Complex Geometry Project

April 24, 2022

Akshunna S. Dogra[a1, 2]

[1]Department of Mathematics, Imperial College London
[2]EPSRC CDT in Mathematics of Random Systems: Analysis, Modelling and Simulation

Imperial College London

[a]adogra@nyu.edu

Abstract

In this essay we go over the proof of the isomorphism between the Čech cohomology $\check{H}(M;\mathbb{R})$ and the de Rham cohomology $H_{\text{dR}}(M;\mathbb{R})$ of a smooth manifold M.

Definition 0.1
- *$\mathscr{A}^p$ sheaf of smooth p-forms on M*
- *$\mathscr{Z}^p$ sheaf of closed p-forms on M*

Remark 0.2 For $p = 0$, $\mathscr{Z}^0$ is simply the constant sheaf $\underline{\mathbb{R}}$.

Lemma 0.3 *For $p > 0$ and all $q \geqslant 0$, the* Čech *cohomology groups of the sheaves $\mathscr{A}^q$ are trivial, i.e. $\check{H}^p(M, \mathscr{A}^q) = 0$.*

Proof. It is enough to prove that this holds for any locally finite cover $\mathcal{U}$ of M. Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be such a cover and let $(\varrho_\alpha)_{\alpha\in I}$ be a corresponding subordinate, i.e. supp$(\varrho_\alpha) \subset U_\alpha$, partition of unity.

Let $\omega \in Z^p(\mathcal{U}, \mathscr{A}^q)$ we have to find a $\psi \in C^{p-1}(\mathcal{U}, \mathscr{A}^q)$ such that $d\psi = \omega$. ω consists of a collection of maps

$$\omega_{\alpha_1\cdots\alpha_p} : \bigcap_{i=1}^{n} U_{\alpha_i} \longrightarrow \mathscr{A}^q\left(\bigcap_{i=1}^{n} U_{\alpha_i}\right)$$

for any combination of p distinct indices $\alpha_1, \ldots, \alpha_p \in I$. For indices $\alpha_1, \ldots, \alpha_{p-1} \in I$ we define ψ to be

$$\psi_{\alpha_1\cdots\alpha_{p-1}} = \sum_{\beta\in I} \varrho_\beta \omega_{\beta\alpha_1\cdots\alpha_{p-1}}.$$

We have to prove that $\omega = d\psi$. On $\bigcap_{i=1}^p U_{\alpha_i}$

$$
\begin{aligned}
(d\psi)_{\alpha_1\cdots\alpha_p} &= -\sum_{i=1}^p (-1)^i \psi_{\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_p}|_{U_{\alpha_i}} \\
&= -\sum_{i=1}^p \sum_{\beta\in I} (-1)^i \varrho_\beta|_{U_{\alpha_i}} \omega_{\beta\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_p}|_{U_{\alpha_i}} \\
&= -\sum_{i=1}^p \sum_{\beta\in I} (-1)^i \varrho_\beta|_{\bigcap_{j=1}^p U_{\alpha_j}} \omega_{\beta\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_p}|_{U_{\alpha_i}} \\
&= -\sum_{\beta\in I} \varrho_\beta|_{\bigcap_{j=1}^p U_{\alpha_j}} \sum_{i=1}^p (-1)^i \omega_{\beta\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_p}|_{U_{\alpha_i}} \\
&= \sum_{\beta\in I} \varrho_\beta|_{\bigcap_{j=1}^p U_{\alpha_j}} \omega_{\alpha_1\cdots\alpha_p}|_{U_\beta} \\
&= \sum_{\beta\in I} \varrho_\beta \omega_{\alpha_1\cdots\alpha_p} = \omega_{\alpha_1\cdots\alpha_p}.
\end{aligned}
$$

where in the third line we used that on $\bigcap_{i=1}^n U_{\alpha_i}$ and all $i \in \{1,\ldots,p\}$

$$\varrho_\beta|_{\bigcap_{j=1}^p U_{\alpha_j}} \equiv \varrho_\beta|_{U_{\alpha_i}},$$

in the forth the fact that $\omega \in Z^p$ and thus

$$\omega_{\alpha_1\cdots\alpha_p}|_{U_\beta} + \sum_{i=1}^p (-1)^i \omega_{\beta\alpha_1\cdots\widehat{\alpha_i}\cdots\alpha_p}|_{U_{\alpha_i}} = 0,$$

in the last line that on $\bigcap_{i=1}^n U_{\alpha_i}$

$$\varrho_\beta|_{\bigcap_{j=1}^p U_{\alpha_j}} \omega_{\alpha_1\cdots\alpha_p}|_{U_\beta} \equiv \varrho_\beta \omega_{\alpha_1\cdots\alpha_p},$$

and that $(\varrho_\beta)_\beta$ is a partition of unity. $\square$

Lemma 0.4 *The sheaf cochain complex*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \cdots$$

is exact, that is it splits into short exact sequences

$$
\begin{array}{c}
0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{Z}^1 \longrightarrow 0 \\
\vdots \\
0 \longrightarrow \mathscr{Z}^p \hookrightarrow \mathscr{A}^p \xrightarrow{d} \mathscr{Z}^{p+1} \longrightarrow 0 \\
\vdots
\end{array}
$$

Proof. By definition, the cochain complex is exact iff it is exact on stalks. Thus, it suffices to prove that for every $x \in M$ there exists $x \in U \subset M$ open and sufficiently small, s.t.

$$\cdots \longrightarrow \mathscr{A}^{p-1}(U) \xrightarrow{d} \mathscr{A}^{p}(U) \xrightarrow{d} \mathscr{A}^{p+1}(U) \xrightarrow{d} \cdots$$

is exact as a cochain complex of Abelian groups. However, this is exactly the content of the classical Poincaré Lemma, see for example [Lee12, Theorem 17.14]. If U is diffeomorphic to a star-shaped open subset of $\mathbb{R}^n$, e.g. diffeomorphic via a chart to an open ball, then every closed differential form is exact. The second statement follows then as ker $\left(d^p|_U\right) = \mathscr{Z}^p(U) = $ im $\left(d^{p+1}|_U\right)$. □

Theorem 0.5 (De Rham's Theorem) *There exists an isomorphism*

$$\check{H}(M;\mathbb{R}) \cong H_{\mathrm{dR}}(M;\mathbb{R}).$$

Proof. From the short exact sequences in 0.4 we can deduce the existence of the following corresponding long exact sequences on cohomology: for all $p, q \in \mathbb{N}$

$$\begin{aligned} \cdots \longrightarrow H^{p-1}(M, \mathscr{A}^{q-1}) \longrightarrow H^{p-1}(M, \mathscr{Z}^{q}) \longrightarrow \\ \longrightarrow H^{p}(M, \mathscr{Z}^{q-1}) \longrightarrow H^{p}(M, \mathscr{A}^{q-1}) \longrightarrow \cdots . \end{aligned}$$

For $p > 1$ the leftmost and rightmost terms are zero thus

$$H^{p-1}(M, \mathscr{Z}^{q}) \cong H^{p}(M, \mathscr{Z}^{q-1}).$$

This gives us the following sequence of isomorphisms

$$\begin{aligned} \check{H}^p(M,\mathbb{R}) := H^p(M,\mathbb{R}) \cong H^{p-1}(M, \mathscr{Z}^1) \cong \\ \cong H^{p-2}(M, \mathscr{Z}^2) \cong \cdots \cong H^1(M, \mathscr{Z}^{p-1}). \end{aligned}$$

For $p = 1$ (in the above sense) we get the following exact sequence

$$H^0(M, \mathscr{A}^{p-1}) \longrightarrow H^0(M, \mathscr{Z}^{p}) \longrightarrow H^1(M, \mathscr{Z}^{p-1}) \longrightarrow 0.$$

Since the 0^{th} order cohomology is simply the space of global sections the above is isomorphic

$$\mathscr{A}^{p-1}(M) \xrightarrow{d} \mathscr{Z}^{p}(M) \longrightarrow H^1(M, \mathscr{Z}^{p-1}) \longrightarrow 0.$$

By exactness it follows therefore that

$$H^1(M, \mathscr{Z}^{p-1}) \cong \mathscr{Z}^p(M)/d\mathscr{A}^{p-1}(M) =: H^p_{\mathrm{dR}}(M,\mathbb{R}).$$

This proves the assertion. □

References

[Lee12] J. LEE. Introduction to Smooth Manifolds. Graduate Texts in Mathematics, 218. Springer New York, New York, NY, 2nd ed., 2012.