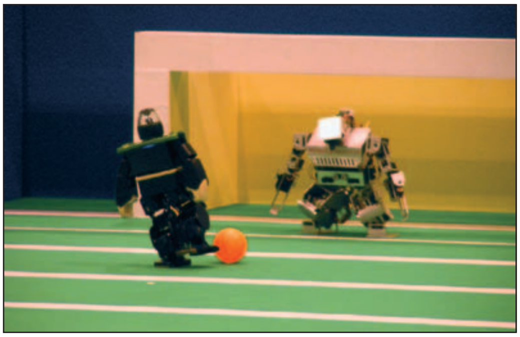
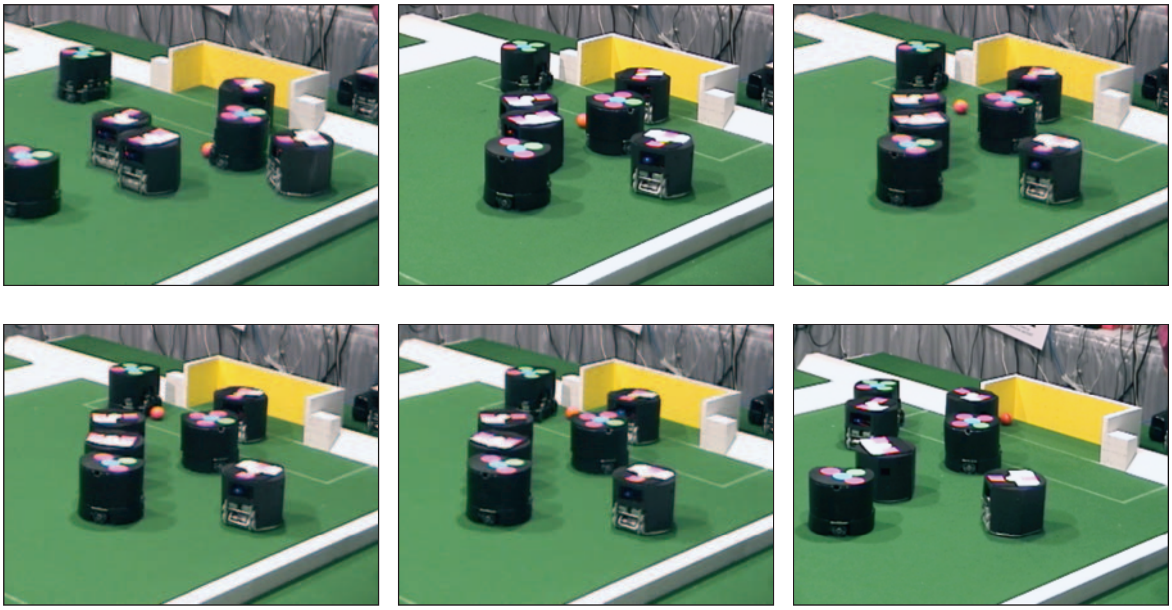
Vector-Valued Functions

CHAPTER 11



RoboCup is the international championship of robot soccer. Unlike the remote-controlled destructive robots that you may have seen on television, the robots in this competition are engineered and programmed to respond automatically to the positions of the ball, goal and other players. Once play starts, the robots are completely on their own to analyze the field of play and use teamwork to outmaneuver their opponents and score goals. RoboCup is a challenge for robotics engineers and artificial intelligence researchers. The competitive aspect of RoboCup focuses teams of researchers, while the annual tournament provides invaluable opportunities for feedback and exchange of information.

The competition is divided into several categories, each with its own unique challenges. Overall, one of the greatest difficulties has been providing the robots with adequate vision. In the small size category, "vision" is provided by overhead cameras, with information relayed wirelessly to the robots. With the formidable sight problem removed, the focus is on effective movement of the robots and on providing artificial intelligence for teamwork. The remarkable abilities of these



robots are demonstrated by the sequence of frames shown here, where a robot on the Cornell Big Red team of 2001 hits a wide-open teammate with a perfect pass that leads to a goal.

There is a considerable amount of mathematics behind this play. To tell whether a teammate is truly open or not, a robot needs to take into account the positions and velocities of each robot, since opponents and teammates could move into the way by the time a pass is executed. In the sequence of photos, notice that all the robots are in motion. Both position and velocity can be described using vectors, but a different vector may be required for each time. In this chapter, we introduce *vector-valued functions,* which assign a vector to each value of the time variable. The calculus introduced in this chapter is essential background knowledge for the programmers of RoboCup.

11.1 VECTOR-VALUED FUNCTIONS

To describe the location of the airplane following the circuitous path indicated in Figure 11.1a, you might consider using a point (x, y, z) in three dimensions. However, it turns out to be more convenient to describe its location at any given time by the endpoint of a vector whose initial point is located at the origin (a **position vector**). (See Figure 11.1b for vectors indicating the location of the plane at a number of times.) Notice that a *function* that gives us a vector in V_3 for each time t would do the job nicely. This is the concept of a vector-valued function, which we define more precisely in Definition 1.1.

DEFINITION 1.1

A **vector-valued function** $\mathbf{r}(t)$ is a mapping from its domain $D \subset \mathbb{R}$ to its range $R \subset V_3$, so that for each t in D, $\mathbf{r}(t) = \mathbf{v}$ for exactly one vector $\mathbf{v} \in V_3$. We can always write a vector-valued function as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \tag{1.1}$$

for some scalar functions f, g and h (called the **component functions** of $\mathbf{r}$).

For each t, we regard $\mathbf{r}(t)$ as a position vector. The endpoint of $\mathbf{r}(t)$ then can be viewed as tracing out a curve, as illustrated in Figure 11.1b. Observe that for $\mathbf{r}(t)$ as defined

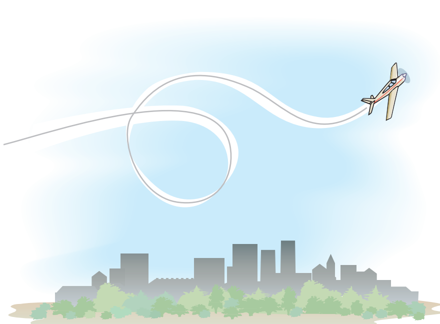


FIGURE 11.1a
Airplane's flight path

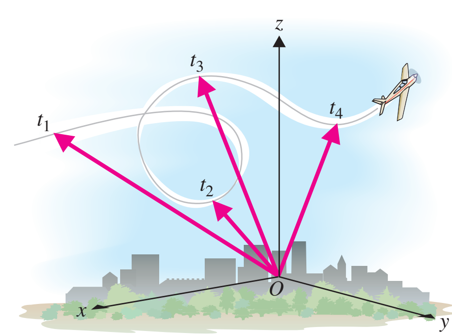


FIGURE 11.1b
Vectors indicating plane's position at several times

in (1.1), this curve is the same as that described by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$. In three dimensions, such a curve is referred to as a **space curve.**

We can likewise define a vector-valued function $\mathbf{r}(t)$ in V_2 by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j},$$

for some scalar functions f and g.

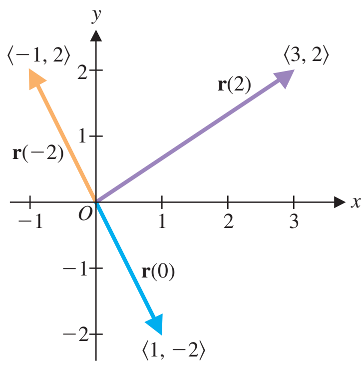


FIGURE 11.2a
Some values of
$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 2)\mathbf{j}$

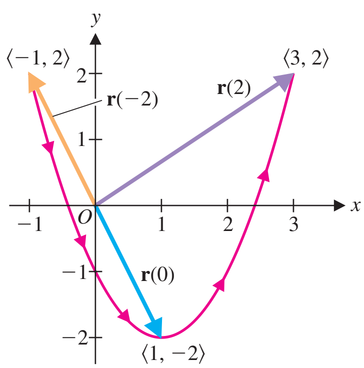


FIGURE 11.2b
Curve defined by
$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 2)\mathbf{j}$

REMARK 1.1

We routinely use the variable t to represent the independent variable for vector-valued functions, since in many applications t represents *time.*

EXAMPLE 1.1 Sketching the Curve Defined by a Vector-Valued Function

Sketch a graph of the curve traced out by the endpoint of the two-dimensional vector-valued function

$$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 2)\mathbf{j}.$$

Solution Substituting some values for t, we have $\mathbf{r}(0) = \mathbf{i} - 2\mathbf{j} = \langle 1, -2\rangle$, $\mathbf{r}(2) = 3\mathbf{i} + 2\mathbf{j} = \langle 3, 2\rangle$ and $\mathbf{r}(-2) = \langle -1, 2\rangle$. We plot these in Figure 11.2a. The endpoints of all position vectors $\mathbf{r}(t)$ lie on the curve C, described parametrically by

$$C\colon x = t + 1, \quad y = t^2 - 2, \quad t \in \mathbb{R}.$$

We can eliminate the parameter by solving for t in terms of x:

$$t = x - 1.$$

The curve is then given by

$$y = t^2 - 2 = (x-1)^2 - 2.$$

Notice that the graph of this is a parabola opening up, with vertex at the point $(1, -2)$, as seen in Figure 11.2b. The small arrows marked on the graph indicate the **orientation,** that is, the direction of increasing values of t. If the curve describes the path of an object, then the orientation indicates the direction in which the object traverses the path. In this case, we can easily determine the orientation from the parametric representation of the curve. Since $x = t + 1$, observe that x increases as t increases. ■

You may recall from your experience with parametric equations in Chapter 9 that eliminating the parameter from the parametric representation of a curve is not always so easy as it was in example 1.1. We illustrate this in example 1.2.

EXAMPLE 1.2 A Vector-Valued Function Defining an Ellipse

Sketch a graph of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = 4\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$, $t \in \mathbb{R}$.

Solution In this case, the curve can be written parametrically as

$$x = 4\cos t, \quad y = -3\sin t, \quad t \in \mathbb{R}.$$

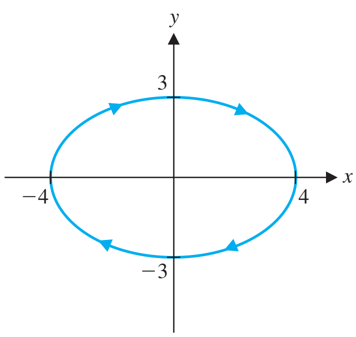


FIGURE 11.3
Curve defined by
$\mathbf{r}(t) = 4\cos t\mathbf{i} - 3\sin t\mathbf{j}$

Instead of solving for the parameter t, it often helps to look for some relationship between the variables. Here,

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$

or

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1,$$

which is the equation of an ellipse (see Figure 11.3). To determine the orientation of the curve here, you'll need to look carefully at both parametric equations. First, fix a starting place on the curve, for convenience, say (4, 0). This corresponds to $t = 0, \pm 2\pi, \pm 4\pi, \ldots$. As t increases, notice that $\cos t$ (and hence, x) decreases initially, while $\sin t$ increases, so that $y = -3\sin t$ decreases (initially). With both x and y decreasing initially, we get the clockwise orientation indicated in Figure 11.3. ■

Just as the endpoint of a vector-valued function in two dimensions traces out a curve, if we were to plot the value of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ for every value of t, the endpoints of the vectors would trace out a curve in three dimensions.

EXAMPLE 1.3 A Vector-Valued Function Defining an Elliptical Helix

Plot the curve traced out by the vector-valued function $\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$, $t \geq 0$.

Solution The curve is given parametrically by

$$x = \sin t, \quad y = -3\cos t, \quad z = 2t, \quad t \geq 0.$$

While most curves in three dimensions are difficult to recognize, you should notice that there is a relationship between x and y here, namely,

$$x^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t = 1. \tag{1.2}$$

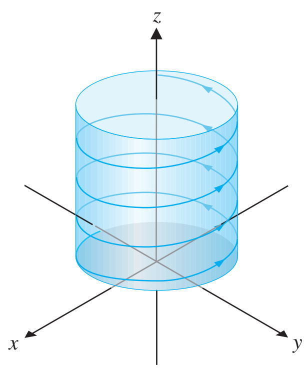


FIGURE 11.4a
Elliptical helix:
$\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$

In two dimensions, this is the equation of an ellipse. In three dimensions, since the equation does not involve z, (1.2) is the equation of an elliptic cylinder whose axis is the z-axis. This says that every point on the curve defined by $\mathbf{r}(t)$ lies on this cylinder. From the parametric equations for x and y (in two dimensions), the ellipse is traversed in the counterclockwise direction. This says that the curve will wrap itself around the cylinder (counterclockwise, as you look down the positive z-axis toward the origin), as t increases. Finally, since $z = 2t$, z will increase as t increases and so, the curve will wind its way up the cylinder, as t increases. We show the curve and the elliptical cylinder in Figure 11.4a. We call this curve an **elliptical helix.** In Figure 11.4b, we display a computer-generated graph of the same helix. There, rather than the usual x-, y- and z-axes, we show a framed graph, where the values of x, y and z are indicated on three adjacent edges of a box containing the graph. ■

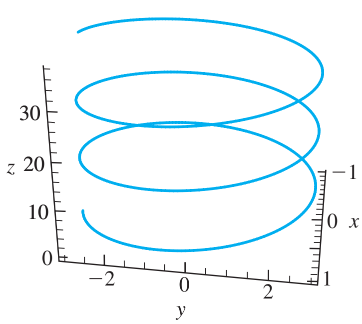


FIGURE 11.4b
Computer sketch:
$\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$

We can use vector-valued functions as a convenient representation of some very familiar curves, as we see in example 1.4.

EXAMPLE 1.4 A Vector-Valued Function Defining a Line

Plot the curve traced out by the vector-valued function

$$\mathbf{r}(t) = \langle 3 + 2t, 5 - 3t, 2 - 4t\rangle, \quad t \in \mathbb{R}.$$

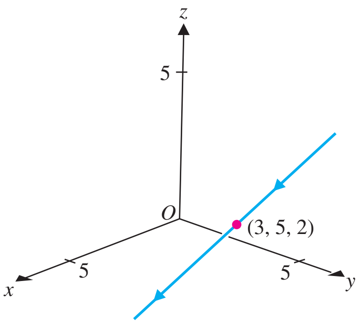


FIGURE 11.5
Straight line:
$\mathbf{r}(t) = \langle 3+2t, 5-3t, 2-4t \rangle$

Solution Notice that the curve is given parametrically by

$$x = 3 + 2t, \quad y = 5 - 3t, \quad z = 2 - 4t, \quad t \in \mathbb{R}.$$

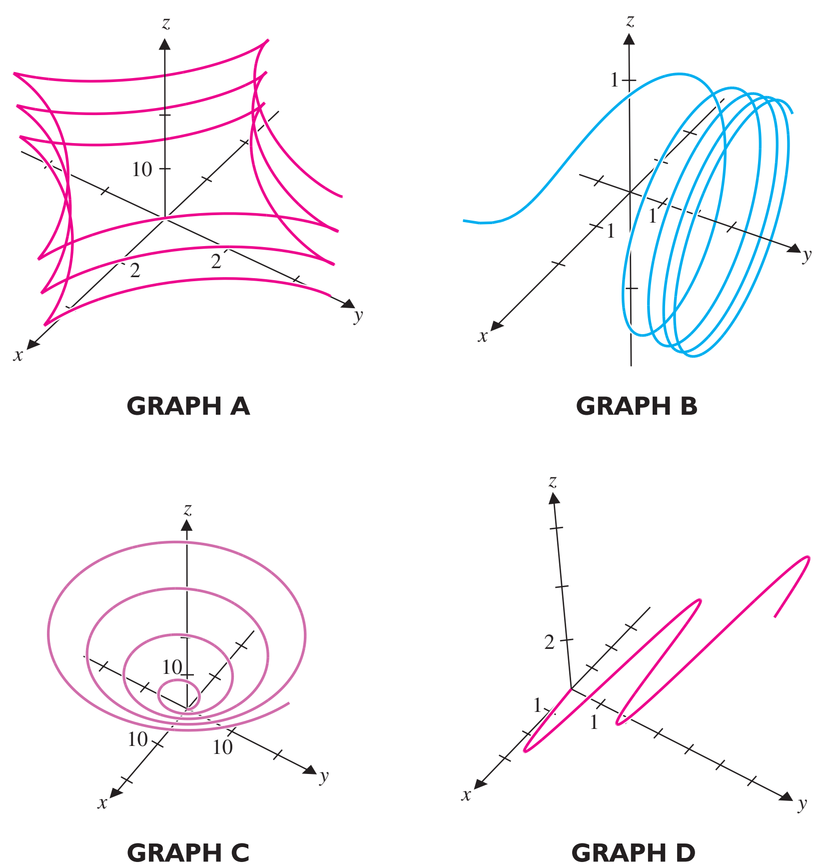
You should recognize these equations as parametric equations for the straight line parallel to the vector $\langle 2, -3, -4 \rangle$ and passing through the point $(3, 5, 2)$, as seen in Figure 11.5. ■

Most three-dimensional graphs are very challenging to sketch by hand. You will probably want to use computer-generated graphics for most sketches. Even so, you will need to be knowledgeable enough to know when to zoom in or out or rotate a graph to uncover a hidden feature. You should be able to draw several basic curves by hand, like those in examples 1.3 and 1.4. More importantly, you should be able to recognize the effects various components have on the graph of a three-dimensional curve. In example 1.5, we walk you through matching four vector-valued functions with their computer-generated graphs.

EXAMPLE 1.5 Matching a Vector-Valued Function to Its Graph

Match each of the vector-valued functions $\mathbf{f}_1(t) = \langle \cos t, \ln t, \sin t \rangle$, $\mathbf{f}_2(t) = \langle t \cos t, t \sin t, t \rangle$, $\mathbf{f}_3(t) = \langle 3 \sin 2t, t, t \rangle$ and $\mathbf{f}_4(t) = \langle 5 \sin^3 t, 5 \cos^3 t, t \rangle$ with the corresponding computer-generated graph.

Solution First, realize that there is no single, correct procedure for solving this problem. Look for familiar functions and match them with familiar graphical properties.



From example 1.3, recall that certain combinations of sines and cosines will produce curves that lie on a cylinder. Notice that for the function $\mathbf{f}_1(t)$, $x = \cos t$ and $z = \sin t$, so that

$$x^2 + z^2 = \cos^2 t + \sin^2 t = 1.$$

This says that every point on the curve lies on the cylinder $x^2 + z^2 = 1$ (the right circular cylinder of radius 1 whose axis is the y-axis). Further, the function $y = \ln t$ tends rapidly to $-\infty$ as $t \to 0$ and increases slowly as t increases beyond $t = 1$. Notice that the curve in Graph B appears to lie on a right circular cylinder and that the spirals get closer together as you move to the right (as $y \to \infty$) and move very far apart as you move to the left (as $y \to -\infty$). At first glance, you might expect the curve traced out by $\mathbf{f}_2(t)$ also to lie on a right circular cylinder, but look more closely. Here, we have $x = t\cos t$, $y = t\sin t$ and $z = t$, so that

$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2.$$

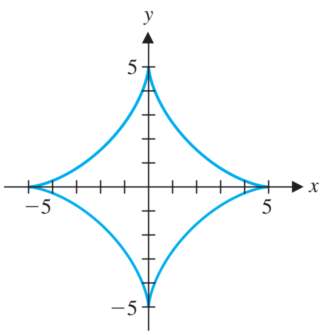


FIGURE 11.6
A cross section of the cylinder $x = 5\sin^3 t$, $y = 5\cos^3 t$

This says that the curve lies on the surface defined by $x^2 + y^2 = z^2$ (a right circular cone with axis along the z-axis). Notice that only the curve shown in Graph C fits this description. Next, notice that for $\mathbf{f}_3(t)$, the y and z components are identical and so, the curve must lie in the plane $y = z$. Replacing t by y, we have $x = 3\sin 2t = 3\sin 2y$, a sine curve lying in the plane $y = z$. Clearly, the curve in Graph D matches this description. Although Graph A is the only curve remaining to match with $\mathbf{f}_4(t)$, notice that if the cosine and sine terms weren't cubed, we'd simply have a helix, as in example 1.3. Since $z = t$, each point on the curve is a point on the cylinder defined parametrically by $x = 5\sin^3 t$ and $y = 5\cos^3 t$. You need only look at the graph of the cross section of the cylinder shown in Figure 11.6 (found by graphing the parametric equations $x = 5\sin^3 t$ and $y = 5\cos^3 t$ in two dimensions) to decide that Graph A is the obvious choice. ■

○ Arc Length in $\mathbb{R}^3$

A natural question to ask about a curve is, "How long is it?" Recall from section 5.4 that if f and f' are continuous on the interval $[a, b]$, then the arc length of the curve $y = f(x)$ on that interval is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

In section 9.3, we extended this to the case of a curve defined parametrically by $x = f(t)$, $y = g(t)$, where f, f', g and g' are all continuous for $t \in [a, b]$. In this case, we showed that if the curve is traversed exactly once as t increases from a to b, then the arc length is given by

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt. \tag{1.3}$$

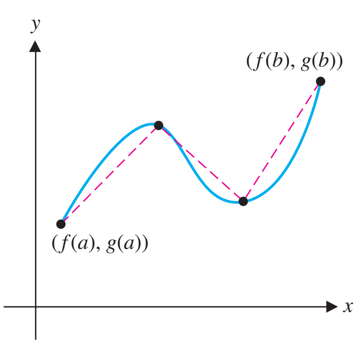


FIGURE 11.7a
Approximate arc length in $\mathbb{R}^2$

In both cases, recall that we developed the arc length formula by first breaking the curve into small pieces (i.e., we *partitioned* the interval $[a, b]$) and then approximating the length with the sum of the lengths of small line segments connecting successive points (see Figure 11.7a). Finally, we made the approximation exact by taking a limit as the number of points in the partition tended to infinity. This says that if the curve C in $\mathbb{R}^2$ is traced out exactly once by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ for $t \in [a, b]$, then the arc length is given by (1.3).

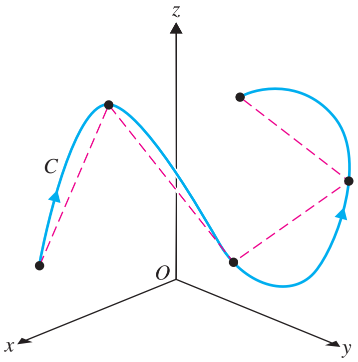


FIGURE 11.7b
Approximate arc length in $\mathbb{R}^3$

The situation in three dimensions is a straightforward extension of the two-dimensional case. Suppose that a curve is traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, f', g, g', h and h' are all continuous for $t \in [a, b]$ and where the curve is traversed exactly once as t increases from a to b. As we have done countless times now, we begin by approximating the quantity of interest, in this case, the arc length. To do this, we partition the interval $[a, b]$ into n subintervals of equal size: $a = t_0 < t_1 < \cdots < t_n = b$, where $t_i - t_{i-1} = \Delta t = \frac{b-a}{n}$, for all $i = 1, 2, \ldots, n$. Next, for each $i = 1, 2, \ldots, n$, we approximate the arc length s_i of that portion of the curve joining the points $(f(t_{i-1}), g(t_{i-1}), h(t_{i-1}))$ and $(f(t_i), g(t_i), h(t_i))$ by the straight-line distance between the points. (See Figure 11.7b for an illustration of the case where $n = 4$.) From the distance formula, we have

$$\begin{aligned} s_i &\approx d\{(f(t_{i-1}), g(t_{i-1}), h(t_{i-1})), (f(t_i), g(t_i), h(t_i))\} \\ &= \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 + [h(t_i) - h(t_{i-1})]^2}. \end{aligned}$$

Applying the Mean Value Theorem three times (why can we do this?), we get

$$\begin{aligned} f(t_i) - f(t_{i-1}) &= f'(c_i)(t_i - t_{i-1}) = f'(c_i)\,\Delta t, \\ g(t_i) - g(t_{i-1}) &= g'(d_i)(t_i - t_{i-1}) = g'(d_i)\,\Delta t \end{aligned}$$

and

$$h(t_i) - h(t_{i-1}) = h'(e_i)(t_i - t_{i-1}) = h'(e_i)\,\Delta t,$$

for some points c_i, d_i and e_i in the interval (t_{i-1}, t_i). This gives us

$$\begin{aligned} s_i &\approx \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 + [h(t_i) - h(t_{i-1})]^2} \\ &= \sqrt{[f'(c_i)\,\Delta t]^2 + [g'(d_i)\,\Delta t]^2 + [h'(e_i)\,\Delta t]^2} \\ &= \sqrt{[f'(c_i)]^2 + [g'(d_i)]^2 + [h'(e_i)]^2}\,\Delta t. \end{aligned}$$

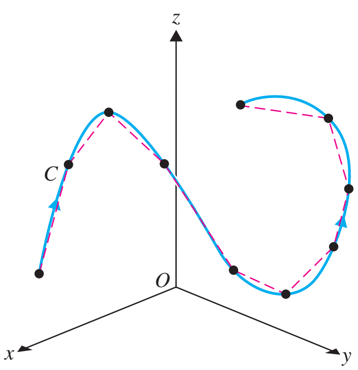


FIGURE 11.7c
Improved arc length approximation

Notice that if Δt is small, then all of c_i, d_i and e_i are very close and we can make the further approximation

$$s_i \approx \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t,$$

for each $i = 1, 2, \ldots, n$. The total arc length is then approximately

$$s \approx \sum_{i=1}^{n} \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t,$$

where the total error in the approximation of arc length tends to 0, as $\Delta t \to 0$. (Carefully consider Figures 11.7b and 11.7c to see why.)

Taking the limit as $n \to \infty$ gives the exact arc length:

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t,$$

provided the limit exists. You should recognize this as the definite integral

Arc length

$$\boxed{s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt.} \tag{1.4}$$

Observe that the arc length formula for a curve in $\mathbb{R}^2$ (1.3) is a special case of (1.4). Unfortunately, the integral in (1.4) can only rarely be computed exactly and we must typically be satisfied with a numerical approximation. Example 1.6 illustrates one of the very few arc lengths in $\mathbb{R}^3$ that can be computed exactly.

EXAMPLE 1.6 Computing Arc Length in $\mathbb{R}^3$

Find the arc length of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle 2t, \ln t, t^2 \rangle$, for $1 \le t \le e$.

Solution First, notice that for $x(t) = 2t$, $y(t) = \ln t$ and $z(t) = t^2$, we have $x'(t) = 2$, $y'(t) = \frac{1}{t}$ and $z'(t) = 2t$, and the curve is traversed exactly once for $1 \le t \le e$. (To see why, observe that $x = 2t$ is an increasing function.) From (1.4), we now have

$$\begin{aligned} s &= \int_1^e \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\, dt = \int_1^e \sqrt{4 + \frac{1}{t^2} + 4t^2}\, dt \\ &= \int_1^e \sqrt{\frac{1 + 4t^2 + 4t^4}{t^2}}\, dt = \int_1^e \sqrt{\frac{(1 + 2t^2)^2}{t^2}}\, dt \\ &= \int_1^e \frac{1 + 2t^2}{t}\, dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt \\ &= \left(\ln |t| + 2\frac{t^2}{2}\right)\Bigg|_1^e = (\ln e + e^2) - (\ln 1 + 1) = e^2. \end{aligned}$$

We show a graph of the curve for $1 \le t \le e$ in Figure 11.8. ■

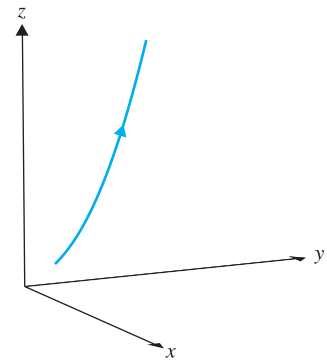


FIGURE 11.8
The curve defined by $\mathbf{r}(t) = \langle 2t, \ln t, t^2 \rangle$

The arc length integral in example 1.7 is typical, in that we need a numerical approximation.

EXAMPLE 1.7 Approximating Arc Length in $\mathbb{R}^3$

Find the arc length of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle e^{2t}, \sin t, t \rangle$, for $0 \le t \le 2$.

Solution First, note that for $x(t) = e^{2t}$, $y(t) = \sin t$ and $z(t) = t$, we have $x'(t) = 2e^{2t}$, $y'(t) = \cos t$ and $z'(t) = 1$, and that the curve is traversed exactly once for $0 \le t \le 2$. From (1.4), we now have

$$s = \int_0^2 \sqrt{(2e^{2t})^2 + (\cos t)^2 + 1^2}\, dt = \int_0^2 \sqrt{4e^{4t} + \cos^2 t + 1}\, dt.$$

Since you don't know how to evaluate this integral exactly (which is typically the case), you can approximate the integral using Simpson's Rule or the numerical integration routine built into your calculator or computer algebra system, to find that the arc length is approximately $s \approx 53.8$. ■

Often, the curve of interest is determined by the intersection of two surfaces. Parametric equations can give us simple representations of many such curves.

EXAMPLE 1.8 Finding Parametric Equations for an Intersection of Surfaces

Find the arc length of the portion of the curve determined by the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $y + z = 2$ in the first octant.

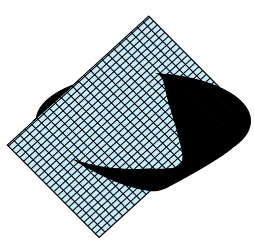


FIGURE 11.9a
Intersection of cone and plane

Solution The cone and plane are shown in Figure 11.9a. From your knowledge of conic sections, note that this curve could be a parabola or an ellipse. Parametric equations for the curve must satisfy both $z = \sqrt{x^2 + y^2}$ and $y + z = 2$. Eliminating z by solving for it in each equation, we get

$$z = \sqrt{x^2 + y^2} = 2 - y.$$

Squaring both sides and gathering terms, we get

$$x^2 + y^2 = (2 - y)^2 = 4 - 4y + y^2$$

or

$$x^2 = 4 - 4y.$$

Solving for y now gives us

$$y = 1 - \frac{x^2}{4},$$

which is clearly the equation of a parabola in two dimensions. To obtain the equation for the three-dimensional parabola, let x be the parameter, which gives us the parametric equations

$$x = t, \quad y = 1 - \frac{t^2}{4} \quad \text{and} \quad z = \sqrt{t^2 + (1 - t^2/4)^2} = 1 + \frac{t^2}{4}.$$

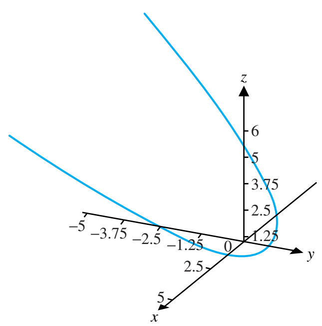


FIGURE 11.9b
Curve of intersection

A graph is shown in Figure 11.9b. The portion of the parabola in the first octant must have $x \geq 0$ (so $t \geq 0$), $y \geq 0$ (so $t^2 \leq 4$) and $z \geq 0$ (always true). This occurs if $0 \leq t \leq 2$. The arc length is then

$$s = \int_0^2 \sqrt{1 + (-t/2)^2 + (t/2)^2}\, dt = \frac{\sqrt{2}}{2} \ln\left(\sqrt{2} + \sqrt{3}\right) + \sqrt{3} \approx 2.54,$$

where we leave the details of the integration to you. ■

BEYOND FORMULAS

If you think that examples 1.1 and 1.2 look very much like parametric equations examples, you're exactly right. The ideas presented there are not new; only the notation and terminology are new. However, the vector notation lets us easily extend these ideas into three dimensions, where the graphs can be more complicated.

EXERCISES 11.1

WRITING EXERCISES

1. Discuss the differences, if any, between the curve traced out by the terminal point of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ and the curve defined parametrically by $x = f(t)$, $y = g(t)$.

2. In example 1.3, describe the "shadow" of the helix in the xy-plane (the shadow created by shining a light down from the "top" of the z-axis). Equivalently, if the helix is collapsed down into the xy-plane, describe the resulting curve. Compare this curve to the ellipse defined parametrically by $x = \sin t$, $y = -3\cos t$.

3. Discuss how you would compute the arc length of a curve in four or more dimensions. Specifically, for the curve traced out by the terminal point of the n-dimensional vector-valued function $\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t)\rangle$ for $n \geq 4$, state the arc

length formula and discuss how it relates to the n-dimensional distance formula.

4. The helix in Figure 11.4a is shown from a standard viewpoint (above the xy-plane, in between the x- and y-axes). Describe what an observer at the point $(0, 0, -1000)$ would see. Also, describe what observers at the points $(1000, 0, 0)$ and $(0, 1000, 0)$ would see.

In exercises 1–4, plot the values of the vector-valued function at the indicated values of t.

1. $\mathbf{r}(t) = \langle 3t, t^2, 2t - 1\rangle, t = 0, t = 1, t = 2$
2. $\mathbf{r}(t) = (4 - t)\mathbf{i} + (1 - t^2)\mathbf{j} + (t^3 - 1)\mathbf{k}, t = -2, t = 0, t = 2$
3. $\mathbf{r}(t) = \langle \cos 3t, 2, \sin 2t - 1\rangle, t = -\frac{\pi}{2}, t = 0, t = \frac{\pi}{2}$
4. $\mathbf{r}(t) = \langle e^{2-t}, 1 - t, 3\rangle, t = -1, t = 0, t = 1$

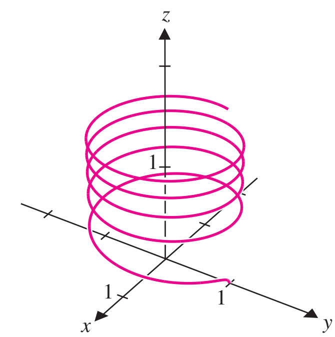
In exercises 5–18, sketch the curve traced out by the given vector-valued function by hand.

5. $\mathbf{r}(t) = \langle 2\cos t, \sin t - 1\rangle$
6. $\mathbf{r}(t) = \langle \sin t - 2, 4\cos t\rangle$
7. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 3\rangle$
8. $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, 1\rangle$
9. $\mathbf{r}(t) = \langle t, t^2 + 1, -1\rangle$
10. $\mathbf{r}(t) = \langle 3, t, t^2 - 1\rangle$
11. $\mathbf{r}(t) = \langle t, 1, 3t^2\rangle$
12. $\mathbf{r}(t) = \langle t + 2, 2t - 1, t + 2\rangle$
13. $\mathbf{r}(t) = \langle 4t - 1, 2t + 1, -6t\rangle$
14. $\mathbf{r}(t) = \langle -2t, 2t, 3 - t\rangle$
15. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$
16. $\mathbf{r}(t) = \langle 2\cos t, \sin t, 3t\rangle$
17. $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 2t\rangle$
18. $\mathbf{r}(t) = \langle -1, 2\cos t, 2\sin t\rangle$

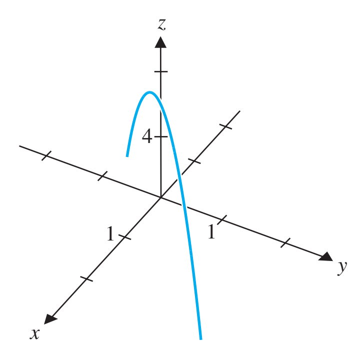
In exercises 19–30, use graphing technology to sketch the curve traced out by the given vector-valued function.

19. $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t\rangle$
20. $\mathbf{r}(t) = \langle 2\cos 3t + \sin 5t, 2\sin 3t + \cos 5t\rangle$
21. $\mathbf{r}(t) = \langle 4\cos 4t - 6\cos t, 4\sin 4t - 6\sin t\rangle$
22. $\mathbf{r}(t) = \langle 8\cos t + 2\cos 7t, 8\sin t + 2\sin 7t\rangle$
23. $\mathbf{r}(t) = \langle t\cos 2t, t\sin 2t, 2t\rangle$
24. $\mathbf{r}(t) = \langle t\cos t, 2t, t\sin t\rangle$
25. $\mathbf{r}(t) = \langle \cos 5t, \sin t, \sin 6t\rangle$
26. $\mathbf{r}(t) = \langle 3\cos 2t, \sin t, \cos 3t\rangle$
27. $\mathbf{r}(t) = \langle t, t, 2t^2 - 1\rangle$
28. $\mathbf{r}(t) = \langle t^3 - t, t^2, 2t - 4\rangle$
29. $\mathbf{r}(t) = \langle \tan t, \sin t^2, \cos t\rangle$
30. $\mathbf{r}(t) = \langle \sin t, -\csc t, \cot t\rangle$

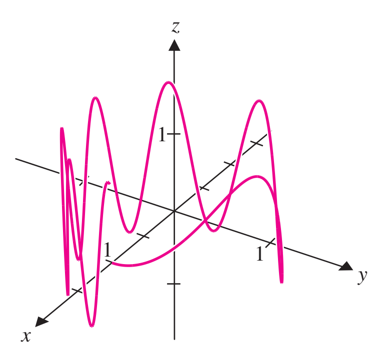
31. In parts a–f, match the vector-valued function with its graph. Give reasons for your choices.
   a. $\mathbf{r}(t) = \langle \cos t^2, t, t\rangle$
   b. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t^2\rangle$
   c. $\mathbf{r}(t) = \langle \sin 16\sqrt{t}, \cos 16\sqrt{t}, t\rangle$
   d. $\mathbf{r}(t) = \langle \sin t^2, \cos t^2, t\rangle$
   e. $\mathbf{r}(t) = \langle t, t, 6 - 4t^2\rangle$
   f. $\mathbf{r}(t) = \langle t^3 - t, 0.5t^2, 2t - 4\rangle$



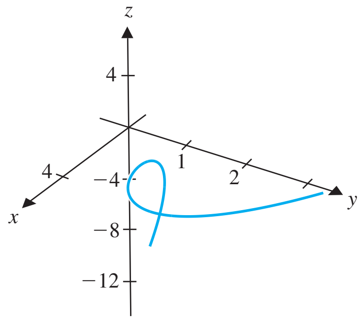
GRAPH A



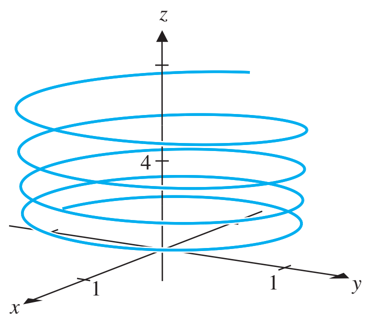
GRAPH B



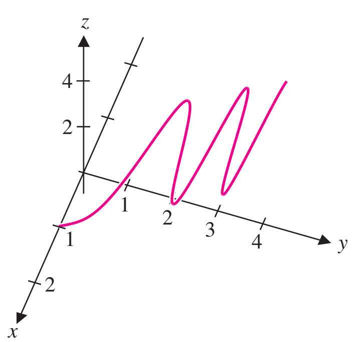
GRAPH C



GRAPH D



GRAPH E



GRAPH F

32. Of the functions in exercise 31, which are periodic? Which are bounded?

In exercises 33–38, use a CAS to sketch the curve and estimate its arc length.

33. $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle, 0 \le t \le 2\pi$

34. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t + \cos t \rangle, 0 \le t \le 2\pi$

35. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, \cos 16t \rangle, 0 \le t \le 2$

36. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, \cos 16t \rangle, 0 \le t \le 4$

37. $\mathbf{r}(t) = \langle t, t^2 - 1, t^3 \rangle, 0 \le t \le 2$

38. $\mathbf{r}(t) = \langle t^2 + 1, 2t, t^2 - 1 \rangle, 0 \le t \le 2$

39. Show that the curve in exercise 33 lies on the hyperbolic paraboloid $z = x^2 - y^2$. Use a CAS to sketch both the surface and the curve.

40. Show that the curve in exercise 34 lies on the plane $z = x + y$. Use a CAS to sketch both the plane and the curve.

41. Show that the curve $\mathbf{r}(t) = \langle 2t, 4t^2 - 1, 8t^3 \rangle, 0 \le t \le 1$, has the same arc length as the curve in exercise 37.

42. Show that the curve $\mathbf{r}(t) = \langle t + 1, 2\sqrt{t}, t - 1 \rangle, 0 \le t \le 4$, has the same arc length as the curve in exercise 38.

43. Compare the graphs of $\mathbf{r}(t) = \langle t, t^2, t^2 \rangle$, $\mathbf{g}(t) = \langle \cos t, \cos^2 t, \cos^2 t \rangle$ and $\mathbf{h}(t) = \langle \sqrt{t}, t, t \rangle$. Explain the similarities and the differences.

44. Compare the graphs of $\mathbf{r}(t) = \langle 2t - 1, t^2, t \rangle$, $\mathbf{g}(t) = \langle 2\sin t - 1, \sin^2 t, \sin t \rangle$ and $\mathbf{h}(t) = \langle 2e^t - 1, e^{2t}, e^t \rangle$. Explain the similarities and the differences.

In exercises 45–48, find parametric equations for the indicated curve. If you have access to a graphing utility, graph the surfaces and the resulting curve.

45. The intersection of $z = \sqrt{x^2 + y^2}$ and $z = 2$

46. The intersection of $z = \sqrt{x^2 + y^2}$ and $y + 2z = 2$

47. The intersection of $x^2 + y^2 = 9$ and $y + z = 2$

48. The intersection of $y^2 + z^2 = 9$ and $x = 2$

49. A spiral staircase makes two complete turns as it rises 10 feet between floors. A handrail at the outside of the staircase is located 3 feet from the center pole of the staircase. Use parametric equations for a helix to compute the length of the handrail.

50. Exercise 49 can be worked without calculus. (You might have suspected this since the integral in exercise 49 simplifies dramatically.) Imagine unrolling the staircase so that the handrail is a line segment. Use the formula for the hypotenuse of a right triangle to compute its length.

51. Use a graphing utility to sketch the graph of $\mathbf{r}(t) = \langle \cos t, \cos t, \sin t \rangle$ with $0 \le t \le 2\pi$. Explain why the graph should be the same with $0 \le t \le T$, for any $T \ge 2\pi$. Try several larger domains ($0 \le t \le 2\pi, 0 \le t \le 10\pi, 0 \le t \le 50\pi$, etc.) with your graphing utility. Eventually, the ellipse should start looking thicker and for large enough domains you will see a mess of jagged lines. Explain what has gone wrong with the graphing utility.

52. It may surprise you that the curve in exercise 51 is not a circle. Show that the shadows in the xz-plane and yz-plane are circles. Show that the curve lies in the plane $x = y$. Sketch a graph showing the plane $x = y$ and a circular shadow in the yz-plane. To draw a curve in the plane $x = y$ with the circular shadow, explain why the curve must be wider in the xy-direction than in the z-direction. In other words, the curve is not circular.

53. Show that the arc length of the helix $\langle \cos t, \sin t, t \rangle$, for $0 \le t \le 2\pi$, is $2\pi\sqrt{2}$, equal to the length of the diagonal of a square of side 2π. Show this graphically.

EXPLORATORY EXERCISES

1. In contrast to exercises 51 and 52, the graph of $\mathbf{r}(t) = \langle \cos t, \cos t, \sqrt{2}\sin t \rangle$ *is* a circle. To verify this, start by showing that $\|\mathbf{r}(t)\| = \sqrt{2}$, for all t. Then observe that the curve lies in the plane $x = y$. Explain why this proves that the graph is a (portion of a) circle. A little more insight can be gained by looking at basis vectors. The circle lies in the plane $x = y$, which contains the vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1, 0 \rangle$. The plane $x = y$ also contains the vector $\mathbf{v} = \langle 0, 0, 1 \rangle$. Show that *any* vector $\mathbf{w}$ in the plane $x = y$ can be written as $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ for some constants c_1 and c_2. Also, show that $\mathbf{r}(t) = (\sqrt{2}\cos t)\mathbf{u} + (\sqrt{2}\sin t)\mathbf{v}$. Recall that in two dimensions, a circle of radius r centered at the origin can be written parametrically as $(r\cos t)\mathbf{i} + (r\sin t)\mathbf{j}$. In general, suppose that $\mathbf{u}$ and $\mathbf{v}$ are any orthogonal unit vectors. If $\mathbf{r}(t) = (r\cos t)\mathbf{u} + (r\sin t)\mathbf{v}$, show that $\mathbf{r}(t)\cdot\mathbf{r}(t) = r^2$.

2. Referring to exercises 21 and 22, examine the graphs of several vector-valued functions of the form $\mathbf{r}(t) = \langle a\cos ct + b\cos dt, a\sin ct + b\sin dt \rangle$, for constants a, b, c and d. Determine the values of these constants that produce graphs of different types. For example, starting with the graph of $\langle 4\cos 4t - 6\cos t, 4\sin 4t - 6\sin t \rangle$, change $c = 4$ to $c = 3$, $c = 5$, $c = 2$, etc. Conjecture a relationship between the number of loops and the difference between c and d. Test this conjecture on other vector-valued functions. Returning to $\langle 4\cos 4t - 6\cos t, 4\sin 4t - 6\sin t \rangle$, change $a = 4$ to other values. Conjecture a relationship between the size of the loops and the value of a.

11.2 THE CALCULUS OF VECTOR-VALUED FUNCTIONS

In this section, we begin to explore the calculus of vector-valued functions, beginning with the notion of limit and progressing to continuity, derivatives and finally, integrals. Take careful note of how our presentation parallels our development of the calculus of scalar functions in Chapters 1, 2 and 4. We follow this same progression again when we examine functions of several variables in Chapter 12. We define everything in this section in terms of vector-valued functions in three dimensions. The definitions can be interpreted for vector-valued functions in two dimensions in the obvious way, by simply dropping the third component everywhere.

For a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, when we write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{u},$$

we mean that as t gets closer and closer to a, the vector $\mathbf{r}(t)$ is getting closer and closer to the vector $\mathbf{u}$. For $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, this means that

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} \langle f(t), g(t), h(t) \rangle = \mathbf{u} = \langle u_1, u_2, u_3 \rangle.$$

Notice that for this to occur, we must have that $f(t)$ is approaching u_1, $g(t)$ is approaching u_2 and $h(t)$ is approaching u_3. In view of this, we make the following definition.

DEFINITION 2.1

For a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, the **limit** of $\mathbf{r}(t)$ as t approaches a is given by

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} \langle f(t), g(t), h(t) \rangle = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle, \tag{2.1}$$

provided *all* of the indicated limits exist. If any of the limits indicated on the right-hand side of (2.1) fail to exist, then $\lim_{t\to a} \mathbf{r}(t)$ **does not exist.**

In example 2.1, we see that calculating a limit of a vector-valued function simply consists of calculating three separate limits of scalar functions.

EXAMPLE 2.1 Finding the Limit of a Vector-Valued Function

Find $\lim_{t\to 0}\langle t^2+1, 5\cos t, \sin t\rangle$.

Solution Here, each of the component functions is continuous (for all t) and so, we can calculate their limits simply by substituting the value for t. We have

$$\begin{aligned}\lim_{t\to 0}\langle t^2+1, 5\cos t, \sin t\rangle &= \left\langle \lim_{t\to 0}(t^2+1), \lim_{t\to 0}(5\cos t), \lim_{t\to 0}\sin t\right\rangle \\ &= \langle 1, 5, 0\rangle.\end{aligned}$$

EXAMPLE 2.2 A Limit That Does Not Exist

Find $\lim_{t\to 0}\langle e^{2t}+5, t^2+2t-3, 1/t\rangle$.

Solution Notice that the limit of the third component is $\lim_{t\to 0}\frac{1}{t}$, which does not exist. So, even though the limits of the first two components exist, the limit of the vector-valued function does not exist.

Recall that for a scalar function f, we say that f is **continuous** at a if and only if

$$\lim_{t\to a} f(t) = f(a).$$

That is, a scalar function is continuous at a point whenever the limit and the value of the function are the same. We define the continuity of vector-valued functions in the same way.

DEFINITION 2.2

The vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is **continuous** at $t = a$ whenever

$$\lim_{t\to a}\mathbf{r}(t) = \mathbf{r}(a)$$

(i.e., whenever the limit exists and equals the value of the vector-valued function).

Notice that in terms of the components of $\mathbf{r}$, this says that $\mathbf{r}(t)$ is continuous at $t = a$ whenever

$$\lim_{t\to a}\langle f(t), g(t), h(t)\rangle = \langle f(a), g(a), h(a)\rangle.$$

Further, since

$$\lim_{t\to a}\langle f(t), g(t), h(t)\rangle = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle,$$

it follows that $\mathbf{r}$ is continuous at $t = a$ if and only if

$$\left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle = \langle f(a), g(a), h(a)\rangle,$$

which occurs if and only if

$$\lim_{t\to a} f(t) = f(a), \quad \lim_{t\to a} g(t) = g(a) \quad \text{and} \quad \lim_{t\to a} h(t) = h(a).$$

Look carefully at what we have just said, and observe that we just proved the following theorem.

THEOREM 2.1

A vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at $t = a$ if and only if *all* of f, g and h are continuous at $t = a$.

Theorem 2.1 says that to determine where a vector-valued function is continuous, you need only check the continuity of each component function (something you already know how to do). We demonstrate this in examples 2.3 and 2.4.

EXAMPLE 2.3 Determining Where a Vector-Valued Function Is Continuous

Determine for what values of t the vector-valued function $\mathbf{r}(t) = \langle e^{5t}, \ln(t+1), \cos t\rangle$ is continuous.

Solution From Theorem 2.1, $\mathbf{r}(t)$ will be continuous wherever *all* its components are continuous. We have: e^{5t} is continuous for all t, $\ln(t+1)$ is continuous for $t > -1$ and $\cos t$ is continuous for all t. So, $\mathbf{r}(t)$ is continuous for $t > -1$. ■

EXAMPLE 2.4 A Vector-Valued Function with Infinitely Many Discontinuities

Determine for what values of t the vector-valued function $\mathbf{r}(t) = \left\langle \tan t, |t+3|, \frac{1}{t-2}\right\rangle$ is continuous.

Solution First, note that $\tan t$ is continuous, except at $t = \dfrac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$ (i.e., except at odd multiples of $\frac{\pi}{2}$). The second component $|t+3|$ is continuous for all t (although it's not differentiable at $t = -3$). Finally, the third component $\dfrac{1}{t-2}$ is continuous except at $t = 2$. Since all three components must be continuous in order for $\mathbf{r}(t)$ to be continuous, we have that $\mathbf{r}(t)$ is continuous, except at $t = 2$ and $t = \dfrac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$. ■

Recall that in Chapter 2, we defined the derivative of a scalar function f to be

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

Replacing h by Δt, we can rewrite this as

$$f'(t) = \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

You may be wondering why we want to change from a perfectly nice variable like h to something more unusual like Δt. The only reason is that we want to use the notation to emphasize that Δt is an *increment* of the variable t. In Chapter 12, we'll be defining partial derivatives of functions of more than one variable, where we'll use this type of notation to make it clear which variable is being incremented.

We now define the derivative of a vector-valued function in the expected way.

DEFINITION 2.3

The **derivative** $\mathbf{r}'(t)$ of the vector-valued function $\mathbf{r}(t)$ is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}, \tag{2.2}$$

for any values of t for which the limit exists. When the limit exists for $t = a$, we say that $\mathbf{r}$ is **differentiable** at $t = a$.

Fortunately, you will not need to learn any new differentiation rules, as the derivative of a vector-valued function is found directly from the derivatives of the individual components, as we see in Theorem 2.2.

THEOREM 2.2

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and suppose that the components f, g and h are all differentiable for some value of t. Then $\mathbf{r}$ is also differentiable at that value of t and its derivative is given by

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle. \tag{2.3}$$

PROOF

From the definition of derivative of a vector-valued function (2.2), we have

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\langle f(t + \Delta t) - f(t), g(t + \Delta t) - g(t), h(t + \Delta t) - h(t) \rangle,
\end{aligned}$$

from the definition of vector subtraction. Distributing the scalar $\frac{1}{\Delta t}$ into each component and using the definition of limit of a vector-valued function (2.1), we have

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\langle f(t + \Delta t) - f(t), g(t + \Delta t) - g(t), h(t + \Delta t) - h(t) \rangle \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t) \rangle,
\end{aligned}$$

where in the last step we recognized the definition of the derivatives of each of the component functions f, g and h. ■

We illustrate this in example 2.5.

EXAMPLE 2.5 Finding the Derivative of a Vector-Valued Function

Find the derivative of $\mathbf{r}(t) = \langle \sin(t^2), e^{\cos t}, t \ln t \rangle$.

Solution Applying the chain rule to the first two components and the product rule to the third, we have (for $t > 0$):

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle \frac{d}{dt}[\sin(t^2)], \frac{d}{dt}(e^{\cos t}), \frac{d}{dt}(t \ln t) \right\rangle \\
&= \left\langle \cos(t^2)\frac{d}{dt}(t^2), e^{\cos t}\frac{d}{dt}(\cos t), \frac{d}{dt}(t) \ln t + t\frac{d}{dt}(\ln t) \right\rangle \\
&= \left\langle \cos(t^2)(2t), e^{\cos t}(-\sin t), (1) \ln t + t\frac{1}{t} \right\rangle \\
&= \langle 2t\cos(t^2), -\sin t\, e^{\cos t}, \ln t + 1 \rangle. \quad \blacksquare
\end{aligned}$$

For the most part, to compute derivatives of vector-valued functions, we need only to use the already familiar rules for differentiation of scalar functions. There are several special derivative rules, however, which we state in Theorem 2.3.

THEOREM 2.3

Suppose that $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector-valued functions, $f(t)$ is a differentiable scalar function and c is any scalar constant. Then

(i) $\dfrac{d}{dt}[\mathbf{r}(t) + \mathbf{s}(t)] = \mathbf{r}'(t) + \mathbf{s}'(t)$

(ii) $\dfrac{d}{dt}[c\,\mathbf{r}(t)] = c\,\mathbf{r}'(t)$

(iii) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$

(iv) $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)] = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$ and

(v) $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$.

Notice that parts (iii), (iv) and (v) are the product rules for the various kinds of products we can define. In each of these three cases, it's important to recognize that these follow the same pattern as the usual product rule for the derivative of the product of two scalar functions.

PROOF

(i) For $\mathbf{r}(t) = \langle f_1(t), g_1(t), h_1(t) \rangle$ and $\mathbf{s}(t) = \langle f_2(t), g_2(t), h_2(t) \rangle$, we have from (2.3) and the rules for vector addition that

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r}(t) + \mathbf{s}(t)] &= \frac{d}{dt}[\langle f_1(t), g_1(t), h_1(t) \rangle + \langle f_2(t), g_2(t), h_2(t) \rangle] \\
&= \frac{d}{dt}\langle f_1(t) + f_2(t), g_1(t) + g_2(t), h_1(t) + h_2(t) \rangle
\end{aligned}$$

$$\begin{aligned}&= \langle f_1'(t) + f_2'(t), g_1'(t) + g_2'(t), h_1'(t) + h_2'(t)\rangle \\ &= \langle f_1'(t), g_1'(t), h_1'(t)\rangle + \langle f_2'(t), g_2'(t), h_2'(t)\rangle \\ &= \mathbf{r}'(t) + \mathbf{s}'(t).\end{aligned}$$

(iv) From the definition of dot product and the usual product rule for the product of two scalar functions, we have

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{s}(t)] &= \frac{d}{dt}[\langle f_1(t), g_1(t), h_1(t)\rangle \cdot \langle f_2(t), g_2(t), h_2(t)\rangle] \\ &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t) + h_1(t)h_2(t)] \\ &= f_1'(t)f_2(t) + f_1(t)f_2'(t) + g_1'(t)g_2(t) + g_1(t)g_2'(t) \\ &\quad + h_1'(t)h_2(t) + h_1(t)h_2'(t) \\ &= [f_1'(t)f_2(t) + g_1'(t)g_2(t) + h_1'(t)h_2(t)] \\ &\quad + [f_1(t)f_2'(t) + g_1(t)g_2'(t) + h_1(t)h_2'(t)] \\ &= \mathbf{r}'(t)\cdot\mathbf{s}(t) + \mathbf{r}(t)\cdot\mathbf{s}'(t).\end{aligned}$$

We leave the proofs of (ii), (iii) and (v) as exercises. ■

We say that the curve traced out by the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ on an interval I is **smooth** if $\mathbf{r}'$ is continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$, except possibly at any endpoints of I. Notice that this says that the curve is smooth provided f', g' and h' are all continuous on I and $f'(t)$, $g'(t)$ and $h'(t)$ are not *all* zero at the same point in I.

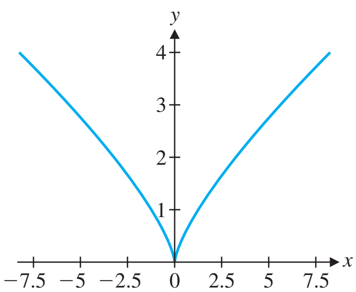


FIGURE 11.10
The curve traced out by $\mathbf{r}(t) = \langle t^3, t^2\rangle$

EXAMPLE 2.6 Determining Where a Curve Is Smooth

Determine where the plane curve traced out by the vector-valued function $\mathbf{r}(t) = \langle t^3, t^2\rangle$ is smooth.

Solution We show a graph of the curve in Figure 11.10.

Here, $\mathbf{r}'(t) = \langle 3t^2, 2t\rangle$ is continuous everywhere and $\mathbf{r}'(t) = \mathbf{0}$ if and only if $t = 0$. This says that the curve is smooth in any interval not including $t = 0$. Referring to Figure 11.10, observe that the curve is smooth except at the cusp located at the origin. ■

We next explore an important graphical interpretation of the derivative of a vector-valued function. First, recall that one interpretation of the derivative of a scalar function is that the value of the derivative at a point gives the slope of the tangent line to the curve at that point. For the case of the vector-valued function $\mathbf{r}(t)$, notice that from (2.2), the derivative of $\mathbf{r}(t)$ at $t = a$ is given by

$$\mathbf{r}'(a) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t}.$$

Again, recall that the endpoint of the vector-valued function $\mathbf{r}(t)$ traces out a curve C in $\mathbb{R}^3$. In Figure 11.11a (on the following page), we show the position vectors $\mathbf{r}(a)$, $\mathbf{r}(a+\Delta t)$ and $\mathbf{r}(a+\Delta t) - \mathbf{r}(a)$, for some fixed $\Delta t > 0$, using our graphical interpretation of vector subtraction, developed in Chapter 10. (How does the picture differ if $\Delta t < 0$?) Notice that for $\Delta t > 0$, the vector $\dfrac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t}$ points in the same direction as $\mathbf{r}(a+\Delta t) - \mathbf{r}(a)$.

If we take smaller and smaller values of Δt, $\dfrac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t}$ will approach $\mathbf{r}'(a)$. We illustrate this graphically in Figures 11.11b and 11.11c.

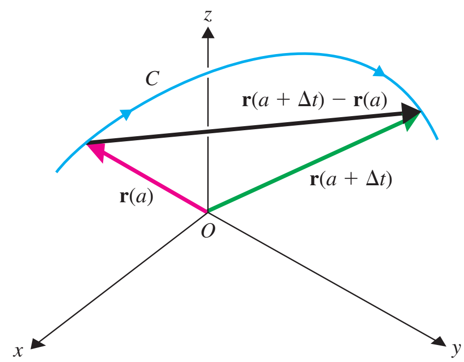


FIGURE 11.11a

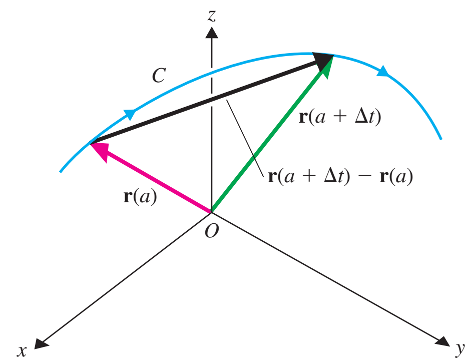


FIGURE 11.11b

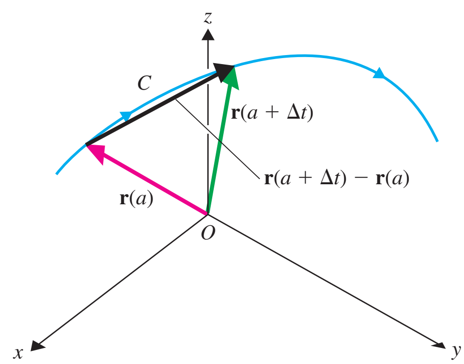


FIGURE 11.11c

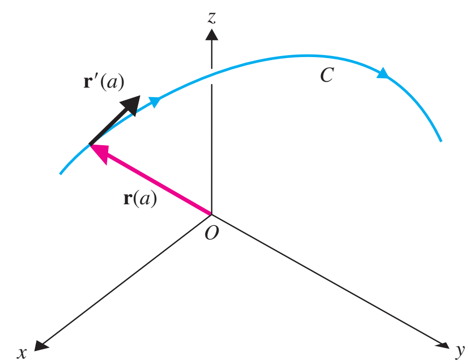


FIGURE 11.11d
The tangent vector $\mathbf{r}'(a)$

As $\Delta t \to 0$, notice that the vector $\dfrac{\mathbf{r}(a+\Delta t)-\mathbf{r}(a)}{\Delta t}$ approaches a vector that is tangent to the curve C at the terminal point of $\mathbf{r}(a)$, as seen in Figure 11.11d. We refer to $\mathbf{r}'(a)$ as a **tangent vector** to the curve C at the point corresponding to $t = a$. Be sure to observe that $\mathbf{r}'(a)$ lies along the tangent line to the curve at $t = a$ and points in the direction of the orientation of C. (Recognize that Figures 11.11a, 11.11b and 11.11c are all drawn so that $\Delta t > 0$. What changes in each of the figures if $\Delta t < 0$?)

We illustrate this notion for a simple curve in $\mathbb{R}^2$ in example 2.7.

EXAMPLE 2.7 Drawing Position and Tangent Vectors

For $\mathbf{r}(t) = \langle -\cos 2t, \sin 2t\rangle$, plot the curve traced out by the endpoint of $\mathbf{r}(t)$ and draw the position vector and tangent vector at $t = \frac{\pi}{4}$.

Solution First, notice that

$$\mathbf{r}'(t) = \langle 2\sin 2t, 2\cos 2t\rangle.$$

Also, the curve traced out by $\mathbf{r}(t)$ is given parametrically by

$$C\colon x = -\cos 2t, \quad y = \sin 2t, \quad t \in \mathbb{R}.$$

Observe that here,

$$x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1,$$

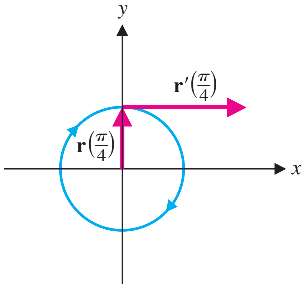


FIGURE 11.12
Position and tangent vectors

so that the curve is the circle of radius 1, centered at the origin. Further, from the parameterization, you can see that the orientation is clockwise. The position and tangent vectors at $t = \frac{\pi}{4}$ are given by

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle -\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right\rangle = \langle 0, 1\rangle$$

and

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle 2\sin\frac{\pi}{2}, 2\cos\frac{\pi}{2}\right\rangle = \langle 2, 0\rangle,$$

respectively. We show the curve, along with the vectors $\mathbf{r}(\frac{\pi}{4})$ and $\mathbf{r}'(\frac{\pi}{4})$ in Figure 11.12. In particular, you might note that

$$\mathbf{r}\left(\frac{\pi}{4}\right) \cdot \mathbf{r}'\left(\frac{\pi}{4}\right) = 0,$$

so that $\mathbf{r}(\frac{\pi}{4})$ and $\mathbf{r}'(\frac{\pi}{4})$ are orthogonal. In fact, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for every t, as follows:

$$\begin{aligned}\mathbf{r}(t) \cdot \mathbf{r}'(t) &= \langle -\cos 2t, \sin 2t\rangle \cdot \langle 2\sin 2t, 2\cos 2t\rangle \\ &= -2\cos 2t \sin 2t + 2\sin 2t \cos 2t = 0. \quad \blacksquare\end{aligned}$$

Were you surprised to find in example 2.7 that the position vector and the tangent vector were orthogonal at every point? As it turns out, this is a special case of a more general result, which we state in Theorem 2.4.

THEOREM 2.4

$\|\mathbf{r}(t)\|$ = constant if and only if $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal, for all t.

PROOF

(i) Suppose that $\|\mathbf{r}(t)\| = c$, for some constant c. Recall that

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = c^2. \tag{2.4}$$

Differentiating both sides of (2.4), we get

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \frac{d}{dt}c^2 = 0.$$

From Theorem 2.3 (iv), we now have

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t),$$

so that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, as desired.

(ii) We leave the proof of the converse as an exercise. ■

Note that in two dimensions, if $\|\mathbf{r}(t)\| = c$ for all t (where c is a constant), then the curve traced out by the position vector $\mathbf{r}(t)$ must lie on the circle of radius c, centered at the origin. Theorem 2.4 then says that the path traced out by $\mathbf{r}(t)$ lies on a circle centered at the origin if and only if the tangent vector is orthogonal to the position vector at every point on the curve. Likewise, in three dimensions, if $\|\mathbf{r}(t)\| = c$ for all t (where c is a constant), the curve traced out by $\mathbf{r}(t)$ lies on the sphere of radius c centered at the origin. In this case, Theorem 2.4 says that the curve traced out by $\mathbf{r}(t)$ lies on a sphere centered at the origin if and only if the tangent vector is orthogonal to the position vector at every point on the curve.

We conclude this section by making a few straightforward definitions. Recall that when we say that the scalar function $F(t)$ is an antiderivative of the scalar function $f(t)$, we mean that F is any function such that $F'(t) = f(t)$. We now extend this notion to vector-valued functions.

DEFINITION 2.4

The vector-valued function $\mathbf{R}(t)$ is an **antiderivative** of the vector-valued function $\mathbf{r}(t)$ whenever $\mathbf{R}'(t) = \mathbf{r}(t)$.

Notice that if $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ and f, g and h have antiderivatives F, G and H, respectively, then

$$\frac{d}{dt}\langle F(t), G(t), H(t)\rangle = \langle F'(t), G'(t), H'(t)\rangle = \langle f(t), g(t), h(t)\rangle.$$

That is, $\langle F(t), G(t), H(t)\rangle$ is an antiderivative of $\mathbf{r}(t)$. In fact, $\langle F(t) + c_1, G(t) + c_2, H(t) + c_3\rangle$ is also an antiderivative of $\mathbf{r}(t)$, for any choice of constants c_1, c_2 and c_3. This leads us to Definition 2.5.

DEFINITION 2.5

If $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$, the **indefinite integral** of $\mathbf{r}(t)$ is defined to be

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

As in the scalar case, $\mathbf{R}(t) + \mathbf{c}$ is the most general antiderivative of $\mathbf{r}(t)$. (Why is that?) Notice that this says that

Indefinite integral of a vector-valued function

$$\int \mathbf{r}(t)\,dt = \int \langle f(t), g(t), h(t)\rangle\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle. \tag{2.5}$$

That is, you integrate a vector-valued function by integrating each of the individual components.

EXAMPLE 2.8 Evaluating the Indefinite Integral of a Vector-Valued Function

Evaluate the indefinite integral $\int \left\langle t^2 + 2, \sin 2t, 4te^{t^2}\right\rangle dt$.

Solution From (2.5), we have

$$\begin{aligned}\int \left\langle t^2 + 2, \sin 2t, 4te^{t^2}\right\rangle dt &= \left\langle \int (t^2+2)\,dt, \int \sin 2t\,dt, \int 4te^{t^2}\,dt \right\rangle \\ &= \left\langle \frac{1}{3}t^3 + 2t + c_1, -\frac{1}{2}\cos 2t + c_2, 2e^{t^2} + c_3 \right\rangle \\ &= \left\langle \frac{1}{3}t^3 + 2t, -\frac{1}{2}\cos 2t, 2e^{t^2} \right\rangle + \mathbf{c},\end{aligned}$$

where $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ is an arbitrary constant vector. ■

Similarly, we define the definite integral of a vector-valued function in the obvious way.

DEFINITION 2.6

For the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, we define the **definite integral** of $\mathbf{r}(t)$ on the interval $[a, b]$ by

$$\int_a^b \mathbf{r}(t)\,dt = \int_a^b \langle f(t), g(t), h(t)\rangle dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle. \tag{2.6}$$

This says simply that the definite integral of a vector-valued function $\mathbf{r}(t)$ is the vector whose components are the definite integrals of the corresponding components of $\mathbf{r}(t)$. With this in mind, we now extend the Fundamental Theorem of Calculus to vector-valued functions.

THEOREM 2.5

Suppose that $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on the interval $[a, b]$. Then,

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a).$$

PROOF

The proof is straightforward and we leave this as an exercise. ■

EXAMPLE 2.9 Evaluating the Definite Integral of a Vector-Valued Function

Evaluate $\int_0^1 \langle \sin \pi t, 6t^2 + 4t\rangle\,dt$.

Solution Notice that an antiderivative for the integrand is

$$\left\langle -\frac{1}{\pi}\cos \pi t, \frac{6t^3}{3} + 4\frac{t^2}{2}\right\rangle = \left\langle -\frac{1}{\pi}\cos \pi t, 2t^3 + 2t^2\right\rangle.$$

From Theorem 2.5, we have that

$$\begin{aligned}\int_0^1 \langle \sin \pi t, 6t^2 + 4t\rangle\,dt &= \left\langle -\frac{1}{\pi}\cos \pi t, 2t^3 + 2t^2\right\rangle\Big|_0^1 \\ &= \left\langle -\frac{1}{\pi}\cos \pi, 2 + 2\right\rangle - \left\langle -\frac{1}{\pi}\cos 0, 0\right\rangle \\ &= \left\langle \frac{1}{\pi} + \frac{1}{\pi}, 4 - 0\right\rangle = \left\langle \frac{2}{\pi}, 4\right\rangle.\end{aligned}$$ ■

BEYOND FORMULAS

Theorem 2.4 illustrates the importance of good notation. While we could have derived the same result using parametric equations, the vector notation greatly simplifies both the statement and proof of the theorem. The simplicity of the notation allows us to make connections and use our geometric intuition, instead of floundering in a mess of equations. We can visualize the graph of a vector-valued function $\mathbf{r}(t)$ more easily than we can try to keep track of separate equations $x(t)$, $y(t)$ and $z(t)$.

EXERCISES 11.2

WRITING EXERCISES

1. Suppose that $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where $\lim_{t\to 0} f(t) = \lim_{t\to 0} g(t) = 0$ and $\lim_{t\to 0} h(t) = \infty$. Describe what is happening graphically as $t \to 0$ and explain why (even though the limits of two of the component functions exist) the limit of $\mathbf{r}(t)$ as $t \to 0$ does not exist.

2. In example 2.3, describe what is happening graphically for $t \le -1$. Explain why we don't say that $\mathbf{r}(t)$ is continuous for $t \le -1$.

3. Suppose that $\mathbf{r}(t)$ is a vector-valued function such that $\mathbf{r}(0) = \langle a, b, c\rangle$ and $\mathbf{r}'(0)$ exists. Imagine zooming in on the curve traced out by $\mathbf{r}(t)$ near the point (a, b, c). Describe what the curve will look like and how it relates to the tangent vector $\mathbf{r}'(0)$.

4. There is a quotient rule corresponding to the product rule in Theorem 2.3, part (iii). State this rule and describe in words how you would prove it. Explain why there isn't a quotient rule corresponding to the product rules in parts (iv) and (v) of Theorem 2.3.

In exercises 1–6, find the limit if it exists.

1. $\lim_{t\to 0}\langle t^2 - 1, e^{2t}, \sin t\rangle$

2. $\lim_{t\to 1}\left\langle t^2, e^{2t}, \sqrt{t^2+2t}\right\rangle$

3. $\lim_{t\to 0}\left\langle \frac{\sin t}{t}, \cos t, \frac{t+1}{t-1}\right\rangle$

4. $\lim_{t\to 1}\left\langle \sqrt{t-1}, t^2+3, \frac{t+1}{t-1}\right\rangle$

5. $\lim_{t\to 0}\left\langle \ln t, \sqrt{t^2+1}, t-3\right\rangle$

6. $\lim_{t\to \pi/2}\langle \cos t, t^2+3, \tan t\rangle$

In exercises 7–12, determine all values of t at which the given vector-valued function is continuous.

7. $\mathbf{r}(t) = \left\langle \frac{t+1}{t-1}, t^2, 2t\right\rangle$

8. $\mathbf{r}(t) = \left\langle \sin t, \cos t, \frac{3}{t}\right\rangle$

9. $\mathbf{r}(t) = \langle \tan t, \sin t^2, \cos t\rangle$

10. $\mathbf{r}(t) = \langle \cos 5t, \tan t, 6\sin t\rangle$

11. $\mathbf{r}(t) = \langle 4\cos t, \sqrt{t}, 4\sin t\rangle$

12. $\mathbf{r}(t) = \langle \sin t, -\csc t, \cot t\rangle$

In exercises 13–18, find the derivative of the given vector-valued function.

13. $\mathbf{r}(t) = \left\langle t^4, \sqrt{t+1}, \frac{3}{t^2}\right\rangle$

14. $\mathbf{r}(t) = \left\langle \frac{t-3}{t+1}, te^{2t}, t^3\right\rangle$

15. $\mathbf{r}(t) = \langle \sin t, \sin t^2, \cos t\rangle$

16. $\mathbf{r}(t) = \langle \cos 5t, \tan t, 6\sin t\rangle$

17. $\mathbf{r}(t) = \langle e^{t^2}, t^2, \sec 2t\rangle$

18. $\mathbf{r}(t) = \left\langle \sqrt{t^2+1}, \cos t, e^{-3t}\right\rangle$

In exercises 19–22, sketch the curve traced out by the endpoint of the given vector-valued function and plot position and tangent vectors at the indicated points.

19. $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$

20. $\mathbf{r}(t) = \langle t, t^2 - 1\rangle$, $t = 0$, $t = 1$, $t = 2$

21. $\mathbf{r}(t) = \langle \cos t, t, \sin t\rangle$, $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$

22. $\mathbf{r}(t) = \langle t, t, t^2 - 1\rangle$, $t = 0$, $t = 1$, $t = 2$

In exercises 23–32, evaluate the given indefinite or definite integral.

23. $\int \left\langle 3t - 1, \sqrt{t}\right\rangle dt$

24. $\int \left\langle \frac{3}{t^2}, \frac{4}{t}\right\rangle dt$

25. $\int \langle \cos 3t, \sin t, e^{4t}\rangle\, dt$

26. $\int \langle e^{-3t}, \sin 5t, t^{3/2}\rangle\, dt$

27. $\int \left\langle te^{t^2}, 3t\sin t, \frac{3t}{t^2+1}\right\rangle dt$

28. $\int \langle e^{-3t}, t^2 \cos t^3, t \cos t \rangle \, dt$

29. $\int_0^1 \langle t^2 - 1, 3t \rangle \, dt$

30. $\int_1^4 \langle \sqrt{t}, 5 \rangle \, dt$

31. $\int_0^2 \left\langle \frac{4}{t+1}, e^{t-2}, te^t \right\rangle dt$

32. $\int_0^4 \left\langle 2te^{4t}, t^2 - 1, \frac{4t}{t^2+1} \right\rangle dt$

In exercises 33–36, find t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular.

33. $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$

34. $\mathbf{r}(t) = \langle 2\cos t, \sin t \rangle$

35. $\mathbf{r}(t) = \langle t, t, t^2 - 1 \rangle$

36. $\mathbf{r}(t) = \langle t^2, t, t^2 - 5 \rangle$

37. In each of exercises 33 and 34, show that there are no values of t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel.

38. In each of exercises 35 and 36, show that there are no values of t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel.

In exercises 39–42, find all values of t such that $\mathbf{r}'(t)$ is parallel to the xy-plane.

39. $\mathbf{r}(t) = \langle t, t, t^3 - 3 \rangle$

40. $\mathbf{r}(t) = \langle t^2, t, \sin t^2 \rangle$

41. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$

42. $\mathbf{r}(t) = \langle \sqrt{t+1}, \cos t, t^4 - 8t^2 \rangle$

43. Prove Theorem 2.3, part (ii).

44. In Theorem 2.3, part (ii), replace the scalar product $c\mathbf{r}(t)$ with the dot product $\mathbf{c} \cdot \mathbf{r}(t)$, for a constant vector $\mathbf{c}$ and prove the results.

45. Prove Theorem 2.3, parts (ii) and (iii).

46. Prove Theorem 2.3, part (v).

47. Label as true or false and explain why. If $\mathbf{u}(t) = \frac{1}{\|\mathbf{r}(t)\|}\mathbf{r}(t)$ and $\mathbf{u}(t) \cdot \mathbf{u}'(t) = 0$ then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

48. Label as true or false and explain why. If $\mathbf{r}(t_0) \cdot \mathbf{r}'(t_0) = 0$ for some t_0, then $\|\mathbf{r}(t)\|$ is constant.

49. Prove that if $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all t, then $\|\mathbf{r}(t)\| = \text{constant}$ [Theorem 2.4, part (ii)].

50. Prove Theorem 2.5.

51. Define the ellipse C with parametric equations $x = a \cos t$ and $y = b \sin t$, for positive constants a and b. For a fixed value of t, define the points $P = (a \cos t, b \sin t)$, $Q = (a \cos(t + \pi/2), b \sin(t + \pi/2))$ and $Q' = (a \cos(t - \pi/2), b \sin(t - \pi/2))$. Show that the vector QQ' (called the **conjugate diameter**) is parallel to the tangent vector to C at the point P. Sketch a graph and show the relationship between P, Q and Q'.

52. Repeat exercise 51 for the general angle θ, so that the points are $P = (a \cos t, b \sin t)$, $Q = (a \cos(t + \theta), b \sin(t + \theta))$ and $Q' = (a \cos(t - \theta), b \sin(t - \theta))$.

53. Find $\frac{d}{dt}[\mathbf{f}(t) \cdot (\mathbf{g}(t) \times \mathbf{h}(t))]$.

54. Determine whether the following is true or false: $\int_a^b \mathbf{f}(t) \cdot \mathbf{g}(t)\, dt = \int_a^b \mathbf{f}(t)\, dt \cdot \int_a^b \mathbf{g}(t)\, dt$.

EXPLORATORY EXERCISES

1. Find all values of t such that $\mathbf{r}'(t) = \mathbf{0}$ for each function: (a) $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle$, (b) $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t \rangle$, (c) $\mathbf{r}(t) = \langle 2\cos 3t + \sin 5t, 2\sin 3t + \cos 5t \rangle$, (d) $\mathbf{r}(t) = \langle t^2, t^4 - 1 \rangle$ and (e) $\mathbf{r}(t) = \langle t^3, t^6 - 1 \rangle$. Based on your results, conjecture the graphical significance of having the derivative of a vector-valued function equal the zero vector. If $\mathbf{r}(t)$ is the position function of some object in motion, explain the physical significance of having a zero derivative. Explain your geometric interpretation in light of your physical interpretation.

2. You may recall that a scalar function has either a discontinuity, a "sharp corner" or a cusp at places where the derivative doesn't exist. In this exercise, we look at the analogous *smoothness* of graphs of vector-valued functions. We have said that a curve C is *smooth* if it is traced out by a vector-valued function $\mathbf{r}(t)$, where $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for all values of t. Sketch the graph of $\mathbf{r}(t) = \langle t, \sqrt[3]{t^2} \rangle$ and explain why we include the requirement that $\mathbf{r}'(t)$ be continuous. Sketch the graph of $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t \rangle$ and show that $\mathbf{r}'(0) = \mathbf{0}$. Explain why we include the requirement that $\mathbf{r}'(t)$ be nonzero. Sketch the graph of $\mathbf{r}(t) = \langle 2\cos 3t + \sin 5t, 2\sin 3t + \cos 5t \rangle$ and show that $\mathbf{r}'(t)$ never equals the zero vector. By zooming in on the edges of the graph, show that this curve is accurately described as smooth. Sketch the graphs of $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle$ and $\mathbf{g}(t) = \langle t^2, t^4 - 1 \rangle$ for $t \geq 0$ and observe that they trace out the same curve. Show that $\mathbf{g}'(0) = \mathbf{0}$, but that the curve is smooth at $t = 0$. Explain why this says that the requirement that $\mathbf{r}'(t) \neq \mathbf{0}$ need not hold for *every* $\mathbf{r}(t)$ tracing out the curve. [This requirement needs to hold for only one such $\mathbf{r}(t)$.] Determine which of the following curves are smooth. If the curve is not smooth, identify the graphical characteristic that is "unsmooth": $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt[3]{t^2} \rangle$, $\mathbf{r}(t) = \langle \tan t, \sin t^2, \cos t \rangle$, $\mathbf{r}(t) = \langle 5\sin^3 t, 5\cos^3 t, t \rangle$ and $\mathbf{r}(t) = \langle \cos t, t^2 e^{-t}, \cos^2 t \rangle$.

11.3 MOTION IN SPACE

We are finally at a point where we have sufficient mathematical machinery to describe the motion of an object in a three-dimensional setting. Problems such as this, dealing with motion, were one of the primary focuses of Newton and many of his contemporaries. Newton used his newly invented calculus to explain all kinds of motion, from the motion of a projectile (such as a ball) hurled through the air, to the motion of the planets. His stunning achievements in this field unlocked mysteries that had eluded the greatest minds for centuries and form the basis of our understanding of mechanics today.

Suppose that an object moves along a curve traced out by the endpoint of the vector-valued function

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where t represents time and where $t \in [a, b]$. We observed in section 11.2 that the value of $\mathbf{r}'(t)$ for any given value of t is a tangent vector pointing in the direction of the orientation of the curve. We can now give another interpretation of this. From (2.3), we have

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

and the magnitude of this vector-valued function is

$$\|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}.$$

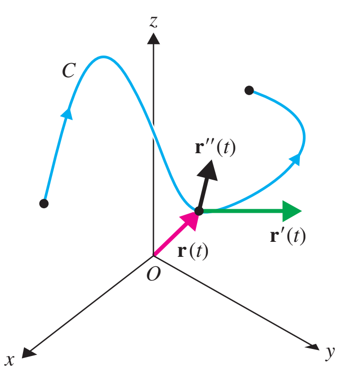


FIGURE 11.13
Position, velocity and acceleration vectors

(Where have you seen this expression before?) Notice that from (1.4), given any number $t_0 \in [a, b]$, the arc length of the portion of the curve from $u = t_0$ up to $u = t$ is given by

$$s(t) = \int_{t_0}^{t} \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du. \tag{3.1}$$

Part II of the Fundamental Theorem of Calculus says that if we differentiate both sides of (3.1), we get

$$s'(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \|\mathbf{r}'(t)\|.$$

Since $s(t)$ represents arc length, $s'(t)$ gives the instantaneous rate of change of arc length with respect to time, that is, the **speed** of the object as it moves along the curve. So, for any given value of t, $\mathbf{r}'(t)$ is a tangent vector pointing in the direction of the orientation of C (i.e., the direction followed by the object) and whose magnitude gives the speed of the object. So, we call $\mathbf{r}'(t)$ the **velocity** vector, denoted $\mathbf{v}(t)$. Finally, we refer to the derivative of the velocity vector $\mathbf{v}'(t) = \mathbf{r}''(t)$ as the **acceleration** vector, denoted $\mathbf{a}(t)$. When drawing the velocity and acceleration vectors, we locate both of their initial points at the terminal point of $\mathbf{r}(t)$ (i.e., at the point on the curve), as shown in Figure 11.13.

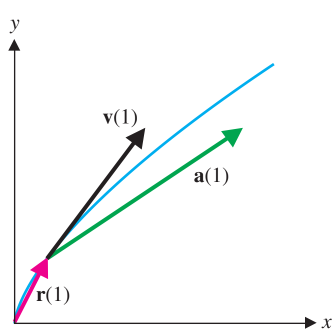


FIGURE 11.14
Position, velocity and acceleration vectors

EXAMPLE 3.1 Finding Velocity and Acceleration Vectors

Find the velocity and acceleration vectors if the position of an object moving in the xy-plane is given by $\mathbf{r}(t) = \langle t^3, 2t^2 \rangle$.

Solution We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3t^2, 4t \rangle \quad \text{and} \quad \mathbf{a}(t) = \mathbf{r}''(t) = \langle 6t, 4 \rangle.$$

In particular, this says that at $t = 1$, we have $\mathbf{r}(1) = \langle 1, 2 \rangle$, $\mathbf{v}(1) = \mathbf{r}'(1) = \langle 3, 4 \rangle$ and $\mathbf{a}(1) = \mathbf{r}''(1) = \langle 6, 4 \rangle$. We plot the curve and these vectors in Figure 11.14. ■

Just as in the case of one-dimensional motion, given the acceleration vector, we can determine the velocity and position vectors, provided we have some additional information.

EXAMPLE 3.2 Finding Velocity and Position from Acceleration

Find the velocity and position of an object at any time t, given that its acceleration is $\mathbf{a}(t) = \langle 6t, 12t + 2, e^t \rangle$, its initial velocity is $\mathbf{v}(0) = \langle 2, 0, 1 \rangle$ and its initial position is $\mathbf{r}(0) = \langle 0, 3, 5 \rangle$.

Solution Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we integrate once to obtain

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt = \int [6t\mathbf{i} + (12t + 2)\mathbf{j} + e^t\mathbf{k}]\,dt \\ &= 3t^2\mathbf{i} + (6t^2 + 2t)\mathbf{j} + e^t\mathbf{k} + \mathbf{c}_1,\end{aligned}$$

where $\mathbf{c}_1$ is an arbitrary constant vector. To determine the value of $\mathbf{c}_1$, we use the initial velocity:

$$\langle 2, 0, 1 \rangle = \mathbf{v}(0) = (0)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} + \mathbf{c}_1,$$

so that $\mathbf{c}_1 = \langle 2, 0, 0 \rangle$. This gives us the velocity

$$\mathbf{v}(t) = (3t^2 + 2)\mathbf{i} + (6t^2 + 2t)\mathbf{j} + e^t\mathbf{k}.$$

Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we integrate again, to obtain

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = \int [(3t^2 + 2)\mathbf{i} + (6t^2 + 2t)\mathbf{j} + e^t\mathbf{k}]\,dt \\ &= (t^3 + 2t)\mathbf{i} + (2t^3 + t^2)\mathbf{j} + e^t\mathbf{k} + \mathbf{c}_2,\end{aligned}$$

where $\mathbf{c}_2$ is an arbitrary constant vector. We can use the initial position to determine the value of $\mathbf{c}_2$, as follows:

$$\langle 0, 3, 5 \rangle = \mathbf{r}(0) = (0)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} + \mathbf{c}_2,$$

so that $\mathbf{c}_2 = \langle 0, 3, 4 \rangle$. This gives us the position vector

$$\mathbf{r}(t) = (t^3 + 2t)\mathbf{i} + (2t^3 + t^2 + 3)\mathbf{j} + (e^t + 4)\mathbf{k}.$$

We show the curve and indicate sample vectors for $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ in Figure 11.15. ■

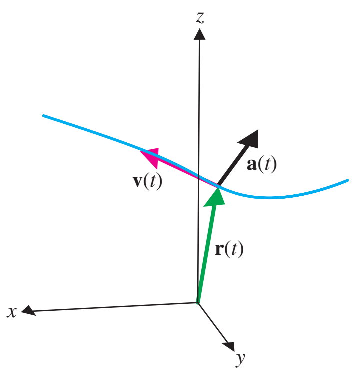


FIGURE 11.15
Position, velocity and acceleration vectors

We have already seen **Newton's second law of motion** several times now. In the case of one-dimensional motion, we had that the net force acting on an object equals the product of the mass and the acceleration ($F = ma$). In the case of motion in two or more dimensions, we have the vector form of Newton's second law:

$$\mathbf{F} = m\mathbf{a}.$$

Here, m is the mass, $\mathbf{a}$ is the acceleration vector and $\mathbf{F}$ is the vector representing the net force acting on the object.

EXAMPLE 3.3 Finding the Force Acting on an Object

Find the force acting on an object moving along a circular path of radius b, with constant angular speed.

Solution For simplicity, we will take the circular path to lie in the xy-plane and have its center at the origin. Here, by constant **angular speed,** we mean that if θ is the angle

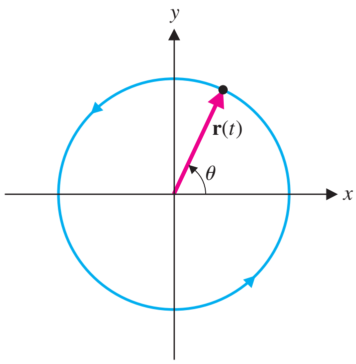


FIGURE 11.16a
Motion along a circle

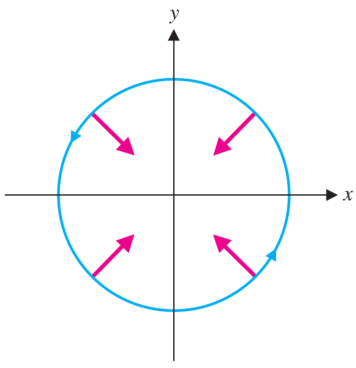


FIGURE 11.16b
Centripetal force

made by the position vector and the positive x-axis and t is time (see Figure 11.16a, where the indicated orientation is for the case where $\omega > 0$), then we have that

$$\frac{d\theta}{dt} = \omega \text{ (constant).}$$

Notice that this says that $\theta = \omega t + c$, for some constant c. Further, we can think of the circular path as the curve traced out by the endpoint of the vector-valued function

$$\mathbf{r}(t) = \langle b\cos\theta, b\sin\theta\rangle = \langle b\cos(\omega t + c), b\sin(\omega t + c)\rangle.$$

Notice that the path is the same for every value of c. (Think about what the value of c affects.) For simplicity, we take $\theta = 0$ when $t = 0$, so that $\theta = \omega t$ and

$$\mathbf{r}(t) = \langle b\cos\omega t, b\sin\omega t\rangle.$$

Now that we know the position at any time t, we can differentiate to find the velocity and acceleration. We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -b\omega\sin\omega t, b\omega\cos\omega t\rangle,$$

so that the speed is $\|\mathbf{v}(t)\| = \omega b$ and

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) &= \langle -b\omega^2\cos\omega t, -b\omega^2\sin\omega t\rangle \\ &= -\omega^2\langle b\cos\omega t, b\sin\omega t\rangle = -\omega^2\mathbf{r}(t).\end{aligned}$$

From Newton's second law of motion, we now have

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2\mathbf{r}(t).$$

Notice that since $m\omega^2 > 0$, this says that the force acting on the object points in the direction opposite the position vector. That is, at any point on the path, it points in toward the origin (see Figure 11.16b). We call such a force a **centripetal** (center-seeking) force. Finally, observe that on this circular path, $\|\mathbf{r}(t)\| = b$, so that at every point on the path, the force vector has constant magnitude:

$$\|\mathbf{F}(t)\| = \|-m\omega^2\mathbf{r}(t)\| = m\omega^2\|\mathbf{r}(t)\| = m\omega^2 b.$$

Notice that one consequence of the result $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$ from example 3.3 is that the magnitude of the force increases as the rotation rate ω increases. You have experienced this if you have been on a roller coaster with tight turns or loops. The faster you are going, the stronger the force that your seat exerts on you. Alternatively, since the speed is $\|\mathbf{v}(t)\| = \omega b$, the tighter the turn (i.e., the smaller b is), the larger ω must be to obtain a given speed. So, on a roller coaster, a tighter turn requires a larger value of ω, which in turn increases the centripetal force.

Just as we did in the one-dimensional case, we can use Newton's second law of motion to determine the position of an object given only a knowledge of the forces acting on it. For instance, an important problem faced by the military is how to aim a projectile (e.g., a missile) so that it will end up hitting its intended target. This problem is harder than it sounds, particularly when the target is an aircraft moving faster than the speed of sound. We present the simplest possible case (where neither the target nor the source of the projectile is moving) in example 3.4.

EXAMPLE 3.4 Analyzing the Motion of a Projectile

A projectile is launched with an initial speed of 140 feet per second from ground level at an angle of $\frac{\pi}{4}$ to the horizontal. Assuming that the only force acting on the object is

TODAY IN MATHEMATICS

Evelyn Granville (1924–)
An American mathematician who has made important contributions to the space program and the teaching of mathematics. Growing up poor, she and her sister "accepted education as the means to rise above the limitations that a prejudiced society endeavored to place upon us." She was the first black American woman to be awarded a Ph.D. in mathematics. Upon graduation, she was employed as a computer programmer and in the early 1960s helped NASA write programs to track the paths of vehicles in space for Project Mercury. She then turned to education, her first love. Granville has coauthored an influential textbook on the teaching of mathematics.

gravity (i.e., there is no air resistance, etc.), find the maximum altitude, the horizontal range and the speed at impact of the projectile.

Solution Notice that here, the motion is in a single plane (so that we need only consider two dimensions) and the only force acting on the object is the force of gravity, which acts straight downward. Although this is not constant, it is nearly so at altitudes reasonably close to sea level. We will assume that

$$\mathbf{F}(t) = -mg\mathbf{j},$$

where g is the constant acceleration due to gravity, $g \approx 32$ feet/second^2. From Newton's second law of motion, we have

$$-mg\mathbf{j} = \mathbf{F}(t) = m\mathbf{a}(t).$$

We now have

$$\mathbf{v}'(t) = \mathbf{a}(t) = -32\mathbf{j}.$$

Integrating this once gives us

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -32t\mathbf{j} + \mathbf{c}_1, \tag{3.2}$$

where $\mathbf{c}_1$ is an arbitrary constant vector. If we knew the initial velocity vector $\mathbf{v}(0)$, we could use this to solve for $\mathbf{c}_1$, but we know only the initial speed (i.e., the magnitude of the velocity vector). Referring to Figure 11.17a, notice that you can read off the components of $\mathbf{v}(0)$, using the definitions of the sine and cosine functions:

$$\mathbf{v}(0) = \left\langle 140\cos\frac{\pi}{4}, 140\sin\frac{\pi}{4}\right\rangle = \langle 70\sqrt{2}, 70\sqrt{2}\rangle.$$

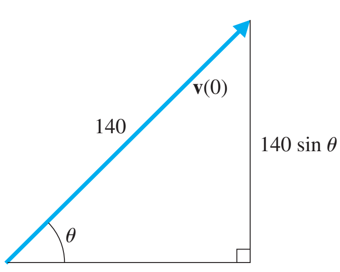


FIGURE 11.17a
Initial velocity vector

From (3.2), we now have

$$\langle 70\sqrt{2}, 70\sqrt{2}\rangle = \mathbf{v}(0) = (-32)(0)\mathbf{j} + \mathbf{c}_1 = \mathbf{c}_1.$$

Substituting this back into (3.2), we have

$$\mathbf{v}(t) = -32t\mathbf{j} + \langle 70\sqrt{2}, 70\sqrt{2}\rangle = \langle 70\sqrt{2}, 70\sqrt{2} - 32t\rangle. \tag{3.3}$$

Integrating (3.3) will give us the position vector

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \langle 70\sqrt{2}t, 70\sqrt{2}t - 16t^2\rangle + \mathbf{c}_2,$$

where $\mathbf{c}_2$ is an arbitrary constant vector. Since the initial location was not specified, we choose it to be the origin (for simplicity). This gives us

$$\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2,$$

so that

$$\mathbf{r}(t) = \langle 70\sqrt{2}t, 70\sqrt{2}t - 16t^2\rangle. \tag{3.4}$$

We show a graph of the path of the projectile in Figure 11.17b (on the following page). Now that we have found expressions for the position and velocity vectors for any time, we can answer the physical questions. Notice that the maximum altitude occurs at the instant when the object stops moving up (just before it starts to fall). This says that the vertical ($\mathbf{j}$) component of velocity must be zero. From (3.3), we get

$$0 = 70\sqrt{2} - 32t,$$

so that the time at the maximum altitude is

$$t = \frac{70\sqrt{2}}{32}.$$

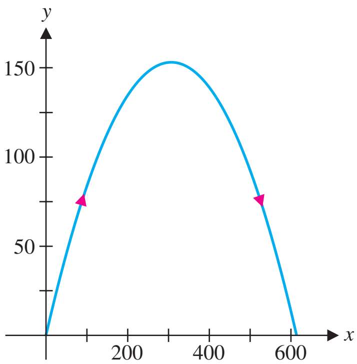


FIGURE 11.17b
Path of a projectile

The maximum altitude is then found from the vertical component of the position vector at this time:

$$\text{Maximum altitude} = 70\sqrt{2}t - 16t^2\Big|_{t=\frac{70\sqrt{2}}{32}} = 70\sqrt{2}\left(\frac{70\sqrt{2}}{32}\right) - 16\left(\frac{70\sqrt{2}}{32}\right)^2$$

$$= \frac{1225}{8} = 153.125 \text{ feet.}$$

To determine the horizontal range, we first need to determine the instant at which the object strikes the ground. Notice that this occurs when the vertical component of the position vector is zero (i.e., when the height above the ground is zero). From (3.4), we see that this occurs when

$$0 = 70\sqrt{2}t - 16t^2 = 2t(35\sqrt{2} - 8t).$$

There are two solutions of this equation: $t = 0$ (the time at which the projectile is launched) and $t = \frac{35\sqrt{2}}{8}$ (the time of impact). The horizontal range is then the horizontal component of position at this time:

$$\text{Range} = 70\sqrt{2}t\Big|_{t=\frac{35\sqrt{2}}{8}} = \left(70\sqrt{2}\right)\left(\frac{35\sqrt{2}}{8}\right) = \frac{1225}{2} = 612.5 \text{ feet.}$$

Finally, the speed at impact is the magnitude of the velocity vector at the time of impact:

$$\left\|\mathbf{v}\left(\frac{35\sqrt{2}}{8}\right)\right\| = \left\|\left\langle 70\sqrt{2}, 70\sqrt{2} - 32\left(\frac{35\sqrt{2}}{8}\right)\right\rangle\right\|$$

$$= \left\|\langle 70\sqrt{2}, -70\sqrt{2}\rangle\right\| = 140 \text{ ft/sec.}$$

You might have noticed in example 3.4 that the speed at impact was the same as the initial speed. Don't expect this to always be the case. Generally, this will be true only for a projectile of constant mass that is fired from ground level and returns to ground level and that is not subject to air resistance or other forces.

Equations of Motion

We now derive the equations of motion for a projectile in a slightly more general setting than that described in example 3.4. Consider a projectile launched from an altitude h above the ground at an angle θ to the horizontal and with initial speed v_0. We can use Newton's second law of motion to determine the position of the projectile at any time t and once we have this, we can answer any questions about the motion.

We again start with Newton's second law and assume that the only force acting on the object is gravity. We have

$$-mg\mathbf{j} = \mathbf{F}(t) = m\mathbf{a}(t).$$

This gives us (as in example 3.4)

$$\mathbf{v}'(t) = \mathbf{a}(t) = -g\mathbf{j}. \tag{3.5}$$

Integrating (3.5) gives us

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -gt\mathbf{j} + \mathbf{c}_1, \tag{3.6}$$

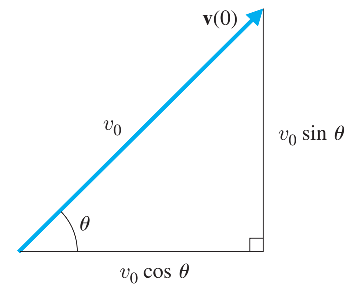


FIGURE 11.18a
Initial velocity

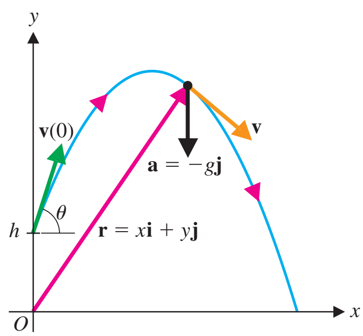


FIGURE 11.18b
Path of the projectile

where $\mathbf{c}_1$ is an arbitrary constant vector. In order to solve for $\mathbf{c}_1$, we need the value of $\mathbf{v}(t)$ for some t, but we are given only the initial speed v_0 and the angle at which the projectile is fired. Notice that from the definitions of sine and cosine, we can read off the components of $\mathbf{v}(0)$ from Figure 11.18a. From this and (3.6), we have

$$\langle v_0 \cos\theta, v_0 \sin\theta\rangle = \mathbf{v}(0) = \mathbf{c}_1.$$

This gives us the velocity vector

$$\mathbf{v}(t) = \langle v_0 \cos\theta, v_0 \sin\theta - gt\rangle. \tag{3.7}$$

Since $\mathbf{r}'(t) = \mathbf{v}(t)$, we integrate (3.7) to get the position:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \left\langle (v_0\cos\theta)t, (v_0\sin\theta)t - \frac{gt^2}{2}\right\rangle + \mathbf{c}_2.$$

To solve for $\mathbf{c}_2$, we want to use the initial position $\mathbf{r}(0)$, but we're not given it. We're told only that the projectile starts from an altitude of h feet above the ground. If we select the origin to be the point on the ground directly below the launching point, we have

$$\langle 0, h\rangle = \mathbf{r}(0) = \mathbf{c}_2,$$

so that

$$\begin{aligned}\mathbf{r}(t) &= \left\langle (v_0\cos\theta)t, (v_0\sin\theta)t - \frac{gt^2}{2}\right\rangle + \langle 0, h\rangle \\ &= \left\langle (v_0\cos\theta)t, h + (v_0\sin\theta)t - \frac{gt^2}{2}\right\rangle. \end{aligned} \tag{3.8}$$

Notice that the path traced out by $\mathbf{r}(t)$ (from $t = 0$ until impact) is a portion of a parabola. (See Figure 11.18b.)

Now that we have derived (3.7) and (3.8), we have all we need to answer any further questions about the motion. For instance, if we need to know the maximum altitude, this occurs at the time at which the vertical component of velocity is zero (i.e., at the time when the projectile stops rising). From (3.7), we solve

$$0 = v_0\sin\theta - gt,$$

so that the time at which the maximum altitude is reached is given by

Time to reach maximum altitude

$$t_{\max} = \frac{v_0\sin\theta}{g}.$$

The maximum altitude itself is the vertical component of the position vector at this time. From (3.8), we have

Maximum altitude

$$\begin{aligned}\text{Maximum altitude} &= h + (v_0\sin\theta)t - \frac{gt^2}{2}\bigg|_{t=t_{\max}} \\ &= h + (v_0\sin\theta)\left(\frac{v_0\sin\theta}{g}\right) - \frac{g}{2}\left(\frac{v_0\sin\theta}{g}\right)^2 \\ &= h + \frac{1}{2}\frac{v_0^2\sin^2\theta}{g}.\end{aligned}$$

To find the horizontal range or the speed at impact, we must first find the time of impact. To get this, we set the vertical component of position to zero. From (3.8), we have

$$0 = h + (v_0\sin\theta)t - \frac{gt^2}{2}.$$

Notice that this is simply a quadratic equation for t. Given v_0, θ and h, we can solve for the time t using the quadratic formula.

In all of the foregoing analysis, we left the constant acceleration due to gravity as g. You will usually use one of the two approximations:

$$g \approx 32 \text{ ft/sec}^2 \quad \text{or} \quad g \approx 9.8 \text{ m/sec}^2.$$

When using any other units, simply adjust the units to feet or meters and the time scale to seconds or make the corresponding adjustments to the value of g.

Just as with bodies moving in a straight line, we can use the calculus to analyze the motion of a body rotating about an axis. It's not hard to see why this is an important problem; just think of a gymnast performing a complicated routine. If the body is considered as a single point moving in three dimensions, the motion can be analyzed as in example 3.4. However, we are also quite interested in the rotational movement of the body. In the case of a gymnast, of course, the twists and turns that are performed are an important consideration. This is an example of rotational motion.

We use a rotational version of Newton's second law to analyze the motion. Torque (denoted by $\boldsymbol{\tau}$) is defined in section 10.4. In the case of an object rotating in two dimensions, the torque has magnitude (denoted by $\tau = \|\boldsymbol{\tau}\|$) given by the product of the force acting in the direction of the motion and the distance from the rotational center. The moment of inertia I of a body is a measure of how much force must be applied to cause the object to start rotating. This is determined by the mass and the distance of the mass from the center of rotation and is examined in some detail in section 13.2. In rotational motion, the primary variable that we track is an angle of displacement, denoted by θ. For a rotating body, the angle measured from some fixed ray changes with time t, so that the angle is a function $\theta(t)$. We define the **angular velocity** to be $\omega(t) = \theta'(t)$ and the **angular acceleration** to be $\alpha(t) = \omega'(t) = \theta''(t)$. The equation of rotational motion is then

$$\tau = I\alpha. \tag{3.9}$$

Notice how closely this resembles Newton's second law, $F = ma$. The calculus used in example 3.5 should look familiar.

EXAMPLE 3.5 The Rotational Motion of a Merry-Go-Round

A stationary merry-go-round of radius 5 feet is started in motion by a push consisting of a force of 10 pounds on the outside edge, tangent to the circular edge of the merry-go-round, for 1 second. The moment of inertia of the merry-go-round is $I = 25$. Find the resulting angular velocity of the merry-go-round.

Solution We first compute the torque of the push. The force is applied 5 feet from the center of rotation, so that the torque has magnitude

$$\tau = (\text{Force})(\text{Distance from axis of rotation}) = (10)(5) = 50 \text{ foot-pounds}.$$

From (3.9), we have

$$50 = 25\alpha,$$

so that $\alpha = 2$. Since the force is applied for one second, this equation holds for $0 \le t \le 1$. Integrating both sides of the equation $\omega' = \alpha$ from $t = 0$ to $t = 1$, we have by the Fundamental Theorem of Calculus that

$$\omega(1) - \omega(0) = \int_0^1 \alpha\, dt = \int_0^1 2\, dt = 2. \tag{3.10}$$

If the merry-go-round is initially stationary, then $\omega(0) = 0$ and (3.10) becomes simply $\omega(1) = 2$ rad/s. ■

Notice that we could draw a more general conclusion from (3.10). Even if the merry-go-round is already in motion, applying a force of 10 pounds tangentially to the edge for 1 second will increase the rotation rate by 2 rad/s.

For rotational motion in three dimensions, the calculations are somewhat more complicated. Recall that we had defined the torque $\boldsymbol{\tau}$ due to a force $\mathbf{F}$ applied at position $\mathbf{r}$ to be

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

Example 3.6 relates torque to angular momentum. The **linear momentum** of an object of mass m with velocity $\mathbf{v}$ is given by $\mathbf{p} = m\mathbf{v}$. The **angular momentum** is defined by $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$.

EXAMPLE 3.6 Relating Torque and Angular Momentum

Show that torque equals the derivative of angular momentum.

Solution From the definition of angular momentum and the product rule for the derivative of a cross product [Theorem 2.3 (v)], we have

$$\begin{aligned} \mathbf{L}'(t) &= \frac{d}{dt}[\mathbf{r}(t) \times m\mathbf{v}(t)] \\ &= \mathbf{r}'(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times m\mathbf{v}'(t) \\ &= \mathbf{v}(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times m\mathbf{a}(t). \end{aligned}$$

Notice that the first term on the right-hand side is the zero vector, since it is the cross product of parallel vectors. From Newton's second law, we have $\mathbf{F}(t) = m\mathbf{a}(t)$, so we have

$$\mathbf{L}'(t) = \mathbf{r}(t) \times m\mathbf{a}(t) = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}. \blacksquare$$

From this result, it is a short step to the principle of **conservation of angular momentum,** which states that, in the absence of torque, angular momentum remains constant. This is left as an exercise.

In example 3.7, we examine a fully three-dimensional projectile motion problem for the first time.

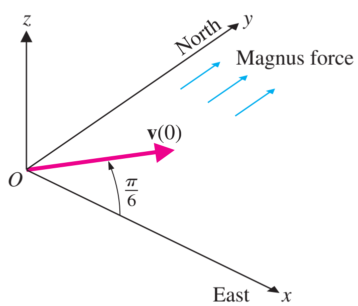


FIGURE 11.19a
The initial velocity and Magnus force vectors

EXAMPLE 3.7 Analyzing the Motion of a Projectile in Three Dimensions

A projectile of mass 1 kg is launched from ground level toward the east at 200 meters/second, at an angle of $\frac{\pi}{6}$ to the horizontal. If the spinning of the projectile applies a steady northerly Magnus force of 2 newtons to the projectile, find the landing location of the projectile and its speed at impact.

Solution Notice that because of the Magnus force, the motion is fully three-dimensional. We orient the x-, y- and z-axes so that the positive y-axis points north, the positive x-axis points east and the positive z-axis points up, as in Figure 11.19a, where we also show the initial velocity vector and vectors indicating the Magnus force. The two forces acting on the projectile are gravity (in the negative z-direction with magnitude $9.8m = 9.8$ newtons) and the Magnus force (in the y-direction with magnitude 2 newtons). Newton's second law is $\mathbf{F} = m\mathbf{a} = \mathbf{a}$. We have

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, -9.8 \rangle$$

Integrating gives us the velocity function

$$\mathbf{v}(t) = \langle 0, 2t, -9.8t \rangle + \mathbf{c}_1, \tag{3.11}$$

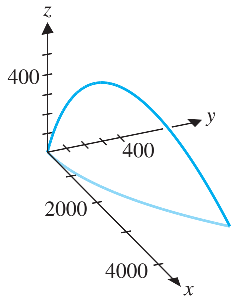


FIGURE 11.19b
Path of the projectile

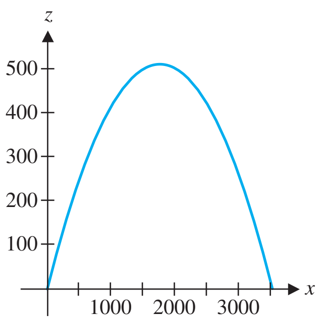


FIGURE 11.19c
Projection of path onto the *xz*-plane

where $\mathbf{c}_1$ is an arbitrary constant vector. Note that the initial velocity is

$$\mathbf{v}(0) = \left\langle 200\cos\frac{\pi}{6}, 0, 200\sin\frac{\pi}{6}\right\rangle = \langle 100\sqrt{3}, 0, 100\rangle.$$

From (3.11), we now have

$$\langle 100\sqrt{3}, 0, 100\rangle = \mathbf{v}(0) = \mathbf{c}_1,$$

which gives us

$$\mathbf{v}(t) = \langle 100\sqrt{3}, 2t, 100 - 9.8t\rangle.$$

We integrate this to get the position vector:

$$\mathbf{r}(t) = \langle 100\sqrt{3}t, t^2, 100t - 4.9t^2\rangle + \mathbf{c}_2,$$

for a constant vector $\mathbf{c}_2$. Taking the initial position to be the origin, we get

$$\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2,$$

so that

$$\mathbf{r}(t) = \langle 100\sqrt{3}t, t^2, 100t - 4.9t^2\rangle. \tag{3.12}$$

Note that the projectile strikes the ground when the $\mathbf{k}$ component of position is zero. From (3.12), we have that this occurs when

$$0 = 100t - 4.9t^2 = t(100 - 4.9t).$$

So, the projectile is on the ground when $t = 0$ (time of launch) and when $t = \frac{100}{4.9} \approx 20.4$ seconds (the time of impact). The location of impact is then the endpoint of the vector $\mathbf{r}\left(\frac{100}{4.9}\right) \approx \langle 3534.8, 416.5, 0\rangle$ and the speed at impact is

$$\left\|\mathbf{v}\left(\frac{100}{4.9}\right)\right\| \approx 204 \text{ m/s}.$$

We show a computer-generated graph of the path of the projectile in Figure 11.19b. In this figure, we also indicate the shadow made by the path of the projectile on the ground. In Figure 11.19c, we show the projection of the projectile's path onto the *xz*-plane. Observe that this parabola is analogous to the parabola shown in Figure 11.17b. ■

EXERCISES 11.3

WRITING EXERCISES

1. Explain why it makes sense in example 3.4 that the speed at impact equals the initial speed. (Hint: What force would slow the object down?) If the projectile were launched from above ground, discuss how the speed at impact would compare to the initial speed.

2. For an actual projectile, taking into account air resistance, explain why the speed at impact would be less than the initial speed.

3. In this section, we assumed that the acceleration due to gravity is constant. By contrast, air resistance is a function of velocity. (The faster the object goes, the more air resistance there is.) Explain why including air resistance in our Newton's law model of projectile motion would make the mathematics *much* more complicated.

4. In example 3.7, use the *x*- and *y*-components of the position function to explain why the projection of the projectile's path onto the *xy*-plane would be a parabola. The projection onto the *xz*-plane is also a parabola. Discuss whether or not the path in Figure 11.19b is a parabola. If you were watching the projectile, would the path appear to be parabolic?

In exercises 1–6, find the velocity and acceleration functions for the given position function.

1. $\mathbf{r}(t) = \langle 5\cos 2t, 5\sin 2t\rangle$

2. $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t\rangle$

3. $\mathbf{r}(t) = \langle 25t, -16t^2 + 15t + 5\rangle$

4. $\mathbf{r}(t) = \langle 25te^{-2t}, -16t^2 + 10t + 20\rangle$

5. $\mathbf{r}(t) = \langle 4te^{-2t}, 2e^{-2t}, -16t^2\rangle$

6. $\mathbf{r}(t) = \langle 3e^{-3t}, \sin 2t, t^3 - 3t\rangle$

In exercises 7–14, find the position function from the given velocity or acceleration function.

7. $\mathbf{v}(t) = \langle 10, -32t + 4\rangle$, $\mathbf{r}(0) = \langle 3, 8\rangle$

8. $\mathbf{v}(t) = \langle 4t, t^2 - 1\rangle$, $\mathbf{r}(0) = \langle 10, -2\rangle$

9. $\mathbf{a}(t) = \langle 0, -32\rangle$, $\mathbf{v}(0) = \langle 5, 0\rangle$, $\mathbf{r}(0) = \langle 0, 16\rangle$

10. $\mathbf{a}(t) = \langle t, \sin t\rangle$, $\mathbf{v}(0) = \langle 2, -6\rangle$, $\mathbf{r}(0) = \langle 10, 4\rangle$

11. $\mathbf{v}(t) = \langle 10, 3e^{-t}, -32t + 4\rangle$, $\mathbf{r}(0) = \langle 0, -6, 20\rangle$

12. $\mathbf{v}(t) = \langle t + 2, t^2, e^{-t/3}\rangle$, $\mathbf{r}(0) = \langle 4, 0, -3\rangle$

13. $\mathbf{a}(t) = \langle t, 0, -16\rangle$, $\mathbf{v}(0) = \langle 12, -4, 0\rangle$, $\mathbf{r}(0) = \langle 5, 0, 2\rangle$

14. $\mathbf{a}(t) = \langle e^{-3t}, t, \sin t\rangle$, $\mathbf{v}(0) = \langle 4, -2, 4\rangle$, $\mathbf{r}(0) = \langle 0, 4, -2\rangle$

In exercises 15–18, find the centripetal force on an object of mass 10 kg with the given position function (in units of meters and seconds).

15. $\mathbf{r}(t) = \langle 4\cos 2t, 4\sin 2t\rangle$

16. $\mathbf{r}(t) = \langle 3\cos 5t, 3\sin 5t\rangle$

17. $\mathbf{r}(t) = \langle 6\cos 4t, 6\sin 4t\rangle$

18. $\mathbf{r}(t) = \langle 2\cos 3t, 2\sin 3t\rangle$

In exercises 19–22, find the force acting on an object of mass 10 kg with the given position function (in units of meters and seconds).

19. $\mathbf{r}(t) = \langle 3\cos 2t, 5\sin 2t\rangle$

20. $\mathbf{r}(t) = \langle 3\cos 4t, 2\sin 5t\rangle$

21. $\mathbf{r}(t) = \langle 3t^2 + t, 3t - 1\rangle$

22. $\mathbf{r}(t) = \langle 20t - 3, -16t^2 + 2t + 30\rangle$

In exercises 23–28, a projectile is fired with initial speed v_0 feet per second from a height of h feet at an angle of θ above the horizontal. Assuming that the only force acting on the object is gravity, find the maximum altitude, horizontal range and speed at impact.

23. $v_0 = 100, h = 0, \theta = \frac{\pi}{3}$

24. $v_0 = 100, h = 0, \theta = \frac{\pi}{6}$

25. $v_0 = 160, h = 10, \theta = \frac{\pi}{4}$

26. $v_0 = 120, h = 10, \theta = \frac{\pi}{3}$

27. $v_0 = 320, h = 10, \theta = \frac{\pi}{4}$

28. $v_0 = 240, h = 10, \theta = \frac{\pi}{3}$

29. Based on your answers to exercises 25 and 27, what effect does doubling the initial speed have on the horizontal range?

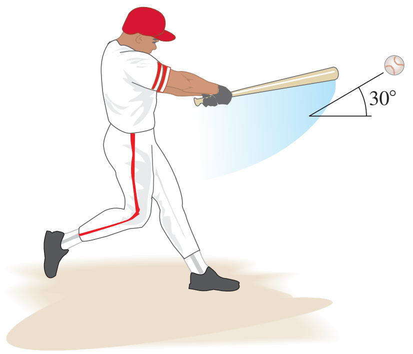
30. The angles $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are symmetric about $\frac{\pi}{4}$; that is, $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{3} - \frac{\pi}{4}$. Based on your answers to exercises 23 and 24, how do horizontal ranges for symmetric angles compare?

31. Beginning with Newton's second law of motion, derive the equations of motion for a projectile fired from altitude h above the ground at an angle θ to the horizontal and with initial speed v_0.

32. For the general projectile of exercise 31, with $h = 0$, (a) show that the horizontal range is $\dfrac{v_0^2 \sin 2\theta}{g}$ and (b) find the angle that produces the maximum horizontal range.

In exercises 33–40, neglect all forces except gravity. In all these situations, the effect of air resistance is actually significant, but your calculations will give a good first approximation.

33. A baseball is hit from a height of 3 feet with initial speed 120 feet per second and at an angle of 30 degrees above the horizontal. Find a vector-valued function describing the position of the ball t seconds after it is hit. To be a home run, the ball must clear a wall that is 385 feet away and 6 feet tall. Determine whether this is a home run.

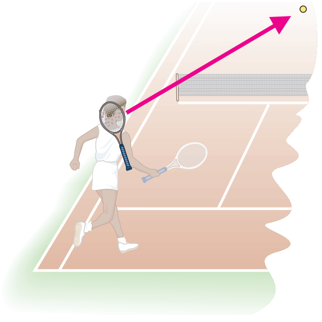


34. Repeat exercise 33 if the ball is launched with an initial angle of 31 degrees.

35. A baseball pitcher throws a pitch horizontally from a height of 6 feet with an initial speed of 130 feet per second. Find a vector-valued function describing the position of the ball t seconds after release. If home plate is 60 feet away, how high is the ball when it crosses home plate?

36. If a person drops a ball from height 6 feet at the same time the pitcher of exercise 35 releases the ball, how high will the dropped ball be when the pitch crosses home plate?

37. A tennis serve is struck horizontally from a height of 8 feet with initial speed 120 feet per second. For the serve to count (be "in"), it must clear a net that is 39 feet away and 3 feet high and must land before the service line 60 feet away. Find a vector function for the position of the ball and determine whether this serve is in or out.



38. Repeat exercise 37 if the ball is struck with an initial speed of (a) 80 ft/s or (b) 65 ft/s.

39. A football punt is launched at an angle of 50 degrees with an initial speed of 55 mph. Assuming the punt is launched from ground level, compute the "hang time" (the amount of time in the air) for the punt.

40. Compute the extra hang time if the punt in exercise 39 has an initial speed of 60 mph.

41. Find the landing point in exercise 23 if the object has mass 1 slug, is launched due east and there is a northerly Magnus force of 8 pounds.

42. Find the landing point in exercise 24 if the object has mass 1 slug, is launched due east and there is a southerly Magnus force of 4 pounds.

43. Suppose an airplane is acted on by three forces: gravity, wind and engine thrust. Assume that the force vector for gravity is $m\mathbf{g} = m\langle 0, 0, -32\rangle$, the force vector for wind is $\mathbf{w} = \langle 0, 1, 0\rangle$ for $0 \le t \le 1$ and $\mathbf{w} = \langle 0, 2, 0\rangle$ for $t > 1$, and the force vector for engine thrust is $\mathbf{e} = \langle 2t, 0, 24\rangle$. Newton's second law of motion gives us $m\mathbf{a} = m\mathbf{g} + \mathbf{w} + \mathbf{e}$. Assume that $m = 1$ and the initial velocity vector is $\mathbf{v}(0) = \langle 100, 0, 10\rangle$. Show that the velocity vector for $0 \le t \le 1$ is $\mathbf{v}(t) = \langle t^2 + 100, t, 10 - 8t\rangle$. For $t > 1$, integrate the equation $\mathbf{a} = \mathbf{g} + \mathbf{w} + \mathbf{e}$, to get $\mathbf{v}(t) = \langle t^2 + a, 2t + b, -8t + c\rangle$, for constants *a, b* and *c*. Explain (on physical grounds) why the function $\mathbf{v}(t)$ should be continuous and find the values of the constants that make it so. Show that $\mathbf{v}(t)$ is not differentiable. Given the nature of the force function, why does this make sense?

44. Find the position function for the airplane in exercise 43.

45. A roller coaster is designed to travel a circular loop of radius 100 feet. If the riders feel weightless at the top of the loop, what is the speed of the roller coaster?

46. A roller coaster travels at variable angular speed $\omega(t)$ and radius $r(t)$ but constant speed $c = \omega(t)r(t)$. For the centripetal force $F(t) = m\omega^2(t)r(t)$, show that $F'(t) = m\omega(t)r(t)\omega'(t)$. Conclude that entering a tight curve with $r'(t) < 0$ but maintaining constant speed, the centripetal force increases.

47. A jet pilot executing a circular turn experiences an acceleration of "5 g's" (that is, $\|\mathbf{a}\| = 5g$). If the jet's speed is 900 km/hr, what is the radius of the turn?

48. For the jet pilot of exercise 47, how many g's would be experienced if the speed were 1800 km/hr?

49. A force of 20 pounds is applied to the outside of a stationary merry-go-round of radius 5 feet for 0.5 second. The moment of inertia is $I = 10$. Find the resultant change in angular velocity of the merry-go-round.

50. A merry-go-round of radius 5 feet and moment of inertia $I = 10$ rotates at 4 rad/s. Find the constant force needed to stop the merry-go-round in 2 seconds.

51. A golfer rotates a club with constant angular acceleration α through an angle of π radians. If the angular velocity increases from 0 to 15 rad/s, find α.

52. For the golf club in exercise 51, find the increase in angular velocity if the club is rotated through an angle of $\frac{3\pi}{2}$ radians with the same angular acceleration. Describe one advantage of a long swing.

53. Softball pitchers such as Jennie Finch often use a double windmill to generate arm speed. At a constant angular acceleration, compare the speeds obtained rotating through an angle of 2π versus rotating through an angle of 4π.

54. As the softball in exercise 53 rotates, its linear speed v is related to the angular velocity ω by $v = r\omega$, where r is the distance of the ball from the center of rotation. The picture shows the pitcher with arm fully extended. Explain why this is a good technique for throwing a fast pitch.

55. Use the result of example 3.6 to prove the **Law of Conservation of Angular Momentum:** if there is zero (net) torque on an object, its angular momentum remains constant.

56. Prove the **Law of Conservation of Linear Momentum:** if there is zero (net) force on an object, its linear momentum remains constant.

57. If acceleration is parallel to position ($\mathbf{a}\|\mathbf{r}$), show that there is no torque. Explain this result in terms of the change in angular momentum. (Hint: If $\mathbf{a}\|\mathbf{r}$, would angular velocity or linear velocity be affected?)

58. If the acceleration $\mathbf{a}$ is constant, show that $\mathbf{L}''' = \mathbf{0}$.

59. Example 3.3 is a model of a satellite orbiting the earth. In this case, the force $\mathbf{F}$ is the gravitational attraction of the earth on the satellite. The magnitude of the force is $\frac{mMG}{b^2}$, where m is the mass of the satellite, M is the mass of the earth and G

is the universal gravitational constant. Using example 3.3, this should be equal to $m\omega^2 b$. For a **geosynchronous orbit,** the frequency ω is such that the satellite completes one orbit in one day. (By orbiting at the same rate as the earth spins, the satellite can remain directly above the same point on the earth.) For a sidereal day of 23 hours, 56 minutes and 4 seconds, find ω. Using $MG \approx 39.87187 \times 10^{13}$ N-m^2/kg, find b for a geosynchronous orbit (the units of b will be m).

60. Example 3.3 can also model a jet executing a turn. For a jet traveling at 1000 km/h, find the radius b such that the pilot feels 7 g's of force; that is, the magnitude of the force is 7 mg.

61. We have seen how we can find the trajectory of a projectile given its initial position and initial velocity. For military personnel tracking an incoming missile, the only data available correspond to various points on the trajectory, while the initial position (where the enemy gun is located) is unknown but very important. Assume that a projectile follows a parabolic path (after launch, the only force is gravity). If the projectile passes through points (x_1, y_1, z_1) at time t_1 and (x_2, y_2, z_2) at time t_2, find the initial position $(x_0, y_0, 0)$.

62. Use the result of exercise 61 to identify the initial velocity and launch position of a projectile that passes through (1, 2, 4) at $t = 1$ and (3, 6, 6) at $t = 2$.

63. For a satellite in earth orbit, the speed v in miles per second is related to the height h miles above the surface of the earth by $v = \sqrt{\frac{95{,}600}{4000+h}}$. Suppose a satellite is in orbit 15,000 miles above the surface of the earth. How much does the speed need to decrease to raise the orbit to a height of 20,000 miles?

EXPLORATORY EXERCISES

1. A ball rolls off a table of height 3 feet. Its initial velocity is horizontal with speed v_0. Determine where the ball hits the ground and the velocity vector of the ball at the moment of impact. Find the angle between the horizontal and the impact velocity vector. Next, assume that the next bounce of the ball starts with the ball being launched from the ground with initial conditions determined by the impact velocity. The launch speed equals 0.6 times the impact speed (so the ball won't bounce forever) and the launch angle equals the (positive) angle between the horizontal and the impact velocity vector. Using these conditions, determine where the ball next hits the ground. Continue on to find the third point at which the ball bounces.

2. In many sports such as golf and ski jumping, it is important to determine the range of a projectile on a slope. Suppose that the ground passes through the origin and slopes at an angle of α to the horizontal. Show that an equation of the ground is $y = -(\tan\alpha)x$. An object is launched at height $h = 0$ with initial speed v_0 at an angle of θ from the horizontal. Referring to exercise 31, show that the landing condition is now $y = -(\tan\alpha)x$. Find the x-coordinate of the landing point and show that the range (the distance along the ground) is given by $R = \frac{2}{g}v_0^2 \sec\alpha\cos\theta(\sin\theta + \tan\alpha\cos\theta)$. Use trigonometric identities to rewrite this as $R = \frac{1}{g}v_0^2\sec^2\alpha[\sin\alpha + \sin(\alpha + 2\theta)]$. Use this formula to find the value of θ that maximizes the range. For flat ground ($\alpha = 0$), the optimal angle is 45°. State an easy way of taking the value of α (say, $\alpha = 10°$ or $\alpha = -8°$) and adjusting from 45° to the optimal angle.

11.4 CURVATURE

Imagine that you are designing a new highway. Nearly all roads have curves, to avoid both natural and human-made obstacles. So that cars are able to maintain a reasonable speed on your new road, you should avoid curves that are too sharp. To do this, it would help to have some concept of how sharp a given curve is. In this section, we develop a measure of how much a curve is twisting and turning at any given point. First, realize that any given curve has infinitely many different parameterizations. For instance, the parametric equations $x = t^2$ and $y = t$ describe a parabola that opens to the right. In fact, for any real number $a > 0$, the equations $x = (at)^2$ and $y = at$ describe the same parabola. So, any measure of how sharp a curve is should be independent of the parameterization. The simplest choice of a parameter (for conceptual purposes, but not for computational purposes) is arc length. Further, observe that this is the correct parameter to use, as we measure how sharp a curve is by seeing how much it twists and turns per unit length. (Think about it this way: a turn of 90° over a quarter mile is not particularly sharp in comparison with a turn of 90° over a distance of 30 feet.)

For the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, for $a \le t \le b$, we define the arc length parameter $s(t)$ to be the arc length of that portion of the curve from $u = a$ up to $u = t$. That is, from (1.4),

$$s(t) = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du.$$

Recognizing that $\sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2} = \|\mathbf{r}'(u)\|$, we can write this more simply as

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du. \tag{4.1}$$

Although explicitly finding an arc length parameterization of a curve is not the central thrust of our discussion here, we briefly pause now to construct such a parameterization, for the purpose of illustration.

EXAMPLE 4.1 Parameterizing a Curve in Terms of Arc Length

Find an arc length parameterization of the circle of radius 4 centered at the origin.

Solution Note that one parameterization of this circle is

$$C: x = f(t) = 4\cos t, \quad y = g(t) = 4\sin t, \quad 0 \le t \le 2\pi.$$

In this case, the arc length from $u = 0$ to $u = t$ is given by

$$\begin{aligned} s(t) &= \int_0^t \sqrt{[f'(u)]^2 + [g'(u)]^2}\, du \\ &= \int_0^t \sqrt{[-4\sin u]^2 + [4\cos u]^2}\, du = 4\int_0^t 1\, du = 4t. \end{aligned}$$

That is, $t = s/4$, so that an arc length parameterization for C is

$$C: x = 4\cos\left(\frac{s}{4}\right), \quad y = 4\sin\left(\frac{s}{4}\right), \quad 0 \le s \le 8\pi.$$

Consider the smooth curve C traced out by the endpoint of the vector-valued function $\mathbf{r}(t)$. Recall that for each t, $\mathbf{v}(t) = \mathbf{r}'(t)$ can be thought of as both the velocity vector and a tangent vector, pointing in the direction of motion (i.e., the orientation of C). Notice that

Unit tangent vector

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}} \tag{4.2}$$

is also a tangent vector, but has length one ($\|\mathbf{T}(t)\| = 1$). We call $\mathbf{T}(t)$ the **unit tangent vector** to the curve C. That is, for each t, $\mathbf{T}(t)$ is a tangent vector of length one pointing in the direction of the orientation of C.

EXAMPLE 4.2 Finding a Unit Tangent Vector

Find the unit tangent vector to the curve determined by $\mathbf{r}(t) = \langle t^2 + 1, t\rangle$.

Solution We have

$$\mathbf{r}'(t) = \langle 2t, 1\rangle,$$

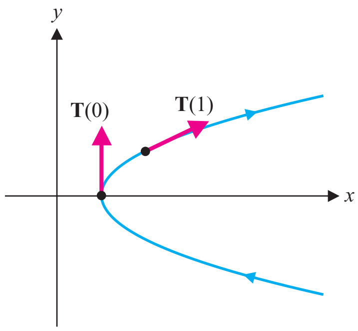


FIGURE 11.20
Unit tangent vectors

so that $$\|\mathbf{r}'(t)\| = \sqrt{(2t)^2 + 1} = \sqrt{4t^2 + 1}.$$

From (4.2), the unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 2t, 1\rangle}{\sqrt{4t^2+1}} = \left\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}} \right\rangle.$$

In particular, we have $\mathbf{T}(0) = \langle 0, 1\rangle$ and $\mathbf{T}(1) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$. We indicate both of these in Figure 11.20. ■

In Figures 11.21a and 11.21b, we show two curves, both connecting the points A and B. Think about driving a car along roads in the shape of these two curves. The curve in Figure 11.21b indicates a much sharper turn than the curve in Figure 11.21a. The question before us is to see how to mathematically describe this degree of "sharpness." You should get an idea of this from Figures 11.21c and 11.21d. These are the same curves as those shown in Figures 11.21a and 11.21b, respectively, but we have drawn in a number of unit tangent vectors at equally spaced points on the curves. Notice that the unit tangent vectors change very slowly along the gentle curve in Figure 11.21c, but twist and turn quite rapidly in the vicinity of the sharp curve in Figure 11.21d. Based on our analysis of Figures 11.21c and 11.21d, notice that the rate of change of the unit tangent vectors with respect to arc length along the curve will give us a measure of sharpness. To this end, we make the following definition.

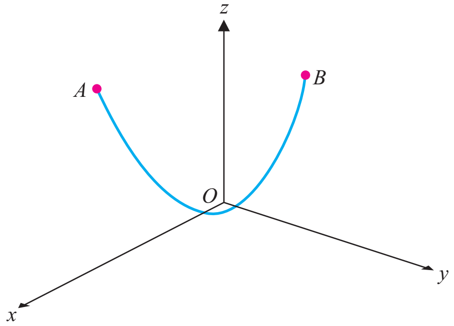


FIGURE 11.21a
Gentle curve

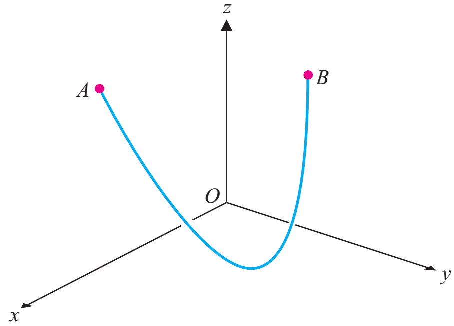


FIGURE 11.21b
Sharp curve

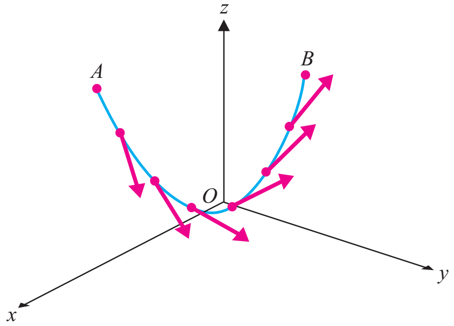


FIGURE 11.21c
Unit tangent vectors

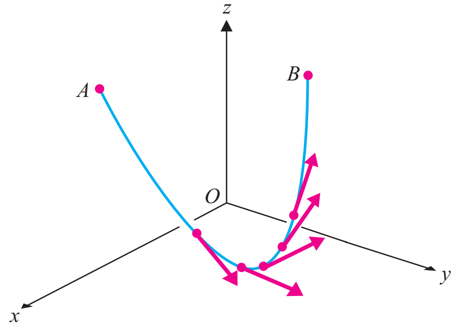


FIGURE 11.21d
Unit tangent vectors

DEFINITION 4.1

The **curvature** κ of a curve is the scalar quantity

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|. \tag{4.3}$$

Note that, while the definition of curvature makes sense intuitively, it is not a simple matter to compute κ directly from (4.3). To do so, we would need to first find the arc length parameter and the unit tangent vector $\mathbf{T}(t)$, rewrite $\mathbf{T}(t)$ in terms of the arc length parameter s and then differentiate with respect to s. This is not usually done. Instead, observe that by the chain rule,

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt},$$

so that when $\frac{ds}{dt} \neq 0$,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{T}'(t)\|}{\left|\frac{ds}{dt}\right|}. \tag{4.4}$$

Now, from (4.1), we had

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du,$$

so that by part II of the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|. \tag{4.5}$$

From (4.4) and (4.5), we now have

Curvature

$$\boxed{\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|},} \tag{4.6}$$

where $\|\mathbf{r}'(t)\| \neq 0$, for a smooth curve. Notice that it should be comparatively simple to use (4.6) to compute the curvature. We illustrate this in example 4.3.

EXAMPLE 4.3 Finding the Curvature of a Straight Line

Find the curvature of a straight line.

Solution First, think about what we're asking. Straight lines are, well, straight, so their curvature should be zero at every point. Let's see. Suppose that the line is traced out by the vector-valued function $\mathbf{r}(t) = \langle at + b, ct + d, et + f \rangle$, for some constants a, b, c, d, e and f (where at least one of a, c or e is nonzero). Then,

$$\mathbf{r}'(t) = \langle a, c, e \rangle$$

and so,

$$\|\mathbf{r}'(t)\| = \sqrt{a^2 + c^2 + e^2} = \text{constant} \neq 0.$$

The unit tangent vector is then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle a, c, e \rangle}{\sqrt{a^2 + c^2 + e^2}},$$

which is a constant vector. This gives us $\mathbf{T}'(t) = \mathbf{0}$, for all t. From (4.6), we now have

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{0}\|}{\sqrt{a^2 + c^2 + e^2}} = 0,$$

as expected. ■

Well, if a line has zero curvature, can you think of a curve with lots of curvature? The first one to come to mind is likely a circle, which we discuss next.

EXAMPLE 4.4 Finding the Curvature of a Circle

Find the curvature for a circle of radius $a > 0$.

Solution We leave it as an exercise to show that the curvature does not depend on the location of the center of the circle. (Intuitively, it certainly should not.) So, for simplicity, we assume that the circle is centered at the origin. Notice that the circle of radius a centered at the origin is traced out by the vector-valued function $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$. Differentiating, we get

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$$

and $$\|\mathbf{r}'(t)\| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a\sqrt{\sin^2 t + \cos^2 t} = a.$$

The unit tangent vector is then given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(\mathbf{t})}{\|\mathbf{r}'(t)\|} = \frac{\langle -a\sin t, a\cos t\rangle}{a} = \langle -\sin t, \cos t\rangle.$$

Differentiating this gives us

$$\mathbf{T}'(t) = \langle -\cos t, -\sin t\rangle$$

and from (4.6), we have

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\langle -\cos t, -\sin t\rangle\|}{a} = \frac{\sqrt{(-\cos t)^2 + (-\sin t)^2}}{a} = \frac{1}{a}.$$ ■

Notice that the result of example 4.4 is consistent with your intuition. First, observe that you should be able to drive a car around a circular track while holding the steering wheel in a fixed position. (That is, the curvature should be constant.) Further, the smaller that the radius of a circular track is, the sharper you will need to turn (that is, the larger the curvature). On the other hand, on a circular track of very large radius, it would seem as if you were driving fairly straight (i.e., the curvature will be close to 0).

You probably noticed that computing the curvature of the curves in examples 4.3 and 4.4 was just the slightest bit tedious. We simplify this process somewhat with the result of Theorem 4.1.

THEOREM 4.1

The curvature of the smooth curve traced out by the vector-valued function $\mathbf{r}(t)$ is given by

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{4.7}$$

The proof of Theorem 4.1 is rather long and involved and so, we omit it at this time, in the interest of brevity. We return to this in section 11.5, where the proof becomes a simple consequence of another result.

Notice that it is a relatively simple matter to use (4.7) to compute the curvature for nearly any three-dimensional curve.

EXAMPLE 4.5 Finding the Curvature of a Helix

Find the curvature of the helix traced out by $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4t\rangle$.

Solution A graph of the helix is indicated in Figure 11.22. We have

$$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 4\rangle$$

and

$$\mathbf{r}''(t) = \langle -2\sin t, -2\cos t, 0\rangle.$$

Now,

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \langle 2\cos t, -2\sin t, 4\rangle \times \langle -2\sin t, -2\cos t, 0\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos t & -2\sin t & 4 \\ -2\sin t & -2\cos t & 0 \end{vmatrix} \\ &= \langle 8\cos t, -8\sin t, -4\cos^2 t - 4\sin^2 t\rangle \\ &= \langle 8\cos t, -8\sin t, -4\rangle.\end{aligned}$$

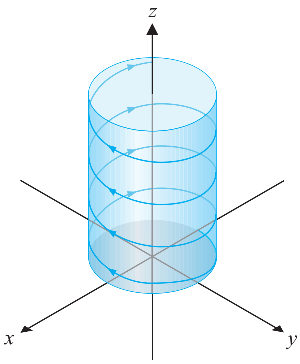


FIGURE 11.22
Circular helix

From (4.7), we get that the curvature is

$$\begin{aligned}\kappa &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \\ &= \frac{\|\langle 8\cos t, -8\sin t, -4\rangle\|}{\|\langle 2\cos t, -2\sin t, 4\rangle\|^3} = \frac{\sqrt{80}}{\left(\sqrt{20}\right)^3} = \frac{1}{10}.\end{aligned}$$

Note that this says that the helix has a constant curvature, as you should suspect from the graph in Figure 11.22. ■

In the case of a plane curve that is the graph of a function, $y = f(x)$, we can derive a particularly simple formula for the curvature. Notice that such a curve is traced out by the vector-valued function $\mathbf{r}(t) = \langle t, f(t), 0\rangle$, where the third component is 0, since the curve lies completely in the xy-plane. Further, $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$ and $\mathbf{r}''(t) = \langle 0, f''(t), 0\rangle$. From (4.7), we have

$$\begin{aligned}\kappa &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\langle 1, f'(t), 0\rangle \times \langle 0, f''(t), 0\rangle\|}{\|\langle 1, f'(t), 0\rangle\|^3} \\ &= \frac{|f''(t)|}{\{1 + [f'(t)]^2\}^{3/2}},\end{aligned}$$

where we have left the calculation of the cross product as a simple exercise. Since the parameter $t = x$, we can write the curvature as

Curvature for the plane curve $y = f(x)$

$$\boxed{\kappa = \frac{|f''(x)|}{\{1 + [f'(x)]^2\}^{3/2}}.} \tag{4.8}$$

EXAMPLE 4.6 Finding the Curvature of a Parabola

Find the curvature of the parabola $y = ax^2 + bx + c$. Also, find the limiting value of the curvature as $x \to \infty$.

Solution Taking $f(x) = ax^2 + bx + c$, we have that $f'(x) = 2ax + b$ and $f''(x) = 2a$. From (4.8), we have that

$$\kappa = \frac{|2a|}{[1 + (2ax + b)^2]^{3/2}}.$$

Taking the limit as $x \to \infty$, we have

$$\lim_{x\to\infty} \kappa = \lim_{x\to\infty} \frac{|2a|}{[1 + (2ax + b)^2]^{3/2}} = 0.$$

In other words, as $x \to \infty$, the parabola straightens out. You've certainly observed this in the graphs of parabolas for some time. Now, we have verified that this is not some sort of optical illusion; it's reality. It is a straightforward exercise to show that the maximum curvature occurs at the vertex of the parabola ($x = -b/2a$). ■

BEYOND FORMULAS

You can think of curvature as being loosely related to concavity, although there are important differences. The precise relationship for curves of the form $y = f(x)$ is given in equation (4.8). Curvature applies to curves in any dimension, whereas concavity applies only to two dimensions. More importantly, curvature measures the amount of curving as you move along the curve, regardless of where the curve goes. Concavity measures curving as you move along the x-axis, not the curve and thus requires that the curve correspond to a function $f(x)$. Given the generality of the curvature measurement, the formulas derived in this section are actually remarkably simple.

EXERCISES 11.4

WRITING EXERCISES

1. Explain what it means for a curve to have zero curvature (a) at a point and (b) on an interval of t-values.

2. Throughout our study of calculus, we have looked at tangent line approximations to curves. Some tangent lines approximate a curve well over a fairly lengthy interval while some stay close to a curve for only very short intervals. If the curvature at $x = a$ is large, would you expect the tangent line at $x = a$ to approximate the curve well over a lengthy interval or a short interval? What if the curvature is small? Explain.

3. Discuss the relationship between curvature and concavity for a function $y = f(x)$.

4. Explain why the curvature $\kappa = \frac{1}{10}$ of the helix in example 4.5 is less than the curvature of the circle $\langle 2\sin t, 2\cos t\rangle$ in two dimensions.

In exercises 1–4, find an arc length parameterization of the given two-dimensional curve.

1. The circle of radius 2 centered at the origin

2. The circle of radius 5 centered at the origin

3. The line segment from the origin to the point (3, 4)

4. The line segment from (1, 2) to the point (5, −2)

In exercises 5–10, find the unit tangent vector to the curve at the indicated points.

5. $\mathbf{r}(t) = \langle 3t, t^2\rangle$, $t = 0$, $t = -1$, $t = 1$

6. $\mathbf{r}(t) = \langle 2t^3, \sqrt{t}\rangle$, $t = 1$, $t = 2$, $t = 3$

7. $\mathbf{r}(t) = \langle 3\cos t, 2\sin t\rangle$, $t = 0$, $t = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$

8. $\mathbf{r}(t) = \langle 4\sin t, 2\cos t\rangle$, $t = -\pi$, $t = 0$, $t = \pi$

9. $\mathbf{r}(t) = \langle 3t, \cos 2t, \sin 2t \rangle$, $t = 0$, $t = -\pi$, $t = \pi$

10. $\mathbf{r}(t) = \langle 4t, 2t, t^2 \rangle$, $t = -1$, $t = 0$, $t = 1$

11. Sketch the curve in exercise 7 along with the vectors $\mathbf{r}(0)$, $\mathbf{T}(0)$, $\mathbf{r}\left(\frac{\pi}{2}\right)$ and $\mathbf{T}\left(\frac{\pi}{2}\right)$.

12. Sketch the curve in exercise 8 along with the vectors $\mathbf{r}(0)$, $\mathbf{T}(0)$, $\mathbf{r}\left(\frac{\pi}{2}\right)$ and $\mathbf{T}\left(\frac{\pi}{2}\right)$.

13. Sketch the curve in exercise 9 along with the vectors $\mathbf{r}(0)$, $\mathbf{T}(0)$, $\mathbf{r}(\pi)$ and $\mathbf{T}(\pi)$.

14. Sketch the curve in exercise 10 along with the vectors $\mathbf{r}(0)$, $\mathbf{T}(0)$, $\mathbf{r}(1)$ and $\mathbf{T}(1)$.

In exercises 15–22, find the curvature at the given point.

15. $\mathbf{r}(t) = \langle e^{-2t}, 2t, 4 \rangle$, $t = 0$

16. $\mathbf{r}(t) = \langle 2, \sin \pi t, \ln t \rangle$, $t = 1$

17. $\mathbf{r}(t) = \langle t, \sin 2t, 3t \rangle$, $t = 0$

18. $\mathbf{r}(t) = \langle t, t^2 + t - 1, t \rangle$, $t = 0$

19. $f(x) = 3x^2 - 1$, $x = 1$

20. $f(x) = x^3 + 2x - 1$, $x = 2$

21. $f(x) = \sin x$, $x = \frac{\pi}{2}$

22. $f(x) = e^{-3x}$, $x = 0$

23. For $f(x) = \sin x$, (see exercise 21), show that the curvature is the same at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Use the graph of $y = \sin x$ to predict whether the curvature would be larger or smaller at $x = \pi$.

24. For $f(x) = e^{-3x}$ (see exercise 22), show that the curvature is larger at $x = 0$ than at $x = 2$. Use the graph of $y = e^{-3x}$ to predict whether the curvature would be larger or smaller at $x = 4$.

In exercises 25–28, sketch the curve and compute the curvature at the indicated points.

25. $\mathbf{r}(t) = \langle 2\cos 2t, 2\sin 2t, 3t \rangle$, $t = 0$, $t = \frac{\pi}{2}$

26. $\mathbf{r}(t) = \langle \cos 2t, 2\sin 2t, 4t \rangle$, $t = 0$, $t = \frac{\pi}{2}$

27. $\mathbf{r}(t) = \langle t, t, t^2 - 1 \rangle$, $t = 0$, $t = 2$

28. $\mathbf{r}(t) = \langle 2t - 1, t + 2, t - 3 \rangle$, $t = 0$, $t = 2$

In exercises 29–32, sketch the curve and find any points of maximum or minimum curvature.

29. $\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle$

30. $\mathbf{r}(t) = \langle 4\cos t, 3\sin t \rangle$

31. $y = 4x^2 - 3$

32. $y = \sin x$

In exercises 33–36, graph the curvature function $\kappa(x)$ and find the limit of the curvature as $x \to \infty$.

33. $y = e^{2x}$

34. $y = e^{-2x}$

35. $y = x^3$

36. $y = \sqrt{x}$

37. Explain how the answers to exercises 33–36 relate to the graphs.

38. Find the curvature of the circular helix $\langle a\cos t, a\sin t, bt \rangle$.

39. Label as true or false and explain: at a relative extremum of $y = f(x)$, the curvature is either a minimum or maximum.

40. Label as true or false and explain: at an inflection point of $y = f(x)$, the curvature is zero.

41. Label as true or false and explain: the curvature of the two-dimensional curve $y = f(x)$ is the same as the curvature of the three-dimensional curve $\mathbf{r}(t) = \langle t, f(t), c \rangle$ for any constant c.

42. Label as true or false and explain: the curvature of the two-dimensional curve $y = f(x)$ is the same as the curvature of the three-dimensional curve $\mathbf{r}(t) = \langle t, f(t), t \rangle$.

43. Show that the curvature of the polar curve $r = f(\theta)$ is given by

$$\kappa = \frac{|2[f'(\theta)]^2 - f(\theta)f''(\theta) + [f(\theta)]^2|}{\{[f'(\theta)]^2 + [f(\theta)]^2\}^{3/2}}.$$

44. If $f(0) = 0$, show that the curvature of the polar curve $r = f(\theta)$ at $\theta = 0$ is given by $\kappa = \dfrac{2}{|f'(0)|}$.

In exercises 45–48, use exercises 43 and 44 to find the curvature of the polar curve at the indicated points.

45. $r = \sin 3\theta$, $\theta = 0$, $\theta = \frac{\pi}{6}$

46. $r = 3 + 2\cos\theta$, $\theta = 0$, $\theta = \frac{\pi}{2}$

47. $r = 3e^{2\theta}$, $\theta = 0$, $\theta = 1$

48. $r = 1 - 2\sin\theta$, $\theta = 0$, $\theta = \frac{\pi}{2}$

49. Find the curvature of the helix traced out by $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 0.4t \rangle$ and compare to the result of example 4.5.

50. Find the limit as $n \to 0$ of the curvature of $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, nt \rangle$ for $n > 0$. Explain this result graphically.

51. The cycloid is a curve with parametric equations $x = t - \sin t$, $y = 1 - \cos t$. Show that the curvature of the cycloid equals $\frac{1}{\sqrt{8y}}$, for $y \neq 0$.

52. Find the curvature of $\mathbf{r}(t) = \langle \cosh t, \sinh t, 2 \rangle$. Find the limit of the curvature as $t \to \infty$. Use a property of hyperbolas to explain this result.

53. For the logarithmic spiral $r = ae^{b\theta}$, show that the curvature equals $\kappa = \dfrac{e^{-b\theta}}{a\sqrt{1+b^2}}$. Show that as $b \to 0$, the spiral approaches a circle.

EXPLORATORY EXERCISES

1. In this exercise, we explore an unusual two-dimensional parametric curve sometimes known as the **Cornu spiral.** Define the vector-valued function $\mathbf{r}(t) = \left\langle \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du \right\rangle$. Use a graphing utility to sketch the graph of $\mathbf{r}(t)$ for $-\pi \le t \le \pi$. Compute the arc length of the curve from $t = 0$ to $t = c$ and compute the curvature at $t = c$. What is the remarkable property that you find?

2. Assume that $f(x)$ has three continuous derivatives. Prove that at a local minimum of $y = f(x)$, $\kappa = f''(x)$ and $\kappa'(x) = f'''(x)$. Prove that at a local maximum of $y = f(x)$, $\kappa = -f''(x)$ and $\kappa'(x) = -f'''(x)$.

11.5 TANGENT AND NORMAL VECTORS

Up to this point, we have used a single frame of reference for all of our work with vectors. That is, we have written all vectors in terms of the standard unit basis vectors **i**, **j** and **k**. However, this is not always the most convenient framework for describing vectors. For instance, such a fixed frame of reference would be particularly inconvenient when investigating the forces acting on an aircraft as it flies across the sky. A much better frame of reference would be one that moves along with the aircraft. As it turns out, such a moving frame of reference sheds light on a wide variety of problems. In this section, we will construct this moving reference frame and see how it immediately provides useful information regarding the forces acting on an object in motion.

Consider an object moving along a smooth curve traced out by the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. If we are to define a reference frame that moves with the object, we will need to have (at each point on the curve) three mutually orthogonal unit vectors. One of these should point in the direction of motion (i.e., in the direction of the orientation of the curve). In section 11.4, we defined the unit tangent vector $\mathbf{T}(t)$ by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Further, recall from Theorem 2.4 that since $\mathbf{T}(t)$ is a unit vector (and consequently has the constant magnitude of 1), $\mathbf{T}(t)$ must be orthogonal to $\mathbf{T}'(t)$ for each t. This gives us a second unit vector in our moving frame of reference, as follows.

DEFINITION 5.1

The **principal unit normal vector** $\mathbf{N}(t)$ is a unit vector having the same direction as $\mathbf{T}'(t)$ and is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}. \tag{5.1}$$

You might wonder about the direction in which $\mathbf{N}(t)$ points. Simply saying that it's orthogonal to $\mathbf{T}(t)$ is not quite enough. After all, in three dimensions, there are infinitely many directions that are orthogonal to $\mathbf{T}(t)$. (In two dimensions, there are only two possible directions.) We can clarify this with the following observation.

Recall that from (4.5), we have that $\frac{ds}{dt} = \|\mathbf{r}'(t)\| > 0$. (This followed from the definition of the arc length parameter in (4.1).) In particular, this says that $\left|\frac{ds}{dt}\right| = \frac{ds}{dt}$. From the chain rule, we have

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

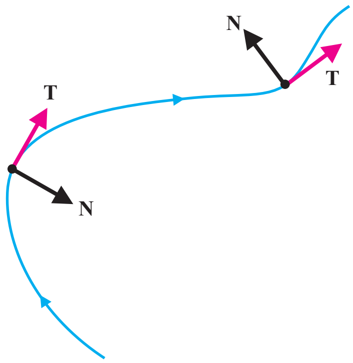


FIGURE 11.23
Principal unit normal vectors

This gives us
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{d\mathbf{T}}{ds}\dfrac{ds}{dt}}{\left\|\dfrac{d\mathbf{T}}{ds}\right\|\left|\dfrac{ds}{dt}\right|} = \frac{\dfrac{d\mathbf{T}}{ds}}{\left\|\dfrac{d\mathbf{T}}{ds}\right\|}$$

or equivalently,
$$\mathbf{N}(t) = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}, \tag{5.2}$$

where we have used the definition of curvature in (4.3), $\kappa = \left\|\dfrac{d\mathbf{T}}{ds}\right\|$.

Although (5.2) is not particularly useful as a formula for computing $\mathbf{N}(t)$ (why not?), we can use it to interpret the meaning of $\mathbf{N}(t)$. Since $\kappa > 0$, in order for (5.2) to make sense, $\mathbf{N}(t)$ will have the same direction as $\dfrac{d\mathbf{T}}{ds}$. Note that $\dfrac{d\mathbf{T}}{ds}$ is the instantaneous rate of change of the unit tangent vector with respect to arc length. This says that $\dfrac{d\mathbf{T}}{ds}$ (and consequently also, $\mathbf{N}$) points in the direction in which $\mathbf{T}$ is turning as arc length increases. That is, $\mathbf{N}(t)$ will always point to the *concave* side of the curve (see Figure 11.23).

EXAMPLE 5.1 Finding Unit Tangent and Principal Unit Normal Vectors

Find the unit tangent and principal unit normal vectors to the curve defined by $\mathbf{r}(t) = \langle t^2, t\rangle$.

Solution Notice that $\mathbf{r}'(t) = \langle 2t, 1\rangle$ and so from (4.2), we have
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 2t, 1\rangle}{\|\langle 2t, 1\rangle\|} = \frac{\langle 2t, 1\rangle}{\sqrt{4t^2+1}}$$
$$= \frac{2t}{\sqrt{4t^2+1}}\mathbf{i} + \frac{1}{\sqrt{4t^2+1}}\mathbf{j}.$$

Using the quotient rule, we have
$$\mathbf{T}'(t) = \frac{2\sqrt{4t^2+1} - 2t\left(\frac{1}{2}\right)(4t^2+1)^{-1/2}(8t)}{4t^2+1}\mathbf{i} - \frac{1}{2}(4t^2+1)^{-3/2}(8t)\mathbf{j}$$
$$= 2(4t^2+1)^{-1/2}\frac{(4t^2+1)-4t^2}{4t^2+1}\mathbf{i} - (4t^2+1)^{-3/2}(4t)\mathbf{j}$$
$$= 2(4t^2+1)^{-3/2}\langle 1, -2t\rangle.$$

Further,
$$\|\mathbf{T}'(t)\| = 2(4t^2+1)^{-3/2}\|\langle 1, -2t\rangle\|$$
$$= 2(4t^2+1)^{-3/2}\sqrt{1+4t^2} = 2(4t^2+1)^{-1}.$$

From (5.1), the principal unit normal is then
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{2(4t^2+1)^{-3/2}\langle 1, -2t\rangle}{2(4t^2+1)^{-1}}$$
$$= (4t^2+1)^{-1/2}\langle 1, -2t\rangle.$$

In particular, for $t = 1$, we get $\mathbf{T}(1) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle$ and $\mathbf{N}(1) = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$. We sketch the curve and these two sample vectors in Figure 11.24. ■

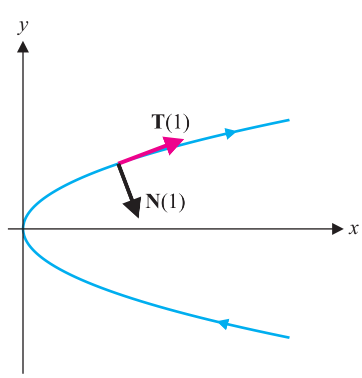


FIGURE 11.24
Unit tangent and principal unit normal vectors

The calculations are similar in three dimensions, as we illustrate in example 5.2.

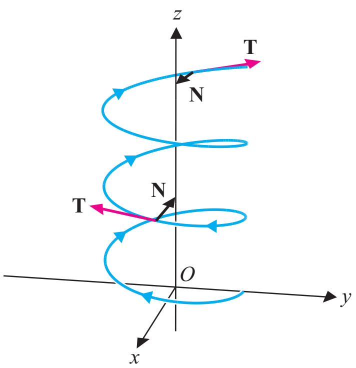


FIGURE 11.25
Unit tangent and principal unit normal vectors

EXAMPLE 5.2 Finding Unit Tangent and Principal Unit Normal Vectors

Find the unit tangent and principal unit normal vectors to the curve determined by $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, t \rangle$.

Solution First, observe that $\mathbf{r}'(t) = \langle 2\cos 2t, -2\sin 2t, 1 \rangle$ and so, we have from (4.2) that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 2\cos 2t, -2\sin 2t, 1\rangle}{\|\langle 2\cos 2t, -2\sin 2t, 1\rangle\|} = \frac{1}{\sqrt{5}}\langle 2\cos 2t, -2\sin 2t, 1\rangle.$$

This gives us
$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -4\sin 2t, -4\cos 2t, 0\rangle$$

and so, from (5.1), the principal unit normal is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{4}\langle -4\sin 2t, -4\cos 2t, 0\rangle = \langle -\sin 2t, -\cos 2t, 0\rangle.$$

Notice that the curve here is a circular helix and that at each point, $\mathbf{N}(t)$ points straight back toward the z-axis (see Figure 11.25). ■

To get a third unit vector orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$, we simply take their cross product.

DEFINITION 5.2

We define the **binormal** vector $\mathbf{B}(t)$ to be

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

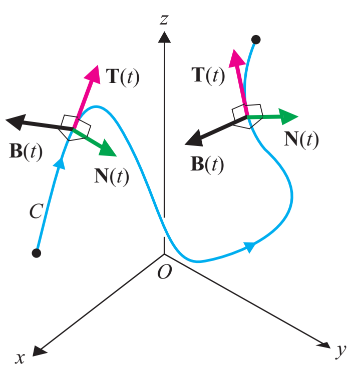


FIGURE 11.26
The **TNB** frame

Notice that by definition, $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and by Theorem 4.4 in Chapter 10, its magnitude is given by

$$\|\mathbf{B}(t)\| = \|\mathbf{T}(t) \times \mathbf{N}(t)\| = \|\mathbf{T}(t)\|\,\|\mathbf{N}(t)\| \sin\theta,$$

where θ is the angle between $\mathbf{T}(t)$ and $\mathbf{N}(t)$. However, since $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are both unit vectors, $\|\mathbf{T}(t)\| = \|\mathbf{N}(t)\| = 1$. Further, $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal, so that $\sin\theta = 1$ and consequently, $\|\mathbf{B}(t)\| = 1$, too. This triple of three unit vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$ forms a frame of reference, called the **TNB frame** (or the **moving trihedral**), that moves along the curve defined by $\mathbf{r}(t)$ (see Figure 11.26). This has particular importance in a branch of mathematics called *differential geometry* and is used in the navigation of spacecraft.

As you can see, the definition of the binormal vector is certainly straightforward. We illustrate this now for the curve from example 5.2.

EXAMPLE 5.3 Finding the Binormal Vector

Find the binormal vector $\mathbf{B}(t)$ for the curve traced out by $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, t \rangle$.

Solution Recall from example 5.2 that the unit tangent vector is given by $\mathbf{T}(t) = \frac{1}{\sqrt{5}}\langle 2\cos 2t, -2\sin 2t, 1\rangle$ and the principal unit normal vector is given by

$\mathbf{N}(t) = \langle -\sin 2t, -\cos 2t, 0\rangle$. The binormal vector is then

$$\begin{aligned}\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) &= \frac{1}{\sqrt{5}}\langle 2\cos 2t, -2\sin 2t, 1\rangle \times \langle -\sin 2t, -\cos 2t, 0\rangle \\ &= \frac{1}{\sqrt{5}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos 2t & -2\sin 2t & 1 \\ -\sin 2t & -\cos 2t & 0 \end{vmatrix} \\ &= \frac{1}{\sqrt{5}}[\mathbf{i}(\cos 2t) - \mathbf{j}(\sin 2t) + \mathbf{k}(-2\cos^2 2t - 2\sin^2 2t)] \\ &= \frac{1}{\sqrt{5}}\langle \cos 2t, -\sin 2t, -2\rangle.\end{aligned}$$

We illustrate the **TNB** frame for this curve in Figure 11.27. ■

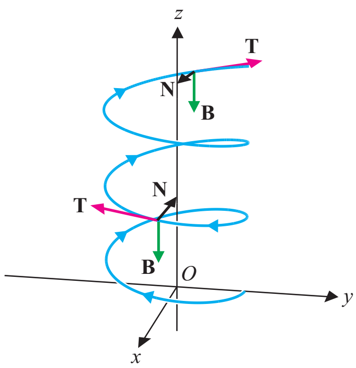


FIGURE 11.27
The **TNB** frame for
$\mathbf{r}(t) = \langle \sin 2t, \cos 2t, t\rangle$

For each point on a curve, the plane passing through that point and determined by $\mathbf{N}(t)$ and $\mathbf{B}(t)$ is called the **normal plane.** Accordingly, the normal plane to a curve at a given point contains all of the lines that are orthogonal to the tangent vector at that point. For each point on a curve, the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ is called the **osculating plane.** For a two-dimensional curve, the osculating plane is simply the xy-plane.

For a given value of t, say $t = t_0$, if the curvature κ of the curve at the point P corresponding to t_0 is nonzero, then the circle of radius $\rho = \frac{1}{\kappa}$ lying completely in the osculating plane and whose center lies a distance of $\frac{1}{\kappa}$ from P along the normal $\mathbf{N}(t)$ is called the **osculating circle** (or the **circle of curvature**). Recall from example 4.4 that the curvature of a circle is the reciprocal of its radius. This says that the osculating circle has the same tangent and curvature at P as the curve. Further, since the normal vector always points to the concave side of the curve, the osculating circle lies on the concave side of the curve. In this sense, then, the osculating circle is the circle that "best fits" the curve at the point P (see Figure 11.28). The radius of the osculating circle is called the **radius of curvature** and the center of the circle is called the **center of curvature.**

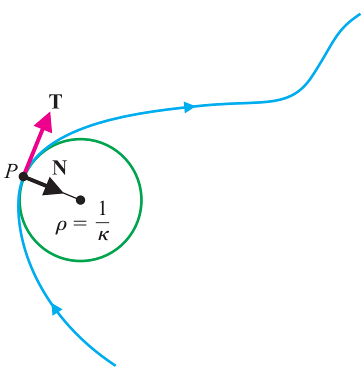


FIGURE 11.28
Osculating circle

EXAMPLE 5.4 Finding the Osculating Circle

Find the osculating circle for the parabola defined by $\mathbf{r}(t) = \langle t^2, t\rangle$ at $t = 0$.

Solution In example 5.1, we found that the unit tangent vector is

$$\begin{aligned}\mathbf{T}(t) &= (4t^2+1)^{-1/2}\langle 2t, 1\rangle, \\ \mathbf{T}'(t) &= 2(4t^2+1)^{-3/2}\langle 1, -2t\rangle\end{aligned}$$

and the principal unit normal is

$$\mathbf{N}(t) = (4t^2+1)^{-1/2}\langle 1, -2t\rangle.$$

So, from (4.6), the curvature is given by

$$\begin{aligned}\kappa(t) &= \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \\ &= \frac{2(4t^2+1)^{-3/2}(1+4t^2)^{1/2}}{(4t^2+1)^{1/2}} = 2(4t^2+1)^{-3/2}.\end{aligned}$$

We now have $\kappa(0) = 2$, so that the radius of curvature for $t = 0$ is $\rho = \frac{1}{\kappa} = \frac{1}{2}$. Further, $\mathbf{N}(0) = \langle 1, 0\rangle$ and $\mathbf{r}(0) = \langle 0, 0\rangle$, so that the center of curvature is located $\rho = \frac{1}{2}$ unit from the origin in the direction of $\mathbf{N}(0)$ (i.e., along the positive x-axis). We draw the curve and the osculating circle in Figure 11.29. ■

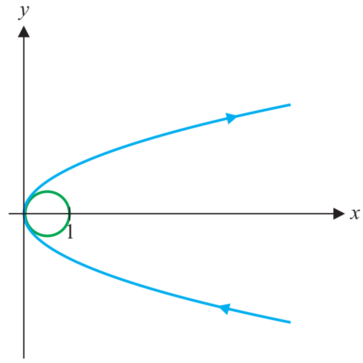


FIGURE 11.29
Osculating circle

Tangential and Normal Components of Acceleration

Now that we have defined the unit tangent and principal unit normal vectors, we can make a remarkable observation about the motion of an object. In particular, we'll see how this observation helps to explain the behavior of an automobile as it travels along a curved stretch of road.

Suppose that an object moves along a smooth curve traced out by the vector-valued function $\mathbf{r}(t)$, where t represents time. Recall from the definition of the unit tangent vector that $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ and from (4.5), $\|\mathbf{r}'(t)\| = \frac{ds}{dt}$, where s represents arc length. The velocity of the object is then given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \|\mathbf{r}'(t)\|\mathbf{T}(t) = \frac{ds}{dt}\mathbf{T}(t).$$

Using the product rule [Theorem 2.3 (iii)], the acceleration is given by

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{T}(t)\right) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t). \tag{5.3}$$

Recall that we had defined the principal unit normal by $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, so that

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t). \tag{5.4}$$

Further, by the chain rule,

$$\begin{aligned} \|\mathbf{T}'(t)\| &= \left\|\frac{d\mathbf{T}}{dt}\right\| = \left\|\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right\| \\ &= \left|\frac{ds}{dt}\right| \left\|\frac{d\mathbf{T}}{ds}\right\| = \kappa\frac{ds}{dt}, \end{aligned} \tag{5.5}$$

where we have also used the definition of the curvature κ given in (4.3) and the fact that $\frac{ds}{dt} > 0$. Putting together (5.4) and (5.5), we now have that

$$\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\,\mathbf{N}(t) = \kappa\frac{ds}{dt}\mathbf{N}(t).$$

Using this together with (5.3), we now get

$$\boxed{\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t).} \tag{5.6}$$

Equation (5.6) provides us with a surprising wealth of insight into the motion of an object. First, notice that since $\mathbf{a}(t)$ is written as a sum of a vector parallel to $\mathbf{T}(t)$ and a vector parallel to $\mathbf{N}(t)$, the acceleration vector always lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$ (i.e., the osculating plane). In particular, this says that the acceleration is always orthogonal to the binormal $\mathbf{B}(t)$. We call the coefficient of $\mathbf{T}(t)$ in (5.6) the **tangential component of**

TODAY IN MATHEMATICS

Edward Witten (1951–)
An American theoretical physicist who is one of the world's experts in string theory. He earned the Fields Medal in 1990 for his contributions to mathematics. Michael Atiyah, a mathematics colleague, wrote, "Although he is definitely a physicist (as his list of publications clearly shows), his command of mathematics is rivalled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique. Time and again he has surprised the mathematical community by his brilliant application of physical insight leading to new and deep mathematical theorems." In addition, Atiyah wrote, "In his hands, physics is once again providing a rich source of inspiration and insight in mathematics."

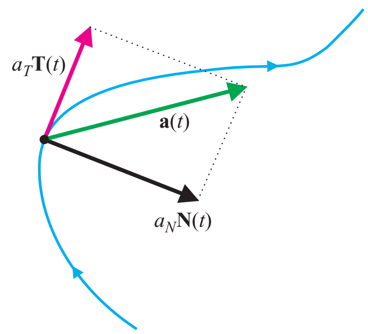


FIGURE 11.30
Tangential and normal components of acceleration

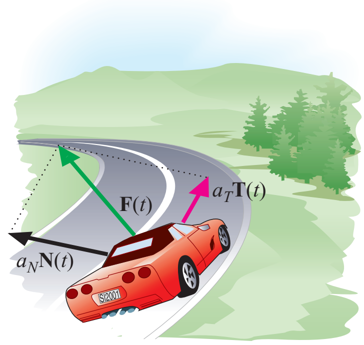


FIGURE 11.31
Driving around a curve

acceleration a_T and the coefficient of $\mathbf{N}(t)$ the **normal component of acceleration** a_N. That is,

$$a_T = \frac{d^2s}{dt^2} \quad \text{and} \quad a_N = \kappa\left(\frac{ds}{dt}\right)^2. \tag{5.7}$$

See Figure 11.30 for a graphical depiction of this decomposition of $\mathbf{a}(t)$ into tangential and normal components.

We now discuss (5.6) in the familiar context of a car driving around a curve in the road (see Figure 11.31). From Newton's second law of motion, the net force acting on the car at any given time t is $\mathbf{F}(t) = m\mathbf{a}(t)$, where m is the mass of the car. From (5.6), we have

$$\mathbf{F}(t) = m\mathbf{a}(t) = m\frac{d^2s}{dt^2}\,\mathbf{T}(t) + m\kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t).$$

Since $\mathbf{T}(t)$ points in the direction of the path of motion, you want the component of the force acting in the direction of $\mathbf{T}(t)$ to be as large as possible compared to the component of the force acting in the direction of the normal $\mathbf{N}(t)$. (If the normal component of the force is too large, it may exceed the normal component of the force of friction between the tires and the highway, causing the car to skid off the road.) Notice that the only way to minimize the force applied in this direction is to make $\left(\frac{ds}{dt}\right)^2$ small, where $\frac{ds}{dt}$ is the speed. So, reducing speed is the only way to reduce the normal component of the force. To have a larger tangential component of the force, you will need to have $\frac{d^2s}{dt^2}$ (the instantaneous rate of change of speed with respect to time) larger. So, to maximize the tangential component of the force, you need to be accelerating while in the curve. You have probably noticed advisory signs on the highway advising you to slow down *before* you enter a sharp curve. Notice that reducing speed (i.e., reducing $\frac{ds}{dt}$) before the curve and then gently accelerating (keeping $\frac{d^2s}{dt^2} > 0$) once you're in the curve keeps the resultant force $\mathbf{F}(t)$ pointing in the general direction you are moving. Alternatively, waiting until you're in the curve to slow down keeps $\frac{d^2s}{dt^2} < 0$, which makes $\frac{d^2s}{dt^2}\mathbf{T}(t)$ point in the *opposite* direction as $\mathbf{T}(t)$. The net force $\mathbf{F}(t)$ will then point away from the direction of motion (see Figure 11.32).

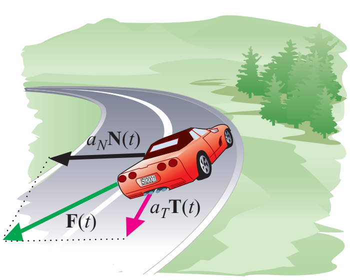


FIGURE 11.32
Net force: $\frac{d^2s}{dt^2} < 0$

EXAMPLE 5.5 Finding Tangential and Normal Components of Acceleration

Find the tangential and normal components of acceleration for an object with position vector $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4t\rangle$.

Solution In example 4.5, we found that the curvature of this curve is $\kappa = \frac{1}{10}$. We also have $\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 4\rangle$, so that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \sqrt{20}$$

and so, $\dfrac{d^2s}{dt^2} = 0$, for all t. From (5.6), we have that the acceleration is

$$\begin{aligned}\mathbf{a}(t) &= \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t)\\ &= (0)\,\mathbf{T}(t) + \frac{1}{10}\left(\sqrt{20}\right)^2\mathbf{N}(t) = 2\mathbf{N}(t).\end{aligned}$$

So, here we have $a_T = 0$ and $a_N = 2$. ■

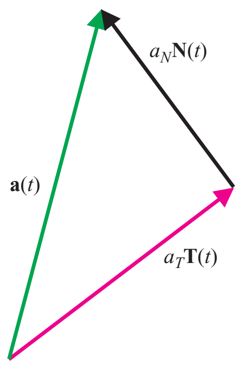


FIGURE 11.33
Components of $\mathbf{a}(t)$

Notice that it's reasonably simple to compute $a_T = \dfrac{d^2s}{dt^2}$. You must only calculate $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$ and then differentiate the result. On the other hand, computing a_N is a bit more complicated, since it requires you to first compute the curvature κ. We can simplify the calculation of a_N with the following observation. From (5.6), we have

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t).$$

This says that $\mathbf{a}(t)$ is the vector resulting from adding the **orthogonal** vectors $a_T\mathbf{T}(t)$ and $a_N\mathbf{N}(t)$. (See Figure 11.33, where we have drawn the vectors so that the initial point of $a_N\mathbf{N}(t)$ is located at the terminal point of $a_T\mathbf{T}(t)$.) From the Pythagorean Theorem, we have that

$$\begin{aligned}\|\mathbf{a}(t)\|^2 &= \|a_T\mathbf{T}(t)\|^2 + \|a_N\mathbf{N}(t)\|^2\\ &= a_T^2 + a_N^2,\end{aligned} \tag{5.8}$$

since $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are unit vectors (i.e., $\|\mathbf{T}(t)\| = \|\mathbf{N}(t)\| = 1$). Solving (5.8) for a_N, we get

$$a_N = \sqrt{\|\mathbf{a}(t)\|^2 - a_T^2}, \tag{5.9}$$

where we have taken the positive root since $a_N = \kappa\left(\dfrac{ds}{dt}\right)^2 \geq 0$. Once you know $\mathbf{a}(t)$ and a_T, you can use (5.9) to quickly calculate a_N, without first computing the curvature. As an alternative, observe that a_T is the component of $\mathbf{a}(t)$ along the velocity vector $\mathbf{v}(t)$. Further, from (5.7) and (5.9), we can compute a_N and κ. This allows us to compute a_T, a_N and κ without first computing the derivative of the speed.

EXAMPLE 5.6 Finding Tangential and Normal Components of Acceleration

Find the tangential and normal components of acceleration for an object whose path is defined by $\mathbf{r}(t) = \langle t, 2t, t^2\rangle$. In particular, find these components at $t = 1$. Also, find the curvature.

Solution First, we compute the velocity $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2, 2t \rangle$ and the acceleration $\mathbf{a}(t) = \langle 0, 0, 2 \rangle$. This gives us

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = \|\langle 1, 2, 2t \rangle\| = \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2}.$$

The tangential component of acceleration a_T is the component of $\mathbf{a}(t) = \langle 0, 0, 2 \rangle$ along $\mathbf{v}(t) = \langle 1, 2, 2t \rangle$:

$$a_T = \langle 0, 0, 2 \rangle \cdot \frac{\langle 1, 2, 2t \rangle}{\sqrt{5 + 4t^2}} = \frac{4t}{\sqrt{5 + 4t^2}}.$$

From (5.9), we have that the normal component of acceleration is

$$a_N = \sqrt{\|\mathbf{a}(t)\|^2 - a_T^2} = \sqrt{2^2 - \frac{16t^2}{5 + 4t^2}}$$

$$= \sqrt{\frac{4(5 + 4t^2) - 16t^2}{5 + 4t^2}} = \frac{\sqrt{20}}{\sqrt{5 + 4t^2}}.$$

Think about computing a_N from its definition in (5.7) and notice how much simpler it was to use (5.9). Further, at $t = 1$, we have

$$a_T = \frac{4}{3} \quad \text{and} \quad a_N = \frac{\sqrt{20}}{3}.$$

Finally, from (5.7), the curvature is

$$\kappa = \frac{a_N}{\left(\frac{ds}{dt}\right)^2} = \frac{\sqrt{20}}{\sqrt{5 + 4t^2}} \frac{1}{\left(\sqrt{5 + 4t^2}\right)^2}$$

$$= \frac{\sqrt{20}}{(5 + 4t^2)^{3/2}}.$$

Notice how easy it was to compute the curvature in this way. In Figure 11.34, we show a plot of the curve traced out by $\mathbf{r}(t)$, along with the tangential and normal components of acceleration at $t = 1$. ■

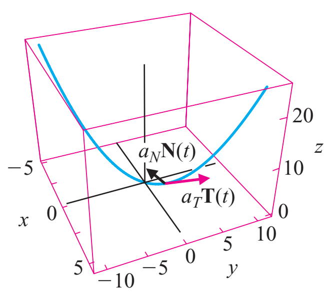


FIGURE 11.34
Tangential and normal components of acceleration at $t = 1$

Equation (5.6) has a wealth of applications. Among many others, it provides us with a relatively simple proof of Theorem 4.1, which we had deferred until now. You may recall that the result says that the curvature of a path traced out by the vector-valued function $\mathbf{r}(t)$ is given by

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}. \tag{5.10}$$

PROOF

From (5.6), we have

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}(t).$$

Taking the cross product of both sides of this equation with $\mathbf{T}(t)$ gives us

$$\mathbf{T}(t) \times \mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) \times \mathbf{T}(t) + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{T}(t) \times \mathbf{N}(t)$$

$$= \kappa\left(\frac{ds}{dt}\right)^2\mathbf{T}(t) \times \mathbf{N}(t),$$

since the cross product of any vector with itself is the zero vector. Taking the magnitude of both sides and recognizing that $\mathbf{T}(t) \times \mathbf{N}(t) = \mathbf{B}(t)$, we get

$$\begin{aligned}\|\mathbf{T}(t) \times \mathbf{a}(t)\| &= \kappa \left(\frac{ds}{dt}\right)^2 \|\mathbf{T}(t) \times \mathbf{N}(t)\| \\ &= \kappa \left(\frac{ds}{dt}\right)^2 \|\mathbf{B}(t)\| = \kappa \left(\frac{ds}{dt}\right)^2,\end{aligned}$$

since the binormal vector $\mathbf{B}(t)$ is a unit vector. Recalling that $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$, $\mathbf{a}(t) = \mathbf{r}''(t)$ and $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$ gives us

$$\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|} = \kappa \|\mathbf{r}'(t)\|^2.$$

Solving this for κ leaves us with (5.10), as desired. ■

Kepler's Laws

We are now in a position to present one of the most profound discoveries ever made by humankind. For hundreds of years, people believed that the sun, the other stars and the planets all revolved around the earth. The year 1543 saw the publication of the astronomer Copernicus' theory that the earth and other planets, in fact, revolved around the sun. Sixty years later, based on a very careful analysis of a massive number of astronomical observations, the German astronomer Johannes Kepler formulated three laws that he reasoned must be followed by every planet. We present these now.

KEPLER'S LAWS OF PLANETARY MOTION

1. Each planet follows an elliptical orbit, with the sun at one focus.
2. The line segment joining the sun to a planet sweeps out equal areas in equal times.
3. If T is the time required for a given planet to make one orbit of the sun and if the length of the major axis of its elliptical orbit is $2a$, then $T^2 = ka^3$, for some constant k (i.e., T^2 is proportional to a^3).

Kepler's exhaustive analysis of the data changed our perception of our place in the universe. While Kepler's work was empirical in nature, Newton's approach to the same problem was not. In 1687, in his book *Principia Mathematica*, Newton showed how to use his calculus to derive Kepler's three laws from two of Newton's laws: his second law of motion and his law of universal gravitation. You should not underestimate the significance of this achievement. With this work, Newton shed light on some of the fundamental physical laws that govern our universe.

In order to simplify our analysis, we assume that we are looking at a solar system consisting of one sun and one planet. This is a reasonable assumption, since the gravitational attraction of the sun is far greater than that of any other body (planet, moon, comet, etc.), owing to the sun's far greater mass. (As it turns out, the gravitational attraction of other bodies does have an effect. In fact, it was an observation of the irregularities in the orbit of Uranus that led astronomers to hypothesize the existence of Neptune before it had ever been observed in a telescope.)

We assume that the center of mass of the sun is located at the origin and that the center of mass of the planet is located at the terminal point of the vector-valued function $\mathbf{r}(t)$. The

HISTORICAL NOTES

Johannes Kepler (1571–1630) German astronomer and mathematician whose discoveries revolutionized Western science. Kepler's remarkable mathematical ability and energy produced connections among many areas of research. A study of observations of the moon led to work in optics that included the first fundamentally correct description of the operation of the human eye. Kepler's model of the solar system used an ingenious nesting of the five Platonic solids to describe the orbits of the six known planets. (Kepler invented the term "satellite" to describe the moon, which his system demoted from planetary status.) A study of the wine casks opened at his wedding led Kepler to compute volumes of solids of revolution, inventing techniques that were vital to the subsequent development of calculus.

velocity vector for the planet is then $\mathbf{v}(t) = \mathbf{r}'(t)$, with the acceleration given by $\mathbf{a}(t) = \mathbf{r}''(t)$. From Newton's second law of motion, we have that the net (gravitational) force $\mathbf{F}(t)$ acting on the planet is

$$\mathbf{F}(t) = m\mathbf{a}(t),$$

where m is the mass of the planet. From Newton's law of universal gravitation, we have that if M is the mass of the sun, then the gravitational attraction between the two bodies satisfies

$$\mathbf{F}(t) = -\frac{GmM}{\|\mathbf{r}(t)\|^2}\frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|},$$

where G is the **universal gravitational constant.**[1] We have written $\mathbf{F}(t)$ in this form so that you can see that at each point, the gravitational attraction acts in the direction **opposite** the position vector $\mathbf{r}(t)$. Further, the gravitational attraction is jointly proportional to the masses of the sun and the planet and inversely proportional to the square of the distance between the sun and the planet. For simplicity, we will let $r = \|\mathbf{r}\|$ and not explicitly indicate the t-variable. Taking $\mathbf{u}(t) = \dfrac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|}$ (a unit vector in the direction of $\mathbf{r}(t)$), we can then write Newton's laws as simply

$$\mathbf{F} = m\mathbf{a} \quad \text{and} \quad \mathbf{F} = -\frac{GmM}{r^2}\mathbf{u}.$$

We begin by demonstrating that the orbit of a planet lies in a plane. Equating the two expressions above for $\mathbf{F}$ and canceling out the common factor of m, we have

$$\mathbf{a} = -\frac{GM}{r^2}\mathbf{u}. \tag{5.11}$$

Notice that this says that the acceleration $\mathbf{a}$ always points in the **opposite** direction from $\mathbf{r}$, so that the force of gravity accelerates the planet toward the sun at all times. Since $\mathbf{a}$ and $\mathbf{r}$ are parallel, we have that

$$\mathbf{r} \times \mathbf{a} = \mathbf{0}. \tag{5.12}$$

Next, from the product rule [Theorem 2.3 (v)], we have

$$\begin{aligned}\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) &= \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{0},\end{aligned}$$

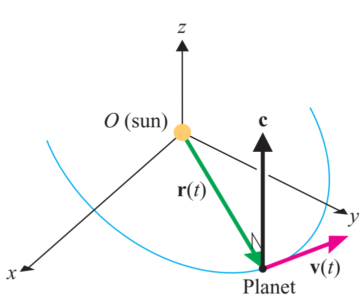


FIGURE 11.35
Position and velocity vectors for planetary motion

in view of (5.12) and since $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. Integrating both sides of this expression gives us

$$\mathbf{r} \times \mathbf{v} = \mathbf{c}, \tag{5.13}$$

for some constant vector $\mathbf{c}$. This says that for each t, $\mathbf{r}(t)$ is orthogonal to the constant vector $\mathbf{c}$. In particular, then, the terminal point of $\mathbf{r}(t)$ (and consequently, the orbit of the planet) lies in the plane orthogonal to the vector $\mathbf{c}$ and containing the origin.

Now that we have established that a planet's orbit lies in a plane, we are in a position to prove Kepler's first law. For the sake of simplicity, we assume that the plane containing the orbit is the xy-plane, so that $\mathbf{c}$ is parallel to the z-axis (see Figure 11.35). Now, observe that since $\mathbf{r} = r\mathbf{u}$, we have by the product rule [Theorem 2.3 (iii)] that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\mathbf{u}) = \frac{dr}{dt}\mathbf{u} + r\frac{d\mathbf{u}}{dt}.$$

[1] If we measure mass in kilograms, force in newtons and distance in meters, G is given approximately by $G \approx 6.672 \times 10^{-11}\ \mathrm{N\ m^2/kg^2}$.

Substituting this into (5.13), and replacing $\mathbf{r}$ by $r\mathbf{u}$, we have

$$\begin{aligned}\mathbf{c} = \mathbf{r} \times \mathbf{v} &= r\mathbf{u} \times \left(\frac{dr}{dt}\mathbf{u} + r\frac{d\mathbf{u}}{dt}\right) \\ &= r\frac{dr}{dt}(\mathbf{u} \times \mathbf{u}) + r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) \\ &= r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right),\end{aligned}$$

since $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Together with (5.11), this gives us

$$\begin{aligned}\mathbf{a} \times \mathbf{c} &= -\frac{GM}{r^2}\mathbf{u} \times r^2\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) \\ &= -GM\mathbf{u} \times \left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right) \\ &= -GM\left[\left(\mathbf{u} \cdot \frac{d\mathbf{u}}{dt}\right)\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\frac{d\mathbf{u}}{dt}\right],\end{aligned} \tag{5.14}$$

where we have rewritten the vector triple product using Theorem 4.3 (vi) in Chapter 10. There are two other things to note here. First, since $\mathbf{u}$ is a unit vector, $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 = 1$. Further, from Theorem 2.4, since $\mathbf{u}$ is a vector-valued function of constant magnitude, $\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0$. Consequently, (5.14) simplifies to

$$\mathbf{a} \times \mathbf{c} = GM\frac{d\mathbf{u}}{dt} = \frac{d}{dt}(GM\mathbf{u}),$$

since G and M are constants. Observe that using the definition of $\mathbf{a}$, we can write

$$\mathbf{a} \times \mathbf{c} = \frac{d\mathbf{v}}{dt} \times \mathbf{c} = \frac{d}{dt}(\mathbf{v} \times \mathbf{c}),$$

since $\mathbf{c}$ is a constant vector. Equating these last two expressions for $\mathbf{a} \times \mathbf{c}$ gives us

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{c}) = \frac{d}{dt}(GM\mathbf{u}).$$

Integrating both sides gives us

$$\mathbf{v} \times \mathbf{c} = GM\mathbf{u} + \mathbf{b}, \tag{5.15}$$

for some constant vector $\mathbf{b}$. Now, note that $\mathbf{v} \times \mathbf{c}$ must be orthogonal to $\mathbf{c}$ and so, $\mathbf{v} \times \mathbf{c}$ must lie in the xy-plane. (Recall that we had chosen the orientation of the xy-plane so that $\mathbf{c}$ was a vector orthogonal to the plane. This says further that every vector orthogonal to $\mathbf{c}$ must lie in the xy-plane.) From (5.15), since $\mathbf{u}$ and $\mathbf{v} \times \mathbf{c}$ lie in the xy-plane, $\mathbf{b}$ must also lie in the same plane. (Think about why this must be so.) Next, align the x-axis so that the positive x-axis points in the same direction as $\mathbf{b}$ (see Figure 11.36). Also, let θ be the angle from the positive x-axis to $\mathbf{r}(t)$, so that (r, θ) are polar coordinates for the endpoint of the position vector $\mathbf{r}(t)$, as indicated in Figure 11.36.

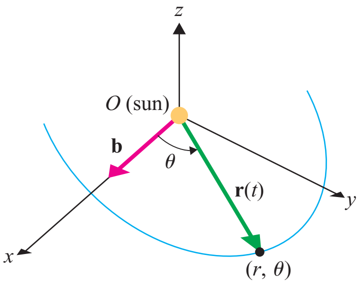


FIGURE 11.36
Polar coordinates for the position of the planet

Next, let $b = \|\mathbf{b}\|$ and $c = \|\mathbf{c}\|$. Then, from (5.13), we have

$$c^2 = \mathbf{c} \cdot \mathbf{c} = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{c} = \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}),$$

where we have rewritten the scalar triple product using Theorem 4.3 (v) in Chapter 10. Putting this together with (5.15), and writing $\mathbf{r} = r\mathbf{u}$, we get

$$\begin{aligned}c^2 = \mathbf{r} \cdot (\mathbf{v} \times \mathbf{c}) &= r\mathbf{u} \cdot (GM\mathbf{u} + \mathbf{b}) \\ &= rGM\mathbf{u} \cdot \mathbf{u} + r\mathbf{u} \cdot \mathbf{b}.\end{aligned} \tag{5.16}$$

Since $\mathbf{u}$ is a unit vector, $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2 = 1$ and by Theorem 3.2 in Chapter 10,

$$\mathbf{u}\cdot\mathbf{b} = \|\mathbf{u}\|\,\|\mathbf{b}\|\cos\theta = b\cos\theta,$$

where θ is the angle between $\mathbf{b}$ and $\mathbf{u}$ (i.e., the angle between the positive x-axis and $\mathbf{r}$). Together with (5.16), this gives us

$$c^2 = rGM + rb\cos\theta = r(GM + b\cos\theta).$$

Solving this for r gives us $$r = \frac{c^2}{GM + b\cos\theta}.$$

Dividing numerator and denominator by GM reduces this to

$$r = \frac{ed}{1 + e\cos\theta}, \tag{5.17}$$

where $e = \dfrac{b}{GM}$ and $d = \dfrac{c^2}{b}$. Recall from Theorem 7.2 in Chapter 9 that (5.17) is a polar equation for a conic section with focus at the origin and eccentricity e. Finally, since the orbit of a planet is a closed curve, this must be the equation of an ellipse, since the other conic sections (parabolas and hyperbolas) are not closed curves. We have now proved that (assuming one sun and one planet and no other celestial bodies), the orbit of a planet is an ellipse with one focus located at the center of mass of the sun.

You may be thinking what a long derivation this was. (We haven't been keeping score, but it was probably one of the longest derivations of anything in this book.) Take a moment, though, to realize the enormity of what we have done. Thanks to the genius of Newton and his second law of motion and his law of universal gravitation, we have in only a few pages used the calculus to settle one of the most profound questions of our existence: How do the mechanics of a solar system work? Through the power of reason and the use of considerable calculus, we have found an answer that is consistent with the observed motion of the planets, first postulated by Kepler. This magnificent achievement came more than 300 years ago and was one of the earliest (and most profound) success stories for the calculus. Since that time, the calculus has proven to be an invaluable tool for countless engineers, physicists, mathematicians and others.

EXERCISES 11.5

WRITING EXERCISES

1. Suppose that you are driving a car, going slightly uphill as the road curves to the left. Describe the directions of the unit tangent, principal unit normal and binormal vectors. What changes if the road curves to the right?

2. If the components of $\mathbf{r}(t)$ are linear functions, explain why you can't compute the principal unit normal vector. Describe graphically why it is impossible to define a single direction for the principal unit normal.

3. Previously in your study of calculus, you have approximated curves with lines and the graphs of other polynomials (Taylor polynomials). Discuss possible circumstances in which the osculating circle would be a better or worse approximation of a curve than the graph of a polynomial.

4. Suppose that you are flying in a fighter jet and an enemy jet is headed straight at you with velocity vector parallel to your principal unit normal vector. Discuss how much danger you are in and what maneuver(s) you might want to make to avoid danger.

In exercises 1–8, find the unit tangent and principal unit normal vectors at the given points.

1. $\mathbf{r}(t) = \langle t, t^2\rangle$ at $t = 0, t = 1$

2. $\mathbf{r}(t) = \langle t, t^3\rangle$ at $t = 0, t = 1$

3. $\mathbf{r}(t) = \langle \cos 2t, \sin 2t\rangle$ at $t = 0, t = \frac{\pi}{4}$

4. $\mathbf{r}(t) = \langle 2\cos t, 3\sin t\rangle$ at $t = 0, t = \frac{\pi}{4}$

5. $\mathbf{r}(t) = \langle \cos 2t, t, \sin 2t\rangle$ at $t = 0, t = \frac{\pi}{2}$

6. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t\rangle$ at $t = 0, t = \frac{\pi}{2}$

7. $\mathbf{r}(t) = \langle t, t^2 - 1, t\rangle$ at $t = 0, t = 1$

8. $\mathbf{r}(t) = \langle t, t, 3\sin 2t\rangle$ at $t = 0, t = -\pi$

In exercises 9–12, find the osculating circle at the given points.

9. $\mathbf{r}(t) = \langle t, t^2\rangle$ at $t = 0$

10. $\mathbf{r}(t) = \langle t, t^3\rangle$ at $t = 0$

11. $\mathbf{r}(t) = \langle \cos 2t, \sin 2t\rangle$ at $t = \frac{\pi}{4}$

12. $\mathbf{r}(t) = \langle 2\cos t, 3\sin t\rangle$ at $t = \frac{\pi}{4}$

In exercises 13–16, find the tangential and normal components of acceleration for the given position functions at the given points.

13. $\mathbf{r}(t) = \langle 8t, 16t - 16t^2\rangle$ at $t = 0, t = 1$

14. $\mathbf{r}(t) = \langle \cos 2t, \sin 2t\rangle$ at $t = 0, t = 2$

15. $\mathbf{r}(t) = \langle \cos 2t, t^2, \sin 2t\rangle$ at $t = 0, t = \frac{\pi}{4}$

16. $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, t^2\rangle$ at $t = 0, t = \frac{\pi}{4}$

17. In exercise 15, determine whether the speed of the object is increasing or decreasing at the given points.

18. In exercise 16, determine whether the speed of the object is increasing or decreasing at the given points.

19. For the circular helix traced out by $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$, find the tangential and normal components of acceleration.

20. For the linear path traced out by $\mathbf{r}(t) = \langle a + bt, c + dt, e + ft\rangle$, find the tangential and normal components of acceleration.

In exercises 21–24, find the binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ at $t = 0$ and $t = 1$. Also, sketch the curve traced out by $\mathbf{r}(t)$ and the vectors T, N and B at these points.

21. $\mathbf{r}(t) = \langle t, 2t, t^2\rangle$

22. $\mathbf{r}(t) = \langle t, 2t, t^3\rangle$

23. $\mathbf{r}(t) = \langle 4\cos\pi t, 4\sin\pi t, t\rangle$

24. $\mathbf{r}(t) = \langle 3\cos 2\pi t, t, \sin 2\pi t\rangle$

In exercises 25–28, label the statement as true (i.e., always true) or false and explain your answer.

25. $\mathbf{T} \cdot \dfrac{d\mathbf{T}}{ds} = 0$

26. $\mathbf{T} \cdot \mathbf{B} = 0$

27. $\dfrac{d}{ds}(\mathbf{T} \cdot \mathbf{T}) = 0$

28. $\mathbf{T} \cdot (\mathbf{N} \times \mathbf{B}) = 1$

The friction force required to keep a car from skidding on a curve is given by $\mathbf{F}_s(t) = ma_N\mathbf{N}(t)$. In exercises 29–32, find the friction force needed to keep a car of mass $m = 100$ (slugs) from skidding.

29. $\mathbf{r}(t) = \langle 100\cos\pi t, 100\sin\pi t\rangle$

30. $\mathbf{r}(t) = \langle 200\cos\pi t, 200\sin\pi t\rangle$

31. $\mathbf{r}(t) = \langle 100\cos 2\pi t, 100\sin 2\pi t\rangle$

32. $\mathbf{r}(t) = \langle 300\cos 2t, 300\sin 2t\rangle$

33. Based on your answers to exercises 29 and 30, how does the required friction force change when the radius of a turn is doubled?

34. Based on your answers to exercises 29 and 31, how does the required friction force change when the speed of a car on a curve is doubled?

35. Compare the radii of the osculating circles for $y = \cos x$ at $x = 0$ and $x = \frac{\pi}{4}$. Compute the concavity of the curve at these points and use this information to explain why one circle is larger than the other.

36. Compare the osculating circles for $y = \cos x$ at $x = 0$ and $x = \pi$. Compute the concavity of the curve at these points and use this information to help explain why the circles have the same radius.

37. For $y = x^2$, show that each center of curvature lies on the curve traced out by $\mathbf{r}(t) = \langle 2t + 4t^3, \frac{1}{2} + 3t^2\rangle$. Graph this curve.

38. For $r = e^{a\theta}$, $a > 0$, show that the radius of curvature is $e^{a\theta}\sqrt{a^2 + 1}$. Show that each center of curvature lies on the curve traced out by $ae^{at}\langle -\sin t, \cos t\rangle$ and graph the curve.

39. In this exercise, we prove Kepler's second law. Denote the (two-dimensional) path of the planet in polar coordinates by $\mathbf{r} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j}$. Show that $\mathbf{r} \times \mathbf{v} = r^2\dfrac{d\theta}{dt}\mathbf{k}$. Conclude that $r^2\dfrac{d\theta}{dt} = \|\mathbf{r} \times \mathbf{v}\|$. Recall that in polar coordinates, the area swept out by the curve $\mathbf{r} = r(\theta)$ is given by $A = \displaystyle\int_a^b \frac{1}{2}r^2 d\theta$ and show that $\dfrac{dA}{dt} = \dfrac{1}{2}r^2\dfrac{d\theta}{dt}$. From $\dfrac{dA}{dt} = \dfrac{1}{2}\|\mathbf{r} \times \mathbf{v}\|$, conclude that equal areas are swept out in equal times.

40. In this exercise, we prove Kepler's third law. Recall that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab. From exercise 39, the rate at which area is swept out is given by $\dfrac{dA}{dt} = \dfrac{1}{2}\|\mathbf{r} \times \mathbf{v}\|$. Conclude that the period of the orbit is $T = \dfrac{\pi ab}{\frac{1}{2}\|\mathbf{r} \times \mathbf{v}\|}$ and so, $T^2 = \dfrac{4\pi^2a^2b^2}{\|\mathbf{r} \times \mathbf{v}\|^2}$. Use (5.17) to show that the minimum value of r is $r_{\min} = \dfrac{ed}{1 + e}$ and that the maximum value of r is $r_{\max} = \dfrac{ed}{1 - e}$. Explain why $2a = r_{\min} + r_{\max}$ and use this to show that $a = \dfrac{ed}{1 - e^2}$. Given that $1 - e^2 = \dfrac{b^2}{a^2}$, show that $\dfrac{b^2}{a} = ed$. From $e = \dfrac{b}{GM}$ and $d = \dfrac{c^2}{b}$, show that $ed = \dfrac{\|\mathbf{r} \times \mathbf{v}\|^2}{GM}$. It then follows that $\dfrac{b^2}{a} = \dfrac{\|\mathbf{r} \times \mathbf{v}\|^2}{GM}$. Finally, show that $T^2 = ka^3$, where the constant $k = \dfrac{4\pi^2}{GM}$ does not depend on the specific orbit of the planet.

41. (a) Show that $\dfrac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{T}$. (b) Show that $\dfrac{d\mathbf{B}}{ds}$ is orthogonal to $\mathbf{B}$.

42. Use the result of exercise 41 to show that $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$, for some scalar τ. (τ is called the **torsion,** which measures how much a curve twists.) Also, show that $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N}$.

43. Show that the torsion for the curve traced out by $\mathbf{r}(t) = \langle f(t), g(t), k\rangle$ is zero for any constant k. (In general, the torsion is zero for any curve that lies in a single plane.)

44. The following three formulas (called the **Frenet-Serret formulas**) are of great significance in the field of differential geometry:

a. $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ [equation (5.2)]

b. $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ (see exercise 42)

c. $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$

Use the fact that $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and the product rule [Theorem 2.3 (v)] to establish (c).

45. Use the Frenet-Serret formulas (see exercise 44) to establish each of the following formulas:

a. $\mathbf{r}''(t) = s''(t)\mathbf{T} + \kappa[s'(t)]^2\mathbf{N}$

b. $\mathbf{r}'(t) \times \mathbf{r}''(t) = \kappa[s'(t)]^3\mathbf{B}$

c. $\mathbf{r}'''(t) = \{s'''(t) - \kappa^2[s'(t)]^3\}\mathbf{T} + \{3\kappa s'(t)s''(t) + \kappa'(t)[s'(t)]^2\}\mathbf{N} + \kappa\tau[s'(t)]^3\mathbf{B}$

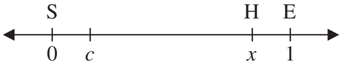
d. $\tau = \dfrac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$

46. Show that the torsion for the helix traced out by $\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ is given by $\tau = \dfrac{b}{a^2 + b^2}$. [Hint: See exercise 45 (d).]

EXPLORATORY EXERCISES

1. In this exercise, we explore some ramifications of the precise form of Newton's law of universal gravitation. Suppose that the gravitational force between objects is $\mathbf{F} = -\dfrac{GMm}{r^n}\mathbf{u}$, for some positive integer $n \geq 1$ (the actual law has $n = 2$). Show that the path of the planet would still be planar and that Kepler's second law still holds. Also, show that the circular orbit $\mathbf{r} = \langle r\cos kt, r\sin kt\rangle$ (where r is a constant) satisfies the equation $\mathbf{F} = m\mathbf{a}$ and hence, is a potential path for the orbit. For this path, find the relationship between the period of the orbit and the radius of the orbit.

2. In this exercise, you will find the locations of three of the five **Lagrange points.** These are equilibrium solutions of the "restricted three-body problem" in which a large body S of mass M_1 is orbited by a smaller body E of mass $M_2 < M_1$. A third object H of very small mass m orbits S such that the relative positions of S, E and H remain constant. Place S at the origin, E at (1, 0) and H at $(x, 0)$ as shown.



Assume that H and E have circular orbits about the center of mass $(c, 0)$. Show that the gravitational force on H is $F = -\dfrac{GM_1m}{x^2} + \dfrac{GM_2m}{(1-x)^2}$. As shown in example 3.3, for circular motion $F = -m\omega^2(x - c)$, where ω is the angular velocity of H. Analyzing the orbit of E, show that $GM_1M_2 = M_2\omega^2(1 - c)$. In particular, explain why E has the same angular velocity ω. Combining the three equations, show that $\dfrac{1}{x^2} - \dfrac{k}{(1-x)^2} = \dfrac{x-c}{1-c}$, where $k = \dfrac{M_2}{M_1}$. Given that $c = \dfrac{M_2}{M_1 + M_2}$, show that

$$(1 + k)x^5 - (3k + 2)x^4 + (3k + 1)x^3 - x^2 + 2x - 1 = 0.$$

For the Sun-Earth system with $k = 0.000002$, estimate x, the location of the L_1 Lagrange point and the location of NASA's SOHO solar observatory satellite. Then derive and estimate solutions for the L_2 Lagrange point with $x > 1$ and L_3 with $x < 0$.

11.6 PARAMETRIC SURFACES

Throughout this chapter, we have emphasized the connection between vector-valued functions and parametric equations. In this section, we extend the notion of parametric equations to those with two independent parameters. This also means that we will be working with simple cases of functions of two variables, which are developed in more detail in Chapter 12. We will make use of parametric surfaces throughout the remainder of the book.

We have already seen the helix defined by the parametric equations $x = \cos t$, $y = \sin t$ and $z = t$. This curve winds around the cylinder $x^2 + y^2 = 1$. Now, suppose that we wanted to obtain parametric equations that describe the entire cylinder. Given $x = \cos t$ and $y = \sin t$, we have that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

So, any point (x, y, z) with x and y defined in this way must lie on the cylinder. To describe the entire cylinder (i.e., every point on the cylinder), we must allow z to be any real number, not just $z = t$. In other words, z needs to be independent of x and y. Assigning z its own parameter will accomplish this. Using the parameters u and v, we have the parametric equations

$$x = \cos u, \quad y = \sin u \quad \text{and} \quad z = v$$

for the cylinder. In general, parametric equations with two independent parameters correspond to a three-dimensional surface. Examples 6.1 through 6.3 explore some basic but important surfaces.

EXAMPLE 6.1 Graphing a Parametric Surface

Identify and sketch a graph of the surface defined by the parametric equations $x = 2\cos u \sin v$, $y = 2\sin u \sin v$ and $z = 2\cos v$.

Solution Given the cosine and sine terms in both parameters, you should be expecting circular cross sections. Notice that we can eliminate the u parameter, by observing that

$$\begin{aligned} x^2 + y^2 &= (2\cos u \sin v)^2 + (2\sin u \sin v)^2 = 4\cos^2 u \sin^2 v + 4\sin^2 u \sin^2 v \\ &= 4(\cos^2 u + \sin^2 u)\sin^2 v = 4\sin^2 v. \end{aligned}$$

So, for each fixed value of z (which also means a fixed value for v), $x^2 + y^2$ is constant. That is, cross sections of the surface parallel to the xy-plane are circular, with radius $|2\sin v|$. Since $z = 2\cos v$, we also have circular cross sections parallel to either of the other two coordinate planes. Of course, one surface that you've seen that has circular cross sections in all directions is a sphere. To determine that the given parametric equations represent a sphere, observe that

$$\begin{aligned} x^2 + y^2 + z^2 &= 4\cos^2 u \sin^2 v + 4\sin^2 u \sin^2 v + 4\cos^2 v \\ &= 4(\cos^2 u + \sin^2 u)\sin^2 v + 4\cos^2 v \\ &= 4\sin^2 v + 4\cos^2 v = 4. \end{aligned}$$

You should recognize $x^2 + y^2 + z^2 = 4$ as an equation of the sphere centered at the origin with radius 2. A computer-generated sketch is shown in Figure 11.37. ■

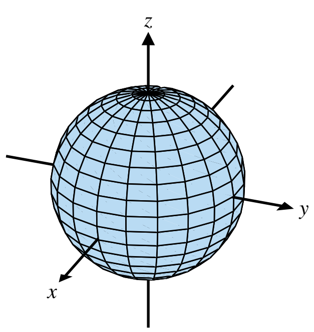


FIGURE 11.37
$x^2 + y^2 + z^2 = 4$

There are several important points to make about example 6.1. First, we did not actually demonstrate that the surface defined by the given parametric equations is the entire sphere. Rather, we showed that points lying on the parametric surface were also on the sphere. In other words, the parametric surface is (at least) part of the sphere. In the exercises, we will supply the missing steps to this puzzle, showing that the parametric equations from example 6.1 do, in fact, describe the entire sphere. In this instance, the equations are a special case of something called **spherical coordinates,** which we will introduce in Chapter 13. An understanding of spherical coordinates makes it simple to find parametric equations for half of a sphere or some other portion of a sphere. Next, as with parametric equations of curves, there are other parametric equations representing the same sphere. In the exercises, we will see that the roles of cosine and sine can be reversed in these equations, with the resulting

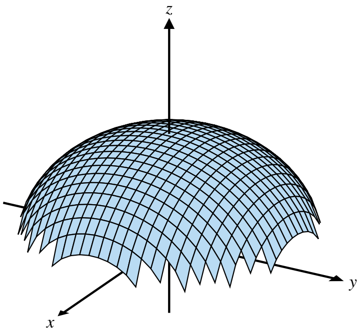


FIGURE 11.38
$z = \sqrt{4 - x^2 - y^2}$

equations still describing a sphere. Finally, to repeat a point made in section 10.6, parametric equations can be used to produce many interesting graphs. Notice the smooth contours and clearly defined circular cross sections in Figure 11.37, compared with the jagged graph in Figure 11.38. Figure 11.38 is a computer-generated graph of $z = \sqrt{4 - x^2 - y^2}$, which should be the top half of the sphere.

For parametric equations of hyperboloids and hyperbolic paraboloids, it is convenient to use the hyperbolic functions $\cosh x$ and $\sinh x$. Recall that

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

A little algebra shows that $\cosh^2 x - \sinh^2 x = 1$, which is an identity needed in example 6.2.

EXAMPLE 6.2 Graphing a Parametric Surface

Sketch the surface defined parametrically by $x = 2\cos u \cosh v$, $y = 2\sin u \cosh v$ and $z = 2\sinh v$, $0 \le u \le 2\pi$ and $-\infty < v < \infty$.

Solution A sketch such as the one we show in Figure 11.39 can be obtained from a computer algebra system.

Notice that this looks like a hyperboloid of one sheet wrapped around the z-axis. To verify that this is correct, observe that

$$\begin{aligned} x^2 + y^2 - z^2 &= 4\cos^2 u \cosh^2 v + 4\sin^2 u \cosh^2 v - 4\sinh^2 v \\ &= 4(\cos^2 u + \sin^2 u)\cosh^2 v - 4\sinh^2 v \\ &= 4\cosh^2 v - 4\sinh^2 v = 4, \end{aligned}$$

where we have used the identities $\cos^2 u + \sin^2 u = 1$ and $\cosh^2 v - \sinh^2 v = 1$. Recall that the graph of $x^2 + y^2 - z^2 = 4$ is indeed a hyperboloid of one sheet. ■

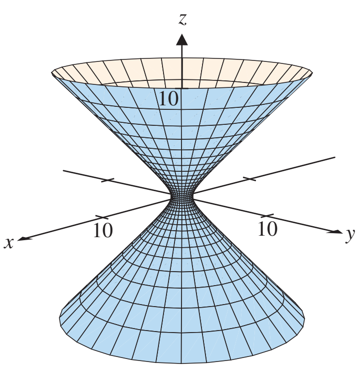


FIGURE 11.39
$x^2 + y^2 - z^2 = 4$

Again, it is instructive to compare the graph in Figure 11.39 with the computer-generated graph of $z = \sqrt{x^2 + y^2 - 4}$ shown in Figure 11.40.

Most of the time when we use parametric representations of surfaces, the task is the opposite of that in example 6.2. That is, given a particular surface, we may need to find a convenient parametric representation of the surface. Example 6.2 gives us an important clue for working example 6.3.

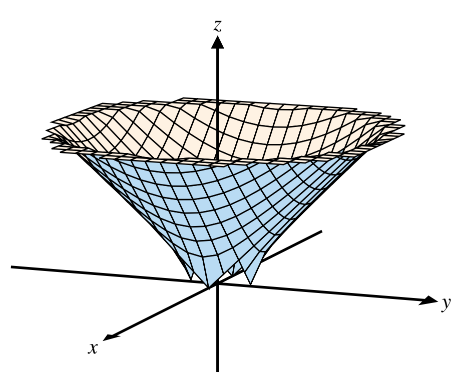


FIGURE 11.40
$z = \sqrt{x^2 + y^2 - 4}$

EXAMPLE 6.3 Finding a Parametric Representation of a Hyperbolic Paraboloid

Find parametric equations for the hyperbolic paraboloid $z = x^2 - y^2$.

Solution It helps to understand the surface with which we are working. Observe that for any value of k, the trace in the plane $z = k$ is a hyperbola. The spread of the hyperbola depends on whether $|k|$ is large or small. To get hyperbolas in x and y, we can start with $x = \cosh u$ and $y = \sinh u$. To enlarge or shrink the hyperbola, we can multiply $\cosh u$ and $\sinh u$ by a constant. We now have $x = v \cosh u$ and $y = v \sinh u$. To get $z = x^2 - y^2$, simply compute

$$x^2 - y^2 = v^2 \cosh^2 u - v^2 \sinh^2 u = v^2(\cosh^2 u - \sinh^2 u) = v^2,$$

since $\cosh^2 u - \sinh^2 u = 1$. This gives us the parametric equations

$$x = v \cosh u, \quad y = v \sinh u \quad \text{and} \quad z = v^2.$$

A graph of the parametric equations is shown in Figure 11.41a. However, notice that this is only the top half of the surface, since $z = v^2 \geq 0$. To get the bottom half of the surface, we set $x = v \sinh u$ and $y = v \cosh u$ so that $z = x^2 - y^2 = -v^2 \leq 0$. Figure 11.41b shows both halves of the surface.

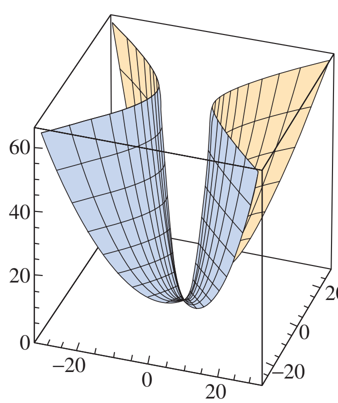


FIGURE 11.41a
Top half of the surface

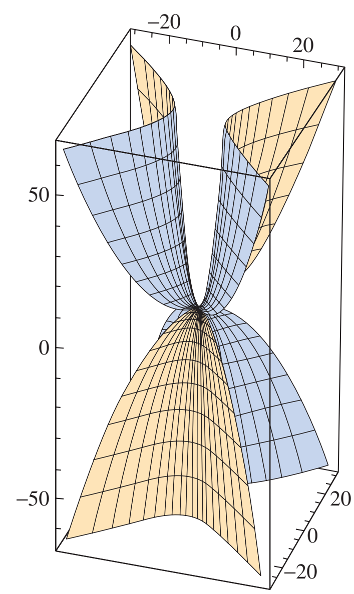


FIGURE 11.41b
$z = x^2 - y^2$

In many cases, the parametric equations that we use are determined by the geometry of the surface. Recall that in two dimensions, certain curves (especially circles) are more easily described in polar coordinates than in rectangular coordinates. We use this fact in example 6.4. Polar coordinates are essentially the parametric equations for circles that we have used over and over again. In particular, the polar coordinates r and θ are related to x and y by

$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{and} \quad r = \sqrt{x^2 + y^2}.$$

So, the equation for the circle $x^2 + y^2 = 4$ can be written in polar coordinates simply as $r = 2$.

EXAMPLE 6.4 Finding Parametric Representations of Surfaces

Find a parametric representation of each surface: (a) the portion of $z = \sqrt{x^2 + y^2}$ inside $x^2 + y^2 = 4$ and (b) the portion of $z = 9 - x^2 - y^2$ above the xy-plane with $y \geq 0$.

Solution For part (a), a graph indicating the cone and the cylinder is shown in Figure 11.42a. Notice that the equations for both surfaces include the term $x^2 + y^2$ and x and y appear only in this combination. This suggests the use of polar coordinates. Taking $x = r\cos\theta$ and $y = r\sin\theta$, the equation of the cone $z = \sqrt{x^2 + y^2}$ becomes $z = r$ and the equation of the cylinder $x^2 + y^2 = 4$ becomes $r = 2$. Since the surface in question is that portion of the cone lying inside the cylinder, every point on the surface lies on the cone. So, every point on the surface satisfies $x = r\cos\theta$, $y = r\sin\theta$ and $z = r$. Observe that the cylinder cuts off the cone, something like a cookie cutter. Instead of all r-values being possible, the cylinder limits us to $r \leq 2$. (Think about why this is so.) A parametric representation for (a) is then

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = r, \quad \text{for} \quad 0 \leq r \leq 2 \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

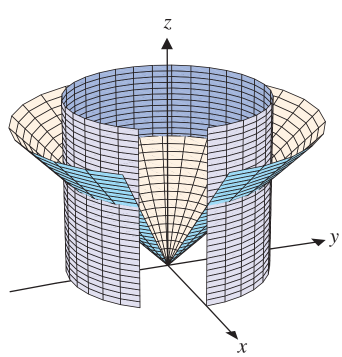


FIGURE 11.42a
Portion of $z = \sqrt{x^2 + y^2}$ inside $x^2 + y^2 = 4$

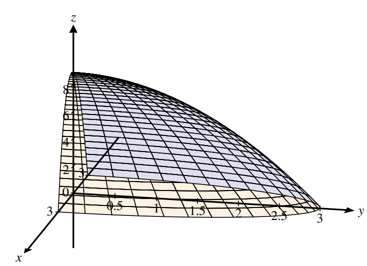


FIGURE 11.42b
Portion of $z = 9 - x^2 - y^2$ above the xy-plane, with $y \geq 0$

For part (b), a graph is shown in Figure 11.42b. Again, the presence of the term $x^2 + y^2$ in the defining equation suggests polar coordinates. Taking $x = r\cos\theta$ and $y = r\sin\theta$, the equation of the paraboloid becomes

$$z = 9 - (x^2 + y^2) = 9 - r^2.$$

To stay above the xy-plane, we need $z > 0$ or $9 - r^2 > 0$ or $|r| < 3$. Choosing positive r-values, we have $0 \leq r < 3$. Then $y \geq 0$, if $\sin\theta \geq 0$. One choice of θ that gives this is $0 \leq \theta \leq \pi$. A parametric representation for the surface is then

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = 9 - r^2, \quad \text{for} \quad 0 \leq r < 3 \quad \text{and} \quad 0 \leq \theta \leq \pi.$$

EXERCISES 11.6

WRITING EXERCISES

1. Suppose that a surface has parametric equations $x = f(u)$, $y = g(u)$ and $z = h(v)$. Explain why the surface must be a cylinder. If the range of the function h consists of all real numbers, discuss whether or not the parametric equations $x = f(u)$, $y = g(u)$ and $z = v$ would describe the same surface.

2. In this exercise, we want to understand why parametric equations with two parameters typically graph as surfaces. If $x = f(u, v)$, $y = g(u, v)$ and $z = h(u, v)$, substitute in some constant $v = v_1$ and let C_1 be the curve with parametric equations $x = f(u, v_1)$, $y = g(u, v_1)$ and $z = h(u, v_1)$. Similarly, for a different constant v_2 let C_2 be the curve with parametric equations $x = f(u, v_2)$, $y = g(u, v_2)$ and $z = h(u, v_2)$. Sketch a picture showing what C_1 and C_2 might look like if v_1 and v_2 are close together. Then fill in what you would expect other curves would look like for v-values between v_1 and v_2. Discuss whether you are generating a surface.

3. To show that two parameters do not always produce a surface, sketch the curve with $x = u + v$, $y = u + v$ and $z = u + v$. Discuss the special feature of these equations that enables us to replace the parameters u and v with a single parameter t.

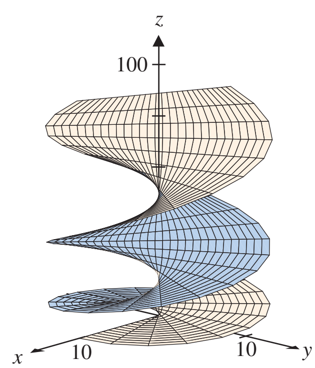
4. The graph of $x = f(t)$ and $y = g(t)$ is typically a curve in two dimensions. The graph of $x = f(u, v)$, $y = g(u, v)$ and $z = h(u, v)$ is typically a surface in three dimensions. Discuss what the graph of $x = f(r, s, t)$, $y = g(r, s, t)$, $z = h(r, s, t)$ and $w = d(r, s, t)$ would look like.

In exercises 1–12, identify and sketch a graph of the parametric surface.

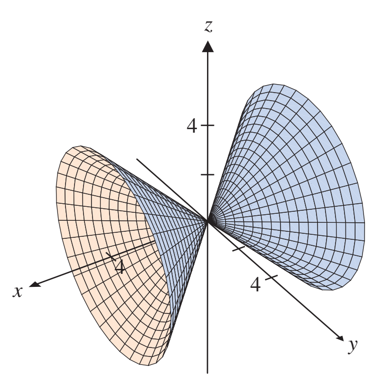
1. $x = u$, $y = v$, $z = u^2 + 2v^2$
2. $x = u$, $y = v$, $z = 4 - u^2 - v^2$
3. $x = u\cos v$, $y = u\sin v$, $z = u^2$
4. $x = u\cos v$, $y = u\sin v$, $z = u$
5. $x = u$, $y = \sin u\cos v$, $z = \sin u\sin v$
6. $x = \cos u\cos v$, $y = u$, $z = \cos u\sin v$
7. $x = 2\sin u\cos v$, $y = 2\sin u\sin v$, $z = 2\cos u$
8. $x = u\cos v$, $y = u\sin v$, $z = v$
9. $x = v\sinh u$, $y = 4v^2$, $z = v\cosh u$
10. $x = \sinh v$, $y = \cos u\cosh v$, $z = \sin u\cosh v$
11. $x = 2\cos u\sinh v$, $y = 2\sin u\sinh v$, $z = 2\cosh v$
12. $x = 2\sinh u$, $y = v$, $z = 2\cosh u$

In exercises 13–22, find a parametric representation of the surface.

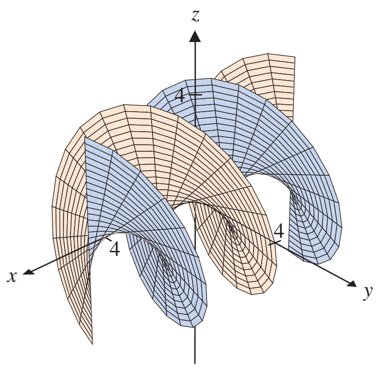
13. $z = 3x + 4y$
14. $x^2 + y^2 + z^2 = 4$
15. $x^2 + y^2 - z^2 = 1$
16. $z^2 = x^2 + y^2$
17. The portion of $x^2 + y^2 = 4$ from $z = 0$ to $z = 2$
18. The portion of $y^2 + z^2 = 9$ from $x = -1$ to $x = 1$
19. The portion of $z = 4 - x^2 - y^2$ above the xy-plane
20. The portion of $z = x^2 + y^2$ below $z = 4$
21. $x^2 - y^2 - z^2 = 1$
22. $x = 4y^2 - z^2$
23. Match the parametric equations with the surface.
    a. $x = u\cos v$, $y = u\sin v$, $z = v^2$
    b. $x = v$, $y = u\cos v$, $z = u\sin v$
    c. $x = u$, $y = u\cos v$, $z = u\sin v$



SURFACE A



SURFACE B



SURFACE C

24. Show that changing the parametric equations of example 6.1 to $x = 2\sin u \cos v$, $y = 2\cos u \cos v$ and $z = 2\sin v$ does not change the fact that $x^2 + y^2 + z^2 = 4$.

25. To show that the surface in example 6.1 is the entire sphere $x^2 + y^2 + z^2 = 4$, start by finding the trace of the sphere in the plane $z = k$ for $-2 \le k \le 2$. If $z = 2\cos v = k$, determine as fully as possible the value of $2\sin v$ and then determine the trace in the plane $z = k$ for $x = 2\cos u \sin v$, $y = 2\sin u \sin v$ and $z = 2\cos v$. If the traces are the same, then the surfaces are the same.

26. Investigate whether $x = 2\sin u \cos v$, $y = 2\cos u \cos v$ and $z = 2\sin u$ also are parametric equations for the sphere $x^2 + y^2 + z^2 = 4$. If not, use computer-generated graphs to describe the surface.

Exercises 27–40 relate to spherical coordinates defined by $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$, where $0 \le \rho$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

27. Replace ρ with $\rho = 3$ and determine the surface with parametric equations $x = 3\cos\theta\sin\phi$, $y = 3\sin\theta\sin\phi$ and $z = 3\cos\phi$.

28. Use the results of example 6.1 and exercise 27 to determine the surface defined by $\rho = k$, where k is some positive constant.

29. Replace ϕ with $\phi = \frac{\pi}{4}$ and determine the surface with parametric equations $x = \rho\cos\theta\sin\frac{\pi}{4}$, $y = \rho\sin\theta\sin\frac{\pi}{4}$ and $z = \rho\cos\frac{\pi}{4}$.

30. Replace ϕ with $\phi = \frac{\pi}{6}$ and determine the surface with parametric equations $x = \rho\cos\theta\sin\frac{\pi}{6}$, $y = \rho\sin\theta\sin\frac{\pi}{6}$ and $z = \rho\cos\frac{\pi}{6}$.

31. Replace θ with $\theta = \frac{\pi}{4}$ and determine the surface with parametric equations $x = \rho\cos\frac{\pi}{4}\sin\phi$, $y = \rho\sin\frac{\pi}{4}\sin\phi$ and $z = \rho\cos\phi$.

32. Replace θ with $\theta = \frac{3\pi}{4}$ and determine the surface with parametric equations $x = \rho\cos\frac{3\pi}{4}\sin\phi$, $y = \rho\sin\frac{3\pi}{4}\sin\phi$ and $z = \rho\cos\phi$.

33. Use the results of exercises 29 and 30 to determine the surface $\phi = k$ for some constant k with $0 < k < \pi$.

34. Use the results of exercises 31 and 32 to determine the surface $\theta = k$ for some constant k with $0 < k < 2\pi$.

35. Use the results of exercises 28, 33 and 34 to find parametric equations for the top half-sphere $z = \sqrt{9 - x^2 - y^2}$.

36. Use the results of exercises 28, 33 and 34 to find parametric equations for the right half-sphere $y = \sqrt{9 - x^2 - z^2}$.

37. Use the results of exercises 28, 33 and 34 to find parametric equations for the cone $z = \sqrt{x^2 + y^2}$.

38. Use the results of exercises 28, 33 and 34 to find parametric equations for the cone $z = -\sqrt{x^2 + y^2}$.

39. Find parametric equations for the region that lies above $z = \sqrt{x^2 + y^2}$ and below $x^2 + y^2 + z^2 = 4$.

40. Find parametric equations for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.

Exercises 41–46 relate to parametric equations of a plane.

41. Sketch the plane with parametric equations $x = 2 + u + 2v$, $y = -1 + 2u - v$ and $z = 3 - 3u + 2v$. Show that the points $(2, -1, 3)$, $(3, 1, 0)$ and $(4, -2, 5)$ are on the plane by finding the correct values of u and v. Sketch these points along with the plane and the displacement vectors $\langle 1, 2, -3\rangle$ and $\langle 2, -1, 2\rangle$.

42. Use the results of exercise 41 to find a normal vector to the plane and an equation of the plane in terms of x and y.

43. The parametric equations of exercise 41 can be written as $\mathbf{r} = \langle 2, -1, 3\rangle + u\langle 1, 2, -3\rangle + v\langle 2, -1, 2\rangle$ for $\mathbf{r} = \langle x, y, z\rangle$. For the general equation $\mathbf{r} = \mathbf{r}_0 + u\mathbf{v}_1 + v\mathbf{v}_2$, sketch a plane and show how the vectors $\mathbf{r}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ would relate to the plane. In terms of $\mathbf{v}_1$ and $\mathbf{v}_2$, find a normal vector to the plane.

44. Find an equation in x and y for the plane defined by $\mathbf{r} = \langle 3, 1, -1\rangle + u\langle 2, -4, 1\rangle + v\langle 2, 0, 1\rangle$.

45. Find parametric equations for the plane through the point $(3, 1, 1)$ and containing the vectors $\langle 2, -1, 3\rangle$ and $\langle 4, 2, 1\rangle$.

46. Find parametric equations for the plane through the point $(0, -1, 2)$ and containing the vectors $\langle -2, 4, 0\rangle$ and $\langle 3, -2, 5\rangle$.

Exercises 47–54 relate to cylindrical coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$.

47. Sketch the two-dimensional polar graph $r = \cos 2\theta$. Sketch the solid in three dimensions defined by $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$ with $r = \cos 2\theta$ and $0 \le z \le 1$, and compare it to the polar graph. Show that parametric equations for the solid are $x = \cos 2u \cos u$, $y = \cos 2u \sin u$ and $z = v$ with $0 \le u \le 2\pi$ and $0 \le v \le 1$.

48. Sketch the solid defined by $x = (2 - 2\cos u)\cos u$, $y = (2 - 2\cos u)\sin u$ and $z = v$ with $0 \le u \le 2\pi$ and $0 \le v \le 1$. (Hint: Use polar coordinates as in exercise 47.)

49. Find parametric equations for the wedge in the first octant bounded by $y = 0$, $y = x$, $x^2 + y^2 = 4$, $z = 0$ and $z = 1$.

50. Find parametric equations for the portion of the parabola $z = x^2 + y^2$ below $z = 4$, with $y \geq 0$.

51. Sketch the (two-dimensional) graph of $f(t) = e^{-t^2}$ for $t \geq 0$. Sketch the surface $z = e^{-x^2-y^2}$ and compare it to the graph of $f(t)$. Show that parametric equations of the surface are $x = u\cos v$, $y = u\sin v$ and $z = e^{-u^2}$.

52. Sketch the surface defined by $x = u\cos v$, $y = u\sin v$ and $z = ue^{-u^2}$. [Hint: Use the graph of $f(t) = te^{-t^2}$.]

53. Find parametric equations for the surface $z = \sin\sqrt{x^2 + y^2}$.

54. Find parametric equations for the surface $z = \cos(x^2 + y^2)$.

EXPLORATORY EXERCISES

1. If $x = 3\sin u\cos v$, $y = 3\cos u$ and $z = 3\sin u\sin v$, show that $x^2 + y^2 + z^2 = 9$. Explain why this equation doesn't guarantee that the parametric surface defined is the entire sphere, but it does guarantee that all points on the surface are also on the sphere. In this case, the parametric surface is the entire sphere. To verify this in graphical terms, sketch a picture showing geometric interpretations of the "spherical coordinates" u and v. To see what problems can occur, sketch the surface defined by $x = 3\sin\dfrac{u^2}{u^2+1}\cos v$, $y = 3\cos\dfrac{u^2}{u^2+1}$ and $z = 3\sin\dfrac{u^2}{u^2+1}\sin v$. Explain why you do not get the entire sphere. To see a more subtle example of the same problem, sketch the surface $x = \cos u\ \cosh v$, $y = \sinh v$, $z = \sin u\cosh v$. Use identities to show that $x^2 - y^2 + z^2 = 1$ and identify the surface. Then sketch the surface $x = \cos u\cosh v$, $y = \cos u\sinh v$, $z = \sin u$ and use identities to show that $x^2 - y^2 + z^2 = 1$. Explain why the second surface is not the entire hyperboloid. Explain in words and pictures exactly what the second surface is.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Vector-valued function	Tangential component	Normal component
		Continuous $\mathbf{F}(x)$
Tangent vector	Arc length	Speed
Angular velocity	Velocity vector	Angular momentum
Arc length parameter	Angular acceleration	Principal unit normal
	Curvature	Osculating circle
Binormal vector	Radius of curvature	Parametric surface

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. The graph of the vector-valued function $\langle \cos t, \sin t, f(t)\rangle$, for some function f lies on the circle $x^2 + y^2 = 1$.

2. For vector-valued functions, derivatives are found component by component and all of the usual rules (product, quotient, chain) apply.

3. The derivative of a vector-valued function gives the slope of the tangent line.

4. Newton's laws apply only to straight-line motion and not to rotational motion.

5. The greater the osculating circle, the greater the curvature.

6. While driving a car, the vector $\mathbf{T}$ would be "straight ahead," $\mathbf{N}$ in the direction you are turning and $\mathbf{B}$ straight up.

7. If parametric equations for a surface are $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ with $x^2 + y^2 + z^2 = 1$, the surface is a sphere.

In exercises 1 and 2, sketch the curve and plot the values of the vector-valued function.

1. $\mathbf{r}(t) = \langle t^2, 2 - t^2, 1\rangle$, $t = 0$, $t = 1$, $t = 2$

2. $\mathbf{r}(t) = \langle \sin t, 2\cos t, 3\rangle$, $t = -\pi$, $t = 0$, $t = \pi$

In exercises 3–12, sketch the curve traced out by the given vector-valued function.

3. $\mathbf{r}(t) = \langle 3\cos t + 1, \sin t\rangle$ **4.** $\mathbf{r}(t) = \langle 2\sin t, \cos t + 2\rangle$

5. $\mathbf{r}(t) = \langle 3\cos t + 2\sin 3t, 3\sin t + 2\cos 3t\rangle$

6. $\mathbf{r}(t) = \langle 3\cos t + \sin 3t, 3\sin t + \cos 3t\rangle$

7. $\mathbf{r}(t) = \langle 2\cos t, 3, 3\sin t\rangle$ **8.** $\mathbf{r}(t) = \langle 3\cos t, -2, 2\sin t\rangle$

9. $\mathbf{r}(t) = \langle 4\cos 3t + 6\cos t, 6\sin t, 4\sin 3t\rangle$

Review Exercises

10. $\mathbf{r}(t) = \langle \sin \pi t, \sqrt{t^2 + t^3}, \cos \pi t \rangle$

11. $\mathbf{r}(t) = \langle \tan t, 4 \cos t, 4 \sin t \rangle$

12. $\mathbf{r}(t) = \langle \cos 5t, \tan t, 6 \sin t \rangle$

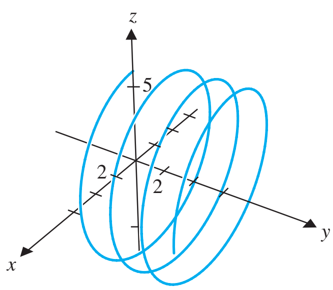
13. In parts (a)–(f), match the vector-valued function with its graph.

a. $\mathbf{r}(t) = \langle \sin t, t, \sin 2t \rangle$ **b.** $\mathbf{r}(t) = \langle t, \sin t, \sin 2t \rangle$

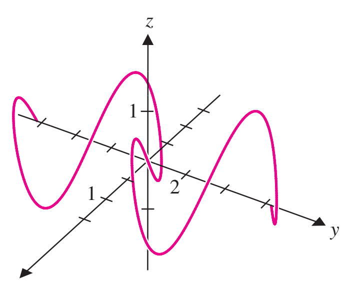
c. $\mathbf{r}(t) = \langle 6 \sin \pi t, t, 6 \cos \pi t \rangle$ **d.** $\mathbf{r}(t) = \langle \sin^5 t, \sin^2 t, \cos t \rangle$

e. $\mathbf{r}(t) = \langle \cos t, 1 - \cos^2 t, \cos t \rangle$

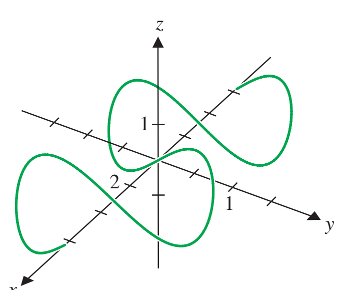
f. $\mathbf{r}(t) = \langle t^2 + 1, t^2 + 2, t - 1 \rangle$



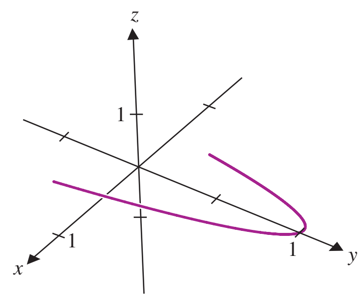
GRAPH A



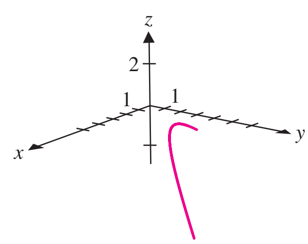
GRAPH B



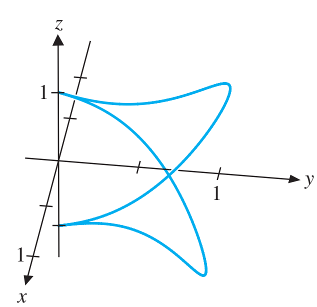
GRAPH C



GRAPH D



GRAPH E



GRAPH F

In exercises 14–16, sketch the curve and find its arc length.

14. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, \cos 4\pi t \rangle, 0 \le t \le 2$

15. $\mathbf{r}(t) = \langle \cos t, \sin t, 6t \rangle, 0 \le t \le 2\pi$

16. $\mathbf{r}(t) = \langle t, 4t - 1, 2 - 6t \rangle, 0 \le t \le 2$

In exercises 17 and 18, find the limit if it exists.

17. $\lim_{t \to 1} \langle t^2 - 1, e^{2t}, \cos \pi t \rangle$

18. $\lim_{t \to 1} \langle e^{-2t}, \csc \pi t, t^3 - 5t \rangle$

Review Exercises

In exercises 19 and 20, determine all values of t at which the given vector-valued function is continuous.

19. $\mathbf{r}(t) = \langle e^{4t}, \ln t^2, 2t\rangle$

20. $\mathbf{r}(t) = \left\langle \sin t, \tan 2t, \dfrac{3}{t^2 - 1}\right\rangle$

In exercises 21 and 22, find the derivative of the given vector-valued function.

21. $\mathbf{r}(t) = \left\langle \sqrt{t^2 + 1}, \sin 4t, \ln 4t\right\rangle$

22. $\mathbf{r}(t) = \langle te^{-2t}, t^3, 5\rangle$

In exercises 23–26, evaluate the given indefinite or definite integral.

23. $\displaystyle\int \left\langle e^{-4t}, \frac{2}{t^3}, 4t - 1\right\rangle dt$

24. $\displaystyle\int \left\langle \frac{2t^2}{t^3 + 2}, \sqrt{t+1}\right\rangle dt$

25. $\displaystyle\int_0^1 \langle \cos \pi t, 4t, 2\rangle\, dt$

26. $\displaystyle\int_0^2 \langle e^{-3t}, 6t^2\rangle\, dt$

In exercises 27 and 28, find the velocity and acceleration vectors for the given position vector.

27. $\mathbf{r}(t) = \langle 4\cos 2t, 4\sin 2t, 4t\rangle$

28. $\mathbf{r}(t) = \langle t^2 + 2, 4, t^3\rangle$

In exercises 29–32, find the position vector from the given velocity or acceleration vector.

29. $\mathbf{v}(t) = \langle 2t + 4, -32t\rangle$, $\mathbf{r}(0) = \langle 2, 1\rangle$

30. $\mathbf{v}(t) = \langle 4, t^2 - 1\rangle$, $\mathbf{r}(0) = \langle -4, 2\rangle$

31. $\mathbf{a}(t) = \langle 0, -32\rangle$, $\mathbf{v}(0) = \langle 4, 3\rangle$, $\mathbf{r}(0) = \langle 2, 6\rangle$

32. $\mathbf{a}(t) = \langle t, e^{2t}\rangle$, $\mathbf{v}(0) = \langle 2, 0\rangle$, $\mathbf{r}(0) = \langle 4, 0\rangle$

In exercises 33 and 34, find the force acting on an object of mass 4 with the given position vector.

33. $\mathbf{r}(t) = \langle 12t, 12 - 16t^2\rangle$

34. $\mathbf{r}(t) = \langle 3\cos 2t, 2\sin 2t\rangle$

In exercises 35 and 36, a projectile is fired with initial speed v_0 feet per second from a height of h feet at an angle of θ above the horizontal. Assuming that the only force acting on the object is gravity, find the maximum altitude, horizontal range and speed at impact.

35. $v_0 = 80$, $h = 0$, $\theta = \frac{\pi}{12}$

36. $v_0 = 80$, $h = 6$, $\theta = \frac{\pi}{4}$

In exercises 37 and 38, find the unit tangent vector to the curve at the indicated points.

37. $\mathbf{r}(t) = \langle e^{-2t}, 2t, 4\rangle$, $t = 0$, $t = 1$

38. $\mathbf{r}(t) = \langle 2, \sin \pi t^2, \ln t\rangle$, $t = 1$, $t = 2$

In exercises 39–42, find the curvature of the curve at the indicated points.

39. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t\rangle$, $t = 0$, $t = \frac{\pi}{4}$

40. $\mathbf{r}(t) = \langle 4\cos 2t, 3\sin 2t\rangle$, $t = 0$, $t = \frac{\pi}{4}$

41. $\mathbf{r}(t) = \langle 4, 3t\rangle$, $t = 0$, $t = 1$

42. $\mathbf{r}(t) = \langle t^2, t^3, t^4\rangle$, $t = 0$, $t = 2$

In exercises 43 and 44, find the unit tangent and principal unit normal vectors at the given points.

43. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t\rangle$ at $t = 0$

44. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t\rangle$ at $t = \frac{\pi}{2}$

In exercises 45 and 46, find the tangential and normal components of acceleration at the given points.

45. $\mathbf{r}(t) = \langle 2t, t^2, 2\rangle$ at $t = 0$, $t = 1$

46. $\mathbf{r}(t) = \langle t^2, 3, 2t\rangle$ at $t = 0$, $t = 2$

In exercises 47 and 48, the friction force required to keep a car from skidding on a curve is given by $\mathbf{F}_s(t) = ma_N\mathbf{N}(t)$. Find the friction force needed to keep a car of mass $m = 120$ (slugs) from skidding for the given position vectors.

47. $\mathbf{r}(t) = \langle 80\cos 6t, 80\sin 6t\rangle$

48. $\mathbf{r}(t) = \langle 80\cos 4t, 80\sin 4t\rangle$

In exercises 49–52, sketch a graph of the parametric surface.

49. $x = 2\sin u$, $y = v^2$, $z = 3\cos u$

50. $x = \cos u \sin v$, $y = \sin u \sin v$, $z = \sin v$

Review Exercises

51. $x = u^2,\ y = v^2,\ z = u + 2v$

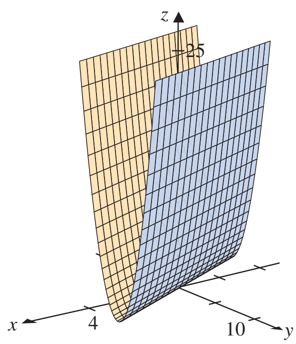
52. $x = (3 + 2\cos u)\cos v,\ y = (3 + 2\cos u)\sin v,\ z = 2\cos v$

53. Match the parametric equations with the surfaces.

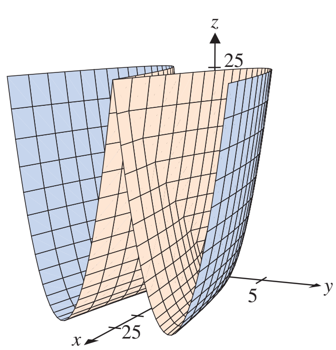
a. $x = u^2,\ y = u + v,\ z = v^2$

b. $x = u^2,\ y = u + v,\ z = v$

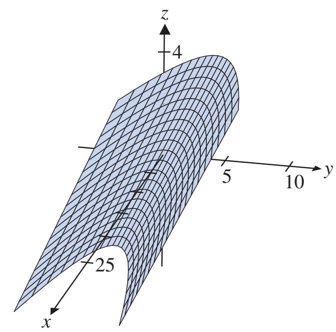
c. $x = u,\ y = u + v,\ z = v^2$



SURFACE A



SURFACE B



SURFACE C

54. Find a parametric representation of $x^2 + y^2 + z^2 = 9$.

EXPLORATORY EXERCISES

1. In this three-dimensional projectile problem, think of the x-axis as pointing to the right, the y-axis as pointing straight ahead and the z-axis as pointing up. Suppose that a projectile of mass $m = 1$ kg is launched from the ground with an initial velocity of 100 m/s in the yz-plane at an angle of $\frac{\pi}{6}$ above the horizontal. The spinning of the projectile produces a constant Magnus force of $\langle 0.1, 0, 0\rangle$ newtons. Find a vector for the position of the projectile at time $t \geq 0$. Assuming level ground, find the time of flight T for the projectile and find its landing place. Find the curvature for the path of the projectile at time $t \geq 0$. Find the times of minimum and maximum curvature of the path for $0 \leq t \leq T$.

2. A tennis serve is struck at an angle θ below the horizontal from a height of 8 feet and with initial speed 120 feet per second. For the serve to count (be "in"), it must clear a net that is 39 feet away and 3 feet high and must land before the service line 60 feet away. Find the range of angles for which the serve is in.

3. A baseball pitcher throws a pitch at an angle θ below the horizontal from a height of 6 feet with an initial speed of 130 feet per second. Home plate is 60 feet away. For the pitch to be a strike, the ball must cross home plate between 20″ and 42″ above the ground. Find the range of angles for which the pitch will be a strike.