Ex9 186 Fall

Assignment group 10

1 Exercise 9.1

1.1 i

First, we use the l'Hopital's rule:

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

but we cannot prove that $\lim_{x\to\infty}\frac{e^x+e^{-x}}{e^x-e^{-x}}$ and $\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$ necessaily exist by just using l'Hopital's rule.
Another approach:
Because $e^{-x}>0$ is always true. So we have

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{e^{2x}-1}{e^{2x}+1}=1$$

1.2 ii

First, we use the l'Hopital's rule:

$$\lim_{x\to 0}\frac{x^2\cos(\frac{1}{x})}{\sin x}=\lim_{x\to 0}\frac{2x\cos(\frac{1}{x})+\sin(\frac{1}{x})}{\cos x}$$

However, denominator $\cos x$ does not converge to 0 or $+\infty$. So, l'Hopital's rule cannot be used.
Another approach:

$$\lim_{x\to 0}\frac{x^2\cos(\frac{1}{x})}{\sin x}=\lim_{x\to 0}(\frac{x}{\sin x})\cdot\lim_{x\to 0}(x\cos(\frac{1}{x}))=1\times 0=0$$

1.3 iii

$$\begin{aligned}\lim_{x\to\infty}\frac{f'(x)}{g'(x)}&=\lim_{x\to\infty}\frac{1+\cos^2 x-\sin^2 x}{e^{\sin x}\cos x f(x)+f'(x)e^{\sin x}}\\&=\lim_{x\to\infty}\frac{2\cos x}{e^{\sin x}(x+\sin x\cos x+2\cos x)}\end{aligned}\tag{1}$$

Because $\lim_{x\to\infty}e^{\sin x}(x+\sin x\cos x+2\cos x)=\lim_{x\to\infty}(e^{\sin x}x+e^{\sin x}(\sin x\cos x+2\cos x))$, and $e^{\sin x}\geq e^{-1}$.
Thus, $\lim_{x\to\infty}(e^{\sin x}x+e^{\sin x}(\sin x\cos x+2\cos x))=\infty$.
So $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=0$
Firstly,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f(x)}{f(x)e^{\sin x}}$$

Because when $x\to\infty, f(x)\neq 0$, So, $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}e^{-\sin x}$. Because $\lim_{x\to\infty}e^{-\sin x}$ does not exist.
Thus, $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ does not exist. And this doesn't contradict with l'Hopital's rule.

2 Exercise 9.2

2.1 i

The function we find is

$$f(x) = \sin(x^2) \quad x \in \mathbb{R}$$

We can know that $\sup\limits_{x\in\mathbb{R}} |f(x)| = |\sin(x^2)| = 1$, and

$$\sup_{x\in\mathbb{R}} |f'(x)| = \sup_{x\in\mathbb{R}} |x^2 \cdot \cos(x^2)| = \infty \quad \text{as } x \to \infty$$

2.2 ii

The function we find is

$$f(x) = \ln(x) \quad x \in \mathbb{R}$$

We can know that $\sup\limits_{x\in\mathbb{R}} |f(x)| = |\ln(x)| = \infty$, and

$$\sup_{x\in\mathbb{R}} |f'(x)| = \sup_{x\in\mathbb{R}} |\frac{1}{x}| = 0 \quad \text{as } x \to \infty$$

3 Exercise 9.3

3.1 i

Prove:

$$\min_{1\le i\le n} x_i = \sum_{i=1}^{n} \lambda_i \min_{1\le i\le n} x_i \le \sum_{i=1}^{n} \lambda_i x_i \le \sum_{i=1}^{n} \lambda_i \max_{1\le i\le n} x_i = \max_{1\le i\le n} x_i$$

3.2 ii

Prove:

$$\min_{1\le i\le n} x_i \le \min_{1\le i\le n-1} x_i = \frac{1}{t}\sum_{i=1}^{n-1} \lambda_i \min_{1\le i\le n-1} x_i \le \frac{1}{t}\sum_{i=1}^{n-1} \lambda_i x_i \le \frac{1}{t}\sum_{i=1}^{n-1} \lambda_i \max_{1\le i\le n-1} x_i = \max_{1\le i\le n-1} x_i \le \max_{1\le i\le n} x_i$$

3.3 iii

Prove:

Note that:

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) = f\left(\lambda_n x_n + \sum_{i=1}^{n-1} \lambda_i x_i\right) = f\left(\lambda_n x_n + (1-\lambda_n)\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i\right)$$

(according to Ex6.4 i) since $\frac{1}{t}\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i \in \mathrm{I}$)

$$\le \lambda_n f(x_n) + (1-\lambda n) f\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i\right)$$

Then use mathematic induction:

1) When n=2, it is the same as Ex 6.4

2) We assume that when n=k it is correct, which means that,

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \le \sum_{i=1}^{k} \lambda_i f(x_i)$$

then,

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right) \leq \lambda_{k+1} f\left(x_{k+1}\right)+(1-\lambda_{k+1}) f\left(\sum_{i=1}^{k}\frac{\lambda_i}{1-\lambda_{k+1}}x_i\right) \leq \lambda_{k+1} f\left(x_{k+1}\right)+(1-\lambda_{k+1})\sum_{i=1}^{k}\frac{\lambda_i}{1-\lambda_{k+1}}f\left(x_i\right) = \sum_{i=1}^{k+1}$$

Then we can conclude that

$$f\left(\sum_{i=1}^{n}\lambda_i x_i\right) \leq \sum_{i=1}^{n}\lambda_i f\left(x_i\right)$$

3.4 iv

We know that the function $f\left(x\right) := -ln(x)$ is a convex function.So,

$$-ln\left(\sum_{i=1}^{n}\lambda_i x_i\right) = f\left(\sum_{i=1}^{n}\lambda_i x_i\right) \leq \sum_{i=1}^{n}\lambda_i f\left(x_i\right) = -\sum_{i=1}^{n}\lambda_i ln\left(x_i\right)$$

$$ln\left(\Pi_{i=1}^{n}x_i^{\lambda_i}\right) = \sum_{i=1}^{n}\lambda_i ln\left(x_i\right) \leq ln\left(\sum_{i=1}^{n}\lambda_i x_i\right)$$

Thus,

$$\Pi_{i=1}^{n}x_i^{\lambda_i} \leq \sum_{i=1}^{n}\lambda_i x_i$$

4 Exercise 9.4

$f(x) = \begin{cases} x^4\sin^2(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

It's apparent that for $\forall x \neq 0$, $f(x) \geq 0$. Because $f(0) = 0$, then according to the definition of minimum points, 0 is a local minimum point of f.

Now,

$$\begin{aligned} f'(x) &= \begin{cases} 4x^3\sin^2(\frac{1}{x}) + x^4\cdot(-1)\frac{1}{x^2}\cdot 2\sin\frac{1}{x}\cos\frac{1}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases} \\ &= \begin{cases} 4x^3\sin^2(\frac{1}{x}) - 2x^2\sin\frac{1}{x}\cos\frac{1}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases} \end{aligned}$$

As $f(0+h) = f(h) = 0 + f(h) \leq 0 + h^4 = 0+\text{o}(h)$ as $h \to 0$, so $f'(0) = 0$.

Finally, $f'(0+h) = f'(h) = 0 + f'(h) \leq 0 + 4x^3 - 2x^2 = 0+\text{o}(h)$ as $x \to 0$, so $f''(0) = 0$.

5 Exercise 9.5

5.1 9.5.1

Because $z^7 = 3 + 4i$, so $|z|^7 = 5$ and we can know $|z| = \sqrt[7]{5} := a$ Then, we assume that $z = a\cos\theta + ai\sin\theta$, θ is a fixed number that lays in the interval of $[0, 2\pi)$. So we can have the equation :

$$7\theta = \arcsin(\frac{4}{5}) + 2k\pi \quad k \in \mathbb{N}$$

$$\theta = \frac{\arcsin(\frac{4}{5}) + 2k\pi}{7} \quad k \in \mathbb{N}$$

We can know there are $k = 0, 1, 2, 3, 4, 5, 6$ that can make the equation correct.

So, we can get the answer:

$z = \sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})}{7} + \sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})}{7}$ or $z = \sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})+\pi}{7} + \sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})+\pi}{7}$

or $z=\sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})+2\pi}{7}+\sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})+2\pi}{7}$ or $z=\sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})+3\pi}{7}+\sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})+3\pi}{7}$ or $z=\sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})+4\pi}{7}+\sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})+4\pi}{7}$ or $z=\sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})+5\pi}{7}+\sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})+5\pi}{7}$ or $z=\sqrt[7]{5}\cos\frac{\arcsin(\frac{4}{5})+6\pi}{7}+\sqrt[7]{5}i\sin\frac{\arcsin(\frac{4}{5})+6\pi}{7}$.

5.2 9.5.2

Assume that $z=:a+bi \quad a,b\in\mathbb{R}$, we can substitute it into the equation and we get

$$(a^2-b^2+b+1)+(2ab-a)i=0$$

Then, we know :

$$\begin{cases} a^2-b^2+b+1=0 \\ 2ab-a=0 \end{cases}$$

1.When $a=0$, we can also know b^2-b-1 and thus $b=\frac{1\pm\sqrt{5}}{2}$.
2.When $a\neq 0$, we can also know $b=\frac{1}{2}$, we can thus get $a^2+\frac{5}{4}=0$, which contradicts with $a\in\mathbb{R}$

$$z=\frac{1+\sqrt{5}}{2}i \quad \text{or} \quad z=\frac{1-\sqrt{5}}{2}i$$

5.3 9.5.3

We assume $z^2=:x+yi \quad (x,y\in\mathbb{C})$ and $z=:a+bi \quad (a,b\in\mathbb{R})$

$$\begin{gathered}(z^2)^2+z^2+1=0 \\ z^2=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\end{gathered} \tag{2}$$

$z^2=(a^2-b^2)+2abi$, so we can know:

$$\begin{cases} a^2-b^2=-\frac{1}{2} \\ ab=\pm\frac{\sqrt{3}}{4} \end{cases}$$

We can know $a=\pm\frac{\sqrt{3}}{4b}$ and substitute it into the equation set. We can know $\begin{cases} a=\frac{1}{2} \\ b=\frac{\sqrt{3}}{2} \end{cases}$ and $\begin{cases} a=-\frac{1}{2} \\ b=-\frac{\sqrt{3}}{2} \end{cases}$

5.4 9.5.4

$$\begin{cases} iz-(1+i)w=3 \\ (2+i)z+iw=4 \end{cases}$$

We can get:

$$\begin{cases} -z-i(1+i)w=3i \\ (1+i)(2+i)z+i(1+i)w=4 \end{cases}$$

We can solve the equation set and get the solution: $\begin{cases} z=\frac{7}{3}-\frac{4}{3}i \\ w=\frac{1}{3}+2i \end{cases}$

6 9.6

6.1 9.6.1

$$\begin{aligned}\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\ &= \frac{\sin x\cos y + \sin y\cos x}{\cos x\cos y - \sin x\sin y} \quad \text{(divide numerator and denominator by } \cos x\cos y) \\ &= \frac{\tan x + \tan y}{1 - \tan x\tan y}\end{aligned} \tag{3}$$

Because $\cos(x+y) \neq 0$ and $\tan(x+y)$ exists. Thus, we can get $x+y \neq \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z})$

6.2 9.6.2

$$\arctan x + \arctan y = \arctan(\frac{x+y}{1-xy})$$

Assume $x := \tan\alpha, y := \tan\beta \quad (\alpha, \beta \in (-\frac{\pi}{2}, \frac{\pi}{2}), x, y \in \mathbb{R}, x \cdot y \neq 1)$, Thus, we can get

$$\arctan(\tan\alpha) + \arctan(\tan\beta) - \arctan(\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}) = \alpha + \beta - \arctan(\tan(\alpha+\beta)) = 0$$

So, we get the proof.

6.3 9.6.3

We first establish that:

$$\frac{d}{dx}(\arctan x + \arctan\frac{1}{x}) = \frac{1}{x^2+1} - \frac{1}{x^2} \cdot \frac{1}{1+(\frac{1}{x})^2} = 0$$

Thus, $\forall x \in \mathbb{R}$, for some $c \in \mathbb{R}$

$$\arctan x + \arctan\frac{1}{x} = c$$

Then, $\forall x \in \mathbb{R}$, since $\arctan 1 = \frac{\pi}{4}$, we see that

$$\arctan x + \arctan\frac{1}{x} = \arctan 1 + \arctan 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

7 Exercise 9.7

7.1 9.7.1

First, we would show that if f is continuous at $a \in \mathbb{R}$ and $f(a) > 0$, then $f > 0$ on some $B_\epsilon(a)$. According to the definition of continuity:

$$\underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\operatorname{dom} f}{\forall}\ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

Let $f(a) = \epsilon > 0$ and $B_\epsilon(a) = (a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2})$ and that proves the lemma.

Now, because $f'(x) > 0$, according to the lemma, $f'(x) > 0$ on some $B_\epsilon(x)$, so on some $B_\epsilon(x)$, $f(x)$ is increasing.

7.2 9.7.2

Because the function, $f(x) = \alpha x + x^2 \sin(\frac{1}{x})$ is an odd function, an "arbitrary interval containing 0" can be reduced to an "arbitrary interval $[0, x)$, where $x > 0$".
We can know that

$$f'(x) = \alpha + 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

and as $x \to 0$, the limit $\lim_{x \to 0} f'(x)$ does not exist, because though α is a constant and $\lim_{x \to 0} 2x \sin(\frac{1}{x})$ converges to 0, $\cos(\frac{1}{x})$ is "fluctuating" between 0 and 1 near 0.

Now select an arbitrary $x_1 \in \mathbb{R}^+$. Because $\frac{1}{x} \to +\infty$ as $x \to 0$, we can select an x_1 such that $\frac{1}{x}$ is an integral multiple of 2π. Similarily, we can select an x_2 such that $\frac{1}{x}$ is an integral multiple of -2π.

Because $\alpha \in (0)$, we can have x_1 such that $f'(x_1) > 0$ and x_2 such that $f'(x_2) < 0$.