Ex7 186 Fall

Assignment group 10

1 Ex. 7.1

1.1 Ex. 7.1.1 Hölder's Inequality

Lemma:
$\forall a, b \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.
Proof: From (1), which is a special case of *Jensen's Inequality*, given in Assignment 6, we have:

$$(a^p)^{\frac{1}{p}} \cdot (b^q)^{\frac{1}{q}} = a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Q.E.D.

Proof of Hölder's Inequality:
Let $a_i = \frac{|x_i|}{||x_i||_p}$ and $b_i = \frac{|y_i|}{||y_i||_p}$, in which i=1,2,3,4...n.
From Young's Inequality, we have:

$$a_i b_i \leq \frac{|x_i|^p}{p||x_i||_p^p} + \frac{|y_i|^q}{q||y_i||_q^q}$$

So,

$$\sum_{i=1}^{n} a_i b_i \leq \frac{1}{p}\sum_{i=1}^{n}\frac{|x_i|^p}{||x_i||_p^p} + \frac{1}{q}\sum_{i=1}^{n}\frac{|y_i|^q}{||y_i||_q^q} = \frac{1}{p} + \frac{1}{q} = 1$$

Q.E.D.

1.2 Ex. 7.1.2 Minkowski's Ineqality

Using *Hölder's Inequality* as a lemma.
Proof of Minkowski's Inequality:

$$\begin{aligned}
\sum_{j=1}^{n}|x_j + y_j|^p &= \sum_{j=1}^{n}|x_j + y_j||x_j + y_j|^{p-1} \\
&\leq \sum_{j=1}^{n}(|x_j| + |y_j|)|x_j + y_j|^{p-1} \\
&= \sum_{j+1}^{n}|x_j||x_j + y_j|^{p-1} + \sum_{j+1}^{n}|y_j||x_j + y_j|^{p-1} \\
&\leq (\sum_{j=1}^{n}|x_j|^p)^{\frac{1}{p}} \cdot (\sum_{j=1}^{n}|x_j + y_j|^{(p-1)q})^{\frac{1}{q}} + (\sum_{j=1}^{n}|y_j|^p)^{\frac{1}{p}} \cdot (\sum_{j=1}^{n}|x_j + y_j|^{(p-1)q})^{\frac{1}{q}} \\
&= (\sum_{j=1}^{n}|x_j + y_j|^p)^{\frac{1}{q}} \cdot [(\sum_{j=1}^{n}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{n}|y_j|^p)^{\frac{1}{p}}]
\end{aligned}$$

Thus,

$$\frac{\sum_{j=1}^{n}|x_j + y_j|^p}{\sum_{j=1}^{n}|x_j + y_j|^p)^{\frac{1}{q}}} = (\sum_{j=1}^{n}|x_j + y_j|^p)^{\frac{1}{p}} \leq [(\sum_{j=1}^{n}|x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{n}|y_j|^p)^{\frac{1}{p}}]$$

for $\forall p \geq 1$, $q \neq 0$.

Q.E.D.

1.3 Ex. 7.1.3

First, for $\forall x \in \mathbb{R}^n$, $||x||_p \geq 0$ and $||x||_p = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_n = 0$.
Besides,

$$||x \cdot y||_p = (\sum_{1 \leq i,j \leq n} |x_j \cdot y_j|^p)^{\frac{1}{p}} = (\sum_{i=1}^{n} |x_i|^p \cdot \sum_{j=1}^{n} |y_j|^p)^{\frac{1}{p}}$$
$$= ||x||_p \cdot ||y||_p \, .$$

Finally,

$$||x + y||_p = (\sum_{j=1}^{n} |x_j + y_j|^p)^{\frac{1}{p}} \leq (\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^{n} |y_j|^p)^{\frac{1}{p}}$$
$$= ||x||_p + ||y||_p \, .$$

So, $|| \cdot ||_p$ defines a norm on $\mathbb{R}^n$ for $\forall p \in \mathbb{N} \backslash \{0\}$.

1.4 Ex. 7.1.4

Let $\xi := ||x||_p$. By the definition of $|| \cdot ||_p$, $\frac{|x_j|}{\xi} \leq 1$ for $j = 1, 2, 3 ... n$, and $(\frac{|x_j|}{\xi})^q \leq (\frac{|x_j|}{\xi})^p$ for $p < q$, $j = 1, 2, 3 ... n$.
So,

$$\sum_{j=1}^{n} (\frac{|x_j|}{\xi})^q \leq \sum_{j=1}^{n} (\frac{|x_j|}{\xi})^p = \frac{\sum_{j=1}^{n} |x_j|^p}{\xi^p} = \frac{\sum_{j=1}^{n} |x_j|^p}{\sum_{j=1}^{n} |x_j|^p} = 1$$

So, $\sum_{j=1}^{n} |x_j|^q \leq \xi^q$, $(\sum_{j=1}^{n} |x_j|^q)^{\frac{1}{q}} \leq \xi$ and thus $||x||_q \leq ||x||_p$.

1.5 Ex. 7.1.5

First for all $x \in \mathbb{R}^n$, $||x||_\infty \geq 0$. Also, $||x||_\infty = 0$ if and only if $\max\limits_{1 \leq j \leq n} |x_j| = 0$, which means $x_1 = x_2 = x_3 = \cdots = x_n = 0$.
Besides, $||x \cdot y||_\infty = \max\limits_{1 \leq i,j \leq n} |x_i y_j| = \max\limits_{1 \leq i \leq n} |x_i| \cdot \max\limits_{1 \leq j \leq n} |y_j| = ||x||_\infty \cdot |||y||_\infty$.
Finally, $||x + y||_\infty = \max\limits_{1 \leq i \leq n} |x_i + y_i| \leq \max\limits_{1 \leq i \leq n} (|x_i| + |y_i|) \leq \max\limits_{1 \leq i \leq n} |x_i| + \max\limits_{1 \leq i \leq n} |y_i| = ||x||_\infty + ||y||_\infty$, which partially finishes our proof.

Without losing generosity, we can assume that $|x_1| = \max |x_i|$ and

$$\lim_{p \to \infty} (\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}} = |x_1| \lim_{p \to \infty} (\sum_{j=1}^{n} (\frac{|x_j|}{|x_1|})^p)^{\frac{1}{p}} \, .$$

Notice that $\lim\limits_{p \to \infty} (\frac{|x_j|}{|x_1|})^p = \begin{cases} 1 & |x_j| = |x_1| \\ 0 & |x_j| \neq |x_1| \end{cases}$ for $j = 1, 2, 3 \cdots n$.
Firstly,

$$(\lim_{p \to \infty} (\sum_{j=1}^{n} |\frac{x_j}{x_1}|)^p)^{\frac{1}{p}} \geq \lim_{p \to \infty} (|\frac{x_1}{x_1}|^p)^{\frac{1}{p}} = 1 \, .$$

Secondly,

$$\lim_{p \to \infty} (\sum_{j=1}^{n} (|\frac{x_j}{x_1}|)^p)^{\frac{1}{p}} \leq \lim_{p \to \infty} (\sum_{j=1}^{n} (|\frac{x_1}{x_1}|)^p)^{\frac{1}{p}} = \lim_{p \to \infty} n^{\frac{1}{p}}$$

for $\lim\limits_{m \to \infty} \sqrt[m]{y} = 1$ for a fixed $y > 0$. (See Assignment 6 Exercise 6.7)
So, $\lim\limits_{p \to \infty} (\sum_{j=1}^{n} (|\frac{x_j}{x_1}|)^p)^{\frac{1}{p}} \leq \lim\limits_{p \to \infty} n^{\frac{1}{p}} = 0$.

Q.E.D.

2 Ex. 7.2

Only the third object define a real vector space. Let's look at them in detail.
For 7.2.1:
Counterexample: Let $\{(x_1, x_2, x_3 \cdots x_n) \in \mathbb{R}^n | x_1 \leq 0\}$ be U. Take $e = (-1, 1, 1, 1 \cdots 1) \in U$. Then, $\lambda \cdot e = (1, -1, -1, -1 \cdots -1) \notin U$ when $\lambda = -1 \in \mathbb{R}$. Hence U is not closed to scalar multiplication, so $(U, +, \cdot)$ is not a subspace of $\mathbb{R}^n$, i.e. not a real vector space.
For 7.2.2:
Counterexample: Let $\{(x_1, x_2, x_3 \cdots x_n) \in \mathbb{R}^n | x_1 x_n \leq 0\}$ be U. Take $e_1 = (0, 1, 1, 1 \cdots 1)$ and $e_2 = (1, 1, 1 \cdots 1, 0) \in U$. Then, $e_1 + e_2 = (1, 2, 2, 2 \cdots 2, 1) \notin U$. Hence U is not closed to pointwise addition, so $(U, +, \cdot)$ is not a subspace of $\mathbb{R}^n$, i.e. not a real vector space.
For 7.3.3:
Proof. Let $\{(x_1, x_2, x_3 \cdots x_n) \in \mathbb{R}^n | x_1 + 5x_2 = 0\}$ be U. First let's show that U is closed to pointwise addition:

For $e_1 = (a_1, a_2, a_3 \cdots a_n)$ and $e_2 = (b_1, b_2, b_3 \cdots b_n) \in U$. Then $e_1 + e_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3 \cdots a_n + b_n) = (c_1, c_2, c_3 \cdots c_n)$, in which $c_1 + 5c_2 = (a_1 + b_1) + 5(a_2 + b_2) = (a_1 + 5a_2) + (b_1 + 5b_2) = 0$.

Now let's show further that U is closed to scalar multiplication, which finishes the proof:

For $e_3 = e_1 + e_2 \in U$, which had been given above, let $\lambda \in \mathbb{R}$. If $\lambda = 0$, it would be apparent that $\lambda e_3 \in U$. If not, $\lambda \cdot e_3 = (\lambda c_1, \lambda c_2, \lambda c_3 \cdots \lambda c_n)$, in which $\lambda c_1 + 5\lambda c_2 = \lambda(c_1 + 5c_2) = \lambda \cdot 0 = 0$.

Q.E.D.

3 Ex. 7.3

3.1 Ex. 7.3.1

We have already known that, $\lim_{n \to +\infty} \sqrt[n]{x} = 0$ when $x = 0$ or 1 as $x > 0$, which had been proven in Assignment 6 Exercise 6.7.
So, the pointwise limit of the sequence of functions $f_n(x)$: $\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \sqrt[n]{x}$ $(x \in [0, 1])$ exist, and the sequence converges to $f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$.
However, the convergence is not uniform, as we shall see: Because $||f_n - f||_\infty = \sup_{[0,1]} |f_n - f| = \sup_{[0,1]} |1 - x^{\frac{1}{n}}| \geq 1 - f(0.1^n) = 0.9 > 0$, the convergence is not uniform.

3.2 Ex. 7.3.2

We have $\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \frac{nx}{1+n+x}$. As $x \in \mathbb{R}$, $\lim_{n \to +\infty} \frac{nx}{1+n+x} = \lim_{n \to +\infty} \frac{nx}{x} = x$.
So the pointwise limit of the sequence of functions $f_n(x) = \frac{nx}{1+n+x}$ exist, and the sequence converges to $f(x) = x$. The convergence is uniform, as we shall see:

$$||f_n(x) - f(x)||_\infty = ||\frac{nx}{1 + n + x} - x||_\infty = ||\frac{-x - x^2}{1 + n + x}||_\infty = \sup_{x \in \mathbb{R}^+ \cup \{0\}} |\frac{-x - x^2}{1 + n + x}| = \sup_{x \in \mathbb{R}^+ \cup \{0\}} \frac{(x + x^2)}{(1 + x) + n}$$

As $x \in \mathbb{R}$, $(-x - x^2)$ and $(1 + x) \in \mathbb{R}$. So, as $n \to +\infty$,

$$\lim_{n \to +\infty} \frac{x + x^2}{(1 + x) + n} = \lim_{n \to +\infty} \frac{1}{n} = 0$$

So, $||f_n(x) - f(x)||_\infty = 0$. That means that the convergence is uniform.

3.3 Ex. 7.3.3

$f(x) = \begin{cases} 0 & x \leq n \\ \frac{1}{x} & x > n \end{cases}$, dom$f = \mathbb{R}$. Because $\lim_{x \to +\infty} f_n(x) = \lim_{x \to +\infty} \frac{1}{x} = 0$, pointwise limit exists and the series of functions converge pointwisely to $f(x) = 0$.

We'll now show that the convergence is uniform.
Because $||f_n(x) - f(x)||_\infty = \sup_{\mathbb{R}} |\frac{1}{x} - 0|$. As $n \to +\infty$, $\frac{1}{x} \to 0$ and $\sup_{\mathbb{R}} |\frac{1}{x} - 0| = \frac{1}{n} - 0 \to 0$. So, the convergence is uniform.

3.4 Ex. 7.3.4

It's apparent that $\lim_{n\to+\infty} f_n(x) = 0$. Pointwise limit exists and the sequence of functions f_n converges pointwisely to $f(x) = 0$.
The convergence is uniform, as we will now show. As

$$||f_n(x) - f(x)||_\infty = ||f_n(x)||_\infty = \sup_{\mathbb{R}^+}(\sqrt{\frac{1}{n} + x} - \sqrt{x})$$

and

$$\lim_{n\to+\infty} \sup_{\mathbb{R}^+}(\sqrt{\frac{1}{n} + x} - \sqrt{x}) = \sup_{\mathbb{R}^+}(\sqrt{x} - \sqrt{x}) = 0\,,$$

$||f_n(x)||_\infty \to 0$ as $n \to +\infty$, so the convergence is uniform.

3.5 Ex. 7.3.5

Because

$$\begin{aligned}\lim_{n\to+\infty} f_n(x) &= \lim_{n\to+\infty} (\sqrt{n^2x + n} - \sqrt{n^2x}) \\ &= \lim_{n\to+\infty} (\sqrt{n^2x + o(n^2)} - \sqrt{n^2x}) \\ &= \lim_{n\to+\infty} (\sqrt{n^2x} - \sqrt{n^2x}) = 0\,,\end{aligned}$$

the pointwise limit of the sequence of functions exists, and $f_n(x)$ converges pointwisely to $f(x) = 0$.
However, we can show that the convergence is not uniform. Because

$$||f_n(x) - f(x)||_\infty = ||f_n(x)||_\infty = \sup_{\mathbb{R}^+} n \cdot (\sqrt{\frac{1}{n} + x} - \sqrt{x})$$

and as n goes to infinity, let's have $x = \frac{1}{n^2}$.

$$\begin{aligned}\sup_{\mathbb{R}^+} n(\sqrt{\frac{1}{n} + x} - \sqrt{x}) \geq n(\sqrt{\frac{1}{n} + \frac{1}{n^2}} - \sqrt{x}) \\ &= n \cdot \sqrt{\frac{1}{n} + \frac{1}{n^2}} - 1 \\ &> n \cdot \sqrt{\frac{1}{n}} - 1 \\ &= \sqrt{n} - 1 \to +\infty \text{ as } n \to +\infty\,.\end{aligned}$$

The norm of $f_n(x) - f(x)$ does not converge, so the sequence of functions does converge uniformly to $f(x) = 0$.

4 Ex7.4

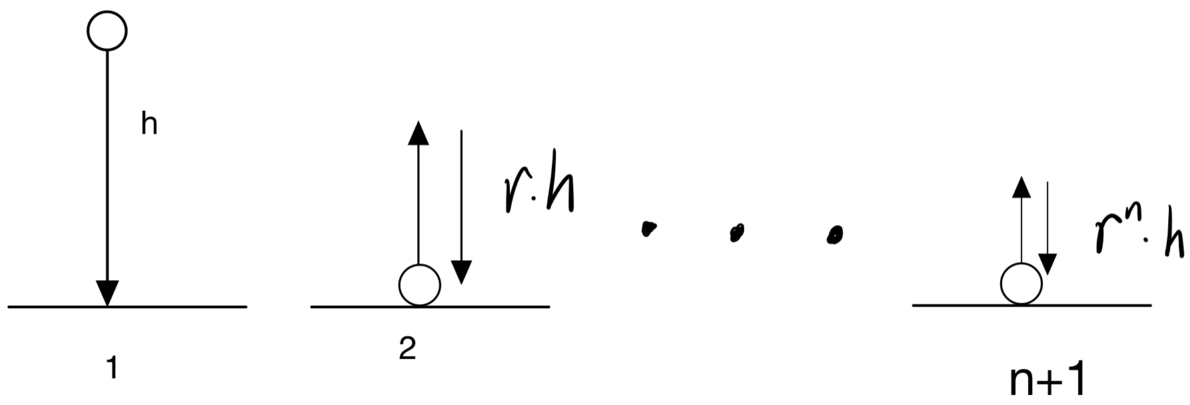

Figure 1: Figure 7.4

The distance can be assessed through the image above (figure Ex7.4).

The first process : h

The second process : $2r \cdot h$

...

The nth process : $2r^{n-1} \cdot h$

Except for the first process is one-way, other process are all double-way Thus, the total distance :

$$\begin{aligned} D &= 2(\sum_{n=1}^{\infty} r^{n-1} \cdot h) - h \\ &= \frac{2h}{1-r} - h \end{aligned} \tag{1}$$

5 Ex7.5

We can know from the question that

$$\sum \frac{1}{n} = \sum_{n \in X} \frac{1}{n} + \sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n}$$

Because

$$\begin{aligned} \sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n} &= \frac{1}{9} + \frac{1}{81} + \cdots + \frac{1}{9^k} + \dots \\ &= \frac{1}{9} \sum \frac{1}{n} \end{aligned} \tag{2}$$

So, $\sum_{n \in \mathbb{N}^* \setminus X} \frac{1}{n}$ diverges.

Then, we can find that $\sum_{n \in X} \frac{1}{n} = \frac{8}{9} \sum \frac{1}{n}$ So, $\sum_{n \in X} \frac{1}{n}$ also diverges.

6 Ex7.6

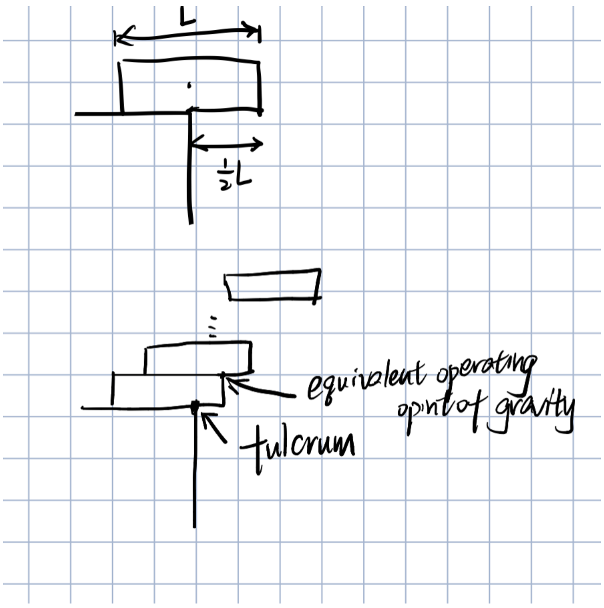

Figure 2: Figure 7.6.1

We can model the question(figure 7.6.1). When there are $n+1$ bricks, assume that the mass of each brick is m, the acceleration of gravity is g, the length of the lever is L and the distance between the middlepoint and the right end is l_{n+1}. We can then simplify it into a lever (show in figure 7.6.2).

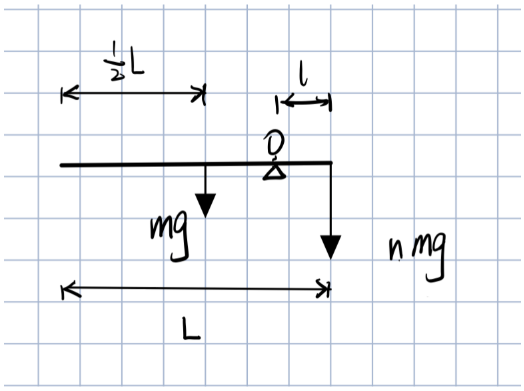

Figure 3: Figure 7.6.2

According to lever principle, we can know that

$$\begin{aligned} mg(\frac{L}{2} - l_{n+1}) &= nmgl_{n+1} \\ \frac{1}{2}mgL - mgl_{n+1} &= nmgl_{n+1} \\ l_{n+1} &= \frac{L}{2(n+1)} \\ l_{n+1} &= \frac{L}{2} \cdot \frac{1}{n+1} \end{aligned} \tag{3}$$

So we can know $\sum_{n=1}^{\infty} l_{n+1}$ diverges. Thus, the tower can extend to infinite far.

7 Ex7.7

7.1 7.7.1

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^{2n}\cdot 3^{3n}}{5^{3n}} &= \sum_{n=1}^{\infty}\frac{4^n27^n}{125^n}\\ &= (\frac{108}{125})^n\\ &= \frac{108}{17}\end{aligned} \tag{4}$$

So, we can know the $\sum_{n=1}^{\infty}\frac{2^{2n}\cdot 3^{3n}}{5^{3n}}$ converges.

7.2 7.7.2

Because we know when $n > 3$, $n^2 - 3n + 1 > 0$ Thus, we can know:

$$\begin{aligned}a_n := \sum_{n=1}^{\infty}\frac{n+4}{n^2-3n+1} &= \sum_{n=1}^{3}\frac{n+4}{n^2-3n+1} + \sum_{n=4}^{\infty}\frac{n+4}{n^2-3n+1}\\ &> \sum_{n=1}^{3}\frac{n+4}{n^2-3n+1} + \sum_{n=4}^{\infty}\frac{n+4}{n^2+16n+16}\\ &= \sum_{n=1}^{3}\frac{n+4}{n^2-3n+1} + \sum_{n=4}^{\infty}\frac{1}{n+4} =: b_n\end{aligned} \tag{5}$$

Because $\sum_{n=4}^{\infty}\frac{1}{n+4}$ diverges, so b_n diverges.
Thus $a_n = \sum_{n=1}^{\infty}\frac{n+4}{n^2-3n+1}$ diverges as $0 < b_n < a_n$.

7.3 7.7.3

Let $a_n := \frac{n^4}{3^n}$. We will use deduction to prove that when $n \geq 32$, $n^4 < 2^n$.
Firstly, when $n = 32$, $(32)^4 = 2^{20} < 2^{32}$
Secondly, assume that when $n = k$,$n \in \mathbb{N}^*, k \geq 32$, we also have $2^k > k^4$.
So, when $n = k + 1$, because

$$\frac{(k+1)^4}{k^4} = (\frac{k+1}{k})^4 < (\frac{33}{32})^4 < 2$$

Thus,

$$2^{k+1} = 2\cdot 2^k > 2\cdot k^n > (k+1)^4$$

So, we have proved that when $n \geq 32$, $n^4 < 2^n$.
We get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{32} a_n + \sum_{n=33}^{\infty}\frac{n^4}{3^n} < \sum_{n=1}^{32} a_n + \sum_{n=33}^{\infty}(\frac{2}{3})^n := b_n$$

Because we know that $\sum_{n=33}^{\infty}(\frac{2}{3})^n$ converges. So, b_n converges. Thus a_n converges as $0 < a_n < b_n$,

7.4 7.7.4

Let $a_n := \frac{2^n}{n!}$, Then,

$$\lim_{x\to\infty}\frac{a_{n+1}}{a_n} = \lim_{x\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{x\to\infty}\frac{2}{n+1} = 0$$

Thus,$a_n = \frac{2^n}{n!}$ converges.

7.5 7.7.5

Let $a_n := \dfrac{2^n}{n^n}$. Then,

$$\lim_{x\to\infty}\frac{a_{n+1}}{a_n}=\lim_{x\to\infty}\frac{2n^n}{(n+1)^(n+1)}=\lim_{x\to\infty}2\cdot(\frac{n}{n+1})^n\cdot(\frac{1}{n+1})=0$$

Thus, $a_n = \dfrac{2^n}{n^n}$ converges.

7.6 7.7.6

We can know from $\displaystyle\sum_{n=1}^{\infty}\frac{n}{10n^3-100}$ that when $n>4$, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{10n^3-100}>0$

And then, we will prove that when $n>4$, $\dfrac{n}{10n^3-100}>0$ strictly decreases.

Proof:

For $\forall n>4$,

$$\begin{aligned}\frac{n+1}{10(n+1)^3-100}-\frac{n}{10n^3-100}&=\frac{1}{10}(\frac{n+1}{(n+1)^3-10}-\frac{1}{10}(\frac{n}{n^3-10})\\&=\frac{1}{10}\cdot\frac{-2n^3-3n^2-n-10}{[(n+1)^3-10](n^3-10)}<0\end{aligned}\tag{6}$$

Thus, $\dfrac{n}{10n^3-100}>0$ strictly decreases.

Then, we will prove $a_n := \sum_{n=1}^{k}\dfrac{n}{10n^3-100}$ is Cauchy squence.

Proof:

For all $n>4$ and fixed p,

$$\sum_{n=1}^{k+p}\frac{n}{10n^3-100}-\sum_{n=1}^{k}\frac{n}{10n^3-100}=\sum_{n=k+1}^{k+p}\frac{n}{10n^3-100}\tag{7}$$

Because

$$\begin{aligned}\left|\sum_{n=k+1}^{k+p}\frac{n}{10n^3-100}-\sum_{n=k+2}^{k+p+1}\frac{n}{10n^3-100}\right|&=\frac{k+p+1}{10(k+p+1)^3-100}-\frac{k+1}{10(k+1)^3-100}\\&<\frac{p}{10(k+1)^3-100}\end{aligned}\tag{8}$$

We can know that $\displaystyle\lim_{k\to\infty}\frac{p}{10(k+1)^3-100}=0$. So $\forall\varepsilon>0,\exists k\in\mathbb{N}^*$, there exists ε that satisfies

$$\left|\sum_{n=1}^{k+p}\frac{n}{10n^3-100}-\sum_{n=1}^{k}\frac{n}{10n^3-100}\right|<\varepsilon$$

We have prove $a_n := \sum_{n=1}^{k}\dfrac{n}{10n^3-100}$ is Cauchy squence. And then we can know $\displaystyle\sum_{n=1}^{\infty}\frac{n}{10n^3-100}$ converges.