Complex Manifolds and Kähler Geometry
Project 4

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Abstract

In this essay we go over the proof of the isomorphism between the Čech cohomology $\check{H}(M;\mathbb{R})$ and the de Rham cohomology $H_{\mathrm{dR}}(M;\mathbb{R})$ of a smooth manifold M based on the exposition in [Gri94].

1 Short Introduction to Sheaves

Let $(X, \mathcal{T})$ be a topological space. We associate with this space the category $\mathbf{Op}(X)$ with its set of objects just being the open sets $\mathcal{T}$ and the morphisms $\mathrm{Hom}(U, V)$ either being solely the inclusion $U \hookrightarrow V$ if $U \subset V$ or otherwise $\emptyset$.

Further, let **Ab** denote the category of Abelian groups.

Definition 1.1 *Let $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{Ab}$ be a contravariant functor. We call elements $\sigma \in \mathcal{F}(V)$ sections of V and the image of the inclusion map $U \hookrightarrow V$ under the functor by $\sigma|_U$, the restriction of σ to U.*

We call such a functor $\mathcal{F}$, a pre-sheaf, if $\mathcal{F}(\emptyset) = \mathbf{0}$, the 0-group.

This definition encodes some of the basic notions one associates with local constructions on a manifold, such as the set of holomorphic functions on an open subset of a complex manifold, or the local vector fields on a manifold. However, these enjoy some further gluing properties. For this reason we add two requirements that allow us to speak of sheaves.

Definition 1.2 *A pre-sheaf $\mathcal{F} : \mathbf{Op}(X) \to \mathbf{Ab}$ is called a sheaf if in addition it satisfies the following two requirements:*

- *For all $U, V \in \mathcal{T}$ and all sections $\sigma \in \mathcal{F}(U)$ and $\tau \in \mathcal{F}(V)$ such that*

$$\sigma|_{U \cap V} = \tau|_{U \cap V}$$

  there exists $\omega \in \mathcal{F}(U \cup V)$ s.t.

$$\sigma = \omega|_U, \qquad \tau = \omega|_V.$$

- *If $\sigma \in \mathcal{F}(U \cup V)$, s.t. $\sigma|_U = \sigma|_V = 0$, then $\sigma = 0$.*

Let G be a fixed Abelian group. One might think that an easy example of sheaf would be the constant sheaf that assigns to every open set U (other than the empty set) the whole group G, i.e. $\mathcal{F}(U) = G$, with all the restriction maps being the identity. However, once the space X has more than one connected one runs into trouble. For suppose that X has two connected components $X_1 \cup X_2 = X$. Then for $g_1, g_2 \in G$ distinct one can set

$$g_1 =: \sigma \in \mathcal{F}(X_1), \qquad g_2 =: \sigma \in \mathcal{F}(X_2).$$

Since $X_1 \cap X_2 = \emptyset$ one trivially has

$$\sigma|_{X_1 \cap X_2} = 0 = \tau|_{X_1 \cap X_2}$$

but there cannot be a $g \in \mathcal{F}(X) = G$ such that its restrictions are equal to g_1 and g_2 for

$$g|_{X_1} = g = g|_{X_2}.$$

This therefore only a pre-sheaf. The correct "sheafification" of the constant pre-sheaf is the constant sheaf which we shall present together with a couple of examples.

Example 1.3
- *The constant sheaf $\underline{G}$ which assigns to $U \in \mathcal{T}$ the group G^n where n is the number of connected components of X U intersects. The restriction maps are the obvious ones.*
- *Let M be a smooth manifold. Let $\mathscr{A}^p$ be the sheaf of smooth p-forms on M, i.e. for $U \subset M$, $\mathscr{A}^p(U)$ is the (vector) space of smooth, real-valued p-forms defined on U.*
- *Analogously to above let $\mathscr{Z}^p$ be the sheaf of closed p-forms on M.*
- *Let (X, J) be a complex manifold. Let $\mathscr{O}$ denote the sheaf of holomorphic functions on X, i.e. for $U \subset X$, $\mathscr{O}(U)$ is the (vector) space of holomorphic functions defined on U.*

Remark 1.4 Note that $\mathscr{A}^0$ is simply the sheaf $\mathscr{C}^\infty$ of smooth functions on M and $\mathscr{Z}^0$ is the constant sheaf $\underline{\mathbb{R}}$.

2 Cohomology of Sheaves

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ a locally finite open cover, that is a cover s.t. for any $U_\alpha \in \mathcal{U}$ there exist only finitely many $U_\beta \in \mathcal{U}$ that are non-zero.

We define the chain complex of $\mathcal{U}$ and $\mathcal{F}$ to be given by the groups

$$C^p(\mathcal{U}, \mathcal{F}) \stackrel{\text{def}}{=} \prod_{\substack{\alpha_0, \dots, \alpha_p \in I \\ \alpha_0 \neq \dots \neq \alpha_p}} \mathcal{F}\big(U_{\alpha_0} \cap \dots \cap U_{\alpha_p}\big)$$

and coboundary operator $d : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$

$$(d\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_p}|_{U_{\alpha_0} \cap \dots \cap U_{\alpha_{p+1}}}$$

where $\widehat{\alpha_i}$ denotes the lack of the index α_i. Elements of $C^p(\mathcal{U}, \mathcal{F})$ are called p-*cochains*. A p-cochain σ will be known as a *cocylce* if $d\sigma = 0$, the set of which shall be denoted by $Z^p(\mathcal{U}, \mathcal{F})$, and a *coboundary* if there exists $p-1$-cochain τ, s.t. $d\tau = 0$.

It is a routine calculation to check that $d^2 = 0$, thus $dC^{p-1} \subset Z^p$ and one can define the cohomology group

$$H^p(\mathcal{U}, \mathcal{F}) := \frac{Z^p(\mathcal{U}, \mathcal{F})}{dC^{p-1}(\mathcal{U}, \mathcal{F})}.$$

However, this group strongly depends on the choice of open cover $\mathcal{U}$. Thus, we define the Čech cohomology group of the sheaf $\mathcal{F}$ over X to be the direct limit over all locally finite open covers of the above group, i.e.

$$H^p(X, \mathcal{F}) := \check{H}^p(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F}).$$

Of course, it is extremely hard to calculate the above direct limit in general. However, often there are conditions under which for a sufficiently fine yet still finite cover $\mathcal{U}$ of a sufficiently nice space X one has

$$H^p(X, \mathcal{F}) \cong H^p(\mathcal{U}, \mathcal{F}).$$

We shall prove that this is the case in the following section for the Čech cohomology of a manifold.

Definition 2.1 (Čech Cohomology of a Smooth Manifold) *Let M be a smooth connected manifold. We call the group $\check{H}^p(M, \mathbb{R})$ the p^{th} Čech cohomology group of M.*

3 De Rham's Theorem

We shall need the following two lemmate to show how one can calculate the Čech cohomology group of a manifold.

Lemma 3.1 *For $p > 0$ and all $q \geqslant 0$, the Čech cohomology groups of the sheaves $\mathscr{A}^q$ are trivial, i.e. $\check{H}^p(M, \mathscr{A}^q) = 0$.*

Proof. It is enough to prove that this holds for any locally finite cover $\mathcal{U}$ of M. Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be such a cover and let $(\varrho_\alpha)_{\alpha \in I}$ be a corresponding subordinate, i.e. supp$(\varrho_\alpha) \subset U_\alpha$, partition of unity.

Let $\omega \in Z^p(\mathcal{U}, \mathscr{A}^q)$ we have to find a $\psi \in C^{p-1}(\mathcal{U}, \mathscr{A}^q)$ such that $d\psi = \omega$. ω consists of a collection of maps

$$\omega_{\alpha_0 \cdots \alpha_p} : \bigcap_{i=1}^{n} U_{\alpha_i} \longrightarrow \mathscr{A}^q \left(\bigcap_{i=1}^{n} U_{\alpha_i} \right)$$

for any combination of p distinct indices $\alpha_1, \ldots, \alpha_p \in I$. For indices $\alpha_1, \ldots, \alpha_{p-1} \in I$ we define ψ to be

$$\psi_{\alpha_0 \cdots \alpha_{p-1}} = \sum_{\beta \in I} \varrho_\beta \omega_{\beta \alpha_0 \cdots \alpha_{p-1}}.$$

We have to prove that $\omega = d\psi$. On $\bigcap_{i=1}^{p} U_{\alpha_i}$

$$\begin{aligned}
(d\psi)_{\alpha_0 \cdots \alpha_p} &= \sum_{i=0}^{p} (-1)^i \psi_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_p}|_{U_{\alpha_i}} \\
&= \sum_{i=0}^{p} \sum_{\beta \in I} (-1)^i \varrho_\beta|_{U_{\alpha_i}} \omega_{\beta \alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_p}|_{U_{\alpha_i}} \\
&= \sum_{i=0}^{p} \sum_{\beta \in I} (-1)^i \varrho_\beta|_{\bigcap_{j=0}^{p} U_{\alpha_j}} \omega_{\beta \alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_p}|_{U_{\alpha_i}} \\
&= \sum_{\beta \in I} \varrho_\beta|_{\bigcap_{j=0}^{p} U_{\alpha_j}} \sum_{i=0}^{p} (-1)^i \omega_{\beta \alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_p}|_{U_{\alpha_i}} \\
&= \sum_{\beta \in I} \varrho_\beta|_{\bigcap_{j=0}^{p} U_{\alpha_j}} \omega_{\alpha_0 \cdots \alpha_p}|_{U_\beta} \\
&= \sum_{\beta \in I} \varrho_\beta \omega_{\alpha_0 \cdots \alpha_p} = \omega_{\alpha_0 \cdots \alpha_p}.
\end{aligned}$$

where in the third line we used that on $\bigcap_{i=0}^{n} U_{\alpha_i}$ and all $i \in \{0, \ldots, p\}$

$$\varrho_\beta|_{\bigcap_{j=0}^{p} U_{\alpha_j}} \equiv \varrho_\beta|_{U_{\alpha_i}},$$

in the forth the fact that $\omega \in Z^p$ and thus

$$\omega_{\alpha_0\cdots\alpha_p}\big|_{U_\beta} - \sum_{i=0}^{p}(-1)^i \omega_{\beta\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p}\big|_{U_{\alpha_i}} = 0,$$

in the last line that on $\bigcap_{i=1}^n U_{\alpha_i}$

$$\varrho_\beta\big|_{\bigcap_{j=0}^p U_{\alpha_j}} \omega_{\alpha_1\cdots\alpha_p}\big|_{U_\beta} \equiv \varrho_\beta \omega_{\alpha_1\cdots\alpha_p},$$

and that $(\varrho_\beta)_\beta$ is a partition of unity. □

Lemma 3.2 *The sheaf cochain complex*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \cdots$$

is exact, that is it splits into short exact sequences

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{Z}^1 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \mathscr{Z}^p \hookrightarrow \mathscr{A}^p \xrightarrow{d} \mathscr{Z}^{p+1} \longrightarrow 0$$

$$\vdots$$

Proof. By definition, the cochain complex is exact iff it is exact on stalks. Thus, it suffices to prove that for every $x \in M$ there exists $x \in U \subset M$ open and sufficiently small, s.t.

$$\cdots \longrightarrow \mathscr{A}^{p-1}(U) \xrightarrow{d} \mathscr{A}^p(U) \xrightarrow{d} \mathscr{A}^{p+1}(U) \xrightarrow{d} \cdots$$

is exact as a cochain complex of Abelian groups. However, this is exactly the content of the classical Poincaré Lemma, see for example [Lee12, Theorem 17.14]. If U is diffeomorphic to a star-shaped open subset of $\mathbb{R}^n$, e.g. diffeomorphic via a chart to an open ball, then every closed differential form is exact. The second statement follows then as ker $\left(d^p|_U\right) = \mathscr{Z}^p(U) =$ im $\left(d^{p+1}|_U\right)$. □

The following important result allows one to relate the Čech cohomology group of M with the de Rham cohomology group and to have a criterion for when a cover is fine enough.

Theorem 3.3 (De Rham's Theorem) *There exists an isomorphism*

$$\check{H}(M;\mathbb{R}) \cong H_{\mathrm{dR}}(M;\mathbb{R}).$$

and in particular

$$\check{H}(M;\mathbb{R}) \cong \check{H}(\mathcal{U};\mathbb{R})$$

for any open cover wherein each set is contractible.

Proof. From the short exact sequences in 3.2 we can deduce the existence of the following corresponding long exact sequences on cohomology: for all $p, q \in \mathbb{N}$

$$\begin{aligned} \cdots \longrightarrow H^{p-1}(M, \mathscr{A}^{q-1}) \longrightarrow H^{p-1}(M, \mathscr{Z}^{q}) \longrightarrow \\ \longrightarrow H^{p}(M, \mathscr{Z}^{q-1}) \longrightarrow H^{p}(M, \mathscr{A}^{q-1}) \longrightarrow \cdots . \end{aligned}$$

For $p > 1$ the leftmost and rightmost terms are zero thus

$$H^{p-1}(M, \mathscr{Z}^{q}) \cong H^{p}(M, \mathscr{Z}^{q-1}).$$

This gives us the following sequence of isomorphisms

$$\begin{aligned} \check{H}^{p}(M, \mathbb{R}) := H^{p}(M, \mathbb{R}) \cong H^{p-1}(M, \mathscr{Z}^{1}) \cong \\ \cong H^{p-2}(M, \mathscr{Z}^{2}) \cong \cdots \cong H^{1}(M, \mathscr{Z}^{p-1}). \end{aligned}$$

For $p = 1$ (in the above sense) we get the following exact sequence

$$H^{0}(M, \mathscr{A}^{p-1}) \longrightarrow H^{0}(M, \mathscr{Z}^{p}) \longrightarrow H^{1}(M, \mathscr{Z}^{p-1}) \longrightarrow 0.$$

Since the 0^{th} order cohomology is simply the space of global sections the above is isomorphic

$$\mathscr{A}^{p-1}(M) \xrightarrow{d} \mathscr{Z}^{p}(M) \longrightarrow H^{1}(M, \mathscr{Z}^{p-1}) \longrightarrow 0.$$

By exactness it follows therefore that

$$H^{1}(M, \mathscr{Z}^{p-1}) \cong \mathscr{Z}^{p}(M)/d\mathscr{A}^{p-1}(M) =: H^{p}_{\text{dR}}(M, \mathbb{R}).$$

This proves the assertion. □

References

[Gri94] P. GRIFFITHS. Principles of algebraic geometry. Wiley classics library. Wiley, New York ;, 1978 - 1994.

[Lee12] J. LEE. Introduction to Smooth Manifolds. Graduate Texts in Mathematics, 218. Springer New York, New York, NY, 2nd ed., 2012.