THE KAKEYA PROBLEM

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Editorial Note. In 1958 a grant from the National Science Foundation enabled the Mathematical Association of America to establish a Committee on Production of Films for the purpose of exploring by experiment the possibilities of mathematical motion pictures. In all, the committee produced four films at different educational levels, employing a variety of production techniques. For information about rental or purchase of these films address Modern Learning Aids, 3 East 54th Street, New York 22, New York.

For the last of these films, the Committee invited Professor A. S. Besicovitch to lecture on his brilliant solution to the Kakeya Problem, first published in 1928. The technique of animation employed in this film is particularly appropriate for the geometric constructions involved. The following article approximates the script of the film.

In my paper "Sur deux questions d'intégrabilité," published in a Russian periodical in 1920, I considered the problem:

Given a function of two variables, Riemann-integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that the repeated simple integration along the two directions exists and gives the value of the integral over the domain?

The problem reduces to that of the existence of a set of Jordan plane measure zero which is the union of segments of all directions each of length $\geqq 1$.

The Russian periodical hardly reached other countries because of the isolation of Russia caused by the civil war and the blockade.

In 1917 a twin problem had been proposed by the Japanese mathematician S. Kakeya:

In the class of figures in which a segment of length 1 *can be turned around through* 360°, *remaining always within the figure, which one has the smallest area?*

This problem of course did not reach Russia. I call the two problems "twin problems" because each of them is concerned with sets containing segments of all directions, with the additional condition in Kakeya's case that there should be a continuous transition, within the set, from one position of the segment to any other one.

Let us look at a few figures of the above class.

Obviously a circle of diameter 1 (Fig. 1) is a figure of the class, for if we place the mid-point of the segment in the center of the circle and rotate the segment about the center through 360°, the segment will always remain within the circle. Another obvious figure of the class is an equilateral triangle ABC of height 1 (Fig. 2). For, placing the segment on the side AC so that one end is at A, we can rotate it about A through 60° bringing it onto AB; then we let it slide along AB until the other end of the segment reaches B, then rotate it about B, and so on.

The areas of the circle and of the triangle are $\pi/4 = .78$ and $.58$ respectively.

A three-cornered hypocycloid inscribed in a circle of diameter 3/2 also belongs to the class (Fig. 3). For it is well known that the tangent line at any point M of the hypocycloid meets the hypocycloid at two other points A and B distant 1 from each other. Thus if we let one end of the segment describe the hypocycloid while keeping the segment touching the hypocycloid, we have the other end of the segment also moving on the hypocycloid and so the whole of the segment remains all the time within the hypocycloid.

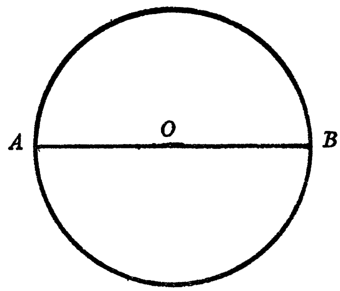

FIG. 1

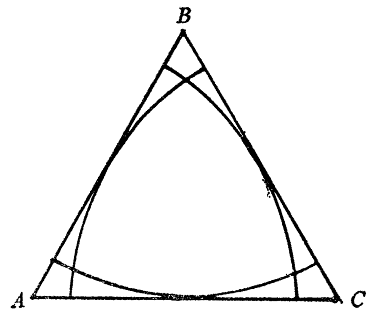

FIG. 2

The area of the hypocycloid is $\pi/8 = .39$, that is exactly half of the area of a circle of diameter 1. It was conjectured that the hypocycloid was the figure of minimum area. The problem aroused great interest. In 1925 G. D. Birkhoff, one of the greatest mathematicians of his time, writing about unsolved problems in his book *The Origin, Nature and Influence of Relativity*, first mentions the four-color problem, then adds: "Of like intriguing simplicity is the question raised a few years ago by the Japanese mathematician Kakeya."

My solution of the Kakeya problem was published in the *Mathematische Zeitschrift* in 1928 and was reported to the American Mathematical Society by J. D. Tamarkin.

The figure arrived at in solving my problem, together with the "joins" suggested by a Hungarian mathematician J. Pal, represent my solution of the Kakeya Problem.

My solution shows that the hypocycloid conjecture is false, and that in fact, there are figures of *arbitrarily small area* which permit a unit segment to change its direction by 360° while moving continuously within them.

The plan of the solution is this. We take a square of side 2 (see Fig. 4) and divide it into four congruent right triangles by joining the center to the vertices. The hypotenuse of each triangle is divided into a large number n of equal parts. Joining each point of division to the center of the square, we have $4n$ "elementary" triangles, each of height 1.

We enumerate the elementary triangles in the order in which they come as we move around the boundary of the square in the counterclockwise direction, starting at the vertex A. The directions of the various segments which

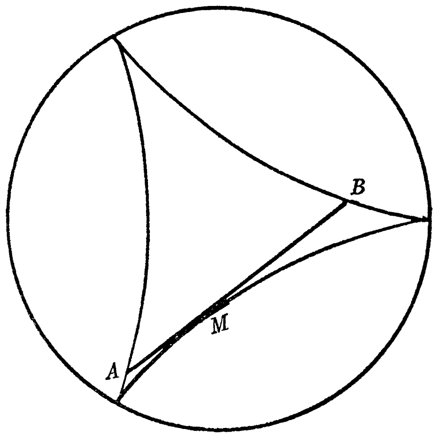

FIG. 3

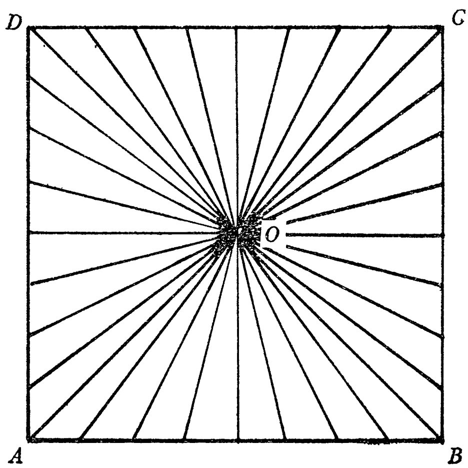

FIG. 4

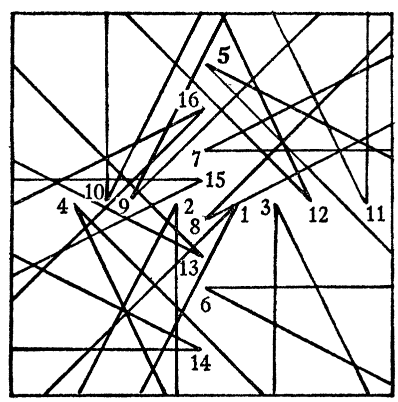

FIG. 4 (a)

join the vertex of each elementary triangle to every point of its base have a range of 360°. The same will remain true if we give arbitrary parallel translations to the elementary triangles (see Fig. 4a). As we shall show, parallel translations can be given to these elementary triangles which achieve such a degree of overlapping that the *total area covered by the triangles in their new position is as small as we please.*

Now if we place an end-point of the unit segment successively at the vertices $O_1, O_2, \cdots$ of the first triangle, the second one, and so on, in their position after translations and in each case rotate it in the positive directions from one side of the triangle to the other, the segment would turn through 360°. But this movement would not be continuous, for in moving from one triangle to the next one the segment would not remain within the area of the figure. We eliminate this difficulty by means of Pal's joins, as follows:

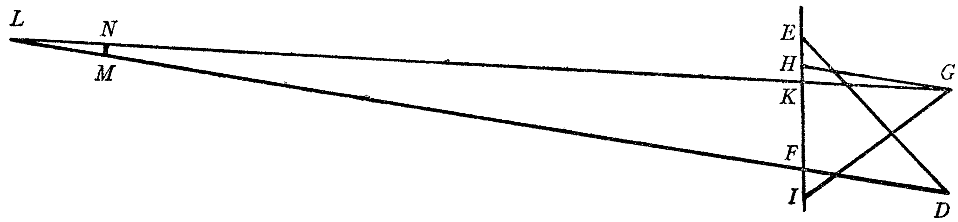

FIG. 5

Let DEF and GHI (Fig. 5) be a pair of consecutive elementary triangles after a parallel translation, and ϵ an arbitrarily small positive number. The sides DF and GH are parallel. Take a point K on HI so that $HK/HI<\epsilon/8$.

Suppose that the lines DF and GK meet in the point L and the triangle LMN is congruent to GHK. We have (denoting area by the sign $|\ |$)

$$|LMN| = |GHK| < \frac{\epsilon}{8}|GHI|.$$

The figure consisting of the lines GL, DL and of the triangle LMN will be called the join. We see that the area of the join is less than $\epsilon/8$ times the area of an elementary triangle. Connecting every pair of consecutive elementary triangles we shall get $4n$ *joins of total area less than* $\epsilon/8$ *times the area of the whole square, i.e.* $<\epsilon/2$. The join added to the triangles DEF, GHI permits the unit segment to come from the triangle DEF to GHI remaining always on the area of the triangles or of the join. For, from the position of the segment on the side DF we let the segment slide down along the line DL until its lower end-point reaches L, then rotate about L until it reaches the side LN and then slide up until its top end reaches G, that is, gets in the second triangle. Thus the problem is reduced *to finding parallel translations of elementary triangles such that the area covered by them be small.*

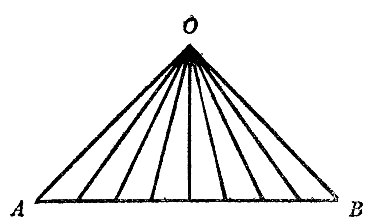

FIG. 6

Perron in 1929 published a new proof of my theorem in which the system of translations while essentially similar to mine is somewhat simpler. Recently Professor I. J. Schoenberg has carried out the Perron construction in the reverse order, which has the advantage of being still simpler in details and easier to visualize. It is essentially this version that we shall adopt here.

We consider the coordinate plane and an integer $p \geqq 2$. We construct the isosceles right triangle $\Delta = OAB$ of our original square with its hypotenuse of length 2 on the x-axis. The base AB of Δ is divided into $n = 2^{p-2}$ equal parts and n elementary triangles with vertex O are constructed. Figure 6 shows the case $p = 5$, $n = 2^{5-2} = 8$.

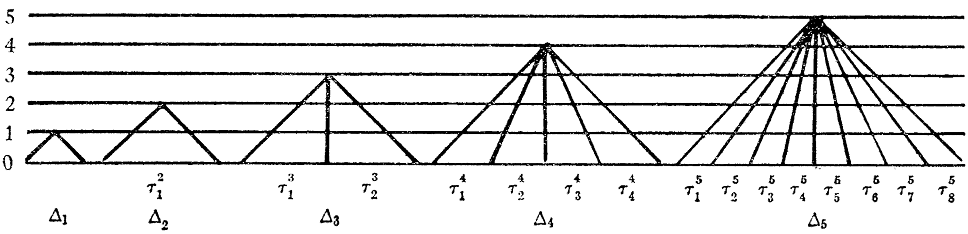

FIG. 7

We next draw the lines $y = k/p$, for $k = 1, 2, \cdots, p$ (Fig. 7) and call the line $y = k/p$ the line of level k, or simply *level* k. We then construct p isosceles right triangles $\Delta_1, \Delta_2, \cdots, \Delta_p$ each with hypotenuse on the x-axis, and the opposite vertices on the $1, 2, \cdots, p$ levels respectively. Note that $\Delta_p = \Delta$. For each k, $k = 3, \cdots, p$, the base of Δ_k is divided into 2^{k-2} equal parts and the elementary triangles are constructed on the subintervals of each base. Notice that Δ_{k+1} is divided into twice as many elementary triangles as Δ_k; Δ_2 is not divided into elementary triangles, Δ_3 is divided into two, Δ_4 into four, Δ_5 into eight, and so on (see Fig. 7).

We shall now assign labels to the elementary triangles in each Δ_k. These will be labeled from left to right as $\tau_1^k, \tau_2^k, \cdots, \tau_j^k, \cdots, \tau_{2^{k-2}}^k$. The superscript k shows that τ_j^k is part of Δ_k and the subscript j says that τ_j^k is the j-th elementary triangle in Δ_k counting from left to right. Δ_2 is not divided into elementary triangles. We shall say it coincides with the elementary triangle τ_1^2; Δ_3 has τ_1^3 and τ_2^3 as elementary triangles; Δ_4 has $\tau_1^4, \tau_2^4, \tau_3^4$, and τ_4^4; and so on.

Note a simple relationship between the elementary triangles of Δ_k and of Δ_{k+1}.

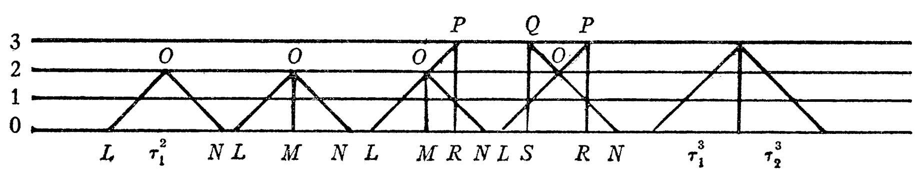

FIG. 8

Let us start with $\tau_1^2 = LON$ (Fig. 8). Bisect it by the median OM into two triangles OLM and ONM and expand them to similar triangles PLR and QNS to the level 3. We shall call this operation the bisection and expansion. The result of this operation is a pair of triangles congruent to the pair τ_1^3 and τ_2^3 of Δ_3. Similarly is defined the operation of bisection and expansion of the triangles τ_j^k for any $k>2$: τ_j^k is bisected into two triangles by the median from its vertex, and each of the triangles is expanded to the next level. The operation transforms τ_j^k into parallel translates of τ_{2j-1}^{k+1} and τ_{2j}^{k+1} (Fig. 10), and applied to the set of all triangles τ_j^k, or to any set of their parallel translates, transforms the set into a set of parallel translates of elementary triangles of Δ_{k+1}. Figure 9 represents a particular case of $k=3$.

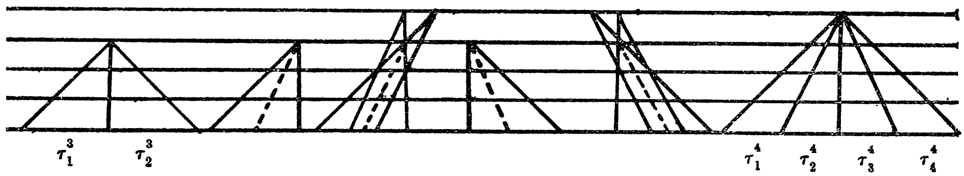

FIG. 9

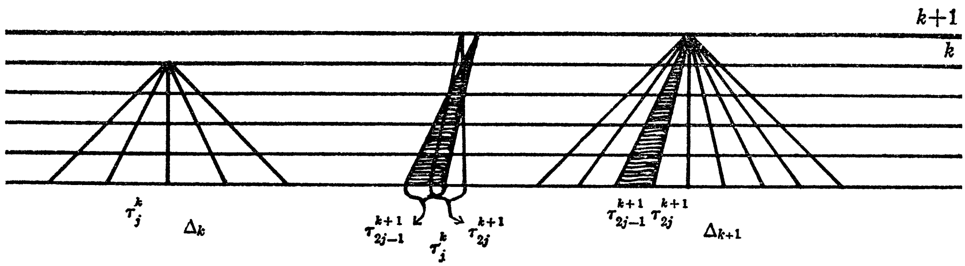

FIG. 10

The part of Δ_k (or of any elementary triangle τ_j^k of Δ_k) which lies between levels $k-1$ and k will be called the "top end" of Δ_k (or of τ_j^k). Notice that the top end of Δ_k is congruent to Δ_1 and that the sum of the areas of the top ends of all elementary triangles of Δ_k is equal to the area of the top of Δ_k, that is to $|\Delta_1|$ (see Fig. 11).

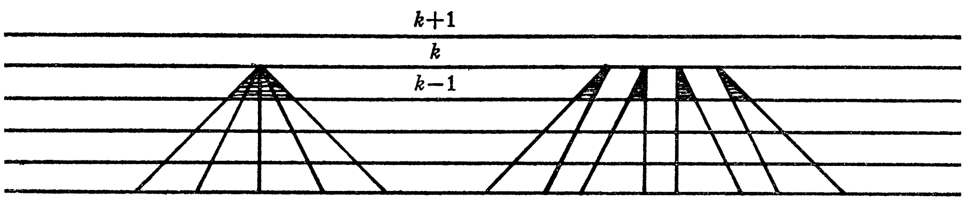

FIG. 11

Now let us look at the change in area when bisection and expansion are applied to a triangle. Consider an elementary triangle $\tau_j^k=LMN$ with vertex N at level k (see Fig. 12). Let NP be the median of LMN, bisecting it into the two subtriangles LPN and MPN. If we expand LPN upwards and to the right to the level $k+1$, we get a similar triangle LRQ. If we expand MPN upward to the left to level $k+1$, we get a similar triangle MTS.

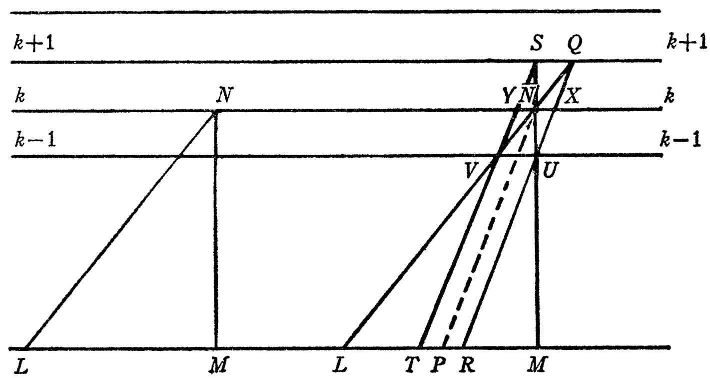

FIG. 12

The two triangles LRQ and MTS together cover the triangle LMN and the two "end pieces" SNV and QNU.

We have

$$|SNV| = |QNU| = |NUV|$$

so that the two overlapping triangles cover an area equal to the area of the original triangle LMN plus twice the area of the top end of LMN.

Observe that for constructing the end pieces to LMN, one merely has to know the top end of the triangle LMN. The rule is this: produce the sides LN and MN to the level $k+1$ and join the end-points to the points of the sides on the level $k-1$.

If we start with a complete set of elementary triangles of Δ_k or of their parallel translates and apply bisection and expansion to each of these triangles, we shall arrive at a set of parallel translates of all elementary triangles of Δ_{k+1} and introduce an increase in area at most equal to twice the sum of the areas of the top ends of these triangles. (The "at most" is necessary here because of possible overlapping.) The sum of the areas of the top ends of a complete set of elementary triangles is equal to the area of Δ_1, so that the total increase in area is at most $2|\Delta_1|$.

This conclusion leads immediately to the complete solution of our problem. We start with the triangle $\Delta_2=\tau_1^2$ and apply to it the bisection and expansion which will transform it into parallel translates of the elementary triangles of Δ_3. Applying the same operation to each of the new triangles we shall get a set of parallel translates of elementary triangles of Δ_4, and so on.

After $p-2$ such operations we shall arrive at a set of parallel translates of elementary triangles of $\Delta_p=\Delta$. As the increment of the area at each operation is $\leqq 2|\Delta_1|$, the area of the final figure will be

$$\leqq |\Delta_2| + 2(p-2)|\Delta_1| = \frac{4}{p^2} + 2(p-2)|\Delta_1| = \frac{4}{p^2} + 2(p-2)\frac{1}{p^2} = \frac{2}{p}\cdot$$

Taking $p>16/\epsilon$ we shall get the area $<\epsilon/8$. With similar translations for the other $3n$ elementary triangles we shall get the total area covered by the translates $<\epsilon/2$. Adding $4n-1$ joins of total area $<\epsilon/2$ we shall get a figure of area $<\epsilon$ on which the unit segment can turn round through 360°, which represents a solution of the problem.

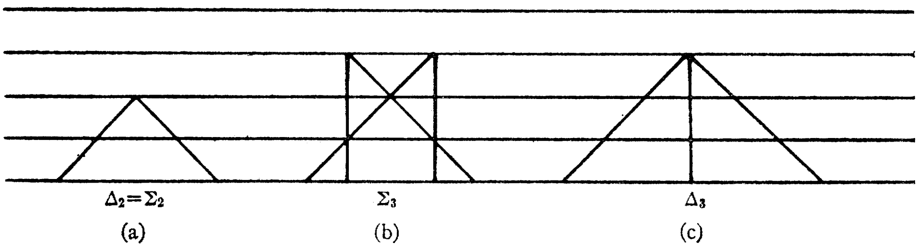

Fig. 13

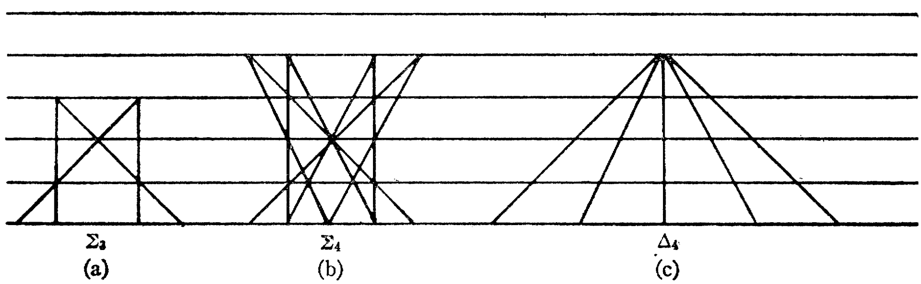

Fig. 14

In order to visualize the final picture of the set of translates of the elementary triangles of Δ we shall go through the process of its construction step by step. Figure 13 shows the beginning of the process of its construction: (a) is the triangle $\Delta_2=\tau_1^2$, we call it the set Σ_2; (b) shows the results of bisection and expansion, it is the set Σ_3 of parallel translates of elementary triangles of Δ_3; (c) represents Δ_3 divided into elementary triangles.

Figure 14 shows the second operation: (a) is Σ_3, (b) is the result of bisection and expansion of the two triangles of Σ_3, it is a set Σ_4 of translates of four elementary triangles of Δ_4, (c) is Δ_4 divided into elementary triangles.

Similarly Figure 15 shows the third operation: (a) is Σ_4, (b) is the result of bisection and expansion of each of the four triangles of Σ_4, it is a set Σ_5 of parallel translates of elementary triangles of Δ_5, (c) is the triangle Δ_5 divided into elementary triangles.

We continue in the same way arriving finally at Σ_p.

Pictures of consecutive Σ_k become more and more complicated. Observe however that for every $k\geqq 2$, Σ_{k+1} is obtained from Σ_k only by constructing the end-pieces of the triangles belonging to Σ_k. These end-pieces are constructed on the top triangles of Σ_k and lie between the levels $k-1$ and $k+1$. Thus each stage of construction is confined only to the two top strips and there is no need for tracing fully all elementary triangles. Figure 16 represents the process.

The figure Σ_p of overlapping triangles has been called by Schoenberg the Perron tree; we shall call it the Perron-Schoenberg tree. The process of its construction can best be described as the "growth of the Perron-Schoenberg tree" (see Fig. 16).

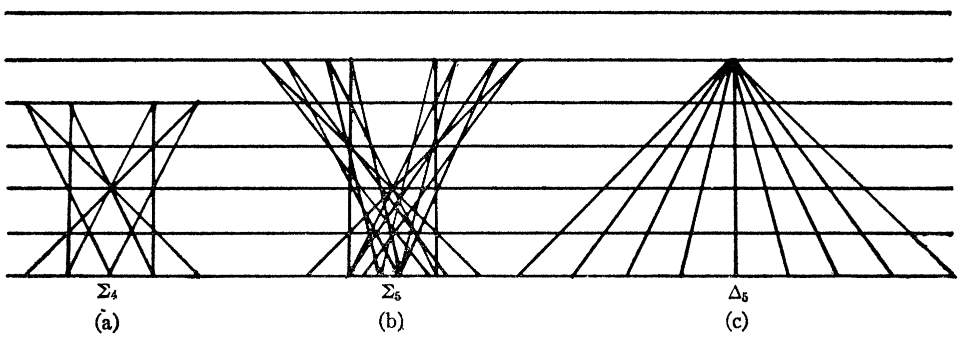

FIG. 15

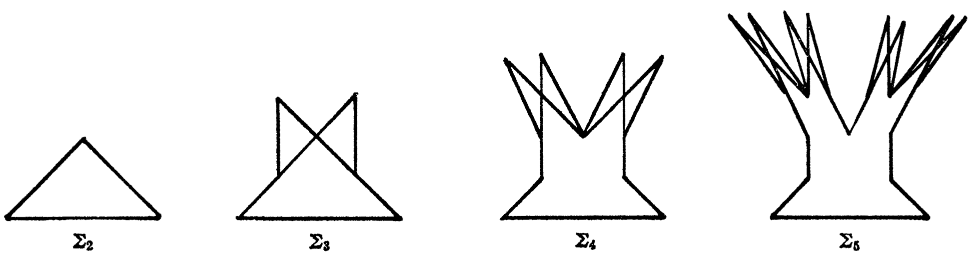

FIG. 16

REMARK. Observe that because of joins our domain is multiply connected, hence there arises a most interesting question: What will the result be if we confine ourselves only to simply connected domains?

I should like to say in conclusion a few words about the place of the Kakeya Problem in mathematics. It is one of the most intriguing problems in the class of extremal problems. On the other hand it is more important as a twin problem to the following problem of fundamental nature in the geometry of general sets of points.

Is the plane measure of a set of unit-segments of all directions in a plane, bounded from below by a positive number?

It is this twin problem that I was concerned with in the paper mentioned at the beginning of this lecture. The answer arrived at was negative.

In fact, I constructed such a set having plane measure zero. At that time one could not suspect that this result was only a first penetration into a much greater problem. My later study of the geometry of sets of points has led me to the solution of the general problem on the plane measure of line-sets.

The study of point-sets has been carried out on the basis of measure. The most interesting class appears to be the class of sets of finite Hausdorff linear measure. A general set of that kind is the sum of a *regular set* and of an *irregular* one, a set being *regular or irregular according as at almost all of its points the density exists or does not exist.* The two classes are fundamentally different. The measure of a set of lines is defined by the measure of the set of the poles with respect to a fixed circle, and a line-set is regular or irregular according as its set of poles is regular or irregular. The problem on the plane measure of a line-set considered as a point-set is solved by the following result.

THEOREM. *The plane measure of any irregular set is* 0, *and of any regular one is* ∞.

The construction of irregular sets containing lines of all directions presents no difficulty.

The occasion of making a film may stimulate others to work on the Kakeya problem. I want only to express a hope that this new work will be crowned by success.

A REMARK ON THE KAKEYA PROBLEM

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Besicovitch [1] poses the question, what is the solution of the Kakeya problem if we restrict ourselves to simply connected domains. It is known, for example, that it is possible to turn a unit segment through 360° within the three-cornered hypocycloid of area $\pi/8$. Insofar as I am aware, no simply connected domain of lesser area is known to satisfy the Kakeya criterion. Here we demonstrate the existence of a class of domains, regular star polygons, which satisfy the Kakeya criterion and for which there exist areas exceeding $\pi/8$ by arbitrarily little. Is $\pi/8$ then the minimum area for simply connected domains, or at least, for star-like domains? If so, we have here a curious instance of a domain functional which is minimized in two entirely different ways.

We construct a sequence of star polygons in the following manner. First, a regular $(2n+1)$-gon is inscribed in a circle of radius $r<1/2$. Each side of the $(2n+1)$-gon is then used as the base of an exterior isosceles triangle with vertex at unit distance from the diametrically opposite vertex of the $(2n+1)$-gon. The domain of interest is the union of the inscribed polygon with the $2n+1$ triangles attached to the sides. In Fig. 1 we depict the case $n=3$.

Let us denote the center of the circle by O and label the vertices of the isosceles triangles by $V_0, V_1, \cdots, V_{2n}, V_{2n+1}, \cdots$, in counter-clockwise order, where if $i \equiv j \pmod{2n+1}$ then $V_i = V_j$. The diametrically opposite vertex of the