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Scott G. Smith

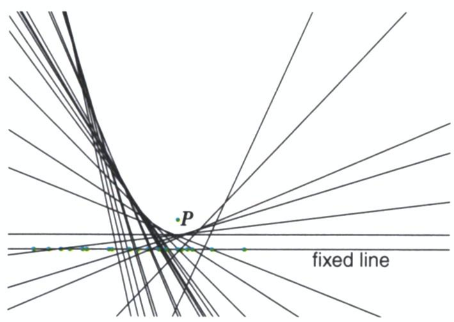
Paper Folding and Conic Sections

One sign of a good problem is that it offers multiple revelations during its investigation. Another is that it can be approached mathematically in more than one way. Three related problems that meet both those criteria involve paper folding and conic sections. Each problem can be demonstrated easily with a sheet of wax paper or emulated by a geometry drawing program like The Geometer's Sketchpad, yet each contains interesting mathematics whose properties are established in nontrivial ways.

The three problems are simple to describe:

1. Draw a fixed line and a fixed point. Fold the fixed point onto a random point on the fixed line, and crease. Repeat many times. The creases form tangents to, and envelop, a *parabola,* as shown in **figure 1.**
2. Draw a circle and a fixed point inside the circle. Fold the fixed point onto a random point of the circle, and crease. Repeat many times. The creases are all tangent to and bound an *ellipse,* as displayed in **figure 2.**

A good problem offers multiple revelations during its investigation



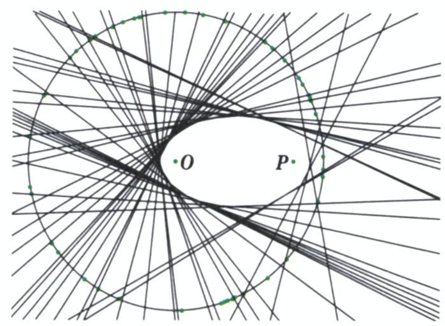
(a)
Parabola enveloped by 44 crease lines

Given:
Straight Object j
Point P

Steps:
1. Let [D] = Random point on Straight Object j.
2. Let [k] = Segment between Point [D] and Point P.
3. Let [E] = Midpoint of Segment [k].
4. Let [l] = Perpendicular to Segment [k] through Midpoint [E].
5. Recurse on j and P.
6. Select j and P.

(b)
The Geometer's Sketchpad script for **figure 1a**

Fig. 1



(a)
Ellipse bounded by 176 crease lines

Given:
Circle c1
Point C

Steps:
1. Let [D] = Random point on Circle c1.
2. Let [j] = Segment between Point [D] and Point C.
3. Let [E] = Midpoint of Segment [j].
4. Let [k] = Perpendicular to Segment [j] through Midpoint [E].
5. Recurse on c1 and C.
6. Select c1 and C.

(b)
The Geometer's Sketchpad script for **figure 2a**

Fig. 2

3. As in the second example, draw a circle and a fixed point, but this time locate the point outside the circle. Fold the fixed point onto a random point on the circle, and crease. When this fold is repeated many times, the lines will all be tangent to a *hyperbola,* as **figure 3** shows.

Although paper folding is a good hands-on activity—particularly if the teacher draws the point, line, or circle with a marker on wax paper ahead of time

Scott Smith, ssmith@us.edu, enjoys mathematical interludes from his duties as technology coordinator and Klein chairholder in computer science at University School's Upper Campus in Hunting Valley, OH 44022.

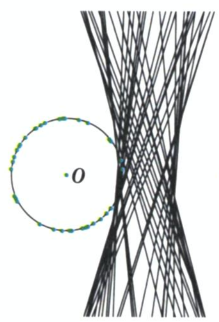


Fig. 3
Hyperbola bounded by 176 crease lines

for the students, it can be emulated by a graphing program like The Geometer's Sketchpad. These sketches are facilitated greatly by the recursive ability (LOOP) of a Sketchpad script. The scripts to produce **figures 1a** and **2a** are given as **figures 1b** and **2b,** respectively. After producing the sketches, we move point P around to see how its position, that is, the focal distance, affects the shape of the particular conic. In **figure 1,** moving P farther from the fixed line produces a broader parabola; moving it closer to the fixed line makes the parabola sharper. Moving P in **figure 2** close to the circle's center makes for a rounded ellipse, whereas moving it close to the circle produces an elongated ellipse. In **figure 3,** the closer P is to the circle, the sharper the hyperbola is.

The proofs showing why these paper folds produce conic sections are accessible to students of second-year algebra. We take each of them in turn.

PARABOLA

Figure 4 shows fixed point P folded onto point P' (and several others) on fixed line L. At P', we construct a perpendicular to line L, hitting the crease line at A. Because the crease line is the perpendicular bisector of $\overline{PP'}$, A is equidistant from the fixed point and the fixed line, hence on a parabola whose focus is P and whose directrix is L. We need to show that the crease line through A is indeed the tangent to the parabola, and we need the parabola's equation.

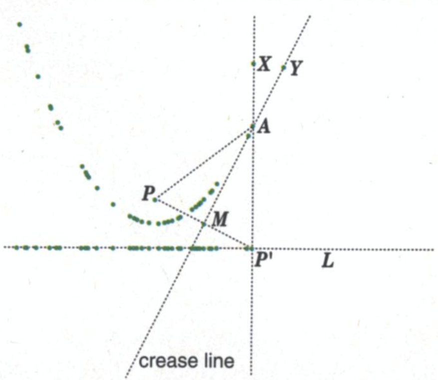


Fig. 4
Crease puts point on parabola.

The standard derivation of a parabola equates the distance from a point $A(x, y)$ to the focus at $P(0, c)$ and to a point on line $y = -c$. Instead, we use the paper-folding properties. See **figure 5.** Let $\overleftrightarrow{MA}$ be the crease line. Then M is the midpoint of $\overline{PP'}$, with coordinates $(x/2, 0)$. The slope of $\overline{PP'}$ is

$$\frac{c-(-c)}{-x},$$

so the slope of $\overleftrightarrow{MA}$ is $x/2c$. The slope of $\overleftrightarrow{MA}$ could also be computed as the slope between A and M,

$$\frac{y-0}{x-\frac{x}{2}} = \frac{2y}{x}.$$

Equating the two yields $x^2 = 4cy$, or

$$(1) \qquad y = \frac{x^2}{4c},$$

the standard form of an up-and-down parabola whose vertex is at the origin. Equation (1) reveals why moving P farther from the fixed line in **figure 1** produces a broader parabola: as c increases, the y-value for the same x decreases.

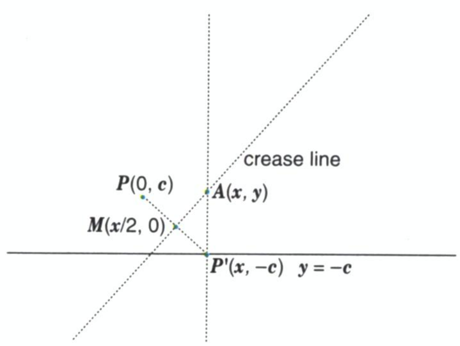


Fig. 5
Coordinates for parabola paper-folding proof

We can next prove that the crease lines are actually tangent to the parabola. If the parabola is given by equation (1), then the slope of the tangent line is $y' = x/2c$, which agrees with the formula derived in the preceding paragraph.

Before we move to the ellipse, we can prove the reflective property of the parabola. We refer to **figure 4.** We know that $\overleftrightarrow{MA}$ is the perpendicular bisector of $\overline{PP'}$. Then by congruent triangles and vertical angles, $\angle PAM \cong \angle XAY$. A light source at point P would therefore reflect off the parabola at multiple points, all in paths parallel to the axis of the

We move point P around to see how its position affects the shape of the conic

parabola ($\overleftrightarrow{P'A}$ is perpendicular to L, as is the axis of the parabola.) Likewise, sound waves coming from afar, parallel to the parabola's axis, would be concentrated at a microphone placed at point P. This result is used in designing mirrors for telescopes, radio telescopes, and solar ovens.

ELLIPSE

Figure 6 shows fixed point P folded onto point P' (and others) of a circle whose center is O. We draw $\overline{OP'}$, intersecting the crease line at A. Therefore, $P'A + AO$ equals the radius of the circle. But because the crease line is the perpendicular bisector of $\overline{PP'}$, AP must equal AP'. Therefore $PA + AO$ equals the radius, which is a fixed value. The sum of the distances from A to points P and O is constant, so by definition, A is on the ellipse with foci O and P. We still need to show that the crease line is a tangent to the ellipse. We therefore need to derive the equation of the ellipse.

The reflective property of the parabola is used in designing parabolas for solar ovens

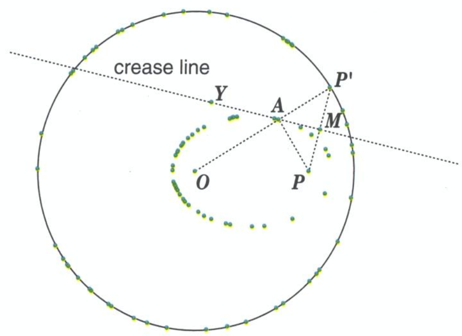


Fig. 6
Crease puts point on ellipse.

The derivation for the ellipse and hyperbola—they are actually the same—and drawing them—are not for those with little practice in or respect for symbol manipulation. In **figure 7,** we note that the circle is centered at the origin and has a radius of r and that fixed point P is located at $(d, 0)$. Let the coordinates of P' on circle O be (a, b). The fold line goes through midpoint M,

$$\left(\frac{a+d}{2}, \frac{b}{2}\right).$$

Depending on the location of P', we draw, or extend, $\overline{OP'}$ to meet the crease line at $A(x, y)$. The slope of $\overline{PP'}$ is $b/(a - d)$, so $\overleftrightarrow{MA}$, with a slope of $(d - a)/b$, could be expressed using point-slope as

$$y - \frac{b}{2} = \frac{d-a}{b}\left(x - \frac{a+d}{2}\right). \tag{2}$$

Because P' is on the circle, $a^2 + b^2 = r^2$, and by similar triangles (dropping perpendiculars from P' and A to the x-axis—not shown in the figure),

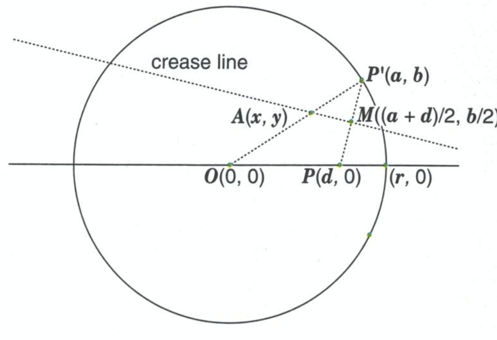


Fig. 7
Coordinates for ellipse paper-folding proof

$$\frac{a}{x} = \frac{r}{\sqrt{x^2+y^2}}$$

and

$$\frac{b}{y} = \frac{r}{\sqrt{x^2+y^2}}.$$

Therefore,

$$\left\{\begin{aligned} a &= \frac{rx}{\sqrt{x^2+y^2}} \\ &\text{and} \\ b &= \frac{ry}{\sqrt{x^2+y^2}}. \end{aligned}\right. \tag{3}$$

Substituting for a and b into equation (2) and several lines of simplification, shown in **figure 8,** lead to the intermediate result

$$2r\sqrt{x^2+y^2} - 2dx = r^2 - d^2. \tag{4}$$

By getting rid of the radical, collecting terms, and completing the square, we can eventually simplify equation (4) to

$$\frac{\left(x - \frac{d}{2}\right)^2}{\frac{r^2}{4}} + \frac{y^2}{\frac{r^2-d^2}{4}} = 1, \tag{5}$$

$$y - \frac{b}{2} = \frac{d-a}{b}\left(x - \frac{a+d}{2}\right)$$
$$2yb - b^2 = (d-a)(2x-a-d)$$
$$2yb - b^2 = 2dx - 2ax + a^2 - d^2$$
Let $z = x^2 + y^2$:
$$2y\left(\frac{ry}{\sqrt{z}}\right) - \left(\frac{ry}{\sqrt{z}}\right)^2 = 2dx - 2x\left(\frac{rx}{\sqrt{z}}\right) + \left(\frac{rx}{\sqrt{z}}\right)^2 - d^2$$
$$\frac{2r}{\sqrt{z}}(x^2+y^2) = \frac{r^2}{z}(x^2+y^2) + 2dx - d^2$$
$$2r\sqrt{z} - 2dx = r^2 - d^2$$
$$2r\sqrt{x^2+y^2} - 2dx = r^2 - d^2$$

Fig. 8
Simplification that leads to equation (4)

the equation of an ellipse whose center is $(d/2, 0)$ and whose major axis is r. Details are left to the reader. (I have waited decades to write that statement.) We have an analytic explanation why moving P close to the circle produced a rounded ellipse. The ratio of the major axis to the minor axis is

$$\frac{r}{\sqrt{r^2-d^2}}.$$

Therefore, as point P approaches the center, the denominator approaches r, and the ellipse is more circular. As P approaches the circle, d approaches r, the denominator approaches 0, and the ellipse is more oblong.

To prove that the crease lines are tangents to the ellipse, we take the derivative of equation (5):

$$\frac{2\left(x-\frac{d}{2}\right)}{\frac{r^2}{4}}+\frac{2yy'}{\frac{r^2-d^2}{4}}=0.$$

Solving for y', we obtain

$$y'=\frac{\left(x-\frac{d}{2}\right)(d^2-r^2)}{yr^2}.$$

From **figure 7,**

$$\text{slope } m(\overline{PP'})=\frac{b-0}{a-d}.$$

Substituting from equations (3) makes

$$m(\overline{PP'})=\frac{ry}{rx-d\sqrt{x^2+y^2}}.$$

Therefore,

$$m(MA)=\frac{d\sqrt{x^2+y^2}-rx}{ry}.$$

It is not at all apparent that $y'=m(MA)$. We use the intermediate result in equation (4) to obtain

$$\sqrt{x^2+y^2}=\frac{r^2-d^2+2dx}{2r}.$$

We multiply both sides by d, subtract rx from both sides, and divide both sides by ry. So

$$\frac{d\sqrt{x^2+y^2}-rx}{ry}=\frac{\frac{d(r^2-d^2+2dx)}{2r}-rx}{ry}.$$

The left-hand side is $m(MA)$. Simplifying the right-hand side does in fact give

$$\frac{\left(x-\frac{d}{2}\right)(d^2-r^2)}{r^2y}=y'.$$

The details are again left to the reader. Although this proof emanated from the figure for the ellipse, the algebra is exactly the same for the hyperbola.

Ellipses have the reflective property that trajectories emanating from one focus reflect off the curve and converge at the other focus. In **figure 6,** because $\overleftrightarrow{MA}$ is the tangent at A, we would need to show that $\angle PAM \cong \angle OAY$. Since A is on the perpendicular bisector of $\overline{PP'}$, $\angle PAM \cong \angle P'AM$ by congruent triangles, $\angle P'AM \cong \angle OAY$ by vertical angles, and transitivity gives the result that we want.

HYPERBOLA

The proof for the paper-folded hyperbola is similar to that for ellipses. In **figure 9,** exterior point P is folded onto P' (and many others) of the circle whose center is O. We draw line OP' until it meets the crease line MA at A. We note that $AO - AP'$ equals the radius of the circle. Again, because the crease line is the perpendicular bisector of $\overline{PP'}$, $AP = AP'$, so $AO - AP$ equals the radius, that is, a fixed value. The difference between the distances from A to two fixed points is a constant, so A is one point of a hyperbola whose foci are O and P. We still need to establish that $\overleftrightarrow{MA}$ is a tangent to the curve. To do so, we need the equation of the hyperbola.

The coordinates of O, P, P', M, and A for **figure 10**'s hyperbola are the same as for **figure 7**'s ellipse, as is the derivation. The only difference is that $d^2 - r^2$ is greater than 0, instead of less than 0, so equation (5) becomes

$$\frac{\left(x-\frac{d}{2}\right)^2}{\frac{r^2}{4}}-\frac{y^2}{\frac{d^2-r^2}{4}}=1, \tag{6}$$

the equation of a hyperbola whose center is $(d/2, 0)$ and whose transverse axis is r. Furthermore, the slope of the asymptotes is

$$\pm\frac{\sqrt{d^2-r^2}}{r}.$$

As P approaches the circle, d approaches r, the slope of the asymptotes approaches 0, and the

The proof for the paper-folded hyperbola is similar to that for the ellipse

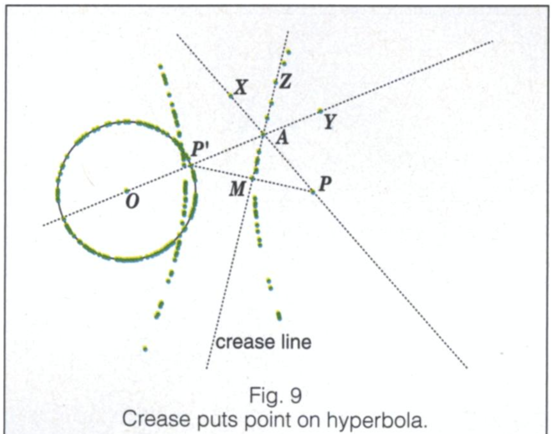


Fig. 9
Crease puts point on hyperbola.

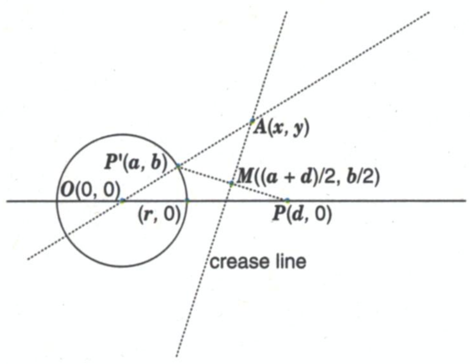


Fig. 10
Coordinates for hyperbola paper-folding proof

hyperbola turns sharply at its vertices. As P moves away from the circle, d approaches infinity, and the asymptotes' slope approaches infinity; this hyperbola has little curve to it. These two extreme cases might have been observed by moving point P in **figure 3** and are suggested in **figures 11** and **12,** where the circle in both is $x^2 + y^2 = 16$; P is located at (5, 0) as P approaches the circle and at (10, 0) as P moves away from the circle.

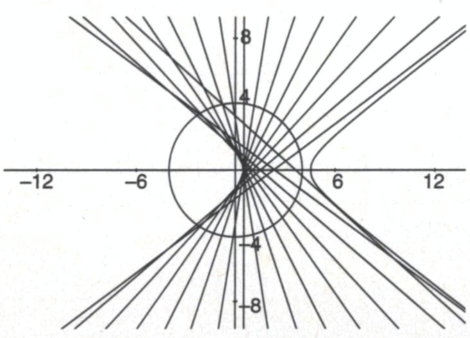


Fig. 11
Hyperbola generated from $x^2 + y^2 = 16$ and $P(5, 0)$

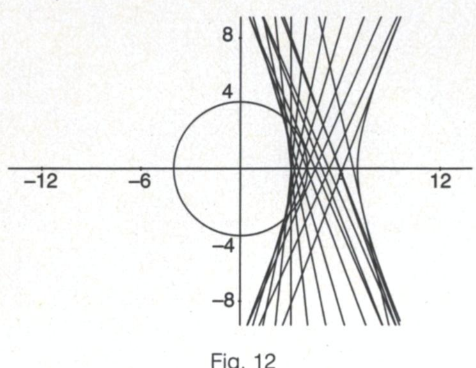


Fig. 12
Hyperbola from $x^2 + y^2 = 16$ and $P(10, 0)$

I noticed that I was obtaining many more points for the left branch than for the right

The proof that crease line MA in **figure 9** is a tangent is the same as for the crease line of the ellipse (they share the same equation).

The reflective property for the hyperbola is not as well known as the reflective property for the parabola and the ellipse. If an object thrown from one of the foci in **figure 9** hits the hyperbola at point A, its reflection is an extension of the line from A to the other focus. If the object is thrown from P in **figure 9,** it would bounce off the hyperbola at A (the crease line is a tangent) toward Y. We need to show that $\angle PAM \cong \angle YAZ$. Again, A's being on the perpendicular bisector of $\overline{PP'}$ and vertical angles provide the answer. If an object is thrown not from the "inside" of the hyperbola but from the "outside" (say, from X in **figure 9**), the reflection would be a path directed at the other focus, O. Here, we would have to show that $\angle XAZ \cong \angle OAM$. The same triangles and a second pair of vertical angles allow us to do so.

RELATED CURIOSITIES

While generating points for the hyperbola in **figure 9,** I noticed that I was obtaining many more points for the left branch than for the right. The closer that P was to the circle, the larger the proportion of points that were on the left branch of the hyperbola. The reason seems to be the points of tangency from P to the points on the circle (Q_1 and Q_2 in **figure 13**). When P is folded onto points in minor arc $Q_1P'Q_2$ (P' is an example), the generated points, for instance, A, are on the right branch. Here, M is the midpoint, $\overleftrightarrow{MA}$ is the crease line, and $\overleftrightarrow{OP'}$ meets $\overleftrightarrow{MA}$ at A. When P is folded onto points in major arc Q_1RQ_2 (R is an example), the generated points, for instance, B, are on the left branch. Here, N is the midpoint, $\overleftrightarrow{NB}$ is the crease line, and $\overleftrightarrow{OR}$ intersects $\overleftrightarrow{NB}$ at B. $|OA - AP| = |BP - BO|$ = fixed distance OP, but A to O is the longer distance in the first case, whereas B to P, the other focus, is the longer distance in the second. This result is consis-

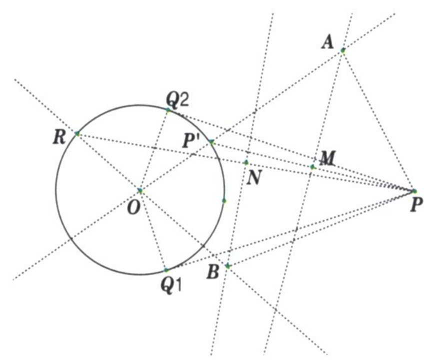


Fig. 13
Points of tangency separate generating regions for left and right branches of hyperbola

tent with analytical derivations of the hyperbola. In addition to separating the two generating regions for the hyperbola, Q_1 and Q_2 share the property that if we try to fold P onto either, the resulting fold line cannot intersect the line containing that point and O to form a point on the hyperbola, because the two lines are parallel. As a matter of fact, these fold lines are the hyperbola's asymptotes. But we can see why a majority of points ended up on the left branch—chosen randomly, more points come from the major arc than from the minor arc. If r is the circle's radius and $OP = d$, then

$$\angle Q_1OQ_2 = 2 \arccos (r/d).$$

For **figure 11,** $\angle Q_1OQ_2 \approx 74°$, so only 74/360, or approximately 20 percent, of points randomly generated are on the right branch. In **figure 13,** $\angle Q_1OQ_2 \approx 133°$, so approximately 37 percent of generated points are on the right branch. The crease lines created when P is folded onto the points of tangency are the hyperbola's asymptotes. Using coordinates O (0, 0), P (d, 0), and Q_2 (a, b), the slopes of perpendicular lines OQ_2 and PQ_2, and midpoint (($a + d$)/2, b/2) of $\overline{PQ_2}$, we can show that the crease lines are the asymptotes

$$y = \frac{\pm\sqrt{d^2 - r^2}}{r}\left(x - \frac{d}{2}\right).$$

A similar observation can be made about the ellipse in **figure 6.** More points appear to be on the left half than on the right. If we label the ends of the major axis B and D, as shown in **figure 14,** the ends of the minor axis A and C, and M the center of the ellipse, we can see that points on the right half of the ellipse are generated only by points of the minor arc subtended by $\angle AOC$. From equation (5), the major axis measures r, the minor axis measures $\sqrt{r^2 - d^2}$, and $\angle AOM = \arctan (AM/OM)$, so

$$\angle AOC = 2 \arctan \frac{\sqrt{r^2 - d^2}}{r}.$$

For instance, if $r = 4$ and $d = 3$, then $\angle AOC \approx$ 67°, so a random point has only a 19 percent chance

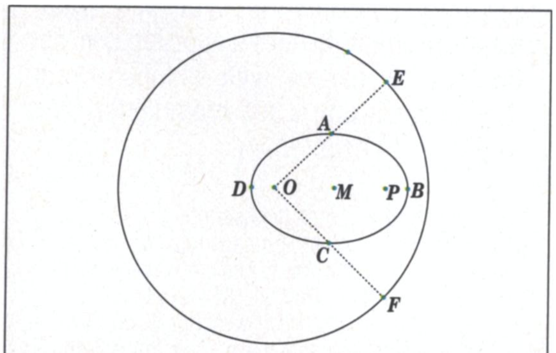


Fig. 14
Paper folding favors the left half of ellipse (opposite *P*).

of being on the right half of the ellipse. As P gets close to the circle, d approaches r, the numerator approaches 0, and the size of the arc from which right-branch points come becomes very small.

A completely propitious discovery occurred when I forgot to have my Sketchpad script erase the midpoints on the crease lines for the hyperbola. Looking at the midpoints in **figure 15,** we see that their locus appears to be a circle. Because each of those points has coordinates

$$\left(\frac{a + d}{2}, \frac{b}{2}\right),$$

and $a^2 + b^2 = r^2$ from **figure 10,** we easily see that those coordinates must satisfy

$$\left(x - \frac{d}{2}\right)^2 + y^2 = \frac{r^2}{4},$$

that is, a circle with center at (d/2, 0) and radius of r/2. This property is true, of course, for the ellipse, as well. The midpoints of the parabola have an uninteresting locus in comparison—they all lie on a line halfway between the focus and the directrix, that is, the x-axis in **figure 5.**

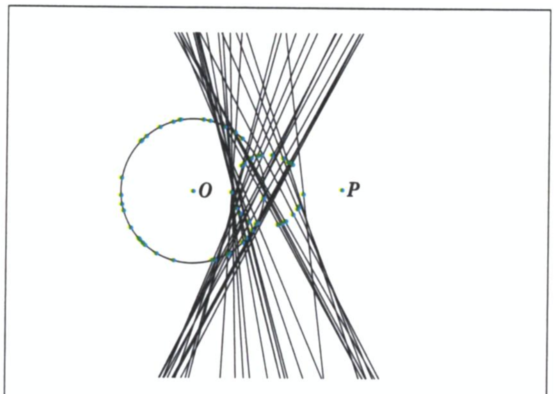


Fig. 15
Paper folding for hyperbola, showing midpoints

CONCLUSION

I was initially charmed by a simple physical activity that hides some interesting mathematics. Pursuing the conic sections from paper folding through The Geometer's Sketchpad, analytic geometry and calculus became an adventure for me. I am always delighted and reassured when the mathematics that I sense is there is finally revealed.

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Swokowski, Earl, and Jeffrey Cole. *Algebra and Trigonometry with Analytic Geometry.* New York: Brooks/Cole Publishing Co., 1997.

The author invites you to visit the following Web sites for non-calculus versions of proofs that the folding lines for the parabola, ellipse, and hyperbola are tangents to the curves:

- upper.us.edu/faculty/smith/foldapp1.htm
- upper.us.edu/faculty/smith/foldapp2.htm