ignore damping. External vibrations create a force of $F(t) = 2\sin \omega t$ lb. Find the natural frequency, the value of ω that produces resonance and a value of ω that produces beats.

32. A mass of 0.4 kg stretches a spring by 3 cm. Ignore damping. External vibrations create a force of $F(t) = 2\sin \omega t$ N. Find the natural frequency, the value of ω that produces resonance and a value of ω that produces beats.

33. In this exercise, we compare solutions where resonance is present and solutions of the same system with a small amount of damping. Start by finding the solution of $y'' + 9y = 12\cos 3t$, $y(0) = 1$, $y'(0) = 0$. Then solve the initial value problem $y'' + 0.1y' + 9y = 12\cos 3t$, $y(0) = 1$, $y'(0) = 0$. Graph both solutions on the same set of axes, and estimate a range of t-values for which the solutions stay close.

34. Repeat exercise 33 for $y'' + 0.01y' + 9y = 12\cos 3t$, $y(0) = 1$, $y'(0) = 0$.

35. For $u'' + 4u = \sin \omega t$, explain why the form of the particular solutions is simply $A\sin \omega t$, for $\omega^2 \neq 4$.

36. For $u'' + 4u = 2t^3$, identify a simplified form of the particular solution.

37. Find the solution of $u'' + 4u = 2\sin(2.1t)$ with $u(0) = u'(0) = 0$.

38. Find the solution of $u'' + 4u = 2\sin 2t$ with $u(0) = u'(0) = 0$. Compare the graphs of the solutions to exercises 37 and 38.

39. For $u'' + 4u = \sin \omega t$, $u(0) = u'(0) = 0$, find the solution as a function of ω. Compare the graphs of the solutions for $\omega = 0.5$, $\omega = 0.9$ and $\omega = 1$.

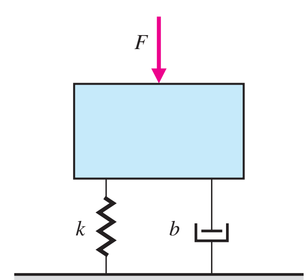
40. For $u'' + 4u = \sin \omega t$, $u(0) = u'(0) = 1$, find the solution as a function of ω. Compare the graphs of the solutions for $\omega = 0.5$, $\omega = 0.9$ and $\omega = 1$. Discuss the effects of the initial conditions by comparing the graphs in exercises 39 and 40.

41. For $u'' + 0.1u' + 4u = \sin \omega t$, find the amplitude of the steady-state solution as a function of ω.

42. For the spring problem in exercise 41, what happens to the steady-state amplitude as ω approaches 0? Explain why this makes sense.

EXPLORATORY EXERCISES

1. A washing machine whose tub spins with rotational speed ω generates a downward force of $f_0 \sin \omega t$ for some constant f_0. If the machine rests on a spring and damping mechanism (see diagram), the vertical motion of the machine satisfies the familiar equation $u'' + cu' + ku = f_0 \sin \omega t$. Explain what happens to the motion as c and k are increased. We now add one layer to the design problem for the machine. The forces absorbed by the spring and damper are transmitted to the floor. That is, $F(t) = cu' + ku$ is the force of the machine on the floor. We would like this to be small. Explain in physical terms why this force increases if c and k increase. So the design of the machine must balance vertical movement versus force transmitted to the floor. Consider the following argument for an equation for $F(t)$.



Machine schematic

Let the symbol D stand for derivative. Then we can write $F = cu' + ku = (cD + k)u$. Solving for u, we get $u = \dfrac{F}{cD + k}$. Now, writing the equation $u'' + cu' + ku = f_0 \sin \omega t$ as $(D^2 + cD + k)u = f_0 \sin \omega t$, we solve for u and get $u = \dfrac{f_0 \sin \omega t}{D^2 + cD + k}$. Setting the two expressions for u equal to each other, we have

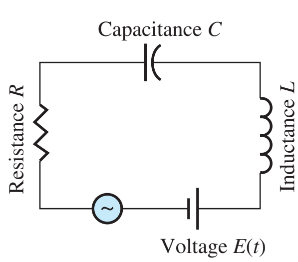
$$\frac{F}{cD + k} = \frac{f_0 \sin \omega t}{D^2 + cD + k}.$$

Multiplying this out, we have
$(D^2 + cD + k)F = (cD + k) f_0 \sin \omega t$.

Show that this gives the correct answer: that is, $F(t)$ satisfies the equation
$F'' + cF' + kF = c(f_0 \sin \omega t)' + k(f_0 \sin \omega t)$.

2. Spring devices are used in a variety of mechanisms, including the railroad car coupler shown in the photo. The coupler allows the railroad cars a certain amount of slack but applies a restoring force if the cars get too close or too far apart. If y measures the displacement of the coupler back and forth, then $y'' = F(y)$, where $F(y)$ is the force produced by the coupler. A simple model is $F(y) = \begin{cases} -y - d & \text{if } y \le -d \\ 0 & \text{if } -d \le y \le d \\ -y + d & \text{if } y \ge d \end{cases}$. This models a restoring force with a dead zone in the middle. Suppose the initial conditions are $y(0) = 0$ and $y'(0) = 1$. That is, the coupler is centered at $y = 0$ and has a positive velocity. At $y = 0$, the coupler is in the dead zone with no forces. Solve $y'' = 0$ with the initial conditions and show that $y(t) = t$ for $0 \le t \le d$. At this point, the coupler leaves the dead zone and we now have $y'' = -y + d$. Explain why initial conditions for this part of the solution are $y(d) = d$ and $y'(d) = 1$. Solve this problem and determine the time at which the coupler reenters the dead zone. Continue in this fashion to construct the solution piece by piece. Describe in words the pattern that emerges. Then, explain in which sense this model ignores damping. Revise the function $F(y)$ to include damping.

15.3 APPLICATIONS OF SECOND-ORDER EQUATIONS



In sections 15.1 and 15.2, we developed models of spring-mass systems with and without external forces. Surprisingly, the charge in a simple electrical circuit can be modeled with the same equation as for the motion of a spring-mass system. An *RLC*-circuit consists of resistors, capacitors, inductors and a voltage source. The net resistance R (measured in ohms), the capacitance C (in farads) and the inductance L (in henrys) are all positive. For now, we will assume that there is no impressed voltage. If $Q(t)$ (coulombs) is the total charge on the capacitor at time t and $I(t)$ is the current, then $I = Q'(t)$. The basic laws of electricity tell us that

the voltage drop across the resistors is IR,
the voltage drop across the capacitor is $\dfrac{Q}{C}$
and the voltage drop across the inductor is $LI'(t)$.

These voltage drops must sum to the impressed voltage. If there is none, then

$$LI'(t) + RI(t) + \frac{1}{C}Q(t) = 0$$

or

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0. \tag{3.1}$$

Observe that this is the same as equation (1.2), except for the names of the constants. Example 3.1 works the same as the examples from section 15.1.

EXAMPLE 3.1 Finding the Charge in an Electrical Circuit

A series circuit has an inductor of 0.2 henry, a resistor of 300 ohms and a capacitor of 10^{-5} farad. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge on the capacitor and the current at any time t.

Solution From (3.1), with $L = 0.2$, $R = 300$ and $C = 10^{-5}$, the equation for the charge is

$$0.2Q''(t) + 300Q'(t) + 100{,}000Q(t) = 0$$

or

$$Q''(t) + 1500Q'(t) + 500{,}000Q(t) = 0.$$

The characteristic equation is then

$$0 = r^2 + 1500r + 500{,}000 = (r + 500)(r + 1000),$$

so that the roots are $r = -500$ and $r = -1000$. The general solution is then

$$Q(t) = c_1 e^{-500t} + c_2 e^{-1000t}. \tag{3.2}$$

The initial conditions are $Q(0) = 10^{-6}$ and $Q'(0) = 0$ [since $Q'(t)$ gives the current]. This gives us

$$10^{-6} = Q(0) = c_1 + c_2$$

and

$$0 = Q'(0) = -500c_1 - 1000c_2,$$

from which we obtain $c_1 = -2c_2$. The first equation now gives us $c_1 = 2 \times 10^{-6}$. The charge function is then

$$Q(t) = 10^{-6}(2e^{-500t} - e^{-1000t}).$$

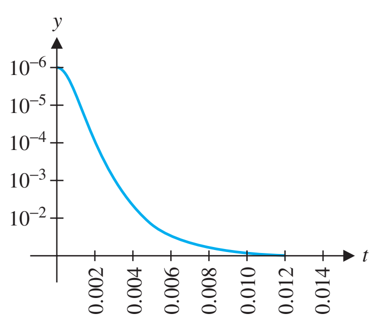


FIGURE 15.10
$Q(t) = 10^{-6}(2e^{-500t} - e^{-1000t})$

The graph in Figure 15.10 shows a rapidly declining charge. The current function is simply the derivative of the charge function. That is,

$$I(t) = -10^{-3}(e^{-500t} - e^{-1000t}). \blacksquare$$

If an impressed voltage $E(t)$ from a power supply is added to the circuit of example 3.1, equation (3.1) is replaced by the nonhomogeneous equation

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t). \tag{3.3}$$

Notice that here, the impressed voltage plays a role equivalent to the external force in a spring-mass system. We can use the techniques of section 15.2 to solve such an equation, as we illustrate in example 3.2.

EXAMPLE 3.2 Finding the Charge in a Circuit with an Impressed Voltage

Suppose that the circuit of example 3.1 is attached to an alternating current power supply with the impressed voltage $E(t) = 170\sin(120\pi t)$ volts. Find the steady-state charge on the capacitor and the steady-state current.

Solution From (3.3) using the values of example 3.1, we obtain the equation for the charge:

$$0.2Q''(t) + 300Q'(t) + 100{,}000Q(t) = 170\sin(120\pi t)$$

or
$$Q''(t) + 1500Q'(t) + 500{,}000Q(t) = 850\sin(120\pi t). \tag{3.4}$$

As in example 3.1, the roots of the characteristic equation are $r = -500$ and $r = -1000$, so that the solution of the homogeneous equation is $c_1e^{-500t} + c_2e^{-1000t}$. Since this part of the solution tends to 0 as t increases, the steady-state solution is simply the particular solution, which here has the form

$$Q_p(t) = A\sin(120\pi t) + B\cos(120\pi t).$$

This gives us

$$Q_p'(t) = 120\pi A\cos(120\pi t) - 120\pi B\sin(120\pi t)$$

and
$$Q_p''(t) = -14{,}400\pi^2 A\sin(120\pi t) - 14{,}400\pi^2 B\cos(120\pi t).$$

Substituting these into (3.4) gives us

$$\begin{aligned}850\sin(120\pi t) &= [-14{,}400\pi^2 A\sin(120\pi t) - 14{,}400\pi^2 B\cos(120\pi t)]\\ &\quad + 1500\,[120\pi A\cos(120\pi t) - 120\pi B\sin(120\pi t)]\\ &\quad + 500{,}000\,[A\sin(120\pi t) + B\cos(120\pi t)]\\ &= [(500{,}000 - 14{,}400\pi^2)A - 180{,}000\pi B]\sin(120\pi t)\\ &\quad + [(500{,}000 - 14{,}400\pi^2)B + 180{,}000A]\cos(120\pi t).\end{aligned}$$

Matching up the coefficients of $\sin(120\pi t)$ and $\cos(120\pi t)$ gives us

$$(500{,}000 - 14{,}400\pi^2)A - 180{,}000\pi B = 850$$

and
$$(500{,}000 - 14{,}400\pi^2)B + 180{,}000\pi A = 0.$$

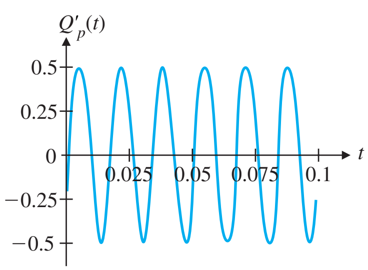


FIGURE 15.11
Steady-state current

Solving for A and B, we get the approximate values $A \approx 0.000679$ and $B \approx -0.00107$. The steady-state charge is then approximately

$$Q_p(t) \approx 0.000679\sin(120\pi t) - 0.00107\cos(120\pi t),$$

which gives us a steady-state current of

$$Q_p'(t) \approx 0.2561\cos(120\pi t) + 0.4046\sin(120\pi t).$$

We show a graph of this in Figure 15.11.

Notice that this is an alternating current with amplitude $\sqrt{(-0.2561)^2 + (0.4046)^2} \approx 0.4788$ and the same 60 hertz (cycles per second) as the power supply. The large resistance in this circuit has greatly reduced the current. ■

The properties of electrical circuits are sometimes summarized in a frequency response curve, as constructed in example 3.3. The numbers are simplified so that we can illustrate a basic principle behind radio reception. For convenience, we use the fact that the solution of the system of equations

$$\begin{aligned}c_1A + c_2B &= d_1\\ c_3A + c_4B &= d_2\end{aligned}$$

can be written in the form

$$A = \frac{c_4d_1 - c_2d_2}{c_1c_4 - c_2c_3} \quad\text{and}\quad B = \frac{c_1d_2 - c_3d_1}{c_1c_4 - c_2c_3}, \tag{3.5}$$

provided $c_1c_4 - c_2c_3 \neq 0$.

EXAMPLE 3.3 A Frequency Response Curve

For a circuit whose charge satisfies $u'' + 8u' + 2532u = \sin \omega t$, find the amplitude of the steady-state solution as a function f of the external frequency ω and plot the resulting **frequency response curve** $y = f(\omega)$. Explain why this circuit could be useful for tuning in a radio station.

Solution We leave it as an exercise to show that the solution of the homogeneous equation tends to 0 as t increases. The steady-state solution is then the particular solution $u_p(t) = A \sin \omega t + B \cos \omega t$. Here, we have

$$u_p'(t) = A\omega \cos \omega t - B\omega \sin \omega t$$

and

$$u_p''(t) = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t.$$

Substituting into the equation, we have

$$\begin{aligned} \sin \omega t &= (-A\omega^2 \sin \omega t - B\omega^2 \cos \omega t) + 8(A\omega \cos \omega t - B\omega \sin \omega t) \\ &\quad + 2532(A \sin \omega t + B \cos \omega t) \\ &= [(2532 - \omega^2)A - 8B\omega] \sin \omega t + [8A\omega + (2532 - \omega^2)B] \cos \omega t. \end{aligned}$$

Equating the coefficients of the sine and cosine terms gives us the system of equations

$$(2532 - \omega^2)A - 8\omega B = 1$$

and

$$8\omega A + (2532 - \omega^2)B = 0.$$

From (3.5), the solution is

$$A = \frac{2532 - \omega^2}{(2532 - \omega^2)^2 + 64\omega^2} \quad \text{and} \quad B = \frac{-8\omega}{(2532 - \omega^2)^2 + 64\omega^2}.$$

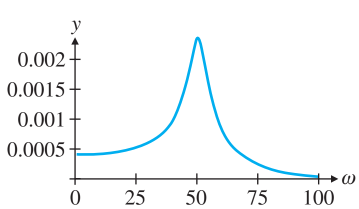


FIGURE 15.12
Frequency response curve $y = f(\omega)$

Without simplifying this, we can write the steady-state solution as $u_p(t) = A \sin \omega t + B \cos \omega t$ as $u_p(t) = \sqrt{A^2 + B^2} \sin(\omega t - \delta)$, for some constant δ, so that the amplitude of the steady-state solution is $\sqrt{A^2 + B^2}$. Notice that since A and B have the same denominator, it factors out of the square root and leaves us with

$$\begin{aligned} \sqrt{A^2 + B^2} &= \frac{1}{(2532 - \omega^2)^2 + 64\omega^2} \sqrt{(2532 - \omega^2)^2 + (-8\omega)^2} \\ &= \frac{1}{\sqrt{(2532 - \omega^2)^2 + 64\omega^2}}. \end{aligned}$$

The frequency response curve is the graph of this function, as shown in Figure 15.12. Notice that the graph has a sharp peak at about $\omega = 50$. Thinking of the right-hand side of the original equation, $\sin \omega t$, as a radio signal, we see that this circuit would "hear" the frequency $\omega = 50$ much better than any other frequency and could thus tune in on a radio station broadcasting at frequency 50. ■

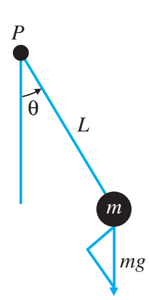


FIGURE 15.13
A simple pendulum

Another basic physical example with a surprising number of applications is the pendulum. In the sketch in Figure 15.13, a weight of mass m is attached to the end of a massless rod of length L that rotates about a pivot point P in two dimensions.

We first model the undamped pendulum, where the only force is due to gravity and the pendulum bob moves along a circular path centered at the pivot point. We can track its position s on the circle by measuring the angle θ from the vertical, where counterclockwise is positive. Since $s = L\theta$, the acceleration is $s'' = L\theta''$. The only force is gravity, which has magnitude mg in the downward direction. The component of gravity along the direction of motion is then $-mg\sin\theta$. Newton's second law of motion $F = ma$ gives us

$$mL\theta''(t) = -mg\sin\theta(t) \quad \text{or} \quad \theta''(t) + \frac{g}{L}\sin\theta(t) = 0. \tag{3.6}$$

Notice that (3.6) is *not* an equation of the form solved in sections 15.1 and 15.2, because of the term $\sin\theta(t)$. However, if we simplify (3.6) by replacing $\sin\theta(t)$ by $\theta(t)$, then (3.6) can be solved quite easily. This replacement is often justified with the statement. "For small angles θ, $\sin\theta$ is approximately equal to θ." As calculus students, you can say more. From the Maclaurin series

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots,$$

it follows that the approximation $\sin\theta \approx \theta$ has an error bounded by $|\theta|^3/6$. So, if $|\theta|^3/6$ is small enough to safely neglect, then you can replace (3.6) with

$$\theta''(t) + \frac{g}{L}\theta(t) = 0. \tag{3.7}$$

This equation is easy to solve, as we see in example 3.4.

EXAMPLE 3.4 The Undamped Pendulum

A pendulum of length 5 cm satisfies equation (3.7). The bob is released from rest from a starting angle $\theta = 0.2$. Find an equation for the position at any time t and find the amplitude and period of the motion.

Solution Taking $g = 9.8$ m/s^2, we convert the length to $L = 0.05$ m. Then (3.7) becomes

$$\theta''(t) + 196\,\theta(t) = 0.$$

The characteristic equation is then $r^2 + 196 = 0$, so that $r = \pm 14i$ and the general solution is

$$\theta(t) = c_1 \sin 14t + c_2 \cos 14t,$$

so that

$$\theta'(t) = 14c_1 \cos 14t - 14c_2 \sin 14t.$$

Since the bob is *released* from rest, it has no initial velocity and so, the initial conditions are $\theta(0) = 0.2$ and $\theta'(0) = 0$. From these, we have that $0.2 = \theta(0) = c_2$ and $0 = \theta'(0) = 14c_1$, so that $c_1 = 0$. The solution is then

$$\theta(t) = 0.2\cos 14t,$$

which has amplitude 0.2 and period $\frac{2\pi}{14} = \frac{\pi}{7}$. A graph of the solution is shown in Figure 15.14. ■

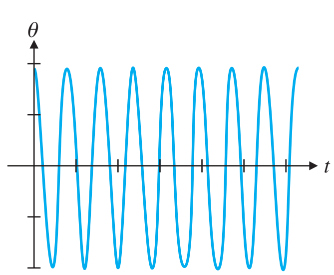


FIGURE 15.14
$\theta(t) = 0.2\cos 14t$

In the exercises, you will show that the period of any solution of (3.7) is $2\pi\sqrt{\frac{L}{g}}$, which gives an approximation of the period of the undamped pendulum. Notice that the period is independent of the mass but depends on the length L.

Observe that the pendulum of example 3.4 oscillates forever. Of course, the motion of a real pendulum dies out due to damping from friction at the pivot and air resistance. The simplest model of the force due to damping effects represents the damping as proportional to the velocity, or $k\theta'(t)$ for some constant k. Retaining the approximation $\sin\theta \approx \theta$ yields the following model for the damped pendulum:

$$\theta''(t) + k\theta'(t) + \frac{g}{L}\theta(t) = 0,$$

for some constant $k > 0$. If we further allow the pendulum to be driven by some external force $F(t)$, we have the more general model

$$\theta''(t) + k\theta'(t) + \frac{g}{L}\theta(t) = \frac{1}{m}F(t). \tag{3.8}$$

Several areas of current biological research involve situations where one periodic quantity serves as input into some other system that is naturally periodic. The effect of sunlight on circadian rhythms and the response of the heart to electrical signals from the sinoatrial node are examples of this phenomenon. In example 3.5, we explore what happens when a small amount of damping is present.

EXAMPLE 3.5 A Damped Forced Pendulum

For a pendulum of weight 2 pounds, length 6 inches, damping constant $k = 0.1$ and forcing function $F(t) = 0.5\sin 4t$, find the amplitude and period of the steady-state motion.

Solution Using $g = 32\text{ ft/s}^2$, we have $L = \frac{1}{2}$ ft and $m = \frac{2}{32}$ slug since weight $= mg$. Equation (3.8) then becomes

$$\theta''(t) + 0.1\theta'(t) + 64\,\theta(t) = 8\sin 4t.$$

We leave it as an exercise to show that the solution of the homogeneous equation approaches 0 as t increases. The steady-state solution is then the particular solution

$$\theta_p(t) = A\sin 4t + B\cos 4t.$$

This gives us

$$\theta_p'(t) = 4A\cos 4t - 4B\sin 4t$$

and

$$\theta_p''(t) = -16A\sin 4t - 16B\cos 4t.$$

Substituting into the differential equation, we have

$$\begin{aligned} 8\sin 4t &= (-16A\sin 4t - 16B\cos 4t) + 0.1(4A\cos 4t - 4B\sin 4t) \\ &\quad + 64\,(A\sin 4t + B\cos 4t) \\ &= (48A - 0.4B)\sin 4t + (0.4A + 48B)\cos 4t. \end{aligned}$$

It follows that

$$48A - 0.4B = 8 \quad \text{and} \quad 0.4A + 48B = 0.$$

The solution of this system is $A = \frac{384}{2304.16} \approx 0.166655$ and $B = -\frac{3.2}{2304.16} \approx -0.001389$. The steady-state solution can now be rewritten as

$$\theta_p(t) = A\sin 4t + B\cos 4t = \sqrt{A^2 + B^2}\sin(4t - \delta) \approx 0.16666\sin(4t - \delta),$$

so that the amplitude is approximately 0.16666 and the period is $\frac{2\pi}{4} = \frac{\pi}{2}$. ■

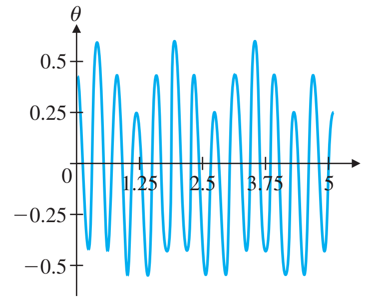


FIGURE 15.15a
$0 \leq t \leq 5$

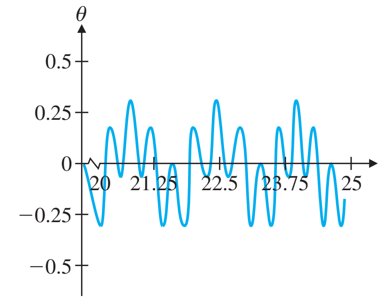


FIGURE 15.15b
$20 \leq t \leq 25$

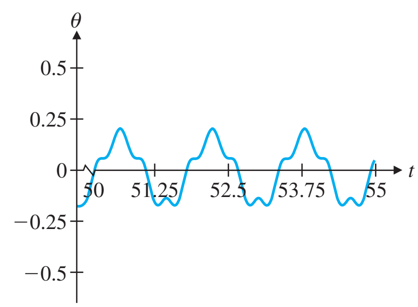


FIGURE 15.15c
$50 \leq t \leq 55$

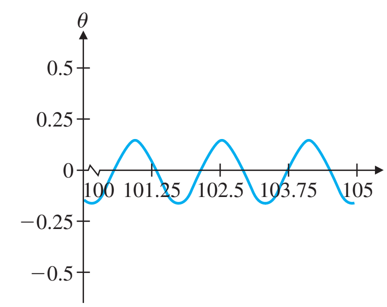


FIGURE 15.15d
$100 \leq t \leq 105$

Notice that the period of the steady-state solution in example 3.5 matches the period of the forcing function $8 \sin 4t$ and not the natural period of the undamped pendulum, $2\pi\sqrt{\frac{L}{g}} = \frac{\pi}{4}$. Keep in mind that the steady-state solution gives the behavior of the solution for very large t. For small t, the motion of the pendulum in example 3.5 can be erratic. For initial conditions $\theta(0) = 0.5$ and $\theta'(0) = 0$, the solution for $0 \leq t \leq 5$ is shown in Figure 15.15a, while Figures 15.15b–15.15d show the solutions for larger values of t. Notice that the solution seems to go through different stages until settling down to the steady-state solution around $t = 100$.

EXERCISES 15.3

WRITING EXERCISES

1. The correspondence between mechanical vibrations and electrical circuits is surprising. To start to understand the correspondence, develop an analogy between the roles of a resistor in a circuit and damping in spring motion. Continue by drawing an analogy between the roles of the spring force and the capacitor in storing and releasing energy.

2. In example 3.3, explain why the sharper the peak is on the frequency response curve, the clearer the radio reception would be.

3. For most objects, the magnitude of air drag is proportional to the *square* of the speed of the object. Explain why we would not want to use that assumption in equation (3.8).

4. To understand why the forced pendulum behaves erratically, consider the case where a child is on a swing and you push the swing. If the swing is coming back at you, does your push increase or decrease the child's speed? If the swing is moving forward away from you, does your push increase or decrease the child's speed? If you push every three seconds and the

swing is not on a three-second cycle, describe how your pushing would affect the movement of the swing.

---

1. A series circuit has an inductor of 0.4 henry, a resistor of 200 ohms and a capacitor of 10^{-4} farad. The initial charge on the capacitor is 10^{-5} coulomb and there is no initial current. Find the charge on the capacitor and the current at any time t.

2. A series circuit has an inductor of 0.4 henry, no resistance and a capacitor of 10^{-4} farad. The initial charge on the capacitor is 10^{-5} coulomb and there is no initial current. Find the charge on the capacitor and the current at any time t. Find the amplitude and phase shift of the charge function. (See exercise 21 in section 15.1.)

3. A series circuit has an inductor of 0.2 henry, no resistance and a capacitor of 10^{-5} farad. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge on the capacitor and the current at any time t. Find the amplitude and phase shift of the charge function. (See exercise 21 in section 15.1.)

4. A series circuit has an inductor of 0.6 henry, a resistor of 400 ohms and a capacitor of 2×10^{-4} farad. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge on the capacitor and the current at any time t.

5. A series circuit has an inductor of 0.5 henry, a resistor of 2 hms and a capacitor of 0.05 farad. The initial charge on the capacitor is zero and the initial current is 1 A. A voltage source of $E(t) = 3\cos 2t$ volts is analogous to an external force. Find the charge on the capacitor and the current at any time t.

6. A series circuit has an inductor of 0.2 henry, a resistor of 20 ohms and a capacitor of 0.1 farad. The initial charge on the capacitor is zero and there is no initial current. A voltage source of $E(t) = 0.4\cos 4t$ volts is analogous to an external force. Find the charge on the capacitor and the current at any time t.

7. A series circuit has an inductor of 1 henry, a resistor of 10 ohms and a capacitor of 0.5 farad. A voltage source of $E(t) = 0.1\cos 2t$ volts is analogous to an external force. Find the steady-state solution and identify its amplitude and phase shift. (See exercise 21 in section 15.1.)

8. A series circuit has an inductor of 0.2 henry, a resistor of 40 ohms and a capacitor of 0.05 farad. A voltage source of $E(t) = 0.2\ \sin 4t$ volts is analogous to an external force. Find the steady-state solution and identify its amplitude and phase shift. (See exercise 21 in section 15.1.)

Exercises 9–16 involve frequency response curves and Bode plots.

9. Suppose that the charge in a circuit satisfies the equation $x''(t) + 2x'(t) + 5x(t) = A_1 \sin \omega t$ for constants A_1 and ω. Find the steady-state solution and rewrite it in the form $A_2 \sin(\omega t + \delta)$, where $A_2 = \dfrac{A_1}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}$. The ratio $\dfrac{A_2}{A_1}$ is called the **gain** of the circuit. Notice that it is independent of the actual value of A_1.

10. Graph the gain function $g(\omega) = \dfrac{1}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}$ from exercise 9 as a function of $\omega > 0$. This is called a **frequency response curve.** Find $\omega > 0$ to maximize the gain by minimizing the function $f(\omega) = (5-\omega^2)^2 + 4\omega^2$. This value of ω is called the **resonant frequency** of the circuit. Also graph the **Bode plot** for this circuit, which is the graph of $20\log_{10} g$ as a function of $\log_{10}\omega$. (In this case, the units of $20\log_{10} g$ are decibels.)

11. The charge in a circuit satisfies the equation $x''(t) + 0.4x'(t) + 4x(t) = A\sin \omega t$. Find the gain function and the value of $\omega > 0$ that maximizes the gain, and graph the Bode plot of $20\log_{10} g$ as a function of $\log_{10}\omega$.

12. The charge in a circuit satisfies the equation $x''(t) + 0.4x'(t) + 5x(t) = A\sin \omega t$. Find the gain function and the value of $\omega > 0$ that maximizes the gain, and graph the Bode plot of $20\log_{10} g$ as a function of $\log_{10}\omega$.

13. The charge in a circuit satisfies the equation $x''(t) + 0.2x'(t) + 4x(t) = A\sin \omega t$. Find the gain function and the value of $\omega > 0$ that maximizes the gain, and graph the Bode plot of $20\log_{10} g$ as a function of $\log_{10}\omega$.

14. Based on your answers to exercises 11–13, which of the constants b, c and A affect the gain in the circuit described by $x''(t) + bx'(t) + cx(t) = A\sin \omega t$?

15. The motion of the arm of a seismometer is modeled by $y'' + by' + cy = \omega^2 \cos \omega t$, where the horizontal shift of the ground during the earthquake is proportional to $\cos \omega t$. (See *Multimedia ODE Architect* for details.) If $b = 1$ and $c = 4$, find the gain function and the value of $\omega > 0$ that maximizes the gain.

16. The amplitude A of the motion of the seismometer in exercise 15 and the distance D of the seismometer from the epicenter of the earthquake determine the Richter measurement M through the formula $M = \log_{10} A + 2.56\log_{10} D - 1.67$. Use the result of exercise 15 to prove that A depends on the frequency of the horizontal motion as well as the actual horizontal distance moved. Explain in terms of the motion of the ground during an earthquake why the frequency affects the amount of damage done.

17. In exercise 10, we sketched the Bode plot of the gain as a function of frequency. The other Bode plot, of phase shift as a function of frequency, is considered here. First, recall that in the general relationship $a\sin\omega t + b\cos\omega t = B\sin(\omega t + \theta)$, we have $a = B\cos\theta$ and $b = B\sin\theta$. We can "solve" for θ as $\cos^{-1}\left(\dfrac{a}{B}\right)$ or $\sin^{-1}\left(\dfrac{b}{B}\right)$ or $\tan^{-1}\left(\dfrac{b}{a}\right)$. In exercise 10, we have $a = \dfrac{(5-\omega^2)A}{(5-\omega^2)^2 + 4\omega^2}$ and $b = \dfrac{-2\omega A}{(5-\omega^2)^2 + 4\omega^2}$, so

that $B = \sqrt{a^2+b^2} = \dfrac{A}{(5-\omega^2)^2+4\omega^2}$. For the frequencies $\omega > 0$, this tells us that $\sin\theta < 0$, so that θ is in quadrant III or IV. Explain why the functions $\sin^{-1}\left(\dfrac{b}{B}\right)$ and $\tan^{-1}\left(\dfrac{b}{a}\right)$ are not convenient for this range of angles. However, $-\cos^{-1}\left(\dfrac{a}{B}\right)$ gives the correct quadrants. Show that $\theta = -\cos^{-1}\left(\dfrac{5-\omega^2}{\sqrt{(5-\omega^2)^2+(2\omega)^2}}\right)$ and sketch the Bode plot.

18. Sketch the plot of phase shift versus frequency for exercise 11.

19. Show that if a, b and c are all positive numbers, then the solutions of $ay'' + by' + cy = 0$ approach 0 as $t \to \infty$.

20. For the electrical charge equation $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0$, if there is nonzero resistance, what is the eventual charge on the capacitor?

21. Show that the gain in the general circuit described by $ax''(t)+bx'(t)+cx(t) = A\sin\omega t$ equals $\dfrac{1}{\sqrt{(c-a\omega^2)^2+(b\omega)^2}}$.

22. Show that in exercise 21 the general resonant frequency equals $\sqrt{\dfrac{2ac-b^2}{2a^2}}$.

23. A pendulum of length 10 cm satisfies equation (3.7). The bob is released from a starting angle $\theta = 0.2$. Find an equation for the position at any time and find the amplitude and period of the motion. Compare your solution to that of example 3.4. What effect does a change in length have?

24. Repeat exercise 23 with a starting angle of $\theta = 0.4$. What effect does doubling the starting angle have?

25. A pendulum of length 10 cm satisfies equation (3.7). The bob is released from a starting angle $\theta = 0$ with an initial angular velocity of $\theta' = 0.1$. Find an equation for the position at any time and find the amplitude and period of the motion.

26. Repeat exercise 25 with initial angular velocity $\theta' = 0.2$. What effect does doubling the initial angular velocity have?

27. For a pendulum of weight 6 pounds, length 8 inches, damping constant $k = 0.2$ and forcing function $F(t) = \cos 3t$, find the amplitude and period of the steady-state motion.

28. For a pendulum of weight 6 pounds, length 8 inches, damping constant $k = 0.2$ and forcing function $F(t) = \cos 6t$, find the amplitude and period of the steady-state motion. Compare your solution to that of exercise 27. Does the frequency of the forcing function affect the amplitude of the motion?

29. In example 3.3, find the general homogeneous solution and show that it approaches 0 as $t \to \infty$.

30. In example 3.5, find the general homogeneous solution and show that it approaches 0 as $t \to \infty$.

31. Use Taylor's Theorem to prove that the error in the approximation $\sin\theta \approx \theta$ is bounded by $|\theta|^3/6$.

32. To keep the error in the approximation $\sin\theta \approx \theta$ less than 0.01, how small does θ need to be?

33. Show that the solution of $\theta'' + \frac{g}{L}\theta = 0$ has period $2\pi\sqrt{\frac{L}{g}}$. Galileo deduced that the square of the period varies directly with the length. Is this consistent with a period of $2\pi\sqrt{\frac{L}{g}}$?

34. Galileo believed that the period of a pendulum is independent of the weight of the bob. Determine whether the model (3.7) is consistent with this prediction.

35. Galileo further believed that the period of a pendulum is independent of its amplitude. Use exercise 24 to determine whether the model (3.7) supports this conjecture.

36. Taking into account damping, Galileo found that a pendulum will eventually come to rest, with lighter ones coming to rest faster than heavy ones. Show that this is implied by (3.8) in that for pendulums of identical length and damping constant c (note that c is different from k) but different masses, the pendulum with the smaller mass will come to rest faster.

37. The gun of a tank is attached to a system with springs and dampers such that the displacement $y(t)$ of the gun after being fired at time 0 is

$$y'' + 2\alpha y' + \alpha^2 y = 0,$$

for some constant α. Initial conditions are $y(0) = 0$ and $y'(0) = 100$. Estimate α such that the quantity $y^2 + (y')^2$ is less than 0.01 at $t = 1$. This enables the gun to be fired again rapidly.

38. Let $G(t)$ be the concentration of glucose in the bloodstream and $g(t) = G(t) - G_0$ the difference between the glucose level and the ideal concentration G_0. Braun derives an equation of the form

$$g''(t) + 2\alpha g'(t) + \omega^2 g(t) = 0$$

for the concentration t hours after a glucose injection. It turns out that if the natural period $\dfrac{2\pi}{\omega}$ of the solution is less than 4 hours, the patient is not likely to be diabetic, whereas $\dfrac{2\pi}{\omega} > 4$ is an indicator of mild diabetes. Using $\alpha = 1$ and initial conditions $g(0) = 10$ and $g'(0) = 0$, compare the graphs of glucose levels for a healthy patient with $\omega = 2$ and a diabetic patient with $\omega = 1$.

39. If $0 < \alpha < \omega$, show that the solution in exercise 38 is a damped exponential. Show that the time between zeros of the solution is greater than $\dfrac{\pi}{\omega}$. Use this result to determine whether the following patient would be suspected of diabetes. The optimal glucose level is 75 mg glucose/100 ml blood. The glucose levels are 90 mg glucose/100 ml blood one hour after an injection, 70 mg glucose/100 ml blood two hours after the injection and 78 mg glucose/100 ml blood three hours after the injection.

40. Show that the data in exercise 39 are inconsistent with the case $0 < \omega < \alpha$.

41. Consider an *RLC*-circuit with capacitance C and charge $Q(t)$ at time t. The energy in the circuit at time t is given by $u(t) = \dfrac{[Q(t)]^2}{2C}$. Show that the charge in a general *RLC*-circuit has the form

$$Q(t) = e^{-(R/L)t/2}|Q_0 \cos \omega t + c_2 \sin \omega t|,$$

where $Q_0 = Q(0)$ and $\omega = \frac{1}{2L}\sqrt{R^2 - 4L/C}$. The relative energy loss from time $t = 0$ to time $t = \frac{2\pi}{\omega}$ is given by $U_{loss} = \dfrac{u(2\pi/\omega) - u(0)}{u(0)}$ and the **inductance quality factor** is defined by $\dfrac{2\pi}{U_{loss}}$. Using a Taylor polynomial approximation of e^x, show that the inductance quality factor is approximately $\omega\dfrac{L}{R}$.

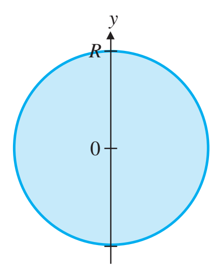
EXPLORATORY EXERCISES

1. In quantum mechanics, the possible locations of a particle are described by its **wave function** $\Psi(x)$. The wave function satisfies **Schrödinger's wave equation**

$$\frac{\hbar}{2m}\Psi''(x) + V(x)\Psi(x) = E\Psi(x).$$

Here, $\hbar$ is Planck's constant, m is mass, $V(x)$ is the potential function for external forces and E is the particle's energy. In the case of a bound particle with an infinite square well of width $2a$, the potential function is $V(x) = 0$ for $-a \le x \le a$. We will show that the particle's energy is quantized by solving the **boundary value problem** consisting of the differential equation $\dfrac{\hbar}{2m}\Psi''(x) + v(x)\Psi(x) = E\Psi(x)$ plus the boundary conditions $\Psi(-a) = 0$ and $\Psi(a) = 0$. The theory of boundary value problems is different from that of the initial value problems in this chapter, which typically have unique solutions. In fact, in this exercise we specifically want more than one solution. Start with the differential equation and show that for $V(x) = 0$; the general solution is $\Psi(x) = c_1 \cos kx + c_2 \sin kx$, where $k = \sqrt{2mE}/\hbar$. Then set up the equations $\Psi(-a) = 0$ and $\Psi(a) = 0$. Both equations are true if $c_1 = c_2 = 0$, but in this case the solution would be $\Psi(x) = 0$. To find **nontrivial solutions** (that is, nonzero solutions), find all values of k such that $\cos ka = 0$ or $\sin ka = 0$. Then, solve for the energy E in terms of a, m and $\hbar$. These are the only allowable energy levels for the particle. Finally, determine what happens to the energy levels as a increases without bound.

2. Imagine a hole drilled through the center of the Earth. What would happen to a ball dropped in the hole? Galileo conjectured that the ball would undergo **simple harmonic motion,** which is the periodic motion of an undamped spring or pendulum. This solution requires no friction and a nonrotating Earth. The force due to gravity of two objects r units apart is $\dfrac{Gm_1m_2}{r^2}$, where G is the universal gravitation constant and m_1 and m_2 are the masses of the objects. Let R be the radius of the Earth and y the displacement from the center of the Earth.



For a ball at position y with $|y| \le R$, the ball is attracted to the center of the Earth as if the Earth were a single particle located at the origin with mass ρv, where ρ is the density of the Earth and v is the volume of the sphere of radius $|y|$. (This assumes a constant density and a spherical Earth.) If M is the mass of the Earth, show that if you neglect damping, the position of the ball satisfies the equation $y'' + \dfrac{GM}{R^3}y = 0$. Use $g = \dfrac{GM}{R^2}$ to simplify this. Find the motion of the ball. Does the period depend on the starting position? Compare the motions of balls dropped simultaneously from the Earth's surface and halfway to the center of the Earth. Explore the motion of a ball thrown from the surface of the Earth at $y = R$ with initial velocity $-R/100$.

15.4 POWER SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

So far in this chapter, we have seen how to solve only those second-order equations with constant coefficients, such as

$$y'' - 6y' + 9y = 0.$$

What if the coefficients aren't constant? For instance, suppose you wanted to solve the equation

$$y'' + 2xy' + 2y = 0.$$

We leave it as an exercise to show that substituting $y = e^{rx}$ in this case does not lead to a solution. However, in many cases such as this, we can find a solution by assuming that the solution can be written as a power series, such as

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The idea is to substitute this series into the differential equation and then use the resulting equation to determine the coefficients, $a_0, a_1, a_2 \ldots, a_n$. Before we see how to do this in general, we illustrate this for a simple equation whose solution is already known, to demonstrate that we arrive at the same solution using either method.

EXAMPLE 4.1 Power Series Solution of a Differential Equation

Use a power series to determine the general solution of

$$y'' + y = 0.$$

Solution First, observe that this equation has constant coefficients and its general solution is

$$y = c_1 \sin x + c_2 \cos x,$$

where c_1 and c_2 are constants.

We now look for a solution of the equation in the form of the power series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

To substitute this into the equation, we first need to obtain representations for y' and y''. Assuming that the power series is convergent and has a positive radius of convergence, recall that we can differentiate term-by-term to obtain the derivatives

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = 2a_2 + 6a_3 x + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these power series into the differential equation, we get

$$0 = y'' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n. \tag{4.1}$$

The immediate objective here is to combine the two series in (4.1) into one power series. Since the powers in the one series are of the form x^{n-2} and in the other series are of the form x^n, we will first need to rewrite one of the two series. Notice that we have

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \end{aligned}$$

REMARK 4.1

Notice that when we change $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ to $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$, the index in the sequence increases by 2 (for example, a_n becomes a_{n+2}), while the initial value of the index decreases by 2.

Substituting this into equation (4.1) gives us

$$0 = y'' + y = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + a_n]x^n. \tag{4.2}$$

Read equation (4.2) carefully; it says that the power series on the right converges to the constant function $f(x) = 0$. In view of this, all of the coefficients must be zero. That is,

$$0 = (n+2)(n+1)a_{n+2} + a_n,$$

for $n = 0, 1, 2, \ldots$. We solve this for the coefficient with the *largest* index, to obtain

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \tag{4.3}$$

for $n = 0, 1, 2, \ldots$. Equation (4.3) is called the **recurrence relation,** which we use to determine all of the coefficients of the series solution. The general idea is to write out (4.3) for a number of specific values of n and then try to recognize a pattern that the coefficients follow. From (4.3), we have for the even-indexed coefficients that

$$\begin{aligned}
a_2 &= \frac{-a_0}{2 \cdot 1} = \frac{-1}{2!}a_0, \\
a_4 &= \frac{-a_2}{4 \cdot 3} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}a_0 = \frac{1}{4!}a_0, \\
a_6 &= \frac{-a_4}{6 \cdot 5} = \frac{-1}{6!}a_0, \\
a_8 &= \frac{-a_6}{8 \cdot 7} = \frac{1}{8!}a_0
\end{aligned}$$

and so on. (Try to write down a_{10} by recognizing the pattern, without referring to the recurrence relation.) Since we can write each even-indexed coefficient as a_{2n}, for some n, we can now write down a simple formula that works for any of these coefficients. We have

$$a_{2n} = \frac{(-1)^n}{(2n)!}a_0,$$

for $n = 0, 1, 2, \ldots$. Similarly, using (4.3), we have that the odd-indexed coefficients are

$$\begin{aligned}
a_3 &= \frac{-a_1}{3 \cdot 2} = \frac{-1}{3!}a_1, \\
a_5 &= \frac{-a_3}{5 \cdot 4} = \frac{1}{5!}a_1, \\
a_7 &= \frac{-a_5}{7 \cdot 6} = \frac{-1}{7!}a_1, \\
a_9 &= \frac{-a_7}{9 \cdot 8} = \frac{1}{9!}a_1
\end{aligned}$$

and so on. Since we can write each odd-indexed coefficient as a_{2n+1} (or alternatively as a_{2n-1}), for some n, note that we have the following simple formula for the odd-indexed coefficients:

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}a_1,$$

for $n = 0, 1, 2, \ldots$. Since we have now written every coefficient in terms of either a_0 or a_1, we can rewrite the solution by separating the a_0 terms from the a_1 terms. We have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + a_1 \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right) \\ &= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{y_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{y_2(x)} \\ &= a_0 y_1(x) + a_1 y_2(x), \end{aligned} \tag{4.4}$$

where $y_1(x)$ and $y_2(x)$ are two solutions of the differential equation (assuming the series converge). At this point, you should be able to easily check that both of the indicated power series converge absolutely for all x, by using the Ratio Test. Beyond this, you might also recognize that the series solutions $y_1(x)$ and $y_2(x)$ that we obtained are, in fact, the Maclaurin series expansions of $\cos x$ and $\sin x$, respectively. In light of this, (4.4) is an equivalent solution to that found by using the methods of section 15.1. ■

The method used to solve the differential equation in example 4.1 is certainly far more complicated than the methods we used in section 15.1 for solving the same equation. However, this new method can be used to solve a wider range of differential equations than those solvable using our earlier methods. We now return to the equation mentioned in the introduction to this section.

EXAMPLE 4.2 Solving a Differential Equation with Variable Coefficients

Find the general solution of the differential equation

$$y'' + 2xy' + 2y = 0.$$

Solution First, observe that since the coefficient of y' is not constant, we have little choice but to look for a series solution of the equation. As in example 4.1, we begin by assuming that we may write the solution as a power series,

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

As before, we have

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these three power series into the equation, we get

$$\begin{aligned} 0 = y'' + 2xy' + 2y &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n, \end{aligned} \tag{4.5}$$

where in the middle term, we moved the x into the series and combined powers of x. In order to combine the three series, we must only rewrite the first series so that its general term is a multiple of x^n, instead of x^{n-2}. As we did in example 4.1, we write

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

and so, from (4.5), we have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 2a_n x^n \\ &= \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2na_n + 2a_n]x^n \\ &= \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+1)a_n]x^n. \end{aligned} \tag{4.6}$$

To get this, we used the fact that $\sum_{n=1}^{\infty} 2na_n x^n = \sum_{n=0}^{\infty} 2na_n x^n$. (Notice that the first term in the series on the right is zero!) Reading equation (4.6) carefully, note that we again have a power series converging to the zero function, from which it follows that all of the coefficients must be zero:

$$0 = (n+2)(n+1)a_{n+2} + 2(n+1)a_n,$$

for $n = 0, 1, 2, \ldots$. Again solving for the coefficient with the largest index, we get the recurrence relation

REMARK 4.2

Always solve for the coefficient with the largest index.

$$a_{n+2} = -\frac{2(n+1)a_n}{(n+2)(n+1)}$$

or

$$a_{n+2} = -\frac{2a_n}{n+2}.$$

Much like we saw in example 4.1, the recurrence relation tells us that all of the even-indexed coefficients are related to a_0 and all of the odd-indexed coefficients are related to a_1. In order to try to recognize the pattern, we write out a number of terms, using the recurrence relation. We have

$$\begin{aligned} a_2 &= -\frac{2}{2}a_0 = -a_0, \\ a_4 &= -\frac{2}{4}a_2 = \frac{1}{2}a_0, \\ a_6 &= -\frac{2}{6}a_4 = -\frac{1}{3!}a_0, \\ a_8 &= -\frac{2}{8}a_6 = \frac{1}{4!}a_0 \end{aligned}$$

and so on. At this point, you should recognize the pattern for these coefficients. (If not, write out a few more terms.) Note that we can write the even-indexed coefficients as

$$a_{2n} = \frac{(-1)^n}{n!}a_0,$$

for $n = 0, 1, 2, \ldots$. Be sure to match this formula against those coefficients calculated above to see that they match. Continuing with the odd-indexed coefficients, we have from the recurrence relation that

$$\begin{aligned} a_3 &= -\frac{2}{3}a_1, \\ a_5 &= -\frac{2}{5}a_3 = \frac{2^2}{5\cdot 3}a_1, \\ a_7 &= -\frac{2}{7}a_5 = -\frac{2^3}{7\cdot 5\cdot 3}a_1, \\ a_9 &= -\frac{2}{9}a_7 = \frac{2^4}{9\cdot 7\cdot 5\cdot 3}a_1 \end{aligned}$$

and so on. While you might recognize the pattern here, it's hard to write down this pattern succinctly. Observe that the products in the denominators are not quite factorials. Rather, each is the product of the first so many odd numbers. The solution to this is to write this as a factorial, but then cancel out all of the even integers in the product. In particular, note that

$$\frac{1}{9\cdot 7\cdot 5\cdot 3} = \frac{\overbrace{8}^{2\cdot 4}\cdot\overbrace{6}^{2\cdot 3}\cdot\overbrace{4}^{2\cdot 2}\cdot\overbrace{2}^{2\cdot 1}}{9!} = \frac{2^4\cdot 4!}{9!},$$

so that a_9 becomes

$$a_9 = \frac{2^4}{9\cdot 7\cdot 5\cdot 3}a_1 = \frac{2^4\cdot 2^4\cdot 4!}{9!}a_1 = \frac{2^{2\cdot 4}\cdot 4!}{9!}a_1.$$

More generally, we now have

$$a_{2n+1} = \frac{(-1)^n 2^{2n} n!}{(2n+1)!}a_1,$$

for $n = 0, 1, 2 \ldots$.

Now that we have expressions for all of the coefficients, we can write the solution of the differential equation as

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(a_{2n}x^{2n} + a_{2n+1}x^{2n+1}) \\ &= a_0 \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}}_{y_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} n!}{(2n+1)!}x^{2n+1}}_{y_2(x)} \\ &= a_0 y_1(x) + a_1 y_2(x), \end{aligned}$$

where y_1 and y_2 are two power series solutions of the differential equation. We leave it as an exercise to use the Ratio Test to show that both of these series converge absolutely for all x. You might recognize $y_1(x)$ as the Maclaurin series expansion for e^{-x^2}, but in practice recognizing series solutions as power series of familiar functions is rather unlikely. To give you an idea of the behavior of these functions, we draw a graph of $y_1(x)$ in Figure 15.16a and of $y_2(x)$ in Figure 15.16b. We obtained the graph of $y_2(x)$ by plotting the partial sums of the series. ■

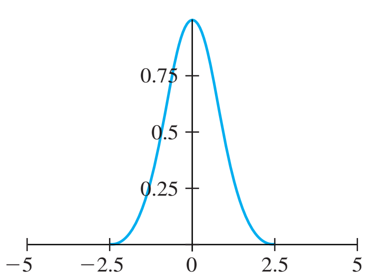


FIGURE 15.16a
$y = y_1(x) = e^{-x^2}$

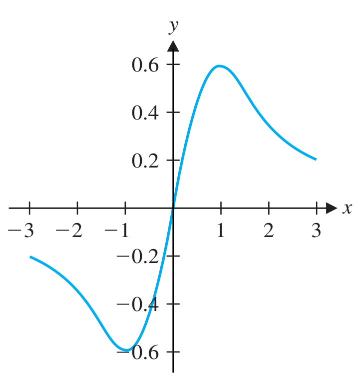


FIGURE 15.16b
10-term approximation to $y = y_2(x)$

From examples 4.1 and 4.2, you might get the idea that if you look for a series solution, you can always recognize the pattern of the coefficients and write the pattern down succinctly. Unfortunately, the pattern is most often difficult to see and even more difficult to write down compactly. Still, series solutions are a valuable means of solving a differential equation. In the worst case, you can always compute a number of the coefficients of the series from the recurrence relation and then use the first so many terms of the series as an approximation to the actual solution.

In example 4.3, we illustrate the more common case where the coefficients are a bit more challenging to find.

EXAMPLE 4.3 A Series Solution Where the Coefficients Are Harder to Find

Use a power series to find the general solution of **Airy's equation**

$$y'' - xy = 0.$$

Solution As before, we assume that we may write the solution as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Again, we have
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Subsituting these power series into the equation, we get

$$\begin{aligned} 0 = y'' - xy &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}. \end{aligned}$$

In order to combine the two preceding series, we must rewrite one or both series so that they both have the same power of x. For simplicity, we rewrite the first series only. We have

$$\begin{aligned} 0 &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= (2)(1)a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= 2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2) a_{n+3} - a_n] x^{n+1}, \end{aligned}$$

where we wrote out the first term of the first series and then combined the two series, once both had an index that started with $n = 0$. Again, this is a power series expansion of the zero function and so, all of the coefficients must be zero. That is,

$$0 = 2a_2 \tag{4.7}$$

and
$$0 = (n+3)(n+2)a_{n+3} - a_n, \tag{4.8}$$
for $n = 0, 1, 2, \ldots$. Equation (4.7) says that $a_2 = 0$ and (4.8) gives us the recurrence relation
$$a_{n+3} = \frac{1}{(n+3)(n+2)}a_n, \tag{4.9}$$
for $n = 0, 1, 2, \ldots$. Notice that here, instead of having all of the even-indexed coefficients related to a_0 and all of the odd-indexed coefficients related to a_1, (4.9) tells us that every *third* coefficient is related. In particular, notice that since $a_2 = 0$, (4.9) now says that
$$a_5 = \frac{1}{5 \cdot 4}a_2 = 0,$$
$$a_8 = \frac{1}{8 \cdot 7}a_5 = 0$$
and so on. So, every third coefficient starting with a_2 is zero. But, how do we concisely write down something like this? Think about the notation a_{2n} and a_{2n+1} that we have used previously. You can view a_{2n} as a representation of every second coefficient starting with a_0. Likewise, a_{2n+1} represents every second coefficient starting with a_1. In the present case, if we want to write down every third coefficient starting with a_2, we write a_{3n+2}. We can now observe that
$$a_{3n+2} = 0,$$
for $n = 0, 1, 2, \ldots$. Continuing on with the remaining coefficients, we have from (4.9) that
$$a_3 = \frac{1}{3 \cdot 2}a_0,$$
$$a_6 = \frac{1}{6 \cdot 5}a_3 = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2}a_0,$$
$$a_9 = \frac{1}{9 \cdot 8}a_6 = \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}a_0$$
and so on. Hopefully, you see the pattern that's developing for these coefficients. The trouble here is that it's not as easy to write down this pattern as it was in the first two examples. Notice that the denominator in the expression for a_9 is almost 9!, but with every third factor in the product deleted. Since we don't have a way of succinctly writing this down, we write the coefficients by indicating the pattern, as follows:
$$a_{3n} = \frac{(3n-2)(3n-5)\cdots 7 \cdot 4 \cdot 1}{(3n)!}a_0,$$
where this is not intended as a literal formula, as explicit substitution of $n = 0$ or $n = 1$ would result in negative values. Rather, this is an indication of the general pattern. Similarly, the recurrence relation gives us
$$a_4 = \frac{1}{4 \cdot 3}a_1,$$
$$a_7 = \frac{1}{7 \cdot 6}a_4 = \frac{1}{7 \cdot 6 \cdot 4 \cdot 3}a_1,$$
$$a_{10} = \frac{1}{10 \cdot 9}a_7 = \frac{1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}a_1$$

and so on. More generally, we can establish the pattern:

$$a_{3n+1} = \frac{(3n-1)(3n-4)\cdots 8\cdot 5\cdot 2}{(3n+1)!}a_1,$$

where again, this is not intended as a literal formula.

Now that we have found all of the coefficients, we can write down the solution, by separately writing out every third term of the series, as follows:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(a_{3n}x^{3n} + a_{3n+1}x^{3n+1} + a_{3n+2}x^{3n+2}\right) \\ &= a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(3n-2)(3n-5)\cdots 7\cdot 4\cdot 1}{(3n)!}x^{3n}}_{y_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(3n-1)(3n-4)\cdots 8\cdot 5\cdot 2}{(3n+1)!}x^{3n+1}}_{y_2(x)} \\ &= a_0 y_1(x) + a_1 y_2(x). \end{aligned}$$

We leave it as an exercise to use the Ratio Test to show that the power series defining y_1 and y_2 are absolutely convergent for all x. ■

You may have noticed that in all three of our examples, we assumed that there was a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots,$$

only to arrive at the general solution

$$y = a_0 y_1(x) + a_1 y_2(x),$$

where y_1 and y_2 were power series solutions of the equation. This is in fact not coincidental. One can show that (at least for certain equations) this is always the case. One clue as to why this might be so lies in the following.

Suppose that we want to solve the initial value problem consisting of a second-order differential equation and the initial conditions $y(0) = A$ and $y'(0) = B$. Taking $y(x) = \sum_{n=0}^{\infty} a_n x^n$ gives us

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots.$$

So, imposing the initial conditions, we have

$$A = y(0) = a_0 + a_1(0) + a_2(0)^2 + \cdots = a_0$$

and

$$B = y'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \cdots = a_1.$$

So, irrespective of the particular equation we're solving, we always have $y(0) = a_0$ and $y'(0) = a_1$.

You might ask what you'd do if the initial conditions were imposed at some point other than at $x = 0$, say at $x = x_0$. In this case, we look for a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

It's easy to show that in this case, we still have $y(x_0) = a_0$ and $y'(x_0) = a_1$.

In the exercises, we explore finding series solutions about a variety of different points.

BEYOND FORMULAS

This section connects two important threads of calculus: solutions of differential equations and infinite series. In Chapter 8, we expressed known functions like $\sin x$ and $\cos x$ as power series. Here, we just extend that idea to unknown solutions of differential equations. For second-order homogeneous equations, keep in mind that the general solution has the format $c_1 y_1(x) + c_2 y_2(x)$. You should think about the problems in this section as following this strategy: write the solution as a power series, substitute into the differential equation and find relationships between the coefficients of the power series, remembering that two of the coefficients will be left as arbitrary constants.

EXERCISES 15.4

WRITING EXERCISES

1. After substituting a power series representation into a differential equation, the next step is always to rewrite one or more of the series, so that all series have the same exponent. (Typically, we want x^n.) Explain why this is an important step. For example, what would we be unable to do if the exponents were not the same?

2. The recurrence relation is typically solved for the coefficient with the largest index. Explain why this is an important step.

3. Explain why you can't solve equations with nonconstant coefficients, such as

$$y'' + 2xy' + 2y = 0,$$

by looking for a solution in the form $y = e^{rx}$.

4. The differential equations solved in this section are actually of a special type, where we find power series solutions centered at what is called an **ordinary point.** For the equation $x^2y'' + y' + 2y = 0$, the point $x = 0$ is not an ordinary point. Discuss what goes wrong here if you look for a power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

In exercises 1–8, find the recurrence relation and general power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

1. $y'' + 2xy' + 4y = 0$
2. $y'' + 4xy' + 8y = 0$
3. $y'' - xy' - y = 0$
4. $y'' - xy' - 2y = 0$
5. $y'' - xy' = 0$
6. $y'' + 2xy = 0$
7. $y'' - x^2y' = 0$
8. $y'' + xy' - 2y = 0$
9. Find a series solution of $y'' + (1 - x)y' - y = 0$ in the form $y = \sum_{n=0}^{\infty} a_n (x-1)^n$.
10. Find a series solution of $y'' + y' + (x - 2)y = 0$ in the form $\sum_{n=0}^{\infty} a_n (x-2)^n$.
11. Find a series solution of Airy's equation $y'' - xy = 0$ in the form $\sum_{n=0}^{\infty} a_n (x-1)^n$. [Hint: First rewrite the equation in the form $y'' - (x - 1)y - y = 0$.]
12. Find a series solution of Airy's equation $y'' - xy = 0$ in the form $\sum_{n=0}^{\infty} a_n (x-2)^n$.
13. Solve the initial value problem $y'' + 2xy' + 4y = 0$, $y(0) = 5$, $y'(0) = -7$. (See exercise 1.)
14. Solve the initial value problem $y'' + 4xy' + 8y = 0$, $y(0) = 2$, $y'(0) = \pi$. (See exercise 2.)
15. Solve the initial value problem $y'' + (1 - x)y' - y = 0$, $y(1) = -3$, $y'(1) = 12$. (See exercise 9.)
16. Solve the initial value problem $y'' + y' + (x - 2)y = 0$, $y(2) = 1$, $y'(2) = -1$. (See exercise 10.)
17. Determine the radius of convergence of the power series solutions about $x_0 = 0$ of $y'' - xy' - y = 0$. (See exercise 3.)
18. Determine the radius of convergence of the power series solutions about $x_0 = 0$ of $y'' - xy' - 2y = 0$. (See exercise 4.)
19. Determine the radius of convergence of the power series solutions about $x_0 = 1$ of $y'' + (1 - x)y' - y = 0$. (See exercise 9.)
20. Determine the radius of convergence of the power series solutions about $x_0 = 1$ of $y'' - xy = 0$ (See exercise 11.)

21. Find a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to the equation $x^2y'' + xy' + x^2y = 0$ (Bessel's equation of order 0).

22. Find a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to the equation $x^2y'' + xy' + (x^2 - 1)y = 0$ (Bessel's equation of order 1).

23. Determine the radius of convergence of the series solution found in example 4.3.

24. Determine the radius of convergence of the series solution found in problem 12.

25. For the initial value problem $y'' + 2xy' - xy = 0$, $y(0) = 2$, $y'(0) = -5$, substitute in $x = 0$ and show that $y''(0) = 0$. Then take $y'' = -2xy' + xy$ and show that $y''' = -2xy'' + (x-2)y' + y$. Conclude that $y'''(0) = 12$. Then compute $y^{(4)}(x)$ and find $y^{(4)}(0)$. Finally, compute $y^{(5)}(x)$ and find $y^{(5)}(0)$. Write out the fifth-degree Taylor polynomial for the solution, $P_5(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2} + y'''(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} + y^{(5)}(0)\frac{x^5}{5!}$.

26. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + x^2y' - (\cos x)y = 0$, $y(0) = 3$, $y'(0) = 2$.

27. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + e^x y' - (\sin x)y = 0$, $y(0) = -2$, $y'(0) = 1$.

28. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + y' - (e^x)y = 0$, $y(0) = 2$, $y'(0) = 0$.

29. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + xy' + (\sin x)y = 0$, $y(\pi) = 0$, $y'(\pi) = 4$.

30. Use the technique of exercise 25 to find the fifth-degree Taylor polynomial for the solution of the initial value problem $y'' + (\cos x)y' + xy = 0$, $y(\frac{\pi}{2}) = 3$, $y'\frac{\pi}{2} = 0$.

EXPLORATORY EXERCISES

1. The equation $y'' - 2xy' + 2ky = 0$ for some integer $k \geq 0$ is known as **Hermite's equation.** Following our procedure for finding series solutions in powers of x, show that, in fact, one of the series solutions is simply a polynomial of degree k. For this polynomial solution, choose the arbitrary constant such that the leading term of the polynomial is $2^k x^k$. The polynomial is called the **Hermite polynomial** $H_k(x)$. Find the Hermite polynomials $H_0(x), H_1(x), \ldots, H_5(x)$.

2. The Chebyshev polynomials are polynomial solutions of the equation $(1 - x^2)y'' - xy' + k^2y = 0$, for some integer $k \geq 0$. Find polynomial solutions for $k = 0, 1, 2$ and 3.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Nonhomogeneous equation	Method of undetermined coefficients	Resonance
Second-order differential equation	Damping	Recurrence relation

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. The form of the solution of $ay'' + by' + cy = 0$ depends on the value of $b^2 - 4ac$.

2. The current in an electrical circuit satisfies the same differential equation as the displacement function for a mass attached to a spring.

3. The particular solution of a nonhomogeneous equation $mu'' + ku' + cu = F$ has the same form as the forcing function F.

4. Resonance cannot occur if there is damping.

5. A recurrence relation can always be solved to find the solution of a differential equation.

In exercises 1–6, find the general solution of the differential equation.

1. $y'' + y' - 12y = 0$

2. $y'' + 4y' + 4y = 0$

3. $y'' + y' + 3y = 0$

4. $y'' + 3y' - 8y = 0$

Review Exercises

5. $y'' - y' - 6y = e^{3t} + t^2 + 1$
6. $y'' - 4y = 2e^{2t} + 16\cos 2t$

In exercises 7–10, solve the initial value problem.

7. $y'' + 2y' - 8y = 0,\ y(0) = 5,\ y'(0) = -2$
8. $y'' + 2y' + 5y = 0,\ y(0) = 2,\ y'(0) = 0$
9. $y'' + 4y = 3\cos t,\ y(0) = 1,\ y'(0) = 2$
10. $y'' - 4y = 2e^{2t} + 16\cos 2t$
11. A spring is stretched 4 inches by a 4-pound weight. The weight is then pulled down an additional 2 inches and released. Neglect damping. Find an equation for the position of the weight at any time t and graph the position function.
12. In exercise 11, if an external force of $4\cos\omega t$ pounds is applied to the weight, find the value of ω that would produce resonance. If instead $\omega = 10$, find and graph the position of the weight.
13. A series circuit has an inductor of 0.2 henry, a resistor of 160 ohms and a capacitor of 10^{-2} farad. The initial charge on the capacitor is 10^{-4} coulomb and there is no initial current. Find the charge on the capacitor and the current at any time t.
14. In exercise 13, if the resistor is removed and an impressed voltage of $2\sin\omega t$ volts is applied, find the value of ω that produces resonance. In this case, what would happen to the circuit?

In exercises 15 and 16, determine the form of a particular solution.

15. $u'' + 2u' + 5u = 2e^{-t}\sin 2t + 4t^3 - 2\cos 2t$
16. $u'' + 2u' - 3u = (3t^2 + 1)e^t - e^{-3t}\cos 2t$
17. A spring is stretched 4 inches by a 4-pound weight. The weight is then pulled down an additional 2 inches and set in motion with a downward velocity of 2 ft/s. A damping force equal to $0.4u'$ slows the motion of the spring. An external force of magnitude 2 sin $2t$ pounds is applied. Completely set up the initial value problem and then find the steady-state motion of the spring.
18. A spring is stretched 2 inches by an 8-pound weight. The weight is then pushed up 3 inches and set in motion with an upward velocity of 1 ft/s. A damping force equal to $0.2u'$ slows the motion of the spring. An external force of magnitude $2\cos 3t$ pounds is applied. Completely set up the initial value problem and then find the steady-state motion of the spring.

In exercises 19 and 20, find the recurrence relation and a general power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$.

19. $y'' - 2xy' - 4y = 0$
20. $y'' + (x-1)y' = 0$

In exercises 21 and 22, find the recurrence relation and a general power series solution of the form $\sum_{n=0}^{\infty} a_n (x-1)^n$.

21. $y'' - 2xy' - 4y = 0$
22. $y'' + (x-1)y' = 0$

In exercises 23 and 24, solve the initial value problem.

23. $y'' - 2xy' - 4y = 0,\ y(0) = 4,\ y'(0) = 2$
24. $y'' - 2xy' - 4y = 0,\ y(1) = 2,\ y'(1) = 4$

EXPLORATORY EXERCISES

1. A pendulum that is free to rotate through 360 degrees has two equilibrium points. One is hanging straight down and the other is pointing straight up. The $\theta = \pi$ equilibrium is unstable and is classified as a saddle point. This means that for *most but not all* initial conditions, solutions that start near $\theta = \pi$ will get farther away. Explain why with initial conditions $\theta(0) = \pi$ and $\theta'(0) = 0$, the solution is exactly $\theta(t) = \pi$. However, explain why initial conditions $\theta(0) = 3.1$ and $\theta'(0) = 0$ would have a solution that gets farther from $\theta = \pi$. For the model $\theta''(t) + \frac{g}{L}\theta(t) = 0$, show that if $v = \pi\sqrt{\frac{g}{L}}$, then the initial conditions $\theta(0) = 0$ and $\theta'(0) = v$ produce a solution that reaches the state $\theta = \pi$ and $\theta' = 0$. Physically, explain why the pendulum would remain at $\theta = \pi$ and then explain why the solution of our model does not get "stuck" at $\theta = \pi$. Explain why for any starting angle θ, there exist two initial angular velocities that will balance the pendulum at $\theta = \pi$. The undamped pendulum model $\theta''(t) + \frac{g}{L}\sin\theta(t) = 0$ is equivalent to the system of equations (with $y_1 = \theta$ and $y_2 = \theta'$)
$$y_1' = y_2,$$
$$y_2' = -\frac{g}{L}\sin y_1.$$
Use a CAS to sketch the phase portrait of this system of equations near the equilibrium point $(\pi, 0)$. Explain why the phase portrait shows an unstable equilibrium point with a small set of initial conditions that lead to the equilibrium point.
2. In exploratory exercise 2 of section 15.3, we investigate the motion of a ball dropped in a hole drilled through a nonrotating Earth. Here, we investigate the motion taking into account the Earth's rotation. (See Andrew Simoson's article in the June 2004 *Mathematics Magazine*.) We describe the motion in polar coordinates with respect to a fixed plane through the equator. Define unit vectors $\mathbf{u}_r = \langle\cos\theta, \sin\theta\rangle$ and

Review Exercises

$\mathbf{u}_\theta = \langle -\sin\theta, \cos\theta\rangle$. If the ball has position vector $r\mathbf{u}_r$, show that its acceleration is given by

$$[r\theta''(t) + 2r'(t)\theta'(t)]\mathbf{u}_\theta + \{r''(t) - r(t)[\theta'(t)]^2\}\mathbf{u}_r.$$

Since gravity acts in the radial direction only, $r\theta''(t) + 2r'(t)\theta'(t) = 0$. Show that this implies that $r^2(t)\theta'(t) = k$ for some constant k. (This is the law of conservation of angular momentum.) If the acceleration due to gravity is $f(r)\mathbf{u}_r$ for some function f, show that

$$r''(t) - \frac{k^2}{r^3} = f(r).$$

Initial conditions are $r(0) = R, r'(0) = 0, \theta(0) = 0$ and $\theta'(0) = \frac{2\pi}{Q}$. Here, Q is the period of one revolution of the Earth and we assume that the ball inherits the initial angular velocity from the rotation of the Earth. For the gravitational force $f(r) = -c^2 r$, show that a solution is

$$r(t) = \sqrt{R^2\cos^2 ct + \frac{k^2}{c^2R^2}\sin^2 ct}$$

or

$$r(\theta) = \frac{1}{\sqrt{\frac{1}{R^2}\cos^2\theta + \frac{c^2R^2}{k^2}\sin^2\theta}}.$$

Show that this converts to

$$\frac{x^2}{R^2} + \frac{y^2}{(k/Rc)^2} = 1$$

and describe the path of the ball.