Ex7 186 Fall

Assignment group 10

1 Ex7.4

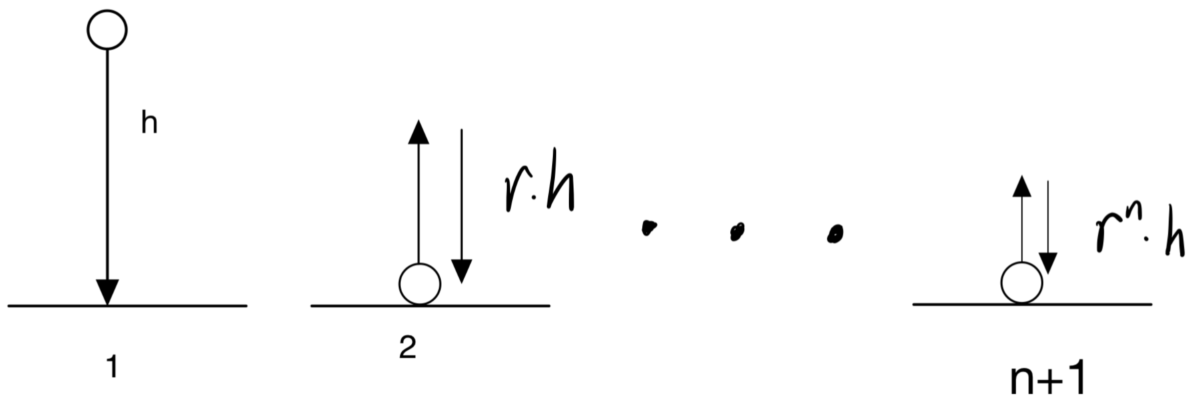

Figure 1: Figure 7.4

The distance can be assessed through the image above (figure Ex7.4).

The first process : h

The second process : $2r \cdot h$

...

The nth process : $2r^{n-1} \cdot h$

Except for the first process is one-way, other process are all double-way Thus, the total distance :

$$\begin{aligned} D &= 2(\sum_{n=1}^{\infty} r^{n-1} \cdot h) - h \\ &= \frac{2h}{1-r} - h \end{aligned} \tag{1}$$

2 Ex7.5

We can know from the question that

$$\sum \frac{1}{n} = \sum_{n \in X} \frac{1}{n} + \sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n}$$

Because

$$\begin{aligned} \sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n} &= \frac{1}{9} + \frac{1}{81} + \cdots + \frac{1}{9^k} + \ldots \\ &= \frac{\frac{1}{9}}{1 - \frac{1}{9}} \\ &= \frac{1}{8} \end{aligned} \tag{2}$$

So, $\sum_{n\in\mathbb{N}^*\setminus X} \frac{1}{n}$ converges.

We suppose that $\sum_{n\in X} \frac{1}{n}$converges. Based on that, we can know $\sum \frac{1}{n} = \sum_{n\in X} \frac{1}{n} + \sum_{n\in\mathbb{N}^*\setminus X} \frac{1}{n}$ also converges, which is contridict with the fact that $\sum \frac{1}{n}$ diverges. So, $\sum_{n\in X} \frac{1}{n}$ diverges.

3 Ex7.6

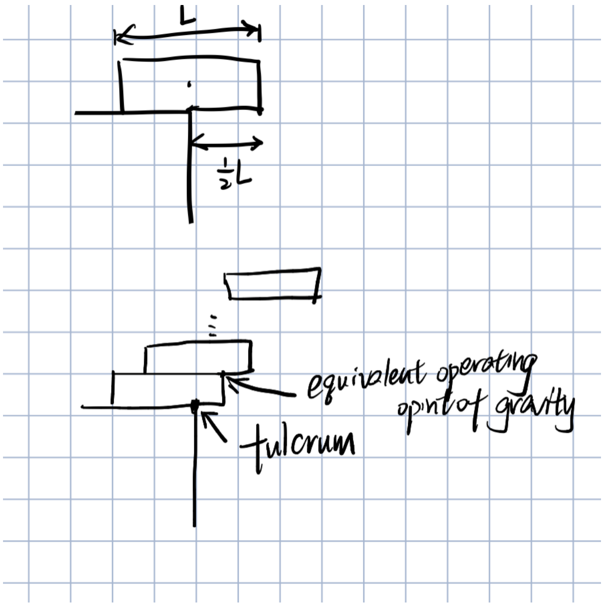

Figure 2: Figure 7.6.1

We can model the question(figure 7.6.1). When there are $n+1$ bricks, assume that the mass of each brick is m, the acceleration of gravity is g, the length of the lever is L and the distance between the middlepoint and the right end is l_{n+1}. We can then simplify it into a lever (show in figure 7.6.2).

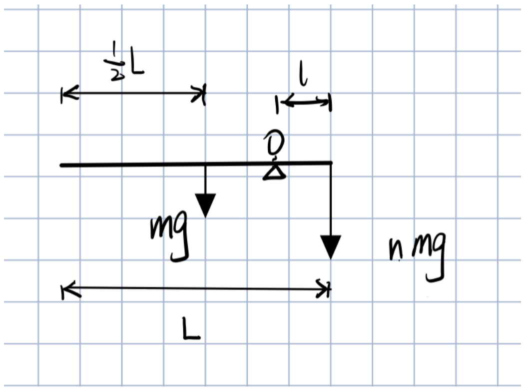

Figure 3: Figure 7.6.2

According to lever principle, we can know that

$$\begin{aligned} mg(\frac{L}{2} - l_{n+1}) &= nmgl_{n+1} \\ \frac{1}{2}mgL - mgl_{n+1} &= nmgl_{n+1} \\ l_{n+1} &= \frac{L}{2(n+1)} \\ l_{n+1} &= \frac{L}{2} \cdot \frac{1}{n+1} \end{aligned} \tag{3}$$

So we can know $\sum_{n=1}^{\infty} l_{n+1}$ diverges. Thus, the tower can extend to infinite far.

4 Ex7.7

4.1 7.7.1

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{2n} \cdot 3^{3n}}{5^{3n}} &= \sum_{n=1}^{\infty} \frac{4^n 27^n}{125^n} \\ &= (\frac{108}{125})^n \\ &= \frac{108}{17}\end{aligned} \tag{4}$$

So, we can know the $\sum_{n=1}^{\infty} \frac{2^{2n} \cdot 3^{3n}}{5^{3n}}$ converges.

4.2 7.7.2

Because we know when $n > 3$, $n^2 - 3n + 1 > 0$ Thus, we can know:

$$\begin{aligned}a_n := \sum_{n=1}^{\infty} \frac{n+4}{n^2 - 3n + 1} &= \sum_{n=1}^{3} \frac{n+4}{n^2 - 3n + 1} + \sum_{n=4}^{\infty} \frac{n+4}{n^2 - 3n + 1} \\ &> \sum_{n=1}^{3} \frac{n+4}{n^2 - 3n + 1} + \sum_{n=4}^{\infty} \frac{n+4}{n^2 + 16n + 16} \\ &= \sum_{n=1}^{3} \frac{n+4}{n^2 - 3n + 1} + \sum_{n=4}^{\infty} \frac{1}{n+4} =: b_n\end{aligned} \tag{5}$$

Because $\sum_{n=4}^{\infty} \frac{1}{n+4}$ diverges, so b_n diverges.
Thus $a_n = \sum_{n=1}^{\infty} \frac{n+4}{n^2 - 3n + 1}$ diverges as $0 < b_n < a_n$.

4.3 7.7.3

Let $a_n := \frac{n^4}{3^n}$. We will use deduction to prove that when $n \geq 32$, $n^4 < 2^n$.
Firstly, when $n = 32$, $(32)^4 = 2^{20} < 2^{32}$
Secondly, assume that when $n = k, n \in \mathbb{N}^*, k \geq 32$, we also have $2^k > k^4$.
So, when $n = k + 1$, because

$$\frac{(k+1)^4}{k^4} = (\frac{k+1}{k})^4 < (\frac{33}{32})^4 < 2$$

Thus,

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k^n > (k+1)^4$$

So, we have proved that when $n \geq 32$, $n^4 < 2^n$.
We get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{32} a_n + \sum_{n=33}^{\infty} \frac{n^4}{3^n} < \sum_{n=1}^{32} a_n + \sum_{n=33}^{\infty} (\frac{2}{3})^n := b_n$$

Because we know that $\sum_{n=33}^{\infty} (\frac{2}{3})^n$ converges. So, b_n converges. Thus a_n converges as $0 < a_n < b_n$,

4.4 7.7.4

Let $a_n := \frac{2^n}{n!}$, Then,

$$\lim_{x\to\infty}\frac{a_{n+1}}{a_n} = \lim_{x\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{x\to\infty}\frac{2}{n+1} = 0$$

Thus,$a_n = \frac{2^n}{n!}$ converges.

4.5 7.7.5

Let $a_n := \frac{2^n}{n^n}$. Then,

$$\lim_{x\to\infty}\frac{a_{n+1}}{a_n} = \lim_{x\to\infty}\frac{2n^n}{(n+1)^(n+1)} = \lim_{x\to\infty} 2\cdot(\frac{n}{n+1})^n\cdot(\frac{1}{n+1}) = 0$$

Thus, $a_n = \frac{2^n}{n^n}$ converges.

4.6 7.7.6

$\sum_{n=1}^{\infty}\frac{n}{10n^3-100}$