250 Homework #3

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P3.1.6 [10 pts]

Two naturals are defined to be **relatively prime** if their greatest common divisor is 1. Note that this can happen even if either or both of the two naturals are composite.

(a) Show that if x is prime, and y is not a multiple of x, then x and y are relatively prime.

(b) Give an example of two composite numbers that are relatively prime.

(c) Recall the example of a hash table, where we have n locations and respond to a collision by moving up k locations. Show that if n and k are not relatively prime, then repeatedly skipping k locations will not reach all possible locations.

Solution:

(a)

(b)

(c)

[*]Collaborated with Nobody.

P3.5.2 [10 pts]

The **Gregorian calendar** (the one in most general use today)[1] is the same as the Julian calendar except that there are 365 days in year x if x is congruent to 100, 200, or 300 modulo 400.

(a) In the Gregorian calendar, as students of World War II may recall, 7 December 1941 was a Sunday. We cannot, as in the case of the Julian calendar, guarantee that 7 December of year x was a Sunday if $x \equiv 1941 \pmod{28}$, but we can guarantee it if $x \equiv 1941 \pmod{c}$ for some value of c. Find the smallest value of c for which this is true.

(b) Determine the day of the week on which you were born, using only the fact that 7 December 1941 was a Sunday. Show all of your reasoning.

(c) What additional complications arise when designing a perpetual Gregorian calendar?

(d) In what years, during the period from 1 to 1941 A.D. (or 1 to 1941 C.E.), have the Gregorian and Julian calendars agreed for the entire year?

Solution:

(a)

(b)

(c)

(d)

[1] Great Britain and its colonies switched from the Julian to Gregorian calendar in 1752, when they were considerably out of step with each other — to see how this was implemented enter `cal 1752` on any Unix machine. George Washington, who was alive at the time of this change, retroactively changed his birthday to 22 February.

P3.6.9 [10 pts]

Because there are infinitely many primes, we can assign each one a number: $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, and so forth. A finite **multiset** of naturals is like an ordinary finite set, except that an element can be included more than once and we care how many times it occurs. Two multisets are defined to be equal if they contain the same number of each natural. So {2, 4, 4, 5}, for example, is equal to {4, 2, 5, 4} but not to {4, 2, 2, 5}. We define a function f so that given any finite multiset S of naturals, $f(S)$ is the product of a prime for each element of S. For example, $f(\{2, 4, 4, 5\})$ is $p_2 p_4 p_4 p_5 = 5 \times 11 \times 11 \times 13 = 7865$.

(a) Prove that f is a bijection from the set of all finite multisets of naturals to the set of positive naturals.

(b) The **union of two multisets** is taken by including all the elements of each, retaining duplicates. For example, if $S = \{1, 2, 2, 5\}$ and $T = \{0, 1, 1, 4\}$, $S \cup T = \{0, 1, 1, 1, 2, 2, 4, 5\}$. How is $f(S \cup T)$ related to $f(S)$ and f(T)?

(c) S is defined to be a **submultiset** of T, written "$S \subseteq T$", if there is some multiset U such that $S \cup U = T$. If $S \subseteq T$, what can we say about $f(S)$ and $f(T)$?

(d) The **intersection of two multisets** consists of the elements that occur in both, with each element occurring the same number of times as it does in the one where it occurs fewer times. For example, if $S = \{0, 1, 1, 2\}$ and $T = \{0, 0, 1, 3\}$, $S \cap T = \{0, 1\}$. How is $f(S \cap T)$ related to $f(S)$ and $f(T)$?

Solution:

(a)

(b)

(c)

(d)

P4.1.5 [10 pts]

(uses Java) Give a recursive definition of the "less than" operator on numbers. (You may refer to equality of numbers in your definition.) Write a static pseudo-Java method "`boolean isLessThan (natural x, natural y)`" that returns `true` if and only if $x < y$ and uses only our given methods. (**Hint:** Follow the example of the functions `plus` and `times` in the text.

Solution:

P4.3.1 [10 pts]

Determine a formula for the sum of the first n positive perfect cubes and prove it correct by induction on all naturals n.

Solution:

P4.3.7 [10 pts]

Determine a formula for the number of size-3 subsets of an n-element set, for any n. Prove your formula correct by induction, using the result of Exercise 4.3.8[2] in your inductive case when $n \geq 3$.

Solution:

[2]For all naturals n, any set of n elements has exactly $n(n\text{-}1)/2$ subsets of size exactly 2.

P4.4.2 [10 pts]

Give a rigorous proof, using strong induction, that every positive natural has at least one factorization into prime numbers.

Solution:

P4.4.6 [10 pts]

I am starting a new plan for the length of my daily dog walks. On Day 0 we walk 3 miles, on Day 1 we walk 2 miles, and for all $n > 0$ the length of our walk on Day $n + 1$ is the average of the lengths of the walks on Days n - 1 and n.

(a) Prove by strong induction for all naturals n that on Day n, we walk $(7 + 2(-1/2)^n)/3$ miles. (**Hint:** Use base cases for $n = 0$ and $n = 1$.)

(b) Give a formula for the total distance that we walk on days 0 through n, and prove your formula correct by strong induction.

Solution:

(a)

(b)

P4.6.5 [10 pts]

Define the exponentiation operator on naturals recursively so that $x^0 = 1$ and $x^{S(y)} = x^y \cdot x$. Prove by induction, using this definition, that for any naturals x, y, and z, $x^{y+z} = x^y \cdot x^z$ and $x^{y \cdot z} = (x^y)^z$.

Solution:

EC: P3.5.10 [10 pts]

Let m and n be two relatively prime positive naturals, and consider what naturals can be expressed as linear combinations $am + bn$ where a and b are *naturals*, not just integers

(a) Show that if $m = 2$ and $n = 3$, any natural except 1 can be so expressed.

(b) Determine which naturals can be expressed if $m = 3$ and $n = 5$.

(c) Argue that for any m and n, there are only a finite number of naturals that cannot be expressed in this way.

Solution:

(a)

(b)

(c)