M235-S19-PP9 sol - problem set solution

Linear Algebra 2 (Hon Math) (University of Waterloo)

1) Orthogonally diagonalize each of the following matrices; that is, find an orthogonal matrix P and diagonal matrix D such that $P^T AP = D$.

(a) $A = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$

Solution: We start with finding the eigenvalues and eigenspaces, as per usual diagonalization.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 32 = (\lambda - 4)(\lambda - 8)$$

$$\Rightarrow \lambda_1 = 4, \ \lambda_2 = 8.$$

<u>For λ_1:</u> $A - 4I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_1}(A) = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

<u>For λ_2 :</u> $A - 8I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_2}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

We now need to orthonormalize each basis from each eigenspace. Since both eigenspaces are 1-dimensional, there's no orthogonalization required. We simply need to normalize the vectors and put them into the columns of a matrix! We have

$$p_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad p_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then taking $P = \begin{bmatrix} p_1 & p_2 \\ \downarrow & \downarrow \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, we have

$$P^T AP = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}.$$

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(b) $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 4 & 4 \\ 3 & 4 & 4 \end{bmatrix}$

Solution: Again, we start with the eigenvalues and eigenspaces.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ 3 & 4-\lambda & 4 \\ 3 & 4 & 4-\lambda \end{vmatrix}$$

$$\underline{\underline{C_3 := C_3 - C_2}} \begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 4-\lambda & \lambda \\ 3 & 4 & -\lambda \end{vmatrix}$$

$$\underline{\underline{R_2 := R_2 + R_3}} \begin{bmatrix} 1-\lambda & 3 & 0 \\ 6 & 8-\lambda & 0 \\ 3 & 4 & -\lambda \end{bmatrix}$$

$$= -\lambda\left[(1-\lambda)(8-\lambda)-18\right] = -\lambda\left[\lambda^2-9\lambda-10\right] = -\lambda(\lambda+1)(\lambda-10)$$

$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 10.$$

<u>For λ_1:</u> $A - 0I = A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 4 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_1}(A) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$.

<u>For λ_2:</u> $A + I = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_2}(A) = \operatorname{span}\left\{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right\}$.

<u>For λ_3:</u> $A - 10I = \begin{bmatrix} -9 & 3 & 3 \\ 3 & -6 & 4 \\ 3 & 4 & -6 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_3}(A) = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}\right\}$.

Again, we are in a situation where each eigenspace is 1-dimensional, so we need only normalize the basis vectors given to create an orthonormal basis of $\mathbb{R}^3$. We have

$$p_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},\quad p_2 = \frac{1}{\sqrt{11}}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix},\quad p_3 = \frac{1}{\sqrt{22}}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}.$$

Then taking $P = \begin{bmatrix} p_1 & p_2 & p_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & -3/\sqrt{11} & 2/\sqrt{22} \\ -1/\sqrt{2} & 1/\sqrt{11} & 3/\sqrt{22} \\ 1/\sqrt{2} & 1/\sqrt{11} & 3/\sqrt{22} \end{bmatrix}$, we have

$$P^TAP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 10 \end{bmatrix}.$$

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(c) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$

$$\underline{\underline{C_3 := C_3 + C_2}} \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 3-\lambda \\ -1 & 1 & 3-\lambda \end{vmatrix}$$

$$\underline{\underline{R_2 := R_2 - R_3}} \begin{vmatrix} 2-\lambda & 1 & 0 \\ 2 & 1-\lambda & 0 \\ -1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(2-\lambda)(1-\lambda)-2\right] = (3-\lambda)\left[\lambda^2-3\lambda\right] = (3-\lambda)\lambda(\lambda-3)$$

$$\Rightarrow \ \lambda_1 = 0, \ \lambda_2 = 3.$$

Aside: Here, our eigenvalue $\lambda_2 = 3$ has algebraic multiplicity 2. Since A is symmetric, we know from the Principal Axis Theorem that A is orthogonally diagonalizable, and thus is diagonalizable, so the geometric multiplicity of $\lambda_2 = 3$ **MUST** be 2. Keep this in mind as you compute the corresponding eigenspace - if your multiplicities mismatch when your original matrix was symmetric, then you have made an error. Same goes for Hermitian matrices in the unitary diagonalization process (by the Spectral Theorem).

<u>For λ_1:</u> $A - 0I = A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, $E_{\lambda_1}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

<u>For λ_2:</u> $A - 3I = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, $E_{\lambda_2}(I) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

Now that we have a multi-dimensional eigenspace, we need to check that the basis is orthogonal, and orthogonalize it if not. We see $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 \neq 0$, so we need to apply the Gram-Schmidt Procedure. We have

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \operatorname{proj}_{v_1}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

As in previous solution sets, we can take $v_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Now we have orthogonal bases for each eigenspace, so we can proceed with normalization. Let

$$p_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad p_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad p_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Then taking $P = \begin{bmatrix} p_1 & p_2 & p_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$, we have

$$P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

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2) Unitarily diagonalize each of the following matrices; that is, find a unitary matrix U and diagonal matrix D such that $U^*AU = D$.

(a) $A = \begin{bmatrix} 4i & 1+3i \\ -1+3i & i \end{bmatrix}$

Solution: As with orthogonal diagonalization, we start with the eigenvalues and eigenspaces.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 4i-\lambda & 1+3i \\ -1+3i & i-\lambda \end{vmatrix} \\ &= (4i-\lambda)(i-\lambda) - (1+3i)(1-3i) \\ &= \left[\lambda^2 - 5i\lambda - 4\right] + 10 \\ &= \lambda^2 - 5i + 6 = (\lambda - 6i)(\lambda + i) \\ \Rightarrow\ \lambda_1 = 6i,\ \lambda_2 = -i. \end{aligned}$$

see factoring trick, top of Page 5.

For λ_1: $A - 6iI = \begin{bmatrix} -2i & 1+3i \\ -1+3i & i \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & \frac{-3}{2} + \frac{i}{2} \\ 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_1}(A) = \operatorname{span}\left\{\begin{bmatrix} 3-i \\ 2 \end{bmatrix}\right\}$.

For λ_2 : $A + i = \begin{bmatrix} 5i & 1+3i \\ -1+3i & 2i \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & \frac{3}{5} - \frac{i}{5} \\ 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_2}(A) = \operatorname{span}\left\{\begin{bmatrix} -3+i \\ 5 \end{bmatrix}\right\}$.

Again, no Gram-Schmidt is required here. Normalizing, we have

$$u_1 = \frac{1}{\sqrt{14}}\begin{bmatrix} 3-i \\ 2 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{35}}\begin{bmatrix} -3+i \\ 5 \end{bmatrix}.$$

Therefore, taking $U = \begin{bmatrix} u_1 & u_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \frac{3-i}{\sqrt{14}} & \frac{-3+i}{\sqrt{35}} \\ \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \end{bmatrix}$, we have

$$U^*AU = \begin{bmatrix} 6i & 0 \\ 0 & -i \end{bmatrix}.$$

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Complex Factoring Trick: In factoring $\lambda^2 - 5i + 6$, the linear coefficient was **purely** imaginary and the others were real. This scenario is not uncommon, so it's nice to have a trick to factor here, as we do for real quadratics.

We can factor by asking "Which two numbers add to -5 and multiply to -6?" (note the negation of the constant term!!) The answer to this is -6 and $+1$. Multiply these by i to factor $\lambda^2 - 5i\lambda + 6 = (\lambda - 6i)(\lambda + i)$. Try this out with other quadratics of the same form!

(b) $A = \begin{bmatrix} 1 & 0 & 1+i \\ 0 & 2 & 0 \\ 1-i & 0 & 0 \end{bmatrix}$

Solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1+i \\ 0 & 2-\lambda & 0 \\ 1-i & 0 & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 1+i \\ 1-i & -\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[\lambda^2 - \lambda - 2\right] = (2-\lambda)(\lambda-2)(\lambda+1)$$

$$\Rightarrow \ \lambda_1 = 2, \quad \lambda_2 = -1.$$

<u>For λ_1:</u> $A - 2I = \begin{bmatrix} -1 & 0 & 1+i \\ 0 & 0 & 0 \\ 1-i & 0 & -2 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 0 & -1-i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus $E_{\lambda_1}(A) = \operatorname{span}\left\{\begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

<u>For λ_2:</u> $A + I = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 3 & 0 \\ 1-i & 0 & 1 \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{2}+\frac{i}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus $E_{\lambda_2}(A) = \operatorname{span}\left\{\begin{bmatrix} -1-i \\ 0 \\ 2 \end{bmatrix}\right\}$.

We can see that the basis for $E_{\lambda_1}(A)$ is already orthogonal. Thus, we need only normalize the vectors. Let

$$u_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1-i \\ 0 \\ 2 \end{bmatrix}.$$

Then taking $U = \begin{bmatrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \frac{1+i}{\sqrt{3}} & 0 & \frac{-1-i}{\sqrt{6}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$, we have

$$U^*AU = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

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(c) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a, b \in \mathbb{R}.$

Solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + a^2 + b^2$$

$$\Rightarrow \lambda = \frac{2a \pm \sqrt{4a^2 - (4a^2 + 4b^2)}}{2} = a \pm bi.$$

For $\lambda_1 = a + bi$: $A - (a+bi)I = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \xrightarrow{\textbf{RREF}} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$. Thus $E_{\lambda_1}(A) = \operatorname{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = a - bi$: As A is a real matrix with imaginary eigenvalues, we can use an old trick from Practice Problem Set 1: real matrices with imaginary eigenvalues have their eigenvalues and eigenvectors come in conjugate pairs! Thus $E_2(A) = \operatorname{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$ as $\lambda_2 = \overline{\lambda_1}$.

Normalizing, let

$$u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Taking $U = \begin{bmatrix} u_1 & u_2 \\ \downarrow & \downarrow \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, we have

$$U^*AU = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}.$$

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Note: Compare this with Practice Problem Set 1 Question 7b.

3) Let $A, B \in M_n(\mathbb{C})$ be Hermitian matrices. Prove that AB is Hermitian if and only if $AB = BA$.

Proof: We have

$$AB \text{ is Hermitian} \iff (AB)^* = AB$$

$$\iff B^*A^* = AB$$

$$\iff BA = AB \quad \textbf{as } A \textbf{ and } B \textbf{ are Hermitian}.$$

$\square$

4) Let $A \in M_n(\mathbb{C})$ be **skew-Hermitian**; that is, $A^* = -A$. Prove that every eigenvalue of A is purely imaginary.

Proof: Let λ be an eigenvalue of A with corresponding eigenvector z. Examining $\lambda \langle z, z \rangle$, we have

$$\begin{aligned}
\lambda \langle z, z \rangle &= \langle \lambda z, z \rangle \\
&= \langle Az, z \rangle \\
&= \langle z, A^* z \rangle \\
&= \langle z, -Az \rangle \\
&= \langle z, -\lambda z \rangle \\
&= -\overline{\lambda} \langle z, z \rangle .
\end{aligned}$$

As z is an eigenvector, $z \neq \vec{0}$, thus $\langle z, z \rangle > 0$. Therefore,

$$\lambda \langle z, z \rangle = -\overline{\lambda} \langle z, z \rangle \quad \Rightarrow \quad -\overline{\lambda} = \lambda, \quad \text{i.e. } \overline{\lambda} = -\lambda.$$

Thus λ is purely imaginary. As λ was chosen arbitrarily, this holds for all eigenvalues of A.

$\square$

5) Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Prove there exists a symmetric $B \in M_n(\mathbb{R})$ such that $B^3 = A$.

Proof: As A is symmetric, there exists an orthogonal matrix $P \in M_n(\mathbb{R})$ such that

$$P^T AP = D = \operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n), \quad \text{i.e. } A = PDP^T.$$

Notation: $\textbf{diag}(\lambda_1, \lambda_2, ..., \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Let $D_1 = \operatorname{diag}\left(\sqrt[3]{\lambda_1}, \sqrt[3]{\lambda_2}, ..., \sqrt[3]{\lambda_n}\right)$. We see that $D_1^3 = D$. Let $B \in M_n(\mathbb{R})$ such that $B = PD_1P^T$. Then

$$B^3 = (PD_1P^T)(PD_1P^T)(PD_1P^T) = PD_1^3P^T = PDP^T = A.$$

$\square$

6) Prove that every matrix $A \in M_n(\mathbb{C})$ is unitarily similar to a lower-triangular matrix $T \in M_n(\mathbb{C})$.

Proof: By Schur's Theorem, we have that $A^* \in M_n(\mathbb{C})$ is unitarily similar to an upper-triangular matrix $T \in M_n(\mathbb{C})$. Let $U \in M_n(\mathbb{C})$ such that

$$U^*A^*U = T.$$

Then

$$T^* = (U^*A^*U)^* = U^*(A^*)^*(U^*)^* = U^*AU.$$

Thus A is unitarily similar to T^*. As T is upper-triangular, T^* is lower-triangular, as required.

□

7) Let $A \in M_n(\mathbb{R})$ be symmetric. Prove that $A^2 = I$ if and only if the only eigenvalues of A are ± 1.

Proof: A is symmetric if and only if it is orthogonally diagonalizable, thus there exists an orthogonal matrix $P \in M_n(\mathbb{R})$ and a diagonal matrix $D \in M_n(\mathbb{R})$ such that

$$P^TAP = D, \quad \text{i.e.} \quad A = PDP^T.$$

From here, we have that $A^2 = PDP^TPDP^T = PD^2P^T$. Therefore,

$$\begin{aligned} A^2 = I &\iff PD^2P^T = I \\ &\iff D^2 = P^TIP = P^TP = I. \end{aligned}$$

As D is diagonal, we have $D^2 = I$ if and only if each of its diagonal entries square to 1. As $D \in M_n(\mathbb{R})$, we have $D^2 = I$ if and only if each of its diagonal entries are 1 or -1. As the diagonal entries of D are precisely the eigenvalues of A, we conclude that for a symmetric matrix A, $A^2 = I$ if and only if the only eigenvalues of A are 1 or -1.

□

8) (a) Prove that every upper-triangular orthogonal matrix $A \in M_3(\mathbb{R})$ is a diagonal matrix.

Proof: Let $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(\mathbb{R})$ be orthogonal. Then the columns of A are an orthonormal set. By normality, we have

$$\left\| \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \right\| = 1 \Rightarrow a = \pm 1.$$

Thus $A = \begin{bmatrix} \pm 1 & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$. Now, as the columns are mutually orthogonal, we have that

$$\begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} = \pm b = 0 \Rightarrow b = 0, \quad \text{and}$$

$$\begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} c \\ e \\ f \end{bmatrix} = \pm c = 0 \ \Rightarrow \ c = 0.$$

So now we know $A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$. Again, the second column must be a unit vector, thus $d = \pm 1$. From a similar argument to the first row, this forces $e = 0$. Finally, yet again, the third column must be a unit vector, thus $f = \pm 1$. Therefore,

$$A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix},$$

which is diagonal.

□

(b) Let $B \in M_3(\mathbb{R})$ be an orthogonal matrix and that all of the eigenvalues of B are real. Prove that B is orthogonally diagonalizable.

Proof: By the Triangularization Theorem, B is orthogonally similar to an upper-triangular matrix $T \in M_3(\mathbb{R})$. Let $Q \in M_3(\mathbb{R})$ be orthogonal such that

$$Q^T BQ = T.$$

As B, Q, and Q^T are all orthogonal matrices, we have that their product is orthogonal. Thus $T \in M_3(\mathbb{R})$ is an upper-triangular orthogonal matrix. From part **(a)**, we have that T a diagonal matrix. Therefore, $Q^T BQ$ is a diagonal matrix, so B is orthogonally diagonalizable.

□

9) Prove or disprove the following statements:

(a) If $P, Q \in M_n(\mathbb{C})$ are Hermitian matrices, then $\langle Q, P \rangle$ is the sum of the eigenvalues of PQ.
Recall: tr(A) is the sum of the n eigenvalues of A (repetition allowed)

Proof: This claim is **TRUE!** We have

$$\langle Q, P \rangle = \operatorname{tr}(P^* Q) = \operatorname{tr}(PQ). \quad \textbf{as } P \textbf{ is Hermitian.}$$

As the trace is the sum of the eigenvalues, we have that $\langle Q, P \rangle$ is the sum of the eigenvalues of PQ.

□

(b) There exists a matrix $A \in M_3(\mathbb{R})$ with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, respectively.

Proof: This claim is **TRUE!** First, note that

$$v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3 = 0,$$

so these vectors form an orthogonal basis of $\mathbb{R}^3$. Normalizing them, we construct the orthogonal matrix

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Let $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) = \operatorname{diag}(-1, 0, 1)$. Then

$$\begin{aligned} PDP^T &= \frac{1}{2} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Taking $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, we have that $P^T AP = D$, so A is diagonalizable, has eigenvalues $-1, 0$, and 1, with corresponding eigenvectors v_1, v_2, and v_3, respectively. $\square$