Folding Quartic Roots

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1. Introduction

The geometric tools of antiquity—straightedge and compass—solve linear and quadratic equations but fail at cubics, to the persistent consternation of angle trisectors. Recent articles show how other geometric systems—origami [**6**] or Mira* [**4**]—go beyond straightedge and compass to solve cubic equations as well; the key is to construct a common tangent to two parabolas. Solving cubics and even equations of higher degree by parabolas or other conics is an old idea ([**7, 1, 3**], and, more recently, [**5**]), but origami and Mira methods share the novel feature of constructing common tangents to parabolas from the parabolas' foci and directrices without needing the parabolas themselves. Thus these methods really use only points and lines.

Since algebra reduces the quartic equation to equations of lower degree, we know in principle how to solve the quartic by tangents to parabolas as well, but translating the algebraic reduction into an origami- or Mira-based procedure is geometrically unmotivated and unappealingly tortuous. Instead, this note shows how to solve certain quartics (to be detailed in a moment) by the common tangents to a parabola and a circle. Granting field operations and two square roots, the actual construction is a geometric hybrid, requiring one compass operation (i.e., a circle) and one origami-type fold. Our tools operate in the Euclidean plane, so of course we are considering real quartics and are constructing real roots.

The path from common tangents to polynomial roots is as follows: Every conic has a so-called dual conic whose points represent the tangents to the original, so a common tangent to two conics leads naturally to a common point of their duals. Having this common point amounts to solving simultaneous quadratic equations; choosing the conics judiciously and substituting one equation into the other shows that in fact we have a solution to a cubic or quartic equation.

Specifically, we will solve a reduced quartic (one with no x^3 term),

$$x^4 + bx^2 + 2cx + d = 0,$$

obtained from the general quartic by translating the variable. We have to impose the conditions

$$c^2 - bd < 0, \quad d < 0;$$

i.e., the quadratic part $bx^2 + 2cx + d$ is negative for all x. While these constraints are unnecessarily restrictive for a real root, they certainly guarantee one and they arise naturally from our construction.

This work stems from the first author's undergraduate thesis under the direction of the second author.

*Mira is a registered trademark of the Mira-Math Company, Willowdale, Ontario, Canada. It refers to a transparent and reflective piece of plastic that allows constructions based on being able to see both a figure and its reflection.

2. Origami folds

In [6], R. Geretschläger lays out a set of axioms for origami-generated geometry. He observes that folding a given point F onto any point Q of a line $\mathcal{D}$ constructs a tangent to the parabola $\mathcal{P}$ having F and $\mathcal{D}$ for its focus and directrix. See FIGURE 1.

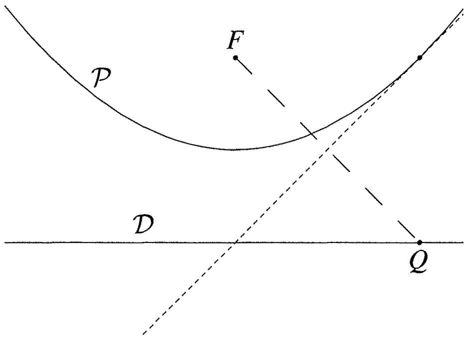

Figure 1 Folding a tangent to a parabola.

Suppose that two parabolas $\mathcal{P}_1$, $\mathcal{P}_2$, with foci F_1, F_2 and directrices $\mathcal{D}_1$, $\mathcal{D}_2$, have common tangents. A continuum of foldings takes F_1 to points of $\mathcal{D}_1$, constructing all tangents to $\mathcal{P}_1$. In particular, sliding F_1 along $\mathcal{D}_1$ until F_2 also lies on $\mathcal{D}_2$ constructs the common tangents to $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus we have the axiom that the common tangents to two parabolas, each specified by its focus and directrix, are constructible by origami when they exist. Of course, they needn't exist at all—consider the case when one parabola lies entirely within the convex hull of the other.

In a similar vein, this paper uses the common tangents to a parabola and a circle. Suppose that a parabola $\mathcal{P}$, with focus F and directrix $\mathcal{D}$, and a circle $\mathcal{C}$, with center O and radius r, have a common tangent. Use a compass to plot the circle $2\mathcal{C}$ with center O and radius $2r$. Then sliding F along $\mathcal{D}$ until O lies on $2\mathcal{C}$ gives all folds tangent to $\mathcal{P}$ and $\mathcal{C}$.

Introducing a compass construction into origami geometry is allowable since origami subsumes ruler and compass [4, 6]. Using the compass may be aesthetically unappealing, but it will let us give a palatable folding procedure for solving the quartic equation.

3. Results from projective space

Working with conics is easy in projective space, where the computations are handled nicely by matrix algebra. (See [8] for a lovely introduction to projective geometry.) Very briefly, homogenizing polynomial equations with an extra variable symmetrizes calculations and corresponds to completing space out at infinity, giving problems the right number of solutions. Specifically, the quadratic equation of a conic in the plane homogenizes to projective form

$$\mathcal{C} : a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2 = 0\,,$$

or, using matrices,

$$\mathcal{C} : v^t M v = 0 \qquad \text{where} \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Note that the symmetric matrix M is defined only up to nonzero scalar multiple. The conic $\mathcal{C}$ is nondegenerate (not the union of two lines) when $\det M \neq 0$ [**8**, Section 1.7], and these are the only conics we consider.

The dual conic of $\mathcal{C}$ is described by the inverse matrix:

$$\widehat{\mathcal{C}} : v^t M^{-1} v = 0.$$

Multiplying by M gives a bijection from $\mathcal{C}$ to $\widehat{\mathcal{C}}$, for if $q = Mp$ with $p \in \mathcal{C}$ then substituting $p = M^{-1}q$ into $p^t M p = 0$ gives $q^t M^{-1} q = 0$.

We are interested in the dual $\widehat{\mathcal{C}}$ because its points describe the tangents to the original conic $\mathcal{C}$. To see this, compute that the partial derivative of $v^t M v$ with respect to the ith variable is $2v^t m_i$ where m_i is the ith column of M; thus the gradient of $\mathcal{C}$ at a point p is $\nabla_p(v^t M v) = 2p^t M$ (as a row vector), and the tangent line to $\mathcal{C}$ at p, being orthogonal to the gradient, is $\mathcal{L} : p^t M v = 0$. Again letting $q = Mp \in \widehat{\mathcal{C}}$, we see that the tangent is then

$$\mathcal{L}_q : q^t \cdot v = 0.$$

Thus the tangent to $\mathcal{C}$ at p is described by the point $q \in \widehat{\mathcal{C}}$, as desired. Decompressing the notation, we see that the tangent line's equation $Ax + By + Cz = 0$ comes directly from the coordinates of $q = [A \quad B \quad C]^t$.

It follows that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are conics then finding their common tangents is equivalent to finding the points where their duals meet. By Bezout's theorem (see, e.g., [**2**]), since the duals have quadratic equations there are four such points in the complex projective plane, counting multiplicity. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ have real matrices, so do their duals and these points occur in complex conjugate pairs, meaning that an even number of them are real. In sum, $\mathcal{C}_1$ and $\mathcal{C}_2$ have zero, two, or four real common tangents, counting multiplicity.

4. Solving the cubic

Geretschläger solves the cubic equation

$$x^3 + bx^2 + cx + d = 0$$

using (essentially—his variables are $-1/2$ times ours) the parabolas

$$\mathcal{P}_1 : (y + c)^2 = -4d(x - b) \quad \text{and} \quad \mathcal{P}_2 : x^2 = -4y.$$

We reproduce his computation but work projectively to foreshadow our methods for the quartic.

The parabolas' matrices are

$$M_1 = \begin{bmatrix} 0 & 0 & 2d \\ 0 & 1 & c \\ 2d & c & c^2 - 4bd \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

with inverses

$$M_1^{-1} = \frac{1}{d} \begin{bmatrix} b & -c/2 & 1/2 \\ -c/2 & d & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}.$$

Inspecting the inverses shows that the duals meet at points $[A \quad B \quad C]^t$ where

$$bA^2 - cAB + AC + dB^2 = 0, \qquad -BC = A^2.$$

For nonvertical affine common tangents of the original pair of parabolas, we normalize to $B = -1$, $z = 1$, de-homogenizing the tangent $Ax + By + Cz = 0$ to $y = Ax + C$. This gives $C = A^2$ in the second relation, and the first becomes

$$A^3 + bA^2 + cA + d = 0.$$

Thus the slopes of common tangents to two parabolas solve the general cubic equation. FIGURE 2 shows this method applied to the cubic equation $x^3 - 2x^2 - x + 2 = 0$, with roots 2, 1, and -1.

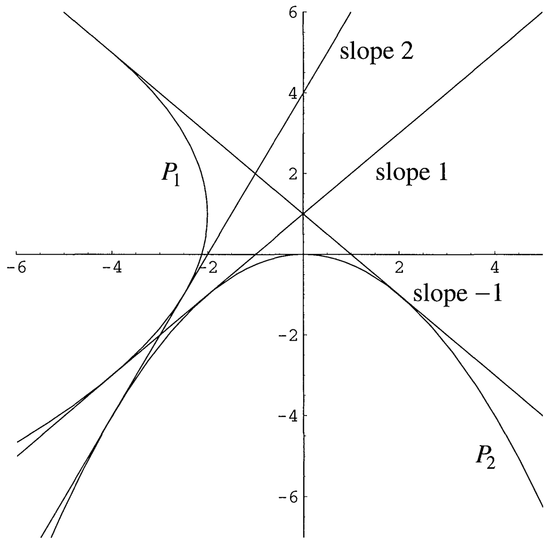

Figure 2 Solving the cubic by slopes of common tangents.

5. Solving the quartic

No construction of common tangents to two parabolas can solve the quartic, even in reduced form

$$x^4 + bx^2 + 2cx + d = 0,$$

because any two parabolas share a projective tangent out at infinity, not leaving enough affine tangents for the four possible roots.

We *can* solve the reduced quartic (assuming its quadratic part $bx^2 + 2cx + d$ has nonzero discriminant $bd - c^2$) by replacing the first parabola in the cubic method with a nonparabolic conic. Set

$$M_1 = \begin{bmatrix} d & c & 0 \\ c & b & 0 \\ 0 & 0 & bd - c^2 \end{bmatrix} \quad \left(\text{so } M_1^{-1} = \frac{1}{bd - c^2} \begin{bmatrix} b & -c & 0 \\ -c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$

and keep M_2 from the cubic case. Then, by the same algebraic method, the slopes A of the common tangents to the conic

$$\mathcal{C} : dx^2 + 2cxy + by^2 + bd - c^2 = 0$$

and the parabola $\mathcal{P}_2 : x^2 = -4y$ solve the reduced quartic equation. That is,

$$A^4 + bA^2 + 2cA + d = 0, \quad \text{assuming} \quad bd - c^2 \neq 0.$$

Unfortunately, geometric methods for constructing the common tangent in question aren't immediately clear (at least to the authors), since the conic $\mathcal{C}$ is not a parabola.

As a second approach—less elegant algebraically but more realizable geometrically—we consider the common tangents to a circle at the origin, constructible by compass once we specify its radius, and to a variable parabola.

Thus, let b, c, d be given with $c^2 - bd < 0$ and $d < 0$. Set

$$e = \pm\frac{\sqrt{bd - c^2}}{d} \text{ (either value)}, \qquad r = |e|\sqrt{-d},$$

so that $-de^2/r^2 = 1$ and $de^2 = b - c^2/d$. Consider the dualized conics $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{P}}$ with matrices

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/r^2 \end{bmatrix} \quad \text{and} \quad M_2^{-1} = \begin{bmatrix} 0 & 0 & de/2 \\ 0 & -d & c/2 \\ de/2 & c/2 & 0 \end{bmatrix}.$$

Computing the matrices M_1 and M_2 shows that the original conics $\mathcal{C}$ and $\mathcal{P}$ are respectively a circle of radius r and a parabola in affine space:

$$\mathcal{C} : x^2 + y^2 = r^2 \quad \text{and} \quad \mathcal{P} : c^2x^2 - 2cdexy + d^2e^2y^2 - 4d^2ex = 0.$$

The duals $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{P}}$ intersect at points $[A \quad B \quad C]^t$ where

$$A^2 + B^2 = (1/r^2)C^2, \qquad deAC = dB^2 - cBC.$$

Again this describes a common tangent line $Ax + By + Cz = 0$ to the original circle and parabola, and setting $B = -1$, $z = 1$ gives the affine form $y = Ax + C$ with y-intercept C,

$$A^2 + 1 = (1/r^2)C^2, \qquad deAC = cC + d.$$

Multiply the first equation by de^2C^2 and square the second to get

$$de^2A^2C^2 + de^2C^2 = (de^2/r^2)C^4, \qquad de^2A^2C^2 = (c^2/d)C^2 + 2cC + d.$$

Substitute the second equation into the first to obtain

$$(-de^2/r^2)C^4 + (de^2 + c^2/d)C^2 + 2cC + d = 0,$$

and recall that $-de^2/r^2 = 1$ and $de^2 = b - c^2/d$, so that

$$C^4 + bC^2 + 2cC + d = 0.$$

Thus the y-intercept solves the reduced quartic when its quadratic part is negative.

This method is really no different algebraically from that used to solve the cubic equation, despite finding roots as intercepts rather than slopes. Projectively the intercepts and slopes are both coefficients; a slope in one affine piece of the projective plane is an intercept in another.

6. The geometric procedure

Translating the preceding algebra into a geometric construction requires locating the focus and directrix of the parabola $\mathcal{P}$. Its equation,

$$\mathcal{P} : c^2x^2 - 2cdexy + d^2e^2y^2 - 4d^2ex = 0,$$

becomes, after a change of variables,

$$\mathcal{P} : z^2 = 4d^3e^2w$$

where

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} c/(bd) & de & c^2e/b^2 \\ -e/b & c & -(bc+cde^2)/b^2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ w \\ 1 \end{bmatrix}.$$

In (z, w)-coordinates, the parabola $\mathcal{P}$ has focus $F = (0, d^3e^2)$ and directrix $\mathcal{D} : w = -d^3e^2$. The (z, w)-origin has (x, y)-coordinates

$$O' = \left(\frac{c^2e}{b^2}, -\frac{bc+cde^2}{b^2}\right),$$

and the focus and directrix in (x, y)-coordinates are

$$F' = O' + d^3e^2(de, c) \quad \text{and} \quad \mathcal{D}' : (x, y) = O' - d^3e^2(de, c) + t(c, -de), \quad t \in \mathbf{R}.$$

With the calculations complete, we have the procedure for solving the reduced quartics whose coefficients satisfy the conditions $c^2 - bd < 0$, $d < 0$. Given b, c, d, compute e and r and use a compass to draw the circle $2\mathcal{C}$ at the origin of radius $2r$. Plot the focus F' and the directrix $\mathcal{D}'$. Fold the paper in a fashion that takes the origin onto the circle and the focus onto the directrix. All such folds are common tangents to $\mathcal{C}$ and $\mathcal{P}$, so their y-intercepts are roots.

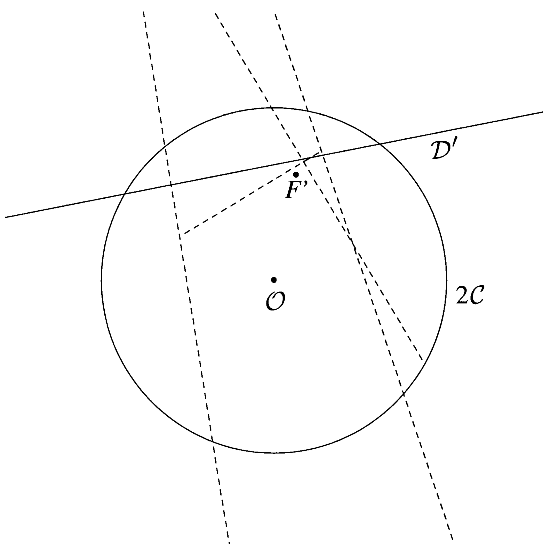

Figure 3 The four origami folds taking $\mathcal{O}$ to $2\mathcal{C}$ and F' to $\mathcal{D}'$.

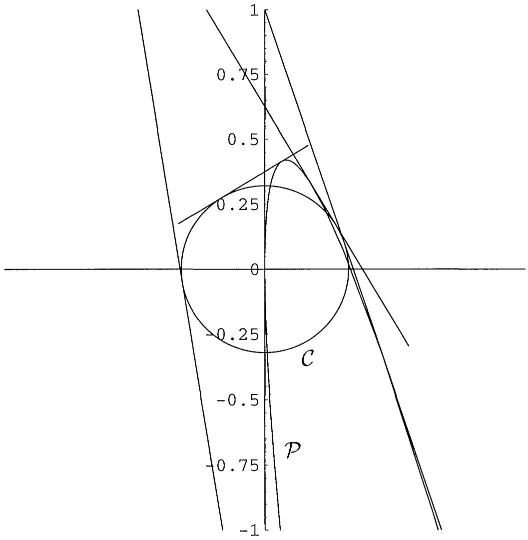

Figure 4 Solving the quartic by y-intercepts of common tangents.

7. An example

Consider the quartic polynomial

$$q(x) = x^4 - \frac{177}{64}x^2 + \frac{143}{64}x - \frac{15}{32} = \left(x - \frac{3}{8}\right)\left(x - \frac{5}{8}\right)(x-1)(x+2).$$

The origin $\mathcal{O}$, the circle $2\mathcal{C}$, the focus F', and the directrix $\mathcal{D}'$ are plotted in FIGURE 3, along with the four folds taking the origin to the circle and the focus to the directrix.

In FIGURE 4, these folds are seen as the common tangents of the circle $\mathcal{C}$ and the parabola $\mathcal{P}$. They visibly intersect the y-axis at three of the roots, $3/8$, $5/8$, 1; the fourth root -2 is out of the plot range.

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