A C^*-Theoretic Anomaly Calculation

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Abstract

In this essay we go over the proof of the isomorphism between the Č cohomology $\check{H}(M;\mathbb{R})$ and the de Rham cohomology $H_{\mathrm{dR}}(M;\mathbb{R})$ of a smooth manifold M.

Definition 0.1
- *$\mathscr{A}^p$ sheaf of smooth p-forms on M*
- *$\mathscr{Z}^p$ sheaf of closed p-forms on M*

Lemma 0.2 *For $p > 0$ and all $q \geqslant 0$, the Čech cohomology groups of the sheaves $\mathscr{A}^q$ are trivial, i.e. $\check{H}^p(M, \mathscr{A}^q) = 0$.*

Lemma 0.3 *The sheaf cochain complex*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \xrightarrow{d} \mathscr{A}^2 \xrightarrow{d} \cdots$$

is exact, that is it splits into short exact sequences

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{Z}^1 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \mathscr{Z}^p \hookrightarrow \mathscr{A}^p \xrightarrow{d} \mathscr{Z}^{p+1} \longrightarrow 0$$

$$\vdots$$

Proof. By definition, the cochain complex is exact iff it is exact on stalks. Thus, it suffices to prove that for every $x \in M$ there exists $x \in U \subset M$ open and sufficiently small, s.t.

$$\cdots \longrightarrow \mathscr{A}^{p-1}(U) \xrightarrow{d} \mathscr{A}^p(U) \xrightarrow{d} \mathscr{A}^{p+1}(U) \xrightarrow{d} \cdots$$

is exact as a cochain complex of Abelian groups. However, this exactly the content of the classical Poincaré Lemma, see for example [Lee12, Theorem 17.14]. If U is diffeomorphic to a star-shaped open subset of $\mathbb{R}^n$, e.g. diffeomorphic via a chart to an open ball, then every closed differential form is exact. $\square$

References

[Lee12] J. LEE. Introduction to Smooth Manifolds. Graduate Texts in Mathematics, 218. Springer New York, New York, NY, 2nd ed., 2012.