# COMPSCI 250 Discussion #2: A Murder Mystery
Individual Handout

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Detectives are famous for the use of deductive reasoning. "When you have eliminated the impossible," said Sherlock Holmes, "whatever remains, however improbable, must be the truth". The case at first appears to offer a vast number of possibilities for what happened, but the clues eliminate some of these possibilities and may allow us to draw some conclusions.

In this discussion we offer some examples of propositional deduction in the form of a murder mystery[1]. In our first example, the possible situations are modeled by four boolean variables, and we are given four clues. It happens that *exactly one* of the $2^4 = 16$ combinations of truth values for the variables is consistent with all the clues. It would be straightforward but tedious to find out which with a truth table – our job is to use the rules of propositional logic to get the answer more quickly.

- The killer was Professor Plum ($p$) or Colonel Mustard ($\neg p$), but not both.
- The murder happened in the kitchen ($k$) or the study ($\neg k$), but not both.
- The murder weapon was a candlestick ($c$), a baseball bat ($b$), both ($c \wedge b$), or neither ($\neg c \wedge \neg b$).
- (I) If Professor Plum did it or if it was with the candlestick, then it was not with the baseball bat ($(p \vee c) \rightarrow \neg b$).
- (II) If it was in the study, the baseball bat was used ($\neg k \rightarrow b$).
- (III) If it was in the kitchen, then it was Colonel Mustard with the candlestick ($k \rightarrow (\neg p \wedge c)$).
- (IV) If it was with the candlestick, then it was Professor Plum in the study ($c \rightarrow (p \wedge \neg k)$).

Where do we start? Since we don't yet know the conclusion of our proof, we can't use the forward-backward method in its standard form. But we can begin with the forward part, looking at what interesting compound statements we might be able to derive from the clues.

Since the clues are in the form of implications, it would be easier if we had some premises to start with. We could start with any tautology. Here's an idea – we have both $k$ and $\neg k$ on the left-hand side of rules. Maybe we can prove some conclusion in the case where $k$ holds *and* in the case where $\neg k$ holds, and thus prove this conclusion by cases. Let's start by seeing what we can derive from $k$:

[1]Inspired by the board game *Clue*, a trademark of Parker Brothers games.

| | |
|---|---|
| $k$ | Premise |
| $\neg p \wedge c$ | Modus Ponens, Clue III |
| $c$ | Right Separation |
| $p \wedge \neg k$ | Modus Ponens, Clue IV |
| $(\neg p \wedge c) \wedge (p \wedge \neg k)$ | Conjunction |
| $(\neg p \wedge p) \wedge (c \wedge \neg k)$ | Commutativity, Associativity of $\wedge$ |
| $0 \wedge (c \wedge \neg k)$ | Variant of Excluded Middle |
| $0$ | Left Zero |

The assumption that it was in the kitchen leads to a contradiction. We've thus proved by contradiction that if the clues hold, the murder was in the study. We can now take this and run with it!

| | |
|---|---|
| $\neg k$ | Conclusion from above |
| $b$ | Modus Ponens, Clue II |

So it was the baseball bat! What more can we learn from this? We don't have $b$ on the left-hand side of a rule, but we have $\neg b$ on the right-hand side, which is just as good if we take the contrapositive:

| | |
|---|---|
| $b \rightarrow \neg(p \vee c)$ | Contrapositive of Rule I |
| $\neg(p \vee c)$ | Modus Ponens, last two lines |
| $\neg p \wedge \neg c$ | DeMorgan Or-to-And |
| $\neg k \wedge b \wedge \neg p \wedge \neg c$ | Conjunction |

So it was Colonel Mustard, in the study, with the baseball bat. Actually there is one other possibility that we should rule out before we make this our final conclusion. It might be that *no* setting of the four variables makes all the clues true[2]. We should check that our deduced setting satisfies all four clues. But it does: the premise $p \vee c$ of Clue I is false, the conclusion $b$ of Clue II is true, and both the premise $k$ of Clue III and the premise $c$ of Clue IV are false. So we have found the *unique* setting of the variables satisfying the clues.

**Writing Exercises:**

You will get two murder mysteries to solve. Here is the first: the other is on your group response sheet.

1. Here the meanings of the four basic propositions $b$, $c$, $k$, and $p$ are exactly the same as in the example above, given by the first three unnumbered statements. Your new clues are:
   (a) Clue I: If the candlestick was used, then Professor Plum did it.
   (b) Clue II: If Professor Plum did it, then he did not use either the baseball bat or the candlestick.
   (c) Clue III: If the baseball bat was not used, then Professor Plum did it in the kitchen.
   (d) Clue IV: If the candlestick was not used, then the baseball bat was used in the study.

[2]This would mean that in the real world, one of the clues is false – detectives do sometimes make mistakes!