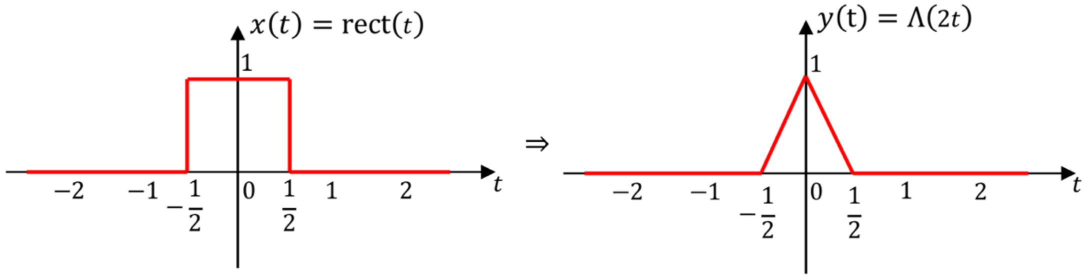
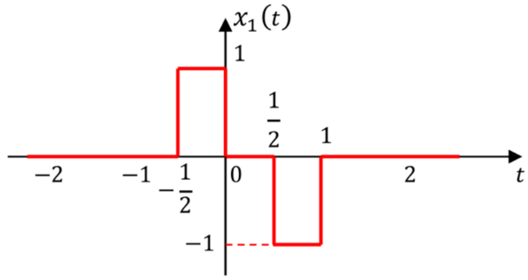
Solutions to Examples from Discussion 4 (ECE 213, Spring 2024)

1. For a certain LTI system, the response to an input $x(t) = \text{rect}(t)$ is $y(t) = \Lambda(2t)$.

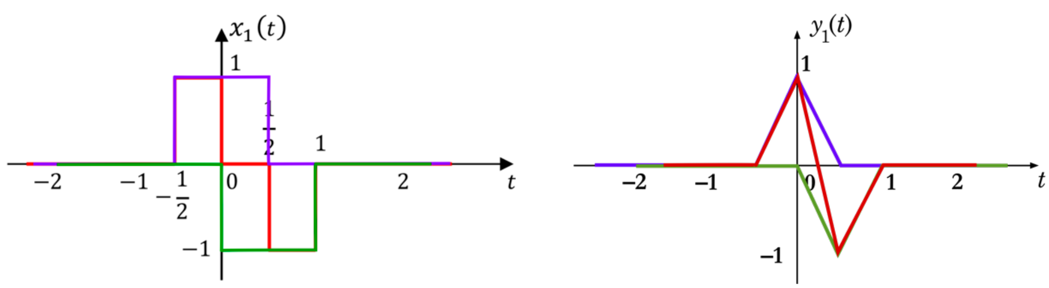


   (a) Compute the response of the system to the input $x_1(t)$.



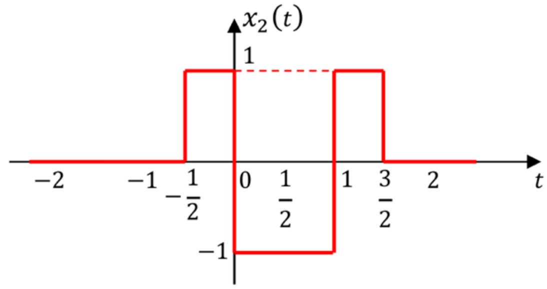
ANSWER:

The input can be written as $x_1(t) = \text{rect}(t) - \text{rect}\left(t - \frac{1}{2}\right) = x(t) - x\left(t - \frac{1}{2}\right)$, as shown below on the left (purple + green). Therefore, the output must be $y_1(t) = y(t) - y\left(t - \frac{1}{2}\right) = \Lambda(2t) - \Lambda\left(2\left(t - \frac{1}{2}\right)\right)$, as shown on the right below. The final result is in red.



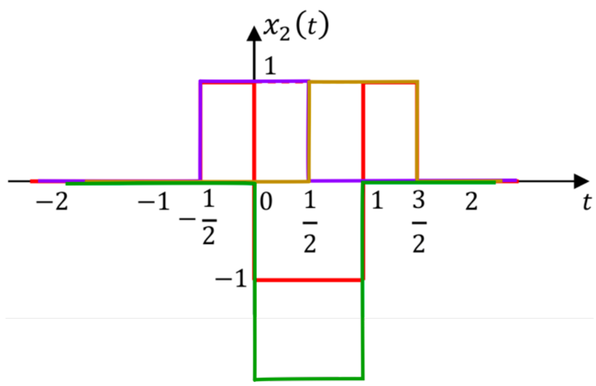
Note that, when evaluating the second term, we replace t with $t - ½$, i.e., we **<u>don't</u>** write $\Lambda\left(2t - \frac{1}{2}\right)$, we write $\Lambda\left(2\left(t - \frac{1}{2}\right)\right)$. Note also that it might be tempting to try to write this input signal as two compressed rect functions, the first one time-shifted to the left by ¼ and the other time-shifted to the right by ¾, but we don't. We choose instead to write the given input signal as two non-compressed / non-expanded rect functions, because it is much easier to find the final result.

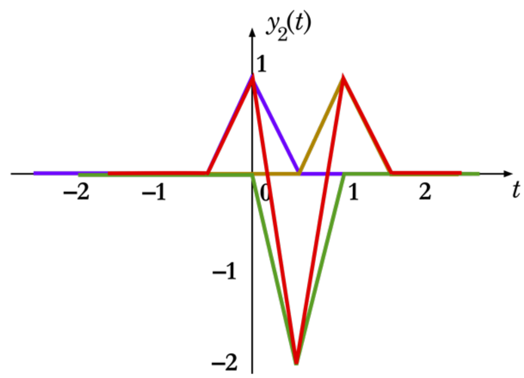
(b) Compute the response of the system to the input $x_2(t)$.



ANSWER:

The input can be written $x_2(t) = \text{rect}(t) - 2\text{rect}\left(t - \frac{1}{2}\right) + \text{rect}(t-1) = x(t) - x\left(t - \frac{1}{2}\right) + x(t-1)$, as shown below on the left (purple + green + brown). Therefore, the output signal must be $y_2(t) = y(t) - 2y\left(t - \frac{1}{2}\right) + y(t-1) = \Lambda(2t) - 2\Lambda\left(2\left(t - \frac{1}{2}\right)\right) + \Lambda\big(2(\text{t}-1)\big)$, as shown on the right below. The final result is in red.





2. Find the impulse response $h(t)$ of the following systems.

(a) $y(t) = \int_t^{\infty} x(\tau)\, e^{2(t-\tau)}\, d\tau$

ANSWER:

To find the impulse response, let $x(t) = \delta(t)$, which means $y(t) = h(t)$, the impulse response…

$$h(t) = \int_t^{\infty} \delta(\tau)\, e^{2(t-\tau)}\, d\tau$$

To take advantage of the full power of the impulse function, change the limits of the integral to be over all time. To do this, we need to find a function $g(\tau)$ that does NOT change the mathematical content of the equation above when the limits are changed…

$$h(t) = \int_{-\infty}^{\infty} g(\tau)\, \delta(\tau)\, e^{2(t-\tau)}\, d\tau$$

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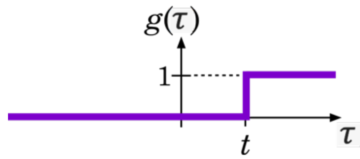
It can be hard to know what $g(\tau)$ should be, but it is relatively easy to draw it first…



In other words, the original integral starts at $\tau = t$, so the "extra" we are adding between $-\infty$ and t must be equal to 0, so $g(\tau) = 0$ until $\tau = t$, as shown. Further, the mathematical content from $\tau = t$ to $\tau = \infty$ also cannot change, so $g(\tau) = 1$, also as shown. This is a known function, a time-shifted unit-step function, $g(\tau) = u(\tau - t)$.

Inserting this we get…

$$h(t) = \int_{-\infty}^{\infty} u(\tau - t)\, \delta(\tau)\, e^{2(t-\tau)}\, d\tau$$

Next, evaluating this integral, we get a final result of…

$$h(t) = u(-t)\, e^{2t}$$

In other words, replace each τ in the integrand with 0, then integrate.

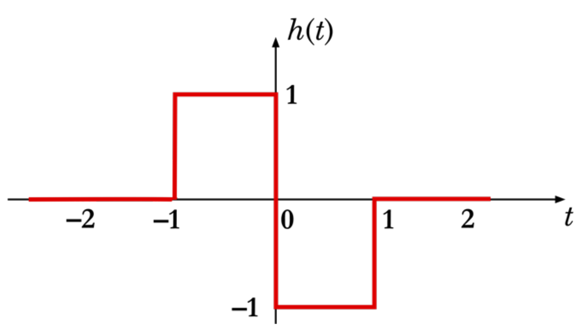
(b) $y(t)$ has the step response $y_{\text{step}}(t) = \Lambda(t)$.

ANSWER:

The step response is the output when the input is $u(t)$. We can show that the impulse response is the derivative of the step response. Therefore…

$$h(t) = \frac{dy_{\text{step}}}{dt} = \begin{cases} 0 & t < -1 \\ 1 & -1 < t < 0 \\ -1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

… which is shown graphically below.



Therefore…

$$h(t) = u(t+1) - 2u(t) + u(t-1)$$

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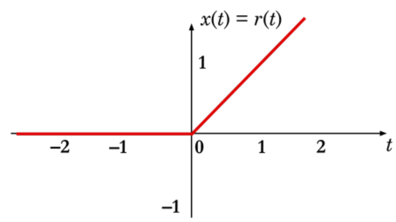
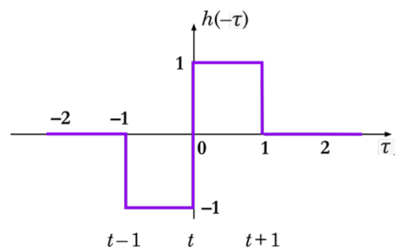
3. Use graphical convolution to evaluate the convolution integral $y(t) = h(t) * x(t)$ for the following.
   (a) Use $h(t)$ from question 2(b) above. Let $x(t) = r(t)$.

ANSWER:

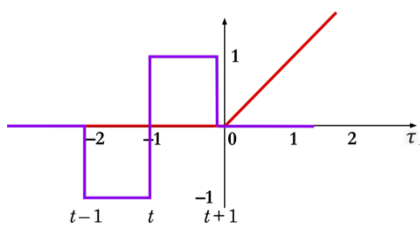


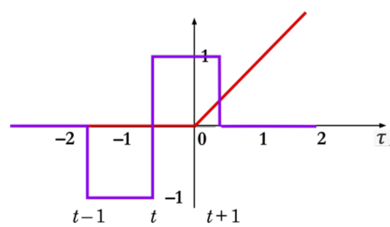
The input signal $x(t)$ is shown to the right. The integral equation we are solving is…

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$



If we plot $h(-\tau)$, as shown to the right, we can see where $t = 0$ is located, i.e., in the middle, which means $t + 1$ is the right edge, and $t - 1$ is the left edge. $h(t-\tau)$ is the time-shifted version of this, so we can imagine moving this along the horizontal axis. The focus is on finding the appropriate limits of integration for each placement of $h(t-\tau)$. For instance, on the left below, $t < -1$, so the product of the two functions is 0, and the resulting integral is also 0. In other words, $h(t-\tau) = 0$ when $r(\tau)$ is non-zero, and $r(\tau) = 0$ when $h(t-\tau)$ is non-zero.



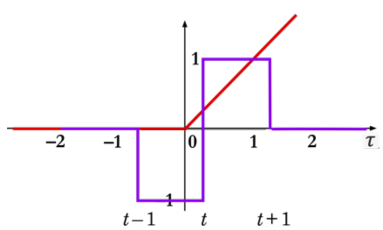


Next, for $-1 < t < 0$, the positive pulse is overlapping with the ramp function, as shown on the right above, so limits are 0 and $t + 1$, and the integral is non-zero…

$$y(t) = \int_{0}^{t+1} 1\,\tau\, d\tau = \frac{(t+1)^2}{2} \qquad [-1 < t < 0]$$

In other words, $r(\tau)$ is 0 for $\tau < 0$, and $h(t-\tau) = 0$ for $\tau > t+1$, so the limits are 0 to $t + 1$.

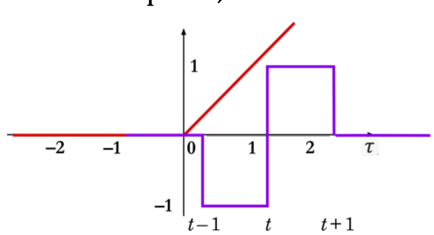
Next, for $0 < t < 1$, the positive pulse is completely to the right of the vertical axis, and part of the negative pulse is also to the right of the vertical axis, as shown below on the left. We need to break this into two integrals, one from 0 to t, and the other from t to t+1. The resulting integral is on the right.



$$y(t) = \int_0^t -1\,\tau\, d\tau + \int_t^{t+1} 1\,\tau\, d\tau$$

$$= -\frac{t^2}{2} + \frac{(t+1)^2}{2} - \frac{t^2}{2} = \frac{(t+1)^2}{2} - t^2 \qquad [0 < t < 1]$$

Finally, for $t > 1$, all of $h(t-\tau)$ is to the right of the vertical axis, and we again need to break up the integral into two parts, the first from $t - 1$ to t, and the other from t to $t + 1$…



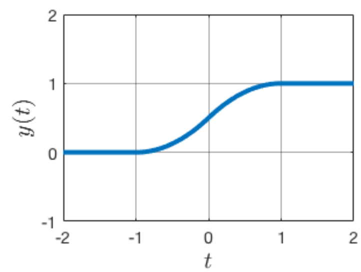
$$y(t) = \int_{t-1}^{t} -1\,\tau\, d\tau + \int_t^{t+1} 1\,\tau\, d\tau$$

$$= -\frac{t^2}{2} + \frac{(t-1)^2}{2} + \frac{(t+1)^2}{2} - t^2$$

$$= 1 \qquad [t > 1]$$

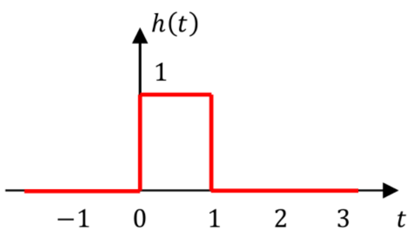
Putting all this together…

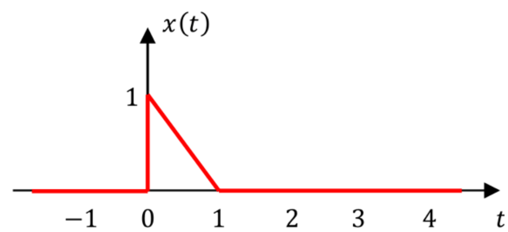
$$y(t) = \begin{cases} 0 & t < -1 \\ \dfrac{(t+1)^2}{2} & -1 < t < 0 \\ \dfrac{(t+1)^2}{2} - t^2 & 0 < t < 1 \\ 1 & t > 1 \end{cases}$$



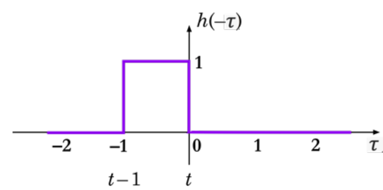
Note that the final result is a continuous function. This is a very useful check on your results, i.e., if you solve a problem like this, you can find your mistake by checking continuity.

(b) Use $h(t)$ and $x(t)$ as defined graphically below.

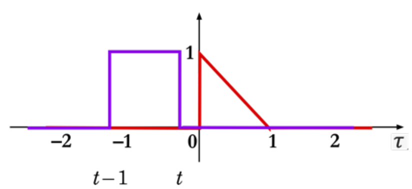


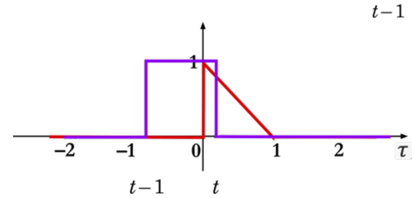


ANSWER:



Again, plotting $h(-\tau)$, as shown to the right, we can see where $t = 0$ is located, which means t is the right edge, and $t - 1$ is the left edge. $h(t-\tau)$ is the time-shifted version of this. On the left below, $t < 0$, so the product of the two functions is 0, and the resulting integral is also 0.



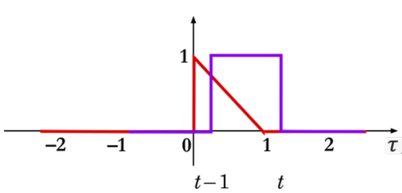


Next, for $0 < t < 1$, the pulse is overlapping with the non-zero part of $x(\tau)$, as shown on the right above, so the limits are 0 and t, and the integral is non-zero…

$$y(t) = \int_0^t 1\,(1-\tau)\,d\tau = t - \frac{t^2}{2} \qquad [0 < t < 1]$$

In other words, $x(\tau)$ is 0 for $\tau < 0$, and $h(t-\tau) = 0$ for $\tau > t$, so the limits are 0 to t.

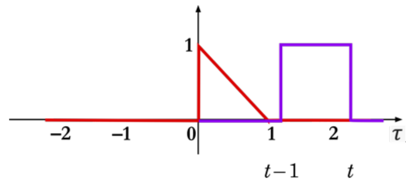
Next, for $1 < t < 2$, the pulse is completely to the right of the vertical axis, as shown on the left below. The limits are now $t - 1$ and 1. The resulting integral is on the right.



$$y(t) = \int_{t-1}^{1} 1\,(1-\tau)\,d\tau$$

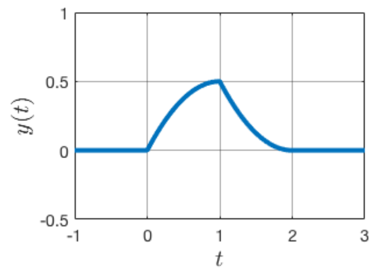
$$= 1 - \frac{1^2}{2} - \left(t - 1 - \frac{(t-1)^2}{2}\right) = 2 - 2t + \frac{t^2}{2} \qquad [1 < t < 2]$$

Finally, for $t > 2$, all of $h(t-\tau)$ is to the right of the signal $x(\tau)$, so the integral is again 0.



Putting all this together…

$$y(t) = \begin{cases} 0 & t < 0 \\ t - \dfrac{t^2}{2} & 0 < t < 1 \\ 2 - 2t + \dfrac{t^2}{2} & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

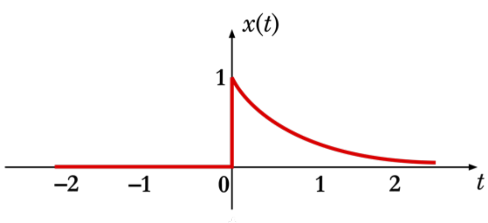


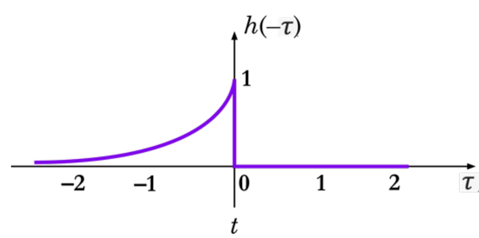
Note that the final result is again a continuous function.

(c) [From Table 2-2, entry 4] Use $h(t) = e^{at}\,u(t),\ x(t) = e^{bt}\,u(t),\ a \neq b$.

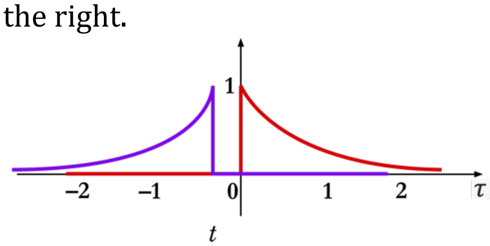
ANSWER:

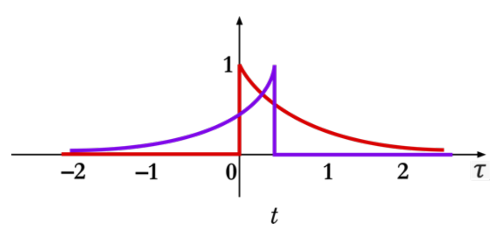
The input signal is shown below on the left. The time-reversed impulse response is shown on the right.





This means time t is the right edge of $h(t-\tau)$. For $t < 0$, the two functions do not overlap, as shown below on the left, so the integral is 0. For $t > 0$, the two functions overlap, but only from 0 to t, as shown below on the right.





Therefore…

$$y(t) = \int_0^t e^{a(t-\tau)}\,e^{b\tau}\,d\tau = \frac{e^{at}}{b-a}\left(e^{(b-a)t} - 1\right) = \frac{e^{bt} - e^{at}}{b-a} = \frac{e^{at} - e^{bt}}{a-b} \qquad [t > 0]$$

… or, since $y(t) = 0$ for $t < 0$…

$$y(t) = \frac{e^{at} - e^{bt}}{a-b}\,u(t)$$