DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

Math 331 **Final Exam** **Fall 2022**

Name: ______________________________ **Student ID Number:** __________________________

Instructor name: _________________________ **Your section number** ___________________

In this exam there are 6 sheets, including this one, and there are 6 problems.

Instructions:

- Calculators and outside notes are **not allowed** to be used during the exam.
- A table of Laplace Transforms is provided to you on the back page.
- You must explain how you arrived at your answers, and show your algebraic calculations.
- You can leave fractions and square roots in your answers – no need to give decimal expansions.
- Be sure that your work on each problem stays inside of the boxed area.
- If you need to use the blank page on the back of the exam to finish your work on a problem, be sure to make a note on the problem that additional work can be found on the blank page and, also, label any/all additional work on the back page by its problem number(s).

Question	Points
1	16
2	16
3	17
4	17
5	17
6	17
Total:	100

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1. (16 points) Find the General Solution to the differential equation:

$$y'' + 2y' + 17y = e^{\alpha t}$$

where α is a real constant. Note that your answer will depend on α.

char eq: $r^2+2r+17=0$

$$r = \frac{-2 \pm \sqrt{4-4\cdot 17}}{2} = \frac{-2 \pm \sqrt{4-68}}{2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i$$

Gen. sln. for H.D.Eq: $y(t) = C_1 e^{-t}\cos(4t) + C_2 e^{-t}\sin(4t)$

$y_p(t) = Ae^{\alpha t}$

$y_p'(t) = \alpha A e^{\alpha t}$

$y_p''(t) = \alpha^2 A e^{\alpha t}$

$$\alpha^2 A e^{\alpha t} + 2\alpha A e^{\alpha t} + 17 A e^{\alpha t} = e^{\alpha t}$$

$$Ae^{\alpha t}(\alpha^2 + 2\alpha + 17) = e^{\alpha t}$$

$$A = \frac{1}{\alpha^2 + 2\alpha + 17}$$

$$y_p(t) = \frac{1}{\alpha^2+2\alpha+17} e^{\alpha t}$$

$$\boxed{\text{Gen. Sol: } y(t) = C_1 e^{-t}\cos(4t) + C_2 e^{-t}\sin(4t) + \frac{1}{\alpha^2+2\alpha+17} e^{\alpha t}}$$

2. (16 points) Find the solution to the initial value problem:

$$y'' + 6y' + 9y = 0 \quad \text{with } y(0) = 3 \text{ and } y'(0) = 7$$

char eq: $r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r = -3$ repeated roots

$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$

$y(0) = C_1 = 3$

$y'(t) = -3C_1 e^{-3t} + \left[C_2 e^{-3t} + C_2 t \left(-3e^{-3t} \right) \right]$

$y'(0) = -3C_1 + C_2 = 7$

$C_2 = 7 + 3C_1 = 7 + 9 = 16$

$\boxed{y(t) = 3e^{-3t} + 16te^{-3t}}$

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3. (17 points) Solve the Initial Value Problem:

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}.$$

$$y' + 2y = u_5(t) \cdot e^{t-5}, \qquad y(0) = 3$$

$$\mathcal{L}\{y'+2y\} = \mathcal{L}\{u_5(t) e^{t-5}\} = e^{-5s}\mathcal{L}\{e^t\}.$$

$$\mathcal{L}\{y'+2y\} = \mathcal{L}\{y'\} + \mathcal{L}\{2y\} = s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\}$$

$$= e^{-5s}\frac{1}{s-1}$$

$$s\mathcal{L}\{y\} - 3 + 2\mathcal{L}\{y\} = \frac{e^{-5s}}{s-1}$$

$$(s+2)\mathcal{L}\{y\} = \frac{e^{-5s}}{s-1} + 3.$$

$$\mathcal{L}\{y\} = \frac{e^{-5s}}{(s-1)(s+2)} + \frac{3}{s+2}.$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-2t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-1)(s+2)}\right\} = u_5(t) f(t-5)$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}.$$

$$\frac{A}{s-1} + \frac{B}{s+2} = \frac{1}{(s-1)(s+2)}$$

$$(s+2)A + (s-1)B = 1$$

$$s=1 \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3}$$

$$s=-2 \Rightarrow -3B = 1 \Rightarrow B = -\frac{1}{3}$$

$$\frac{1}{(s-1)(s-2)} = \frac{1}{3}\left(\frac{1}{s-1} - \frac{1}{s+2}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \frac{1}{3}\left(e^{t} - e^{-2t}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-1)(s-2)}\right\} = \frac{1}{3}u_5(t)\left(e^{t-5} - e^{-2(t-5)}\right)$$

$$\boxed{y = \frac{1}{3}u_5(t)\left(e^{t-5} - e^{-2(t-5)}\right) + 3e^{-2t}}$$

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4. (17 points) Compute the Inverse Laplace Transform:

$$\mathcal{L}^{-1}\left\{\frac{s+1+e^{-7s}}{s^2-4s+13}\right\}$$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

$F(s) = \mathcal{L}\{f(t)\}$.

$$\mathcal{L}^{-1}\left\{\frac{s+1+e^{-7s}}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+13}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-7s}}{s^2-4s+13}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s-2)^2+9}\right\}.$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2+2+1}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\}.$$

$$= e^{2t}\left(\cos(3t) + \sin(3t)\right)$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s-2)^2+9}\right\} = u_7(t)\,f(t-7) \qquad f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}.$$

$$= \frac{1}{3}u_7(t)\,e^{2(t-7)}\sin(3(t-7)) \qquad = \frac{1}{3}e^{2t}\sin(3t)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s+1+e^{-7s}}{s^2-4s+13}\right\} = e^{2t}\left(\cos(3t)+\sin(3t)\right) + \frac{1}{3}u_7(t)\,e^{2(t-7)}\sin(3(t-7))}$$

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5. (17 points) Find the solution to the Initial Value Problem:

$$\vec{Y}' = A\vec{Y}, \quad \text{with} \quad A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \vec{Y}(0) = \begin{pmatrix} 9 \\ 5 \end{pmatrix}.$$

char poly: $\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - (a+d)\lambda + ad - bc$

char poly: $\det(A-\lambda I) = \det\begin{pmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 - 1$

$= \lambda^2 - 8\lambda + 16 - 1$

$= \lambda^2 - 8\lambda + 15$

$= (\lambda-3)(\lambda-5)$

$\lambda_1 = 3 \quad \lambda_1 = 5$

$\lambda = 3$:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 3\begin{bmatrix} a \\ b \end{bmatrix}$$

$4a + b = 3a$

$b = -a$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 5$:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 5\begin{bmatrix} a \\ b \end{bmatrix}$$

$4a + b = 5a$

$b = a$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{Y} = c_1 e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{Y}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

$$\boxed{\vec{Y} = 2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 7e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\begin{cases} c_1 + c_2 = 9 & (1) \\ -c_1 + c_2 = 5 & (2) \end{cases}$

(1) + (2): $\cancel{c_1} + c_2 - \cancel{c_1} + c_2 = 14$

$2c_2 = 14$

$\boxed{c_2 = 7}$

(1) $\cdot$ $c_1 + 7 = 9 \Rightarrow \boxed{c_1 = 2}$

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6. Consider the following system of differential equations:

$$\vec{Y}' = A\vec{Y}, \quad \text{with} \quad A = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}$$

(a) (9 points) Find the General Solution to the system of differential equations.

$tr(A) = -3 + 1 = -2 \qquad det(A) = -3 - (-8) = 5$

char poly: $\lambda^2 + 2\lambda + 5 = 0.$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4\cdot 5}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$\lambda = -1 + 2i$

$$\begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = (-1+2i)\begin{bmatrix} a \\ b \end{bmatrix}$$

$$-3a + 4b = (-1+2i)a$$

$$4b = (2+2i)a$$

$$2b = (1+i)a$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Y_c(t) = e^{(-1+2i)t}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{-t}\left(\cos(2t) + i\sin(2t)\right)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

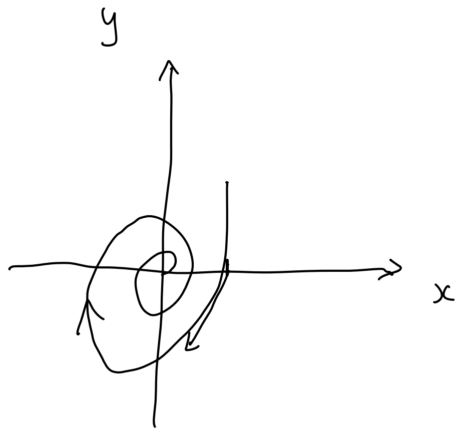
$$= e^{-t}\left[\left(\cos(2t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + i\left(\sin(2t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \cos(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]$$

$$= e^{-t}\begin{bmatrix} 2\cos(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix} + i e^{-t}\begin{bmatrix} 2\sin(2t) \\ \sin(2t) + \cos(2t) \end{bmatrix}$$

$$\boxed{Y(t) = C_1 e^{-t}\begin{bmatrix} 2\cos(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 2\sin(2t) \\ \sin(2t) + \cos(2t) \end{bmatrix}}$$

(b) (6 points) Sketch the phase portrait in the $xy - plane$.

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad Y\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$



$\lambda = -1 \pm 2i$

$Re(\lambda) = -1 < 0$ sink.

(c) (2 points) Classify the equilibrium solution at the origin. Justify your classification.

our eigenvalue $\lambda = -1 \pm 2i$ are complex numbers

$Re(\lambda) = -1 < 0$, so we have a spiral sink.

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Table of Laplace Transforms

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\delta(t-a)$	e^{-as}
$u_a(t)$	$\frac{e^{-as}}{s}$

$f(t)$	$\mathcal{L}(f(t))$
	Derivatives
y	$\mathcal{L}(y)$
y'	$s\mathcal{L}(y)-y(0)$
y''	$s^2\mathcal{L}(y)-sy(0)-y'(0)$
	t-Shift
$f(t)$	$F(s)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$
	s-Shift
$f(t)$	$F(s)$
$e^{at}f(t)$	$F(s-a)$