Ex7 186 Fall

Assignment group 10

1 Ex7.4

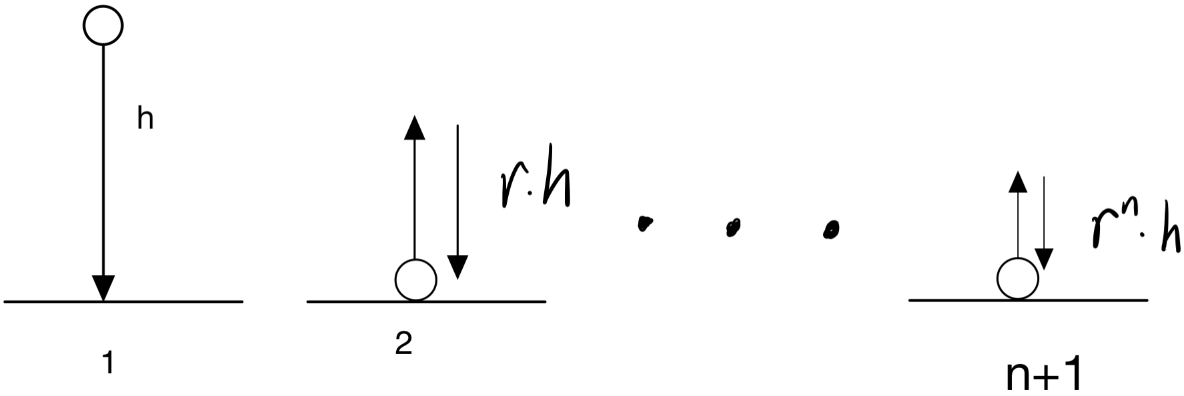

Figure 1: Figure 7.4

The distance can be assessed through the image above (figure Ex7.4).

The first process : h

The second process : $2r \cdot h$

...

The nth process : $2r^{n-1} \cdot h$

Except for the first process is one-way, other process are all double-way Thus, the total distance :

$$
\begin{aligned}
D &= 2(\sum_{n=1}^{\infty} r^{n-1} \cdot h) - h \\
&= \frac{2h}{1-r} - h
\end{aligned} \tag{1}
$$

2 Ex7.5

We can know from the question that

$$\sum \frac{1}{n} = \sum_{n \in X} \frac{1}{n} + \sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n}$$

Because

$$
\begin{aligned}
\sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n} &= \frac{1}{9} + \frac{1}{81} + \cdots + \frac{1}{9^k} + \dots \\
&= \frac{1}{9} \sum \frac{1}{n}
\end{aligned} \tag{2}
$$

So, $\sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n}$ diverges.

Then, we can find that $\sum_{n \in X} \frac{1}{n} = \frac{8}{9} \sum \frac{1}{n}$ So, $\sum_{n \in X} \frac{1}{n}$ also diverges.

3 Ex7.6

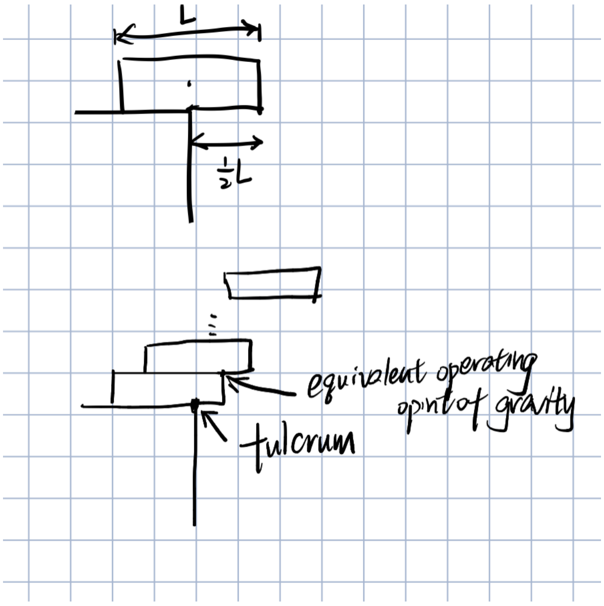

Figure 2: Figure 7.6.1

We can model the question(figure 7.6.1). When there are $n+1$ bricks, assume that the mass of each brick is m, the acceleration of gravity is g, the length of the lever is L and the distance between the middlepoint and the right end is l_{n+1}. We can then simplify it into a lever (show in figure 7.6.2).

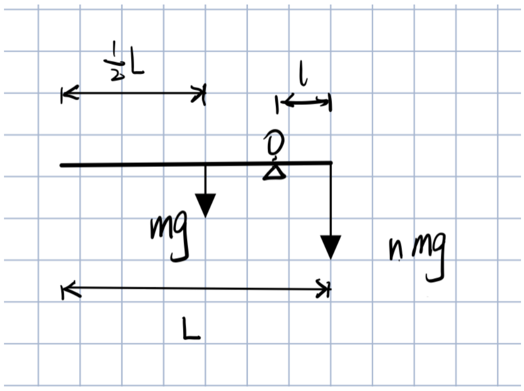

Figure 3: Figure 7.6.2

According to lever principle, we can know that

$$\begin{aligned} mg(\frac{L}{2} - l_{n+1}) &= nmgl_{n+1} \\ \frac{1}{2}mgL - mgl_{n+1} &= nmgl_{n+1} \\ l_{n+1} &= \frac{L}{2(n+1)} \\ l_{n+1} &= \frac{L}{2} \cdot \frac{1}{n+1} \end{aligned} \tag{3}$$

So we can know $\sum_{n=1}^{\infty} l_{n+1}$ diverges. Thus, the tower can extend to infinite far.

4 Ex7.7

4.1 7.7.1

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^{2n} \cdot 3^{3n}}{5^{3n}} &= \sum_{n=1}^{\infty} \frac{4^n 27^n}{125^n} \\ &= (\frac{108}{125})^n \\ &= \frac{108}{17} \end{aligned} \tag{4}$$

So, we can know the $\sum_{n=1}^{\infty} \frac{2^{2n} \cdot 3^{3n}}{5^{3n}}$ converges.

4.2 7.7.2

Because we know when $n > 3$, $n^2 - 3n + 1 > 0$ Thus, we can know:

$$\begin{aligned} a_n := \sum_{n=1}^{\infty} \frac{n+4}{n^2 - 3n + 1} &= \sum_{n=1}^{3} \frac{n+4}{n^2 - 3n + 1} + \sum_{n=4}^{\infty} \frac{n+4}{n^2 - 3n + 1} \\ &> \sum_{n=1}^{3} \frac{n+4}{n^2 - 3n + 1} + \sum_{n=4}^{\infty} \frac{n+4}{n^2 + 16n + 16} \\ &= \sum_{n=1}^{3} \frac{n+4}{n^2 - 3n + 1} + \sum_{n=4}^{\infty} \frac{1}{n+4} =: b_n \end{aligned} \tag{5}$$

Because $\sum_{n=4}^{\infty} \frac{1}{n+4}$ diverges, so b_n diverges.
Thus $a_n = \sum_{n=1}^{\infty} \frac{n+4}{n^2 - 3n + 1}$ diverges as $0 < b_n < a_n$.

4.3 7.7.3

Let $a_n := \frac{n^4}{3^n}$. We will use deduction to prove that when $n \geq 32$, $n^4 < 2^n$.
Firstly, when $n = 32$, $(32)^4 = 2^{20} < 2^{32}$
Secondly, assume that when $n = k$,$n \in \mathbb{N}^*, k \geq 32$, we also have $2^k > k^4$.
So, when $n = k + 1$, because

$$\frac{(k+1)^4}{k^4} = (\frac{k+1}{k})^4 < (\frac{33}{32})^4 < 2$$

Thus,

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot k^n > (k+1)^4$$

So, we have proved that when $n \geq 32$, $n^4 < 2^n$.
We get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{32} a_n + \sum_{n=33}^{\infty} \frac{n^4}{3^n} < \sum_{n=1}^{32} a_n + \sum_{n=33}^{\infty} (\frac{2}{3})^n := b_n$$

Because we know that $\sum_{n=33}^{\infty} (\frac{2}{3})^n$ converges. So, b_n converges. Thus a_n converges as $0 < a_n < b_n$,

4.4 7.7.4

Let $a_n := \frac{2^n}{n!}$, Then,

$$\lim_{x \to \infty} \frac{a_{n+1}}{a_n} = \lim_{x \to \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{x \to \infty} \frac{2}{n+1} = 0$$

Thus,$a_n = \frac{2^n}{n!}$ converges.

4.5 7.7.5

Let $a_n := \dfrac{2^n}{n^n}$. Then,

$$\lim_{x\to\infty} \frac{a_{n+1}}{a_n} = \lim_{x\to\infty} \frac{2n^n}{(n+1)^(n+1)} = \lim_{x\to\infty} 2 \cdot (\frac{n}{n+1})^n \cdot (\frac{1}{n+1}) = 0$$

Thus, $a_n = \dfrac{2^n}{n^n}$ converges.

4.6 7.7.6

We can know from $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10n^3 - 100}$ that when $n > 4$, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10n^3 - 100} > 0$

And then, we will prove that when $n > 4$, $\dfrac{n}{10n^3 - 100} > 0$ strictly decreases.

Proof:

For $\forall n > 4$,

$$\begin{aligned} \frac{n+1}{10(n+1)^3 - 100} - \frac{n}{10n^3 - 100} &= \frac{1}{10}(\frac{n+1}{(n+1)^3 - 10} - \frac{1}{10}(\frac{n}{n^3 - 10}) \\ &= \frac{1}{10} \cdot \frac{-2n^3 - 3n^2 - n - 10}{[(n+1)^3 - 10](n^3 - 10)} < 0 \end{aligned} \tag{6}$$

Thus, $\dfrac{n}{10n^3 - 100} > 0$ strictly decreases.

Then, we will prove $a_n := \sum_{n=1}^{k} \dfrac{n}{10n^3 - 100}$ is Cauchy squence.

Proof:

For all $n > 4$ and fixed p,

$$\sum_{n=1}^{k+p} \frac{n}{10n^3 - 100} - \sum_{n=1}^{k} \frac{n}{10n^3 - 100} = \sum_{n=k+1}^{k+p} \frac{n}{10n^3 - 100} \tag{7}$$

Because

$$\begin{aligned} \left| \sum_{n=k+1}^{k+p} \frac{n}{10n^3 - 100} - \sum_{n=k+2}^{k+p+1} \frac{n}{10n^3 - 100} \right| &= \frac{k+p+1}{10(k+p+1)^3 - 100} - \frac{k+1}{10(k+1)^3 - 100} \\ &< \frac{p}{10(k+1)^3 - 100} \end{aligned} \tag{8}$$

We can know that $\displaystyle\lim_{k\to\infty} \frac{p}{10(k+1)^3 - 100} = 0$. So $\forall \varepsilon > 0, \exists k \in \mathbb{N}^*$, there exists ε that satisfies

$$\left| \sum_{n=1}^{k+p} \frac{n}{10n^3 - 100} - \sum_{n=1}^{k} \frac{n}{10n^3 - 100} \right| < \varepsilon$$

We have prove $a_n := \sum_{n=1}^{k} \dfrac{n}{10n^3 - 100}$ is Cauchy squence. And then we can know $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10n^3 - 100}$ converges.