Announcements, Goals, and Reading

Announcements:

- HW03 due Tuesday October 4^{th}, 11:59 pm on Mastering Physics
- HW02 was due yesterday. Grace period ends Friday.

Goals for Today:

- Projectile Motion
- Demonstration: Howitzer under the bridge
- Demonstration: Monkey hunter

Reading (Physics for Scientists and Engineers 4/e by Knight)

- Chapter 4: Kinematics in 2D

Projectile Motion

Special case of 2D motion with constant acceleration

Acceleration due to gravity in vertical direction, no acceleration in horizontal

x = direction of horizontal motion

y = height above ground

$$a_x = 0$$
$$a_y = -g$$

$$x(t) = x_0 + v_{0x}\,t + \frac{1}{2}a_x\,t^2$$

$$y(t) = y_0 + v_{0y}\,t + \frac{1}{2}a_y\,t^2$$

Plug into the general equations for motion with constant acceleration

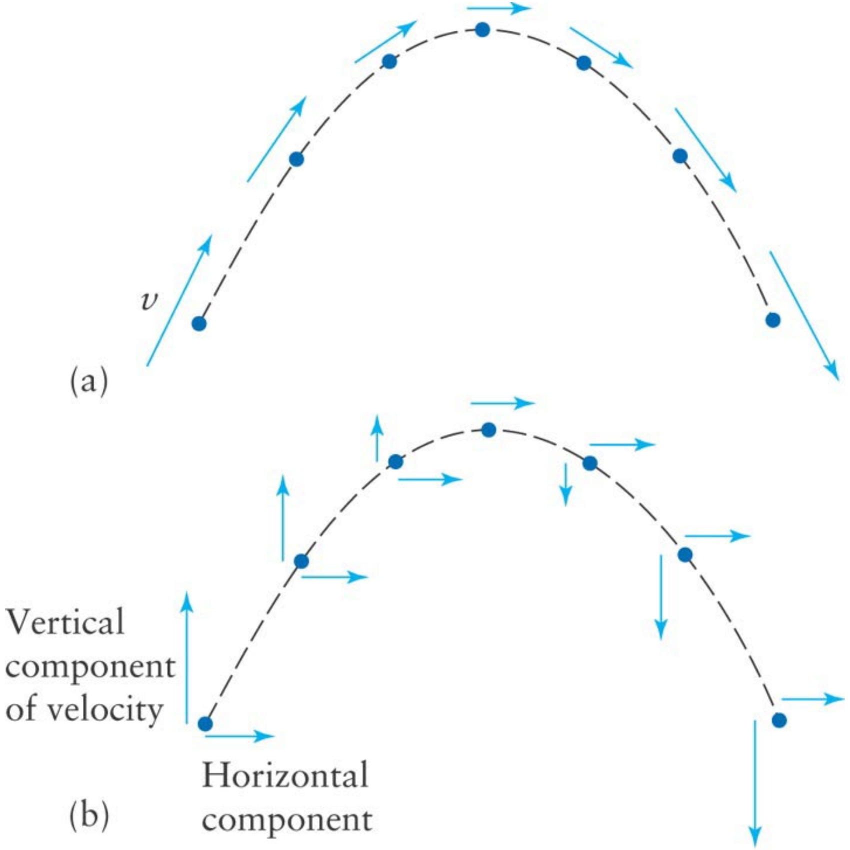
Equations for projectile motion

$$x(t) = x_0 + v_{0x}t$$
$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$v_x(t) = v_{0x}$$
$$v_y(t) = v_{0y} - gt$$

Horizontal velocity stays constant

Projectile Motion

- Curved path is a combination of motion in the horizontal and vertical directions
- We will ignore air resistance for this discussion



KEY POINT :

The horizontal and vertical motions are completely independent

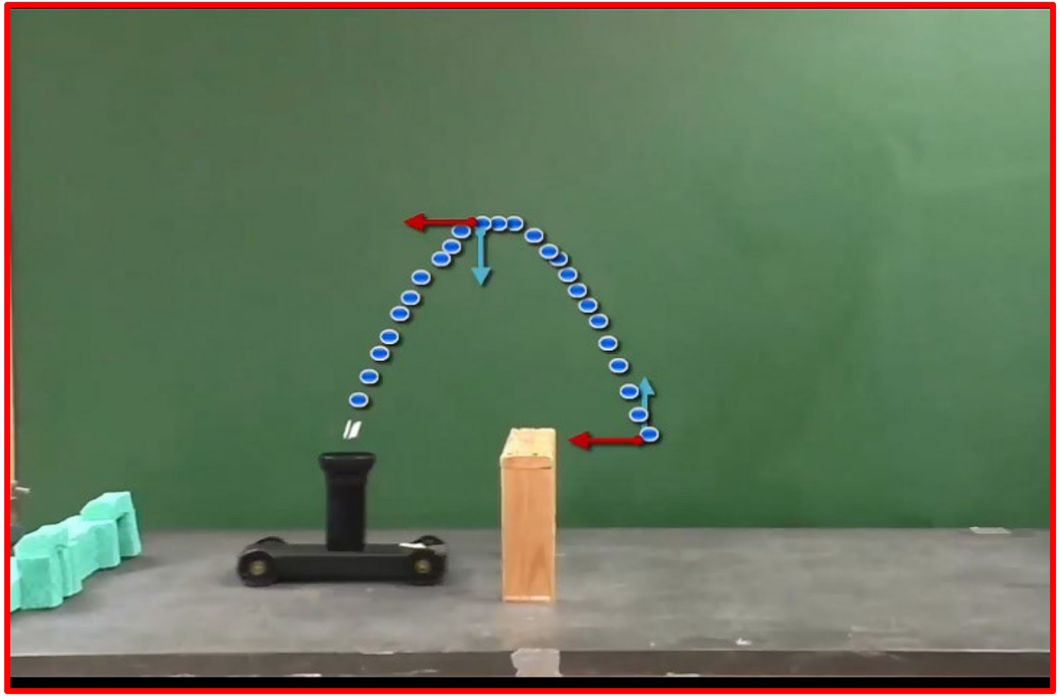
- **Gravity changes *vertical* component of velocity**
- **Horizontal component of velocity is not affected - it doesn't change**

Projectile Motion: Cannon under the Bridge

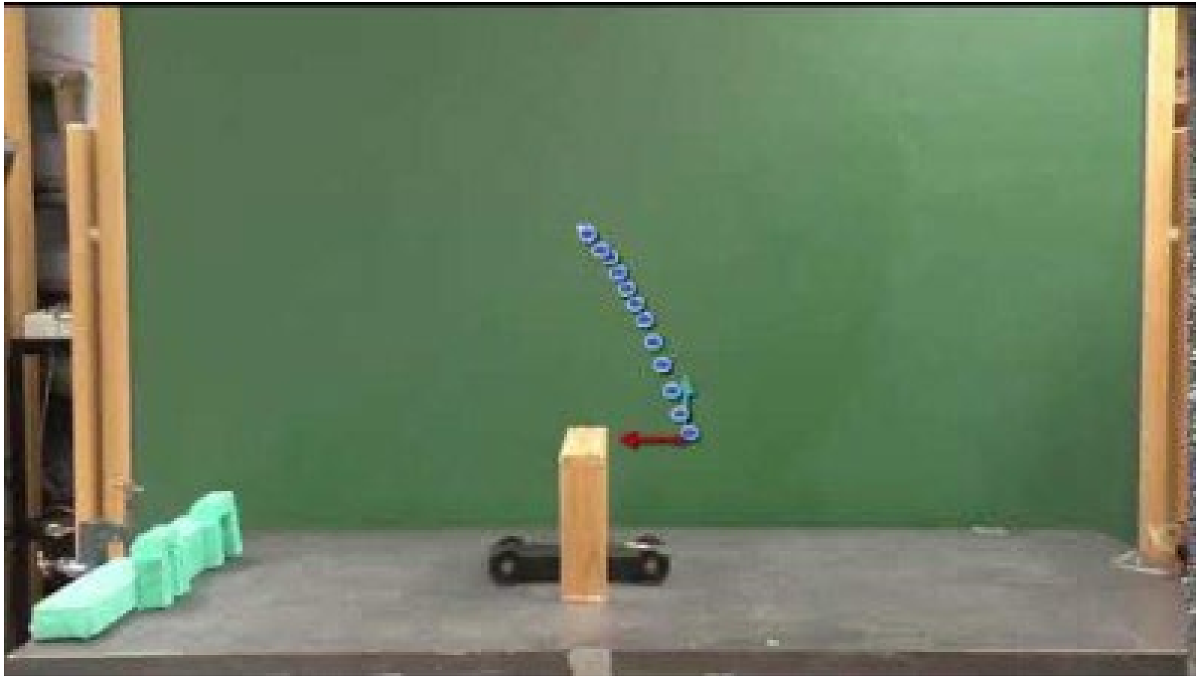
-Is horizontal motion of ball at constant velocity?

-Is horizontal motion of cannon at constant velocity?

-Are these velocities the same? Will ball land back in cannon?

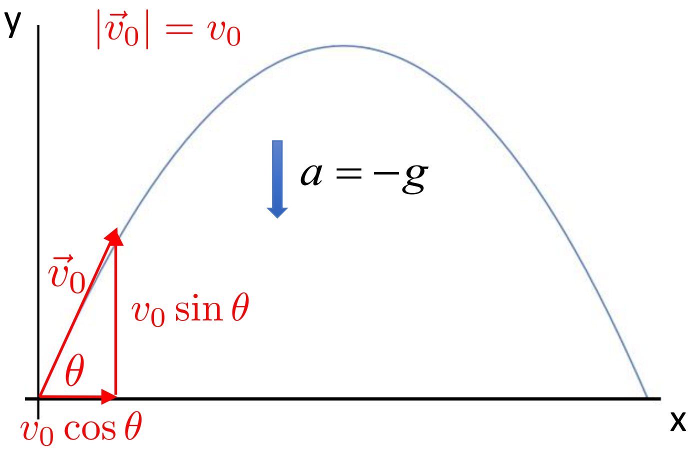


Projectile Motion: Cannon under the bridge
Slow motion



Questions about projectile motion

A projectile is launched with speed v_0 at angle θ with respect to the ground.



How high does it go?
How far does it go?
How long does it take to land?

$$x(t) = x_0 + v_{0x} t$$
$$y(t) = y_0 + v_{0y} t - \frac{1}{2} g t^2$$

plug in initial conditions

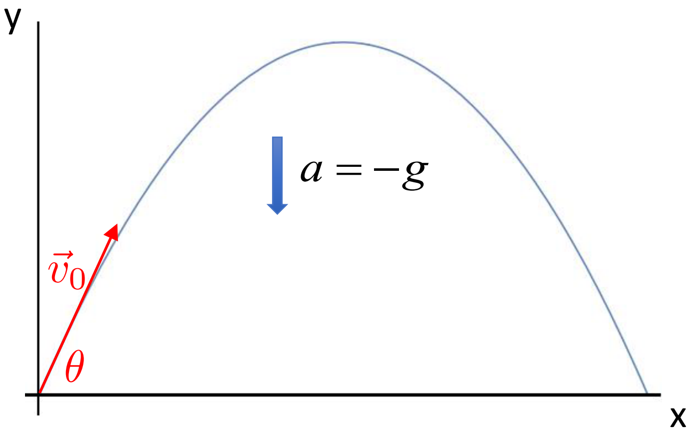
$x_0 = y_0 = 0$ Position & velocity at t=0

$v_{0x} = v_0 \cos\theta$ $v_{0y} = v_0 \sin\theta$

Arrive at...

$$x(t) = v_0 \cos\theta \, t$$
$$y(t) = v_0 \sin\theta \, t - \frac{1}{2} g t^2$$

$$x(t) = v_0 \cos\theta\, t$$
$$y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2$$



Can also express velocity components..

$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - gt$$

Assume projectile reaches top at time T

$$v_y(T) = v_0 \sin\theta - gT = 0$$

Solve to get $T = \dfrac{v_0 \sin\theta}{g}$

Plug into y(T) to get maximum height

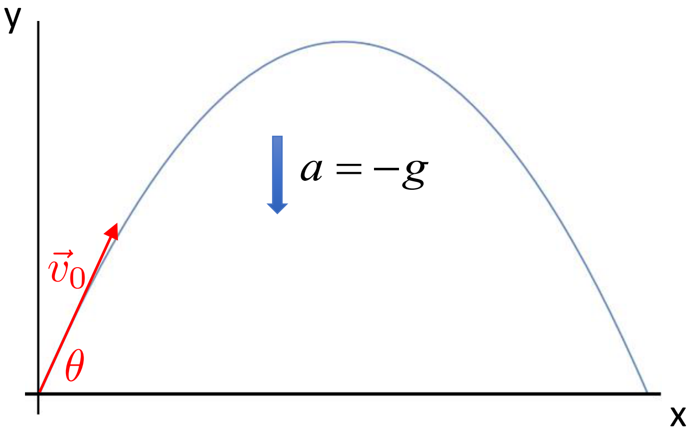
$$y(T) = v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2 = \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(t) = v_0 \cos\theta \, t$$
$$y(t) = v_0 \sin\theta \, t - \frac{1}{2} g t^2$$

$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - gt$$

$$T = \frac{v_0 \sin\theta}{g}$$ Time to top

$$y(T) = \frac{v_0^2 \sin^2\theta}{2g}$$ Max height



Check...

$$0 = v_0 \sin\theta \, t - \frac{1}{2} a t^2 \rightarrow t = \frac{2 v_0 \sin\theta}{a} = 2T$$

Distance travelled → "Range"

Total time for trajectory = 2T

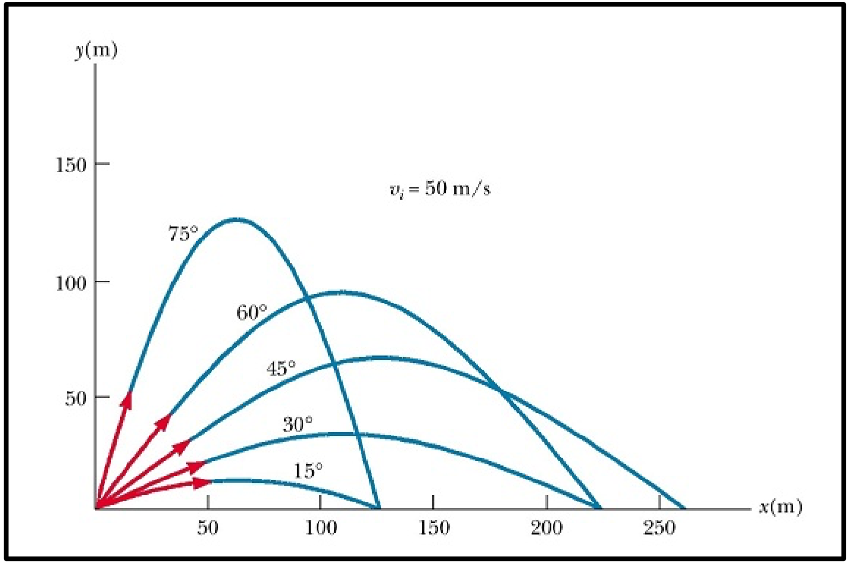
Going down takes same time as going up!
Time reversal symmetry

$$x(2T) = v_0 \cos\theta \left(\frac{2 v_0 \sin\theta}{g} \right)$$
$$= \frac{2 v_0^2 \sin\theta \cos\theta}{g}$$
$$= \frac{v_0^2 \sin(2\theta)}{g}$$ Range

Using trigonometric identity

Range formula

$$x(2T) = \frac{v_0^2 \sin(2\theta)}{g}$$



If we fix v_0 and vary launch angle, what angle maximizes range?

Think about maximum value of sine function

$$\theta = 45^o \rightarrow \sin(2\theta) = 1$$

Optimum tradeoff between horizontal velocity and time of flight

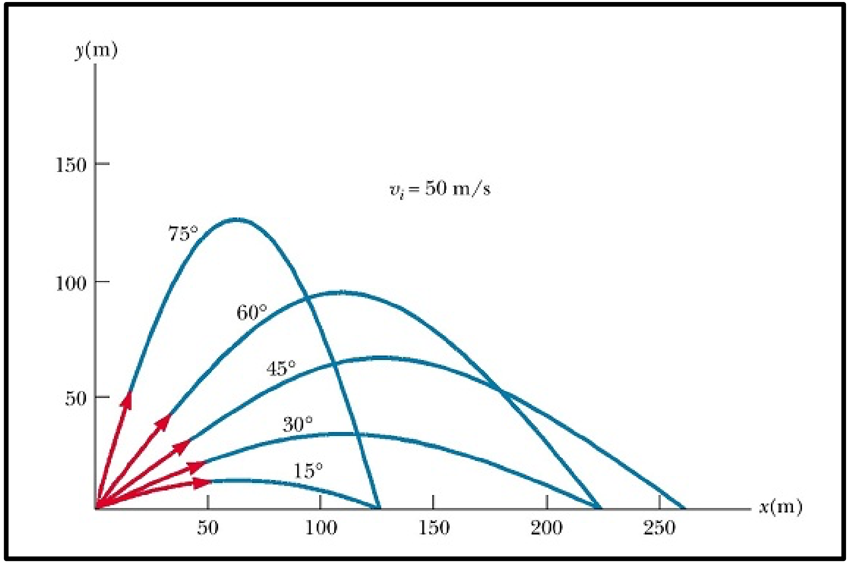
Gives maximum range $$x_{max} = \frac{v_0^2}{g}$$

Steeper angle → More time in air, but not as much horizontal velocity

Shallower angle → Larger horizontal velocity, but not as much time in air

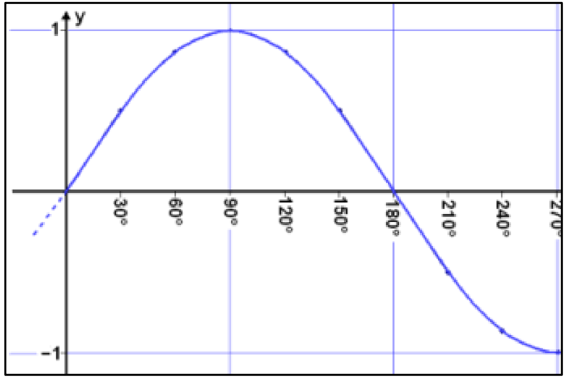
Range formula

$$x(2T) = \frac{v_0^2 \sin(2\theta)}{g}$$



Interesting feature can be seen from graph:
Can get same range for 2 different launch angles!

Range (90° - θ) = Range (θ)



Follow from property of Sine function

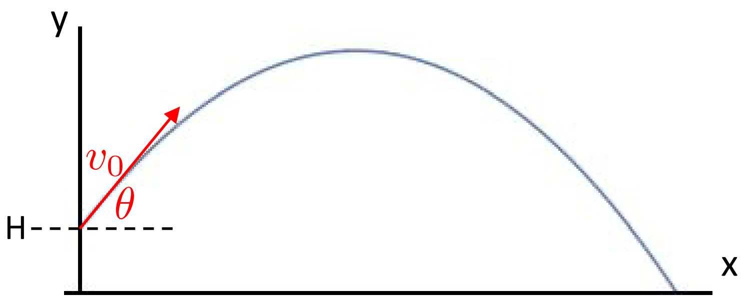
$$\sin(180^o - \phi) = \sin(\phi)$$

$$\phi = 2\theta \Rightarrow \sin(2(90^o - \theta)) = \sin(2\theta)$$

One possible variant of basic range problem

Launch projectile from height H, rather than from the ground

How far does it go?
What angle gives maximum range?



$$x(t) = x_0 + v_{0x}t$$
$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$v_x(t) = v_{0x}$$
$$v_y(t) = v_{0y} - gt$$

Plug in initial conditions…

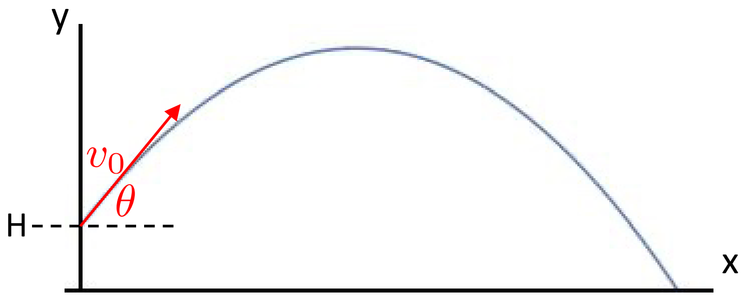
$$x_0 = 0 \qquad v_{0x} = v_o \cos\theta$$
$$y_0 = H \qquad v_{0y} = v_o \sin\theta$$

$$x(t) = v_0 \cos\theta\, t$$
$$y(t) = H + v_0 \sin\theta t - \frac{1}{2}gt^2$$
$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - gt$$

Projectile Equations with these initial conditions

How far does it go?
What angle gives maximum range?

$$x(t) = v_0 \cos\theta\, t$$
$$y(t) = H + v_0 \sin\theta t - \frac{1}{2} g t^2$$
$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - g t$$



Hitting ground $\longrightarrow$ $y(T) = 0 \longrightarrow H + v_0 \sin\theta\, T - \frac{1}{2} g T^2 = 0$

Use quadratic formula: $\underbrace{a}_{-\frac{1}{2}g} T^2 + \underbrace{b}_{v_0 \sin\theta} T + \underbrace{c}_{H} = 0 \longrightarrow T = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

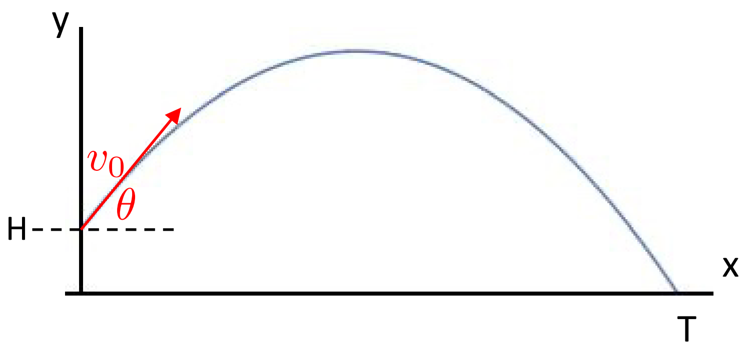
$$T = \frac{v_0 \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$

Other root gives T<0

How far does it go?
What angle gives maximum range?

$$x(t) = v_0 \cos\theta\, t$$

$$T = \frac{v_0 \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$



Range = x(T)

$$x(T) = \frac{v_0^2 \sin(2\theta)}{2g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$

Looks like a mess!
Check some basics...

$$H = 0 \Rightarrow x(T) = \frac{v_0^2 \sin(2\theta)}{g} \quad \checkmark$$

Gives back original range formula

Increasing H with fixed angle gives longer range ✓

Intuition about optimal angle

Longer free fall time makes larger horizontal velocity preferable

Larger H ➡ Smaller optimal angle

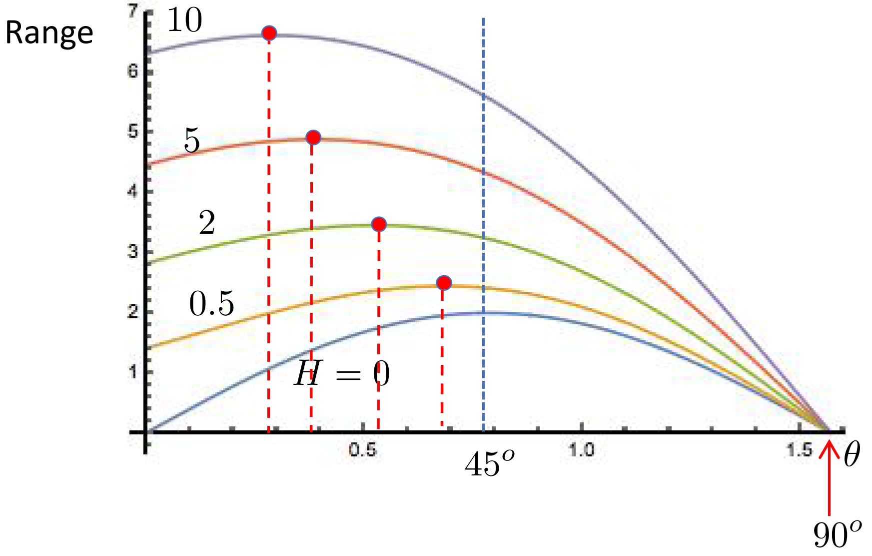
What angle gives maximum range?

$$x(T) = \frac{v_0^2 \sin(2\theta)}{2g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$

Intuition about optimal angle

Longer free fall time makes Larger horizontal velocity will be preferable

Larger H → Smaller optimal angle



Plot range vs. Angle

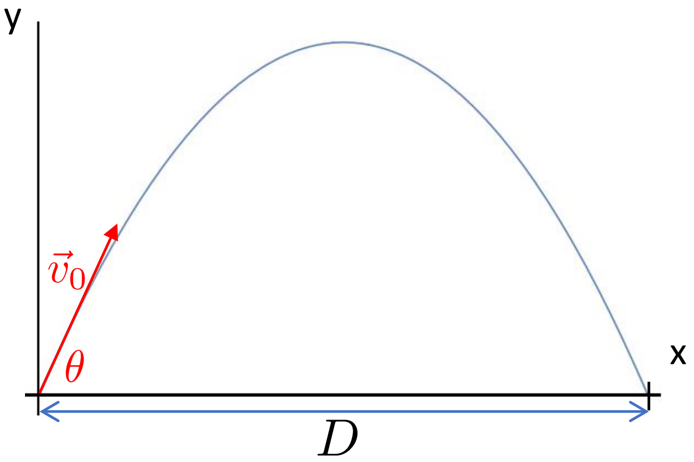
Set $\dfrac{v_0^2}{2g} = 1m$

See that as H is increased, maximum range shifts to smaller angles

Shooting straight up gives zero range

Back to range formula…

$$\boxed{D = \frac{v_0^2 \sin(2\theta)}{g}} \qquad v_0 = |\vec{v}_0|$$



Evel Knievel wants to jump over 100m worth of school buses

His motorcycle can go 50 m/s

What is the longest jump Evel can make with this motorcycle?

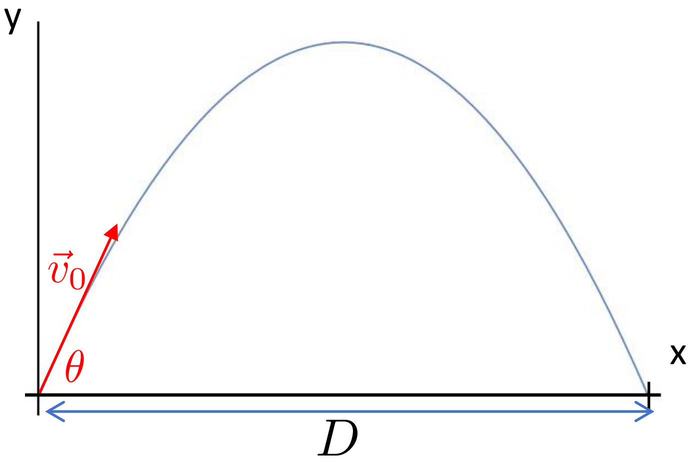
$$\theta = 45^o \Longleftrightarrow \sin(2\theta) = 1$$

Gives maximum distance

$$D_{max} = \frac{v_0^2}{g} = \frac{(50m/s)^2}{9.8m/s^2} = 255m$$

Back to range formula…

$$\boxed{D = \frac{v_0^2 \sin(2\theta)}{g}} \qquad v_0 = |\vec{v}_0|$$



Evel Knievel wants to jump over 100m worth of school buses

His motorcycle can go 50 m/s

To what angle should his ramp be set?

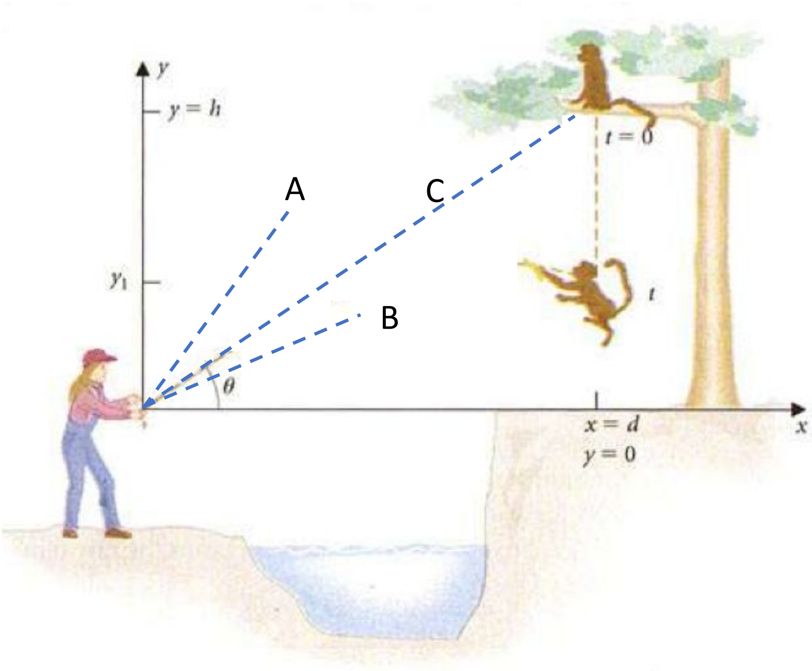
Assuming no air resistance

$$\sin(2\theta) = \frac{gD}{v_0^2} = 0.39$$

$$\Rightarrow \theta = 11^o$$

Second solution: $\theta = 90^0 - 11.5^0 = 78.5^0$ Check: D=50^2 x sin(2 x 78.5^0)/9.8 = 100 m

Projectile Motion : Feeding smart money a banana



You are trying to feed the world's smartest monkey, hanging from a tree. The smart monkey likes to play and always lets go and falls to the ground the instant you throw a banana. Knowing this, to get banana straight to the monkey should you aim:

A) Above the monkey B) Below the monkey C) Right at the monkey

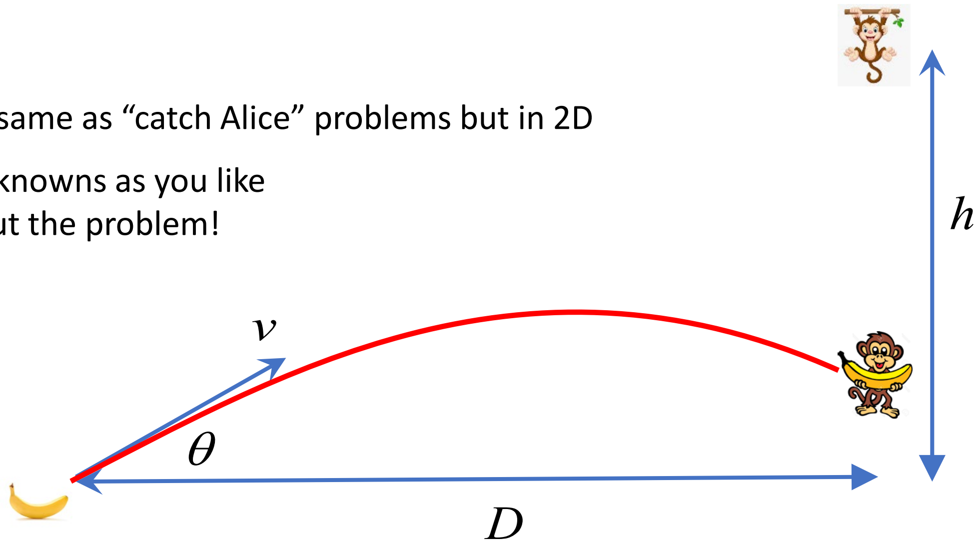
Projectile Motion : Projectiles Launched at an Angle

Projectile Motion : Feeding smart money a banana

Methodical solution: same as “catch Alice” problems but in 2D

Introduce as many unknowns as you like
to help you think about the problem!

D
H
v
t
…



Write down the desired result for **TWO** components in 2D: same location at time t

y(t)= $v\sin(\theta)t - \cancel{\frac{1}{2}gt^2} = h - \cancel{\frac{1}{2}gt^2}$

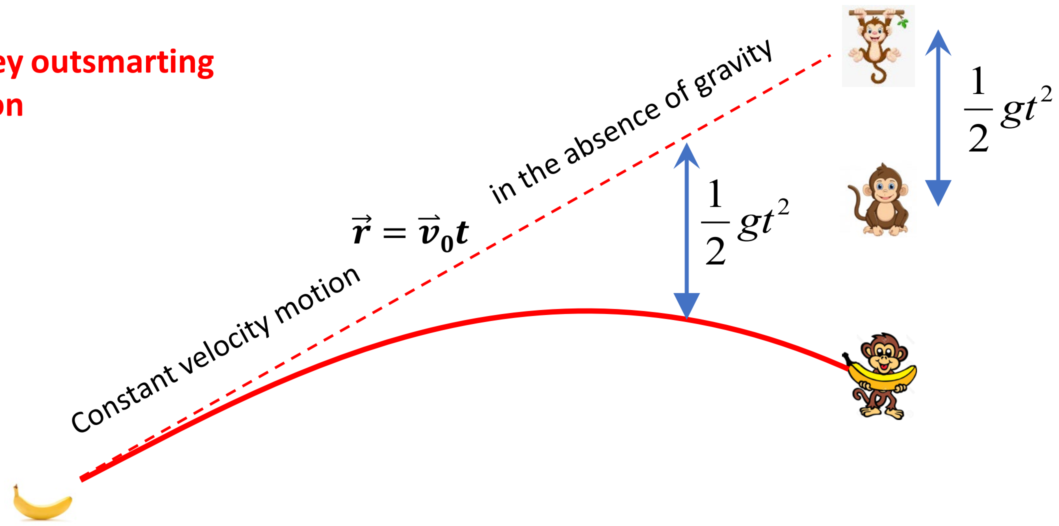
x(t)= $v\cos(\theta)t = D$

$$\Rightarrow \quad v\sin(\theta)t = h, \qquad v\cos(\theta)t = D \quad \Rightarrow \quad \tan(\theta) = \frac{h}{D}$$

C) Right at the monkey ✓

Projectile Motion : Feeding smart money a banana

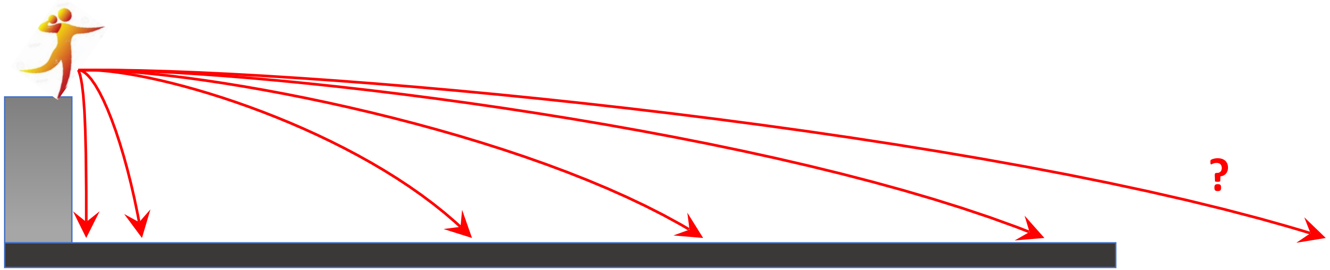
Monkey outsmarting solution



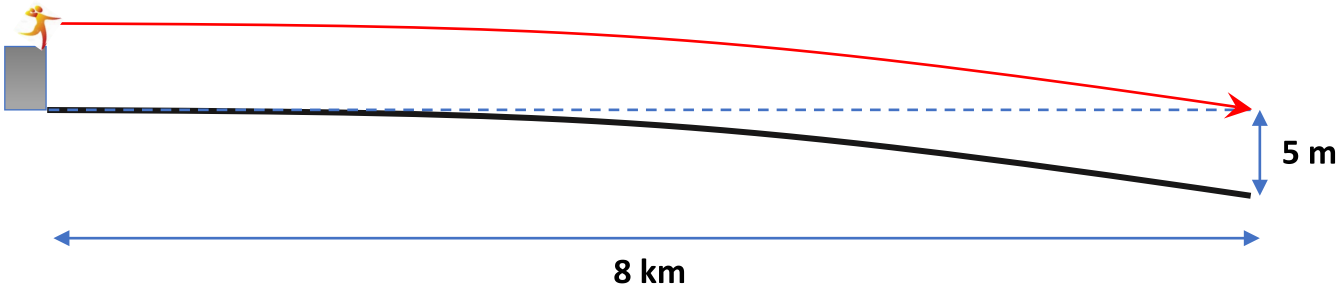
- If no gravity, banana would follow imaginary dashed line
- On Earth, banana falls beneath imaginary dashed line until it hits the ground
- Vertical distance it falls below imaginary line is the same distance it would fall if dropped from rest
- When horizontal displacement is the distance to the tree, the banana falls by exactly the same amount as monkey (all objects under gravity fall the same way!)

  → success

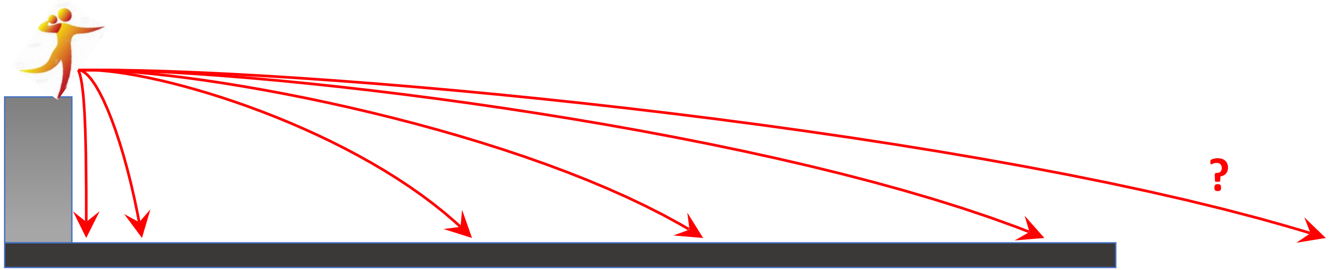
Fast Projectile Motion: Satellites



- After 1 second, ball falls 5 m, ***independent* of initial horizontal velocity**
- Suppose we throw ball with incredible velocities – few km/second – what changes?
- **Earths curvature**: surface falls 5 m for every 8000 m tangent to the surface



Fast Projectile Motion: Satellites



- After 1 second, ball falls 5 m, ***independent* of initial horizontal velocity**
- Suppose we throw ball with incredible velocities – few km/second – what changes?

- **Earths curvature**: surface falls 5 m for every 8000 m tangent to the surface
- If you launch an object so that it goes 8000 m in 1 second, it would fall 5 m in the same time and match the earth's surface
- **A satellite constantly *falls* while orbiting around the earth without ever crashing into it – falling distance must match earth's curvature**
- 8 km/s = 18 000 mph! At sea level air resistance would burn object – need to put satellites many miles up
- Gravity is still nearly as strong 200 km up – we usually don't put satellites up high to get away from Earth gravity