Math 235 Section 8 Exercise Sheet 2

February 24, 2023

1 Basis Practice

For each of the following sets, find a basis. Prove that you found a basis (demonstrate that the two conditions hold).

1. The image of the matrix
$$\begin{pmatrix} 1 & -2 & 0 \\ 6 & -6 & -12 \\ 1 & 1 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

2. The kernel of the function
$$T(x_1, x_2, x_3) = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$$

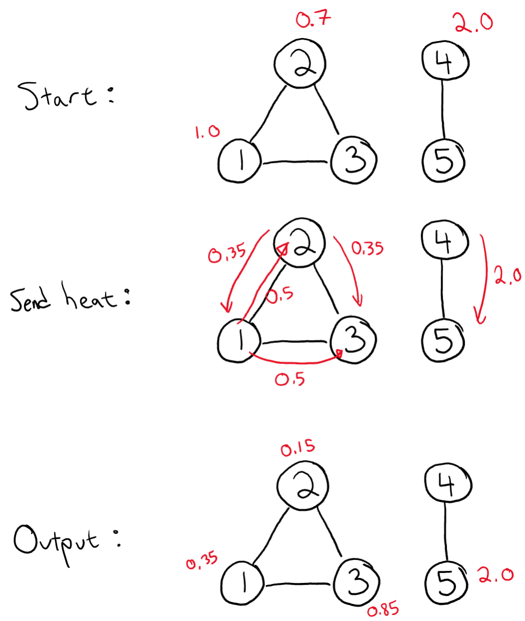
3. The subset of $\mathbb{R}^5$ defined by the intersection of the two hyperplanes with equations:
$$\begin{array}{cccccc} x_1 & +2x_2 & -3x_3 & +x_4 & +x_5 & = 0 \\ & 5x_2 & & +x_4 & -x_5 & = 0 \end{array}$$

2 Applications of Linear Algebra: The Heat Equation

A graph is built out of vertices and edges. Here's a game we can play on a graph:

- To start, put a heat value (real number) on every vertex. This is the input data.
- Vertices send their heat to their neighbors, splitting it equally. That is, if a vertex u has neighbors $w_1, \ldots, w_k$, add $\frac{1}{k}$ times the old heat of u to the new heat of w_1, and to the new heat of w_2, and so on through w_k.
- Output the new heat values.

This game is *linear*: the new heat values are a linear function of the old heat values. Think of the game as a function T, from old heats to new heats. Here is an example of playing the game, on a specific graph:



Let's analyze how the game works for this specific graph:

1. If $x_1, \dots, x_5$ are arbitrary input heat values for vertices 1-5, and $y_1, \dots, y_5$ are the corresponding output heat values, write down the equations for T so that
$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}\right) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}.$$

2. Consider the set V_1 of vectors $\vec{v}$ such that $T(\vec{v}) = \vec{v}$. Find a basis for V_1, and show what happens when I use those vectors to play the game. Hint: $T(\vec{v}) = \vec{v}$ is another linear equation, saying $T(\vec{v}) - \vec{v} = 0$. Write the equations out, and solve the system.

3. Consider the set $V_{-1/2}$ of vectors $\vec{v}$ such that $T(\vec{v}) = -\frac{1}{2}\vec{v}$. Find a basis for $V_{-1/2}$, and show what happens when I use those vectors to play the game.