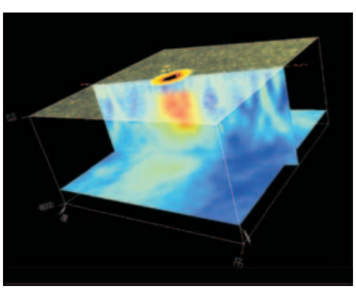
CHAPTER 3

Applications of Differentiation

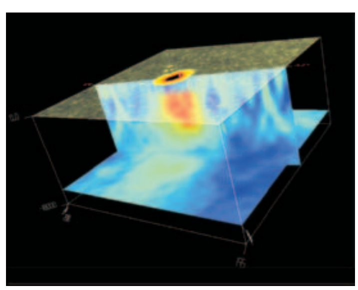
The Solar and Heliospheric Observatory (SOHO) is an international project for the observation and exploration of the Sun. The National Aeronautics and Space Administration (NASA) is responsible for operations of the SOHO spacecraft, including periodic adjustments to the spacecraft's location to maintain its position directly between the Earth and the Sun. With an uninterrupted view of the sun, SOHO can collect data to study the internal structure of the Sun, its outer atmosphere and the solar wind. SOHO has produced numerous unique and important images of the Sun, including the discovery of acoustic solar waves moving through the interior and false color images showing the velocity patterns on the surface of the Sun.

SOHO is in orbit around the Sun, located at a relative position called the L_1 Lagrange point for the Sun-Earth system. This is one of five points at which the gravitational pulls of the Sun and the Earth combine to maintain a satellite's relative position to the Sun and Earth. In the case of the L_1 point, that position is on a line between the Sun and the Earth, giving the SOHO spacecraft (see above) a direct view of the Sun and a direct line of communication back to the Earth. Because gravity causes the L_1 point to rotate in step with the Sun and Earth, little fuel is needed to keep the SOHO spacecraft in the proper location.

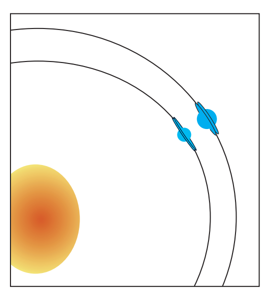
Lagrange points are solutions of "three-body" problems, in which there are three objects with vastly different masses. The Sun, the Earth and a spacecraft comprise one example, but other systems also have significance for space exploration. The Earth, the Moon and a space lab is another system of interest; the Sun,



Wave inside the Sun



Velocity on the Sun



L_1 orbit

Jupiter and an asteroid is a third system. The clusters of asteroids at the L4 and L5 Lagrange points of the Sun-Jupiter system are called Trojan asteroids.

For a given system, the locations of the five Lagrange points can be determined by solving equations. As you will see in the section 3.1 exercises, the equation for the location of SOHO is a difficult fifth-order polynomial equation. For a fifth-order equation, we usually are forced to gather graphical and numerical evidence to approximate solutions. The graphing and analysis of complicated functions and the solution of equations involving these functions are the emphases of this chapter.

3.1 LINEAR APPROXIMATIONS AND NEWTON'S METHOD

There are two distinctly different tasks for which you use scientific calculators. First, they perform basic arithmetic operations much faster than any of us could. We all know how to multiply 1024 by 1673, but it is time-consuming to carry out this calculation with pencil and paper. For such problems, calculators are a tremendous convenience. More significantly, we also use calculators to compute values of transcendental functions such as sine, cosine, tangent, exponentials and logarithms. In this case, the calculator is much more than a mere convenience.

How would you calculate sin(1.2345678) without a calculator? Don't worry if you don't know how to do this. The problem is that the sine function is not *algebraic*. That is, there is no formula for $\sin x$ involving only the arithmetic operations. So, how does your calculator "know" that $\sin(1.2345678) \approx 0.9440056953$? In short, it doesn't *know* this at all. Rather, the calculator has a built-in program that generates *approximate* values of the sine and other transcendental functions.

In this section, we develop a simple approximation method. Although somewhat crude, it points the way toward more sophisticated approximation techniques to follow later in the text.

Linear Approximations

Suppose we wanted to find an approximation for $f(x_1)$, where $f(x_1)$ is unknown, but where $f(x_0)$ is known for some x_0 "close" to x_1. For instance, the value of cos(1) is unknown, but we do know that $\cos(\pi/3) = \frac{1}{2}$ (exactly) and $\pi/3 \approx 1.047$ is "close" to 1. While we could use $\frac{1}{2}$ as an approximation to cos(1), we can do better.

Referring to Figure 3.1, notice that if x_1 is "close" to x_0 and we follow the tangent line at $x = x_0$ to the point corresponding to $x = x_1$, then the y-coordinate of that point (y_1) should be "close" to the y-coordinate of the point on the curve $y = f(x)$ [i.e., $f(x_1)$].

Since the slope of the tangent line to $y = f(x)$ at $x = x_0$ is $f'(x_0)$, the equation of the tangent line to $y = f(x)$ at $x = x_0$ is found from

$$m_{\tan} = f'(x_0) = \frac{y - f(x_0)}{x - x_0}. \tag{1.1}$$

Solving equation (1.1) for y gives us

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{1.2}$$

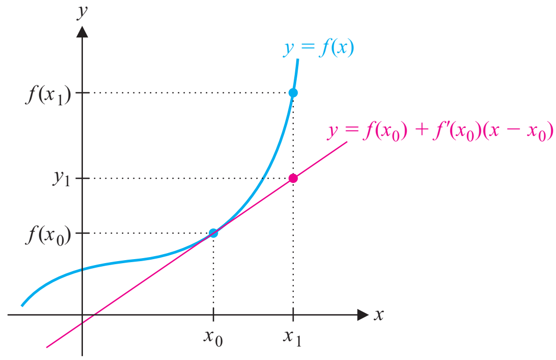


FIGURE 3.1
Linear approximation of $f(x_1)$

Notice that (1.2) is the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$. We give the linear function defined by this equation a name, as follows.

DEFINITION 1.1

The **linear** (or **tangent line**) **approximation** of $f(x)$ at $x = x_0$ is the function $L(x) = f(x_0) + f'(x_0)(x - x_0)$.

Observe that the y-coordinate y_1 of the point on the tangent line corresponding to $x = x_1$ is simply found by substituting $x = x_1$ in equation (1.2). This gives us

$$y_1 = f(x_0) + f'(x_0)(x_1 - x_0). \tag{1.3}$$

We define the **increments** Δx and Δy by

$$\Delta x = x_1 - x_0$$

and

$$\Delta y = f(x_1) - f(x_0).$$

Using this notation, equation (1.3) gives us the approximation

$$\boxed{f(x_1) \approx y_1 = f(x_0) + f'(x_0)\Delta x.} \tag{1.4}$$

We illustrate this in Figure 3.2. We sometimes rewrite (1.4) by subtracting $f(x_0)$ from both sides, to yield

$$\Delta y = f(x_1) - f(x_0) \approx f'(x_0)\,\Delta x = dy, \tag{1.5}$$

where $dy = f'(x_0)\Delta x$ is called the **differential** of y. When using this notation, we also define dx, the differential of x, by $dx = \Delta x$, so that by (1.5),

$$dy = f'(x_0)\,dx.$$

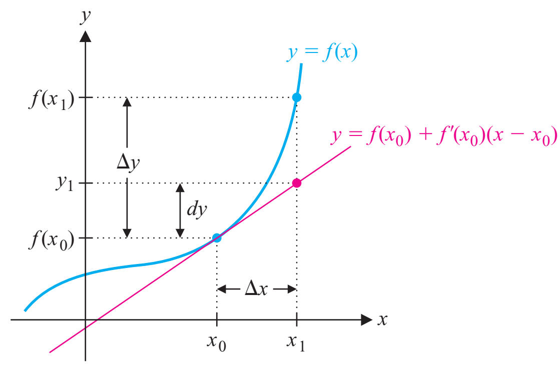


FIGURE 3.2
Increments and differentials

We can use linear approximations to produce approximate values of transcendental functions, as in example 1.1.

EXAMPLE 1.1 Finding a Linear Approximation

Find the linear approximation to $f(x) = \cos x$ at $x_0 = \pi/3$ and use it to approximate $\cos(1)$.

Solution From Definition 1.1, the linear approximation is defined as $L(x) = f(x_0) + f'(x_0)(x - x_0)$. Here, $x_0 = \pi/3$, $f(x) = \cos x$ and $f'(x) = -\sin x$. So, we have

$$L(x) = \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).$$

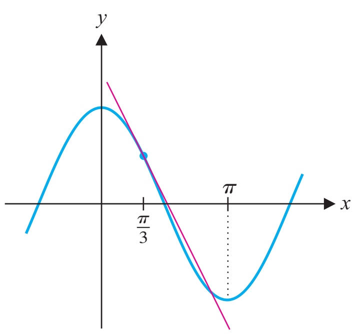


FIGURE 3.3
$y = \cos x$ and its linear approximation at $x_0 = \pi/3$

In Figure 3.3, we show a graph of $y = \cos x$ and the linear approximation to $\cos x$ for $x_0 = \pi/3$. Notice that the linear approximation (i.e., the tangent line at $x_0 = \pi/3$) stays close to the graph of $y = \cos x$ only for x close to $\pi/3$. In fact, for $x < 0$ or $x > \pi$, the linear approximation is obviously quite poor. It is typical of linear approximations (tangent lines) to stay close to the curve only nearby the point of tangency.

Observe that we chose $x_0 = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is the value closest to 1 at which we know the value of the cosine exactly. Finally, an estimate of $\cos(1)$ is

$$L(1) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(1 - \frac{\pi}{3}\right) \approx 0.5409.$$

Your calculator gives you $\cos(1) \approx 0.5403$ and so, we have found a fairly good approximation to the desired value. ■

In example 1.2, we derive a useful approximation to $\sin x$, valid for x close to 0. This approximation is often used in applications in physics and engineering to simplify equations involving $\sin x$.

EXAMPLE 1.2 Linear Approximation of sin *x*

Find the linear approximation of $f(x) = \sin x$, for x close to 0.

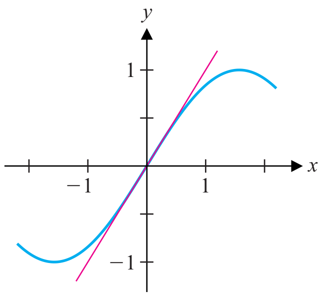


FIGURE 3.4
$y = \sin x$ and $y = x$

Solution Here, $f'(x) = \cos x$, so that from Definition 1.1, we have

$$\sin x \approx L(x) = f(0) + f'(0)(x - 0) = \sin 0 + \cos 0\,(x) = x.$$

This says that for x close to 0, $\sin x \approx x$. We illustrate this in Figure 3.4, where we show graphs of both $y = \sin x$ and $y = x$. ■

Observe from Figure 3.4 that the graph of $y = x$ stays close to the graph of $y = \sin x$ only in the vicinity of $x = 0$. Thus, the approximation $\sin x \approx x$ is valid only for x close to 0. Also note that the farther x gets from 0, the worse the approximation becomes. This becomes even more apparent in example 1.3, where we also illustrate the use of the increments Δx and Δy.

EXAMPLE 1.3 Linear Approximation to Some Cube Roots

Use a linear approximation to approximate $\sqrt[3]{8.02}$, $\sqrt[3]{8.07}$, $\sqrt[3]{8.15}$ and $\sqrt[3]{25.2}$.

Solution Here we are approximating values of the function $f(x) = \sqrt[3]{x} = x^{1/3}$. So, $f'(x) = \frac{1}{3}x^{-2/3}$. The closest number to any of 8.02, 8.07 or 8.15 whose cube root we know exactly is 8. So, we write

$$\begin{aligned} f(8.02) &= f(8) + [f(8.02) - f(8)] \qquad \text{Add and substract } f(8). \\ &= f(8) + \Delta y. \end{aligned} \tag{1.6}$$

From (1.5), we have

$$\begin{aligned} \Delta y \approx dy &= f'(8)\Delta x \\ &= \left(\frac{1}{3}\right) 8^{-2/3}(8.02 - 8) = \frac{1}{600}. \qquad \text{Since } \Delta x = 8.02 - 8. \end{aligned} \tag{1.7}$$

Using (1.6) and (1.7), we get

$$f(8.02) \approx f(8) + dy = 2 + \frac{1}{600} \approx 2.0016667,$$

while your calculator accurately returns $\sqrt[3]{8.02} \approx 2.0016653$. Similarly, we get

$$f(8.07) \approx f(8) + \frac{1}{3}8^{-2/3}(8.07 - 8) \approx 2.0058333$$

and
$$f(8.15) \approx f(8) + \frac{1}{3}8^{-2/3}(8.15 - 8) \approx 2.0125,$$

while your calculator returns $\sqrt[3]{8.07} \approx 2.005816$ and $\sqrt[3]{8.15} \approx 2.012423$, respectively.

To approximate $\sqrt[3]{25.2}$, observe that 8 is not the closest number to 25.2 whose cube root we know exactly. Since 25.2 is much closer to 27 than to 8, we write

$$f(25.2) = f(27) + \Delta y \approx f(27) + dy = 3 + dy.$$

In this case,

$$dy = f'(27)\Delta x = \frac{1}{3}27^{-2/3}(25.2 - 27) = \frac{1}{3}\left(\frac{1}{9}\right)(-1.8) = -\frac{1}{15}$$

and we have
$$f(25.2) \approx 3 + dy = 3 - \frac{1}{15} \approx 2.9333333,$$

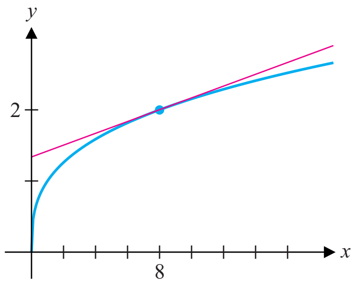


FIGURE 3.5
$y = \sqrt[3]{x}$ and the linear approximation at $x_0 = 8$

compared to the value of 2.931794, produced by your calculator. It is important to recognize here that the farther the value of x gets from the point of tangency, the worse the approximation tends to be. You can see this clearly in Figure 3.5. ■

Our first three examples were intended to familiarize you with the technique and to give you a feel for how good (or bad) linear approximations tend to be. In example 1.4, there is no exact answer to compare with the approximation. Our use of the linear approximation here is referred to as **linear interpolation.**

EXAMPLE 1.4 Using a Linear Approximation to Perform Linear Interpolation

x	6	10	14
$f(x)$	84	60	32

The price of an item affects consumer demand for that item. Suppose that based on market research, a company estimates that $f(x)$ thousand small cameras can be sold at the price of $\$x$, as given in the accompanying table. Estimate the number of cameras that can be sold at \$7.

Solution The closest x-value to $x = 7$ in the table is $x = 6$. [In other words, this is the closest value of x at which we know the value of $f(x)$.] The linear approximation of $f(x)$ at $x = 6$ would look like

$$L(x) = f(6) + f'(6)(x - 6).$$

From the table, we know that $f(6) = 84$, but we do not know $f'(6)$. Further, we can't compute $f'(x)$, since we don't have a formula for $f(x)$. The best we can do with the given data is to approximate the derivative by

$$f'(6) \approx \frac{f(10) - f(6)}{10 - 6} = \frac{60 - 84}{4} = -6.$$

The linear approximation is then

$$L(x) \approx 84 - 6(x - 6).$$

Using this, we estimate that the number of cameras sold at $x = 7$ would be $L(7) \approx 84 - 6 = 78$ thousand. That is, we would expect to sell approximately 78 thousand cameras at a price of \$7. We show a graphical interpretation of this in Figure 3.6, where the straight line is the linear approximation (in this case, the secant line joining the first two data points). ■

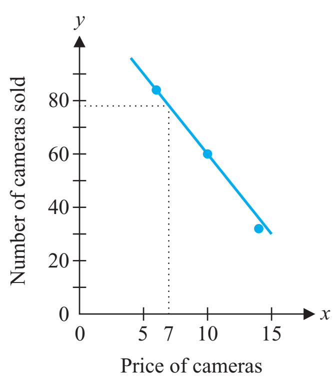


FIGURE 3.6
Linear interpolation

Newton's Method

We now return to the question of finding zeros of a function. In section 1.4, we introduced the method of bisections as one procedure for finding zeros of a continuous function. Here, we explore a method that is usually much more efficient than bisections. We are again looking for values of x such that $f(x) = 0$. These values are called **roots** of the equation $f(x) = 0$ or **zeros** of the function f. It's easy to find the zeros of

$$f(x) = ax^2 + bx + c,$$

but how would you find zeros of

$$f(x) = \tan x - x?$$

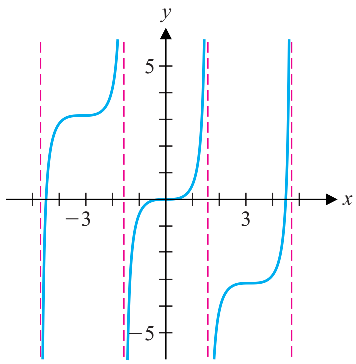


FIGURE 3.7
$y = \tan x - x$

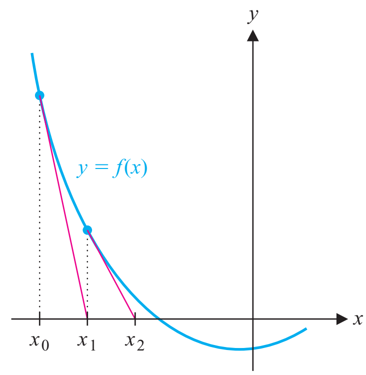


FIGURE 3.8
Newton's method

This function is not algebraic and there are no formulas available for finding the zeros. Even so, we can clearly see zeros in Figure 3.7. (In fact, there are infinitely many of them.) The question is, how are we to *find* them?

In general, to find approximate solutions to $f(x) = 0$, we first make an **initial guess,** denoted x_0, of the location of a solution. Since the tangent line to $y = f(x)$ at $x = x_0$ tends to hug the curve, we follow the tangent line to where it intersects the x-axis (see Figure 3.8).

This appears to provide an improved approximation to the zero. The equation of the tangent line to $y = f(x)$ at $x = x_0$ is given by the linear approximation at x_0 [see equation (1.2)],

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{1.8}$$

We denote the x-intercept of the tangent line by x_1 [found by setting $y = 0$ in (1.8)]. We then have

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

and, solving this for x_1, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If we repeat this process, using x_1 as our new guess, we should produce a further improved approximation,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and so on (see Figure 3.8). In this way, we generate a sequence of **successive approximations** determined by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{for } n = 0, 1, 2, 3, \ldots.} \tag{1.9}$$

HISTORICAL NOTES

Sir Isaac Newton (1642–1727) An English mathematician and scientist known as the co-inventor of calculus. In a 2-year period from 1665 to 1667, Newton made major discoveries in several areas of calculus, as well as optics and the law of gravitation. Newton's mathematical results were not published in a timely fashion. Instead, techniques such as Newton's method were quietly introduced as useful tools in his scientific papers. Newton's *Mathematical Principles of Natural Philosophy* is widely regarded as one of the greatest achievements of the human mind.

This procedure is called the **Newton-Raphson method,** or simply **Newton's method.** If Figure 3.8 is any indication, x_n should get closer and closer to a zero as n increases.

Newton's method is generally a very fast, accurate method for approximating the zeros of a function, as we illustrate with example 1.5.

EXAMPLE 1.5 Using Newton's Method to Approximate a Zero

Find a zero of $f(x) = x^5 - x + 1$.

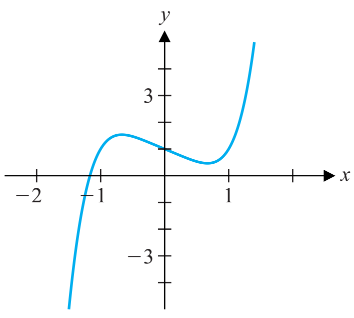


FIGURE 3.9
$y = x^5 - x + 1$

Solution From Figure 3.9, it appears as if the only zero of f is located between $x = -2$ and $x = -1$. We further observe that $f(-1) = 1 > 0$ and $f(-2) = -29 < 0$. Since f is continuous, the Intermediate Value Theorem (Theorem 4.4 in section 1.4) says that f must have a zero on the interval $(-2, -1)$. Because the zero appears to be closer to $x = -1$, we make the initial guess $x_0 = -1$. Finally, $f'(x) = 5x^4 - 1$ and so, Newton's method gives us

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^5 - x_n + 1}{5x_n^4 - 1}, \qquad n = 0, 1, 2, \ldots.$$

Using the initial guess $x_0 = -1$, we get

$$x_1 = -1 - \frac{(-1)^5 - (-1) + 1}{5(-1)^4 - 1}$$
$$= -1 - \frac{1}{4} = -\frac{5}{4}.$$

Likewise, from $x_1 = -\frac{5}{4}$, we get the improved approximation

$$x_2 = -\frac{5}{4} - \frac{\left(-\frac{5}{4}\right)^5 - \left(-\frac{5}{4}\right) + 1}{5\left(-\frac{5}{4}\right)^4 - 1}$$
$$\approx -1.178459394$$

and so on. We find that $x_3 \approx -1.167537389$,

$x_4 \approx -1.167304083$

and $x_5 \approx -1.167303978 \approx x_6$.

Since $x_5 = x_6$, we will make no further progress by calculating additional steps. As a final check on the accuracy of our approximation, we compute

$$f(x_6) \approx 1 \times 10^{-13}.$$

Since this is very close to zero, we say that $x_6 \approx -1.167303978$ is an **approximate zero** of f. ■

You can bring Newton's method to bear on a variety of approximation problems. As we illustrate in example 1.6, you may first need to rephrase the problem as a rootfinding problem.

EXAMPLE 1.6 Using Newton's Method to Approximate a Cube Root

Use Newton's method to approximate $\sqrt[3]{7}$.

Solution Recall that we can use a linear approximation to do this. On the other hand, Newton's method is used to solve equations of the form $f(x) = 0$. We can rewrite the current problem in this form, as follows. Suppose $x = \sqrt[3]{7}$. Then, $x^3 = 7$, which can be rewritten as

$$f(x) = x^3 - 7 = 0.$$

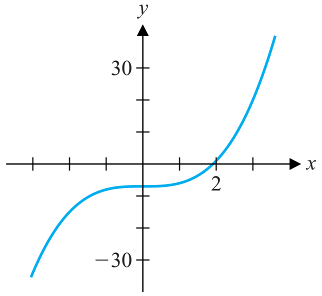


FIGURE 3.10
$y = x^3 - 7$

Here, $f'(x) = 3x^2$ and we obtain an initial guess from a graph of $y = f(x)$ (see Figure 3.10). Notice that there is a zero near $x = 2$ and so we take $x_0 = 2$. Newton's method then yields

$$x_1 = 2 - \frac{2^3 - 7}{3(2^2)} = \frac{23}{12} \approx 1.916666667.$$

Continuing this process, we have

$$x_2 \approx 1.912938458$$

and $$x_3 \approx 1.912931183 \approx x_4.$$

Further, $$f(x_4) \approx 1 \times 10^{-13}$$

and so, x_4 is an approximate zero of f. This also says that

$$\sqrt[3]{7} \approx 1.912931183,$$

which compares very favorably with the value of $\sqrt[3]{7}$ produced by your calculator. ■

REMARK 1.1

Although it seemed to be very efficient in examples 1.5 and 1.6, Newton's method does not always work. We urge you to make sure that the values coming from the method are getting progressively closer and closer together (zeroing in, we hope, on the desired solution). Don't stop until you've reached the limits of accuracy of your computing device. Also, be sure to compute the value of the function at the suspected approximate zero. If the function value is not close to zero, do not accept the value as an approximate zero.

As we illustrate in example 1.7, Newton's method needs a good initial guess to find an accurate approximation.

EXAMPLE 1.7 The Effect of a Bad Guess on Newton's Method

Use Newton's method to find an approximate zero of $f(x) = x^3 - 3x^2 + x - 1$.

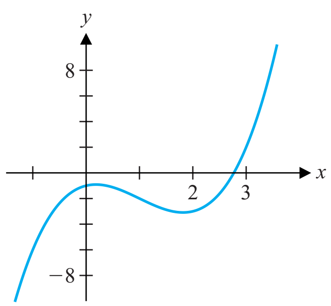


FIGURE 3.11
$y = x^3 - 3x^2 + x - 1$

Solution From the graph in Figure 3.11, there appears to be a zero on the interval $(2, 3)$. Using the (not particularly good) initial guess $x_0 = 1$, we get $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and so on. Try this for yourself. Newton's method is sensitive to the initial guess and $x_0 = 1$ is just a bad initial guess. If we had instead started with the improved initial guess $x_0 = 2$, Newton's method would have quickly converged to the approximate zero 2.769292354. (Again, try this for yourself.) ■

n	x_n
1	−9.5
2	−65.9
3	−2302
4	−2,654,301
5	-3.5×10^{12}
6	-6.2×10^{24}

Newton's method iterations for $x_0 = -2$

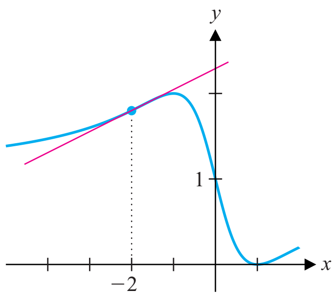


FIGURE 3.12
$y = \dfrac{(x-1)^2}{x^2+1}$ and the tangent line at $x = -2$

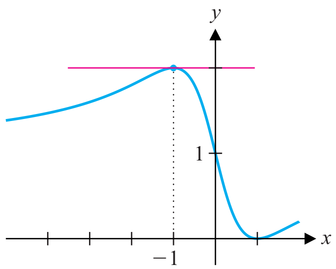


FIGURE 3.13
$y = \dfrac{(x-1)^2}{x^2+1}$ and the tangent line at $x = -1$

As we see in example 1.7, making a good initial guess is essential with Newton's method. However, this alone will not guarantee rapid convergence. By rapid convergence, we mean that it takes only a few iterations to obtain an accurate approximation.

EXAMPLE 1.8 Unusually Slow Convergence for Newton's Method

Use Newton's method with (a) $x_0 = -2$, (b) $x_0 = -1$ and (c) $x_0 = 0$ to try to locate the zero of $f(x) = \dfrac{(x-1)^2}{x^2+1}$.

Solution Of course, there's no mystery here: f has only one zero, located at $x = 1$. However, watch what happens when we use Newton's method with the specified guesses.

(a) If we take $x_0 = -2$, and apply Newton's method, we obtain the values in the table found in the margin. Obviously, the successive iterations are blowing up with this initial guess. To see why, look at Figure 3.12, which shows the graphs of both $y = f(x)$ and the tangent line at $x = -2$. If you follow the tangent line to where it intersects the x-axis, you will be going *away* from the zero (far away). Since all of the tangent lines for $x \le -2$ have positive slope [compute $f'(x)$ to see why this is true], each subsequent step takes you farther from the zero.

(b) If we use the improved initial guess $x_0 = -1$, we cannot even compute x_1. In this case, $f'(x_0) = 0$ and so, Newton's method fails. Graphically, this means that the tangent line to $y = f(x)$ at $x = -1$ is horizontal (see Figure 3.13), so that the tangent line never intersects the x-axis.

(c) With the even better initial guess $x_0 = 0$, we obtain the successive approximations in the following table.

n	x_n
1	0.5
2	0.70833
3	0.83653
4	0.912179
5	0.95425
6	0.976614

n	x_n
7	0.9881719
8	0.9940512
9	0.9970168
10	0.9985062
11	0.9992525
12	0.9996261

Newton's method iterations for $x_0 = 0$

Finally, we happened upon an initial guess for which Newton's method converges to the root $x = 1$. What is unusual here is that the successive approximations shown in the table are converging to 1 much more slowly than in previous examples. By comparison, note that in example 1.5, the iterations stop changing at x_5. Here, x_5 is not particularly close to the desired zero of $f(x)$. In fact, in this example, x_{12} is not as close to the zero as x_5 is in example 1.5. We look further into this type of behavior in the exercises. ■

Despite the minor problems experienced in examples 1.7 and 1.8, you should view Newton's method as a generally reliable and efficient method of locating zeros approximately. Just use a bit of caution and common sense. If the successive approximations are converging to some value that does not appear consistent with the graph, then you need to scrutinize your results more carefully and perhaps try some other initial guesses.

BEYOND FORMULAS

Approximations are at the heart of calculus. To find the slope of a tangent line, for example, we start by approximating the tangent line with secant lines. The fact that there are numerous simple derivative formulas to help us compute exact slopes is an unexpected bonus. In this section, the tangent line is thought of as an approximation of a curve and is used to approximate solutions of equations for which algebra fails. Although this doesn't provide us the luxury of simply computing an exact answer, we *can* make the approximation as accurate as we like and so, for most practical purposes, we can "solve" the equation. Think about a situation where you need the time of day. How often do you need the *exact* time?

EXERCISES 3.1

WRITING EXERCISES

1. Briefly explain in terms of tangent lines why the approximation in example 1.3 gets worse as x gets farther from 8.

2. We constructed a variety of linear approximations in this section. Approximations can be "good" approximations or "bad" approximations. Explain why it can be said that $y = x$ is a good approximation to $y = \sin x$ near $x = 0$ but $y = 1$ is not a good approximation to $y = \cos x$ near $x = 0$. (Hint: Look at the graphs of $y = \sin x$ and $y = x$ on the same axes, then do the same with $y = \cos x$ and $y = 1$.)

3. In example 1.6, we mentioned that you might think of using a linear approximation instead of Newton's method. Discuss the relationship between a linear approximation to $\sqrt[3]{7}$ and a Newton's method approximation to $\sqrt[3]{7}$. (Hint: Compare the first step of Newton's method to a linear approximation.)

4. Explain why Newton's method fails computationally if $f'(x_0) = 0$. In terms of tangent lines intersecting the x-axis, explain why having $f'(x_0) = 0$ is a problem.

In exercises 1–6, find the linear approximation to $f(x)$ at $x = x_0$. Graph the function and its linear approximation.

1. $f(x) = \sqrt{x}$, $x_0 = 1$

2. $f(x) = (x+1)^{1/3}$, $x_0 = 0$

3. $f(x) = \sqrt{2x+9}$, $x_0 = 0$

4. $f(x) = 2/x$, $x_0 = 1$

5. $f(x) = \sin 3x$, $x_0 = 0$

6. $f(x) = \sin x$, $x_0 = \pi$

7. (a) Find the linear approximation at $x = 0$ to each of $f(x) = (x+1)^2$, $g(x) = 1 + \sin(2x)$ and $h(x) = e^{2x}$. Compare your results.
   (b) Graph each function in part (a) together with its linear approximation derived in part (a). Which function has the closest fit with its linear approximation?

8. (a) Find the linear approximation at $x = 0$ to each of $f(x) = \sin x$, $g(x) = \tan^{-1} x$ and $h(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$. Compare your results.
   (b) Graph each function in part (a) together with its linear approximation derived in part (a). Which function has the closest fit with its linear approximation?

In exercises 9 and 10, use linear approximations to estimate the quantity.

9. (a) $\sqrt[4]{16.04}$ (b) $\sqrt[4]{16.08}$ (c) $\sqrt[4]{16.16}$

10. (a) $\sin(0.1)$ (b) $\sin(1.0)$ (c) $\sin\left(\frac{9}{4}\right)$

11. For exercise 9, compute the errors (the absolute value of the difference between the exact values and the linear approximations).

12. Thinking of exercises 9a–9c as numbers of the form $\sqrt[4]{16 + \Delta x}$, denote the errors as $e(\Delta x)$ (where $\Delta x = 0.04$, $\Delta x = 0.08$ and $\Delta x = 0.16$). Based on these three computations, determine a constant c such that $e(\Delta x) \approx c(\Delta x)^2$.

In exercises 13–16, use linear interpolation to estimate the desired quantity.

13. A company estimates that $f(x)$ thousand software games can be sold at the price of \$$x$ as given in the table.

x	20	30	40
$f(x)$	18	14	12

Estimate the number of games that can be sold at (a) \$24 and (b) \$36.

14. A vending company estimates that $f(x)$ cans of soft drink can be sold in a day if the temperature is $x°$F as given in the table.

x	60	80	100
$f(x)$	84	120	168

Estimate the number of cans that can be sold at (a) 72° and (b) 94°.

15. An animation director enters the position $f(t)$ of a character's head after t frames of the movie as given in the table.

t	200	220	240
$f(t)$	128	142	136

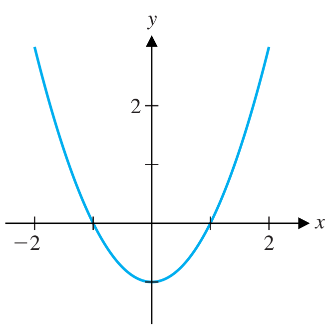
If the computer software uses interpolation to determine the intermediate positions, determine the position of the head at frame numbers (a) 208 and (b) 232.

16. A sensor measures the position $f(t)$ of a particle t microseconds after a collision as given in the table.

t	5	10	15
$f(t)$	8	14	18

Estimate the position of the particle at times (a) $t = 8$ and (b) $t = 12$.

17. Given the graph of $y = f(x)$, draw in the tangent lines used in Newton's method to determine x_1 and x_2 after starting at $x_0 = 2$. Which of the zeros will Newton's method converge to?



18. Repeat exercise 17 with $x_0 = -2$ and $x_0 = 0.4$.

19. What would happen to Newton's method in exercise 17 if you had a starting value of $x_0 = 0$?

20. Consider the use of Newton's method in exercise 17 with $x_0 = 0.2$ and $x_0 = 10$. Obviously, $x_0 = 0.2$ is much closer to a zero of the function, but which initial guess would work better in Newton's method? Explain.

In exercises 21–24, use Newton's method with the given x_0 to (a) compute x_1 and x_2 by hand and (b) use a computer or calculator to find the root to at least five decimal places of accuracy.

21. $x^3 + 3x^2 - 1 = 0,\ x_0 = 1$

22. $x^3 + 4x^2 - x - 1 = 0,\ x_0 = -1$

23. $x^4 - 3x^2 + 1 = 0,\ x_0 = 1$

24. $x^4 - 3x^2 + 1 = 0,\ x_0 = -1$

In exercises 25–32, use Newton's method to find an approximate root (accurate to six decimal places). Sketch the graph and explain how you determined your initial guess.

25. $x^3 + 4x^2 - 3x + 1 = 0$

26. $x^4 - 4x^3 + x^2 - 1 = 0$

27. $x^5 + 3x^3 + x - 1 = 0$

28. $\cos x - x = 0$

29. $\sin x = x^2 - 1$

30. $\cos x^2 = x$

31. $e^x = -x$

32. $e^{-x} = \sqrt{x}$

33. Show that Newton's method applied to $x^2 - c = 0$ (where $c > 0$ is some constant) produces the iterative scheme $x_{n+1} = \frac{1}{2}(x_n + c/x_n)$ for approximating $\sqrt{c}$. This scheme has been known for over 2000 years. To understand why it works, suppose that your initial guess (x_0) for $\sqrt{c}$ is a little too small. How would c/x_0 compare to $\sqrt{c}$? Explain why the average of x_0 and c/x_0 would give a better approximation to $\sqrt{c}$.

34. Show that Newton's method applied to $x^n - c = 0$ (where n and c are positive constants) produces the iterative scheme $x_{n+1} = \frac{1}{n}[(n-1)x_n + cx_n^{1-n}]$ for approximating $\sqrt[n]{c}$.

In exercises 35–40, use Newton's method [state the function $f(x)$ you use] to estimate the given number. (Hint: See exercises 33 and 34.)

35. $\sqrt{11}$

36. $\sqrt{23}$

37. $\sqrt[3]{11}$

38. $\sqrt[3]{23}$

39. $\sqrt[4.4]{24}$

40. $\sqrt[4.6]{24}$

In exercises 41–46, Newton's method fails. Explain why the method fails and, if possible, find a root by correcting the problem.

41. $4x^3 - 7x^2 + 1 = 0,\ x_0 = 0$

42. $4x^3 - 7x^2 + 1 = 0,\ x_0 = 1$

43. $x^2 + 1 = 0,\ x_0 = 0$

44. $x^2 + 1 = 0,\ x_0 = 1$

45. $\dfrac{4x^2 - 8x + 1}{4x^2 - 3x - 7} = 0,\ x_0 = -1$

46. $\left(\dfrac{x+1}{x-2}\right)^{1/3} = 0,\ x_0 = 0.5$

47. Use Newton's method with (a) $x_0 = 1.2$ and (b) $x_0 = 2.2$ to find a zero of $f(x) = x^3 - 5x^2 + 8x - 4$. Discuss the difference in the rates of convergence in each case.

48. Use Newton's method with (a) $x_0 = 0.2$ and (b) $x_0 = 3.0$ to find a zero of $f(x) = x \sin x$. Discuss the difference in the rates of convergence in each case.

49. Use Newton's method with (a) $x_0 = -1.1$ and (b) $x_0 = 2.1$ to find a zero of $f(x) = x^3 - 3x - 2$. Discuss the difference in the rates of convergence in each case.

50. Factor the polynomials in exercises 47 and 49. Find a relationship between the factored polynomial and the rate at which Newton's method converges to a zero. Explain how the function in exercise 48, which does not factor, fits into this relationship. (Note: The relationship will be explored further in exploratory exercise 2.)

In exercises 51–54, find the linear approximation at $x = 0$ to show that the following commonly used approximations are valid for "small" x. Compare the approximate and exact values for $x = 0.01$, $x = 0.1$ and $x = 1$.

51. $\tan x \approx x$

52. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$

53. $\sqrt{4+x} \approx 2 + \frac{1}{4}x$

54. $e^x \approx 1 + x$

55. Use a computer algebra system (CAS) to determine the range of x's in exercise 51 for which the approximation is accurate to within 0.01. That is, find x such that $|\tan x - x| < 0.01$.

56. Use a CAS to determine the range of x's in exercise 54 for which the approximation is accurate to within 0.01. That is, find x such that $|e^x - (1+x)| < 0.01$.

57. A water wave of length L meters in water of depth d meters has velocity v satisfying the equation

$$v^2 = \frac{4.9L}{\pi}\frac{e^{2\pi d/L} - e^{-2\pi d/L}}{e^{2\pi d/L} + e^{-2\pi d/L}}.$$

Treating L as a constant and thinking of v^2 as a function $f(d)$, use a linear approximation to show that $f(d) \approx 9.8d$ for small values of d. That is, for small depths, the velocity of the wave is approximately $\sqrt{9.8d}$ and is independent of the wavelength L.

58. Planck's law states that the energy density of blackbody radiation of wavelength x is given by

$$f(x) = \frac{8\pi hcx^{-5}}{e^{hc/(kTx)} - 1}.$$

Use the linear approximation in exercise 54 to show that $f(x) \approx 8\pi kT/x^4$, which is known as the Rayleigh-Jeans law.

59. Suppose that a species reproduces as follows: with probability p_0, an organism has no offspring; with probability p_1, an organism has one offspring; with probability p_2, an organism has two offspring and so on. The probability that the species goes extinct is given by the smallest nonnegative solution of the equation $p_0 + p_1x + p_2x^2 + \cdots = x$ (see Sigmund's *Games of Life*). Find the positive solutions of the equations $0.1 + 0.2x + 0.3x^2 + 0.4x^3 = x$ and $0.4 + 0.3x + 0.2x^2 + 0.1x^3 = x$. Explain in terms of species going extinct why the first equation has a smaller solution than the second.

60. In Einstein's theory of relativity, the length of an object depends on its velocity. If L_0 is the length of the object at rest, v is the object's velocity and c is the speed of light, the **Lorentz contraction** formula for the length of the object is $L = L_0\sqrt{1 - v^2/c^2}$. Treating L as a function of v, find the linear approximation of L at $v = 0$.

61. The spruce budworm is an enemy of the balsam fir tree. In one model of the interaction between these organisms, possible long-term populations of the budworm are solutions of the equation $r(1 - x/k) = x/(1 + x^2)$, for positive constants r and k (see Murray's *Mathematical Biology*). Find all positive solutions of the equation with $r = 0.5$ and $k = 7$.

62. Repeat exercise 61 with $r = 0.5$ and $k = 7.5$. For a small change in the environmental constant k (from 7 to 7.5), how did the solution change from exercise 61 to exercise 62? The largest solution corresponds to an "infestation" of the spruce budworm.

63. Newton's theory of gravitation states that the weight of a person at elevation x feet above sea level is $W(x) = PR^2/(R + x)^2$, where P is the person's weight at sea level and R is the radius of the earth (approximately 20,900,000 feet). Find the linear approximation of $W(x)$ at $x = 0$. Use the linear approximation to estimate the elevation required to reduce the weight of a 120-pound person by 1%.

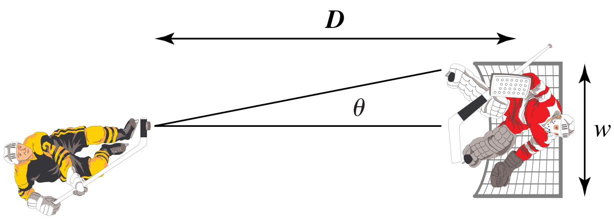
64. One important aspect of Einstein's theory of relativity is that mass is not constant. For a person with mass m_0 at rest, the mass will equal $m = m_0/\sqrt{1 - v^2/c^2}$ at velocity v (where c is the speed of light). Thinking of m as a function of v, find the linear approximation of $m(v)$ at $v = 0$. Use the linear approximation to show that mass is essentially constant for small velocities.

65. In the figure, a beam is clamped at its left end and simply supported at its right end. A force P (called an **axial load**) is applied at the right end.

If enough force is applied, the beam will buckle. Obviously, it is important for engineers to be able to compute this **buckling load.** For certain types of beams, the buckling load is the smallest positive solution of the equation $\tan\sqrt{x} = \sqrt{x}$. The shape of the buckled beam is given by $y = \sqrt{L} - \sqrt{L}x - \sqrt{L}\cos\sqrt{L}x + \sin\sqrt{L}x$, where L is the buckling load. Find L and the shape of the buckled beam.

66. The spectral radiancy S of an ideal radiator at constant temperature can be thought of as a function $S(f)$ of the radiant frequency f. The function $S(f)$ attains its maximum when $3e^{-cf} + cf - 3 = 0$ for the constant $c = 10^{-13}$. Use Newton's method to approximate the solution.

67. In the diagram (on the following page), a hockey player is D feet from the net on the central axis of the rink. The goalie blocks off a segment of width w and stands d feet from the net. The shooting angle to the left of the goalie is given by $\phi = \tan^{-1}\left[\frac{3(1 - d/D) - w/2}{D - d}\right]$. Use a linear approximation of $\tan^{-1}x$ at $x = 0$ to show that if $d = 0$, then $\phi \approx \frac{3 - w/2}{D}$. Based on this, describe how ϕ changes if there is an increase in (a) w or (b) D.



Exercise 67

68. The shooter in exercise 67 is assumed to be in the center of the ice. Suppose that the line from the shooter to the center of the goal makes an angle of θ with center line. For the goalie to completely block the goal, he must stand d feet away from the net where $d = D(1 - w/6\cos\theta)$. Show that for small angles, $d \approx D(1 - w/6)$.

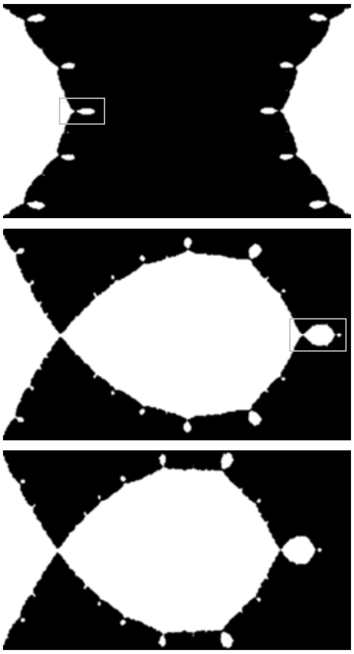
In exercises 69 and 70, we explore the convergence of Newton's method for $f(x) = x^3 - 3x^2 + 2x$.

69. The zeros of f are $x = 0$, $x = 1$ and $x = 2$. Determine which of the three zeros Newton's method iterates converge to for (a) $x_0 = 0.1$, (b) $x_0 = 1.1$ and (c) $x_0 = 2.1$.

70. Determine which of the three zeros Newton's method iterates converge to for (a) $x_0 = 0.54$, (b) $x_0 = 0.55$ and (c) $x_0 = 0.56$.

EXPLORATORY EXERCISES

1. In this exercise, you will extend the work of exercises 69 and 70. First, a definition: The **basin of attraction** of a zero is the set of starting values x_0 for which Newton's method iterates converge to the zero. As exercises 69 and 70 indicate, the basin boundaries are more complicated than you might expect. For example, you have seen that the interval [0.54, 0.56] contains points in all three basins of attraction. Show that the same is true of the interval [0.552, 0.553]. The picture gets even more interesting when you use complex numbers. These are numbers of the form $a + bi$ where $i = \sqrt{-1}$. The remainder of the exercise requires a CAS or calculator that is programmable and performs calculations with complex numbers. First, try Newton's method with starting point $x_0 = 1 + i$. The formula is exactly the same! Use your computer to show that $x_1 = x_0 - \dfrac{x_0^3 - 3x_0^2 + 2x_0}{3x_0^2 - 6x_0 + 2} = 1 + \frac{1}{2}i$. Then verify that $x_2 = 1 + \frac{1}{7}i$ and $x_3 = 1 + \frac{1}{182}i$. It certainly appears that the iterates are converging to the zero $x = 1$. Now, for some programming: set up a double loop with the parameter a running from 0 to 2 in steps of 0.02 and b running from -1 to 1 in steps of 0.02. Within the double loop, set $x_0 = a + bi$ and compute 10 Newton's method iterates. If x_{10} is close to 0, say $|x_{10} - 0| < 0.1$, then we can conjecture that the iterates converge to 0. (Note: For complex numbers, $|a + bi| = \sqrt{a^2 + b^2}$.) Color the pixel at the point (a, b) black if the iterates converge to 0 and white if not. You can change the ranges of a and b and the step size to "zoom in" on interesting regions. The accompanying pictures show the basin of attraction (in black) for $x = 1$. In the first figure, we display the region with $-1.5 \le x \le 3.5$. In the second figure, we have zoomed in to the portion for $0.26 \le x \le 0.56$. The third shows an even tighter zoom: $0.5 \le x \le 0.555$.

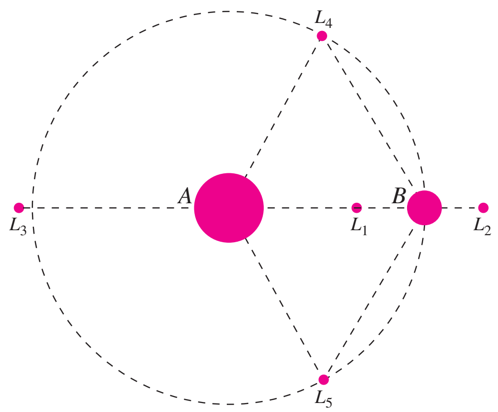


2. Another important question involving Newton's method is how fast it converges to a given zero. Intuitively, we can distinguish between the rate of convergence for $f(x) = x^2 - 1$ (with $x_0 = 1.1$) and that for $g(x) = x^2 - 2x + 1$ (with $x_0 = 1.1$). But how can we measure this? One method is to take successive approximations x_{n-1} and x_n and compute the difference $\Delta_n = x_n - x_{n-1}$. To discover the importance of this quantity, run Newton's method with $x_0 = 1.5$ and then compute the ratios Δ_3/Δ_2, Δ_4/Δ_3, Δ_5/Δ_4 and so on, for each of the following functions:

$$F_1(x) = (x-1)(x+2)^3 = x^4 + 5x^3 + 6x^2 - 4x - 8,$$
$$F_2(x) = (x-1)^2(x+2)^2 = x^4 + 2x^3 - 3x^2 - 4x + 4,$$
$$F_3(x) = (x-1)^3(x+2) = x^4 - x^3 - 3x^2 + 5x - 2 \text{ and}$$
$$F_4(x) = (x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1.$$

In each case, conjecture a value for the limit $r = \lim_{n\to\infty} \dfrac{\Delta_{n+1}}{\Delta_n}$. If the limit exists and is nonzero, we say that Newton's method **converges linearly.** How does r relate to your intuitive sense of how fast the method converges? For $f(x) = (x-1)^4$, we say that the zero $x = 1$ has **multiplicity** 4. For $f(x) = (x-1)^3(x+2)$, $x = 1$ has multiplicity 3 and so on. How does r relate to the multiplicity of the zero? Based on this analysis, why did Newton's method converge faster for $f(x) = x^2 - 1$ than for $g(x) = x^2 - 2x + 1$? Finally, use Newton's method to compute the rate r and hypothesize the multiplicity of the zero $x = 0$ for $f(x) = x \sin x$ and $g(x) = x \sin x^2$.

3. This exercise looks at a special case of the **three-body problem,** in which there is a large object A of mass m_A, a much smaller object B of mass $m_B \ll m_A$ and an object C of negligible mass. (Here, $a \ll b$ means that a is *much smaller than* b.) Assume that object B orbits in a circular path around the common center of mass. There are five circular orbits for object C that maintain constant relative positions of the three objects. These are called **Lagrange points** L_1, L_2, L_3, L_4 and L_5, as shown in the figure.



To derive equations for the Lagrange points, set up a coordinate system with object A at the origin and object B at the point $(1, 0)$. Then L_1 is at the point $(x_1, 0)$, where x_1 is the solution of

$$(1+k)x^5 - (3k+2)x^4 + (3k+1)x^3 - x^2 + 2x - 1 = 0;$$

L_2 is at the point $(x_2, 0)$, where x_2 is the solution of

$$(1+k)x^5 - (3k+2)x^4 + (3k+1)x^3 - (2k+1)x^2 + 2x - 1 = 0$$

and L_3 is at the point $(-x_3, 0)$, where x_3 is the solution of

$$(1+k)x^5 + (3k+2)x^4 + (3k+1)x^3 - x^2 - 2x - 1 = 0,$$

where $k = \dfrac{m_B}{m_A}$. Use Newton's method to find approximate solutions of the following.

(a) Find L_1 for the Earth-Sun system with $k = 0.000002$. This point has an uninterrupted view of the sun and is the location of the solar observatory SOHO.

(b) Find L_2 for the Earth-Sun system with $k = 0.000002$. This is the future location of NASA's Microwave Anistropy Probe.

(c) Find L_3 for the Earth-Sun system with $k = 0.000002$. This point is invisible from the Earth and is the location of Planet X in many science fiction stories.

(d) Find L_1 for the Moon-Earth system with $k = 0.01229$. This point has been suggested as a good location for a space station to help colonize the moon.

(e) The points L_4 and L_5 form equilateral triangles with objects A and B. Explain why this means that polar coordinates for L_4 are $(r, \theta) = \left(1, \frac{\pi}{6}\right)$. Find (x, y)-coordinates for L_4 and L_5. In the Jupiter-Sun system, these are locations of numerous **Trojan asteroids.**

3.2 INDETERMINATE FORMS AND L'HÔPITAL'S RULE

In this section, we reconsider the problem of computing limits. You have frequently seen limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ (or $-\infty$). Recall that from either of these forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right.$, called **indeterminate forms**$\left.\right)$, we cannot determine the value of the limit, or even whether the limit exists. For instance, note that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} \frac{x+1}{1} = \frac{2}{1} = 2,$$

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$$

and $$\lim_{x\to 1}\frac{x-1}{x^2-2x+1}=\lim_{x\to 1}\frac{x-1}{(x-1)^2}=\lim_{x\to 1}\frac{1}{x-1},$$ which does not exist,

even though all three limits initially look like $\frac{0}{0}$. The lesson here is that the expression $\frac{0}{0}$ is mathematically meaningless. It indicates only that both the numerator and denominator tend to zero and that we'll need to dig deeper to find the value of the limit.

Similarly, each of the following limits has the indeterminate form $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty}\frac{x^2+1}{x^3+5}=\lim_{x\to\infty}\frac{(x^2+1)\left(\frac{1}{x^3}\right)}{(x^3+5)\left(\frac{1}{x^3}\right)}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^3}}{1+\frac{5}{x^3}}=\frac{0}{1}=0,$$

$$\lim_{x\to\infty}\frac{x^3+5}{x^2+1}=\lim_{x\to\infty}\frac{(x^3+5)\left(\frac{1}{x^2}\right)}{(x^2+1)\left(\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{x+\frac{5}{x^2}}{1+\frac{1}{x^2}}=\infty$$

and

$$\lim_{x\to\infty}\frac{2x^2+3x-5}{x^2+4x-11}=\lim_{x\to\infty}\frac{(2x^2+3x-5)\left(\frac{1}{x^2}\right)}{(x^2+4x-11)\left(\frac{1}{x^2}\right)}=\lim_{x\to\infty}\frac{2+\frac{3}{x}-\frac{5}{x^2}}{1+\frac{4}{x}-\frac{11}{x^2}}=\frac{2}{1}=2.$$

CAUTION

We will frequently write $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ next to an expression, for instance,

$$\lim_{x\to 1}\frac{x-1}{x^2-1}\quad\left(\frac{0}{0}\right).$$

We use this shorthand to indicate that the limit has the indicated indeterminate form. This notation *does not* mean that the value of the limit is $\frac{0}{0}$. You should take care to avoid writing $\lim_{x\to a} f(x)=\frac{0}{0}$ or $\frac{\infty}{\infty}$, as these are *meaningless* expressions.

So, as with limits of the form $\frac{0}{0}$, if a limit has the form $\frac{\infty}{\infty}$, we must dig deeper to determine the value of the limit or whether the limit even exists. Unfortunately, limits with indeterminate forms are frequently more difficult than those just given. For instance, back at section 2.6, where we struggled with the limit $\lim_{x\to 0}\frac{\sin x}{x}$, ultimately resolving it only with an intricate geometric argument. This limit has the indeterminate form $\frac{0}{0}$, but there is no way to manipulate the numerator or denominator to simplify the expression. In the case of $\lim_{x\to c}\frac{f(x)}{g(x)}$, where $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=0$, we can use linear approximations to suggest a solution, as follows.

If both functions are differentiable at $x=c$, then they are also continuous at $x=c$, so that $f(c)=\lim_{x\to c} f(x)=0$ and $g(c)=\lim_{x\to c} g(x)=0$. We now have the linear approximations

$$f(x)\approx f(c)+f'(c)(x-c)=f'(c)(x-c)$$

and $$g(x)\approx g(c)+g'(c)(x-c)=g'(c)(x-c),$$

since $f(c)=0$ and $g(c)=0$. As we have seen, the approximation should improve as x approaches c, so we would expect that if the limits exist,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(c)(x-c)}{g'(c)(x-c)}=\lim_{x\to c}\frac{f'(c)}{g'(c)}=\frac{f'(c)}{g'(c)},$$

assuming that $g'(c)\neq 0$. Note that if $f'(x)$ and $g'(x)$ are continuous at $x=c$ and $g'(c)\neq 0$, then $\frac{f'(c)}{g'(c)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$. This suggests the following result.

HISTORICAL NOTES

Guillaume de l'Hôpital (1661–1704) A French mathematician who first published the result now known as l'Hôpital's Rule. Born into nobility, l'Hôpital was taught calculus by the brilliant mathematician Johann Bernoulli. A competent mathematician, l'Hôpital is best known as the author of the first calculus textbook. L'Hôpital was a friend and patron of many of the top mathematicians of the seventeenth century.

THEOREM 2.1 (L'Hôpital's Rule)

Suppose that f and g are differentiable on the interval (a, b), except possibly at some fixed point $c \in (a, b)$ and that $g'(x) \neq 0$ on (a, b), except possibly at c. Suppose further that $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ has the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and that $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L$ (or $\pm\infty$). Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

PROOF

Here, we prove only the $\frac{0}{0}$ case where f, f', g and g' are all continuous on all of (a, b) and $g'(c) \neq 0$, while leaving the more intricate general $\frac{0}{0}$ case for Appendix A. First, recall the alternative form of the definition of derivative (found in section 2.2):

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

Working backward, we have by continuity that

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)} = \frac{\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}}{\lim\limits_{x\to c} \dfrac{g(x) - g(c)}{x - c}} = \lim_{x\to c} \frac{\dfrac{f(x) - f(c)}{x - c}}{\dfrac{g(x) - g(c)}{x - c}} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)}.$$

Further, since f and g are continuous at $x = c$, we have that

$$f(c) = \lim_{x\to c} f(x) = 0 \quad \text{and} \quad g(c) = \lim_{x\to c} g(x) = 0.$$

It now follows that

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{f(x)}{g(x)},$$

which is what we wanted. ■

We leave the proof for the $\frac{\infty}{\infty}$ case to more advanced texts.

REMARK 2.1

The conclusion of Theorem 2.1 also holds if $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ is replaced with any of the limits $\lim\limits_{x\to c^+} \dfrac{f(x)}{g(x)}$, $\lim\limits_{x\to c^-} \dfrac{f(x)}{g(x)}$, $\lim\limits_{x\to \infty} \dfrac{f(x)}{g(x)}$ or $\lim\limits_{x\to -\infty} \dfrac{f(x)}{g(x)}$. (In each case, we must make appropriate adjustments to the hypotheses.)

EXAMPLE 2.1 The Indeterminate Form $\frac{0}{0}$

Evaluate $\lim\limits_{x\to 0} \dfrac{1 - \cos x}{\sin x}$.

Solution This has the indeterminate form $\frac{0}{0}$, and both $(1 - \cos x)$ and $\sin x$ are continuous and differentiable everywhere. Further, $\frac{d}{dx} \sin x = \cos x \neq 0$ in some

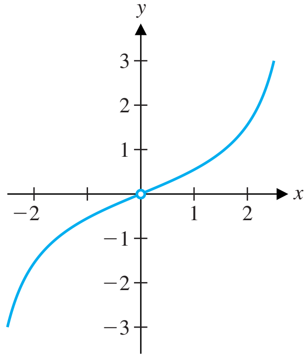


FIGURE 3.14
$y = \dfrac{1 - \cos x}{\sin x}$

interval containing $x = 0$. (Can you determine one such interval?) From the graph of $f(x) = \dfrac{1 - \cos x}{\sin x}$ seen in Figure 3.14, it appears that $f(x) \to 0$, as $x \to 0$. We can confirm this with l'Hôpital's Rule, as follows:

$$\lim_{x\to 0} \frac{1 - \cos x}{\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0.$$

L'Hôpital's Rule is equally easy to apply with limits of the form $\frac{\infty}{\infty}$.

EXAMPLE 2.2 The Indeterminate Form $\frac{\infty}{\infty}$

Evaluate $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$.

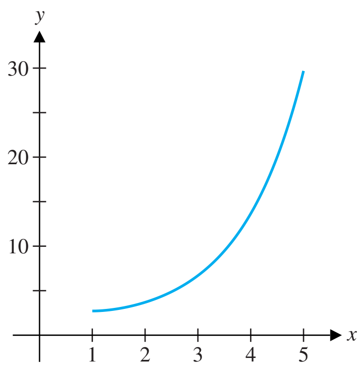


FIGURE 3.15
$y = \dfrac{e^x}{x}$

Solution This has the form $\frac{\infty}{\infty}$ and from the graph in Figure 3.15, it appears that the function grows larger and larger, without bound, as $x \to \infty$. Applying l'Hôpital's Rule confirms our suspicions, as

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x)} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

For some limits, you may need to apply l'Hôpital's Rule repeatedly. Just be careful to verify the hypotheses at each step.

EXAMPLE 2.3 A Limit Requiring Two Applications of L'Hôpital's Rule

Evaluate $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x}$.

Solution First, note that this limit has the form $\frac{\infty}{\infty}$. From the graph in Figure 3.16, it seems that the function tends to 0 as $x \to \infty$ (and does so very rapidly, at that). Applying l'Hôpital's Rule twice, we get

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

as expected.

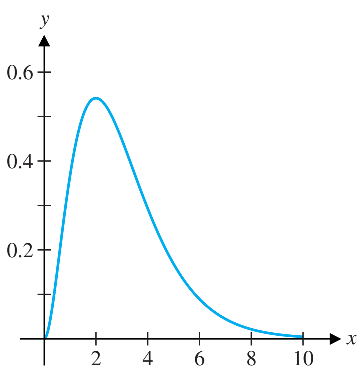


FIGURE 3.16
$y = \dfrac{x^2}{e^x}$

REMARK 2.2

A very common error is to apply l'Hôpital's Rule indiscriminately, without first checking that the limit has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Students also sometimes incorrectly compute the derivative of the quotient, rather than the quotient of the derivatives. Be very careful here.

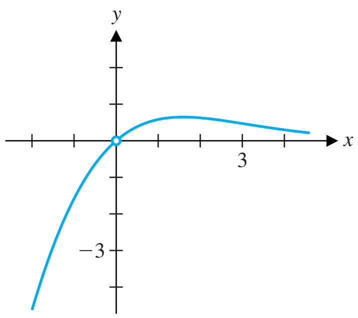


FIGURE 3.17

$y = \dfrac{x^2}{e^x - 1}$

EXAMPLE 2.4 An Erroneous Use of L'Hôpital's Rule

Find the mistake in the string of equalities

$$\lim_{x\to 0}\frac{x^2}{e^x-1} = \lim_{x\to 0}\frac{2x}{e^x} = \lim_{x\to 0}\frac{2}{e^x} = \frac{2}{1} = 2.$$

Solution From the graph in Figure 3.17, we can see that the limit is approximately 0, so 2 appears to be incorrect. The first limit, $\lim\limits_{x\to 0}\dfrac{x^2}{e^x-1}$, has the form $\frac{0}{0}$ and the functions $f(x) = x^2$ and $g(x) = e^x - 1$ satisfy the hypotheses of l'Hôpital's Rule. Therefore, the first equality, $\lim\limits_{x\to 0}\dfrac{x^2}{e^x-1} = \lim\limits_{x\to 0}\dfrac{2x}{e^x}$, holds. However, notice that $\lim\limits_{x\to 0}\dfrac{2x}{e^x} = \dfrac{0}{1} = 0$ and l'Hôpital's Rule does *not* apply here. The correct evaluation is then

$$\lim_{x\to 0}\frac{x^2}{e^x-1} = \lim_{x\to 0}\frac{2x}{e^x} = \frac{0}{1} = 0.$$

Sometimes an application of l'Hôpital's Rule must be followed by some simplification, as we see in example 2.5.

EXAMPLE 2.5 Simplification of the Indeterminate Form $\frac{\infty}{\infty}$

Evaluate $\lim\limits_{x\to 0^+}\dfrac{\ln x}{\csc x}$.

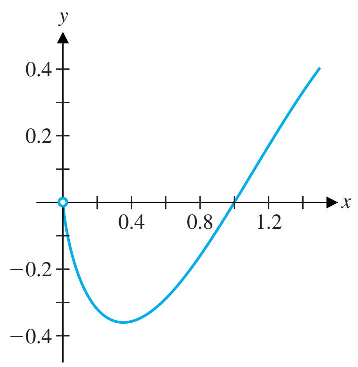


FIGURE 3.18

$y = \dfrac{\ln x}{\csc x}$

Solution First, notice that this limit has the form $\frac{\infty}{\infty}$. From the graph in Figure 3.18, it appears that the function tends to 0 as $x \to 0^+$. Applying l'Hôpital's Rule, we have

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\csc x)} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x} \quad \left(\frac{\infty}{\infty}\right).$$

This last limit still has the indeterminate form $\frac{\infty}{\infty}$, but rather than apply l'Hôpital's Rule again, observe that we can rewrite the expression. (Do this wherever possible when a limit expression gets too complicated.) We have

$$\lim_{x\to 0^+}\frac{\ln x}{\csc x} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x} = \lim_{x\to 0^+}\left(-\frac{\sin x}{x}\tan x\right) = (-1)(0) = 0,$$

as expected, where we have used the fact (established in section 2.6) that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

(You can also establish this by using l'Hôpital's Rule.) Notice that if we had simply blasted away with further applications of l'Hôpital's Rule to $\lim\limits_{x\to 0^+}\dfrac{\frac{1}{x}}{-\csc x\cot x}$, we would *never* have resolved the limit. (Why not?)

Other Indeterminate Forms

There are five additional indeterminate forms to consider: $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞ and ∞^0. Look closely at each of these to see why they are indeterminate. When evaluating a limit of this type, the objective is to somehow reduce it to one of the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, at which point we can apply l'Hôpital's Rule.

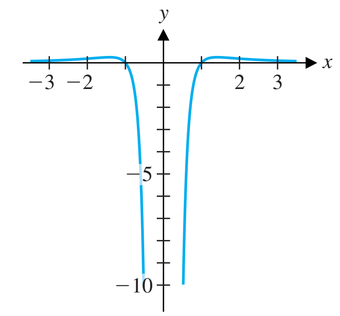


FIGURE 3.19
$y = \dfrac{1}{x^2} - \dfrac{1}{x^4}$

EXAMPLE 2.6 The Indeterminate Form $\infty - \infty$

Evaluate $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4} \right)$.

Solution First, notice that the limit has the form $(\infty - \infty)$. From the graph of the function in Figure 3.19, it appears that the function values tend to $-\infty$ as $x \to 0$. If we add the fractions, we get

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4} \right) = \lim_{x \to 0} \left(\frac{x^2 - 1}{x^4} \right) = -\infty,$$

as conjectured, where we have resolved the limit without using l'Hôpital's Rule, which does not apply here. (Why not?) ■

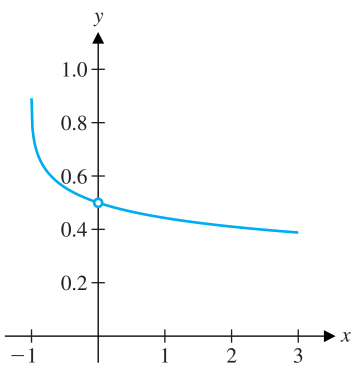


FIGURE 3.20
$y = \dfrac{1}{\ln(x+1)} - \dfrac{1}{x}$

EXAMPLE 2.7 The Indeterminate Form $\infty - \infty$

Evaluate $\lim\limits_{x \to 0} \left[\dfrac{1}{\ln(x+1)} - \dfrac{1}{x} \right]$.

Solution In this case, the limit has the form $(\infty - \infty)$. From the graph in Figure 3.20, it appears that the limit is somewhere around 0.5. If we add the fractions, we get a form to which we can apply l'Hôpital's Rule. We have

$$\begin{aligned}
\lim_{x \to 0} \left[\frac{1}{\ln(x+1)} - \frac{1}{x} \right] &= \lim_{x \to 0} \frac{x - \ln(x+1)}{\ln(x+1)x} \qquad \left(\frac{0}{0}\right) \\
&= \lim_{x \to 0} \frac{\frac{d}{dx}[x - \ln(x+1)]}{\frac{d}{dx}[\ln(x+1)x]} \qquad \text{By l'Hôpital's Rule.} \\
&= \lim_{x \to 0} \frac{1 - \frac{1}{x+1}}{\left(\frac{1}{x+1}\right)x + \ln(x+1)(1)} \qquad \left(\frac{0}{0}\right).
\end{aligned}$$

Rather than apply l'Hôpital's Rule to this last expression, we first simplify the expression, by multiplying top and bottom by $(x + 1)$. We now have

$$\begin{aligned}
\lim_{x \to 0} \left[\frac{1}{\ln(x+1)} - \frac{1}{x} \right] &= \lim_{x \to 0} \frac{1 - \frac{1}{x+1}}{\left(\frac{1}{x+1}\right)x + \ln(x+1)(1)} \left(\frac{x+1}{x+1}\right) \\
&= \lim_{x \to 0} \frac{(x+1) - 1}{x + (x+1)\ln(x+1)} \qquad \left(\frac{0}{0}\right) \\
&= \lim_{x \to 0} \frac{\frac{d}{dx}(x)}{\frac{d}{dx}[x + (x+1)\ln(x+1)]} \qquad \text{By l'Hôpital's Rule.} \\
&= \lim_{x \to 0} \frac{1}{1 + (1)\ln(x+1) + (x+1)\frac{1}{(x+1)}} = \frac{1}{2},
\end{aligned}$$

which is consistent with Figure 3.20. ■

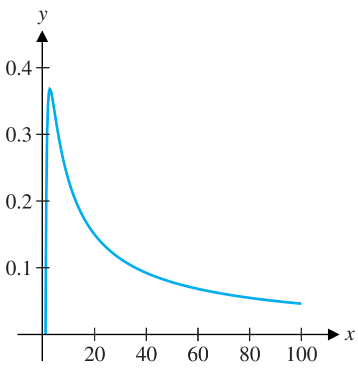


FIGURE 3.21

$y = \frac{1}{x} \ln x$

EXAMPLE 2.8 The Indeterminate Form $0 \cdot \infty$

Evaluate $\lim_{x\to\infty} \left(\frac{1}{x} \ln x\right)$.

Solution This limit has the indeterminate form $(0 \cdot \infty)$. From the graph in Figure 3.21, it appears that the function is decreasing very slowly toward 0 as $x \to \infty$. It's easy to rewrite this in the form $\frac{\infty}{\infty}$, from which we can use l'Hôpital's Rule. Note that

$$\lim_{x\to\infty} \left(\frac{1}{x} \ln x\right) = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x} \quad \text{By l'Hôpital's Rule.}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0. \blacksquare$$

Note: If $\lim_{x\to c} [f(x)]^{g(x)}$ has one of the indeterminate forms 0^0, ∞^0 or 1^∞, then, letting $y = [f(x)]^{g(x)}$, we have for $f(x) > 0$ that

$$\ln y = \ln[f(x)]^{g(x)} = g(x) \ln [f(x)],$$

so that $\lim_{x\to c} \ln y = \lim_{x\to c} \{g(x) \ln [f(x)]\}$ will have the indeterminate form $0 \cdot \infty$, which we can deal with as in example 2.8.

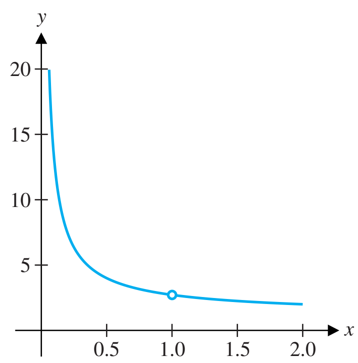


FIGURE 3.22

$y = x^{\frac{1}{x-1}}$

EXAMPLE 2.9 The Indeterminate Form 1^∞

Evaluate $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$.

Solution First, note that this limit has the indeterminate form (1^∞). From the graph in Figure 3.22, it appears that the limit is somewhere around 3. We define $y = x^{\frac{1}{x-1}}$, so that

$$\ln y = \ln x^{\frac{1}{x-1}} = \frac{1}{x-1} \ln x.$$

We now consider the limit

$$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{1}{x-1} \ln x \quad (\infty \cdot 0)$$

$$= \lim_{x\to 1^+} \frac{\ln x}{x-1} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1^+} \frac{x^{-1}}{1} = 1. \quad \text{By l'Hôpital's Rule.}$$

Be careful; we have found that $\lim_{x\to 1^+} \ln y = 1$, but this is not the original limit. We want

$$\lim_{x\to 1^+} y = \lim_{x\to 1^+} e^{\ln y} = e^1,$$

which is consistent with Figure 3.22. $\blacksquare$

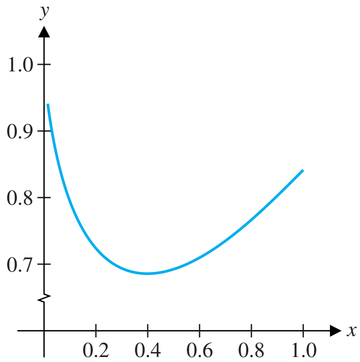


FIGURE 3.23
$y = (\sin x)^x$

Often, the computation of a limit in this form requires several applications of l'Hôpital's Rule. Just be careful (in particular, verify the hypotheses at every step) and do not lose sight of the original problem.

EXAMPLE 2.10 The Indeterminate Form 0^0

Evaluate $\lim\limits_{x \to 0^+} (\sin x)^x$.

Solution This limit has the indeterminate form (0^0). In Figure 3.23, it appears that the limit is somewhere around 1. We let $y = (\sin x)^x$, so that

$$\ln y = \ln (\sin x)^x = x \ln (\sin x).$$

Now consider the limit

$$\begin{aligned}
\lim_{x \to 0^+} \ln y &= \lim_{x \to 0^+} \ln (\sin x)^x = \lim_{x \to 0^+} [x \ln (\sin x)] \quad (0 \cdot \infty) \\
&= \lim_{x \to 0^+} \frac{\ln (\sin x)}{\left(\frac{1}{x}\right)} \quad \left(\frac{\infty}{\infty}\right) \\
&= \lim_{x \to 0^+} \frac{\frac{d}{dx}[\ln (\sin x)]}{\frac{d}{dx}(x^{-1})} \quad \text{By l'Hôpital's Rule.} \\
&= \lim_{x \to 0^+} \frac{(\sin x)^{-1} \cos x}{-x^{-2}} \quad \left(\frac{\infty}{\infty}\right).
\end{aligned}$$

As we have seen earlier, we should rewrite the expression before proceeding. Here, we multiply top and bottom by $x^2 \sin x$ to get

$$\begin{aligned}
\lim_{x \to 0^+} \ln y &= \lim_{x \to 0^+} \frac{(\sin x)^{-1} \cos x}{-x^{-2}} \left(\frac{x^2 \sin x}{x^2 \sin x}\right) \\
&= \lim_{x \to 0^+} \frac{-x^2 \cos x}{\sin x} \quad \left(\frac{0}{0}\right) \\
&= \lim_{x \to 0^+} \frac{\frac{d}{dx}(-x^2 \cos x)}{\frac{d}{dx}(\sin x)} \quad \text{By l'Hôpital's Rule.} \\
&= \lim_{x \to 0^+} \frac{-2x \cos x + x^2 \sin x}{\cos x} = \frac{0}{1} = 0.
\end{aligned}$$

Again, we have not yet found the original limit. However,

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^0 = 1,$$

which is consistent with Figure 3.23. ■

EXAMPLE 2.11 The Indeterminate Form ∞^0

Evaluate $\lim\limits_{x \to \infty} (x + 1)^{2/x}$.

Solution This limit has the indeterminate form (∞^0). From the graph in Figure 3.24, it appears that the function tends to a limit around 1 as $x \to \infty$. We let $y = (x + 1)^{2/x}$

TODAY IN MATHEMATICS

Vaughan Jones (1952–)
A New Zealand mathematician whose work has connected apparently disjoint areas of mathematics. He was awarded the Fields Medal in 1990 for mathematics that was described by peers as 'astonishing'. One of his major accomplishments is a discovery in knot theory that has given biologists insight into the replication of DNA. A strong supporter of science and mathematics education in New Zealand, Jones' "style of working is informal, and one which encourages the free and open interchange of ideas . . . His openness and generosity in this regard have been in the best tradition and spirit of mathematics." His ideas have "served as a rich source of ideas for the work of others."

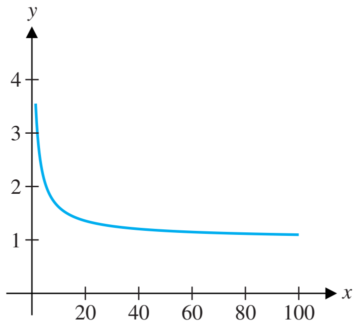


FIGURE 3.24
$y = (x+1)^{2/x}$

and consider

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \ln (x+1)^{2/x} = \lim_{x\to\infty} \left[\frac{2}{x} \ln (x+1)\right] \quad (0\cdot\infty)$$

$$= \lim_{x\to\infty} \frac{2\ln(x+1)}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}[2\ln(x+1)]}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{2(x+1)^{-1}}{1} \qquad \text{By l'Hôpital's Rule.}$$

$$= \lim_{x\to\infty} \frac{2}{x+1} = 0.$$

We now have that
$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = 1,$$

as expected. ■

BEYOND FORMULAS

On several occasions, we have had the benefit of rewriting a function or an equation into a more convenient form. L'Hôpital's Rule gives us a way to rewrite limits that are indeterminate. As we have seen, sometimes just the act of rewriting a function as a single fraction lets us determine the limit. What are other examples where an equation is made easier by rewriting, perhaps with a trigonometric identity or exponential rule?

EXERCISES 3.2

WRITING EXERCISES

1. L'Hôpital's Rule states that, in certain situations, the ratios of function values approach the same limits as the ratios of corresponding derivatives (rates of change). Graphically, this may be hard to understand. To get a handle on this, consider $\dfrac{f(x)}{g(x)}$ where both $f(x) = ax + b$ and $g(x) = cx + d$ are linear functions. Explain why the value of $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ should depend on the relative sizes of the slopes of the lines; that is, it should be equal to $\lim\limits_{x\to\infty} \dfrac{f'(x)}{g'(x)}$.

2. Think of a limit of 0 as actually meaning "getting very small" and a limit of ∞ as meaning "getting very large." Discuss whether the following limit forms are indeterminate or not and explain your answer: $\infty - \infty$, $\frac{1}{0}$, $0\cdot\infty$, $\infty\cdot\infty$, ∞^0, 0^∞ and 0^0.

3. A friend is struggling with l'Hôpital's Rule. When asked to work a problem, your friend says, "First, I plug in for x and get 0 over 0. Then I use the quotient rule to take the derivative. Then I plug x back in." Explain to your friend what the mistake is and how to correct it.

4. Suppose that two runners begin a race from the starting line, with one runner initially going twice as fast as the other. If $f(t)$ and $g(t)$ represent the positions of the runners at time $t \ge 0$, explain why we can assume that $f(0) = g(0) = 0$ and $\lim\limits_{t\to 0^+} \dfrac{f'(t)}{g'(t)} = 2$. Explain in terms of the runners' positions why l'Hôpital's Rule holds: that is, $\lim\limits_{t\to 0} \dfrac{f(t)}{g(t)} = 2$.

In exercises 1–38, find the indicated limits.

1. $\lim\limits_{x\to -2} \dfrac{x+2}{x^2-4}$

2. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-3x+2}$

3. $\lim\limits_{x\to\infty} \dfrac{3x^2+2}{x^2-4}$

4. $\lim\limits_{x\to -\infty} \dfrac{x+1}{x^2+4x+3}$

5. $\lim_{x\to 0} \frac{e^{2x}-1}{x}$

6. $\lim_{x\to 0} \frac{\sin x}{e^{3x}-1}$

7. $\lim_{x\to 0} \frac{\tan^{-1} x}{\sin x}$

8. $\lim_{x\to 0} \frac{\sin x}{\sin^{-1} x}$

9. $\lim_{x\to \pi} \frac{\sin 2x}{\sin x}$

10. $\lim_{x\to -1} \frac{\cos^{-1} x}{x^2-1}$

11. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

12. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

13. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

14. $\lim_{x\to 1} \frac{\ln x}{x-1}$

15. $\lim_{x\to \infty} \frac{x^3}{e^x}$

16. $\lim_{x\to \infty} \frac{e^x}{x^4}$

17. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x \sin^2 x}$

18. $\lim_{x\to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$

19. $\lim_{x\to 1} \frac{\sin \pi x}{x-1}$

20. $\lim_{x\to 1} \frac{e^{x-1}-1}{x^2-1}$

21. $\lim_{x\to \infty} \frac{\ln x}{x^2}$

22. $\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

23. $\lim_{x\to \infty} x e^{-x}$

24. $\lim_{x\to \infty} x \sin(1/x)$

25. $\lim_{x\to 1} \frac{\ln(\ln x)}{\ln x}$

26. $\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}$

27. $\lim_{x\to 0^+} x \ln x$

28. $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$

29. $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$

30. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\ln x}$

31. $\lim_{x\to \infty} (\sqrt{x^2+1} - x)$

32. $\lim_{x\to \infty} (\ln x - x)$

33. $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x$

34. $\lim_{x\to \infty} \left|\frac{x+1}{x-2}\right|^{\sqrt{x^2-4}}$

35. $\lim_{x\to 0^+} \left(\frac{1}{\sqrt{x}} - \sqrt{\frac{x}{x+1}}\right)$

36. $\lim_{x\to 1} \frac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$

37. $\lim_{x\to 0^+} (1/x)^x$

38. $\lim_{x\to 0^+} (\cos x)^{1/x}$

In exercises 39 and 40, find the error(s) in the incorrect calculations.

39. $\lim_{x\to 0} \frac{\cos x}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$

40. $\lim_{x\to 0} \frac{e^x-1}{x^2} = \lim_{x\to 0} \frac{e^x}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

41. Find all errors in the string

$$\lim_{x\to 0} \frac{x^2}{\ln x^2} = \lim_{x\to 0} \frac{x^2}{2\ln x} = \lim_{x\to 0} \frac{2x}{2/x} = \lim_{x\to 0} \frac{2}{-2/x^2} = \lim_{x\to 0}(-x^2) = 0.$$

Then, determine the correct value of the limit.

42. Find all errors in the string

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

Then, determine the correct value of the limit.

43. Starting with $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$, cancel sin to get $\lim_{x\to 0} \frac{3x}{2x}$, then cancel x's to get $\frac{3}{2}$. This answer is correct. Is either of the steps used valid? Use linear approximations to argue that the first step is likely to give a correct answer.

44. Evaluate $\lim_{x\to 0} \frac{\sin nx}{\sin mx}$ for nonzero constants n and m.

45. (a) Compute $\lim_{x\to 0} \frac{\sin x^2}{x^2}$ and compare your result to $\lim_{x\to 0} \frac{\sin x}{x}$.

(b) Compute $\lim_{x\to 0} \frac{1-\cos x^2}{x^4}$ and compare your result to $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

46. Use your results from exercise 45 to evaluate $\lim_{x\to 0} \frac{\sin x^3}{x^3}$ and $\lim_{x\to 0} \frac{1-\cos x^3}{x^6}$ without doing any calculations.

47. Show that $\lim_{x\to 0} \frac{\sin kx^2}{x^2}$ has the indeterminate form $\frac{0}{0}$ and then evaluate the limit (where k is some real number). What is the range of values that a limit of the indeterminate form $\frac{0}{0}$ can have?

48. Show that $\lim_{x\to 0} \frac{\cot kx^2}{\csc x^2}$ has the indeterminate form $\frac{\infty}{\infty}$ and then evaluate the limit (where k is some real number). What is the range of values that a limit of the indeterminate form $\frac{\infty}{\infty}$ can have?

In exercises 49 and 50, determine which function "dominates," where we say that the function $f(x)$ dominates the function $g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and either $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ or $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.

49. e^x or x^n ($n =$ any positive integer)

50. $\ln x$ or x^p (for any number $p > 0$)

51. Evaluate $\lim_{x\to 0} \frac{e^{cx}-1}{x}$ for any constant c.

52. Evaluate $\lim_{x\to 0} \frac{\tan cx - cx}{x^3}$ for any constant c.

53. If $\lim_{x\to 0} \frac{f(x)}{g(x)} = L$, what can be said about $\lim_{x\to 0} \frac{f(x^2)}{g(x^2)}$? Explain why knowing that $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ for $a \neq 0, 1$ does not tell you anything about $\lim_{x\to a} \frac{f(x^2)}{g(x^2)}$.

54. Give an example of functions f and g for which $\lim_{x\to 0} \frac{f(x^2)}{g(x^2)}$ exists, but $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist.

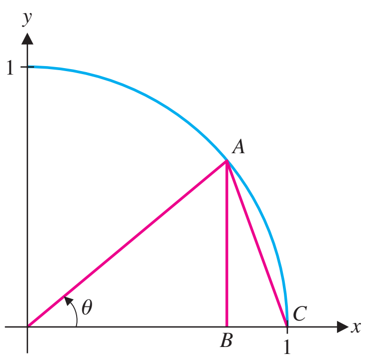
55. In section 1.2, we briefly discussed the position of a baseball thrown with the unusual knuckleball pitch. The left/right position (in feet) of a ball thrown with spin rate ω and a particular grip at time t seconds is $f(\omega) = (2.5/\omega)t - (2.5/4\omega^2)\sin 4\omega t$. Treating t as a constant and ω as the variable (change to x if you like), show that $\lim_{\omega \to 0} f(\omega) = 0$ for any value of t. (Hint: Find a common denominator and use l'Hôpital's Rule.) Conclude that this pitch does not move left or right at all.

56. In this exercise, we look at a knuckleball thrown with a different grip than that of exercise 55. The left or right position (in feet) of a ball thrown with spin rate ω and this new grip at time t seconds is $f(\omega) = (2.5/4\omega^2) - (2.5/4\omega^2)\sin(4\omega t + \pi/2)$. Treating t as a constant and ω as the variable (change to x if you like), find $\lim_{\omega \to 0} f(\omega)$. Your answer should depend on t. By graphing this function of t, you can see the path of the pitch (use a domain of $0 \le t \le 0.68$). Describe this pitch.

57. Find functions f such that $\lim_{x \to \infty} f(x)$ has the indeterminate form $\frac{\infty}{\infty}$, but where the limit (a) does not exist; (b) equals 0; (c) equals 3 and (d) equals -4.

58. Find functions f such that $\lim_{x \to \infty} f(x)$ has the indeterminate form $\infty - \infty$, but where the limit (a) does not exist; (b) equals 0 and (c) equals 2.

59. In the figure shown here, a section of the unit circle is determined by angle θ. Region 1 is the triangle ABC. Region 2 is bounded by the line segments AB and BC and the arc of the circle. As the angle θ decreases, the difference between the two regions decreases, also. You might expect that the areas of the regions become nearly equal, in which case the ratio of the areas approaches 1. To see what really happens, show that the area of region 1 divided by the area of region 2 equals $\frac{(1 - \cos\theta)\sin\theta}{\theta - \cos\theta \sin\theta} = \frac{\sin\theta - \frac{1}{2}\sin 2\theta}{\theta - \frac{1}{2}\sin 2\theta}$ and find the limit of this expression as $\theta \to 0$. Surprise!



Exercise 59

60. The size of an animal's pupils expand and contract depending on the amount of light available. Let $f(x) = \frac{160x^{-0.4} + 90}{8x^{-0.4} + 10}$ be the size in mm of the pupils at light intensity x. Find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to \infty} f(x)$, and argue that these represent the largest and smallest possible sizes of the pupils, respectively.

EXPLORATORY EXERCISES

1. In this exercise, you take a quick look at what we call **Taylor series** in Chapter 8. Start with the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Briefly explain why this means that for x close to 0, $\sin x \approx x$. Graph $y = \sin x$ and $y = x$ to see why this is true. If you look far enough away from $x = 0$, the graph of $y = \sin x$ eventually curves noticeably. Let's see if we can find polynomials of higher order to match this curving. Show that $\lim_{x \to 0} \frac{\sin x - x}{x^2} = 0$. This means that $\sin x - x \approx 0$ or (again) $\sin x \approx x$. Show that $\lim_{x \to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}$. This says that if x is close to 0, then $\sin x - x \approx -\frac{1}{6}x^3$ or $\sin x \approx x - \frac{1}{6}x^3$. Graph these two functions to see how well they match up. To continue, compute $\lim_{x \to 0} \frac{\sin x - (x - x^3/6)}{x^4}$ and $\lim_{x \to 0} \frac{\sin x - f(x)}{x^5}$ for the appropriate approximation $f(x)$. At this point, look at the pattern of terms you have (Hint: $6 = 3!$ and $120 = 5!$). Using this pattern, approximate $\sin x$ with an 11th-degree polynomial and graph the two functions.

2. A *zero* of a function $f(x)$ is a solution of the equation $f(x) = 0$. Clearly, not all zeros are created equal. For example, $x = 1$ is a zero of $f(x) = x - 1$, but in some ways it seems that $x = 1$ should count as two zeros of $f(x) = (x - 1)^2$. To quantify this, we say that $x = 1$ is a **zero of multiplicity 2** of $f(x) = (x - 1)^2$. The precise definition is: $x = c$ is a **zero of multiplicity** n of $f(x)$ if $f(c) = 0$ and $\lim_{x \to c} \frac{f(x)}{(x - c)^n}$ exists and is nonzero. Thus, $x = 0$ is a zero of multiplicity 2 of $x \sin x$ since $\lim_{x \to 0} \frac{x \sin x}{x^2} = \lim_{x \to 0} \frac{\sin x}{x} = 1$. Find the multiplicity of each zero of the following functions: $x^2 \sin x$, $x \sin x^2$, $x^4 \sin x^3$, $(x - 1)\ln x$, $\ln(x - 1)^2$, $e^x - 1$ and $\cos x - 1$.

3.3 MAXIMUM AND MINIMUM VALUES

To remain competitive in today's global economy, businesses need to minimize waste and maximize the return on their investment. In the extremely competitive personal computer industry, companies must continually evaluate how low they can afford to set their prices and still earn a profit. With this backdrop, it should be apparent that it is increasingly important to use mathematical methods to maximize and minimize various quantities of interest. In this section, we investigate the notion of maximum and minimum from a purely mathematical standpoint. In section 3.7, we examine how to apply these notions to problems of an applied nature.

We begin by giving careful mathematical definitions of some familiar terms.

DEFINITION 3.1

For a function f defined on a set S of real numbers and a number $c \in S$,

(i) $f(c)$ is the **absolute maximum** of f on S if $f(c) \geq f(x)$ for all $x \in S$ and
(ii) $f(c)$ is the **absolute minimum** of f on S if $f(c) \leq f(x)$ for all $x \in S$.

An absolute maximum or an absolute minimum is referred to as an **absolute extremum.** If a function has more than one extremum, we refer to these as **extrema** (the plural form of extremum).

The first question you might ask is whether every function has an absolute maximum and an absolute minimum. The answer is no, as we can see from Figures 3.25a and 3.25b.

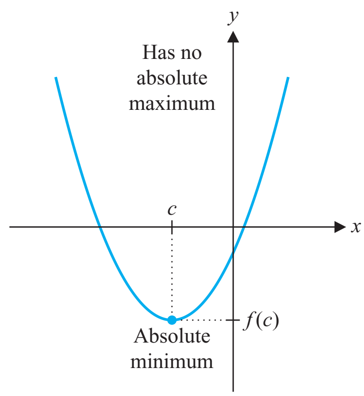


FIGURE 3.25a

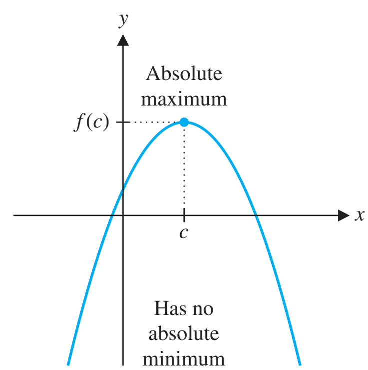


FIGURE 3.25b

EXAMPLE 3.1 Absolute Maximum and Minimum Values

(a) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-\infty, \infty)$. (b) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-3, 3)$. (c) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $[-3, 3]$.

Solution (a) In Figure 3.26, notice that f has an absolute minimum value of $f(0) = -9$, but has no absolute maximum value.

(b) In Figure 3.27a, we see that f has an absolute minimum value of $f(0) = -9$, but still has no absolute maximum value. Your initial reaction might be to say that f has an absolute maximum of 0, but $f(x) \neq 0$ for any $x \in (-3, 3)$, since this is an open interval and hence, does not include the endpoints -3 and 3.

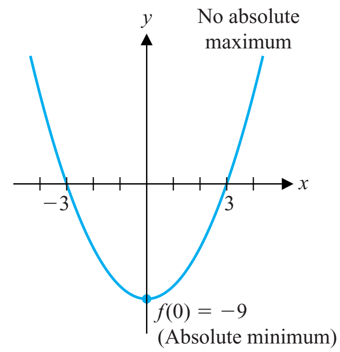


FIGURE 3.26
$y = x^2 - 9$ on $(-\infty, \infty)$

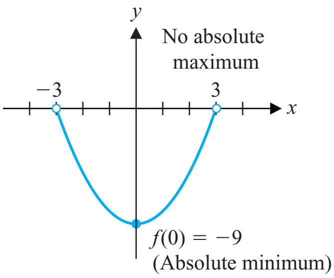


FIGURE 3.27a
$y = x^2 - 9$ on $(-3, 3)$

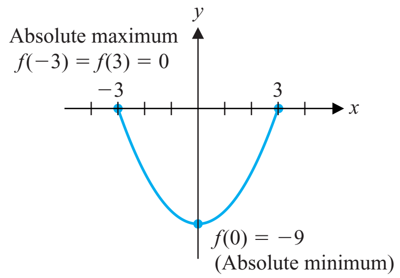


FIGURE 3.27b
$y = x^2 - 9$ on $[-3, 3]$

(c) In this case, the endpoints 3 and −3 are in the interval $[-3, 3]$. Here, f assumes its absolute maximum at two points: $f(3) = f(-3) = 0$ (see Figure 3.27b). ■

We have seen that a function may or may not have absolute extrema, depending on the interval on which we're looking. In example 3.1, the function failed to have an absolute maximum, except on the closed, bounded interval $[-3, 3]$. This provides some clues, and example 3.2 provides another piece of the puzzle.

EXAMPLE 3.2 A Function with No Absolute Maximum or Minimum

Locate any absolute extrema of $f(x) = 1/x$, on the interval $[-3, 3]$.

Solution From the graph in Figure 3.28, f clearly fails to have either an absolute maximum or an absolute minimum on $[-3, 3]$. The following table of values for $f(x)$ for x close to 0 suggests the same conclusion.

x	$1/x$
1	1
0.1	10
0.01	100
0.001	1000
0.0001	10,000
0.00001	100,000
0.000001	1,000,000

x	$1/x$
−1	−1
−0.1	−10
−0.01	−100
−0.001	−1000
−0.0001	−10,000
−0.00001	−100,000
−0.000001	−1,000,000

■

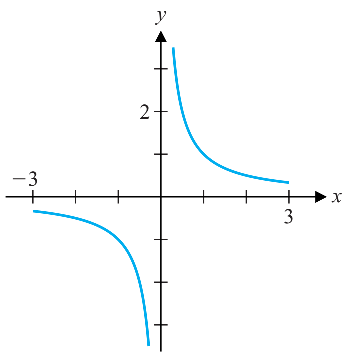


FIGURE 3.28
$y = 1/x$

The most obvious difference between the functions in examples 3.1 and 3.2 is that $f(x) = 1/x$ is discontinuous at a point in the interval $[-3, 3]$. We offer the following theorem without proof.

THEOREM 3.1 (Extreme Value Theorem)

A continuous function f defined on a *closed, bounded* interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.

While you do not need to have a continuous function or a closed interval to have an absolute extremum, Theorem 3.1 says that continuous functions are *guaranteed* to have an absolute maximum and an absolute minimum on a closed, bounded interval.

In example 3.3, we revisit the function from example 3.2, but look on a different interval.

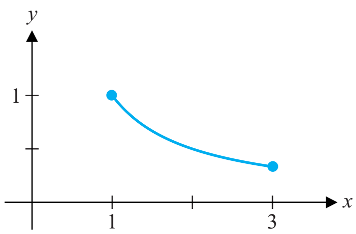


FIGURE 3.29
$y = 1/x$ on $[1, 3]$

EXAMPLE 3.3 Finding Absolute Extrema of a Continuous Function

Find the absolute extrema of $f(x) = 1/x$ on the interval $[1, 3]$.

Solution Notice that on the interval $[1, 3]$, f is continuous. Consequently, the Extreme Value Theorem guarantees that f has both an absolute maximum and an absolute minimum on $[1, 3]$. Judging from the graph in Figure 3.29, it appears that $f(x)$ reaches its maximum value of 1 at $x = 1$ and its minimum value of $1/3$ at $x = 3$. ■

Our objective is to determine how to locate the absolute extrema of a given function. Before we do this, we need to consider an additional type of extremum.

DEFINITION 3.2

(i) $f(c)$ is a **local maximum** of f if $f(c) \geq f(x)$ for all x in some *open* interval containing c.

(ii) $f(c)$ is a **local minimum** of f if $f(c) \leq f(x)$ for all x in some *open* interval containing c.

In either case, we call $f(c)$ a **local extremum** of f.

Local maxima and minima (the plural forms of maximum and minimum, respectively) are sometimes referred to as **relative** maxima and minima, respectively.

Notice from Figure 3.30 that each local extremum seems to occur either at a point where the tangent line is horizontal [i.e., where $f'(x) = 0$], at a point where the tangent line is vertical [where $f'(x)$ is undefined] or at a corner [again, where $f'(x)$ is undefined]. We can see this behavior quite clearly in examples 3.4 and 3.5.

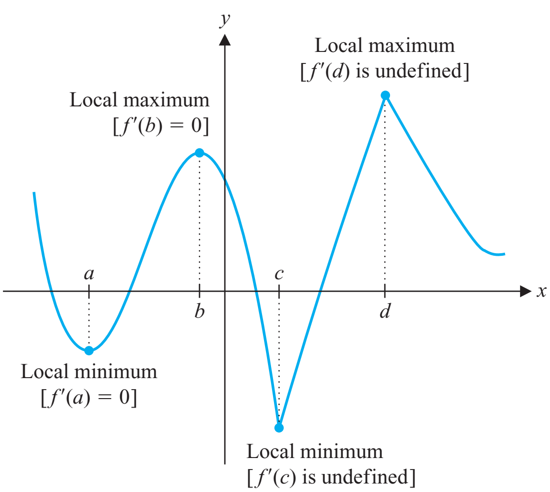


FIGURE 3.30
Local extrema

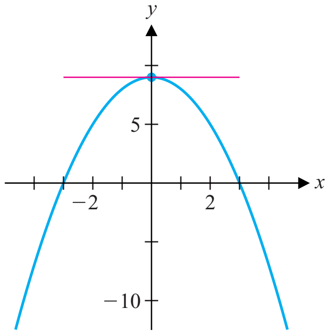


FIGURE 3.31
$y = 9 - x^2$ and the tangent line at $x = 0$

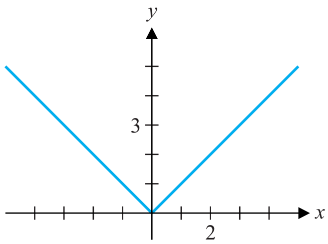


FIGURE 3.32
$y = |x|$

EXAMPLE 3.4 A Function with a Zero Derivative at a Local Maximum

Locate any local extrema for $f(x) = 9 - x^2$ and describe the behavior of the derivative at the local extremum.

Solution We can see from Figure 3.31 that there is a local maximum at $x = 0$. Further, note that $f'(x) = -2x$ and so, $f'(0) = 0$. Note that this says that the tangent line to $y = f(x)$ at $x = 0$ is horizontal, as indicated in Figure 3.31. ■

EXAMPLE 3.5 A Function with an Undefined Derivative at a Local Minimum

Locate any local extrema for $f(x) = |x|$ and describe the behavior of the derivative at the local extremum.

Solution We can see from Figure 3.32 that there is a local minimum at $x = 0$. As we have noted in section 2.1, the graph has a corner at $x = 0$ and hence, $f'(0)$ is undefined. [See example 1.7 in Chapter 2.] ■

The graphs shown in Figures 3.30–3.32 are not unusual. Here is a small challenge: spend a little time now drawing graphs of functions with local extrema. It should not take long to convince yourself that local extrema occur only at points where the derivative is either zero or undefined. Because of this, we give these points a special name.

DEFINITION 3.3

A number c in the domain of a function f is called a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

It turns out that our earlier observation regarding the location of extrema is correct. That is, local extrema occur *only* at points where the derivative is zero or undefined. We state this formally in Theorem 3.2.

THEOREM 3.2 (Fermat's Theorem)

Suppose that $f(c)$ is a local extremum (local maximum or local minimum). Then c must be a critical number of f.

PROOF

Suppose that f is differentiable at $x = c$. (If not, c is a critical number of f and we are done.) Suppose further that $f'(c) \neq 0$. Then, either $f'(c) > 0$ or $f'(c) < 0$.

If $f'(c) > 0$, we have by the definition of derivative that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} > 0.$$

So, for all h sufficiently small,

$$\frac{f(c+h) - f(c)}{h} > 0. \tag{3.1}$$

HISTORICAL NOTES

Pierre de Fermat (1601–1665) A French mathematician who discovered many important results, including the theorem named for him. Fermat was a lawyer and member of the Toulouse supreme court, with mathematics as a hobby. The "Prince of Amateurs" left an unusual legacy by writing in the margin of a book that he had discovered a wonderful proof of a clever result, but that the margin of the book was too small to hold the proof. Fermat's Last Theorem confounded many of the world's best mathematicians for more than 300 years before being proved by Andrew Wiles in 1995.

TODAY IN MATHEMATICS

Andrew Wiles (1953–)
A British mathematician who in 1995 published a proof of Fermat's Last Theorem, the most famous unsolved problem of the 20th century. Fermat's Last Theorem states that there is no integer solution x, y and z of the equation $x^n + y^n = z^n$ for integers $n > 2$. Wiles had wanted to prove the theorem since reading about it as a 10-year-old. After more than ten years as a successful research mathematician, Wiles isolated himself from colleagues for seven years as he developed the mathematics needed for his proof. "I realised that talking to people casually about Fermat was impossible because it generated too much interest. You cannot focus yourself for years unless you have this kind of undivided concentration which too many spectators would destroy." The last step of his proof came, after a year of intense work on this one step, as "this incredible revelation" that was "so indescribably beautiful, it was so simple and elegant."

For $h > 0$, (3.1) says that
$$f(c+h) - f(c) > 0$$
and so,
$$f(c+h) > f(c).$$
Thus, $f(c)$ is not a local maximum.

For $h < 0$, (3.1) says that
$$f(c+h) - f(c) < 0$$
and so,
$$f(c+h) < f(c).$$
Thus, $f(c)$ is not a local minimum.

Since we had assumed that $f(c)$ was a local extremum, this is a contradiction. This rules out the possibility that $f'(c) > 0$.

Similarly, if $f'(c) < 0$, we obtain the same contradiction. This is left as an exercise. The only remaining possibility is to have $f'(c) = 0$ and this proves the theorem. ■

We can use Fermat's Theorem and calculator- or computer-generated graphs to find local extrema, as in examples 3.6 and 3.7.

EXAMPLE 3.6 Finding Local Extrema of a Polynomial

Find the critical numbers and local extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$.

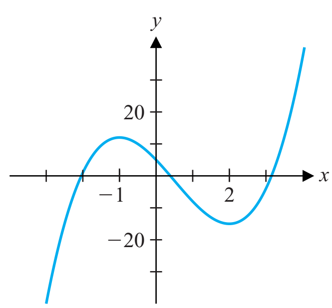


FIGURE 3.33
$y = 2x^3 - 3x^2 - 12x + 5$

Solution Here,
$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$$
$$= 6(x-2)(x+1).$$
Thus, f has two critical numbers, $x = -1$ and $x = 2$. Notice from the graph in Figure 3.33 that these correspond to the locations of a local maximum and a local minimum, respectively. ■

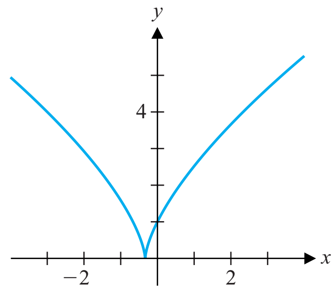


FIGURE 3.34
$y = (3x+1)^{2/3}$

EXAMPLE 3.7 An Extremum at a Point Where the Derivative Is Undefined

Find the critical numbers and local extrema of $f(x) = (3x+1)^{2/3}$.

Solution Here, we have

$$f'(x) = \frac{2}{3}(3x+1)^{-1/3}(3) = \frac{2}{(3x+1)^{1/3}}.$$

Of course, $f'(x) \neq 0$ for all x, but $f'(x)$ is undefined at $x = -\frac{1}{3}$. Be sure to note that $-\frac{1}{3}$ is in the domain of f. Thus, $x = -\frac{1}{3}$ is the only critical number of f. From the graph in Figure 3.34, we see that this corresponds to the location of a local minimum (also the absolute minimum). If you use your graphing utility to try to produce a graph of $y = f(x)$, you may get only half of the graph displayed in Figure 3.34. The reason is that the algorithms used by most calculators and many computers will return a complex number (or an error) when asked to compute certain fractional powers of negative numbers. While this annoying shortcoming presents only occasional difficulties, we mention this here only so that you are aware that technology has limitations. ■

REMARK 3.1

Fermat's Theorem says that local extrema can occur only at critical numbers. This does not say that there is a local extremum at *every* critical number. In fact, this is false, as we illustrate in examples 3.8 and 3.9.

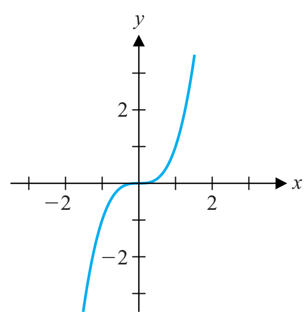


FIGURE 3.35
$y = x^3$

EXAMPLE 3.8 A Horizontal Tangent at a Point That Is Not a Local Extremum

Find the critical numbers and local extrema of $f(x) = x^3$.

Solution It should be clear from Figure 3.35 that f has no local extrema. However, $f'(x) = 3x^2 = 0$ for $x = 0$ (the only critical number of f). In this case, f has a horizontal tangent line at $x = 0$, but does not have a local extremum there. ■

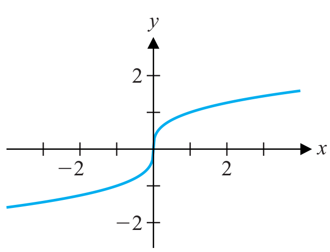


FIGURE 3.36
$y = x^{1/3}$

EXAMPLE 3.9 A Vertical Tangent at a Point That Is Not a Local Extremum

Find the critical numbers and local extrema of $f(x) = x^{1/3}$.

Solution As in example 3.8, f has no local extrema (see Figure 3.36). Here, $f'(x) = \frac{1}{3}x^{-2/3}$ and so, f has a critical number at $x = 0$ (in this case the derivative is undefined at $x = 0$). However, f does not have a local extremum at $x = 0$. ■

You should always check that a given value is in the domain of the function before declaring it a critical number, as in example 3.10.

EXAMPLE 3.10 Finding Critical Numbers of a Rational Function

Find all the critical numbers of $f(x) = \dfrac{2x^2}{x+2}$.

Solution You should note that the domain of f consists of all real numbers other than $x = -2$. Here, we have

$$f'(x) = \frac{4x(x+2) - 2x^2(1)}{(x+2)^2} \qquad \text{From the quotient rule.}$$

$$= \frac{2x(x+4)}{(x+2)^2}.$$

Notice that $f'(x) = 0$ for $x = 0, -4$ and $f'(x)$ is undefined for $x = -2$. However, -2 is not in the domain of f and consequently, the only critical numbers are $x = 0$ and $x = -4$. ■

REMARK 3.2

When we use the terms maximum, minimum or extremum without specifying absolute or local, we will *always* be referring to absolute extrema.

We have observed that local extrema occur only at critical numbers and that continuous functions must have an absolute maximum and an absolute minimum on a closed, bounded interval. However, we haven't yet really been able to say how to find these extrema. Theorem 3.3 is particularly useful.

THEOREM 3.3

Suppose that f is continuous on the closed interval $[a, b]$. Then, the absolute extrema of f must occur at an endpoint (a or b) or at a critical number.

PROOF

First, recall that by the Extreme Value Theorem, f will attain its maximum and minimum values on $[a, b]$, since f is continuous. Let $f(c)$ be an absolute extremum. If c is *not* an endpoint (i.e., $c \neq a$ and $c \neq b$), then c must be in the open interval (a, b). Thus, $f(c)$ must be a local extremum, also. Finally, by Fermat's Theorem, c must be a critical number, since local extrema occur only at critical numbers. ■

REMARK 3.3

Theorem 3.3 gives us a simple procedure for finding the absolute extrema of a continuous function on a closed, bounded interval:

1. Find all critical numbers in the interval and compute function values at these points.
2. Compute function values at the endpoints.
3. The largest function value is the absolute maximum and the smallest function value is the absolute minimum.

We illustrate Theorem 3.3 for the case of a polynomial function in example 3.11.

EXAMPLE 3.11 Finding Absolute Extrema on a Closed Interval

Find the absolute extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$ on the interval $[-2, 4]$.

Solution From the graph in Figure 3.37, the maximum appears to be at the endpoint $x = 4$, while the minimum appears to be at a local minimum near $x = 2$. In example 3.6, we found that the critical numbers of f are $x = -1$ and $x = 2$. Further, both of these are in the interval $[-2, 4]$. So, we compare the values at the endpoints:

$$f(-2) = 1 \quad \text{and} \quad f(4) = 37,$$

and the values at the critical numbers:

$$f(-1) = 12 \quad \text{and} \quad f(2) = -15.$$

Since f is continuous on $[-2, 4]$, Theorem 3.3 says that the absolute extrema must be among these four values. Thus, $f(4) = 37$ is the absolute maximum and $f(2) = -15$ is the absolute minimum. Note that these values are consistent with what we see in the graph in Figure 3.37. ■

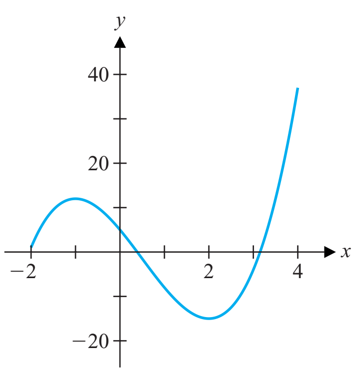


FIGURE 3.37
$y = 2x^3 - 3x^2 - 12x + 5$

Of course, most real problems of interest are unlikely to result in derivatives with integer zeros. Consider the following somewhat less user-friendly example.

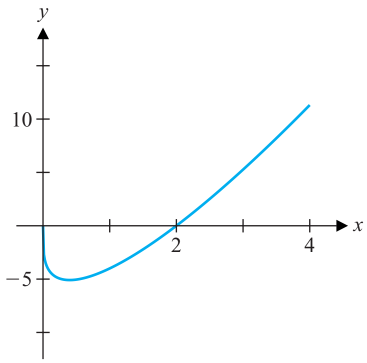


FIGURE 3.38
$y = 4x^{5/4} - 8x^{1/4}$

EXAMPLE 3.12 Finding Extrema for a Function with Fractional Exponents

Find the absolute extrema of $f(x) = 4x^{5/4} - 8x^{1/4}$ on the interval [0, 4].

Solution First, we draw a graph of the function to get an idea of where the extrema are located (see Figure 3.38). From the graph, it appears that the maximum occurs at the endpoint $x = 4$ and the minimum near $x = \frac{1}{2}$. Next, observe that

$$f'(x) = 5x^{1/4} - 2x^{-3/4} = \frac{5x - 2}{x^{3/4}}.$$

Thus, the critical numbers are $x = \frac{2}{5}$ $\left[\text{since } f'\left(\frac{2}{5}\right) = 0\right]$ and $x = 0$ (since $f'(0)$ is undefined and 0 is in the domain of f). We now need only compare

$$f(0) = 0, \quad f(4) \approx 11.3137 \quad \text{and} \quad f\left(\tfrac{2}{5}\right) \approx -5.0897.$$

So, the absolute maximum is $f(4) \approx 11.3137$ and the absolute minimum is $f\left(\frac{2}{5}\right) \approx -5.0897$, which is consistent with what we expected from Figure 3.38. ■

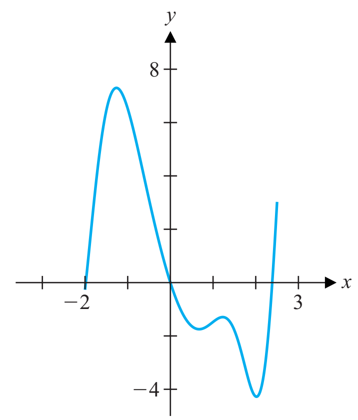


FIGURE 3.39
$y = f(x) = x^3 - 5x + 3\sin x^2$

In practice, the critical numbers are not always as easy to find as they were in examples 3.11 and 3.12. In example 3.13, it is not even known *how many* critical numbers there are. We can, however, estimate the number and locations of these from a careful analysis of computer-generated graphs.

EXAMPLE 3.13 Finding Absolute Extrema Approximately

Find the absolute extrema of $f(x) = x^3 - 5x + 3\sin x^2$ on the interval $[-2, 2.5]$.

Solution We first draw a graph to get an idea of where the extrema will be located (see Figure 3.39). From the graph, we can see that the maximum seems to occur near $x = -1$, while the minimum seems to occur near $x = 2$. Next, we compute

$$f'(x) = 3x^2 - 5 + 6x\cos x^2.$$

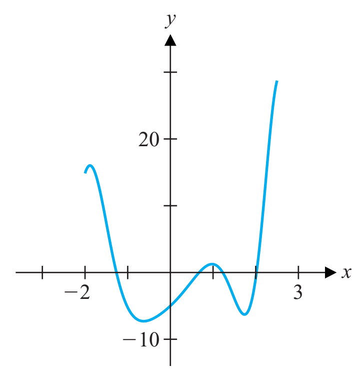


FIGURE 3.40
$y = f'(x) = 3x^2 - 5 + 6x\cos x^2$

Unlike examples 3.11 and 3.12, there is no algebra we can use to find the zeros of f'. Our only alternative is to find the zeros approximately. You can do this by using Newton's method to solve $f'(x) = 0$. (You can also use any other rootfinding method built into your calculator or computer.) First, we'll need adequate initial guesses. We obtain these from the graph of $y = f'(x)$ found in Figure 3.40. From the graph, it appears that there are four zeros of $f'(x)$ on the interval in question, located near $x = -1.3$, 0.7, 1.2 and 2.0. Further, referring back to Figure 3.39, these four zeros correspond with the four local extrema seen in the graph of $y = f(x)$. We now apply Newton's method to solve $f'(x) = 0$, using the preceding four values as our initial guesses. This leads us to four approximate critical numbers of f on the interval $[-2, 2.5]$. We have

$$a \approx -1.26410884789, \quad b \approx 0.674471354085,$$

$$c \approx 1.2266828947 \quad \text{and} \quad d \approx 2.01830371473.$$

We now need only compare the values of f at the endpoints and the approximate critical numbers:

$$f(a) \approx 7.3, \quad f(b) \approx -1.7, \quad f(c) \approx -1.3$$
$$f(d) \approx -4.3, \quad f(-2) \approx -0.3 \quad \text{and} \quad f(2.5) \approx 3.0.$$

Thus, the absolute maximum is approximately $f(-1.26410884789) \approx 7.3$ and the absolute minimum is approximately $f(2.01830371473) \approx -4.3$.

It is important (especially in light of how much of our work here was approximate and graphical) to verify that the approximate extrema correspond with what we expect from the graph of $y = f(x)$. Since these correspond closely, we have great confidence in their accuracy. ■

We have now seen how to locate the absolute extrema of a continuous function on a closed interval. In section 3.4, we see how to find local extrema.

BEYOND FORMULAS

The Extreme Value Theorem is an important but subtle result. Think of it this way. If the hypotheses of the theorem are met, you will never waste your time looking for the maximum of a function that does not have a maximum. That is, the problem is always solvable. The technique described in Remark 3.3 *always works.* If you are asked to find a novel with a certain plot, does it help to know that there actually is such a novel?

EXERCISES 3.3

WRITING EXERCISES

1. Using one or more graphs, explain why the Extreme Value Theorem is true. Is the conclusion true if we drop the hypothesis that f is a continuous function? Is the conclusion true if we drop the hypothesis that the interval is closed?

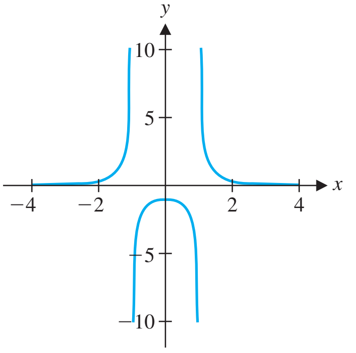
2. Using one or more graphs, argue that Fermat's Theorem is true. Discuss how Fermat's Theorem is used. Restate the theorem in your own words to make its use clearer.

3. Suppose that $f(t)$ represents your elevation after t hours on a mountain hike. If you stop to rest, explain why $f'(t) = 0$. Discuss the circumstances under which you would be at a local maximum, local minimum or neither.

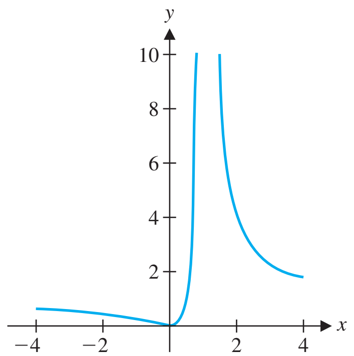
4. Mathematically, an if/then statement is usually strictly one-directional. When we say "If A, then B" it is generally *not* the case that "If B, then A" is also true: when both are true, we say "A if and only if B," which is abbreviated to "A iff B." Unfortunately, common English usage is not always this precise. This occasionally causes a problem interpreting a mathematical theorem. To get this straight, consider the statement, "*If* you wrote a best-selling book, *then* you made a lot of money." Is this true? How does this differ from its **converse,** "*If* you made a lot of money, *then* you wrote a best-selling book." Is the converse always true? Sometimes true? Apply this logic to both the Extreme Value Theorem and Fermat's Theorem: state the converse and decide whether it is sometimes true or always true.

In exercises 1–4, use the graph to locate the absolute extrema (if they exist) of the function on the given interval.

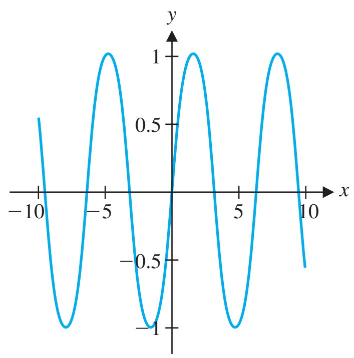
1. $f(x) = \dfrac{1}{x^2 - 1}$ on (a) $(-\infty, \infty)$, (b) $[-1, 1]$ and (c) $(0, 1)$



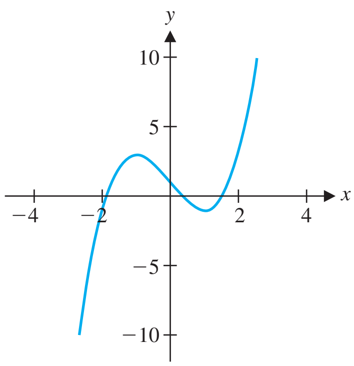
2. $f(x) = \dfrac{x^2}{(x-1)^2}$ on (a) $(-\infty, \infty)$, (b) $[-1, 1]$ and (c) $(0, 1)$



3. $f(x) = \sin x$ on (a) $(-\infty, \infty)$, (b) $\left[0, \frac{\pi}{4}\right]$ and (c) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$



4. $f(x) = x^3 - 3x + 1$ on (a) $(-\infty, \infty)$, (b) $[-2, 2]$ and (c) $(0, 2)$



In exercises 5–10, find all critical numbers by hand. Use your knowledge of the type of graph (i.e., parabola or cubic) to determine whether the critical number represents a local maximum, local minimum or neither.

5. $f(x) = x^2 + 5x - 1$

6. $f(x) = -x^2 + 4x + 2$

7. $f(x) = x^3 - 3x + 1$

8. $f(x) = -x^3 + 6x^2 + 2$

9. $f(x) = x^3 - 3x^2 + 6x$

10. $f(x) = x^3 - 3x^2 + 3x$

In exercises 11–30, find all critical numbers by hand. If available, use graphing technology to determine whether the critical number represents a local maximum, local minimum or neither.

11. $f(x) = x^4 - 3x^3 + 2$

12. $f(x) = x^4 + 6x^2 - 2$

13. $f(x) = x^{3/4} - 4x^{1/4}$

14. $f(x) = (x^{2/5} - 3x^{1/5})^2$

15. $f(x) = \sin x \cos x$, $[0, 2\pi]$

16. $f(x) = \sqrt{3} \sin x + \cos x$

17. $f(x) = \dfrac{x^2 - 2}{x + 2}$

18. $f(x) = \dfrac{x^2 - x + 4}{x - 1}$

19. $f(x) = \dfrac{x}{x^2 + 1}$

20. $f(x) = \dfrac{3x}{x^2 - 1}$

21. $f(x) = \frac{1}{2}(e^x + e^{-x})$

22. $f(x) = xe^{-2x}$

23. $f(x) = x^{4/3} + 4x^{1/3} + 4x^{-2/3}$

24. $f(x) = x^{7/3} - 28x^{1/3}$

25. $f(x) = 2x\sqrt{x + 1}$

26. $f(x) = x/\sqrt{x^2 + 1}$

27. $f(x) = |x^2 - 1|$

28. $f(x) = \sqrt[3]{x^3 - 3x^2 + 2x}$

29. $f(x) = \begin{cases} x^2 + 2x - 1 & \text{if } x < 0 \\ x^2 - 4x + 3 & \text{if } x \geq 0 \end{cases}$

30. $f(x) = \begin{cases} \sin x & \text{if } -\pi < x < \pi \\ -\tan x & \text{if } |x| \geq \pi \end{cases}$

In exercises 31–38, find the absolute extrema of the given function on each indicated interval.

31. $f(x) = x^3 - 3x + 1$ on (a) $[0, 2]$ and (b) $[-3, 2]$

32. $f(x) = x^4 - 8x^2 + 2$ on (a) $[-3, 1]$ and (b) $[-1, 3]$

33. $f(x) = x^{2/3}$ on (a) $[-4, -2]$ and (b) $[-1, 3]$

34. $f(x) = \sin x + \cos x$ on (a) $[0, 2\pi]$ and (b) $[\pi/2, \pi]$

35. $f(x) = e^{-x^2}$ on (a) $[0, 2]$ and (b) $[-3, 2]$

36. $f(x) = x^2 e^{-4x}$ on (a) $[-2, 0]$ and (b) $[0, 4]$

37. $f(x) = \dfrac{3x^2}{x - 3}$ on (a) $[-2, 2]$ and (b) $[2, 8]$

38. $f(x) = \tan^{-1}(x^2)$ on (a) $[0, 1]$ and (b) $[-3, 4]$

In exercises 39–44, numerically estimate the absolute extrema of the given function on the indicated intervals.

39. $f(x) = x^4 - 3x^2 + 2x + 1$ on (a) $[-1, 1]$ and (b) $[-3, 2]$

40. $f(x) = x^6 - 3x^4 - 2x + 1$ on (a) $[-1, 1]$ and (b) $[-2, 2]$

41. $f(x) = x^2 - 3x \cos x$ on (a) $[-2, 1]$ and (b) $[-5, 0]$

42. $f(x) = xe^{\cos 2x}$ on (a) $[-2, 2]$ and (b) $[2, 5]$

43. $f(x) = x \sin x + 3$ on (a) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and (b) $[0, 2\pi]$

44. $f(x) = x^2 + e^x$ on (a) $[0, 1]$ and (b) $[-2, 2]$

45. Repeat exercises 31–38, except instead of finding extrema on the closed interval, find the extrema on the open interval, if they exist.

46. Briefly outline a procedure for finding extrema on an open interval (a, b), a procedure for the half-open interval $(a, b]$ and a procedure for the half-open interval $[a, b)$.

47. Sketch a graph of a function f such that the absolute maximum of $f(x)$ on the interval $[-2, 2]$ equals 3 and the absolute minimum does not exist.

48. Sketch a graph of a continuous function f such that the absolute maximum of $f(x)$ on the interval $(-2, 2)$ does not exist and the absolute minimum equals 2.

49. Sketch a graph of a continuous function f such that the absolute maximum of $f(x)$ on the interval $(-2, 2)$ equals 4 and the absolute minimum equals 2.

50. Sketch a graph of a function f such that the absolute maximum of $f(x)$ on the interval $[-2, 2]$ does not exist and the absolute minimum does not exist.

51. Give an example showing that the following statement is false (not always true): between any two local minima of $f(x)$ there is a local maximum.

52. If you have won three out of four matches against someone, does that mean that the probability that you will win the next one is $\frac{3}{4}$? In general, if you have a probability p of winning each match, the probability of winning m out of n matches is $f(p) = \dfrac{n!}{(n-m)!\,m!}p^m(1-p)^{n-m}$. Find p to maximize f. This value of p is called the **maximum likelihood estimator** of the probability. Briefly explain why your answer makes sense.

53. In this exercise, we will explore the family of functions $f(x) = x^3 + cx + 1$, where c is constant. How many and what types of local extrema are there? (Your answer will depend on the value of c.) Assuming that this family is indicative of all cubic functions, list all types of cubic functions.

54. Prove that any fourth-order polynomial must have at least one local extremum and can have a maximum of three local extrema. Based on this information, sketch several possible graphs of fourth-order polynomials.

55. Show that $f(x) = x^3 + bx^2 + cx + d$ has both a local maximum and a local minimum if $c < 0$.

56. In exercise 55, show that the sum of the critical numbers is $-\frac{2b}{3}$.

57. For the family of functions $f(x) = x^4 + cx^2 + 1$, find all local extrema (your answer will depend on the value of the constant c).

58. For the family of functions $f(x) = x^4 + cx^3 + 1$, find all local extrema. (Your answer will depend on the value of the constant c.)

59. If two soccer teams each score goals at a rate of r goals per minute, the probability that n goals will be scored in t minutes is $P = \frac{(rt)^n}{n!}e^{-rt}$. Take $r = \frac{1}{25}$. Show that for a 90-minute game, P is maximized with $n = 3$. Briefly explain why this makes sense.

60. In the situation of exercise 59, find t to maximize the probability that exactly 1 goal has been scored. Briefly explain why your answer makes sense.

61. If f is differentiable on the interval $[a, b]$ and $f'(a) < 0 < f'(b)$, prove that there is a c with $a < c < b$ for which $f'(c) = 0$. (Hint: Use the Extreme Value Theorem and Fermat's Theorem.)

62. Sketch a graph showing that $y = f(x) = x^2 + 1$ and $y = g(x) = \ln x$ do not intersect. Find x to minimize $f(x) - g(x)$. At this value of x, show that the tangent lines to $y = f(x)$ and $y = g(x)$ are parallel. Explain graphically why it makes sense that the tangent lines are parallel.

63. Sketch a graph of $f(x) = \dfrac{x^2}{x^2+1}$ for $x > 0$ and determine where the graph is steepest. (That is, find where the slope is a maximum.)

64. Sketch a graph of $f(x) = e^{-x^2}$ and determine where the graph is steepest. (Note: This is an important problem in probability theory.)

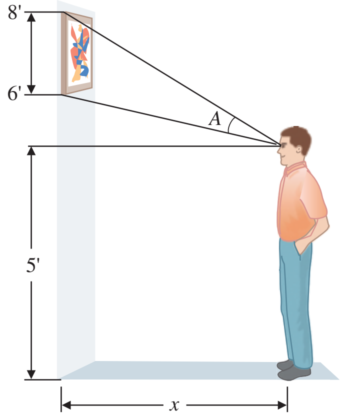
65. A section of roller coaster is in the shape of $y = x^5 - 4x^3 - x + 10$, where x is between -2 and 2. Find all local extrema and explain what portions of the roller coaster they represent. Find the location of the steepest part of the roller coaster.

66. Suppose a large computer file is sent over the Internet. If the probability that it reaches its destination without any errors is x, then the probability that an error is made is $1 - x$. The field of information theory studies such situations. An important quantity is **entropy** (a measure of unpredictability), defined by $H = -x \ln x - (1-x)\ln(1-x)$, for $0 < x < 1$. Find the value of x that maximizes this quantity. Explain why this probability would maximize the measure of unpredictability of errors. (Hint: If $x = 0$ or $x = 1$, are errors unpredictable?)

67. Researchers in a number of fields (including population biology, economics and the study of animal tumors) make use of the Gompertz growth curve, $W(t) = ae^{-be^{-t}}$. As $t \to \infty$, show that $W(t) \to a$ and $W'(t) \to 0$. Find the maximum growth rate.

68. The rate R of an enzymatic reaction as a function of the substrate concentration $[S]$ is given by $R = \dfrac{[S]R_m}{K_m + [S]}$, where R_m and K_m are constants. K_m is called the Michaelis constant and R_m is referred to as the maximum reaction rate. Show that R_m is not a proper maximum in that the reaction rate can never be equal to R_m.

69. Suppose a painting hangs on a wall as in the figure. The frame extends from 6 feet to 8 feet above the floor. A person whose eyes are 5 feet above the ground stands x feet from the wall and views the painting, with a viewing angle A formed by the ray from the person's eye to the top of the frame and the ray from the person's eye to the bottom of the bottom of the frame. Find the value of x that maximizes the viewing angle A.



70. What changes in exercise 69 if the person's eyes are 6 feet above the ground?

EXPLORATORY EXERCISES

1. Explore the graphs of e^{-x}, xe^{-x}, x^2e^{-x} and x^3e^{-x}. Find all local extrema and use l'Hôpital's Rule to determine the behavior as $x \to \infty$. You can think of the graph of x^ne^{-x} as showing the results of a tug-of-war: $x^n \to \infty$ as $x \to \infty$ but $e^{-x} \to 0$ as $x \to \infty$. Describe the graph of x^ne^{-x} in terms of this tug-of-war.

2. Johannes Kepler (1571–1630) is best known as an astronomer, especially for his three laws of planetary motion. However, his discoveries were primarily due to his brilliance as a mathematician. While serving in Austrian Emperor Matthew I's court, Kepler observed the ability of Austrian vintners to quickly and mysteriously compute the capacities of a variety of wine casks. Each cask (barrel) had a hole in the middle of its side (see Figure a). The vintner would insert a rod in the hole until it hit the far corner and then announce the volume. Kepler first analyzed the problem for a cylindrical barrel (see Figure b). The volume of a cylinder is $V = \pi r^2 h$. In Figure b, $r = y$ and $h = 2x$ so $V = 2\pi y^2 x$. Call the rod measurement z. By the Pythagorean Theorem, $x^2 + (2y)^2 = z^2$. Kepler's mystery was how to compute V given only z. The key observation made by Kepler was that Austrian wine casks were made with the same height-to-diameter ratio (for us, x/y). Let $t = x/y$ and show that $z^2/y^2 = t^2 + 4$. Use this to replace y^2 in the volume formula. Then replace x with $\sqrt{z^2 + 4y^2}$. Show that $V = \dfrac{2\pi z^3 t}{(4 + t^2)^{3/2}}$. In this formula, t is a constant, so the vintner could measure z and quickly estimate the volume. We haven't told you yet what t equals. Kepler assumed that the vintners would have made a smart choice for this ratio. Find the value of t that maximizes the volume for a given z. This is, in fact, the ratio used in the construction of Austrian wine casks!

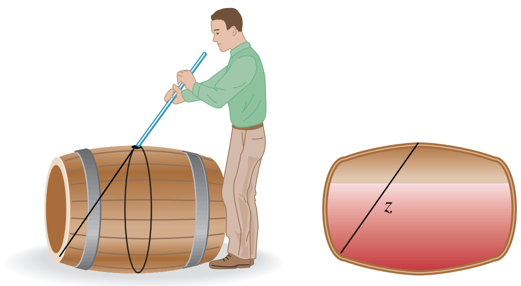


FIGURE a

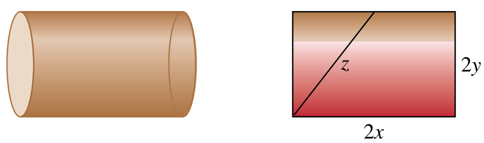


FIGURE b

3. Suppose that a hockey player is shooting at a 6-foot-wide net from a distance of d feet away from the goal line and 4 feet to the side of the center line. (a) Find the distance d that maximizes the shooting angle. (b) Repeat part (a) with the shooter 2 feet to the side of the center line. Explain why the answer is so different. (c) Repeat part (a) with the goalie blocking all but the far 2 feet of the goal.

3.4 INCREASING AND DECREASING FUNCTIONS

In section 3.3, we determined that local extrema occur only at critical numbers. However, not all critical numbers correspond to local extrema. In this section, we see how to determine which critical numbers correspond to local extrema. At the same time, we'll learn more about the connection between the derivative and graphing.

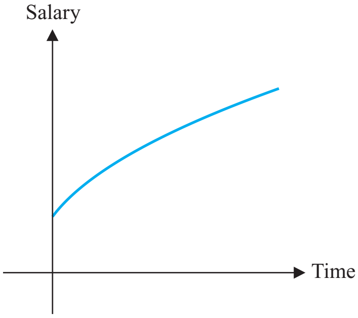


FIGURE 3.41
Increasing salary

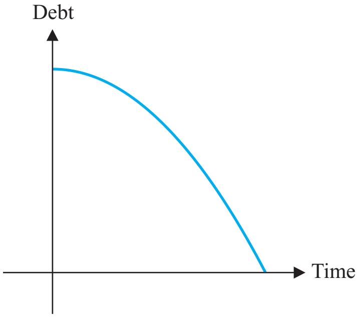


FIGURE 3.42
Decreasing debt

We are all familiar with the terms *increasing* and *decreasing*. If your employer informs you that your salary will be increasing steadily over the term of your employment, you have in mind that as time goes on, your salary will rise something like Figure 3.41. If you take out a loan to purchase a car, once you start paying back the loan, your indebtedness will decrease over time. If you plotted your debt against time, the graph might look something like Figure 3.42.

We now carefully define these notions. Notice that Definition 4.1 is merely a formal statement of something you already understand.

DEFINITION 4.1

A function f is **(strictly) increasing** on an interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) < f(x_2)$ [i.e., $f(x)$ gets larger as x gets larger].

A function f is **(strictly) decreasing** on the interval I if for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) > f(x_2)$ [i.e., $f(x)$ gets smaller as x gets larger].

Why do we bother with such an obvious definition? Of course, anyone can look at a graph of a function and immediately see where that function is increasing and decreasing. The real challenge is to determine where a function is increasing and decreasing, given only a mathematical formula for the function. For example, can you determine where $f(x) = x^2 \sin x$ is increasing and decreasing, *without* looking at a graph? Look carefully at Figure 3.43 to see if you can notice what happens at every point at which the function is increasing or decreasing.

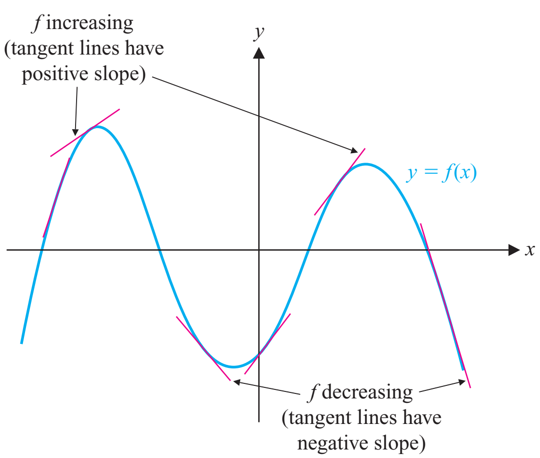


FIGURE 3.43
Increasing and decreasing

Observe that on intervals where the tangent lines have positive slope, f is increasing, while on intervals where the tangent lines have negative slope, f is decreasing. Of course, the slope of the tangent line at a point is given by the value of the derivative at that point. So, whether a function is increasing or decreasing on an interval seems to be connected to the sign of its derivative on that interval. This conjecture, although it's based on only a single picture, sounds like a theorem and it is.

THEOREM 4.1

Suppose that f is differentiable on an interval I.

(i) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.
(ii) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

PROOF

(i) Pick any two points $x_1, x_2 \in I$, with $x_1 < x_2$. Applying the Mean Value Theorem (Theorem 9.4 in section 2.9) to f on the interval (x_1, x_2), we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \tag{4.1}$$

for some number $c \in (x_1, x_2)$. (Why can we apply the Mean Value Theorem here?) By hypothesis, $f'(c) > 0$ and since $x_1 < x_2$ (so that $x_2 - x_1 > 0$), we have from (4.1) that

$$0 < f(x_2) - f(x_1)$$

or

$$f(x_1) < f(x_2). \tag{4.2}$$

Since (4.2) holds for all $x_1 < x_2$, f is increasing on I.

The proof of (ii) is nearly identical and is left as an exercise. ■

What You See May Not Be What You Get

One aim here and in sections 3.5 and 3.6 is to learn how to draw representative graphs of functions (i.e., graphs that display all of the significant features of a function: where it is increasing or decreasing, any extrema, asymptotes and two features we'll introduce in section 3.5, concavity and inflection points). When we draw a graph, we are drawing in a particular viewing *window* (i.e., a particular range of x- and y-values). In the case of computer- or calculator-generated graphs, the window is often chosen by the machine. So, how do we know when significant features are hidden outside of a given window? Further, how do we determine the precise locations of features that we can see in a given window? As we'll discover, the only way we can resolve these questions is with some calculus.

EXAMPLE 4.1 Drawing a Graph

Draw a graph of $f(x) = 2x^3 + 9x^2 - 24x - 10$ showing all local extrema.

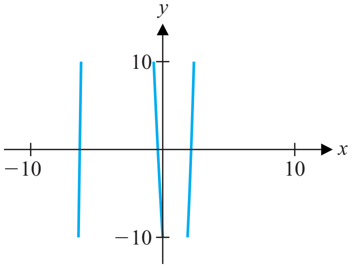
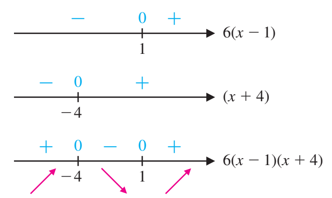


FIGURE 3.44
$y = 2x^3 + 9x^2 - 24x - 10$

Solution Many graphing calculators use the default window defined by $-10 \le x \le 10$ and $-10 \le y \le 10$. Using this window, the graph of $y = f(x)$ looks like that displayed in Figure 3.44, although the three segments shown are not particularly revealing. Instead of blindly manipulating the window in the hope that a reasonable graph will magically appear, we stop briefly to determine where the function is increasing and decreasing. First, observe that

$$\begin{aligned} f'(x) &= 6x^2 + 18x - 24 = 6(x^2 + 3x - 4) \\ &= 6(x - 1)(x + 4). \end{aligned}$$



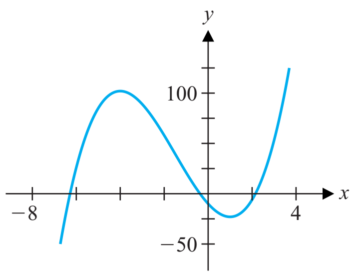


FIGURE 3.45a
$y = 2x^3 + 9x^2 - 24x - 10$

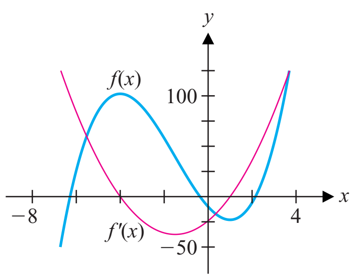


FIGURE 3.45b
$y = f(x)$ and $y = f'(x)$

Note that the critical numbers (1 and -4) are the only possible locations for local extrema. We can see where the two factors and consequently the derivative are positive and negative from the number lines displayed in the margin. From this, note that

$$f'(x) > 0 \text{ on } (-\infty, -4) \cup (1, \infty) \qquad f \text{ increasing.}$$

and

$$f'(x) < 0 \text{ on } (-4, 1). \qquad f \text{ decreasing.}$$

For convenience, we have placed arrows indicating where the function is increasing and decreasing beneath the last number line. In Figure 3.45a, we redraw the graph in the window defined by $-8 \le x \le 4$ and $-50 \le y \le 125$. Here, we have selected the y-range so that the critical points $(-4, 102)$ and $(1, -23)$ are displayed. Since f is increasing on all of $(-\infty, -4)$, we know that the function is still increasing to the left of the portion displayed in Figure 3.45a. Likewise, since f is increasing on all of $(1, \infty)$, we know that the function continues to increase to the right of the displayed portion. In Figure 3.45b, we have plotted both $y = f(x)$ (shown in blue) and $y = f'(x)$ (shown in red). Notice the connection between the two graphs. When $f'(x) > 0$, f is increasing; when $f'(x) < 0$, f is decreasing and also notice what happens to $f'(x)$ at the local extrema of f. (We'll say more about this shortly.) ■

You may be tempted to think that you can draw graphs by machine and with a little fiddling with the graphing window, get a reasonable looking graph. Unfortunately, this frequently isn't enough. For instance, while it's clear that the graph in Figure 3.44 is incomplete, the initial graph in example 4.2 has a familiar shape and may look reasonable, but it is incorrect. The calculus tells you what features you should expect to see in a graph. Without it, you're simply taking a shot in the dark.

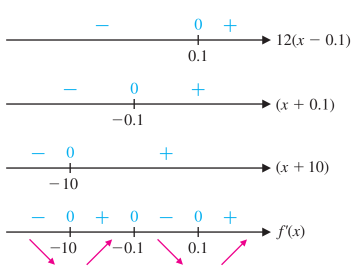
EXAMPLE 4.2 Uncovering Hidden Behavior in a Graph

Graph $f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$ showing all local extrema.

Solution We first show the default graph drawn by our computer algebra system (see Figure 3.46a). We show a common default graphing calculator graph in Figure 3.46b. You can certainly make Figure 3.46b look more like Figure 3.46a by adjusting the window some. But with some calculus, you can discover features of the graph that are hidden in both graphs.

First, notice that

$$\begin{aligned} f'(x) &= 12x^3 + 120x^2 - 0.12x - 1.2 \\ &= 12(x^2 - 0.01)(x + 10) \\ &= 12(x - 0.1)(x + 0.1)(x + 10). \end{aligned}$$



We show number lines for the three factors in the margin. Observe that

$$f'(x) \begin{cases} > 0 \text{ on } (-10, -0.1) \cup (0.1, \infty) & f \text{ increasing.} \\ < 0 \text{ on } (-\infty, -10) \cup (-0.1, 0.1). & f \text{ decreasing.} \end{cases}$$

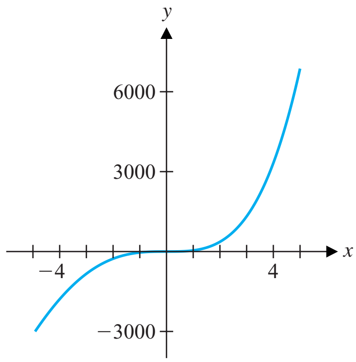


FIGURE 3.46a
Default CAS graph of
$y = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$

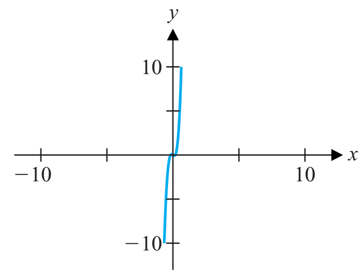


FIGURE 3.46b
Default calculator graph of
$y = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$

This says that neither of the machine-generated graphs seen in Figures 3.46a or 3.46b is adequate, as the behavior on $(-\infty, -10) \cup (-0.1, 0.1)$ cannot be seen in either graph. As it turns out, no single graph captures all of the behavior of this function. However, by increasing the range of x-values to the interval $[-15, 5]$, we get the graph seen in Figure 3.47a. This shows the *big picture,* what we refer to as the **global** behavior of the function. Here, you can see the local minimum at $x = -10$, which was missing in our earlier graphs, but the behavior for values of x close to zero is not clear. To see this, we need a separate graph, restricted to a smaller range of x-values, as seen in Figure 3.47b.

Notice that here, we can see the behavior of the function for x close to zero quite clearly. In particular, the local maximum at $x = -0.1$ and the local minimum at $x = 0.1$ are clearly visible. We often say that a graph such as Figure 3.47b shows the **local** behavior of the function. In Figures 3.48a and 3.48b, we show graphs indicating the global and local behavior of $f(x)$ (in blue) and $f'(x)$ (in red) on the same set of axes. Pay particular attention to the behavior of $f'(x)$ in the vicinity of local extrema of $f(x)$.

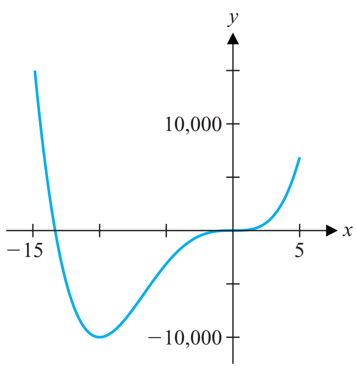


FIGURE 3.47a
The global behavior of
$f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$

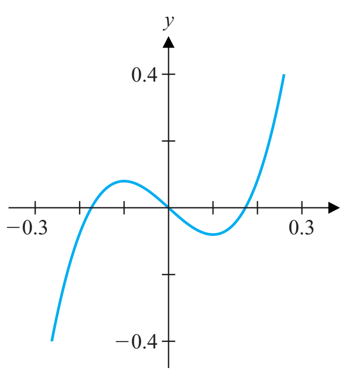


FIGURE 3.47b
Local behavior of
$f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$

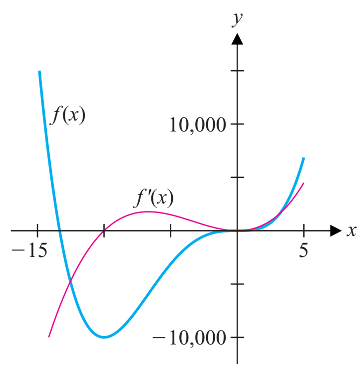


FIGURE 3.48a
$y = f(x)$ and $y = f'(x)$
(global behavior)

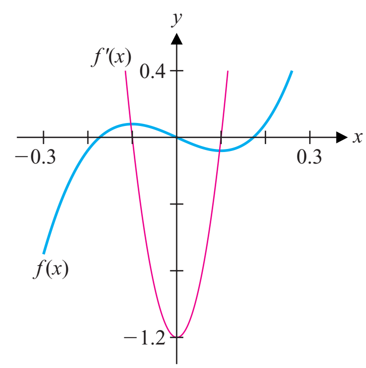


FIGURE 3.48b
$y = f(x)$ and $y = f'(x)$
(local behavior)

You may have already noticed a connection between local extrema and the intervals on which a function is increasing and decreasing. We state this in Theorem 4.2.

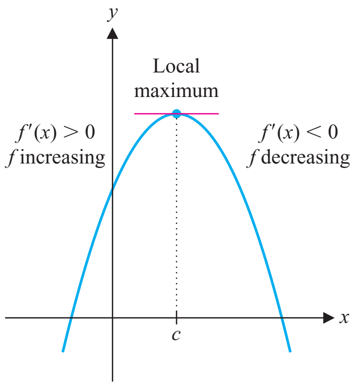


FIGURE 3.49a
Local maximum

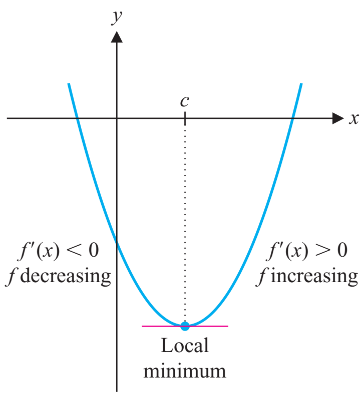


FIGURE 3.49b
Local minimum

THEOREM 4.2 (First Derivative Test)

Suppose that f is continuous on the interval $[a, b]$ and $c \in (a, b)$ is a critical number.

(i) If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$ (i.e., f changes from increasing to decreasing at c), then $f(c)$ is a local maximum.
(ii) If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$ (i.e., f changes from decreasing to increasing at c), then $f(c)$ is a local minimum.
(iii) If $f'(x)$ has the *same* sign on (a, c) and (c, b), then $f(c)$ is *not* a local extremum.

It's easiest to think of this result graphically. If f is increasing to the left of a critical number and decreasing to the right, then there must be a local maximum at the critical number (see Figure 3.49a). Likewise, if f is decreasing to the left of a critical number and increasing to the right, then there must be a local minimum at the critical number (see Figure 3.49b). This suggests a proof of the theorem; the job of writing out all of the details is left as an exercise.

EXAMPLE 4.3 Finding Local Extrema Using the First Derivative Test

Find the local extrema of the function from example 4.1, $f(x) = 2x^3 + 9x^2 - 24x - 10$.

Solution We had found in example 4.1 that

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, -4) \cup (1, \infty) & f \text{ increasing.} \\ < 0, \text{ on } (-4, 1). & f \text{ decreasing.} \end{cases}$$

It now follows from the First Derivative Test that f has a local maximum located at $x = -4$ and a local minimum located at $x = 1$. ■

Theorem 4.2 works equally well for a function with critical points where the derivative is undefined.

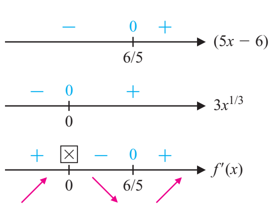
EXAMPLE 4.4 Finding Local Extrema of a Function with Fractional Exponents

Find the local extrema of $f(x) = x^{5/3} - 3x^{2/3}$.

Solution We have

$$f'(x) = \frac{5}{3}x^{2/3} - 3\left(\frac{2}{3}\right)x^{-1/3} = \frac{5x-6}{3x^{1/3}},$$

so that the critical numbers are $\frac{6}{5}$ $[f'\left(\frac{6}{5}\right) = 0]$ and 0 $[f'(0)$ is undefined$]$. Again drawing number lines for the factors, we determine where f is increasing and decreasing. Here, we have placed an ☒ above the 0 on the number line for $f'(x)$ to indicate that $f'(x)$ is not defined at $x = 0$. From this, we can see at a glance where f' is positive and negative:



$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, 0) \cup \left(\frac{6}{5}, \infty\right) & f \text{ increasing.} \\ < 0, \text{ on } \left(0, \frac{6}{5}\right). & f \text{ decreasing.} \end{cases}$$

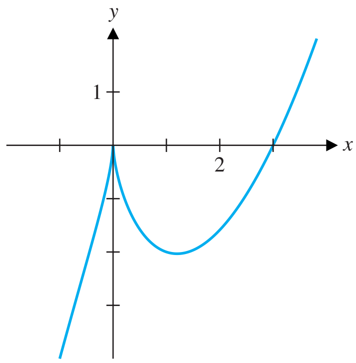


FIGURE 3.50
$y = x^{5/3} - 3x^{2/3}$

Consequently, f has a local maximum at $x = 0$ and a local minimum at $x = \frac{6}{5}$. These local extrema are both clearly visible in the graph in Figure 3.50. ■

EXAMPLE 4.5 Finding Local Extrema Approximately

Find the local extrema of $f(x) = x^4 + 4x^3 - 5x^2 - 31x + 29$ and draw a graph.

Solution A graph of $y = f(x)$ using the most common graphing calculator default window appears in Figure 3.51. Without further analysis, we do not know whether this graph shows all of the significant behavior of the function. [Note that some fourth-degree polynomials (e.g., $f(x) = x^4$) have graphs that look very much like the one in Figure 3.51.] First, we compute

$$f'(x) = 4x^3 + 12x^2 - 10x - 31.$$

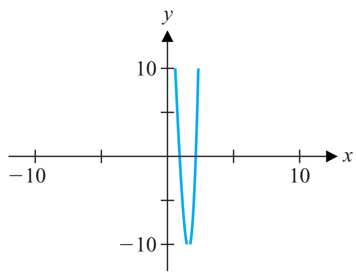


FIGURE 3.51
$f(x) = x^4 + 4x^3 - 5x^2 - 31x + 29$

However, this derivative does not easily factor. A graph of $y = f'(x)$ (see Figure 3.52) reveals three zeros, one near each of $x = -3, -1.5$ and 1.5. Since a cubic polynomial has at most three zeros, there are no others. Using Newton's method or some other rootfinding method [applied to $f'(x)$], we can find approximations to the three zeros of f'. We get $a \approx -2.96008$, $b \approx -1.63816$ and $c \approx 1.59824$. From Figure 3.52, we can see that

$$f'(x) > 0 \text{ on } (a, b) \cup (c, \infty) \qquad f \text{ increasing.}$$

and

$$f'(x) < 0 \text{ on } (-\infty, a) \cup (b, c). \qquad f \text{ decreasing.}$$

You can quickly read off the local extrema: a local minimum at $a \approx -2.96008$, a local maximum at $b \approx -1.63816$ and a local minimum at $c \approx 1.59824$. Since only the local minimum at $x = c$ is visible in the graph in Figure 3.51, this graph is clearly not representative of the behavior of the function. By narrowing the range of displayed x-values and widening the range of displayed y-values, we obtain the far more useful graph seen in Figure 3.53. You should convince yourself, using the preceding analysis, that the local minimum at $x = c \approx 1.59824$ is also the *absolute* minimum.

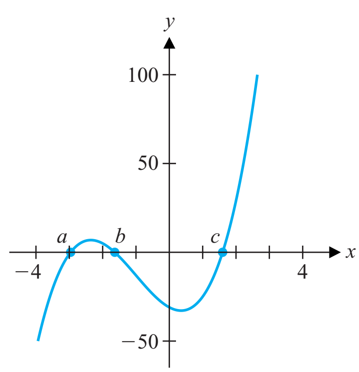


FIGURE 3.52
$f'(x) = 4x^3 + 12x^2 - 10x - 31$

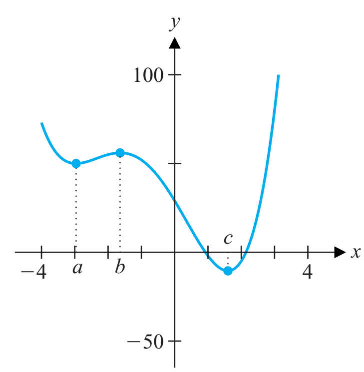


FIGURE 3.53
$f(x) = x^4 + 4x^3 - 5x^2 - 31x + 29$

■

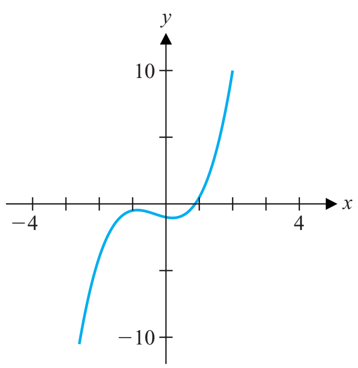
EXERCISES 3.4

WRITING EXERCISES

1. Suppose that $f(0) = 2$ and f is an increasing function. To sketch the graph of $y = f(x)$, you could start by plotting the point (0, 2). Filling in the graph to the left, would you move your pencil up or down? How does this fit with the definition of increasing?

2. Suppose you travel east on an east-west interstate highway. You reach your destination, stay a while and then return home. Explain the First Derivative Test in terms of your velocities (positive and negative) on this trip.

3. Suppose that you have a differentiable function f with two critical numbers. Your computer has shown you a graph that looks like the one in the figure.



Discuss the possibility that this is a representative graph: that is, is it possible that there are any important points not shown in this window?

4. Suppose that the function in exercise 3 has three critical numbers. Explain why the graph is not a representative graph. Explain how you would change the graphing window to show the rest of the graph.

In exercises 1–10, find (by hand) the intervals where the function is increasing and decreasing. Use this information to sketch a graph.

1. $y = x^3 - 3x + 2$
2. $y = x^3 + 2x^2 + 1$
3. $y = x^4 - 8x^2 + 1$
4. $y = x^3 - 3x^2 - 9x + 1$
5. $y = (x + 1)^{2/3}$
6. $y = (x - 1)^{1/3}$
7. $y = \sin x + \cos x$
8. $y = \sin^2 x$
9. $y = e^{x^2-1}$
10. $y = \ln(x^2 - 1)$

In exercises 11–20, find (by hand) all critical numbers and use the First Derivative Test to classify each as the location of a local maximum, local minimum or neither.

11. $y = x^4 + 4x^3 - 2$
12. $y = x^5 - 5x^2 + 1$
13. $y = xe^{-2x}$
14. $y = x^2e^{-x}$
15. $y = \tan^{-1}(x^2)$
16. $y = \sin^{-1}\left(1 - \frac{1}{x^2}\right)$
17. $y = \dfrac{x}{1 + x^3}$
18. $y = \dfrac{x}{1 + x^4}$
19. $y = \sqrt{x^3 + 3x^2}$
20. $y = x^{4/3} + 4x^{1/3}$

In exercises 21–26, find (by hand) all asymptotes and extrema, and sketch a graph.

21. $y = \dfrac{x}{x^2 - 1}$
22. $y = \dfrac{x^2}{x^2 - 1}$
23. $y = \dfrac{x^2}{x^2 - 4x + 3}$
24. $y = \dfrac{x}{1 - x^4}$
25. $y = \dfrac{x}{\sqrt{x^2 + 1}}$
26. $y = \dfrac{x^2 + 2}{(x + 1)^2}$

In exercises 27–34, find the x-coordinates of all extrema and sketch graphs showing global and local behavior of the function.

27. $y = x^3 - 13x^2 - 10x + 1$
28. $y = x^3 + 15x^2 - 70x + 2$
29. $y = x^4 - 15x^3 - 2x^2 + 40x - 2$
30. $y = x^4 - 16x^3 - 0.1x^2 + 0.5x - 1$
31. $y = x^5 - 200x^3 + 605x - 2$
32. $y = x^4 - 0.5x^3 - 0.02x^2 + 0.02x + 1$
33. $y = (x^2 + x + 0.45)e^{-2x}$
34. $y = x^5 \ln 8x^2$

In exercises 35–38, sketch a graph of a function with the given properties.

35. $f(0) = 1$, $f(2) = 5$, $f'(x) < 0$ for $x < 0$ and $x > 2$, $f'(x) > 0$ for $0 < x < 2$.

36. $f(-1) = 1$, $f(2) = 5$, $f'(x) < 0$ for $x < -1$ and $x > 2$, $f'(x) > 0$ for $-1 < x < 2$, $f'(-1) = 0$, $f'(2)$ does not exist.

37. $f(3) = 0$, $f'(x) < 0$ for $x < 0$ and $x > 3$, $f'(x) > 0$ for $0 < x < 3$, $f'(3) = 0$, $f(0)$ and $f'(0)$ do not exist.

38. $f(1) = 0$, $\lim\limits_{x\to\infty} f(x) = 2$, $f'(x) < 0$ for $x < 1$, $f'(x) > 0$ for $x > 1$, $f'(1) = 0$.

In exercises 39–42, estimate critical numbers and sketch graphs showing both global and local behavior.

39. $y = \dfrac{x - 30}{x^4 - 1}$

40. $y = \dfrac{x^2 - 8}{x^4 - 1}$

41. $y = \dfrac{x + 60}{x^2 + 1}$

42. $y = \dfrac{x - 60}{x^2 - 1}$

43. For $f(x) = \begin{cases} x + 2x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ show that $f'(0) > 0$, but that f is not increasing in any interval around 0. Explain why this does not contradict Theorem 4.1.

44. For $f(x) = x^3$, show that f is increasing in any interval around 0, but $f'(0) = 0$. Explain why this does not contradict Theorem 4.1.

45. Prove Theorem 4.2 (the First Derivative Test).

46. Give a graphical argument that if $f(a) = g(a)$ and $f'(x) > g'(x)$ for all $x > a$, then $f(x) > g(x)$ for all $x > a$. Use the Mean Value Theorem to prove it.

In exercises 47–50, use the result of exercise 46 to verify the inequality.

47. $2\sqrt{x} > 3 - \dfrac{1}{x}$ for $x > 1$

48. $x > \sin x$ for $x > 0$

49. $e^x > x + 1$ for $x > 0$

50. $x - 1 > \ln x$ for $x > 1$

51. Give an example showing that the following statement is false. If $f(0) = 4$ and $f(x)$ is a decreasing function, then the equation $f(x) = 0$ has exactly one solution.

52. Assume that f is an increasing function with inverse function f^{-1}. Show that f^{-1} is also an increasing function.

53. State the domain for $\sin^{-1} x$ and determine where it is increasing and decreasing.

54. State the domain for $\sin^{-1}\left(\frac{2}{\pi}\tan^{-1} x\right)$ and determine where it is increasing and decreasing.

55. If f and g are both increasing functions, is it true that $f(g(x))$ is also increasing? Either prove that it is true or give an example that proves it false.

56. If f and g are both increasing functions with $f(5) = 0$, find the maximum and minimum of the following values: $g(1)$, $g(4)$, $g(f(1))$, $g(f(4))$.

57. Suppose that the total sales of a product after t months is given by $s(t) = \sqrt{t + 4}$ thousand dollars. Compute and interpret $s'(t)$.

58. In exercise 57, show that $s'(t) > 0$ for all $t > 0$. Explain in business terms why it is impossible to have $s'(t) < 0$.

59. In this exercise, you will play the role of professor and construct a tricky graphing exercise. The first goal is to find a function with local extrema so close together that they're difficult to see. For instance, suppose you want local extrema at $x = -0.1$ and $x = 0.1$. Explain why you could start with $f'(x) = (x - 0.1)(x + 0.1) = x^2 - 0.01$. Look for a function whose derivative is as given. Graph your function to see if the extrema are "hidden." Next, construct a polynomial of degree 4 with two extrema very near $x = 1$ and another near $x = 0$.

60. Suppose that f and g are differentiable functions and $x = c$ is a critical number of both functions. Either prove (if it is true) or disprove (with a counterexample) that the composition $f \circ g$ also has a critical number at $x = c$.

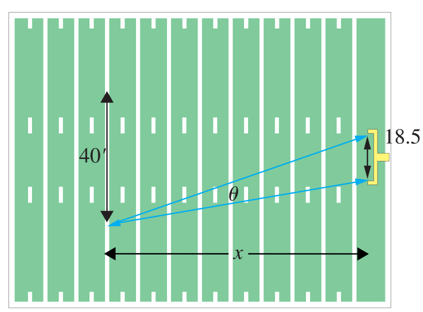
61. Show that $f(x) = x^3 + bx^2 + cx + d$ is an increasing function if $b^2 \leq 3c$.

62. Find a condition on the coefficients b and c similar to exercise 61 that guarantees that $f(x) = x^5 + bx^3 + cx + d$ is an increasing function.

63. The table shows the coefficient of friction μ of ice as a function of temperature. The lower μ is, the more "slippery" the ice is. Estimate $\mu'(C)$ at (a) $C = -10$ and (b) $C = -6$. If skating warms the ice, does it get easier or harder to skate? Briefly explain.

$°C$	-12	-10	-8	-6	-4	-2
μ	0.0048	0.0045	0.0043	0.0045	0.0048	0.0055

64. For a college football field with the dimensions shown, the angle θ for kicking a field goal from a (horizontal) distance of x feet from the goal post is given by $\theta(x) = \tan^{-1}(29.25/x) - \tan^{-1}(10.75/x)$. Show that $f(t) = \dfrac{t}{a^2 + t^2}$ is increasing for $a > t$ and use this fact to show that $\theta(x)$ is a decreasing function for $x \geq 30$. Announcers often say that for a short field goal ($50 \leq x \leq 60$), a team can improve the angle by backing up 5 yards with a penalty. Is this true?



EXPLORATORY EXERCISES

1. In this exercise, we look at the ability of fireflies to synchronize their flashes. (To see a remarkable demonstration of this ability, see David Attenborough's video series *Trials of Life*.) Let the function f represent an individual firefly's rhythm, so that the firefly flashes whenever $f(t)$ equals an integer. Let $e(t)$ represent the rhythm of a neighboring firefly, where again $e(t) = n$, for some integer n, whenever the neighbor flashes. One model of the interaction between fireflies is $f'(t) = \omega + A \sin [e(t) - f(t)]$ for constants ω and A. If the fireflies are synchronized $[e(t) = f(t)]$, then $f'(t) = \omega$, so the fireflies flash every $1/\omega$ time units. Assume that the difference between $e(t)$ and $f(t)$ is less than π. Show that if $f(t) < e(t)$, then $f'(t) > \omega$. Explain why this means that the individual firefly is speeding up its flash to match its neighbor. Similarly, discuss what happens if $f(t) > e(t)$.

2. The HIV virus attacks specialized T-cells that trigger the human immune system response to a foreign substance. If $T(t)$ is the population of uninfected T-cells at time t (days) and $V(t)$ is the population of infectious HIV in the bloodstream, a model that has been used to study AIDS is given by the following **differential equation** that describes the rate at which the population of T cells changes.

$$T'(t) = 10\left[1 + \frac{1}{1+V(t)}\right] - 0.02T(t) + 0.01\frac{T(t)V(t)}{100+V(t)} - 0.000024\, T(t)V(t).$$

If there is no HIV present [that is, $V(t) = 0$] and $T(t) = 1000$, show that $T'(t) = 0$. Explain why this means that the T-cell count will remain constant at 1000 (cells per cubic mm). Now, suppose that $V(t) = 100$. Show that if $T(t)$ is small enough, then $T'(t) > 0$ and the T-cell population will increase. On the other hand, if $T(t)$ is large enough, then $T'(t) < 0$ and the T-cell population will decrease. For what value of $T(t)$ is $T'(t) = 0$? Even though this population would remain stable, explain why this would be bad news for the infected human.

3. In a sport like soccer or hockey where ties are possible, the probability that the stronger team wins depends in an interesting way on the number of goals scored. Suppose that at any point, the probability that team A scores the next goal is p, where $0 < p < 1$. If 2 goals are scored, a 1-1 tie could result from team A scoring first (probability p) and then team B tieing the score (probability $1 - p$), or vice versa. The probability of a tie in a 2-goal game is then $2p(1 - p)$. Similarly, the probability of a 2-2 tie in a 4-goal game is $\frac{4\cdot 3}{2\cdot 1}p^2(1-p)^2$, the probability of a 3-3 tie in a 6-goal game is $\frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}p^3(1-p)^3$ and so on. As the number of goals increases, does the probability of a tie increase or decrease? To find out, first show that $\frac{(2x+2)(2x+1)}{(x+1)^2} < 4$ for $x > 0$ and $x(1-x) \le \frac{1}{4}$ for $0 \le x \le 1$. Use these inequalities to show that the probability of a tie decreases as the (even) number of goals increases. In a 1-goal game, the probability that team A wins is p. In a 2-goal game, the probability that team A wins is p^2. In a 3-goal game, the probability that team A wins is $p^3 + 3p^2(1-p)$. In a 4-goal game, the probability that team A wins is $p^4 + 4p^3(1-p)$. In a 5-goal game, the probability that team A wins is $p^5 + 5p^4(1-p) + \frac{5\cdot 4}{2\cdot 1}p^3(1-p)^2$. Explore the extent to which the probability that team A wins increases as the number of goals increases.

3.5 CONCAVITY AND THE SECOND DERIVATIVE TEST

In section 3.4, we saw how to determine where a function is increasing and decreasing and how this relates to drawing a graph of the function. First, recognize that simply knowing where a function increases and decreases is not sufficient to draw a good graph. In Figures 3.54a and 3.54b, we show two very different shapes of increasing functions joining the same two points.

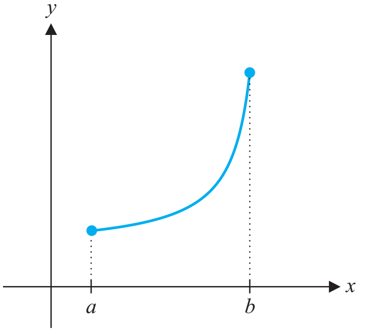


FIGURE 3.54a
Increasing function

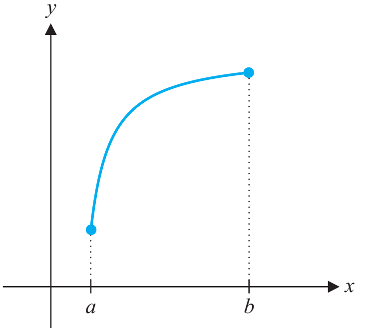


FIGURE 3.54b
Increasing function

Given that a curve joins two particular points and is increasing, we need further information to determine which of the two shapes shown (if either) we should draw. Realize that this is an important distinction to make. For example, suppose that Figure 3.54a or 3.54b depicts the balance in your bank account. Both indicate a balance that is growing. However, the rate of growth in Figure 3.54a, is increasing, while the rate of growth depicted in Figure 3.54b is decreasing. Which would you want to have describe your bank balance? Why?

Figures 3.55a and 3.55b are the same as Figures 3.54a and 3.54b, respectively, but with a few tangent lines drawn in.

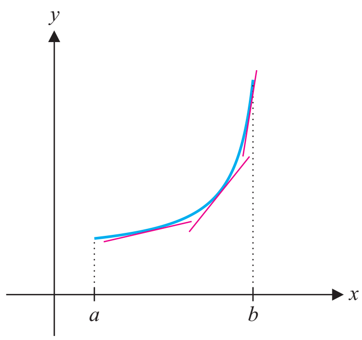


FIGURE 3.55a
Concave up, increasing

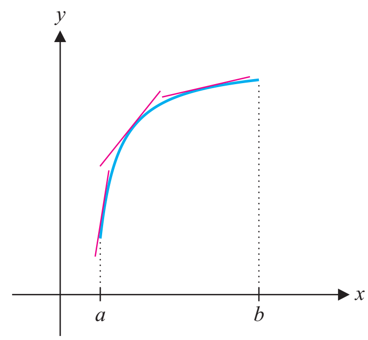


FIGURE 3.55b
Concave down, increasing

Although all of the tangent lines have positive slope [since $f'(x) > 0$], the slopes of the tangent lines in Figure 3.55a are increasing, while those in Figure 3.55b are decreasing. We call the graph in Figure 3.55a **concave up** and the graph in Figure 3.55b **concave down.** The situation is similar for decreasing functions. In Figures 3.56a and 3.56b, we show two different shapes of decreasing functions. The one shown in Figure 3.56a is concave up (slopes of tangent lines increasing) and the one shown in Figure 3.56b is concave down (slopes of tangent lines decreasing). We summarize this in Definition 5.1.

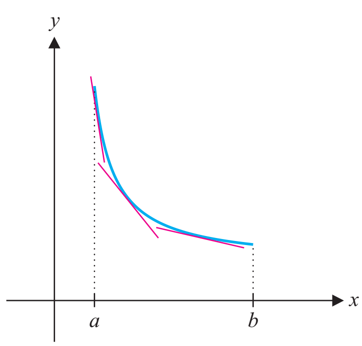


FIGURE 3.56a
Concave up, decreasing

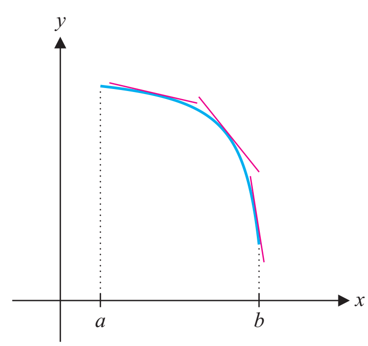


FIGURE 3.56b
Concave down, decreasing

DEFINITION 5.1

For a function f that is differentiable on an interval I, the graph of f is

(i) **concave up** on I if f' is increasing on I or
(ii) **concave down** on I if f' is decreasing on I.

Note that you can tell when f' is increasing or decreasing from the derivative of f' (i.e., f''). Theorem 5.1 connects concavity with what we already know about increasing and decreasing functions. The proof is a straightforward application of Theorem 4.1 to Definition 5.1.

THEOREM 5.1

Suppose that f'' exists on an interval I.

(i) If $f''(x) > 0$ on I, then the graph of f is concave up on I.
(ii) If $f''(x) < 0$ on I, then the graph of f is concave down on I.

EXAMPLE 5.1 Determining Concavity

Determine where the graph of $f(x) = 2x^3 + 9x^2 - 24x - 10$ is concave up and concave down, and draw a graph showing all significant features of the function.

Solution Here, we have $\quad f'(x) = 6x^2 + 18x - 24$

and from our work in example 4.3, we have

$$f'(x) \begin{cases} > 0 \text{ on } (-\infty, -4) \cup (1, \infty) & f \text{ increasing.} \\ < 0 \text{ on } (-4, 1). & f \text{ decreasing.} \end{cases}$$

Further, we have $\quad f''(x) = 12x + 18 \begin{cases} > 0, \text{ for } x > -\frac{3}{2} & \text{Concave up.} \\ < 0, \text{ for } x < -\frac{3}{2}. & \text{Concave down.} \end{cases}$

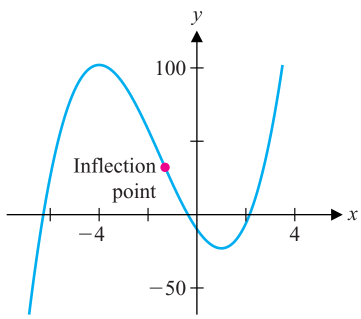


FIGURE 3.57
$y = 2x^3 + 9x^2 - 24x - 10$

Using all of this information, we are able to draw the graph shown in Figure 3.57. Notice that at the point $\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right)$, the graph changes from concave down to concave up. Such points are called *inflection points,* which we define more precisely in Definition 5.2. ■

NOTES

If $(c, f(c))$ is an inflection point, then either $f''(c) = 0$ or $f''(c)$ is undefined. So, finding all points where $f''(x)$ is zero or is undefined gives you all possible candidates for inflection points. But beware: not all points where $f''(x)$ is zero or undefined correspond to inflection points.

DEFINITION 5.2

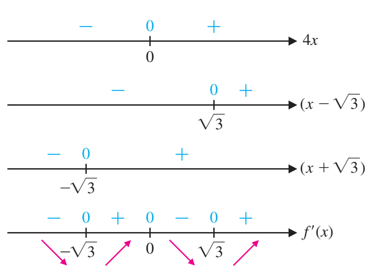
Suppose that f is continuous on the interval (a, b) and that the graph changes concavity at a point $c \in (a, b)$ (i.e., the graph is concave down on one side of c and concave up on the other). Then, the point $(c, f(c))$ is called an **inflection point** of f.

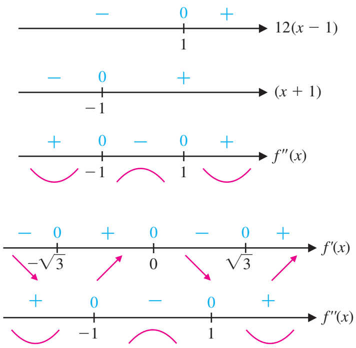
EXAMPLE 5.2 Determining Concavity and Locating Inflection Points

Determine where the graph of $f(x) = x^4 - 6x^2 + 1$ is concave up and concave down, find any inflection points and draw a graph showing all significant features.

Solution Here, we have

$$\begin{aligned} f'(x) &= 4x^3 - 12x = 4x(x^2 - 3) \\ &= 4x(x - \sqrt{3})(x + \sqrt{3}). \end{aligned}$$





We have drawn number lines for the factors of $f'(x)$ in the margin. From this, we can see that

$$f'(x)\begin{cases} > 0, \text{ on } (-\sqrt{3}, 0) \cup (\sqrt{3}, \infty) & f \text{ increasing.} \\ < 0, \text{ on } (-\infty, -\sqrt{3}) \cup (0, \sqrt{3}). & f \text{ decreasing.} \end{cases}$$

Next, we have $$f''(x) = 12x^2 - 12 = 12(x-1)(x+1).$$

We have drawn number lines for the two factors in the margin. From this, we can see that

$$f''(x)\begin{cases} > 0, \text{ on } (-\infty, -1) \cup (1, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-1, 1). & \text{Concave down.} \end{cases}$$

For convenience, we have indicated the concavity below the bottom number line, with small concave up and concave down segments. Finally, observe that since the graph changes concavity at $x = -1$ and $x = 1$, there are inflection points located at $(-1, -4)$ and $(1, -4)$. Using all of this information, we are able to draw the graph shown in Figure 3.58. For your convenience, we have reproduced the number lines for $f'(x)$ and $f''(x)$ together above the figure. ■

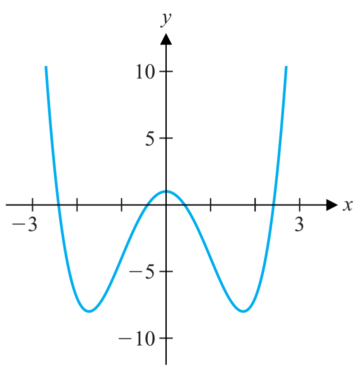


FIGURE 3.58
$y = x^4 - 6x^2 + 1$

As we see in example 5.3, having $f''(x) = 0$ does not imply the existence of an inflection point.

EXAMPLE 5.3 A Graph with No Inflection Points

Determine the concavity of $f(x) = x^4$ and locate any inflection points.

Solution There's nothing tricky about this function. We have $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Since $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$, we know that f is increasing for $x > 0$ and decreasing for $x < 0$. Further, $f''(x) > 0$ for all $x \neq 0$, while $f''(0) = 0$. So, the graph is concave up for $x \neq 0$. Further, even though $f''(0) = 0$, there is *no* inflection point at $x = 0$. We show a graph of the function in Figure 3.59. ■

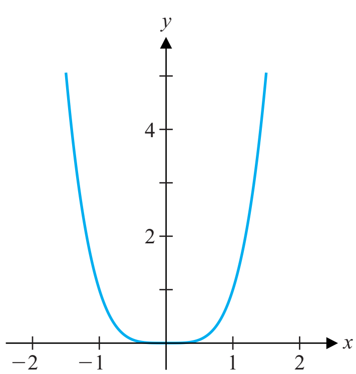


FIGURE 3.59
$y = x^4$

We now explore a connection between second derivatives and extrema. Suppose that $f'(c) = 0$ and that the graph of f is concave down in some open interval containing c. Then, near $x = c$, the graph looks like that in Figure 3.60a and hence, $f(c)$ is a local maximum. Likewise, if $f'(c) = 0$ and the graph of f is concave up in some open interval containing c, then near $x = c$, the graph looks like that in Figure 3.60b and hence, $f(c)$ is a local minimum.

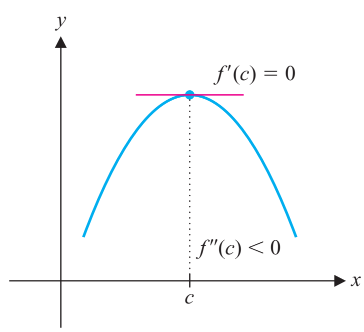


FIGURE 3.60a
Local maximum

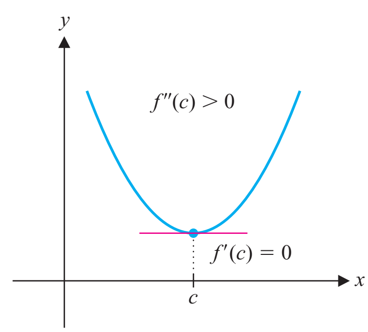


FIGURE 3.60b
Local minimum

We state this more precisely in Theorem 5.2.

THEOREM 5.2 (Second Derivative Test)

Suppose that f is continuous on the interval (a, b) and $f'(c) = 0$, for some number $c \in (a, b)$.

(i) If $f''(c) < 0$, then $f(c)$ is a local maximum.
(ii) If $f''(c) > 0$, then $f(c)$ is a local minimum.

We leave a formal proof of this theorem as an exercise. When applying the theorem, simply think about Figures 3.60a and 3.60b.

EXAMPLE 5.4 Using the Second Derivative Test to Find Extrema

Use the Second Derivative Test to find the local extrema of $f(x) = x^4 - 8x^2 + 10$.

Solution Here,

$$\begin{aligned} f'(x) &= 4x^3 - 16x = 4x(x^2 - 4) \\ &= 4x(x-2)(x+2). \end{aligned}$$

Thus, the critical numbers are $x = 0$, 2 and -2. We also have

$$f''(x) = 12x^2 - 16$$

and so,

$$\begin{aligned} f''(0) &= -16 < 0, \\ f''(-2) &= 32 > 0 \end{aligned}$$

and

$$f''(2) = 32 > 0.$$

So, by the Second Derivative Test, $f(0)$ is a local maximum and $f(-2)$ and $f(2)$ are local minima. We show a graph of $y = f(x)$ in Figure 3.61. ■

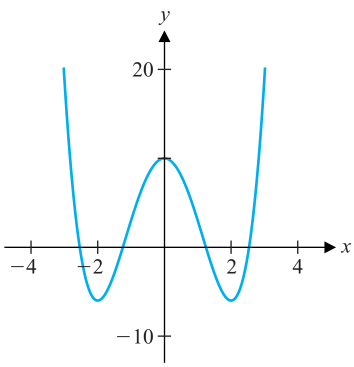


FIGURE 3.61
$y = x^4 - 8x^2 + 10$

REMARK 5.1

If $f''(c) = 0$ or $f''(c)$ is undefined, the Second Derivative Test yields no conclusion. That is, $f(c)$ may be a local maximum, a local minimum or neither. In this event, we must rely solely on first derivative information (i.e., the First Derivative Test) to determine whether $f(c)$ is a local extremum. We illustrate this with example 5.5.

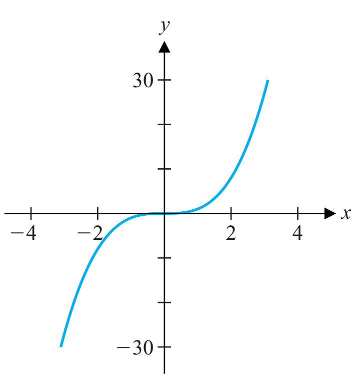


FIGURE 3.62a
$y = x^3$

EXAMPLE 5.5 Functions for Which the Second Derivative Test Is Inconclusive

Use the Second Derivative Test to try to classify any local extrema for (a) $f(x) = x^3$, (b) $g(x) = x^4$ and (c) $h(x) = -x^4$.

Solution (a) Note that $f'(x) = 3x^2$ and $f''(x) = 6x$. So, the only critical number is $x = 0$ and $f''(0) = 0$, also. We leave it as an exercise to show that the point (0, 0) is not a local extremum (see Figure 3.62a).

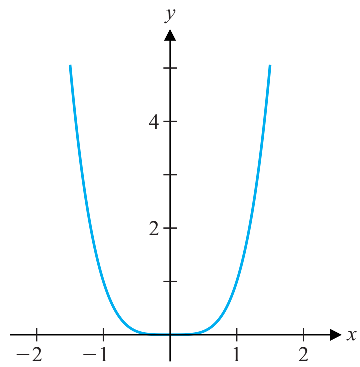


FIGURE 3.62b
$y = x^4$

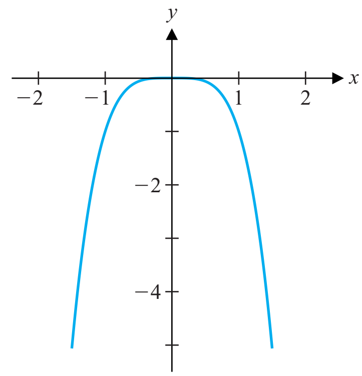


FIGURE 3.62c
$y = -x^4$

(b) We have $g'(x) = 4x^3$ and $g''(x) = 12x^2$. Again, the only critical number is $x = 0$ and $g''(0) = 0$. In this case, though, $g'(x) < 0$ for $x < 0$ and $g'(x) > 0$ for $x > 0$. So, by the First Derivative Test, $(0, 0)$ is a local minimum (see Figure 3.62b).

(c) Finally, we have $h'(x) = -4x^3$ and $h''(x) = -12x^2$. Once again, the only critical number is $x = 0$, $h''(0) = 0$ and we leave it as an exercise to show that $(0, 0)$ is a local maximum for h (see Figure 3.62c). ■

We can use first and second derivative information to help produce a meaningful graph of a function, as in example 5.6.

EXAMPLE 5.6 Drawing a Graph of a Rational Function

Draw a graph of $f(x) = x + \dfrac{25}{x}$, showing all significant features.

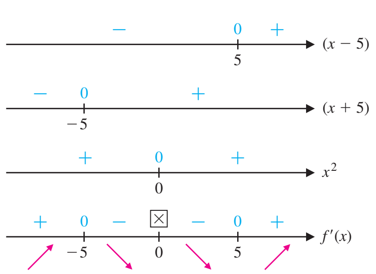
Solution The domain of f consists of all real numbers other than $x = 0$. Then,

$$f'(x) = 1 - \frac{25}{x^2} = \frac{x^2 - 25}{x^2} \qquad \text{Add the fractions.}$$
$$= \frac{(x-5)(x+5)}{x^2}.$$

So, the only two critical numbers are $x = -5$ and $x = 5$. (Why is $x = 0$ *not* a critical number?)

Looking at the three factors in $f'(x)$, we get the number lines shown in the margin. Thus,

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, -5) \cup (5, \infty) & f \text{ increasing.} \\ < 0, \text{ on } (-5, 0) \cup (0, 5). & f \text{ decreasing.} \end{cases}$$



Further,

$$f''(x) = \frac{50}{x^3} \begin{cases} > 0, \text{ on } (0, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-\infty, 0). & \text{Concave down.} \end{cases}$$

Be careful here. There is *no* inflection point on the graph, even though the graph is concave up on one side of $x = 0$ and concave down on the other. (Why not?) We can now use either the First Derivative Test or the Second Derivative Test to determine the local extrema. Since

$$f''(5) = \frac{50}{125} > 0$$

and

$$f''(-5) = -\frac{50}{125} < 0,$$

there is a local minimum at $x = 5$ and a local maximum at $x = -5$ by the Second Derivative Test. Finally, before we can draw a representative graph, we need to know what happens to the graph near $x = 0$, since 0 is not in the domain of f. We have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x + \frac{25}{x}\right) = \infty$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x + \frac{25}{x}\right) = -\infty,$$

so that there is a vertical asymptote at $x = 0$. Putting together all of this information, we get the graph shown in Figure 3.63. ■

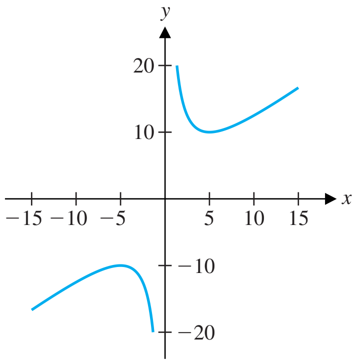


FIGURE 3.63
$y = x + \frac{25}{x}$

In example 5.6, we computed $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ to uncover the behavior of the function near $x = 0$, since $x = 0$ was not in the domain of f. In example 5.7, we'll see that since $x = -2$ is not in the domain of f' (although it *is* in the domain of f), we must compute $\lim_{x\to -2^+} f'(x)$ and $\lim_{x\to -2^-} f'(x)$ to uncover the behavior of the tangent lines near $x = -2$.

EXAMPLE 5.7 A Function with a Vertical Tangent Line at an Inflection Point

Draw a graph of $f(x) = (x + 2)^{1/5} + 4$, showing all significant features.

Solution First, notice that the domain of f is the entire real line. We also have

$$f'(x) = \frac{1}{5}(x + 2)^{-4/5} > 0, \text{ for } x \neq -2.$$

So, f is increasing everywhere, except at $x = -2$ [the only critical number, where $f'(-2)$ is undefined]. This also says that f has no local extrema. Further,

$$f''(x) = -\frac{4}{25}(x + 2)^{-9/5} \begin{cases} > 0, & \text{on } (-\infty, -2) \quad \text{Concave up.} \\ < 0, & \text{on } (-2, \infty). \quad \text{Concave down.} \end{cases}$$

So, there is an inflection point at $x = -2$. In this case, $f'(x)$ is undefined at $x = -2$. Since -2 is in the domain of f, but not in the domain of f', we consider

$$\lim_{x\to -2^-} f'(x) = \lim_{x\to -2^-} \frac{1}{5}(x + 2)^{-4/5} = \infty$$

and

$$\lim_{x\to -2^+} f'(x) = \lim_{x\to -2^+} \frac{1}{5}(x + 2)^{-4/5} = \infty.$$

This says that the graph has a vertical tangent line at $x = -2$. Putting all of this information together, we get the graph shown in Figure 3.64. ■

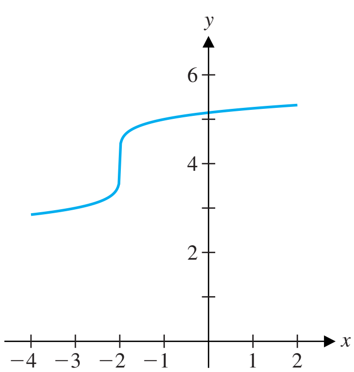


FIGURE 3.64
$y = (x + 2)^{1/5} + 4$

EXERCISES 3.5

WRITING EXERCISES

1. It is often said that a graph is concave up if it "holds water." This is certainly true for parabolas like $y = x^2$, but is it true for graphs like $y = 1/x^2$? It is always helpful to put a difficult concept into everyday language, but the danger is in oversimplification. Do you think that "holds water" is helpful or can it be confusing? Give your own description of concave up, using everyday language. (Hint: One popular image involves smiles and frowns.)
2. Find a reference book with the population of the United States since 1800. From 1800 to 1900, the numerical increase by decade increased. Argue that this means that the population curve is concave up. From 1960 to 1990, the numerical increase by decade has been approximately constant. Argue that this means that the population curve is near a point where the curve is neither concave up nor concave down. Why does this not necessarily mean that we are at an inflection point? Argue that we should hope, in order to avoid overpopulation, that it is indeed an inflection point.
3. The goal of investing in the stock market is to buy low and sell high. But, how can you tell whether a price has peaked or not? Once a stock price goes down, you can see that it *was* at a peak, but then it's too late to do anything about it! Concavity can help. Suppose a stock price is increasing and the price curve is concave up. Why would you suspect that it will continue to rise? Is this a good time to buy? Now, suppose the price is increasing but the curve is concave down. Why should you be preparing to sell? Finally, suppose the price is decreasing. If the curve is concave up, should you buy or sell? What if the curve is concave down?
4. Suppose that $f(t)$ is the amount of money in your bank account at time t. Explain in terms of spending and saving what would cause $f(t)$ to be decreasing and concave down; increasing and concave up; decreasing and concave up.

In exercises 1–8, determine the intervals where the graph of the given function is concave up and concave down.

1. $f(x) = x^3 - 3x^2 + 4x - 1$
2. $f(x) = x^4 - 6x^2 + 2x + 3$
3. $f(x) = x + 1/x$
4. $f(x) = x + 3(1 - x)^{1/3}$
5. $f(x) = \sin x - \cos x$
6. $f(x) = \tan^{-1}(x^2)$
7. $f(x) = x^{4/3} + 4x^{1/3}$
8. $f(x) = xe^{-4x}$

In exercises 9–14, find all critical numbers and use the Second Derivative Test to determine all local extrema.

9. $f(x) = x^4 + 4x^3 - 1$
10. $f(x) = x^4 + 4x^2 + 1$
11. $f(x) = xe^{-x}$
12. $f(x) = e^{-x^2}$
13. $f(x) = \dfrac{x^2 - 5x + 4}{x}$
14. $f(x) = \dfrac{x^2 - 1}{x}$

In exercises 15–26, determine all significant features by hand and sketch a graph.

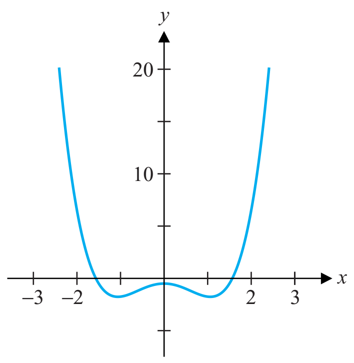
15. $f(x) = (x^2 + 1)^{2/3}$
16. $f(x) = x \ln x$
17. $f(x) = \dfrac{x^2}{x^2 - 9}$
18. $f(x) = \dfrac{x}{x + 2}$
19. $f(x) = \sin x + \cos x$
20. $f(x) = e^{-x} \sin x$
21. $f(x) = x^{3/4} - 4x^{1/4}$
22. $f(x) = x^{2/3} - 4x^{1/3}$
23. $f(x) = x|x|$
24. $f(x) = x^2|x|$
25. $f(x) = x^{1/5}(x + 1)$
26. $f(x) = \dfrac{\sqrt{x}}{1 + \sqrt{x}}$

In exercises 27–36, determine all significant features (approximately if necessary) and sketch a graph.

27. $f(x) = x^4 - 26x^3 + x$
28. $f(x) = 2x^4 - 11x^3 + 17x^2$
29. $f(x) = \sqrt[3]{2x^2 - 1}$
30. $f(x) = \sqrt{x^3 + 1}$
31. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
32. $f(x) = x^4 + 32x^3 - 0.02x^2 - 0.8x$
33. $f(x) = x\sqrt{x^2 - 4}$
34. $f(x) = \dfrac{2x}{\sqrt{x^2 + 4}}$
35. $f(x) = \tan^{-1}\left(\dfrac{1}{x^2 - 1}\right)$
36. $f(x) = e^{-2x} \cos x$

In exercises 37–40, estimate the intervals where the function is concave up and concave down. (Hint: Estimate where the slope is increasing and decreasing.)

37.



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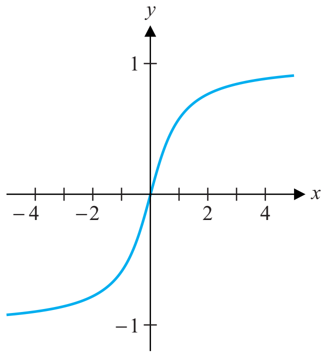

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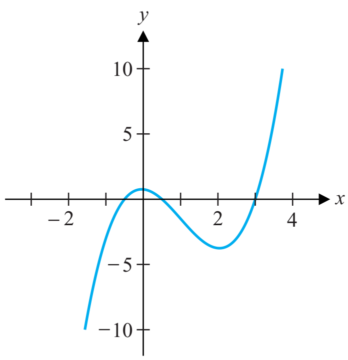

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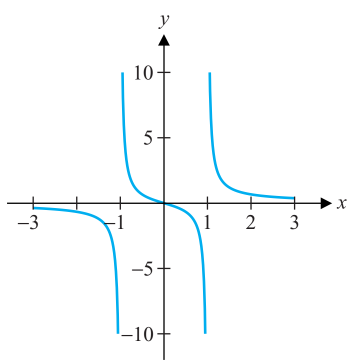


In exercises 41–44, sketch a graph with the given properties.

41. $f(0) = 0$, $f'(x) > 0$ for $x < -1$ and $-1 < x < 1$, $f'(x) < 0$ for $x > 1$, $f''(x) > 0$ for $x < -1$, $0 < x < 1$ and $x > 1$, $f''(x) < 0$ for $-1 < x < 0$

42. $f(0) = 2$, $f'(x) > 0$ for all x, $f'(0) = 1$, $f''(x) > 0$ for $x < 0$, $f''(x) < 0$ for $x > 0$

43. $f(0) = 0$, $f(-1) = -1$, $f(1) = 1$, $f'(x) > 0$ for $x < -1$ and $0 < x < 1$, $f'(x) < 0$ for $-1 < x < 0$ and $x > 1$, $f''(x) < 0$ for $x < 0$ and $x > 0$

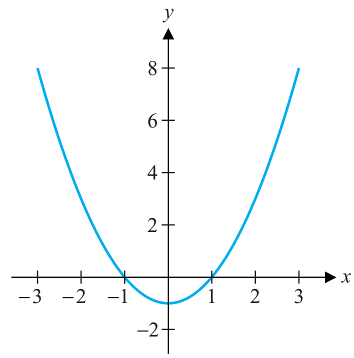
44. $f(1) = 0$, $f'(x) < 0$ for $x < 1$, $f'(x) > 0$ for $x > 1$, $f''(x) < 0$ for $x < 1$ and $x > 1$

45. Show that any cubic $f(x) = ax^3 + bx^2 + cx + d$ has one inflection point. Find conditions on the coefficients a–e that guarantee that the quartic $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ has two inflection points.

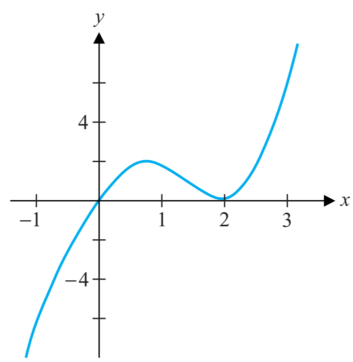
46. If f and g are functions with two derivatives for all x, $f(0) = g(0) = f'(0) = g'(0) = 0$, $f''(0) > 0$ and $g''(0) < 0$, state as completely as possible what can be said about whether $f(x) > g(x)$ or $f(x) < g(x)$.

In exercises 47 and 48, estimate the intervals of increase and decrease, the locations of local extrema, intervals of concavity and locations of inflection points.

47.



48.



49. Repeat exercises 47 and 48 if the given graph is of $f'(x)$ instead of $f(x)$.

50. Repeat exercises 47 and 48 if the given graph is of $f''(x)$ instead of $f(x)$.

51. Suppose that $w(t)$ is the depth of water in a city's water reservoir at time t. Which would be better news at time $t = 0$, $w''(0) = 0.05$ or $w''(0) = -0.05$, or would you need to know the value of $w'(0)$ to determine which is better?

52. Suppose that $T(t)$ is a sick person's temperature at time t. Which would be better news at time t, $T''(0) = 2$ or $T''(0) = -2$, or would you need to know the value of $T'(0)$ and $T(0)$ to determine which is better?

53. Suppose that a company that spends \$$x$ thousand on advertising sells \$$s(x)$ of merchandise, where $s(x) = -3x^3 + 270x^2 - 3600x + 18{,}000$. Find the value of x that maximizes the rate of change of sales. (Hint: Read the question carefully!) Find the inflection point and explain why in advertising terms this is the "point of diminishing returns."

54. The number of units Q that a worker has produced in a day is related to the number of hours t since the work day began. Suppose that $Q(t) = -t^3 + 6t^2 + 12t$. Explain why $Q'(t)$ is a measure of the efficiency of the worker at time t. Find the time at which the worker's efficiency is a maximum. Explain why it is reasonable to call the inflection point the "point of diminishing returns."

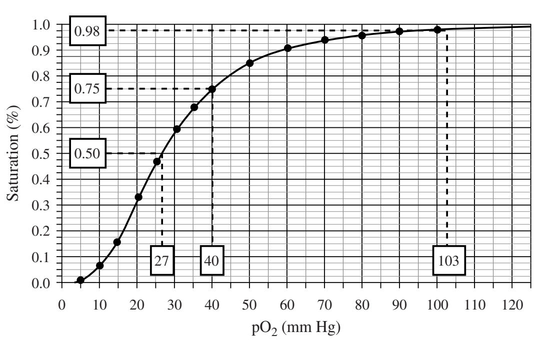
55. Suppose that it costs a company $C(x) = 0.01x^2 + 40x + 3600$ dollars to manufacture x units of a product. For this **cost function,** the **average cost function** is $\overline{C}(x) = \dfrac{C(x)}{x}$. Find the value of x that minimizes the average cost. The cost function can be related to the efficiency of the production process. Explain why a cost function that is concave down indicates better efficiency than a cost function that is concave up.

56. The antiarrhythmic drug lidocaine slowly decays after entering the bloodstream. The plasma concentration t minutes after administering the drug can be modeled by

$$c(t) = 92.8\left(-0.129e^{-t/6.55} + 0.218e^{-t/65.7} - 0.089e^{-t/13.3}\right).$$

Use a CAS to estimate the time of maximum concentration and inflection point ($t > 0$). Suppose that $f(t)$ represents the concentration of another drug. If the graph of $f(t)$ has a similar shape and the same maximum point as the graph of $c(t)$ but the inflection point occurs at a larger value of t, would this drug be more or less effective than lidocaine? Briefly explain.

57. The plot shows the relationship between the specific partial pressure of oxygen (pO_2, measured in mm Hg) and the saturation level of hemoglobin ($y = 1$ would mean that no more oxygen can bind).



Determine whether $f_1(x) = \dfrac{x}{27 + x}$ or $f_2(x) = \dfrac{x^3}{27^3 + x^3}$ is a better model for these data by finding extrema, inflection points and asymptotes (for $x \geq 0$) for each function and comparing to the graph. Each of these functions is called a **Hill function.**

58. Two functions with similar properties to the Hill functions of exercise 57 are $g_1(x) = \tan^{-1} x$ and $g_2(x) = \dfrac{1}{1 + 99e^{-x/2}}$. As in exercise 57, test these functions for use as a model of the hemoglobin data.

59. Give an example of a function showing that the following statement is false. If the graph of $y = f(x)$ is concave down for all x, the equation $f(x) = 0$ has at least one solution.

60. Determine whether the following statement is true or false. If $f(0) = 1$, $f''(x)$ exists for all x and the graph of $y = f(x)$ is concave down for all x, the equation $f(x) = 0$ has at least one solution.

61. A basic principle of physics is that light follows the path of minimum time. Assuming that the speed of light in the earth's atmosphere decreases as altitude decreases, argue that the path that light follows is concave down. Explain why this means that the setting sun appears higher in the sky than it really is.

62. Prove Theorem 5.2 (the Second Derivative Test). (Hint: Think about what the definition of $f''(c)$ says when $f''(c) > 0$ or $f''(c) < 0$.)

EXPLORATORY EXERCISES

1. The linear approximation that we defined in section 3.1 is the line having the same location and the same slope as the function being approximated. Since two points determine a line, two requirements (point, slope) are all that a linear function can satisfy. However, a quadratic function can satisfy three requirements, since three points determine a parabola (and there are three constants in a general quadratic function $ax^2 + bx + c$). Suppose we want to define a **quadratic approximation** to $f(x)$ at $x = a$. Building on the linear approximation, the general form is $g(x) = f(a) + f'(a)(x - a) + c(x - a)^2$ for some constant c to be determined. In this way, show that $g(a) = f(a)$ and $g'(a) = f'(a)$. That is, $g(x)$ has the right position and slope at $x = a$. The third requirement is that $g(x)$ have the right concavity at $x = a$, so that $g''(a) = f''(a)$. Find the constant c that makes this true. Then, find such a quadratic approximation for each of the functions $\sin x$, $\cos x$ and e^x at $x = 0$. In each case, graph the original function, linear approximation and quadratic approximation, and describe how close the approximations are to the original functions.

2. In this exercise, we explore a basic problem in genetics. Suppose that a species reproduces according to the following probabilities: p_0 is the probability of having no children, p_1 is the probability of having one offspring, p_2 is the probability of having two offspring, ..., p_n is the probability of having n offspring and n is the largest number of offspring possible. Explain why for each i, we have $0 \leq p_i \leq 1$ and $p_0 + p_1 + p_2 + \cdots + p_n = 1$. We define the function $F(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$. The smallest nonnegative solution of the equation $F(x) = x$ for $0 \leq x \leq 1$ represents the probability that the species becomes extinct. Show graphically that if $p_0 > 0$ and $F'(1) > 1$, then there is a solution of $F(x) = x$ with $0 < x < 1$. Thus, there is a positive probability of survival. However, if $p_0 > 0$ and $F'(1) < 1$, show that there are no solutions of $F(x) = x$ with $0 < x < 1$. (Hint: First show that F is increasing and concave up.)

3. Give as complete a description of the graph of $f(x) = \dfrac{x + c}{x^2 - 1}$ as possible. In particular, find the values of c for which there are two critical points (or one critical point, or no critical points) and identify any extrema. Similarly, determine how the existence or not of inflection points depends on the value of c.

3.6 OVERVIEW OF CURVE SKETCHING

Graphing calculators and computer algebra systems are powerful tools in the study or application of mathematics. However, they do not actually draw graphs. What they do is plot points (albeit lots of them) and then connect the points as smoothly as possible. While this is very helpful, it often leaves something to be desired. The problem boils down to knowing the window in which you should draw a given graph and how many points you plot in that window. The only way to know how to choose this is to use the calculus to determine the properties of the graph that you are interested in seeing. We have already made this point a number of times.

We begin this section by summarizing the various tests that you should perform on a function when trying to draw a graph of $y = f(x)$.

- **Domain:** You should always determine the domain of f *first*.
- **Vertical Asymptotes:** For any isolated point not in the domain of f, check the limiting value of the function as x approaches that point, to see if there is a vertical asymptote or a jump or removable discontinuity at that point.
- **First Derivative Information:** Determine where f is increasing and decreasing, and find any local extrema.
- **Vertical Tangent Lines:** At any isolated point not in the domain of f', *but in the domain of f*, check the limiting values of $f'(x)$, to determine whether there is a vertical tangent line at that point.
- **Second Derivative Information:** Determine where the graph is concave up and concave down, and locate any inflection points.
- **Horizontal Asymptotes:** Check the limit of $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
- **Intercepts:** Locate x- and y-intercepts, if any. If this can't be done exactly, then do so approximately (e.g., using Newton's method).

We start with a very straightforward example.

EXAMPLE 6.1 Drawing a Graph of a Polynomial

Draw a graph of $f(x) = x^4 + 6x^3 + 12x^2 + 8x + 1$, showing all significant features.

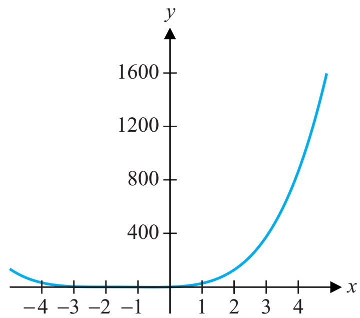


FIGURE 3.65a
$y = x^4 + 6x^3 + 12x^2 + 8x + 1$
(one view)

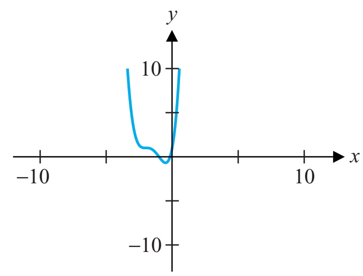


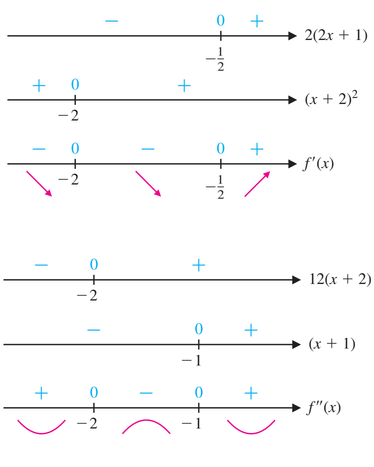
FIGURE 3.65b
$y = x^4 + 6x^3 + 12x^2 + 8x + 1$
(standard calculator view)

Solution Computer algebra systems and graphing calculators usually do one of two things to determine the window in which they will display a graph. One method is to compute a set number of function values over a given standard range of x-values. The y-range is then chosen so that all of the calculated points can be displayed. This might result in a graph that looks like the one in Figure 3.65a. Another method is to draw a graph in a fixed, default window. For instance, most graphing calculators use the default window defined by

$$-10 \le x \le 10 \quad \text{and} \quad -10 \le y \le 10.$$

Using this window, we get the graph shown in Figure 3.65b. Of course, these two graphs are very different. Without the calculus, it's difficult to tell which, if either, of these is truly representative of the behavior of f. Some analysis will clear up the situation. First, note that the domain of f is the entire real line. Further, since $f(x)$ is a polynomial, it doesn't have any vertical or horizontal asymptotes. Next, note that

$$f'(x) = 4x^3 + 18x^2 + 24x + 8 = 2(2x + 1)(x + 2)^2.$$



Drawing number lines for the individual factors of $f'(x)$, we have that

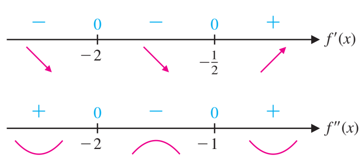
$$f'(x)\begin{cases} > 0, \text{ on } \left(-\frac{1}{2}, -\infty\right) & f \text{ increasing.} \\ < 0, \text{ on } (-\infty, -2) \cup \left(-2, -\frac{1}{2}\right). & f \text{ decreasing.} \end{cases}$$

This also tells us that there is a local minimum at $x = -\frac{1}{2}$ and that there are no local maxima. Next, we have

$$f''(x) = 12x^2 + 36x + 24 = 12(x + 2)(x + 1).$$

Drawing number lines for the factors of $f''(x)$, we have

$$f''(x)\begin{cases} > 0, \text{ on } (-\infty, -2) \cup (-1, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-2, -1). & \text{Concave down.} \end{cases}$$



From this, we see that there are inflection points at $x = -2$ and at $x = -1$. Finally, to find the x-intercepts, we need to solve $f(x) = 0$ approximately. Doing this (we leave the details as an exercise: use Newton's method or your calculator's solver), we find that there are two x-intercepts: $x = -1$ (exactly) and $x \approx -0.160713$. Notice that the significant x-values that we have identified are $x = -2$, $x = -1$ and $x = -\frac{1}{2}$. Computing the corresponding y-values from $y = f(x)$, we get the points $(-2, 1)$, $(-1, 0)$ and $\left(-\frac{1}{2}, -\frac{11}{16}\right)$. We summarize the first and second derivative information in the number lines in the margin. In Figure 3.66, we include all of these important points by setting the x-range to be $-4 \le x \le 1$ and the y-range to be $-2 \le y \le 8$. ■

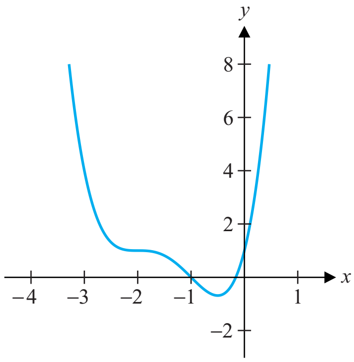


FIGURE 3.66
$y = x^4 + 6x^3 + 12x^2 + 8x + 1$

In example 6.2, we examine a function that has local extrema, inflection points and both vertical and horizontal asymptotes.

EXAMPLE 6.2 Drawing a Graph of a Rational Function

Draw a graph of $f(x) = \dfrac{x^2 - 3}{x^3}$, showing all significant features.

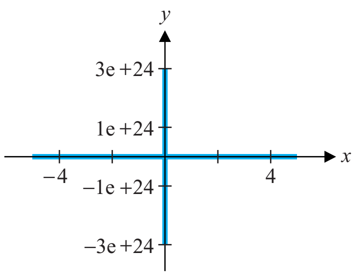


FIGURE 3.67a
$y = \dfrac{x^2 - 3}{x^3}$

Solution The default graph drawn by our computer algebra system appears in Figure 3.67a. Notice that this doesn't seem to be a particularly useful graph, since very little is visible (or at least distinguishable from the axes). The graph drawn using the most common graphing calculator default window is seen in Figure 3.67b. This is arguably an improvement over Figure 3.67a, but does this graph convey all that it could about the function (e.g., about local extrema, inflection points, etc.)? We can answer this question only after we do some calculus. We follow the outline given at the beginning of the section.

First, observe that the domain of f includes all real numbers $x \ne 0$. Since $x = 0$ is an isolated point not in the domain of f, we scrutinize the limiting behavior of f as x approaches 0. We have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\overset{-}{x^2 - 3}}{\underset{+}{x^3}} = -\infty \tag{6.1}$$

and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\overset{-}{x^2 - 3}}{\underset{-}{x^3}} = \infty. \tag{6.2}$$

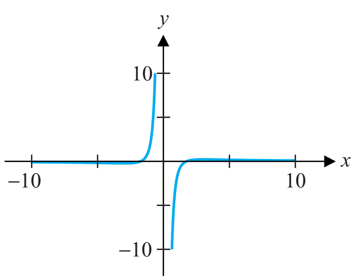
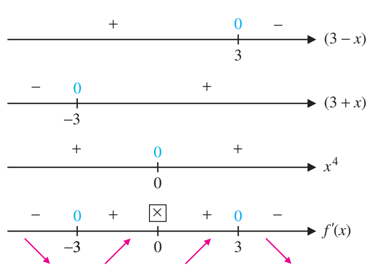


FIGURE 3.67b
$y = \dfrac{x^2 - 3}{x^3}$

From (6.1) and (6.2), we see that the graph has a vertical asymptote at $x = 0$.

Next, we look for whatever information the first derivative will yield. We have

$$\begin{aligned} f'(x) &= \frac{2x(x^3) - (x^2 - 3)(3x^2)}{(x^3)^2} && \text{Quotient rule.} \\ &= \frac{x^2[2x^2 - 3(x^2 - 3)]}{x^6} && \text{Factor out } x^2. \\ &= \frac{9 - x^2}{x^4} && \text{Combine terms.} \\ &= \frac{(3 - x)(3 + x)}{x^4}. && \text{Factor difference of two squares.} \end{aligned}$$



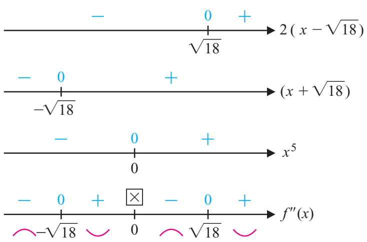
Looking at the individual factors in $f'(x)$, we have the number lines shown in the margin. Thus,

$$f'(x) \begin{cases} > 0, & \text{on } (-3, 0) \cup (0, 3) & f \text{ increasing.} \\ < 0, & \text{on } (-\infty, -3) \cup (3, \infty). & f \text{ decreasing.} \end{cases} \tag{6.3}$$

Note that this says that f has a local minimum at $x = -3$ and a local maximum at $x = 3$.

Next, we look at

$$f''(x) = \frac{-2x(x^4) - (9 - x^2)(4x^3)}{(x^4)^2} \qquad \text{Quotient rule.}$$
$$= \frac{-2x^3[x^2 + (9 - x^2)(2)]}{x^8} \qquad \text{Factor out } -2x^3.$$
$$= \frac{-2(18 - x^2)}{x^5} \qquad \text{Combine terms.}$$
$$= \frac{2(x - \sqrt{18})(x + \sqrt{18})}{x^5}. \qquad \text{Factor difference of two squares.}$$



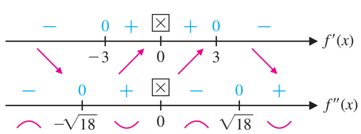
Looking at the individual factors in $f''(x)$, we obtain the number lines shown in the margin. Thus, we have

$$f''(x) \begin{cases} > 0, \text{ on } (-\sqrt{18}, 0) \cup (\sqrt{18}, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-\infty, -\sqrt{18}) \cup (0, \sqrt{18}). & \text{Concave down.} \end{cases} \tag{6.4}$$

This says that there are inflection points at $x = \pm\sqrt{18}$. (Why is there no inflection point at $x = 0$?)

To determine the limiting behavior as $x \to \pm\infty$, we consider

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2 - 3}{x^3}$$
$$= \lim_{x\to\infty} \left(\frac{1}{x} - \frac{3}{x^3}\right) = 0. \tag{6.5}$$



Likewise, we have
$$\lim_{x\to-\infty} f(x) = 0. \tag{6.6}$$

So, the line $y = 0$ is a horizontal asymptote both as $x \to \infty$ and as $x \to -\infty$. Finally, the x-intercepts are where

$$0 = f(x) = \frac{x^2 - 3}{x^3},$$

that is, at $x = \pm\sqrt{3}$. Notice that there are no y-intercepts, since $x = 0$ is not in the domain of the function. We now have all of the information that we need to draw a representative graph. With some experimentation, you can set the x- and y-ranges so that most of the significant features of the graph (i.e., vertical and horizontal asymptotes, local extrema, inflection points, etc.) are displayed, as in Figure 3.68, which is consistent with all of the information that we accumulated about the function in (6.1)–(6.6). Although the existence of the inflection points is clearly indicated by the change in concavity, their precise location is as yet a bit fuzzy in this graph. Notice, however, that both vertical and horizontal asymptotes and the local extrema are clearly indicated, something that cannot be said about either of Figures 3.67a or 3.67b. ■

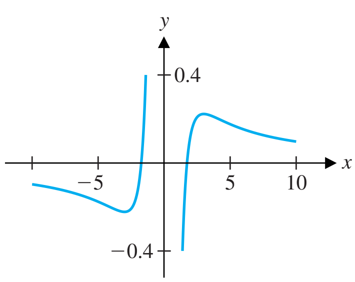


FIGURE 3.68

$$y = \frac{x^2 - 3}{x^3}$$

In example 6.3, there are multiple vertical asymptotes, only one extremum and no inflection points.

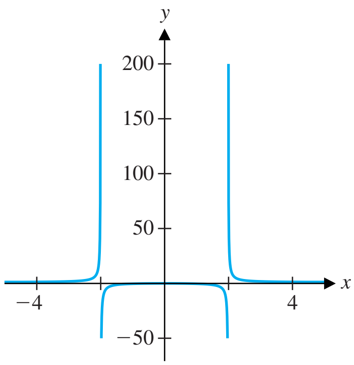


FIGURE 3.69a

$y = \dfrac{x^2}{x^2 - 4}$

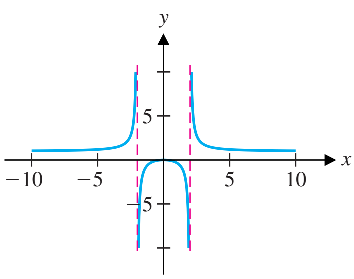


FIGURE 3.69b

$y = \dfrac{x^2}{x^2 - 4}$

EXAMPLE 6.3 A Graph with Two Vertical Asymptotes

Draw a graph of $f(x) = \dfrac{x^2}{x^2 - 4}$ showing all significant features.

Solution The default graph produced by our computer algebra system is seen in Figure 3.69a, while the default graph drawn by most graphing calculators looks like the graph seen in Figure 3.69b. Notice that the domain of f includes all x except $x = \pm 2$ (since the denominator is zero at $x = \pm 2$). Figure 3.69b suggests that there are vertical asymptotes at $x = \pm 2$, but let's establish this carefully. We have

$$\lim_{x \to 2^+} \frac{x^2}{x^2 - 4} = \lim_{x \to 2^+} \frac{\overset{+}{x^2}}{\underset{+}{(x - 2)}\,\underset{+}{(x + 2)}} = \infty. \tag{6.7}$$

Similarly, we get

$$\lim_{x \to 2^-} \frac{x^2}{x^2 - 4} = -\infty, \qquad \lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = -\infty \tag{6.8}$$

and

$$\lim_{x \to -2^-} \frac{x^2}{x^2 - 4} = \infty. \tag{6.9}$$

Thus, there are vertical asymptotes at $x = \pm 2$. Next, we have

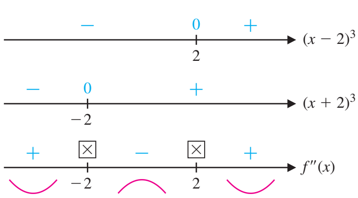
$$f'(x) = \frac{2x(x^2 - 4) - x^2(2x)}{(x^2 - 4)^2} = \frac{-8x}{(x^2 - 4)^2}.$$

Since the denominator is positive for $x \neq \pm 2$, it is a simple matter to see that

$$f'(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \cup (-2, 0) & f \text{ increasing.} \\ < 0, \text{ on } (0, 2) \cup (2, \infty). & f \text{ decreasing.} \end{cases} \tag{6.10}$$

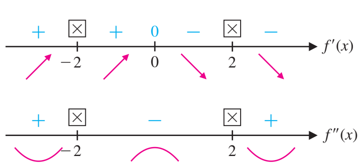
In particular, notice that the only critical number is $x = 0$ (since $x = -2, 2$ are not in the domain of f). Thus, the only local extremum is the local maximum located at $x = 0$. Next, we have

$$\begin{aligned} f''(x) &= \frac{-8(x^2 - 4)^2 + (8x)2(x^2 - 4)^1(2x)}{(x^2 - 4)^4} && \text{Quotient rule.} \\ &= \frac{8(x^2 - 4)[-(x^2 - 4) + 4x^2]}{(x^2 - 4)^4} && \text{Factor out } 8(x^2 - 4). \\ &= \frac{8(3x^2 + 4)}{(x^2 - 4)^3} && \text{Combine terms.} \\ &= \frac{8(3x^2 + 4)}{(x - 2)^3(x + 2)^3}. && \text{Factor difference of two squares.} \end{aligned}$$



Since the numerator is positive for all x, we need only consider the terms in the denominator, as seen in the margin. We then have

$$f''(x) \begin{cases} > 0, \text{ on } (-\infty, -2) \cup (2, \infty) & \text{Concave up.} \\ < 0, \text{ on } (-2, 2). & \text{Concave down.} \end{cases} \tag{6.11}$$



However, since $x = 2, -2$ are not in the domain of f, there are no inflection points. It is an easy exercise to verify that

$$\lim_{x\to\infty} \frac{x^2}{x^2-4} = 1 \tag{6.12}$$

and

$$\lim_{x\to-\infty} \frac{x^2}{x^2-4} = 1. \tag{6.13}$$

From (6.12) and (6.13), we have that $y = 1$ is a horizontal asymptote, both as $x \to \infty$ and as $x \to -\infty$. Finally, we observe that the only x-intercept is at $x = 0$. We summarize the information in (6.7)–(6.13) in the graph seen in Figure 3.70. ■

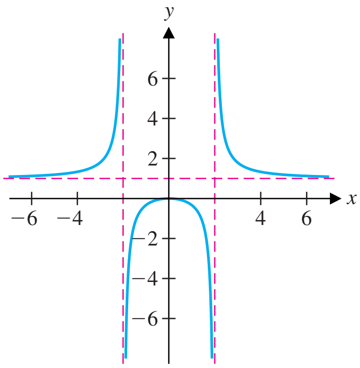


FIGURE 3.70

$y = \dfrac{x^2}{x^2-4}$

In example 6.4, we need to use computer-generated graphs, as well as a rootfinding method to determine the behavior of the function.

EXAMPLE 6.4 Graphing Where the Domain and Extrema Must Be Approximated

Draw a graph of $f(x) = \dfrac{1}{x^3+3x^2+3x+3}$ showing all significant features.

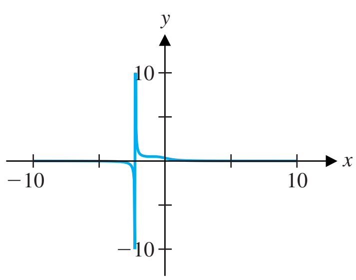


FIGURE 3.71

$y = \dfrac{1}{x^3+3x^2+3x+3}$

Solution The default graph drawn by most graphing calculators and computer algebra systems looks something like the one shown in Figure 3.71. As we have already seen, we can only determine all the significant features by doing some calculus.

Since f is a rational function, it is defined for all x, except for where the denominator is zero, that is, where

$$x^3 + 3x^2 + 3x + 3 = 0.$$

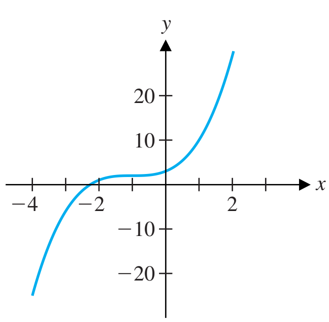


FIGURE 3.72

$y = x^3 + 3x^2 + 3x + 3$

If you don't see how to factor the expression to find the zeros exactly, you must rely on approximate methods. First, to get an idea of where the zero(s) might be, draw a graph of the cubic (see Figure 3.72). The graph does not need to be elaborate, merely detailed enough to get an idea of where and how many zeros there are. In the present case, we see that there is only one zero, around $x = -2$. We can verify that this is the only zero, since

$$\frac{d}{dx}(x^3+3x^2+3x+3) = 3x^2+6x+3 = 3(x+1)^2 \geq 0.$$

Since the derivative is never negative, observe that the function cannot decrease to cross the x-axis a second time. You can get the approximate zero $x = a \approx -2.25992$ using Newton's method or your calculator's solver. We can use the graph in Figure 3.72 to help us compute the limits

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \frac{\overset{+}{1}}{\underset{+}{x^3+3x^2+3x+3}} = \infty \tag{6.14}$$

and

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \frac{\overset{+}{1}}{\underset{-}{x^3+3x^2+3x+3}} = -\infty. \tag{6.15}$$

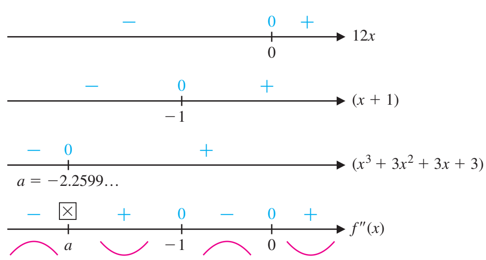
From (6.14) and (6.15), f has a vertical asymptote at $x = a$. Turning to the derivative information, we have

$$
\begin{aligned}
f'(x) &= -(x^3+3x^2+3x+3)^{-2}(3x^2+6x+3) \\
&= -3\left[\frac{(x+1)^2}{(x^3+3x^2+3x+3)^2}\right] \\
&= -3\left(\frac{x+1}{x^3+3x^2+3x+3}\right)^2 \\
&< 0, \text{ for } x \neq a \text{ or } -1
\end{aligned}
\tag{6.16}
$$

and $f'(-1) = 0$. Thus, f is decreasing for $x < a$ and $x > a$. Also, notice that the only critical number is $x = -1$, but since f is decreasing everywhere except at $x = a$, there are no local extrema. Turning to the second derivative, we get

$$
\begin{aligned}
f''(x) &= -6\left(\frac{x+1}{x^3+3x^2+3x+3}\right)\frac{1(x^3+3x^2+3x+3)-(x+1)(3x^2+6x+3)}{(x^3+3x^2+3x+3)^2} \\
&= \frac{-6(x+1)}{(x^3+3x^2+3x+3)^3}(-2x^3-6x^2-6x) \\
&= \frac{12x(x+1)(x^2+3x+3)}{(x^3+3x^2+3x+3)^3}.
\end{aligned}
$$

Since $(x^2+3x+3) > 0$ for all x (why is that?), we need not consider this factor. Considering the remaining factors, we have the number lines shown here.



Thus, we have that

$$
f''(x)\begin{cases} > 0, & \text{on } (a,-1)\cup(0,\infty) \quad \text{Concave up.} \\ < 0, & \text{on } (-\infty,a)\cup(-1,0) \quad \text{Concave down.} \end{cases}
\tag{6.17}
$$

It now follows that there are inflection points at $x = 0$ and at $x = -1$. Notice that in Figure 3.71, the concavity information is not very clear and the inflection points are difficult to discern.

We note the obvious fact that the function is never zero and hence, there are no x-intercepts. Finally, we consider the limits

$$
\lim_{x\to\infty}\frac{1}{x^3+3x^2+3x+3} = 0 \tag{6.18}
$$

and

$$
\lim_{x\to-\infty}\frac{1}{x^3+3x^2+3x+3} = 0. \tag{6.19}
$$

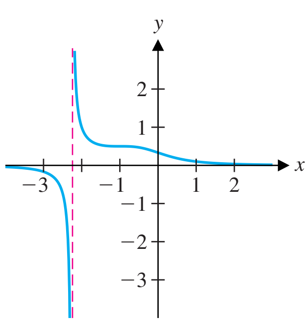


FIGURE 3.73

$y = \dfrac{1}{x^3+3x^2+3x+3}$

Using all of the information in (6.14)–(6.19), we draw the graph seen in Figure 3.73. Here, we can clearly see the vertical and horizontal asymptotes, the inflection points and the fact that the function is decreasing across its entire domain. ■

In example 6.5, we consider the graph of a transcendental function with a vertical asymptote.

EXAMPLE 6.5 Graphing Where Some Features Are Difficult to See

Draw a graph of $f(x) = e^{1/x}$ showing all significant features.

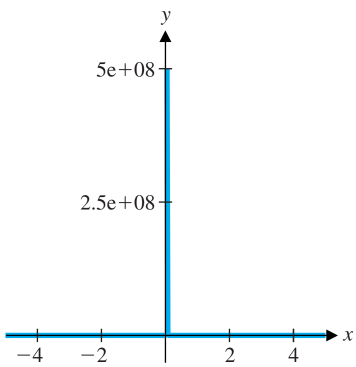


FIGURE 3.74a
$y = e^{1/x}$

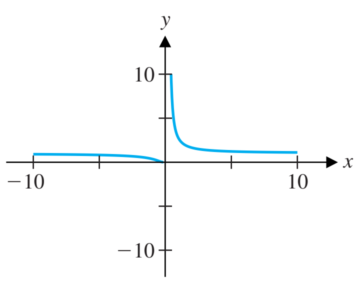


FIGURE 3.74b
$y = e^{1/x}$

Solution The default graph produced by our computer algebra system is not particularly helpful (see Figure 3.74a). The default graph produced by most graphing calculators (see Figure 3.74b) is certainly better, but we can't be sure this is adequate without further analysis. First, notice that the domain of f is $(-\infty, 0) \cup (0, \infty)$. Thus, we consider

$$\lim_{x \to 0^+} e^{1/x} = \infty, \tag{6.20}$$

since $1/x \to \infty$ as $x \to 0^+$. Also, since $1/x \to -\infty$ as $x \to 0^-$ (and $e^t \to 0$ as $t \to -\infty$), we have

$$\lim_{x \to 0^-} e^{1/x} = 0. \tag{6.21}$$

From (6.20) and (6.21), there is a vertical asymptote at $x = 0$, but an unusual one, in that $f(x) \to \infty$ on one side of 0 and $f(x) \to 0$ on the other side. Next,

$$\begin{aligned} f'(x) &= e^{1/x} \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= e^{1/x}\left(\frac{-1}{x^2}\right) < 0, \text{ for all } x \neq 0, \end{aligned} \tag{6.22}$$

since $e^{1/x} > 0$, for all $x \neq 0$. From (6.22), we have that f is decreasing for all $x \neq 0$. We also have

$$\begin{aligned} f''(x) &= e^{1/x}\left(\frac{-1}{x^2}\right)\left(\frac{-1}{x^2}\right) + e^{1/x}\left(\frac{2}{x^3}\right) \\ &= e^{1/x}\left(\frac{1}{x^4} + \frac{2}{x^3}\right) = e^{1/x}\left(\frac{1+2x}{x^4}\right) \\ &\begin{cases} < 0, \text{ on } \left(-\infty, -\frac{1}{2}\right) & \text{Concave down.} \\ > 0, \text{ on } \left(-\frac{1}{2}, 0\right) \cup (0, \infty). & \text{Concave up.} \end{cases} \end{aligned} \tag{6.23}$$

Since $x = 0$ is not in the domain of f, the only inflection point is at $x = -\frac{1}{2}$. Next, note that

$$\lim_{x \to \infty} e^{1/x} = 1, \tag{6.24}$$

since $1/x \to 0$ as $x \to \infty$ and $e^t \to 1$ as $t \to 0$. Likewise,

$$\lim_{x \to -\infty} e^{1/x} = 1. \tag{6.25}$$

From (6.24) and (6.25), $y = 1$ is a horizontal asymptote, both as $x \to \infty$ and as $x \to -\infty$. Finally, since

$$e^{1/x} > 0,$$

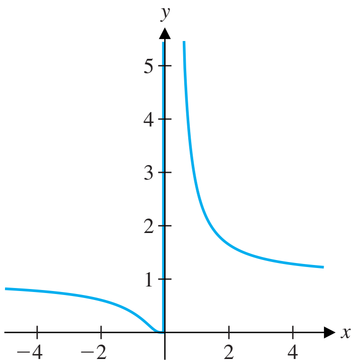


FIGURE 3.75
$y = e^{1/x}$

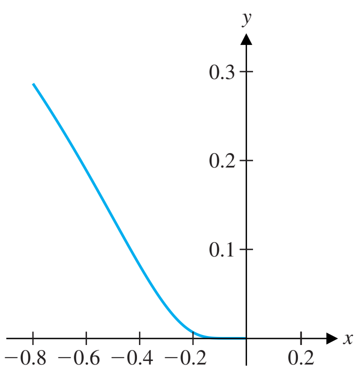


FIGURE 3.76
$y = e^{1/x}$

for all $x \neq 0$, there are no x-intercepts. It is worthwhile noting that all of these features of the graph were discernible from Figure 3.74b, except for the inflection point at $x = -\frac{1}{2}$. Note that in almost any graph you draw, it is difficult to see all of the features of the function. This happens because the inflection point $\left(-\frac{1}{2}, e^{-2}\right)$ or $(-0.5, 0.135335\ldots)$ is so close to the x-axis. Since the horizontal asymptote is the line $y = 1$, it is difficult to see both of these features on the same graph (without drawing the graph on a very large piece of paper). We settle for the graph seen in Figure 3.75, which shows all of the features except the inflection point and the concavity on the interval $\left(-\frac{1}{2}, 0\right)$. To clearly see the behavior near the inflection point, we draw a graph that is zoomed-in on the area of the inflection point (see Figure 3.76). Here, while we have resolved the problem of the concavity near $x = 0$ and the inflection point, we have lost the details of the "big picture." ■

In our final example, we consider the graph of a function that is the sum of a trigonometric function and a polynomial.

EXAMPLE 6.6 Graphing the Sum of a Polynomial and a Trigonometric Function

Draw a graph of $f(x) = \cos x - x$, showing all significant features.

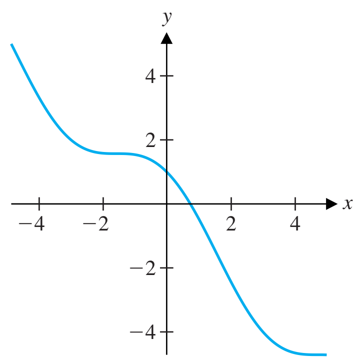


FIGURE 3.77a
$y = \cos x - x$

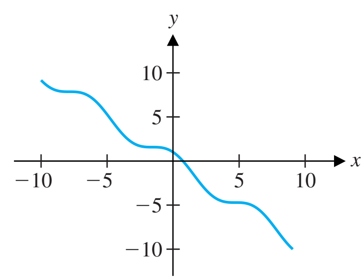


FIGURE 3.77b
$y = \cos x - x$

Solution The default graph provided by our computer algebra system can be seen in Figure 3.77a. The graph produced by most graphing calculators looks like that in Figure 3.77b. As always, we will use the calculus to determine the behavior of the function more precisely. First, notice that the domain of f is the entire real line. Consequently, there are no vertical asymptotes. Next, we have

$$f'(x) = -\sin x - 1 \leq 0, \quad \text{for all } x. \tag{6.26}$$

Further, $f'(x) = 0$ if and only if $\sin x = -1$. So, there are critical numbers (here, these are all locations of horizontal tangent lines), but since $f'(x)$ does not change sign, there are *no* local extrema. Even so, it is still of interest to find the locations of the horizontal tangent lines. Recall that

$$\sin x = -1 \quad \text{for } x = \frac{3\pi}{2}$$

and more generally, for $$x = \frac{3\pi}{2} + 2n\pi,$$

for any integer n. Next, we see that

$$f''(x) = -\cos x$$

and on the interval $[0, 2\pi]$, we have

$$\cos x \begin{cases} > 0, & \text{on } \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] \\ < 0, & \text{on } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \end{cases}$$

So, $$f''(x) = -\cos x \begin{cases} < 0, & \text{on } \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] & \text{Concave down.} \\ > 0, & \text{on } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). & \text{Concave up.} \end{cases} \tag{6.27}$$

Outside of $[0, 2\pi]$, $f''(x)$ simply repeats this pattern. In particular, this says that the graph has infinitely many inflection points, located at odd multiples of $\pi/2$.

To determine the behavior as $x \to \pm\infty$, we examine the limits

$$\lim_{x\to\infty} (\cos x - x) = -\infty \tag{6.28}$$

and $$\lim_{x\to-\infty} (\cos x - x) = \infty, \tag{6.29}$$

since $-1 \le \cos x \le 1$, for all x and since $\lim_{x\to\infty} x = \infty$.

Finally, to determine the x-intercept(s), we need to solve

$$f(x) = \cos x - x = 0.$$

This can't be solved exactly, however. Since $f'(x) \le 0$ for all x and Figures 3.77a and 3.77b show a zero around $x = 1$, there is only one zero and we must approximate this. (Use Newton's method or your calculator's solver.) We get $x \approx 0.739085$ as an approximation to the only x-intercept. Assembling all of the information in (6.26)–(6.29), we can draw the graph seen in Figure 3.78. Notice that Figure 3.77b shows the behavior just as clearly as Figure 3.78, but for a smaller range of x- and y-values. Which of these is more "representative" is open to discussion. ■

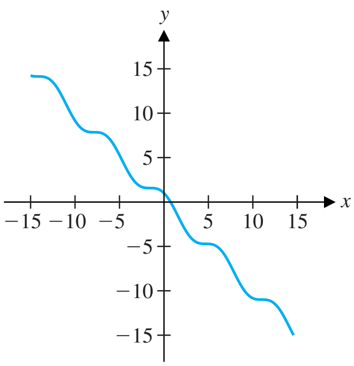


FIGURE 3.78
$y = \cos x - x$

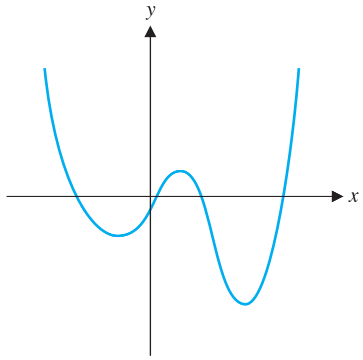
BEYOND FORMULAS

The main characteristic of the examples in sections 3.4–3.6 is the interplay between graphing and equation solving. To analyze the graph of a function, you will go back and forth several times between solving equations (for critical numbers and inflection points and so on) and identifying graphical features of interest. Even if you have access to graphing technology, the equation solving may lead you to uncover hidden features of the graph. What types of graphical features can sometimes be hidden?

EXERCISES 3.6

WRITING EXERCISES

1. We have talked about sketching representative graphs, but it is often impossible to draw a graph correctly to scale that shows all of the properties we might be interested in. For example, try to generate a computer or calculator graph that shows all three local extrema of $x^4 - 25x^3 - 2x^2 + 80x - 3$. When two extrema have y-coordinates of approximately -60 and 50, it takes a very large graph to also show a point with $y = -40{,}000$! If an accurate graph cannot show all the points of interest, perhaps a freehand sketch like the one shown below is needed.



There is no scale shown on the graph because we have distorted different portions of the graph in an attempt to show all of the interesting points. Discuss the relative merits of an "honest" graph with a consistent scale but not showing all the points of interest versus a caricature graph that distorts the scale but does show all the points of interest.

2. While studying for a test, a friend of yours says that a graph is not allowed to intersect an asymptote. While it is often the case that graphs don't intersect asymptotes, there is definitely not any rule against it. Explain why graphs can intersect a horizontal asymptote any number of times (Hint: Look at the graph of $e^{-x}\sin x$), but can't pass through a vertical asymptote.

3. Explain why polynomials never have vertical or horizontal asymptotes.

4. Explain how the graph of $f(x) = \cos x - x$ in example 6.6 relates to the graphs of $y = \cos x$ and $y = -x$. Based on this discussion, explain how to sketch the graph of $y = x + \sin x$.

In exercises 1–18, graph the function and completely discuss the graph as in example 6.2.

1. $f(x) = x^3 - 3x^2 + 3x$
2. $f(x) = x^4 - 3x^2 + 2x$
3. $f(x) = x^5 - 2x^3 + 1$
4. $f(x) = \sin x - \cos x$
5. $f(x) = x + \dfrac{4}{x}$
6. $f(x) = \dfrac{x^2 - 1}{x}$
7. $f(x) = x \ln x$
8. $f(x) = x \ln x^2$
9. $f(x) = \sqrt{x^2 + 1}$
10. $f(x) = \sqrt{2x - 1}$
11. $f(x) = \dfrac{4x}{x^2 - x + 1}$
12. $f(x) = \dfrac{4x^2}{x^2 - x + 1}$
13. $f(x) = \sqrt[3]{x^3 - 3x^2 + 2x}$
14. $f(x) = \sqrt{x^3 - 3x^2 + 2x}$
15. $f(x) = x^5 - 5x$
16. $f(x) = x^3 - \dfrac{3}{400}x$
17. $f(x) = e^{-2/x}$
18. $f(x) = e^{1/x^2}$

In exercises 19–32, determine all significant features (approximately if necessary) and sketch a graph.

19. $f(x) = (x^3 - 3x^2 + 2x)^{2/3}$
20. $f(x) = x^6 - 10x^5 - 7x^4 + 80x^3 + 12x^2 - 192x$
21. $f(x) = \dfrac{x^2 + 1}{3x^2 - 1}$
22. $f(x) = \dfrac{2x^2}{x^3 + 1}$
23. $f(x) = \dfrac{5x}{x^3 - x + 1}$
24. $f(x) = \dfrac{4x}{x^2 + x + 1}$
25. $f(x) = x^2\sqrt{x^2 - 9}$
26. $f(x) = \sqrt[3]{2x^2 - 1}$
27. $f(x) = e^{-2x}\sin x$
28. $f(x) = \sin x - \dfrac{1}{2}\sin 2x$
29. $f(x) = x^4 - 16x^3 + 42x^2 - 39.6x + 14$
30. $f(x) = x^4 + 32x^3 - 0.02x^2 - 0.8x$
31. $f(x) = \dfrac{25 - 50\sqrt{x^2 + 0.25}}{x}$
32. $f(x) = \tan^{-1}\left(\dfrac{1}{x^2 - 1}\right)$

In exercises 33–38, the "family of functions" contains a parameter c. The value of c affects the properties of the functions. Determine what differences, if any, there are for c being zero, positive or negative. Then determine what the graph would look like for very large positive c's and for very large negative c's.

33. $f(x) = x^4 + cx^2$
34. $f(x) = x^4 + cx^2 + x$
35. $f(x) = \dfrac{x^2}{x^2 + c^2}$
36. $f(x) = e^{-x^2/c}$
37. $f(x) = \sin(cx)$
38. $f(x) = x^2\sqrt{c^2 - x^2}$

39. In a variety of applications, researchers model a phenomenon whose graph starts at the origin, rises to a single maximum and then drops off to a horizontal asymptote of $y = 0$. For example, the probability density function of events such as the time from conception to birth of an animal and the amount of time surviving after contracting a fatal disease might have these properties. Show that the family of functions xe^{-bx} has these properties for all positive constants b. What effect does

b have on the location of the maximum? In the case of the time since conception, what would b represent? In the case of survival time, what would b represent?

40. The "FM" in FM radio stands for **frequency modulation,** a method of transmitting information encoded in a radio wave by modulating (or varying) the frequency. A basic example of such a modulated wave is $f(x) = \cos(10x + 2\cos x)$. Use computer-generated graphs of $f(x)$, $f'(x)$ and $f''(x)$ to try to locate all local extrema of $f(x)$.

41. A rational function is a function of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Is it true that all rational functions have vertical asymptotes? Is it true that all rational functions have horizontal asymptotes?

42. It can be useful to identify asymptotes other than vertical and horizontal. For example, the parabola $y = x^2$ is an asymptote of $f(x)$ if $\lim_{x\to\infty}[f(x) - x^2] = 0$ and/or $\lim_{x\to-\infty}[f(x) - x^2] = 0$. Show that x^2 is an asymptote of $f(x) = \frac{x^4 - x^2 + 1}{x^2 - 1}$. Graph $y = f(x)$ and zoom out until the graph looks like a parabola. (Note: The effect of zooming out is to emphasize large values of x.)

A function f has a slant asymptote $y = mx + b\ (m \neq 0)$ if $\lim_{x\to\infty}[f(x) - (mx + b)] = 0$ and/or $\lim_{x\to-\infty}[f(x) - (mx + b)] = 0$. In exercises 43–48, find the slant asymptote. (Use long division to rewrite the function.) Then, graph the function and its asymptote on the same axes.

43. $f(x) = \frac{3x^2 - 1}{x}$

44. $f(x) = \frac{3x^2 - 1}{x - 1}$

45. $f(x) = \frac{x^3 - 2x^2 + 1}{x^2}$

46. $f(x) = \frac{x^3 - 1}{x^2 - 1}$

47. $f(x) = \frac{x^4}{x^3 + 1}$

48. $f(x) = \frac{x^4 - 1}{x^3 + x}$

In exercises 49–52, find a function whose graph has the given asymptotes.

49. $x = 1$, $x = 2$ and $y = 3$

50. $x = -1$, $x = 1$ and $y = 0$

51. $x = -1$, $x = 1$, $y = -2$ and $y = 2$

52. $x = 1$, $y = 2$ and $x = 3$

53. Find all extrema and inflection points, and sketch the graphs of $y = \sinh x = \frac{e^x - e^{-x}}{2}$ and $y = \cosh x = \frac{e^x + e^{-x}}{2}$.

54. On the same axes as the graphs of exercise 53, sketch in the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$. Explain why these graphs serve as an envelope for the graphs in exercise 53. (Hint: As $x \to \pm\infty$, what happens to e^x and e^{-x}?)

EXPLORATORY EXERCISES

1. For each function, find a polynomial $p(x)$ such that $\lim_{x\to\infty}[f(x) - p(x)] = 0$.

(a) $\frac{x^4}{x + 1}$ (b) $\frac{x^5 - 1}{x + 1}$ (c) $\frac{x^6 - 2}{x + 1}$

Show by zooming out that $f(x)$ and $p(x)$ look similar for large x. The *first term* of a polynomial is the term with the highest power (e.g., x^3 is the first term of $x^3 - 3x + 1$). Can you zoom out enough to make the graph of $f(x)$ look like the first term of its polynomial asymptote? State a very quick rule enabling you to look at a rational function and determine the first term of its polynomial asymptote (if one exists).

2. One of the natural enemies of the balsam fir tree is the spruce budworm, which attacks the leaves of the fir tree in devastating outbreaks. Define $N(t)$ to be the number of worms on a particular tree at time t. A mathematical model of the population dynamics of the worm must include a term to indicate the worm's death rate due to its predators (e.g., birds). The form of this term is often taken to be $\frac{B[N(t)]^2}{A^2 + [N(t)]^2}$ for positive constants A and B. Graph the functions $\frac{x^2}{4 + x^2}$, $\frac{2x^2}{1 + x^2}$, $\frac{x^2}{9 + x^2}$ and $\frac{3x^2}{1 + x^2}$ for $x > 0$. Based on these graphs, discuss why $\frac{B[N(t)]^2}{A^2 + [N(t)]^2}$ is a plausible model for the death rate by predation. What role do the constants A and B play? The possible stable population levels for the spruce budworms are determined by intersections of the graphs of $y = r(1 - x/k)$ and $y = \frac{x}{1 + x^2}$. Here, $x = N/A$, r is proportional to the birthrate of the budworms and k is determined by the amount of food available to the budworms. Note that $y = r(1 - x/k)$ is a line with y-intercept r and x-intercept k. How many solutions are there to the equation $r(1 - x/k) = \frac{x}{1 + x^2}$? (Hint: The answer depends on the values of r and k.) One current theory is that outbreaks are caused in situations where there are three solutions and the population of budworms jumps from a small population to a large population.

3. Suppose that $f(x)$ is a function with two derivatives and that $f(a) = f'(a) = 0$ but $f''(a) \neq 0$ for some number a. Show that $f(x)$ has a local extremum at $x = a$. Next, suppose that $f(x)$ is a function with three derivatives and that $f(a) = f'(a) = f''(a) = 0$ but $f'''(a) \neq 0$ for some number a. Show that $f(x)$ does *not* have a local extremum at $x = a$. Generalize your work to the case where $f^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n - 1$, but $f^{(n)}(a) \neq 0$, keeping in mind that there are different conclusions depending on whether n is odd or even. Use this result to determine whether $f(x) = x\sin x^2$ or $g(x) = x^2\sin(x^2)$ has a local extremum at $x = 0$.

3.7 OPTIMIZATION

Everywhere in business and industry today, we see people struggling to minimize waste and maximize productivity. We are now in a position to bring the power of the calculus to bear on problems involving finding a maximum or a minimum. This section contains a number of illustrations of such problems. Pay close attention to how we solve the problems. We start by giving a few general guidelines. (Notice that we said *guidelines* and not *rules.*) Do not memorize them, but rather keep these in mind as you work through the section.

- If there's a picture to draw, draw it! Don't try to visualize how things look in your head. Put a picture down on paper and label it.
- Determine what the variables are and how they are related.
- Decide what quantity needs to be maximized or minimized.
- Write an expression for the quantity to be maximized or minimized in terms of only *one* variable. To do this, you may need to solve for any other variables in terms of this one variable.
- Determine the minimum and maximum allowable values (if any) of the variable you're using.
- Solve the problem. (Be sure to answer the question that is asked.)

We begin with a simple example where the goal is to accomplish what businesses face every day: getting the most from limited resources.

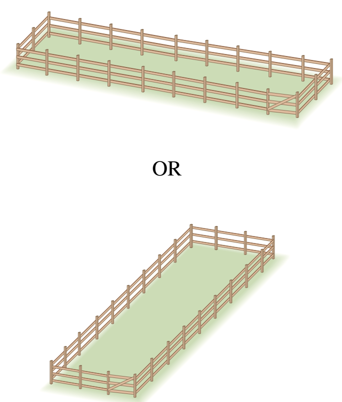


FIGURE 3.79
Possible plots

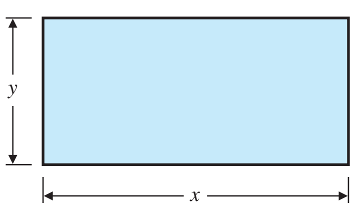


FIGURE 3.80
Rectangular plot

EXAMPLE 7.1 Constructing a Rectangular Garden of Maximum Area

You have 40 (linear) feet of fencing with which to enclose a rectangular space for a garden. Find the *largest* area that can be enclosed with this much fencing and the dimensions of the corresponding garden.

Solution First, note that there are lots of possibilities. We could enclose a plot that is very long but narrow, or one that is very wide but not very long (see Figure 3.79). How are we to decide which configuration is optimal? We first draw a picture and label it appropriately (see Figure 3.80). The variables for a rectangular plot are length and width, which we name x and y, respectively.

We want to find the dimensions of the largest possible area, that is, maximize

$$A = xy.$$

Notice immediately that this function has two variables and so, cannot be dealt with via the means we have available. However, there is another piece of information that we can use here. If we want the maximum area, then *all* of the fencing must be used. This says that the perimeter of the resulting fence must be $40'$ and hence,

$$40 = perimeter = 2x + 2y. \tag{7.1}$$

Notice that we can use (7.1) to solve for one variable (either one) in terms of the other. We have

$$2y = 40 - 2x$$

and hence,
$$y = 20 - x.$$

Substituting for y, we get that

$$A = xy = x(20 - x).$$

So, our job is to find the maximum value of the function

$$A(x) = x(20 - x).$$

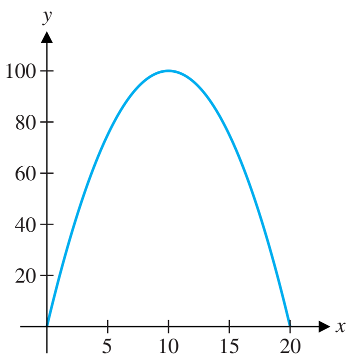


FIGURE 3.81
$y = x(20 - x)$

Hold it! You may want to multiply out this expression. Unless you have a clear picture of how it will affect your later work, leave the function alone!

Before we maximize $A(x)$, we need to determine the interval in which x must lie. Since x is a distance, we must have $0 \le x$. Further, since the perimeter is 40′, we must have $x \le 20$. (Why don't we have $x \le 40$?) So, we want to find the maximum value of $A(x)$ on the closed interval [0, 20]. This is now a simple problem. As a check on what a reasonable answer should be, we draw a graph of $y = A(x)$ (see Figure 3.81) on the interval [0, 20]. The maximum value appears to occur around $x = 10$. Now, let's analyze the problem carefully. We have

$$\begin{aligned} A'(x) &= 1(20 - x) + x(-1) \\ &= 20 - 2x \\ &= 2(10 - x). \end{aligned}$$

So, the only critical number is $x = 10$ and this is in the interval under consideration. Recall that the maximum and minimum values of a continuous function on a closed and bounded interval must occur at either the endpoints or a critical number. This says that we need only compare the function values

$$A(0) = 0, \quad A(20) = 0 \quad \text{and} \quad A(10) = 100.$$

Thus, the maximum area that can be enclosed with 40′ of fencing is 100 ft^2. We also want the dimensions of the plot. (This result is only of theoretical value if we don't know how to construct the rectangle with the maximum area.) We have that $x = 10$ and

$$y = 20 - x = 10.$$

That is, the rectangle of perimeter 40′ with maximum area is a *square* 10′ on a side. ■

A more general problem that you can now solve is to show that (given a fixed perimeter) the rectangle of maximum area is a square. This is virtually identical to example 7.1 and is left as an exercise. It's worth noting here that this more general problem is one that cannot be solved by simply using a calculator to draw a graph. You'll need to use some calculus here.

Manufacturing companies routinely make countless decisions that affect the efficiency of their production processes. One decision that is surprisingly important is how to economically package products for shipping. Example 7.2 provides a simple illustration of this problem.

EXAMPLE 7.2 Constructing a Box of Maximum Volume

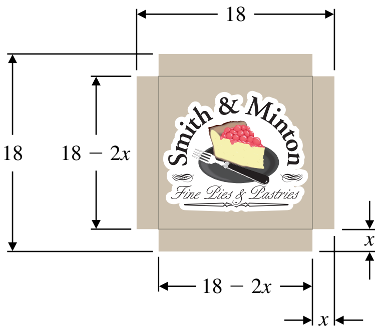


FIGURE 3.82a
A sheet of cardboard

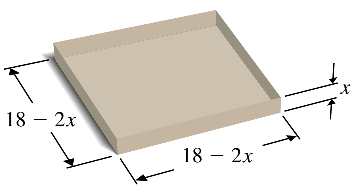


FIGURE 3.82b
Rectangular box

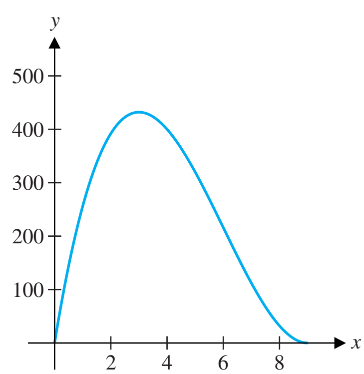


FIGURE 3.83
$y = 4x(9 - x)^2$

A square sheet of cardboard 18″ on a side is made into an open box (i.e., there's no top), by cutting squares of equal size out of each corner (see Figure 3.82a) and folding up the sides along the dotted lines (see Figure 3.82b). Find the dimensions of the box with the maximum volume.

Solution Recall that the volume of a rectangular parallelepiped (a box) is given by

$$V = l \times w \times h.$$

From Figure 3.82b, we can see that the height is $h = x$, while the length and width are $l = w = 18 - 2x$. Thus, we can write the volume in terms of the one variable x as

$$V = V(x) = (18 - 2x)^2(x) = 4x(9 - x)^2.$$

Once again, don't multiply this out, just out of habit. Notice that since x is a distance, we have $x \geq 0$. Further, we have $x \leq 9$, since cutting squares of side 9 out of each corner will cut up the entire sheet of cardboard. Thus, we are faced with finding the absolute maximum of the continuous function

$$V(x) = 4x(9 - x)^2$$

on the closed interval $0 \leq x \leq 9$.

This should be a simple matter. The graph of $y = V(x)$ on the interval [0, 9] is seen in Figure 3.83. From the graph, the maximum volume seems to be somewhat over 400 and seems to occur around $x = 3$. Now, we solve the problem precisely. We have

$$\begin{aligned} V'(x) &= 4(9 - x)^2 + 4x(2)(9 - x)(-1) && \text{Product rule and chain rule.} \\ &= 4(9 - x)[(9 - x) - 2x] && \text{Factor out } 4(9 - x). \\ &= 4(9 - x)(9 - 3x). \end{aligned}$$

So, V has two critical numbers, 3 and 9, and these are both in the interval under consideration. We now need only compare the value of the function at the endpoints and the critical numbers. We have

$$V(0) = 0, \quad V(9) = 0 \quad \text{and} \quad V(3) = 432.$$

Obviously, the maximum possible volume is 432 cubic inches. We can achieve this volume if we cut squares of side 3″ out of each corner. You should note that this corresponds with what we expected from the graph of $y = V(x)$ in Figure 3.83. Finally, observe that the dimensions of this optimal box are 12″ long by 12″ wide by 3″ deep. ■

When a new building is built, it must be connected to existing telephone and power cables, water and sewer lines and paved roads. If the cables, water or sewer lines or road are curved, then it may not be obvious how to make the shortest (i.e., least expensive) connection possible. In examples 7.3 and 7.4, we consider the general problem of finding the shortest distance from a point to a curve.

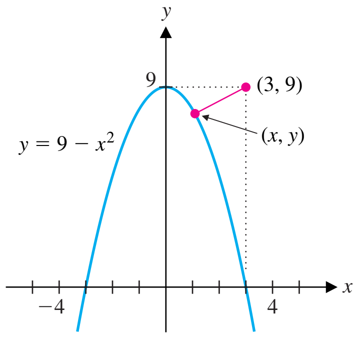


FIGURE 3.84
$y = 9 - x^2$

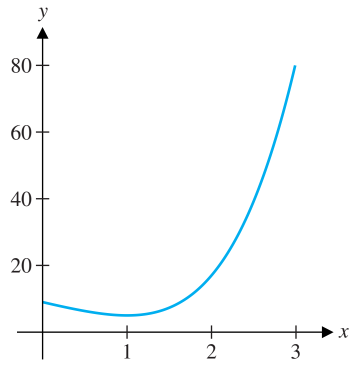


FIGURE 3.85
$y = (x - 3)^2 + x^4$

EXAMPLE 7.3 Finding the Closest Point on a Parabola

Find the point on the parabola $y = 9 - x^2$ closest to the point (3, 9) (see Figure 3.84).

Solution Using the usual distance formula, we find that the distance between the point (3, 9) and any point (x, y) is

$$d = \sqrt{(x-3)^2 + (y-9)^2}.$$

If the point (x, y) is on the parabola, note that its coordinates satisfy the equation $y = 9 - x^2$ and so, we can write the distance in terms of the single variable x as follows

$$\begin{aligned} d(x) &= \sqrt{(x-3)^2 + [(9-x^2) - 9]^2} \\ &= \sqrt{(x-3)^2 + (-x^2)^2} \\ &= \sqrt{(x-3)^2 + x^4}. \end{aligned}$$

Although we can certainly solve the problem in its present form, we can simplify our work by observing that $d(x)$ is minimized if and only if the quantity under the square root is minimized. (We leave it as an exercise to show why this is true.) So, instead of minimizing $d(x)$ directly, we minimize the *square* of $d(x)$:

$$f(x) = [d(x)]^2 = (x-3)^2 + x^4$$

instead. Notice from Figure 3.84 that any point on the parabola to the left of the y-axis is farther away from (3, 9) than is the point (0, 9). Likewise, any point on the parabola below the x-axis is farther from (3, 9) than is the point (3, 0). So, it suffices to look for the closest point with

$$0 \le x \le 3.$$

See Figure 3.85 for a graph of $y = f(x)$ over the interval of interest. Observe that the minimum value of f (the square of the distance) seems to be around 5 and seems to occur near $x = 1$. We have

$$f'(x) = 2(x-3)^1 + 4x^3 = 4x^3 + 2x - 6.$$

Notice that $f'(x)$ factors. [One way to see this is to recognize that $x = 1$ is a zero of $f'(x)$, which makes $(x - 1)$ a factor.] We have

$$f'(x) = 2(x-1)(2x^2 + 2x + 3).$$

So, $x = 1$ is a critical number. In fact, it's the only critical number, since $(2x^2 + 2x + 3)$ has no zeros. (Why not?) We now need only compare the value of f at the endpoints and the critical number. We have

$$f(0) = 9, \quad f(3) = 81 \quad \text{and} \quad f(1) = 5.$$

Thus, the minimum value of $f(x)$ is 5. This says that the minimum distance from the point (3, 9) to the parabola is $\sqrt{5}$ and the closest point on the parabola is (1, 8). Again, notice that this corresponds with what we expected from the graph of $y = f(x)$. ■

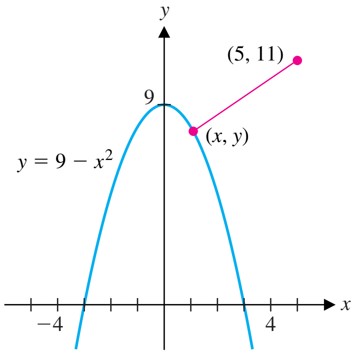


FIGURE 3.86
$y = 9 - x^2$

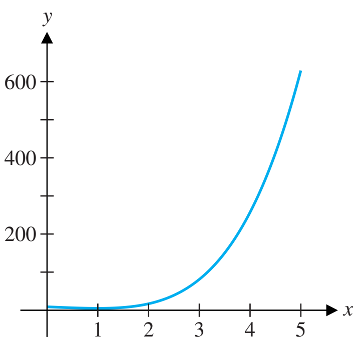


FIGURE 3.87
$y = f(x) = [d(x)]^2$

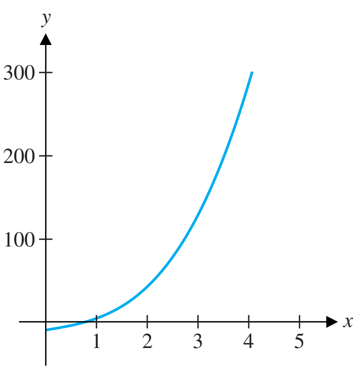


FIGURE 3.88
$y = f'(x)$

Example 7.4 is very similar to example 7.3, except that we need to use approximate methods to find the critical number.

EXAMPLE 7.4 Finding Minimum Distance Approximately

Find the point on the parabola $y = 9 - x^2$ closest to the point (5, 11) (see Figure 3.86).

Solution As in example 7.3, we want to minimize the distance from a fixed point [in this case, the point (5, 11)] to a point (x, y) on the parabola. Using the distance formula, the distance from any point (x, y) on the parabola to the point (5, 11) is

$$\begin{aligned} d &= \sqrt{(x-5)^2 + (y-11)^2} \\ &= \sqrt{(x-5)^2 + [(9-x^2) - 11]^2} \\ &= \sqrt{(x-5)^2 + (x^2+2)^2}. \end{aligned}$$

Again, it is equivalent (and simpler) to minimize the quantity under the square root:

$$f(x) = [d(x)]^2 = (x-5)^2 + (x^2+2)^2.$$

As in example 7.3, we can see from Figure 3.86 that any point on the parabola to the left of the y-axis is farther from (5, 11) than is (0, 9). Likewise, any point on the parabola to the right of $x = 5$ is farther from (5, 11) than is (5, −16). Thus, we minimize $f(x)$ for $0 \le x \le 5$. In Figure 3.87 we see a graph of $y = f(x)$ on the interval of interest. The minimum value of f seems to occur around $x = 1$. We can make this more precise as follows:

$$\begin{aligned} f'(x) &= 2(x-5) + 2(x^2+2)(2x) \\ &= 4x^3 + 10x - 10. \end{aligned}$$

Unlike in example 7.3, the expression for $f'(x)$ has no obvious factorization. Our only choice then is to find zeros of $f'(x)$ approximately. First, we draw a graph of $y = f'(x)$ on the interval of interest (see Figure 3.88). The only zero appears to be slightly less than 1. Using $x_0 = 1$ as an initial guess in Newton's method (applied to $f'(x) = 0$) or using your calculator's solver, you should get the approximate root $x_c \approx 0.79728$. We now compare function values:

$$f(0) = 29, \quad f(5) = 729 \quad \text{and} \quad f(x_c) \approx 24.6.$$

Thus, the minimum distance from (5, 11) to the parabola is approximately $\sqrt{24.6} \approx 4.96$ and the closest point on the parabola is located at approximately (0.79728, 8.364). ■

Notice that in both Figures 3.84 and 3.86, the shortest path appears to be perpendicular to the tangent line to the curve at the point where the path intersects the curve. We leave it as an exercise to prove that this is always the case. This observation is an important geometric principle that applies to many problems of this type.

REMARK 7.1

At this point you might be tempted to forgo the comparison of function values at the endpoints and at the critical numbers. After all, in all of the examples we have seen so far, the desired maximizer or minimizer (i.e., the point at which the maximum or minimum occurred) was the only critical number in the interval under consideration. You might just suspect that if there is only one critical number, it will correspond to the maximizer or minimizer for which you are searching. Unfortunately, this is not always the case. In 1945, two prominent aeronautical engineers derived a function to model the range of an aircraft. Their intention was to use this function to discover how to maximize the range. They found a critical number of this function (corresponding to distributing virtually all of the plane's weight in the wings) and reasoned that it gave the *maximum* range. The result was the famous "Flying Wing" aircraft. Some years later, it was argued that the critical number they found corresponded to a local *minimum* of the range function. In the engineers' defense, they did not have easy, accurate computational power at their fingertips, as we do today. Remarkably, this design strongly resembles the modern B-2 Stealth bomber. This story came out as controversy brewed over the production of the B-2 (see *Science,* **244,** pp. 650–651, May 12, 1989; also see the *Monthly* of the Mathematical Association of America, October, 1993, pp. 737–738). The moral should be crystal clear: check the function values at the critical numbers and at the endpoints. Do not simply *assume* (even by virtue of having only one critical number) that a given critical number corresponds to the extremum you are seeking.

Next, we consider an optimization problem that cannot be restricted to a closed interval. We will use the fact that for a continuous function, a single local extremum must be an absolute extremum. (Think about why this is true.)

EXAMPLE 7.5 Designing a Soda Can That Uses a Minimum Amount of Material

A soda can is to hold 12 fluid ounces. Find the dimensions that will minimize the amount of material used in its construction, assuming that the thickness of the material is uniform (i.e., the thickness of the aluminum is the same everywhere in the can).

FIGURE 3.89
Soda can

Solution First, we draw and label a picture of a typical soda can (see Figure 3.89). Here we have drawn a right circular cylinder of height h and radius r. Assuming uniform thickness of the aluminum, notice that we minimize the amount of material by minimizing the surface area of the can. We have

$$\begin{aligned} \textit{area} &= \textit{area of top} + \textit{area of bottom} + \textit{curved surface area} \\ &= 2\pi r^2 + 2\pi r h. \end{aligned} \tag{7.2}$$

We can eliminate one of the variables by using the fact that the volume (using 1 fluid ounce ≈ 1.80469 in.^{3}) must be

$$12 \text{ fluid ounces} \approx 12 \text{ fl oz} \times 1.80469 \frac{\text{in.}^3}{\text{fl oz}} = 21.65628 \text{ in.}^3.$$

Further, the volume of a right circular cylinder is

$$vol = \pi r^2 h$$

and so,
$$h = \frac{vol}{\pi r^2} \approx \frac{21.65628}{\pi r^2}. \tag{7.3}$$

Thus, from (7.2) and (7.3), the surface area is approximately

$$\begin{aligned} A(r) &= 2\pi r^2 + 2\pi r \frac{21.65628}{\pi r^2} \\ &= 2\pi \left(r^2 + \frac{21.65628}{\pi r}\right). \end{aligned}$$

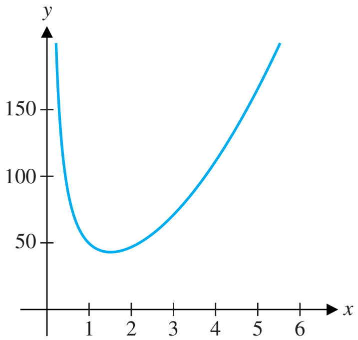


FIGURE 3.90
$y = A(r)$

So, our job is to minimize $A(r)$, but here, there is no closed and bounded interval of allowable values. In fact, all we can say is that $r > 0$. We can have r as large or small as you can imagine, simply by taking h to be correspondingly small or large, respectively. That is, we must find the absolute minimum of $A(r)$ on the open and unbounded interval $(0, \infty)$. To get an idea of what a plausible answer might be, we graph $y = A(r)$ (see Figure 3.90). There appears to be a local minimum (slightly less than 50) located between $r = 1$ and $r = 2$. Next, we compute

$$\begin{aligned} A'(r) &= \frac{d}{dr}\left[2\pi \left(r^2 + \frac{21.65628}{\pi r}\right)\right] \\ &= 2\pi \left(2r - \frac{21.65628}{\pi r^2}\right) \\ &= 2\pi \left(\frac{2\pi r^3 - 21.65628}{\pi r^2}\right). \end{aligned}$$

Notice that the only critical numbers are those for which the numerator of the fraction is zero:

$$0 = 2\pi r^3 - 21.65628.$$

This occurs if and only if
$$r^3 = \frac{21.65628}{2\pi}$$

and hence, the only critical number is

$$r = r_c = \sqrt[3]{\frac{21.65628}{2\pi}} \approx 1.510548.$$

Further, notice that for $0 < r < r_c$, $A'(r) < 0$ and for $r_c < r$, $A'(r) > 0$. That is, $A(r)$ is decreasing on the interval $(0, r_c)$ and increasing on the interval (r_c, ∞). Thus, $A(r)$ has not only a local minimum, but also an absolute minimum at $r = r_c$. Notice, too, that this corresponds with what we expected from the graph of $y = A(r)$ in Figure 3.90. This says that the can that uses a minimum of material has radius $r_c \approx 1.510548$ and height

$$h = \frac{21.65628}{\pi r_c^2} \approx 3.0211.$$

Note that the optimal can from example 7.5 is "square," in the sense that the height (h) equals the diameter ($2r$). Also, we should observe that example 7.5 is not completely realistic. A standard 12-ounce soda can has a radius of about 1.156″. You should review example 7.5 to find any unrealistic assumptions we made. We study the problem of designing a soda can further in the exercises.

In our final example, we consider a problem where most of the work must be done numerically and graphically.

EXAMPLE 7.6 Minimizing the Cost of Highway Construction

The state wants to build a new stretch of highway to link an existing bridge with a turnpike interchange, located 8 miles to the east and 8 miles to the south of the bridge. There is a 5-mile-wide stretch of marshland adjacent to the bridge that must be crossed (see Figure 3.91). Given that the highway costs \$10 million per mile to build over the marsh and only \$7 million to build over dry land, how far to the east of the bridge should the highway be when it crosses out of the marsh?

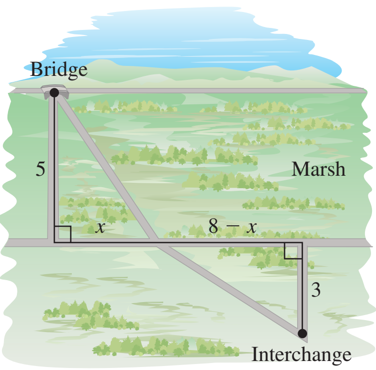


FIGURE 3.91
A new highway

Solution You might guess that the highway should cut directly across the marsh, so as to minimize the amount built over marshland. We will use the calculus to decide this question. We let x represent the distance in question (see Figure 3.91). Then, the interchange lies $(8 - x)$ miles to the east of the point where the highway leaves the marsh. Thus, the total cost (in millions of dollars) is

$$\textit{cost} = 10(\textit{distance across marsh}) + 7(\textit{distance across dry land}).$$

Using the Pythagorean Theorem on the two right triangles seen in Figure 3.91, we get the cost function

$$C(x) = 10\sqrt{x^2 + 25} + 7\sqrt{(8 - x)^2 + 9}.$$

Observe from Figure 3.91 that we must have $0 \le x \le 8$. So, we have the routine problem of minimizing a continuous function $C(x)$ over the closed and bounded interval $[0, 8]$. Or is it really that routine? First, we draw a graph of $y = C(x)$ on the interval in question to get an idea of a plausible answer (see Figure 3.92). From the graph, the minimum appears to be slightly less than 100 and occurs around $x = 4$.

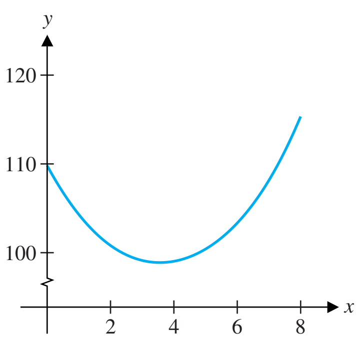


FIGURE 3.92
$y = C(x)$

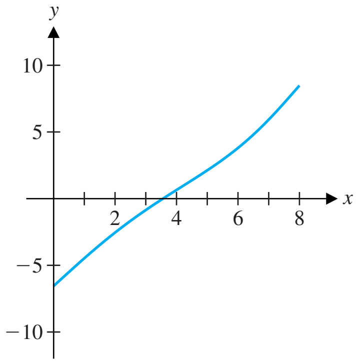


FIGURE 3.93
$y = C'(x)$

We now compare

$$\begin{aligned} C'(x) &= \frac{d}{dx}\left[10\sqrt{x^2+25} + 7\sqrt{(8-x)^2+9}\right] \\ &= 5(x^2+25)^{-1/2}(2x) + \frac{7}{2}[(8-x)^2+9]^{-1/2}(2)(8-x)^1(-1) \\ &= \frac{10x}{\sqrt{x^2+25}} - \frac{7(8-x)}{\sqrt{(8-x)^2+9}}. \end{aligned}$$

First, note that the only critical numbers are where $C'(x) = 0$. (Why?) The only way to find these is to approximate them. From the graph of $y = C'(x)$ seen in Figure 3.93, the only zero of $C'(x)$ on the interval $[0, 8]$ appears to be between $x = 3$ and $x = 4$. We approximate this zero numerically (e.g., with bisections or your calculator's solver), to obtain the approximate critical number

$$x_c \approx 3.560052.$$

Now, we need only compare the value of $C(x)$ at the endpoints and at this one critical number:

$$C(0) \approx \$109.8 \text{ million},$$
$$C(8) \approx \$115.3 \text{ million}$$

and

$$C(x_c) \approx \$98.9 \text{ million}.$$

So, by using a little calculus, we can save the taxpayers more than \$10 million over cutting directly across the marsh and more than \$16 million over cutting diagonally across the marsh (not a bad reward for a few minutes of work). ■

The examples that we've presented in this section together with the exercises should give you the basis for solving a wide range of applied optimization problems. When solving these problems, be careful to draw good pictures, as well as graphs of the functions involved. Make sure that the answer you obtain computationally is consistent with what you expect from the graphs. If not, further analysis is required to see what you have missed. Also, make sure that the solution makes physical sense, when appropriate. All of these multiple checks on your work will reduce the likelihood of error.

EXERCISES 3.7

WRITING EXERCISES

1. Suppose some friends complain to you that they can't work any of the problems in this section. When you ask to see their work, they say that they couldn't even get started. In the text, we have emphasized sketching a picture and defining variables. Part of the benefit of this is to help you get started writing something (anything) down. Do you think this advice helps? What do you think is the most difficult aspect of these problems? Give your friends the best advice you can.

2. We have neglected one important aspect of optimization problems, an aspect that might be called "common sense." For example, suppose you are finding the optimal dimensions for a fence and the mathematical solution is to build a square fence of length $10\sqrt{5}$ feet on each side. At the meeting with the carpenter who is going to build the fence, what length fence do you order? Why is $10\sqrt{5}$ probably not the best way to express the length? We can approximate $10\sqrt{5} \approx 22.36$. What would you

tell the carpenter? Suppose the carpenter accepts measurements down to the inch. Assuming that the building constraint was that the perimeter of the fence could not exceed a certain figure, why should you truncate to 22′4″ instead of rounding up to 22′5″?

3. In example 7.3, we stated that $d(x) = \sqrt{f(x)}$ is minimized by exactly the same x-value(s) as $f(x)$. Use the fact that $\sqrt{x}$ is an increasing function to explain why this is true.

4. Suppose that $f(x)$ is a continuous function with a single critical number and $f(x)$ has a local minimum at that critical number. Explain why $f(x)$ also has an absolute minimum at the critical number.

---

1. Give an example showing that $f(x)$ and $\sin(f(x))$ need not be minimized by the same x-values.

2. True or false: $e^{f(x)}$ is minimized by exactly the same x-value(s) as $f(x)$.

3. A three-sided fence is to be built next to a straight section of river, which forms the fourth side of a rectangular region. The enclosed area is to equal 1800 ft^2. Find the minimum perimeter and the dimensions of the corresponding enclosure.

4. A three-sided fence is to be built next to a straight section of river, which forms the fourth side of a rectangular region. There is 96 feet of fencing available. Find the maximum enclosed area and the dimensions of the corresponding enclosure.

5. A two-pen corral is to be built. The outline of the corral forms two identical adjoining rectangles. If there is 120 ft of fencing available, what dimensions of the corral will maximize the enclosed area?

6. A showroom for a department store is to be rectangular with walls on three sides, 6-ft door openings on the two facing sides and a 10-ft door opening on the remaining wall. The showroom is to have 800 ft^2 of floor space. What dimensions will minimize the length of wall used?

7. Show that the rectangle of maximum area for a given perimeter P is always a square.

8. Show that the rectangle of minimum perimeter for a given area A is always a square.

9. Find the point on the curve $y = x^2$ closest to the point (0, 1).

10. Find the point on the curve $y = x^2$ closest to the point (3, 4).

11. Find the point on the curve $y = \cos x$ closest to the point (0, 0).

12. Find the point on the curve $y = \cos x$ closest to the point (1, 1).

13. In exercises 9 and 10, find the slope of the line through the given point and the closest point on the given curve. Show that in each case, this line is perpendicular to the tangent line to the curve at the given point.

14. Sketch the graph of some function $y = f(x)$ and mark a point not on the curve. Explain why the result of exercise 13 is true. (Hint: Pick a point for which the joining line is *not* perpendicular and explain why you can get closer.)

15. A box with no top is to be built by taking a 6″-by-10″ sheet of cardboard and cutting x-in. squares out of each corner and folding up the sides. Find the value of x that maximizes the volume of the box.

16. A box with no top is to be built by taking a 12″-by-16″ sheet of cardboard and cutting x-in. squares out of each corner and folding up the sides. Find the value of x that maximizes the volume of the box.

17. A water line runs east-west. A town wants to connect two new housing developments to the line by running lines from a single point on the existing line to the two developments. One development is 3 miles south of the existing line; the other development is 4 miles south of the existing line and 5 miles east of the first development. Find the place on the existing line to make the connection to minimize the total length of new line.

18. A company needs to run an oil pipeline from an oil rig 25 miles out to sea to a storage tank that is 5 miles inland. The shoreline runs east-west and the tank is 8 miles east of the rig. Assume it costs \$50 thousand per mile to construct the pipeline under water and \$20 thousand per mile to construct the pipeline on land. The pipeline will be built in a straight line from the rig to a selected point on the shoreline, then in a straight line to the storage tank. What point on the shoreline should be selected to minimize the total cost of the pipeline?

19. A city wants to build a new section of highway to link an existing bridge with an existing highway interchange, which lies 8 miles to the east and 10 miles to the south of the bridge. The first 4 miles south of the bridge is marshland. Assume that the highway costs \$5 million per mile over marsh and \$2 million per mile over dry land. The highway will be built in a straight line from the bridge to the edge of the marsh, then in a straight line to the existing interchange. At what point should the highway emerge from the marsh in order to minimize the total cost of the new highway? How much is saved over building the new highway in a straight line from the bridge to the interchange? (Hint: Use similar triangles to find the point on the boundary corresponding to a straight path and evaluate your cost function at that point.)

20. After construction has begun on the highway in exercise 19, the cost per mile over marshland is reestimated at \$6 million. Find the point on the marsh/dry land boundary that would minimize the total cost of the highway with the new cost function. If the construction is too far along to change paths, how much extra cost is there in using the path from exercise 19?

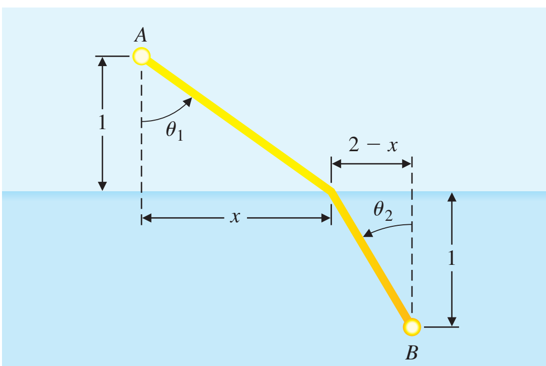
21. After construction has begun on the highway in exercise 19, the cost per mile over dry land is reestimated at \$3 million. Find the point on the marsh/dry land boundary that would minimize the total cost of the highway with the new cost function. If the construction is too far along to change paths, how much extra cost is there in using the path from exercise 19?

22. In an endurance contest, contestants 2 miles at sea need to reach a location 2 miles inland and 3 miles east (the shoreline runs east-west). Assume a contestant can swim 4 mph and run 10 mph. To what point on the shoreline should the person swim to minimize the total time? Compare the amount of time spent in the water and the amount of time spent on land.

23. Suppose that light travels from point A to point B as shown in the figure. (Recall that light always follows the path that minimizes time.) Assume that the velocity of light above the boundary line is v_1 and the velocity of light below the boundary is v_2. Show that the total time to get from point A to point B is

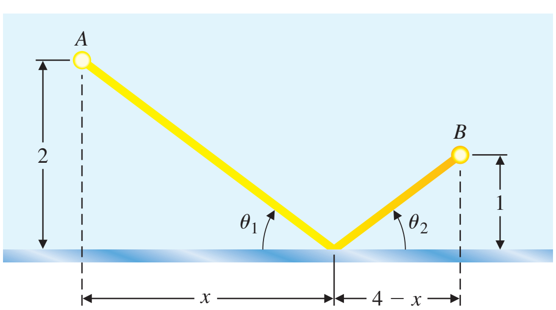
$$T(x) = \frac{\sqrt{1+x^2}}{v_1} + \frac{\sqrt{1+(2-x)^2}}{v_2}.$$

Write out the equation $T'(x) = 0$, replace the square roots using the sines of the angles in the figure and derive Snell's Law $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$.



Exercise 23

24. Suppose that light reflects off a mirror to get from point A to point B as indicated in the figure. Assuming a constant velocity of light, we can minimize time by minimizing the distance traveled. Find the point on the mirror that minimizes the distance traveled. Show that the angles in the figure are equal (the angle of incidence equals the angle of reflection).



Exercise 24

25. A soda can is to hold 12 fluid ounces. Suppose that the bottom and top are twice as thick as the sides. Find the dimensions of the can that minimize the amount of material used. (Hint: Instead of minimizing surface area, minimize the cost, which is proportional to the product of the thickness and the area.)

26. Following example 7.5, we mentioned that real soda cans have a radius of about 1.156″. Show that this radius minimizes the cost if the top and bottom are 2.23 times as thick as the sides.

27. The human cough is intended to increase the flow of air to the lungs, by dislodging any particles blocking the windpipe and changing the radius of the pipe. Suppose a windpipe under no pressure has radius r_0. The velocity of air through the windpipe at radius r is approximately $V(r) = cr^2(r_0 - r)$ for some constant c. Find the radius that maximizes the velocity of air through the windpipe. Does this mean the windpipe expands or contracts?

28. To supply blood to all parts of the body, the human artery system must branch repeatedly. Suppose an artery of radius r branches off from an artery of radius R ($R > r$) at an angle θ. The energy lost due to friction is approximately

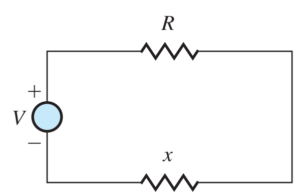
$$E(\theta) = \frac{\csc\theta}{r^4} + \frac{1-\cot\theta}{R^4}.$$

Find the value of θ that minimizes the energy loss.

29. In an electronic device, individual circuits may serve many purposes. In some cases, the flow of electricity must be controlled by reducing the power instead of amplifying it. In the circuit shown here, a voltage V volts and resistance R ohms are given. We want to determine the size of the remaining resistor (x ohms). The power absorbed by the circuit is

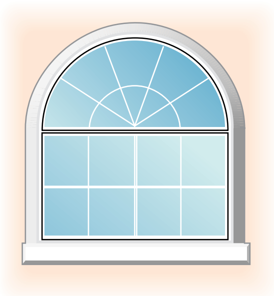
$$p(x) = \frac{V^2 x}{(R+x)^2}.$$

Find the value of x that maximizes the power absorbed.



30. In an AC circuit with voltage $V(t) = v\sin 2\pi ft$, a voltmeter actually shows the average (root-mean-square) voltage of $v/\sqrt{2}$. If the frequency is $f = 60$ (Hz) and the meter registers 115 volts, find the maximum voltage reached. [Hint: This is "obvious" if you determine v and think about the graph of $V(t)$.]

31. A Norman window has the outline of a semicircle on top of a rectangle, as shown in the figure. Suppose there is $8+\pi$ feet of wood trim available. Discuss why a window designer might want to maximize the area of the window. Find the dimensions of the rectangle (and, hence, the semicircle) that will maximize the area of the window.

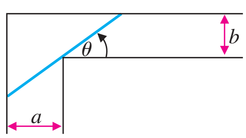


32. Suppose a wire 2 ft long is to be cut into two pieces, each of which will be formed into a square. Find the size of each piece to maximize the total area of the two squares.

33. An advertisement consists of a rectangular printed region plus 1-in. margins on the sides and 2-in. margins at top and bottom. If the area of the printed region is to be 92 in.^{2}, find the dimensions of the printed region and overall advertisement that minimize the total area.

34. An advertisement consists of a rectangular printed region plus 1-in. margins on the sides and 1.5-in. margins at top and bottom. If the total area of the advertisement is to be 120 in.^{2}, what dimensions should the advertisement be to maximize the area of the printed region?

35. A hallway of width $a = 5$ ft meets a hallway of width $b = 4$ ft at a right angle. Find the length of the longest ladder that could be carried around the corner. (Hint: Express the length of the ladder as a function of the angle θ in the figure.)



36. In exercise 35, show that the maximum ladder length for general a and b equals $(a^{2/3} + b^{2/3})^{3/2}$.

37. In exercise 35, suppose that $a = 5$ and the ladder is 8 ft long. Find the minimum value of b such that the ladder can turn the corner.

38. Solve exercise 37 for a general a and ladder length L.

39. A company's revenue for selling x (thousand) items is given by $R(x) = \dfrac{35x - x^2}{x^2 + 35}$. Find the value of x that maximizes the revenue and find the maximum revenue.

40. The function in exercise 39 has a special form. For any positive constant c, find x to maximize $R(x) = \dfrac{cx - x^2}{x^2 + c}$.

41. In t hours, a worker makes $Q(t) = -t^3 + 12t^2 + 60t$ items. Graph $Q'(t)$ and explain why it can be interpreted as the **efficiency** of the worker. Find the time at which the worker's efficiency is maximum.

42. Suppose that $Q(t)$ represents the output of a worker, as in exercise 41. If T is the length of a workday, then the graph of $Q(t)$ should be increasing for $0 \le t \le T$. Suppose that the graph of $Q(t)$ has a single inflection point for $0 \le t \le T$, called the **point of diminishing returns.** Show that the worker's efficiency is maximized at the point of diminishing returns.

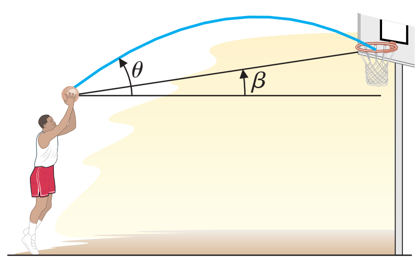
43. Suppose that group tickets to a concert are priced at \$40 per ticket if 20 tickets are ordered, but cost \$1 per ticket less for each extra ticket ordered, up to a maximum of 50 tickets. (For example, if 22 tickets are ordered, the price is \$38 per ticket.) Find the number of tickets that maximizes the total cost of the tickets.

44. In exercise 43, if management wanted the solution to be 50 (that is, ordering the maximum number of tickets produces the maximum cost), how much should the price be discounted for extra tickets ordered?

45. In sports where balls are thrown or hit, the ball often finishes at a different height than it starts. Examples include a downhill golf shot and a basketball shot. In the diagram, a ball is released at an angle θ and finishes at an angle β above the horizontal (for downhill trajectories, β would be negative). Neglecting air resistance and spin, the horizontal range is given by

$$R = \frac{2v^2 \cos^2 \theta}{g}(\tan\theta - \tan\beta)$$

if the initial velocity is v and g is the gravitational constant. In the following cases, find θ to maximize R (treat v and g as constants): (a) $\beta = 10^\circ$, (b) $\beta = 0^\circ$ and (c) $\beta = -10^\circ$. Verify that $\theta = 45^\circ + \beta^\circ/2$ maximizes the range.



46. For your favorite sport in which it is important to throw or hit a ball a long way, explain the result of exercise 45 in the language of your sport.

47. A ball is thrown from $s = b$ to $s = a$ (where $a < b$) with initial speed v_0. Assuming that air resistance is proportional to speed, the time it takes the ball to reach $s = a$ is

$$T = -\frac{1}{c}\ln\left(1 - c\,\frac{b-a}{v_0}\right),$$

where c is a constant of proportionality. A baseball player is 300 ft from home plate and throws a ball directly toward home plate with an initial speed of 125 ft/s. Suppose that $c = 0.1$. How long does it take the ball to reach home plate? Another player standing x feet from home plate has the option of catching the ball and then, after a delay of 0.1 s, relaying the ball toward home plate with an initial speed of 125 ft/s. Find x to minimize the total time for the ball to reach home plate. Is the straight throw or the relay faster? What, if anything, changes if the delay is 0.2 s instead of 0.1 s?

48. For the situation in exercise 47, for what length delay is it equally fast to have a relay and not have a relay? Do you think that you could catch and throw a ball in such a short time? Why do you think it is considered important to have a relay option in baseball?

49. Repeat exercises 47 and 48 if the second player throws the ball with initial speed 100 ft/s.

50. For a delay of 0.1 s in exercise 47, find the value of the initial speed of the second player's throw for which it is equally fast to have a relay and not have a relay.

51. The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ defines an ellipse with $-a \le x \le a$ and $-b \le y \le b$. The area enclosed by the ellipse equals πab. Find the maximum area of a rectangle inscribed in the ellipse (that is, a rectangle with sides parallel to the x-axis and y-axis and vertices on the ellipse). Show that the ratio of the maximum inscribed area to the area of the ellipse to the area of the circumscribed rectangle is $1 : \frac{\pi}{2} : 2$.

52. Show that the maximum volume enclosed by a right circular cylinder inscribed in a sphere equals $\frac{1}{\sqrt{3}}$ times the volume of the sphere.

53. Find the maximum area of an isosceles triangle of given perimeter p. [Hint: Use Heron's formula for the area of a triangle of sides a, b and c : $A=\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$.]

EXPLORATORY EXERCISES

1. In exploratory exercise 2 in section 3.3, you did a preliminary investigation of Kepler's wine cask problem. You showed that a height-to-diameter ratio (x/y) of $\sqrt{2}$ for a cylindrical barrel will maximize the volume (see Figure a). However, real wine casks are bowed out (like beer kegs). Kepler continued his investigation of wine cask construction by approximating a cask with the straight-sided barrel in Figure b. It can be shown (we told you Kepler was good!) that the volume of this barrel is $V=\frac{2}{3}\pi[y^2+(w-y)^2+y(w-y)]\sqrt{z^2-w^2}$. Treating w and z as constants, show that $V'(y)=0$ if $y=w/2$. Recall that such a critical point can correspond to a maximum or minimum of $V(y)$, but it also could correspond to something else (e.g., an inflection point). To discover which one we have here, redraw Figure b to scale (show the correct relationship between $2y$ and w). In physical terms (think about increasing and decreasing y), argue that this critical point is neither a maximum nor minimum. Interestingly enough, such a nonextreme critical point would have a definite advantage to the Austrian vintners. Recall that their goal was to convert the measurement z into an estimate of the volume. The vintners would hope that small imperfections in the dimensions of the cask would have little effect on the volume. Explain why $V'(y)=0$ means that small variations in y would convert to small errors in the volume V.

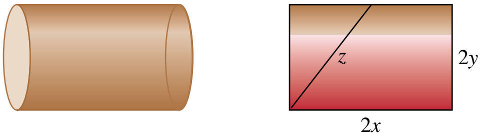


FIGURE a

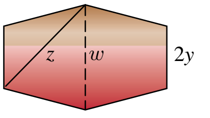
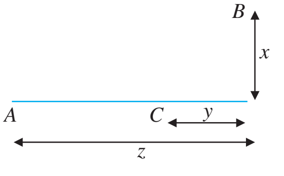


FIGURE b

2. The following problem is fictitious, but involves the kind of ambiguity that can make technical jobs challenging. The Band Candy Company decides to liquidate one of its candies. The company has 600,000 bags in inventory that it wants to sell. The candy had cost 35 cents per bag to manufacture and originally sold for 90 cents per bag. A marketing study indicates that if the candy is priced at p cents per bag, approximately $Q(p)=-p^2+40p+250$ thousand bags will be sold. Your task as consultant is to recommend the best selling price for the candy. As such, you should do the following: (a) find p to maximize $Q(p)$; (b) find p to maximize $10p\,Q(p)$, which is the actual revenue brought in by selling the candy. Then form your opinion, based on your evaluation of the relative importance of getting rid of as much candy as possible and making the most money possible.

3. A wonderful article by Timothy J. Pennings in the May 2003 issue of *The College Mathematics Journal* asks the question, "Do Dogs Know Calculus?" A slightly simplified version of exercises 17–22 can be solved in a very general form. Suppose that a ball is thrown from point A on the edge of the water and lands at point B, which is x meters into the water and z meters downshore from point A. (See the diagram.) At what point C should a dog enter the water to minimize the time to reach the ball? Assume that the dog's running speed is r m/s and the dog's swimming speed is s m/s. Find y as a function of x to minimize the time to reach the ball. Show that the answer is independent of z! Explain why your solution is invalid if $r \le s$ and explain what the dog should do in this case. Dr. Pennings' dog Elvis was clocked at $r=6.4$ m/s and $s=0.9$ m/s. Show that Elvis should follow the rule $y=0.144x$. In fact, Elvis' actual entry points are very close to these values for a variety of throws!



3.8 RELATED RATES

In this section, we present a group of problems known as **related rates** problems. The common thread in each problem is an equation relating two or more quantities that are all changing with time. In each case, we will use the chain rule to find derivatives of all terms in the equation (much as we did in section 2.8 with implicit differentiation). The differentiated equation allows us to determine how different derivatives (rates) are related.

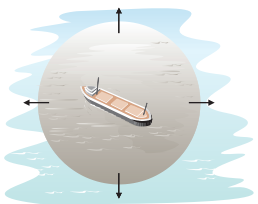


FIGURE 3.94
Oil spill

EXAMPLE 8.1 A Related Rates Problem

An oil tanker has an accident and oil pours out at the rate of 150 gallons per minute. Suppose that the oil spreads onto the water in a circle at a thickness of $\frac{1}{10}''$ (see Figure 3.94). Given that 1 ft^3 equals 7.5 gallons, determine the rate at which the radius of the spill is increasing when the radius reaches 500 feet.

Solution Since the area of a circle of radius r is πr^2, the volume of oil is given by

$$V = (\text{depth})(\text{area}) = \frac{1}{120}\pi r^2,$$

since the depth is $\frac{1}{10}'' = \frac{1}{120}$ ft. Both volume and radius are functions of time, so

$$V(t) = \frac{\pi}{120}[r(t)]^2.$$

Differentiating both sides of the equation with respect to t, we get

$$V'(t) = \frac{\pi}{120}2r(t)r'(t).$$

We are given a radius of 500 feet. The volume increases at a rate of 150 gallons per minute, or $\frac{150}{7.5} = 20$ ft^3/min. Substituting in $V'(t) = 20$ and $r = 500$, we have

$$20 = \frac{\pi}{120}2(500)r'(t).$$

Finally, solving for $r'(t)$, we find that the radius is increasing at the rate of $\frac{2.4}{\pi} \approx 0.76394$ feet per minute. ■

Although the details change from problem to problem, the general pattern of solution is the same for all related rates problems. Looking back, you should be able to identify each of the following steps in example 8.1.

1. Make a simple sketch, if appropriate.
2. Set up an equation relating all of the relevant quantities.
3. Differentiate (implicitly) both sides of the equation with respect to time (t).
4. Substitute in values for all known quantities and derivatives.
5. Solve for the remaining rate.

EXAMPLE 8.2 A Sliding Ladder

A 10-foot ladder leans against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2 ft/sec, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 feet off the ground?

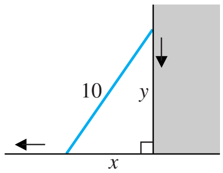


FIGURE 3.95
Sliding ladder

Solution First, we make a sketch of the problem, as seen in Figure 3.95. We have denoted the height of the top of the ladder as y and the distance from the wall to the bottom of the ladder as x. Since the ladder is sliding *down* the wall at the rate of 2 ft/sec, we must have that $\frac{dy}{dt} = -2$. (Note the minus sign here.) Observe that both x and y are functions of time, t. We can relate the variables by observing that, since the ladder is 10 feet long, the Pythagorean Theorem gives us

$$[x(t)]^2 + [y(t)]^2 = 100.$$

Differentiating both sides of this equation with respect to time gives us

$$\begin{aligned} 0 = \frac{d}{dt}(100) &= \frac{d}{dt}\left\{[x(t)]^2 + [y(t)]^2\right\} \\ &= 2x(t)x'(t) + 2y(t)y'(t). \end{aligned}$$

We can solve for $x'(t)$, to obtain

$$x'(t) = -\frac{y(t)}{x(t)}y'(t).$$

To make use of this, we need values for $x(t)$, $y(t)$ and $y'(t)$ at the point in question. Since we know that the height above ground of the top of the ladder is 8 feet, we have $y = 8$ and from the Pythagorean Theorem, we get

$$100 = x^2 + 8^2,$$

so that $x = 6$. We now have that at the point in question,

$$x'(t) = -\frac{y(t)}{x(t)}y'(t) = -\frac{8}{6}(-2) = \frac{8}{3}.$$

So, the bottom of the ladder is sliding away from the building at the rate of $\frac{8}{3}$ ft/sec. ■

EXAMPLE 8.3 Another Related Rates Problem

A car is traveling at 50 mph due south at a point $\frac{1}{2}$ mile north of an intersection. A police car is traveling at 40 mph due west at a point $\frac{1}{4}$ mile east of the same intersection. At that instant, the radar in the police car measures the rate at which the distance between the two cars is changing. What does the radar gun register?

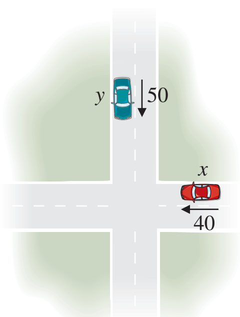


FIGURE 3.96
Cars approaching an intersection

Solution First, we sketch a picture and denote the vertical distance of the first car from the center of the intersection y and the horizontal distance of the police car x (see Figure 3.96). Notice that this says that $\frac{dx}{dt} = -40$, since the police car is moving in the direction of the negative x-axis and $\frac{dy}{dt} = -50$, since the other car is moving in the direction of the negative y-axis. From the Pythagorean Theorem, the distance between the two cars is $d = \sqrt{x^2 + y^2}$. Since all quantities are changing with time, our equation is

$$d(t) = \sqrt{[x(t)]^2 + [y(t)]^2} = \{[x(t)]^2 + [y(t)]^2\}^{1/2}.$$

Differentiating both sides with respect to t, we have by the chain rule that

$$d'(t) = \frac{1}{2}\{[x(t)]^2 + [y(t)]^2\}^{-1/2}\ 2[x(t)x'(t) + y(t)y'(t)]$$
$$= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{[x(t)]^2 + [y(t)]^2}}.$$

Substituting in $x(t) = \frac{1}{4}$, $x'(t) = -40$, $y(t) = \frac{1}{2}$ and $y'(t) = -50$, we have

$$d'(t) = \frac{\frac{1}{4}(-40) + \frac{1}{2}(-50)}{\sqrt{\frac{1}{4} + \frac{1}{16}}} = \frac{-140}{\sqrt{5}} \approx -62.6,$$

so that the radar gun registers 62.6 mph. Note that this is a poor estimate of the car's actual speed. For this reason, police nearly always take radar measurements from a stationary position. ■

In some problems, the variables are not related by a geometric formula, in which case you will not need to follow the first two steps of our outline. In example 8.4, the third step is complicated by the lack of a given value for one of the rates of change.

EXAMPLE 8.4 Estimating a Rate of Change in Economics

A small company estimates that when it spends x thousand dollars for advertising in a year, its annual sales will be described by $s = 60 - 40e^{-0.05x}$ thousand dollars. The four most recent annual advertising totals are given in the following table.

Year	1	2	3	4
Dollars	14,500	16,000	18,000	20,000

Estimate the current (year 4) value of $x'(t)$ and the current rate of change of sales.

Solution From the table, we see that the recent trend is for advertising to increase by \$2000 per year. A good estimate is then $x'(4) \approx 2$. Starting with the sales equation

$$s(t) = 60 - 40e^{-0.05x(t)},$$

we use the chain rule to obtain

$$s'(t) = -40e^{-0.05x(t)}[-0.05x'(t)] = 2x'(t)e^{-0.05x(t)}.$$

Using our estimate that $x'(4) \approx 2$ and since $x(4) = 20$, we get $s'(4) \approx 2(2)e^{-1} \approx 1.472$. Thus, sales are increasing at the rate of approximately \$1472 per year. ■

Notice that example 8.5 is similar to example 8.6 of section 2.8.

EXAMPLE 8.5 Tracking a Fast Jet

A spectator at an air show is trying to follow the flight of a jet. The jet follows a straight path in front of the observer at 540 mph. At its closest approach, the jet passes 600 feet in front of the person. Find the maximum rate of change of the angle between the spectator's line of sight and a line perpendicular to the flight path, as the jet flies by.

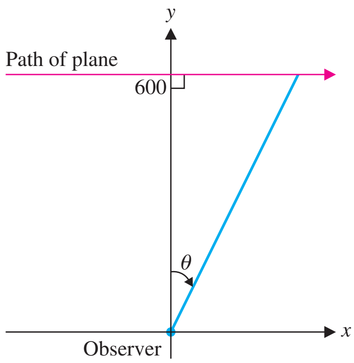


FIGURE 3.97
Path of jet

Solution Place the spectator at the origin (0, 0) and the jet's path left to right on the line $y = 600$, and call the angle between the positive y-axis and the line of sight θ (see Figure 3.97). If we measure distance in feet and time in seconds, we first need to convert the jet's speed to feet per second. We have

$$540\tfrac{\text{mi}}{\text{h}} = \left(540\tfrac{\text{mi}}{\text{h}}\right)\left(5280\tfrac{\text{ft}}{\text{mi}}\right)\left(\tfrac{1}{3600}\tfrac{\text{h}}{\text{s}}\right) = 792\tfrac{\text{ft}}{\text{s}}.$$

From triangle trigonometry (see Figure 3.97), an equation relating the angle θ with x and y is $\tan\theta = \dfrac{x}{y}$. Be careful with this; since we are measuring θ from the vertical, this equation may not be what you expect. Since all quantities are changing with time, we have

$$\tan\theta(t) = \frac{x(t)}{y(t)}.$$

Differentiating both sides with respect to time, we have

$$[\sec^2\theta(t)]\,\theta'(t) = \frac{x'(t)y(t) - x(t)y'(t)}{[y(t)]^2}.$$

With the jet moving left to right along the line $y = 600$, we have $x'(t) = 792$, $y(t) = 600$ and $y'(t) = 0$. Substituting these quantities, we have

$$[\sec^2\theta(t)]\,\theta'(t) = \frac{792(600)}{600^2} = 1.32.$$

Solving for the rate of change $\theta'(t)$, we get

$$\theta'(t) = \frac{1.32}{\sec^2\theta(t)} = 1.32\cos^2\theta(t).$$

Observe that the rate of change is a maximum when $\cos^2\theta(t)$ is a maximum. Since the maximum of the cosine function is 1, the maximum value of $\cos^2\theta(t)$ is 1, occurring when $\theta = 0$. We conclude that the maximum rate of angle change is 1.32 radians/second. This occurs when $\theta = 0$, that is, when the jet reaches its closest point to the observer. (Think about this; it should match your intuition!) Since humans can track objects at up to about 3 radians/second, this means that we can visually follow even a fast jet at a very small distance. ■

EXERCISES 3.8

WRITING EXERCISES

1. As you read examples 8.1–8.3, to what extent do you find the pictures helpful? In particular, would it be clear what x and y represent in example 8.3 without a sketch? Also, in example 8.3 explain why the derivatives $x'(t)$, $y'(t)$ and $d'(t)$ are all negative. Does the sketch help in this explanation?

2. In example 8.4, the increase in advertising dollars from year 1 to year 2 was \$1500. Explain why this amount is not especially relevant to the approximation of $s'(4)$.

---

1. Oil spills out of a tanker at the rate of 120 gallons per minute. The oil spreads in a circle with a thickness of $\frac{1}{4}''$. Given that 1 ft^3 equals 7.5 gallons, determine the rate at which the radius of the spill is increasing when the radius reaches (a) 100 ft and (b) 200 ft. Explain why the rate decreases as the radius increases.

2. Oil spills out of a tanker at the rate of 90 gallons per minute. The oil spreads in a circle with a thickness of $\frac{1}{8}''$. Determine the rate at which the radius of the spill is increasing when the radius reaches 100 feet.

3. Oil spills out of a tanker at the rate of g gallons per minute. The oil spreads in a circle with a thickness of $\frac{1}{4}''$. Given that the radius of the spill is increasing at a rate of

0.6 ft/min when the radius equals 100 feet, determine the value of g.

4. In exercises 1–3 and example 8.1, if the thickness of the oil is doubled, how does the rate of increase of the radius change?

5. Assume that the infected area of an injury is circular. If the radius of the infected area is 3 mm and growing at a rate of 1 mm/hr, at what rate is the infected area increasing?

6. For the injury of exercise 5, find the rate of increase of the infected area when the radius reaches 6 mm. Explain in commonsense terms why this rate is larger than that of exercise 5.

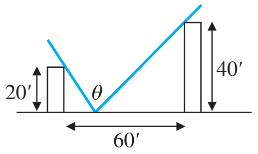
7. Suppose that a raindrop evaporates in such a way that it maintains a spherical shape. Given that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$, if the radius changes in time, show that $V' = Ar'$. If the rate of evaporation (V') is proportional to the surface area, show that the radius changes at a constant rate.

8. Suppose a forest fire spreads in a circle with radius changing at a rate of 5 feet per minute. When the radius reaches 200 feet, at what rate is the area of the burning region increasing?

9. A 10-foot ladder leans against the side of a building as in example 8.2. If the bottom of the ladder is pulled away from the wall at the rate of 3 ft/s and the ladder remains in contact with the wall, find the rate at which the top of the ladder is dropping when the bottom is 6 feet from the wall.

10. In exercise 9, find the rate at which the angle between the ladder and the horizontal is changing when the bottom of the ladder is 6 feet from the wall.

11. Two buildings of height 20 feet and 40 feet, respectively, are 60 feet apart. Suppose that the intensity of light at a point between the buildings is proportional to the angle θ in the figure. If a person is moving from right to left at 4 ft/s, at what rate is θ changing when the person is exactly halfway between the two buildings?



12. Find the location in exercise 11 where the angle θ is maximum.

13. A plane is located $x = 40$ miles (horizontally) away from an airport at an altitude of h miles. Radar at the airport detects that the distance $s(t)$ between the plane and airport is changing at the rate of $s'(t) = -240$ mph. If the plane flies toward the airport at the constant altitude $h = 4$, what is the speed $|x'(t)|$ of the airplane?

14. Repeat exercise 13 with a height of 6 miles. Based on your answers, how important is it to know the actual height of the airplane?

15. Rework example 8.3 if the police car is not moving. Does this make the radar gun's measurement more accurate?

16. Show that the radar gun of example 8.3 gives the correct speed if the police car is located at the origin.

17. Show that the radar gun of example 8.3 gives the correct speed if the police car is at $x = \frac{1}{2}$ moving at a speed of $(\sqrt{2} - 1)$ 50 mph.

18. Find a position and speed for which the radar gun of example 8.3 has a slower reading than the actual speed.

19. Suppose that the average yearly cost per item for producing x items of a business product is $\overline{C}(x) = 10 + \frac{100}{x}$. If the current production is $x = 10$ and production is increasing at a rate of 2 items per year, find the rate of change of the average cost.

20. Suppose that the average yearly cost per item for producing x items of a business product is $\overline{C}(x) = 12 + \frac{94}{x}$. The three most recent yearly production figures are given in the table.

Year	0	1	2
Prod. (x)	8.2	8.8	9.4

Estimate the value of $x'(2)$ and the current (year 2) rate of change of the average cost.

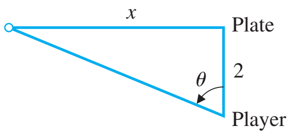
21. For a small company spending $\$x$ thousand per year in advertising, suppose that annual sales in thousands of dollars equal $s = 60 - 40e^{-0.05x}$. The three most recent yearly advertising figures are given in the table.

Year	0	1	2
Adver.	16,000	18,000	20,000

Estimate the value of $x'(2)$ and the current (year 2) rate of change of sales.

22. For a small company spending $\$x$ thousand per year in advertising, suppose that annual sales in thousands of dollars equal $s = 80 - 20e^{-0.04x}$. If the current advertising budget is $x = 40$ and the budget is increasing at a rate of $1500 per year, find the rate of change of sales.

23. A baseball player stands 2 feet from home plate and watches a pitch fly by. In the diagram, x is the distance from the ball to home plate and θ is the angle indicating the direction of the player's gaze. Find the rate θ' at which his eyes must move to watch a fastball with $x'(t) = -130$ ft/s as it crosses home plate at $x = 0$.

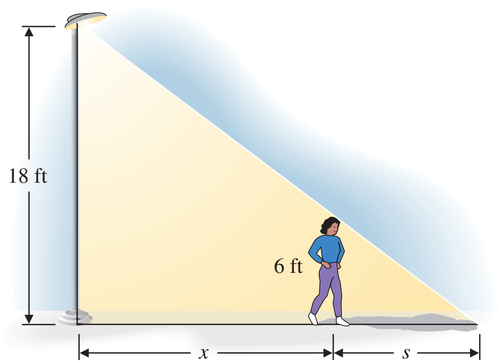


24. In the situation of exercise 23, humans can maintain focus only when $\theta' \le 3$ (see Watts and Bahill's book *Keep Your Eye on the Ball*). Find the fastest pitch that you could actually watch cross home plate.

25. A camera tracks the launch of a vertically ascending spacecraft. The camera is located at ground level 2 miles from the launchpad. If the spacecraft is 3 miles up and traveling at 0.2 mile per second, at what rate is the camera angle (measured from the horizontal) changing?

26. Repeat exercise 25 for the spacecraft at 1 mile up (assume the same velocity). Which rate is higher? Explain in commonsense terms why it is larger.

27. Suppose a 6-ft-tall person is 12 ft away from an 18-ft-tall lamppost (see the figure). If the person is moving away from the lamppost at a rate of 2 ft/s, at what rate is the length of the shadow changing? $\left(\text{Hint: Show that } \frac{x+s}{18} = \frac{s}{6}.\right)$



Exercise 27

28. Rework exercise 27 if the person is 6 ft away from the lamppost and is walking toward the lamppost at a rate of 3 ft/s.

29. Boyle's law for a gas at constant temperature is $PV = c$, where P is pressure, V is volume and c is a constant. Assume that both P and V are functions of time. Show that $P'(t)/V'(t) = -c/V^2$.

30. In exercise 29, solve for P as a function of V. Treating V as an independent variable, compute $P'(V)$. Compare $P'(V)$ and $P'(t)/V'(t)$ from exercise 29.

31. A dock is 6 feet above water. Suppose you stand on the edge of the dock and pull a rope attached to a boat at the constant rate of 2 ft/s. Assume that the boat remains at water level. At what speed is the boat approaching the dock when it is 20 feet from the dock? 10 feet from the dock? Isn't it surprising that the boat's speed is not constant?

32. Sand is poured into a conical pile with the height of the pile equalling the diameter of the pile. If the sand is poured at a constant rate of 5 m^3/s, at what rate is the height of the pile increasing when the height is 2 meters?

33. The frequency at which a guitar string vibrates (which determines the pitch of the note we hear) is related to the tension T to which the string is tightened, the density ρ of the string and the effective length L of the string by the equation $f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$. By running his finger along a string, a guitarist can change L by changing the distance between the bridge and his finger. Suppose that $L = \frac{1}{2}$ ft and $\sqrt{\frac{T}{\rho}} = 220$ ft/s so that the units of f are Hertz (cycles per second). If the guitarist's hand slides so that $L'(t) = -4$, find $f'(t)$. At this rate, how long will it take to raise the pitch one octave (that is, double f)?

34. Suppose that you are blowing up a balloon by adding air at the rate of 1 ft^3/s. If the balloon maintains a spherical shape, the volume and radius are related by $V = \frac{4}{3}\pi r^3$. Compare the rate at which the radius is changing when $r = 0.01$ ft versus when $r = 0.1$ ft. Discuss how this matches the experience of a person blowing up a balloon.

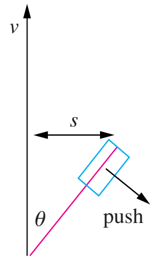
35. Water is being pumped into a spherical tank of radius 60 feet at the constant rate of 10 ft^3/s. Find the rate at which the radius of the top level of water in the tank changes when the tank is half full.

36. For the water tank in exercise 35, find the height at which the height of the water in the tank changes at the same rate as the radius.

37. Sand is dumped such that the shape of the sandpile remains a cone with height equal to twice the radius. If the sand is dumped at the constant rate of 20 ft^3/s, find the rate at which the radius is increasing when the height reaches 6 feet.

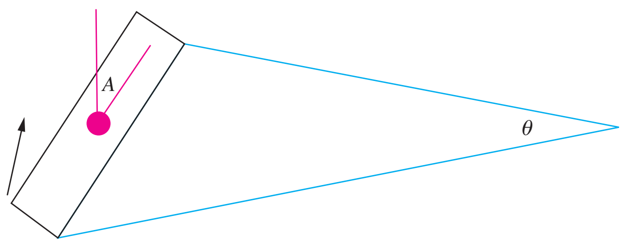
38. Repeat exercise 37 for a sandpile for which the edge of the sandpile forms an angle of 45° with the horizontal.

39. To start skating, you must angle your foot and push off the ice. Alain Haché's *The Physics of Hockey* derives the relationship between the skate angle θ, the sideways stride distance s, the stroke period T and the forward speed v of the skater, with $\theta = \tan^{-1}\left(\frac{2s}{vT}\right)$. For $T = 1$ second, $s = 60$ cm and an acceleration of 1 m/s^2, find the rate of change of the angle θ when the skater reaches (a) 1 m/s and (b) 2 m/s. Interpret the sign and size of θ' in terms of skating technique.

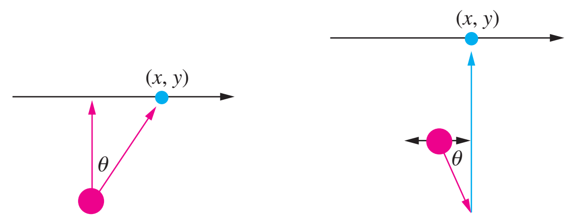


EXPLORATORY EXERCISES

1. A thin rectangular advertising sign rotates clockwise at the rate of 6 revolutions per minute. The sign is 4 feet wide and an observer stands 40 feet away. Find the rate of change of the viewing angle θ as a function of the sign angle A. At which position of the sign is the rate of change maximum?



2. Vision has proved to be the biggest challenge for building functional robots. Robot vision can either be designed to mimic human vision or follow a different design. Two possibilities are analyzed here. In the diagram below, a camera follows an object directly from left to right. If the camera is at the origin, the object moves with speed 1 m/s and the line of motion is at $y = c$, find an expression for θ' as a function of the position of the object. In the diagram to the right, the camera looks down into a parabolic mirror and indirectly views the object. If the mirror has polar coordinates (in this case, the angle θ is measured from the horizontal) equation $r = \dfrac{1 - \sin\theta}{2\cos^2\theta}$ and $x = r\cos\theta$, find an expression for θ' as a function of the position of the object. Compare values of θ' at $x = 0$ and other x-values. If a large value of θ' causes the image to blur, which camera system is better? Does the distance $y = c$ affect your preference?



3. A particle moves down a ramp subject only to the force of gravity. Let y_0 be the maximum height of the particle. Then conservation of energy gives

$$\frac{1}{2}mv^2 + mgy = mgy_0$$

(a) From the definition $v(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}$, conclude that $|y'(t)| \le |v(t)|$.

(b) Show that $|v'(t)| \le g$.

(c) What shape must the ramp have to get equality in part (b)? Briefly explain in physical terms why g is the maximum value of $|v'(t)|$.

3.9 RATES OF CHANGE IN ECONOMICS AND THE SCIENCES

It has often been said that mathematics is the language of nature. Today, the concepts of calculus are being applied in virtually every field of human endeavor. The applications in this section represent but a small sampling of some elementary uses of the derivative. These are not all of the uses of the derivative nor are they necessarily the most important uses, but rather, represent some interesting applications in a variety of fields.

Recall that the derivative of a function gives the instantaneous rate of change of that function. So, when you see the word *rate,* you should be thinking *derivative*. You can hardly pick up a newspaper without finding reference to some rates (e.g., inflation rate, interest rate, unemployment rate, etc.). These can be thought of as derivatives. There are also many quantities with which you are familiar, but that you might not recognize as rates of change. Our first example, which comes from economics, is of this type.

In economics, the term **marginal** is used to indicate a rate. Thus, **marginal cost** is the derivative of the cost function, **marginal profit** is the derivative of the profit function and so on. We introduce marginal cost in some detail here, with further applications given in the exercises.

Suppose that you are manufacturing an item, where your start-up costs are \$4000 and productions costs are \$2 per item. The total cost of producing x items would then be $4000 + 2x$. Of course, the assumption that the cost per item is constant is unrealistic. Efficient mass-production techniques could reduce the cost per item, but machine maintenance, labor, plant expansion and other factors could drive costs up as production (x) increases. In example 9.1, a quadratic cost function is used to take into account some of these extra factors. In practice, you would find a cost function by making some observations of the cost of producing a number of different quantities and then fitting the data to the graph of a known function. (This is one way in which the calculus is brought to bear on real-world problems.)

When the cost per item is not constant, an important question for managers to answer is how much it will cost to increase production. This is the idea behind marginal cost.

EXAMPLE 9.1 Analyzing the Marginal Cost of Producing a Commercial Product

Suppose that

$$C(x) = 0.02x^2 + 2x + 4000$$

is the total cost (in dollars) for a company to produce x units of a certain product. Compute the marginal cost at $x = 100$ and compare this to the actual cost of producing the 100th unit.

Solution The marginal cost function is the derivative of the cost function:

$$C'(x) = 0.04x + 2$$

and so, the marginal cost at $x = 100$ is $C'(100) = 4 + 2 = 6$ dollars per unit. On the other hand, the actual cost of producing item number 100 would be $C(100) - C(99)$. (Why?) We have

$$\begin{aligned} C(100) - C(99) &= 200 + 200 + 4000 - (196.02 + 198 + 4000) \\ &= 4400 - 4394.02 = 5.98 \text{ dollars.} \end{aligned}$$

Note that this is very close to the marginal cost of \$6. Also notice that the marginal cost is easier to compute. ■

Another quantity that businesses use to analyze production is **average cost.** You can easily remember the formula for average cost by thinking of an example. If it costs a total of \$120 to produce 12 items, then the average cost would be \$10 $\left(\$\frac{120}{12}\right)$ per item. In general, the total cost is given by $C(x)$ and the number of items by x, so average cost is defined by

$$\overline{C}(x) = \frac{C(x)}{x}.$$

A business manager would want to know the level of production that minimizes average cost.

EXAMPLE 9.2 Minimizing the Average Cost of Producing a Commercial Product

Suppose that
$$C(x) = 0.02x^2 + 2x + 4000$$
is the total cost (in dollars) for a company to produce x units of a certain product. Find the production level x that minimizes the average cost.

Solution The average cost function is given by

$$\overline{C}(x) = \frac{0.02x^2 + 2x + 4000}{x} = 0.02x + 2 + 4000x^{-1}.$$

To minimize $\overline{C}(x)$, we start by finding critical numbers in the domain $x > 0$. We have

$$\overline{C}'(x) = 0.02 - 4000x^{-2} = 0 \quad \text{if}$$
$$4000x^{-2} = 0.02 \quad \text{or}$$
$$\frac{4000}{0.02} = x^2.$$

Then $x^2 = 200{,}000$ or $x = \pm\sqrt{200{,}000} \approx \pm 447$. Since $x > 0$, the only relevant critical number is at approximately $x = 447$. Further, $\overline{C}'(x) < 0$ if $x < 447$ and $\overline{C}'(x) > 0$ if $x > 447$, so this critical number is the location of the absolute minimum on the domain $x > 0$. A graph of the average cost function (see Figure 3.98) shows the minimum. ■

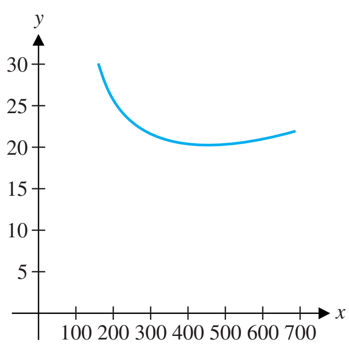


FIGURE 3.98
Average cost function

Our third example also comes from economics. This time, we will explore the relationship between price and demand. Clearly, in most cases, a higher price will lower the demand for a product. However, if sales do not decrease significantly, a company may actually increase revenue despite a price increase. As we will see, an analysis of the *elasticity of demand* can give us important information about revenue.

Suppose that the demand x for an item is a function of its price p. That is, $x = f(p)$. If the price changes by a small amount Δp, then the **relative change in price** equals $\frac{\Delta p}{p}$. However, the change in price would create a change in demand Δx, with a **relative change in demand** of $\frac{\Delta x}{x}$. Economists define the **elasticity of demand at price *p*** to be the relative change in demand divided by the relative change in price for very small changes in price. As calculus students, you can define the elasticity E as a limit:

$$E = \lim_{\Delta p \to 0} \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}}.$$

In the case where x is a function of p, we write $\Delta p = (p + h) - p = h$ for some small h and then $\Delta x = f(p + h) - f(p)$. We then have

$$E = \lim_{h \to 0} \frac{\frac{f(p+h) - f(p)}{f(p)}}{\frac{h}{p}} = \frac{p}{f(p)} \lim_{h \to 0} \frac{f(p+h) - f(p)}{h} = \frac{p}{f(p)} f'(p),$$

assuming that f is differentiable. In example 9.3, we analyze elasticity of demand and revenue. Recall that if $x = f(p)$ items are sold at price p, then the revenue equals $pf(p)$.

EXAMPLE 9.3 Computing Elasticity of Demand and Changes in Revenue

Suppose that
$$f(p) = 400(20 - p)$$
is the demand for an item at price p (in dollars) with $p < 20$. (a) Find the elasticity of demand. (b) Find the range of prices for which $E < -1$. Compare this to the range of prices for which revenue is a decreasing function of p.

Solution The elasticity of demand is given by

$$E = \frac{p}{f(p)} f'(p) = \frac{p}{400(20-p)}(-400) = \frac{p}{p-20}.$$

We show a graph of $E = \dfrac{p}{p-20}$ in Figure 3.99. Observe that $E < -1$ if

$$\frac{p}{p-20} < -1$$

or
$$p > -(p - 20). \quad \text{Since } p - 20 < 0.$$

Solving this gives us
$$2p > 20$$

or
$$p > 10.$$

To analyze revenue, we compute $R = pf(p) = p(8000 - 400p) = 8000p - 400p^2$. Revenue decreases if $R'(p) < 0$. From $R'(p) = 8000 - 800p$, we see that $R'(p) = 0$ if $p = 10$ and $R'(p) < 0$ if $p > 10$. Of course, this says that the revenue decreases if the price exceeds 10. ■

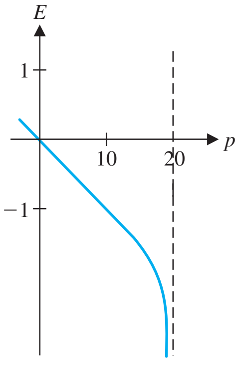


FIGURE 3.99
$E = \dfrac{p}{p-20}$

Notice in example 9.3 that the prices for which $E < -1$ (in this case, we say that the demand is **elastic**) correspond exactly to the prices for which an increase in price will decrease revenue. In the exercises, we will find that this is not a coincidence.

The next example we offer comes from chemistry. It is very important for chemists to have a handle on the rate at which a given reaction proceeds. Reaction rates give chemists information about the nature of the chemical bonds being formed and broken, as well as information about the type and quantity of product to expect. A simple situation is depicted in the schematic

$$A + B \longrightarrow C,$$

which indicates that chemicals A and B (the *reactants*) combine to form chemical C (the *product*). Let $[C](t)$ denote the concentration (in moles per liter) of the product. The average reaction rate between times t_1 and t_2 is

$$\frac{[C](t_2) - [C](t_1)}{t_2 - t_1}.$$

The instantaneous reaction rate at any given time t_1 is then given by

$$\lim_{t \to t_1} \frac{[C](t) - [C](t_1)}{t - t_1} = \frac{d[C]}{dt}(t_1).$$

Depending on the details of the reaction, it is often possible to write down an equation relating the reaction rate $\frac{d[C]}{dt}$ to the concentrations of the reactants, $[A]$ and $[B]$.

EXAMPLE 9.4 Modeling the Rate of a Chemical Reaction

In an **autocatalytic** chemical reaction, the reactant and the product are the same. The reaction continues until some saturation level is reached. From experimental evidence, chemists know that the reaction rate is jointly proportional to the amount of the product present and the difference between the saturation level and the amount of the product. If the initial concentration of the chemical is 0 and the saturation level is 1 (corresponding to 100%), this means that the concentration $x(t)$ of the chemical satisfies the equation

$$x'(t) = rx(t)[1 - x(t)],$$

where $r > 0$ is a constant.

Find the concentration of chemical for which the reaction rate $x'(t)$ is a maximum.

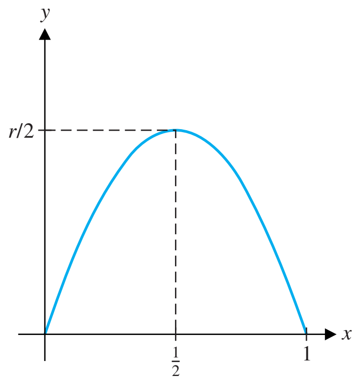


FIGURE 3.100
$y = rx(1 - x)$

Solution To clarify the problem, we write the reaction rate as

$$f(x) = rx(1 - x).$$

Our aim is then to find $x \geq 0$ that maximizes $f(x)$. From the graph of $y = f(x)$ shown in Figure 3.100, the maximum appears to occur at about $x = \frac{1}{2}$. We have

$$\begin{aligned} f'(x) &= r(1)(1 - x) + rx(-1) \\ &= r(1 - 2x) \end{aligned}$$

and so, the only critical number is $x = \frac{1}{2}$. Notice that the graph of $y = f(x)$ is a parabola opening downward and hence, the critical number must correspond to the absolute maximum. Although the mathematical problem here was easy to solve, the result gives a chemist some precise information. At the time the reaction rate reaches a maximum, the concentration of chemical equals exactly half of the saturation level. ■

Our second example from chemistry involves the titration of a weak acid and a strong base. In this type of titration, a strong base is slowly added to a weak acid. The pH of the mixture is monitored by observing the color of some pH indicator, which changes dramatically at what is called the **equivalence point.** The equivalence point is then typically used to compute the concentration of the base. A generalized titration curve is shown in Figure 3.101, where the horizontal axis indicates the amount of base added to the mixture and the vertical axis shows the pH of the mixture. Notice the nearly vertical rise of the graph at the equivalence point.

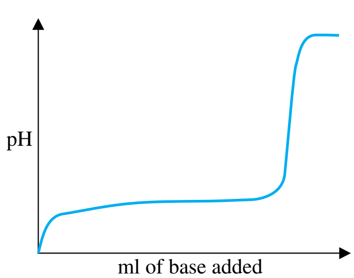
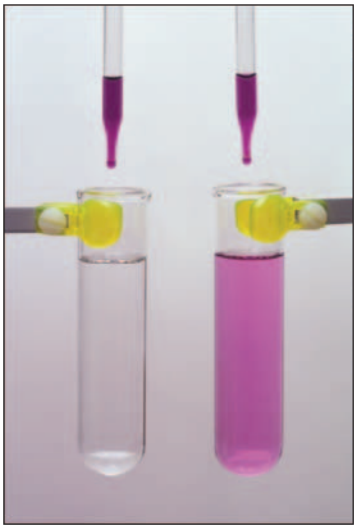


FIGURE 3.101
Acid titration

Let x be the fraction $(0 < x < 1)$ of base added (equal to the fraction of converted acid; see Harris' *Quantitative Chemical Analysis* for more details), with $x = 1$ representing the equivalence point. Then the pH is approximated by $c + \ln \frac{x}{1 - x}$, where c is a constant closely related to the **acid dissociation constant.**



EXAMPLE 9.5 Analyzing a Titration Curve

Find the value of x at which the rate of change of pH is the smallest. Identify the corresponding point on the titration curve in Figure 3.101.

Solution The pH is given by the function $p(x) = c + \ln \dfrac{x}{1-x}$. The rate of change of pH is then given by the derivative $p'(x)$. To make this computation easier, we write $p(x) = c + \ln x - \ln(1-x)$. The derivative is

$$p'(x) = \frac{1}{x} - \frac{1}{1-x}(-1) = \frac{1}{x(1-x)} = \frac{1}{x-x^2}.$$

The problem then is to minimize the function $g(x) = \dfrac{1}{x-x^2} = (x-x^2)^{-1}$, with $0 < x < 1$. Critical points come from the derivative

$$g'(x) = -(x-x^2)^{-2}(1-2x) = \frac{2x-1}{(x-x^2)^2}.$$

Notice that $g'(x)$ does not exist if $x - x^2 = 0$, which occurs when $x = 0$ or $x = 1$, neither of which is in domain $0 < x < 1$. Further, $g'(x) = 0$ if $x = \frac{1}{2}$, which is in the domain. You should check that $g'(x) < 0$ if $0 < x < \frac{1}{2}$ and $g'(x) > 0$ if $\frac{1}{2} < x < 1$, which proves that the minimum of $g(x)$ occurs at $x = \frac{1}{2}$. Although the horizontal axis in Figure 3.101 is not labeled, observe that we can still locate this point on the graph. We found the solution of $g'(x) = 0$. Since $g(x) = p'(x)$, we have $p''(x) < 0$ for $0 < x < \frac{1}{2}$ and $p''(x) > 0$ for $\frac{1}{2} < x < 1$, so that the point of minimum change is an inflection point of the original graph. ■

Calculus and elementary physics are quite closely connected historically. It should come as no surprise, then, that physics provides us with such a large number of important applications of the calculus. We have already explored the concepts of velocity and acceleration. Another important application in physics where the derivative plays a role involves density. There are many different kinds of densities that we could consider. For example, we could study population density (number of people per unit area) or color density (depth of color per unit area) used in the study of radiographs. However, the most familiar type of density is **mass density** (mass per unit volume). You probably already have some idea of what we mean by this, but how would you define it? If an object of interest is made of some homogeneous material (i.e., the mass of any portion of the object of a given volume is the same), then the mass density is simply

$$\textit{mass density} = \frac{\textit{mass}}{\textit{volume}}$$

and this quantity is constant throughout the object. However, if the mass of a given volume varies in different parts of the object, then this formula only calculates the *average density* of the object. In example 9.6 we find a means of computing the mass density at a specific point in a nonhomogeneous object.

Suppose that the function $f(x)$ gives us the mass (in kilograms) of the first x meters of a thin rod (see Figure 3.102).

The total mass between marks x and x_1 $(x > x_1)$ is given by $[f(x) - f(x_1)]$ kg. The **average linear density** (i.e., mass per unit length) between x and x_1 is then defined as

$$\frac{f(x) - f(x_1)}{x - x_1}.$$

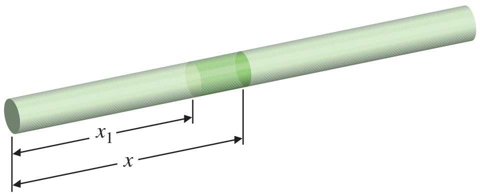


FIGURE 3.102
A thin rod

Finally, the **linear density** at $x = x_1$ is defined as

$$\rho(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} = f'(x_1), \tag{9.1}$$

where we have recognized the alternative definition of derivative discussed in section 2.2.

EXAMPLE 9.6 Density of a Thin Rod

Suppose that the mass of the first x meters of a thin rod is given by $f(x) = \sqrt{2x}$. Compute the linear density at $x = 2$ and at $x = 8$, and compare the densities at the two points.

Solution From (9.1), we have

$$\rho(x) = f'(x) = \frac{1}{2\sqrt{2x}}(2) = \frac{1}{\sqrt{2x}}.$$

Thus, $\rho(2) = 1/\sqrt{4} = 1/2$ and $\rho(8) = 1/\sqrt{16} = 1/4$. Notice that this says that the rod is *nonhomogeneous* (i.e., the mass density in the rod is not constant). Specifically, we have that the rod is less dense at $x = 8$ than at $x = 2$. ■

The next example also comes from physics, in particular from the study of electromagnetism.

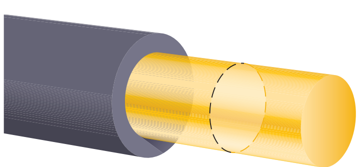


FIGURE 3.103
An electrical wire

Suppose that $Q(t)$ represents the electrical charge in a wire at time t. Then, the derivative $Q'(t)$ gives the **current** flowing through the wire. To see this, consider the cross section of a wire as shown in Figure 3.103. Between times t_1 and t_2, the net charge passing through such a cross section is $Q(t_2) - Q(t_1)$. The **average current** (charge per unit time) over this time interval is then defined as

$$\frac{Q(t_2) - Q(t_1)}{t_2 - t_1}.$$

The **instantaneous current** $I(t)$ at any time t_1 can then be found by computing the limit

$$I(t_1) = \lim_{t \to t_1} \frac{Q(t) - Q(t_1)}{t - t_1} = Q'(t_1), \tag{9.2}$$

since (9.2) is again the alternative definition of derivative.

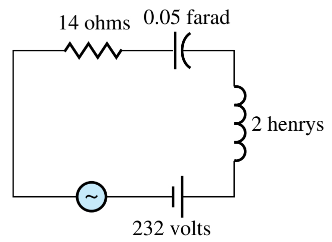


FIGURE 3.104
A simple electrical circuit

EXAMPLE 9.7 Modeling Electrical Current in a Wire

The electrical circuit shown in Figure 3.104 includes a 14-ohm resistor, a 2-henry inductor, a 0.05-farad capacitor and a battery supplying 232 volts of AC current modeled by the oscillating function $232 \sin 2t$, where t is measured in seconds. Find the current in the circuit at any time t.

Solution It can be shown (using the elementary laws of electricity) that the charge in this circuit is given by

$$Q(t) = 10e^{-5t} + 2te^{-2t} + 3\sin 2t - 7\cos 2t \text{ coulombs.}$$

The current is then

$$Q'(t) = -50e^{-5t} + 2e^{-2t} - 4te^{-2t} + 6\cos 2t + 14\sin 2t \text{ amps (coulombs per second).}$$

In section 2.1, we briefly explored the rate of growth of a population. Population dynamics is an area of biology that makes extensive use of calculus. We examine population models in some detail in sections 7.1 and 7.2. For now, we explore one aspect of a basic model of population growth called the **logistic equation.** This states that if $p(t)$ represents population (measured as a fraction of the maximum sustainable population), then the rate of change of the population satisfies the equation

$$p'(t) = rp(t)[1 - p(t)],$$

for some constant r. A typical solution [for $r = 1$ and $p(0) = 0.1$] is shown in Figure 3.105. Although we won't learn how to compute a solution until sections 7.1 and 7.2, we can determine some of the mathematical properties that all solutions must possess.

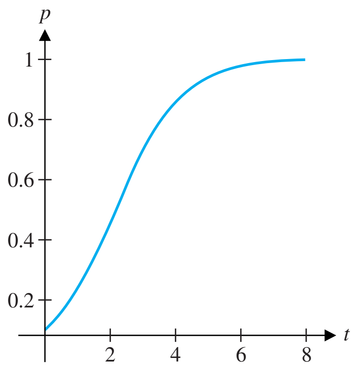


FIGURE 3.105
Logistic growth

EXAMPLE 9.8 Finding the Maximum Rate of Population Growth

Suppose that a population grows according to the equation $p'(t) = 2p(t)[1 - p(t)]$ (the logistic equation with $r = 2$). Find the population for which the growth rate is a maximum. Interpret this point graphically.

Solution To clarify the problem, we write the population growth rate as

$$f(p) = 2p(1 - p).$$

Our aim is then to find the population $p \geq 0$ that maximizes $f(p)$. We have

$$\begin{aligned} f'(p) &= 2(1)(1 - p) + 2p(-1) \\ &= 2(1 - 2p) \end{aligned}$$

and so, the only critical number is $p = \frac{1}{2}$. Notice that the graph of $y = f(p)$ is a parabola opening downward and hence, the critical number must correspond to the absolute maximum. In Figure 3.105, observe that the height $p = \frac{1}{2}$ corresponds to the portion of the graph with maximum slope. Also, notice that this point is an inflection point on the graph. We can verify this by noting that we solved the equation $f'(p) = 0$, where $f(p)$ equals $p'(t)$. Therefore, $p = \frac{1}{2}$ is the p-value corresponding to the solution of $p''(t) = 0$. This fact can be of value to population biologists. If they are tracking a population that reaches an inflection point, then (assuming that the logistic equation gives an accurate model) the population will eventually double in size. ■

Notice the similarities between examples 9.4 and 9.8. One reason that mathematics has such great value is that seemingly unrelated physical processes often have the same mathematical description. Comparing examples 9.4 and 9.8, we learn that the underlying mechanisms for autocatalytic reactions and population growth are identical.

We have now discussed examples of eight rates of change drawn from economics and the sciences. Add these to the applications that we have seen in previous sections and we have an impressive list of applications of the derivative. Even so, we have barely begun to scratch the surface. In any field where it is possible to quantify and analyze the properties of a function, calculus and the derivative are powerful tools. This list includes at least some aspect of nearly every major field of study. The continued study of calculus will give you the ability to read (and understand) technical studies in a wide variety of fields and to see (as we have in this section) the underlying unity that mathematics brings to a broad range of human endeavors.

EXERCISES 3.9

WRITING EXERCISES

1. The **logistic equation** $x'(t) = x(t)[1 - x(t)]$ is used to model many important phenomena (see examples 9.4 and 9.8). The equation has two competing contributions to the rate of change $x'(t)$. The term $x(t)$ by itself would mean that the larger $x(t)$ is, the faster the population (or concentration of chemical) grows. This is balanced by the term $1 - x(t)$, which indicates that the closer $x(t)$ gets to 1, the slower the population growth is. With these two terms together, the model has the property that for small $x(t)$, slightly larger $x(t)$ means greater growth, but as $x(t)$ approaches 1, the growth tails off. Explain in terms of population growth and the concentration of a chemical why the model is reasonable.

2. Corporate deficits and debt are frequently in the news, but the terms are often confused with each other. To take an example, suppose a company finishes a fiscal year owing \$5000. That is their **debt.** Suppose that in the following year the company has revenues of \$106,000 and expenses of \$109,000. The company's **deficit** for the year is \$3000, and the company's debt has increased to \$8000. Briefly explain why deficit can be thought of as the derivative of debt.

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1. If the cost of manufacturing x items is $C(x) = x^3 + 20x^2 + 90x + 15$, find the marginal cost function and compare the marginal cost at $x = 50$ with the actual cost of manufacturing the 50th item.

2. If the cost of manufacturing x items is $C(x) = x^4 + 14x^2 + 60x + 35$, find the marginal cost function and compare the marginal cost at $x = 50$ with the actual cost of manufacturing the 50th item.

3. If the cost of manufacturing x items is $C(x) = x^3 + 21x^2 + 110x + 20$, find the marginal cost function and compare the marginal cost at $x = 100$ with the actual cost of manufacturing the 100th item.

4. If the cost of manufacturing x items is $C(x) = x^3 + 11x^2 + 40x + 10$, find the marginal cost function and compare the marginal cost at $x = 100$ with the actual cost of manufacturing the 100th item.

5. Suppose the cost of manufacturing x items is $C(x) = x^3 - 30x^2 + 300x + 100$ dollars. Find the inflection point and discuss the significance of this value in terms of the cost of manufacturing.

6. A baseball team owner has determined that if tickets are priced at \$10, the average attendance at a game will be 27,000 and if tickets are priced at \$8, the average attendance will be 33,000. Using a linear model,we would then estimate that tickets priced at \$9 would produce an average attendance of 30,000. Discuss whether you think the use of a linear model here is reasonable. Then, using the linear model, determine the price at which the revenue is maximized.

In exercises 7–10, find the production level that minimizes the average cost.

7. $C(x) = 0.1x^2 + 3x + 2000$

8. $C(x) = 0.2x^3 + 4x + 4000$

9. $C(x) = 10e^{0.02x}$

10. $C(x) = \sqrt{x^3 + 800}$

11. Let $C(x)$ be the cost function and $\overline{C}(x)$ be the average cost function. Suppose that $C(x) = 0.01x^2 + 40x + 3600$. Show that $C'(100) < \overline{C}(100)$ and show that increasing the production (x) by 1 will decrease the average cost.

12. For the cost function in exercise 11, show that $C'(1000) > \overline{C}(1000)$ and show that increasing the production (x) by 1 will increase the average cost.

13. For the cost function in exercise 11, prove that average cost is minimized at the x-value where $C'(x) = \overline{C}(x)$.

14. If the cost function is linear, $C(x) = a + bx$ with a and b positive, show that there is no minimum average cost and that $C'(x) \neq \overline{C}(x)$ for all x.

15. Let $R(x)$ be the revenue and $C(x)$ be the cost from manufacturing x items. **Profit** is defined as $P(x) = R(x) - C(x)$. Show that at the value of x that maximizes profit, marginal revenue equals marginal cost.

16. Find the maximum profit if $R(x) = 10x - 0.001x^2$ dollars and $C(x) = 2x + 5000$ dollars.

In exercises 17–20, find (a) the elasticity of demand and (b) the range of prices for which the demand is elastic ($E < -1$).

17. $f(p) = 200(30 - p)$

18. $f(p) = 200(20 - p)$

19. $f(p) = 100p(20 - p)$

20. $f(p) = 60p(10 - p)$

21. Suppose that at price $p = 15$ dollars the demand for a product is elastic. If the price is raised, what will happen to revenue?

22. Suppose that at price $p = 10$ dollars the demand for a product is inelastic. If the price is raised, what will happen to the revenue?

23. If the demand function f is differentiable, prove that $[pf(p)]' < 0$ if and only if $\frac{p}{f(p)} f'(p) < -1$. (That is, revenue decreases if and only if demand is elastic.)

24. The term **income elasticity of demand** is defined as the percentage change in quantity purchased divided by the percentage change in real income. If I represents income and $Q(I)$ is demand as a function of income, derive a formula for the income elasticity of demand.

25. If the concentration of a chemical changes according to the equation $x'(t) = 2x(t)[4 - x(t)]$, find the concentration $x(t)$ for which the reaction rate is a maximum.

26. If the concentration of a chemical changes according to the equation $x'(t) = 0.5x(t)[5 - x(t)]$, find the concentration $x(t)$ for which the reaction rate is a maximum.

27. Show that in exercise 25, the limiting concentration is 4 as $t \to \infty$. Find the limiting concentration in exercise 26.

28. Find the equation for an autocatalytic reaction in which the maximum concentration is $x(t) = 16$ and the reaction rate equals 12 when $x(t) = 8$.

29. Mathematicians often study equations of the form $x'(t) = rx(t)[1 - x(t)]$, instead of the more complicated $x'(t) = cx(t)[K - x(t)]$, justifying the simplification with the statement that the second equation "reduces to" the first equation. Starting with $y'(t) = cy(t)[K - y(t)]$, substitute $y(t) = Kx(t)$ and show that the equation reduces to the form $x'(t) = rx(t)[1 - x(t)]$. How does the constant r relate to the constants c and K?

30. Suppose a chemical reaction follows the equation $x'(t) = cx(t)[K - x(t)]$. Suppose that at time $t = 4$ the concentration is $x(4) = 2$ and the reaction rate is $x'(4) = 3$. At time $t = 6$, suppose that the concentration is $x(6) = 4$ and the reaction rate is $x'(6) = 4$. Find the values of c and K for this chemical reaction.

31. In a general *second-order chemical reaction,* chemicals A and B (the **reactants**) combine to form chemical C (the **product**). If the initial concentrations of the reactants A and B are a and b, respectively, then the concentration $x(t)$ of the product satisfies the equation $x'(t) = [a - x(t)][b - x(t)]$. What is the rate of change of the product when $x(t) = a$? At this value, is the concentration of the product increasing, decreasing or staying the same? Assuming that $a < b$ and there is no product present when the reaction starts, explain why the maximum concentration of product is $x(t) = a$.

32. For the second-order reaction defined in exercise 31, find the (mathematical) value of $x(t)$ that minimizes the reaction rate. Show that the reaction rate for this value of $x(t)$ is negative. Explain why the concentration $x(t)$ would never get this large, so that this mathematical solution is not physically relevant. Explain why $x(t)$ must be between 0 and a and find the maximum and minimum reaction rates on this closed interval.

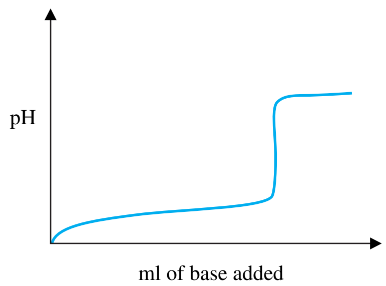
33. It can be shown that a solution of the equation $x'(t) = [a - x(t)][b - x(t)]$ is given by

$$x(t) = \frac{a[1 - e^{-(b-a)t}]}{1 - (a/b)e^{-(b-a)t}}.$$

Find $x(0)$, the initial concentration of chemical and $\lim_{t \to \infty} x(t)$, the limiting concentration of chemical (assume $a < b$). Graph $x(t)$ on the interval $[0, \infty)$ and describe in words how the concentration of chemical changes over time.

34. For the solution in exercise 33, find and graph $x'(t)$. Compute $\lim_{t \to \infty} x'(t)$ and describe in words how the reaction rate changes over time.

35. In example 9.5, you found the significance of one inflection point of a titration curve. A second inflection point, called the **equivalence point,** corresponds to $x = 1$. In the generalized titration curve shown on the following page, identify on the graph both inflection points and briefly explain why chemists prefer to measure the equivalence point and not the inflection point of example 9.5. (Note: The horizontal axis of a titration curve indicates the amount of base added to the mixture. This is directly proportional to the amount of converted acid in the region where $0 < x < 1$.)



36. In the titration of a weak acid and strong base, the pH is given by $c + \ln \frac{x}{1-x}$, where f is the fraction $(0 < x < 1)$ of converted acid. What happens to the rate of change of pH as x approaches 1?

37. The rate R of an enzymatic reaction is given by $R = \frac{rx}{k+x}$, where k is the Michaelis constant and x is the substrate concentration. Determine whether there is a maximum rate of the reaction.

38. The relationship among the pressure P, volume V and temperature T of a gas or liquid is given by van der Waals' equation $\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$, for positive constants n, a, b and R. For constant temperatures, find and interpret $\frac{dV}{dP}$.

39. In an adiabatic chemical process, there is no net change in heat, so pressure and volume are related by an equation of the form $PV^{1.4} = c$, for some positive constant c. Find and interpret $\frac{dV}{dP}$.

40. If the equation in exercise 39 holds and atmospheric pressure decreases as altitude increases, what will happen to a rising balloon?

In exercises 41–44, the mass of the first x meters of a thin rod is given by the function $m(x)$ on the indicated interval. Find the linear density function for the rod. Based on what you find, briefly describe the composition of the rod.

41. $m(x) = 4x - \sin x$ grams for $0 \le x \le 6$

42. $m(x) = (x-1)^3 + 6x$ grams for $0 \le x \le 2$

43. $m(x) = 4x$ grams for $0 \le x \le 2$

44. $m(x) = 4x^2$ grams for $0 \le x \le 2$

45. Suppose that the charge in an electrical circuit is $Q(t) = e^{-2t}(\cos 3t - 2\sin 3t)$ coulombs. Find the current.

46. Suppose that the charge in an electrical circuit is $Q(t) = e^{t}(3\cos 2t + \sin 2t)$ coulombs. Find the current.

47. Suppose that the charge at a particular location in an electrical circuit is $Q(t) = e^{-3t}\cos 2t + 4\sin 3t$ coulombs. What happens to this function as $t \to \infty$? Explain why the term $e^{-3t}\cos 2t$ is called a **transient** term and $4\sin 3t$ is known as the **steady-state** or **asymptotic** value of the charge function. Find the transient and steady-state values of the current function.

48. As in exercise 47, find the steady-state and transient values of the current function if the charge function is given by $Q(t) = e^{-2t}(\cos t - 2\sin t) + te^{-3t} + 2\cos 4t$.

49. Suppose that a population grows according to the logistic equation $p'(t) = 4p(t)[5 - p(t)]$. Find the population at which the population growth rate is a maximum.

50. Suppose that a population grows according to the logistic equation $p'(t) = 2p(t)[7 - 2p(t)]$. Find the population at which the population growth rate is a maximum.

51. It can be shown that solutions of the logistic equation have the form $p(t) = \frac{B}{1 + Ae^{-kt}}$, for constants B, A and k. Find the rate of change of the population and find the limiting population, that is, $\lim_{t\to\infty} p(t)$.

52. In exercise 51, suppose you are studying the growth of a population and your data indicate an inflection point at $p = 120$. Use this value to determine the constant B. In your study, the initial population is $p(0) = 40$. Use this value to determine the constant A. If your current measurement is $p(12) = 160$, use this value to determine the constant k.

53. The function $f(t) = a/(1 + 3e^{-bt})$ has been used to model the spread of a rumor. Suppose that $a = 70$ and $b = 0.2$. Compute $f(2)$, the percentage of the population that has heard the rumor after 2 hours. Compute $f'(2)$ and describe what it represents. Compute $\lim_{t\to\infty} f(t)$ and describe what it represents.

54. After an injection, the concentration of drug in a muscle is given by a function of time, $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Determine the time when the maximum concentration of drug occurs.

55. Suppose that the size of the pupil of an animal is given by $f(x)$ (mm), where x is the intensity of the light on the pupil. If

$$f(x) = \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15},$$

show that $f(x)$ is a decreasing function. Interpret this result in terms of the response of the pupil to light.

56. Suppose that the body temperature 1 hour after receiving x mg of a drug is given by $T(x) = 102 - \frac{1}{6}x^2(1 - x/9)$ for $0 \le x \le 6$. The absolute value of the derivative, $|T'(x)|$, is defined as the **sensitivity** of the body to the drug dosage. Find the dosage that maximizes sensitivity.

57. Referring to exercise 15, explain why a value of x for which marginal revenue equals marginal cost does not necessarily maximize profit.

58. Referring to exercise 15, explain why the conditions $R'(x_0) = C'(x_0)$ and $R''(x_0) < C''(x_0)$ will guarantee that profit is maximized at x_0.

59. A fish swims at velocity v upstream from point A to point B, against a current of speed c. Explain why we must have $v > c$. The energy consumed by the fish is given by $E = \frac{kv^2}{v - c}$, for some constant $k > 1$. Show that E has one critical number. Does it represent a maximum or a minimum?

60. The power required for a bird to fly at speed v is proportional to $P = \frac{1}{v} + cv^3$, for some positive constant c. Find v to minimize the power.

EXPLORATORY EXERCISES

1. Epidemiology is the study of the spread of infectious diseases. A simple model for the spread of fatal diseases such as AIDS divides people into the categories of susceptible (but not exposed), exposed (but not infected) and infected. The proportions of people in each category at time t are denoted $S(t)$, $E(t)$ and $I(t)$, respectively. The general equations for this model are

$$S'(t) = mI(t) - bS(t)I(t),$$
$$E'(t) = bS(t)I(t) - aE(t),$$
$$I'(t) = aE(t) - mI(t),$$

where m, b and a are positive constants. Notice that each equation gives the rate of change of one of the categories. Each rate of change has both a positive and negative term. Explain why the positive term represents people who are entering the category and the negative term represents people who are leaving the category. In the first equation, the term $mI(t)$ represents people who have died from the disease (the constant m is the reciprocal of the life expectancy of someone with the disease). This term is slightly artificial: the assumption is that the population is constant, so that when one person dies, a baby is born who is not exposed or infected. The dynamics of the disease are such that susceptible (healthy) people get infected by contact with infected people. Explain why the number of contacts between susceptible people and infected people is proportional to $S(t)$ and $I(t)$. The term $bS(t)I(t)$, then, represents susceptible people who have been exposed by contact with infected people. Explain why this same term shows up as a positive in the second equation. Explain the rest of the remaining two equations in this fashion. (Hint: The constant a represents the reciprocal of the average latency period. In the case of AIDS, this would be how long it takes an HIV-positive person to actually develop AIDS.)

2. Without knowing how to solve differential equations (we hope you will go far enough in your study of mathematics to learn to do so!), we can nonetheless deduce some important properties of the solutions of differential equations. For example, consider the equation for an autocatalytic reaction $x'(t) = x(t)[1 - x(t)]$. Suppose $x(0)$ lies between 0 and 1. Show that $x'(0)$ is positive, by determining the possible values of $x(0)[1 - x(0)]$. Explain why this indicates that the value of $x(t)$ will increase from $x(0)$ and will continue to increase as long as $0 < x(t) < 1$. Explain why if $x(0) < 1$ and $x(t) > 1$ for some $t > 0$, then it must be true that $x(t) = 1$ for some $t > 0$. However, if $x(t) = 1$, then $x'(t) = 0$ and the solution $x(t)$ stays constant (equal to 1). Therefore, we can conjecture that $\lim_{t\to\infty} x(t) = 1$. Similarly, show that if $x(0) > 1$, then $x(t)$ decreases and we could again conjecture that $\lim_{t\to\infty} x(t) = 1$. Changing equations, suppose that $x'(t) = -0.05x(t) + 2$. This is a model of an experiment in which a radioactive substance is decaying at the rate of 5% but the substance is being replenished at the constant rate of 2. Find the value of $x(t)$ for which $x'(t) = 0$. Pick various starting values of $x(0)$ less than and greater than the constant solution and determine whether the solution $x(t)$ will increase or decrease. Based on these conclusions, conjecture the value of $\lim_{t\to\infty} x(t)$, the limiting amount of radioactive substance in the experiment.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Linear approximation	Newton's method	Critical number
Absolute extremum	Local extremum	First Derivative Test
Inflection points	Concavity	Second Derivative Test
Marginal cost	Current	Related rates
l'Hôpital's Rule	Extreme Value Theorem	Fermat's Theorem

Review Exercises

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. Linear approximations give good approximations of function values for x's close to the point of tangency.
2. The closer the initial guess is to the solution, the faster Newton's method converges.
3. L'Hôpital's Rule states that the limit of the derivative equals the limit of the function.
4. If there is a maximum of $f(x)$ at $x = a$, then $f'(a) = 0$.
5. An absolute extremum must occur at either a critical number or an endpoint.
6. If $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$, then $f(a)$ is a local maximum.
7. If $f''(a) = 0$, then $y = f(x)$ has an inflection point at $x = a$.
8. If there is a vertical asymptote at $x = a$, then either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = -\infty$.
9. In a maximization problem, if f has only one critical number, then it is the maximum.
10. If the population $p(t)$ has a maximum growth rate at $t = a$, then $p''(a) = 0$.
11. If $f'(a) = 2$ and $g'(a) = 4$, then $\dfrac{dg}{df} = 2$ and g is increasing twice as fast as f.

In exercises 1 and 2, find the linear approximation to $f(x)$ at x_0.

1. $f(x) = e^{3x}$, $x_0 = 0$
2. $f(x) = \sqrt{x^2+3}$, $x_0 = 1$

In exercises 3 and 4, use a linear approximation to estimate the quantity.

3. $\sqrt[3]{7.96}$
4. $\sin 3$

In exercises 5 and 6, use Newton's method to find an approximate root.

5. $x^3 + 5x - 1 = 0$
6. $x^3 = e^{-x}$

7. Explain why, in general, if $y = f(x)$ has an inflection point at $x = a$ and does not have an inflection point at $x = b$, then the linear approximation of $f(x)$ at $x = a$ will tend to be more accurate for a larger set of x's than the linear approximation of $f(x)$ at $x = b$.

8. Show that the approximation $\dfrac{1}{(1-x)} \approx 1 + x$ is valid for "small" x.

In exercises 9–16, find the limit.

9. $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$
10. $\lim_{x\to 0} \dfrac{\sin x}{x^2+3x}$
11. $\lim_{x\to\infty} \dfrac{e^{2x}}{x^4+2}$
12. $\lim_{x\to\infty} (x^2 e^{-3x})$
13. $\lim_{x\to 2^+} \left|\dfrac{x+1}{x-2}\right|^{\sqrt{x^2-4}}$
14. $\lim_{x\to\infty} x\ln(1+1/x)$
15. $\lim_{x\to 0^+} (\tan x \ln x)$
16. $\lim_{x\to 0} \dfrac{\tan^{-1} x}{\sin^{-1} x}$

In exercises 17–26, do the following by hand. (a) Find all critical numbers, (b) identify all intervals of increase and decrease, (c) determine whether each critical number represents a local maximum, local minimum or neither, (d) determine all intervals of concavity and (e) find all inflection points.

17. $f(x) = x^3 + 3x^2 - 9x$
18. $f(x) = x^4 - 4x + 1$
19. $f(x) = x^4 - 4x^3 + 2$
20. $f(x) = x^3 - 3x^2 - 24x$
21. $f(x) = xe^{-4x}$
22. $f(x) = x^2 \ln x$
23. $f(x) = \dfrac{x-90}{x^2}$
24. $f(x) = (x^2-1)^{2/3}$
25. $f(x) = \dfrac{x}{x^2+4}$
26. $f(x) = \dfrac{x}{\sqrt{x^2+2}}$

In exercises 27–30, find the absolute extrema of the given function on the indicated interval.

27. $f(x) = x^3 + 3x^2 - 9x$ on $[0, 4]$
28. $f(x) = \sqrt{x^3 - 3x^2 + 2x}$ on $[-1, 3]$
29. $f(x) = x^{4/5}$ on $[-2, 3]$
30. $f(x) = x^2 e^{-x}$ on $[-1, 4]$

In exercises 31–34, find the x-coordinates of all local extrema.

31. $f(x) = x^3 + 4x^2 + 2x$
32. $f(x) = x^4 - 3x^2 + 2x$
33. $f(x) = x^5 - 2x^2 + x$
34. $f(x) = x^5 + 4x^2 - 4x$

Review Exercises

35. Sketch a graph of a function with $f(-1) = 2$, $f(1) = -2$, $f'(x) < 0$ for $-2 < x < 2$ and $f'(x) > 0$ for $x < -2$ and $x > 2$.

36. Sketch a graph of a function with $f'(x) > 0$ for $x \neq 0$, $f'(0)$ undefined, $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$.

In exercises 37–46, sketch a graph of the function and completely discuss the graph.

37. $f(x) = x^4 + 4x^3$

38. $f(x) = x^4 + 4x^2$

39. $f(x) = x^4 + 4x$

40. $f(x) = x^4 - 4x^2$

41. $f(x) = \dfrac{x}{x^2 + 1}$

42. $f(x) = \dfrac{x}{x^2 - 1}$

43. $f(x) = \dfrac{x^2}{x^2 + 1}$

44. $f(x) = \dfrac{x^2}{x^2 - 1}$

45. $f(x) = \dfrac{x^3}{x^2 - 1}$

46. $f(x) = \dfrac{4}{x^2 - 1}$

47. Find the point on the graph of $y = 2x^2$ that is closest to $(2, 1)$.

48. Show that the line through the two points of exercise 47 is perpendicular to the tangent line to $y = 2x^2$ at $(2, 1)$.

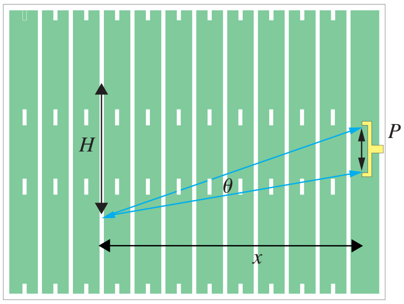
49. A city is building a highway from point A to point B, which is 4 miles east and 6 miles south of point A. The first 4 miles south of point A is swampland, where the cost of building the highway is \$6 million per mile. On dry land, the cost is \$2 million per mile. Find the point on the boundary of swampland and dry land to which the highway should be built to minimize the total cost.

50. If a muscle contracts at speed v, the force produced by the muscle is proportional to $e^{-v/2}$. Show that the greater the speed of contraction, the less force produced. However, the power produced by the contracting muscle is proportional to $ve^{-v/2}$. Determine the speed that maximizes the power.

51. A soda can in the shape of a cylinder is to hold 16 fluid ounces. Find the dimensions of the can that minimize the surface area of the can.

52. Suppose that $C(x) = 0.02x^2 + 4x + 1200$ is the cost of manufacturing x items. Show that $C'(x) > 0$ and explain in business terms why this has to be true. Show that $C''(x) > 0$ and explain why this indicates that the manufacturing process is not very efficient.

53. The diagram shows a football field with hash marks H feet apart and goalposts P feet apart. If a field goal is to be tried from a (horizontal) distance of x feet from the goalposts, the angle θ gives the margin of error for that direction. Find x to maximize θ.



54. In the situation of exercise 53, sports announcers often say that for a short field goal $(50 \leq x \leq 60)$, a team can improve the angle by backing up 5 yards with a penalty. Determine whether this is true for high school ($H = 53\frac{1}{3}$ and $P = 23\frac{1}{3}$), college ($H = 40$ and $P = 18\frac{1}{2}$) or pros ($H = 18\frac{1}{2}$ and $P = 18\frac{1}{2}$).

55. The charge in an electrical circuit at time t is given by $Q(t) = e^{-3t} \sin 2t$ coulombs. Find the current.

56. If the concentration $x(t)$ of a chemical in a reaction changes according to the equation $x'(t) = 0.3x(t)[4 - x(t)]$, find the concentration at which the reaction rate is a maximum.

57. Suppose that the mass of the first x meters of a thin rod is given by $m(x) = 20 + x^2$ for $0 \leq x \leq 4$. Find the density of the rod and briefly describe the composition of the rod.

58. A person scores $f(t) = 90/(1 + 4e^{-0.4t})$ points on a test after t hours of studying. What does the person score without studying at all? Compute $f'(0)$ and estimate how many points 1 hour of studying will add to the score.

59. The cost of manufacturing x items is given by $C(x) = 0.02x^2 + 20x + 1800$. Find the marginal cost function. Compare the marginal cost at $x = 20$ to the actual cost of producing the 20th item.

60. For the cost function in exercise 59, find the value of x that minimizes the average cost $\overline{C}(x) = C(x)/x$.

EXPLORATORY EXERCISES

1. Let $n(t)$ be the number of photons in a laser field. One model of the laser action is $n'(t) = an(t) - b[n(t)]^2$, where a and b are positive constants. If $n(0) = a/b$, what is $n'(0)$? Based on this calculation, would $n(t)$ increase, decrease or neither? If $n(0) > a/b$, is $n'(0)$ positive or negative? Based on this calculation, would $n(t)$ increase, decrease or neither? If $n(0) < a/b$, is $n'(0)$ positive or negative? Based on this calculation, would $n(t)$ increase, decrease or neither? Putting this

Review Exercises

information together, conjecture the limit of $n(t)$ as $t \to \infty$. Repeat this analysis under the assumption that $a < 0$. [Hint: Because of its definition, $n(t)$ is positive, so ignore any negative values of $n(t)$.]

2. One way of numerically approximating a derivative is by computing the slope of a secant line. For example, $f'(a) \approx \dfrac{f(b) - f(a)}{b - a}$, if b is close enough to a. In this exercise, we will develop an analogous approximation to the second derivative. Graphically, we can think of the secant line as an approximation of the tangent line. Similarly, we can match the second derivative behavior (concavity) with a parabola. Instead of finding the secant line through two points on the curve, we find the parabola through three points on the curve. The second derivative of this approximating parabola will serve as an approximation of the second derivative of the curve. The first step is messy, so we recommend using a CAS if one is available. Find a function of the form $g(x) = ax^2 + bx + c$ such that $g(x_1) = y_1$, $g(x_2) = y_2$ and $g(x_3) = y_3$. Since $g''(x) = 2a$, you actually only need to find the constant a. The so-called **second difference** approximation to $f''(x)$ is the value of $g''(x) = 2a$ using the three points $x_1 = x - \Delta x$ [$y_1 = f(x_1)$], $x_2 = x$ [$y_2 = f(x_2)$] and $x_3 = x + \Delta x$ [$y_3 = f(x_3)$]. Find the second difference for $f(x) = \sqrt{x + 4}$ at $x = 0$ with $\Delta x = 0.5$, $\Delta x = 0.1$ and $\Delta x = 0.01$. Compare to the exact value of the second derivative, $f''(0)$.

3. The technique of **Picard iteration** is very effective for estimating solutions of complicated equations. For equations of the form $f(x) = 0$, start with an initial guess x_0. For $g(x) = f(x) + x$, compute the iterates $x_1 = g(x_0)$, $x_2 = g(x_1)$ and so on. Show that this makes it so that if the iterates repeat (i.e., $g(x_{n+1}) = g(x_n)$) at x_n, then $f(x_n) = 0$. Compute iterates starting at $x_0 = -1$ for (a) $f(x) = x^3 - x^2 + 3$, (b) $f(x) = -x^3 + x^2 - 3$ and (c) $f(x) = -\dfrac{x^3}{11} + \dfrac{x^2}{11} - \dfrac{3}{11}$. To see what is going on, suppose that $f(x_c) = 0$, $x_0 < x_c$ and $f(x_0) < 0$. Show that x_1 is farther from the solution x_c than is x_0. Continue in this fashion and show that Picard iteration does not converge to x_c if $f'(x_c) > 0$ [this explains the failure in part (a)]. Investigate the effect of $f'(x_c)$ on the behavior of Picard iteration and explain why the function in part (c) is better than the function in part (b).

4. For the hyperbolic tangent function $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, show that $(\tanh x)' > 0$. Conclude that $\tanh(x)$ has an inverse function and find the derivative of the inverse function.