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Infinite Series as Sums of Triangular Areas

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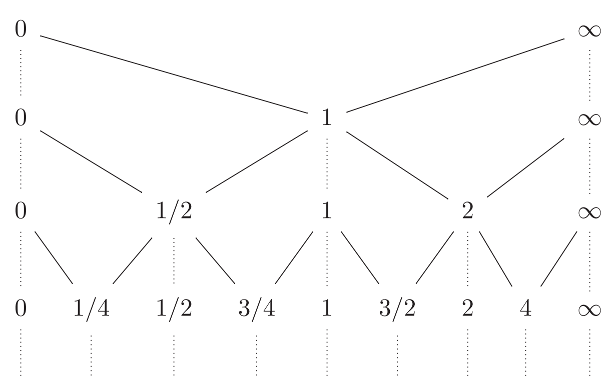
The Riemann approach to integration associates the area under an integrable function's graph with rectangles. In this paper we take a different approach, exhausting the area using triangles. The methods are straightforward, involving only basic ideas from calculus and vector calculus, and the change in perspective is interesting in its own right. Because progress in mathematics often comes by reframing a problem, this particular example of a shifted perspective on a familiar topic may be instructive to anyone who is familiar with the varied historical approaches to integration.

Our approach relies on generating a dense subset of the interval of integration through the use of an infinite binary tree. This tree generates a series, and because the terms of the series arise from both the function defining the region and the choice for the nodes of the tree, it becomes possible to generate both familiar and exotic series and to know readily whether they diverge or converge and, if convergent, to what value. A particularly noteworthy example comes from using the Stern-Brocot tree, in which case connections arise with certain types of zeta functions. This connection provides a hint at the way the Riemann hypothesis is connected with topics such as Farey sequences.

A motivating example

To illustrate a few key ideas, consider the function $f(x) = 1/x$ on $(0, \infty)$. Imagine generating a countable dense subset of $(0, \infty)$ by starting with 0 and ∞ and placing between them the additional point 1. Between each of these three, generate two additional points: $1/2$ and 2. These five points are used to create four more by dividing the finite intervals into halves and adding the next power of 2 "between" the current largest finite value and ∞. The general process can be illustrated via an infinite tree:



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We now view the x and y-axes as the tangent lines to the graph of f "at" the points 0 and ∞, and we generate a new tangent line to the graph of f at the first point in the tree which lies between 0 and ∞. The resulting three tangent lines intersect, forming a triangle that lies below the graph of f, covering a portion of the area in the first quadrant. We repeat the process, except now forming triangles in the so-far uncovered area by considering the tangent lines at the next two points in the tree: $1/2$ and 2. This is shown in Figure 1.

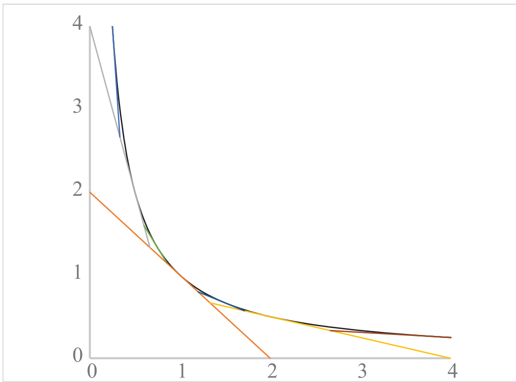


Figure 1 The first three tangent lines used in our triangular partition of the area under the curve.

In the limit, these triangles will exhaust the entire area under the curve. To further illustrate our point, if we consider the right-most or left-most triangles at each iteration, we see they all have an area of $2/3$—and thus can readily (and unsurprisingly) conclude that the area is infinite because the corresponding infinite series diverges. In this way we can establish a connection between the area under the curve and the infinite series corresponding to the combined area of the covering triangles. We investigate the nature of these series for different functions and different countable dense subsets.

Generalizing the approach

The standard approach to the Riemann integral amounts to utilizing a sequence of successively finer partitions to generate approximating areas whose limit is the area under the curve. Our approach is similar, except that the sequence of partitions must be regulated enough to ensure the approximating triangles can be conveniently defined. This is where the infinite binary tree comes in—it ensures partitions of sufficient regularity.

With such a sequence of partitions in hand, the process for generating the triangles which exhaust the area under the graph of a function f relies only on a few key properties:

1. The function must be defined on an interval of the form $(0, \infty)$, $[0, \infty)$, $(0, a]$, or $[0, a]$ for $0 < a < \infty$ and be tangent to the x- and y-axes at 0 and (if appropriate) a, respectively, or have one or both of the axes as asymptotes.
2. The function must be twice continuously differentiable, strictly monotonically decreasing, and strictly concave up, i.e., $f'(x) < 0$ and $f''(x) > 0$ for all $x \in A$ where A is the interval specified in Property 1.
3. The set generated by the tree must be a dense subset of A.

These three conditions are sufficient. Property 1 establishes that two of the legs of the first triangle will be the two axes, while Property 2 ensures that each newly-generated tangent line is distinct from those generated in previous steps of the process and thus yields the third side of a new triangle which does not overlap (except at the boundaries) any of the previous triangles. It also implies the triangle so formed lies below the curve. Finally, Property 3 means that every point (x_0, y_0) in the first quadrant and below the graph of the function will be in at least one triangle. In particular, by choosing a point on the tree sufficiently close to x_0 to ensure that the resulting tangent line lies above y_0 one ensures (x_0, y_0) is included in the covering.

Calculating the area of any one of these triangles is an application of basic calculus. For instance, if $0 < x_1 < x_{1,2} < x_2$ are three consecutive points at a given level of the tree, with $x_{1,2}$ having been generated from the two previously consecutive points x_1 and x_2, then one can find the tangent lines to the graph of f at each of x_1, $x_{1,2}$ and x_2, use them to find the corresponding three vertices of the resulting triangle, and—by finding the cross product of the vectors describing two of the triangle's sides—obtain an area formula of the form Area $= A^2/2B$ where

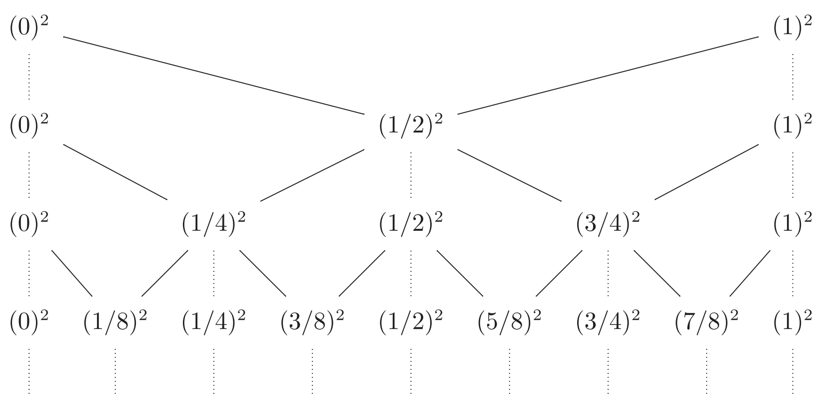
$$\begin{aligned} A = {} & f'(x_1)f'(x_{1,2})(x_1 - x_{1,2}) + f'(x_1)f'(x_2)(x_2 - x_1) \\ & + f'(x_2)f'(x_{1,2})(x_{1,2} - x_2) + f(x_1)(f'(x_2) - f'(x_{1,2})) \\ & + f(x_2)(f'(x_{1,2}) - f'(x_1)) + f(x_{1,2})(f'(x_1) - f'(x_2)) \end{aligned}$$

and

$$B = (f'(x_1) - f'(x_{1,2}))(f'(x_1) - f'(x_2))(f'(x_2) - f'(x_{1,2})).$$

An illustrative example

Consider the function $g(x) = (1 - \sqrt{x})^2$ on $[0, 1]$ and the tree



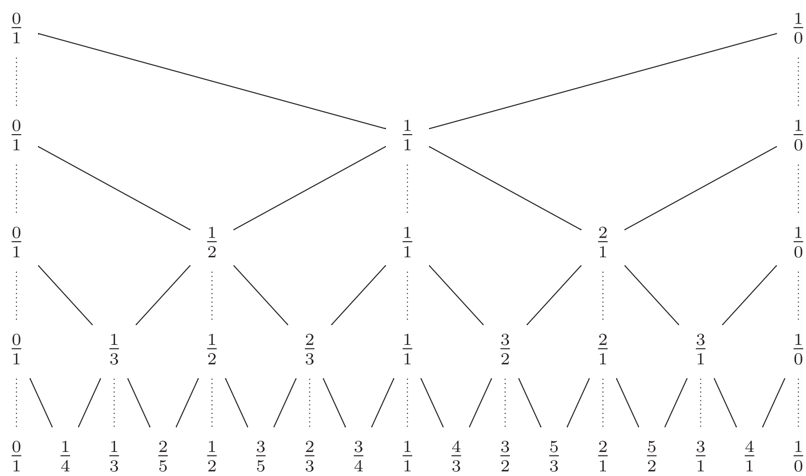
which represents a modified version of the tree presented above. This situation meets the three properties specified earlier, and we find that if we consider level n of this tree, letting k range between 0 and $2^n - 1$, it is possible to calculate the area of each triangle generated at the n-th tree level. These areas turn out to be independent of k, all equaling $1/2^{3n+3}$. Thus, for each $n \geq 0$ and each $0 \leq k \leq 2^n - 1$ the area of the corresponding triangle is $1/2^{3n+3}$. Thus, the area under the graph of $g(x)$ is given by

$$\sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \frac{1}{2^{3n+3}} = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{2^{3n+3}} = \frac{1}{6}.$$

It should be noted that this is in no way a surprising conclusion, as the area may also be calculated by evaluating $\int_0^1 g(x)\,dx$ directly. However, this example illustrates the variety of ways one may conveniently generate the dense subset A of the domain of the function. Modifications such as this can be useful for simplifying the form of the terms in the corresponding series.

Generating dense subsets using the Stern-Brocot tree

The example of $g(x)$ on [0, 1] becomes more profound when used in conjunction with a dense subset generated by the Stern-Brocot tree:



The most general form of the tree is generated by starting with the rationals $0/1$ and $1/0$ and generating successive vertices by forming the *mediant* of those adjacent rationals already formed as part of the iterative process. The mediant is simply found by separately adding numerators and denominators of the two given rationals. It turns out (see, e.g., Graham, Knuth, and Patashnik [2]) that if m/n and m'/n' are adjacent rationals at some level in the Stern-Brocot tree, then $mn' - m'n = -1$. If we use this fact, focus on the left half of the Stern-Brocot tree (that generated by starting with $0/1$ and $1/1$), and apply the area formula given above to $g(x)$, the area may be simplified to

$$\frac{1}{2n^2(n')^2(n+n')^2}$$

provided we take the dense subset A of [0, 1] in this case to be generated by squaring the value at each vertex of the tree. Hence, given our conclusion about the total area from the previous section, we arrive at

$$\sum_{l=0}^{\infty}\sum_{\forall n_l}\frac{1}{2n_l^2(n_l')^2(n_l+n_l')^2}=\frac{1}{6},$$

where n_l and n_l' are adjacent denominators at level l of the tree. Thus, we sum over all levels and all pairs of adjacent denominators in each level.

Note how changing the nature of the points constituting the subset A leads to a different form for the terms of the infinite series. In fact, this last result, with additional details, can also be found in Kramer and von Pippich [3], where it is noted that this series is a special instance of a more general type of function called a Mordell-Tornheim Zeta Function,

$$\sum_{(m,n)=1}^{\infty} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}}.$$

Remarkably, related ideas are connected, via Farey Sequences, to the Riemann Zeta Function and the Riemann Hypothesis [1,4,5].

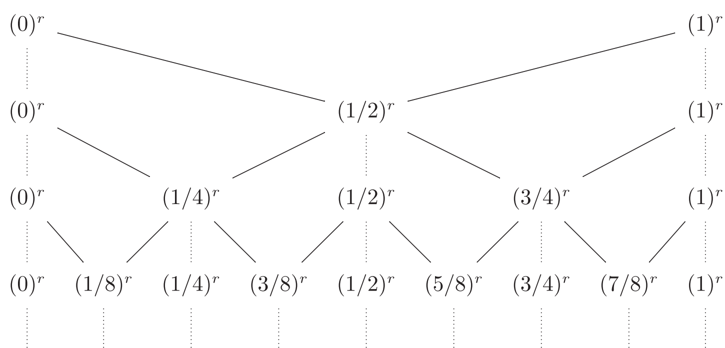
Thus, via the same function $g(x)$ and domain $[0, 1]$, we obtain—by altering the process through which the points of A are generated—a much different, much less easily evaluated, convergent series.

A class of convergent series

Applying the above ideas to $y = 1/x$ using either the tree based on powers of 2 or the Stern-Brocot tree yields a simple formula for the corresponding divergent series, although in neither case is the conclusion of divergence surprising given the form of the series' terms – which allow for the conclusion of divergence in a more direct way. The reader may also reasonably wonder about situations in which the portion of the unit circle $(x-1)^2 + (y-1)^2 = 1$ for $0 \le x, y \le 1$ or a function like $y = (1 - \sqrt[3]{x})^3$ are used on the interval $[0, 1]$ to generate convergent series. In both cases straightforward modifications of the terms constituting the tree make helpful simplifications to the calculations necessary to determine the areas of the covering triangles. In neither case, however, is the result a series with an "elegant" or simple form.

On the other hand, taking r to be a natural number greater than 1 and forming the function $f(x) = x^{-1/r}$ on $[0, 1]$ does allow for a conveniently expressible form for the resulting family of infinite series, and illustrates one additional generalization to the ideas outlined in this paper.

Focusing first on the tree:



it is possible to conclude a general triangular area is given by the fraction A/B, where

$$\begin{aligned} A &= (1+r)^2 \left[(k+1)^r (2k+1)^r + k^r (2k+1)^r - 2^{r+1} k^r (k+1)^r\right]^2 \\ B &= r 2^{nr-n+1} \left[(2k+2)^{r+1} - (2k+1)^{r+1}\right] \\ &\qquad \times \left[(2k+1)^{r+1} - (2k)^{r+1}\right]\left[(k+1)^{r+1} - k^{r+1}\right], \end{aligned}$$

and thus, for instance, with $r = 2$ we have the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1}\frac{9(6k^2+6k+1)^2}{2^{n+2}(3k^2+3k+1)(12k^2+6k+1)(12k^2+18k+7)}.$$

No doubt the reader has noted that this example violates one part of the three properties outlined earlier in the paper—namely that $f(x)$ be tangent to both axes on [0, 1]. Thus, it is not possible to conclude directly that the above series converges to the value of $\int_0^1 f(x)\,dx$. However, the necessary adjustment is straightforward and easily discerned by examining Figure 2, showing which portion of the area under the graph of f is not covered by triangular areas. In particular, for the case where $r = 2$ we see that

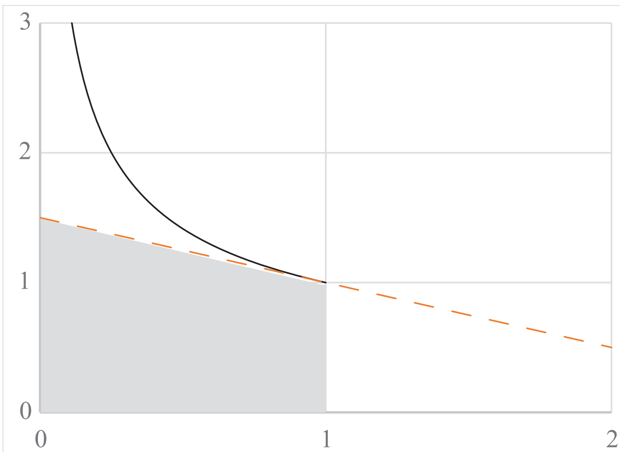


Figure 2 The region under the graph of f not covered by triangular areas.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{2^n-1}\frac{9(6k^2+6k+1)^2}{2^{n+2}(3k^2+3k+1)(12k^2+6k+1)(12k^2+18k+7)}.$$

converges to the value of

$$\int_0^1 \frac{1}{x^{1/2}}\,dx - \frac{5}{4} = \frac{3}{4},$$

where $5/4$ is the area of the shaded trapezoid. More generally, if we consider the triangular area corresponding to $f(x) = x^{-1/r}$, the series whose terms are the fractions A/B given above converges to

$$\int_0^1 \frac{1}{x^{1/r}}\,dx - \frac{2r+1}{2r} = \frac{r+1}{2r(r-1)}.$$

Summary

Rather than the standard approach using rectangles, we instead approximate the area under the graphs of certain functions using triangles generated by well-regulated partitions which arise from binary trees. The series which arise have forms that depend on the structure of the tree, and examples are given illustrating how markedly different series turn out to correspond to the same definite integral. The interested reader may want to consider things from the opposite direction; namely, given a series, under what circumstances will there exist a function for which the techniques of this paper lead to that series?

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Summary. A method is developed for exhausting the area under certain curves using triangles generated by an infinite binary tree. The associated series depend on the choice of binary tree, and examples are given to illustrate how this method can be used to evaluate certain series.

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