ECE 213 – Continuous-Time Signals and Systems
Spring 2022

Midterm 2 Solution
April 5, 2022

1. (25 pts) Consider an LTI system described by the impulse response

$$h(t) = \delta(t) + e^{-t}u(t).$$

(a) (5 pts) Find the transfer function $H(s)$.

(b) (10 pts) Find the output $y(t)$ when the input is $x(t) = \cos t$.

(c) (10 pts) Find the output $y(t)$ when the input is $x(t) = \cos(t)u(t)$.

Solution:
(a) Taking the Laplace transform of $h(t)$,

$$H(s) = 1 + \frac{1}{s+1} = \frac{s+2}{s+1}. \tag{1}$$

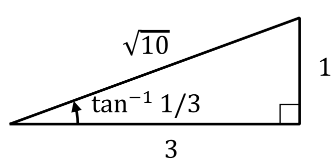
(b) For this noncausal sinusoidal input, we want to evaluate the frequency response function at $\omega = 1$.

$$H(1) = H(s)\Big|_{s=j} = \frac{2+j}{1+j} = \frac{(2+j)(1-j)}{(1+j)(1-j)} = \frac{3-j}{2} = \frac{\sqrt{10}}{2}e^{-j\tan^{-1}(1/3)} = |H(1)|e^{j\theta}. \tag{2}$$

Hence,

$$y(t) = |H(1)|\cos(t+\theta) = \frac{\sqrt{10}}{2}\cos\left(t - \tan^{-1}\frac{1}{3}\right). \tag{3}$$

Using the right triangle figure below,



we find

$$\cos\left(\tan^{-1}\frac{1}{3}\right) = \frac{3}{\sqrt{10}}, \ \sin\left(\tan^{-1}\frac{1}{3}\right) = \frac{1}{\sqrt{10}}. \tag{4}$$

Hence, $y(t)$ can also be written as

$$y(t) = \frac{\sqrt{10}}{2}\left(\cos t \times \frac{3}{\sqrt{10}} + \sin t \times \frac{1}{\sqrt{10}}\right) = \frac{3}{2}\cos t + \frac{1}{2}\sin t. \tag{5}$$

(c) For this causal input, we have

$$X(s) = \frac{s}{s^2+1}. \tag{6}$$

The output has

$$Y(s) = H(s)X(s) = \frac{s(s+2)}{(s+1)(s^2+1)} = \frac{s(s+2)}{(s+1)(s+j)(s-j)} = \frac{A_1}{s+1} + \frac{A_2}{s+\underbrace{j}_{a+jb}} + \frac{A_3}{s-j}. \tag{7}$$

The coefficients are

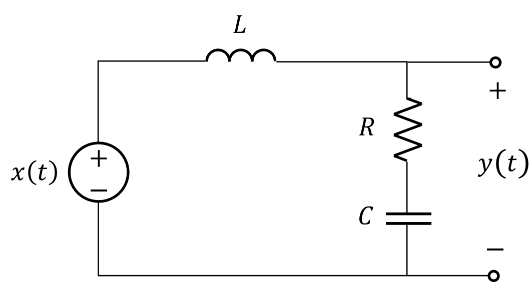
$$A_1 = (s+1)Y(s)\Big|_{s=-1} = \frac{-1}{(-1+j)(-1-j)} = -\frac{1}{2}, \tag{8}$$

$$\begin{aligned} A_2 &= (s+j)Y(s)\Big|_{s=-j} = \frac{-j(2-j)}{(1-j)(-j2)} = \frac{1}{2}\frac{1+j2}{1+j} = \frac{1}{2}\frac{(1+j2)(1-j)}{2} \\ &= \frac{1}{4}(3+j) = \frac{\sqrt{10}}{4}e^{j\tan^{-1}(1/3)} = Ae^{j\theta}. \end{aligned} \tag{9}$$

Inverting (7) using $a = 0$, $b = 1$, $A = \sqrt{10}/4$, and $\theta = \tan^{-1}(1/3)$,

$$\begin{aligned} y(t) &= -\frac{1}{2}e^{-t}u(t) + 2Ae^{-at}\cos(bt-\theta)u(t) = -\frac{1}{2}e^{-t}u(t) + \frac{\sqrt{10}}{2}\cos\left(t - \tan^{-1}\frac{1}{3}\right)u(t) \\ &= -\frac{1}{2}e^{-t}u(t) + \left(\frac{3}{2}\cos t + \frac{1}{2}\sin t\right)u(t). \end{aligned} \tag{10}$$

2. (25 pts) Consider the RLC circuit shown below with $R = 3\ \Omega$, $L = 1$ H, and $C = 0.5$ F.

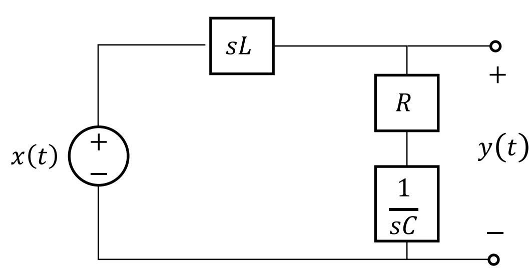


(a) (7 pts) Using the s-domain circuit analysis, find the transfer function $H(s)$.

(b) (9 pts) Find the impulse response $h(t)$.

(c) (9 pts) Find the output $y(t)$ when the input is $x(t) = e^{-3t}u(t)$.

Solution:
(a) The s-domain circuit is shown below.



Using voltage division,

$$Y(s) = X(s)\frac{R + \frac{1}{sC}}{sL + R + \frac{1}{sC}} = X(s)\frac{sRC + 1}{s^2LC + sRC + 1}. \tag{11}$$

Using $R = 3\ \Omega$, $L = 1$ H, and $C = 0.5$ F,

$$H(s) = \frac{1.5s + 1}{0.5s^2 + 1.5s + 1} = \frac{3s + 2}{s^2 + 3s + 2}. \tag{12}$$

(b) The partial fraction for $H(s)$ is

$$H(s) = \frac{3s + 2}{(s + 1)(s + 2)} = \frac{A_1}{s + 1} + \frac{A_2}{s + 2}. \tag{13}$$

The coefficients are

$$A_1 = (s + 1)H(s)\Big|_{s=-1} = \frac{3(-1) + 2}{(-1) + 2} = -1, \tag{14}$$

$$A_2 = (s + 2)H(s)\Big|_{s=-2} = \frac{3(-2) + 2}{(-2) + 1} = 4. \tag{15}$$

Hence,

$$h(t) = (-e^{-t} + 4e^{-2t})u(t). \tag{16}$$

(c) For input $x(t) = e^{-3t}u(t)$,

$$X(s) = \frac{1}{s + 3}. \tag{17}$$

The output has

$$Y(s) = H(s)X(s) = \frac{3s + 2}{(s + 1)(s + 2)(s + 3)} = \frac{A_1}{s + 1} + \frac{A_2}{s + 2} + \frac{A_3}{s + 3}. \tag{18}$$

The coefficients are

$$A_1 = (s + 1)Y(s)\Big|_{s=-1} = \frac{-1}{1 \times 2} = -\frac{1}{2}, \tag{19}$$

$$A_2 = (s + 2)Y(s)\Big|_{s=-2} = \frac{-4}{(-1)1} = 4, \tag{20}$$

$$A_3 = (s + 3)Y(s)\Big|_{s=-3} = \frac{-7}{(-2)(-1)} = -\frac{7}{2}. \tag{21}$$

Hence, the output is

$$y(t) = \left(-\frac{1}{2}e^{-t} + 4e^{-2t} - \frac{7}{2}e^{-3t}\right)u(t). \tag{22}$$

3. (25 pts) An LTI system is described by the input-output LCCDE

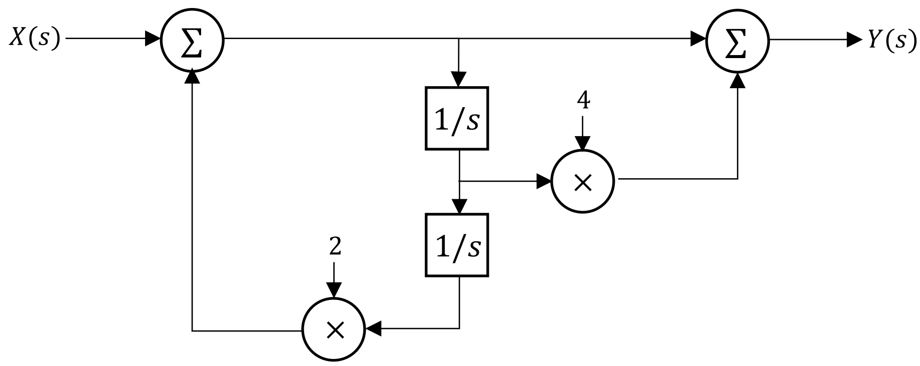
$$2\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} - y(t) = \frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt}.$$

This LCCDE applies to parts (a) and (b) below.

(a) (5 pts) Find the transfer function $H(s)$ of the system.

(b) (10 pts) Draw the Direct Form II implementation.

(c) (10 pts) Write the input-output LCCDE for the system having the Direct Form II implementation shown below.



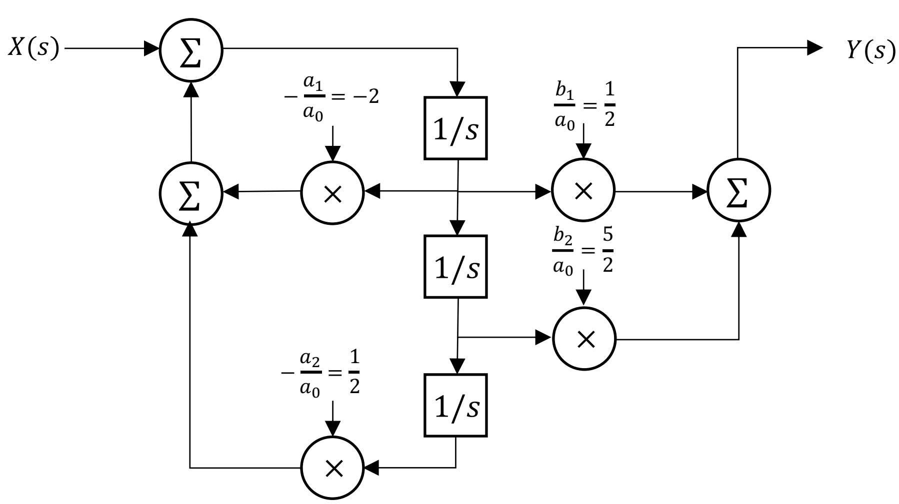
Solution:
(a) From the LCCDE, we recognize that $n = 3$. We identify the coefficients as

$$a_0 = 2,\ a_1 = 4,\ a_2 = 0, a_3 = -1,\ b_0 = 0,\ b_1 = 1,\ b_2 = 5,\ b_3 = 0. \tag{23}$$

The transfer function is

$$H(s) = \frac{s^2 + 5s}{2s^3 + 4s^2 - 1}. \tag{24}$$

(b) From the general DFII implementation diagram from the class notes or Discussion 8 notes, the DFII implementation for the given LCCDE is



(c) We note $n = 2$. From the DFII implementation, we read

$$-\frac{a_1}{a_0} = 0,\ -\frac{a_2}{a_0} = 2,\ \frac{b_0}{a_0} = 1,\ \frac{b_1}{a_0} = 4,\ \frac{b_2}{a_0} = 0. \tag{25}$$

Choosing $a_0 = 1$, we find

$$a_1 = 0,\ a_2 = -2,\ b_0 = 1,\ b_1 = 4, b_2 = 0. \tag{26}$$

The associated LCCDE is then

$$\frac{d^2y(t)}{dt^2} - 2y(t) = \frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt}. \tag{27}$$

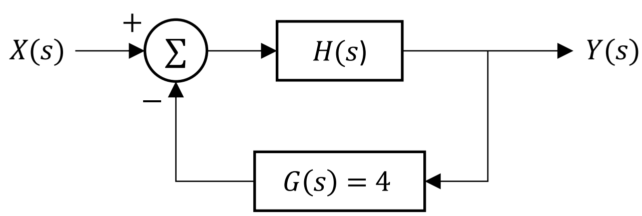
4. (25 pts) Consider an LTI system described by the transfer function

$$H(s) = \frac{s}{s^2 - 2s + 2}.$$

(a) (7 pts) Draw the pole-zero plot of $H(s)$. Remember that a pole-zero plot is the complex-s plane with zeros and poles indicated with '$\circ$' and '$\times$' symbols, respectively.

(b) (6 pts) Is the system BIBO stable? Justify your answer for full credit.

(c) (7 pts) For the proportional feedback system with $G(s) = 4$ below, find the closed-loop transfer function $Q(s) = Y(s)/X(s)$.



(d) (5 pts) Draw the pole-zero plot of $Q(s)$.

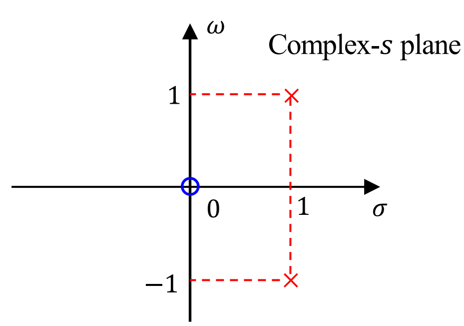
Solution:
(a) There is one zero at

$$z_1 = 0. \tag{28}$$

Setting the denominator of $H(s)$ to zero, we find the two poles at

$$p_{1,2} = 1 \pm \sqrt{1^2 - 1 \times 2} = 1 \pm j. \tag{29}$$

The pole-zero plot is



(b) The number of zeros is one ($m = 1$) and the number of poles is two ($n = 2$). Since $m < n$, $H(s)$ is a strictly proper rational function of s. In this case, all poles should have negative real parts for stability. However, both p_1 and p_2 have positive real parts. Hence, the system is not BIBO stable.

(c) Writing $H(s) = N(s)/D(s)$, we recognize

$$N(s) = s, \;\; D(s) = s^2 - 2s + 2. \tag{30}$$

For the feedback system with $G(s) = 4$, the closed-loop transfer function is

$$Q(s) = \frac{N(s)}{D(s) + G(s)N(s)} = \frac{s}{s^2 - 2s + 2 + 4 \times s} = \frac{s}{s^2 + 2s + 2}. \tag{31}$$

(d) The zero is unchanged from (28). Setting the denominator of $Q(s)$, the new poles are at

$$p_{1,2} = -1 \pm \sqrt{(-1)^2 - 1 \times 2} = -1 \pm j. \tag{32}$$

The pole-zero plot of $Q(s)$ is shown below.