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Announcements, Goals, and Reading

Announcements:

- HW03 due Tuesday October 4^{th}, 11:59 pm on Mastering Physics

Goals for Today:

- Block on a Plane
- Projectile Motion

Reading (Physics for Scientists and Engineers 4/e by Knight)

- Chapter 3: Vectors
- Chapter 4: Kinematics in 2D

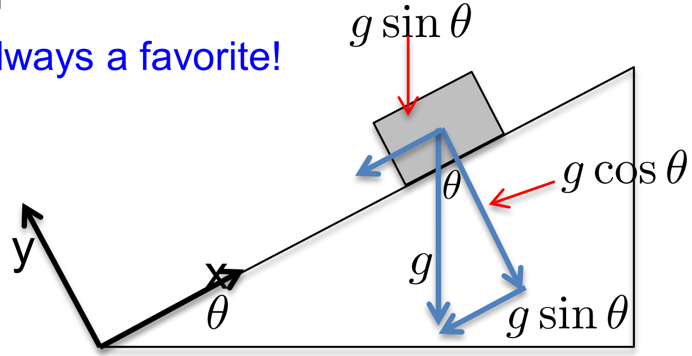
Block on Inclined Plane

Use coordinate x to measure distance up the plane

Use trigonometry to figure out acceleration of block down plane

$$a = -g \sin\theta$$

Always a favorite!



y-component of gravitational acceleration is "blocked" by the plane
NO FRICTION HERE!

Motion along the plane has constant acceleration

$$x = x_0 + v_0 t - \frac{1}{2}(g \sin\theta)\, t^2$$

$$v = v_0 - (g \sin\theta)\, t$$

$$D = x - x_0 = \frac{1}{2a}(v_1^2 - v_0^2)$$

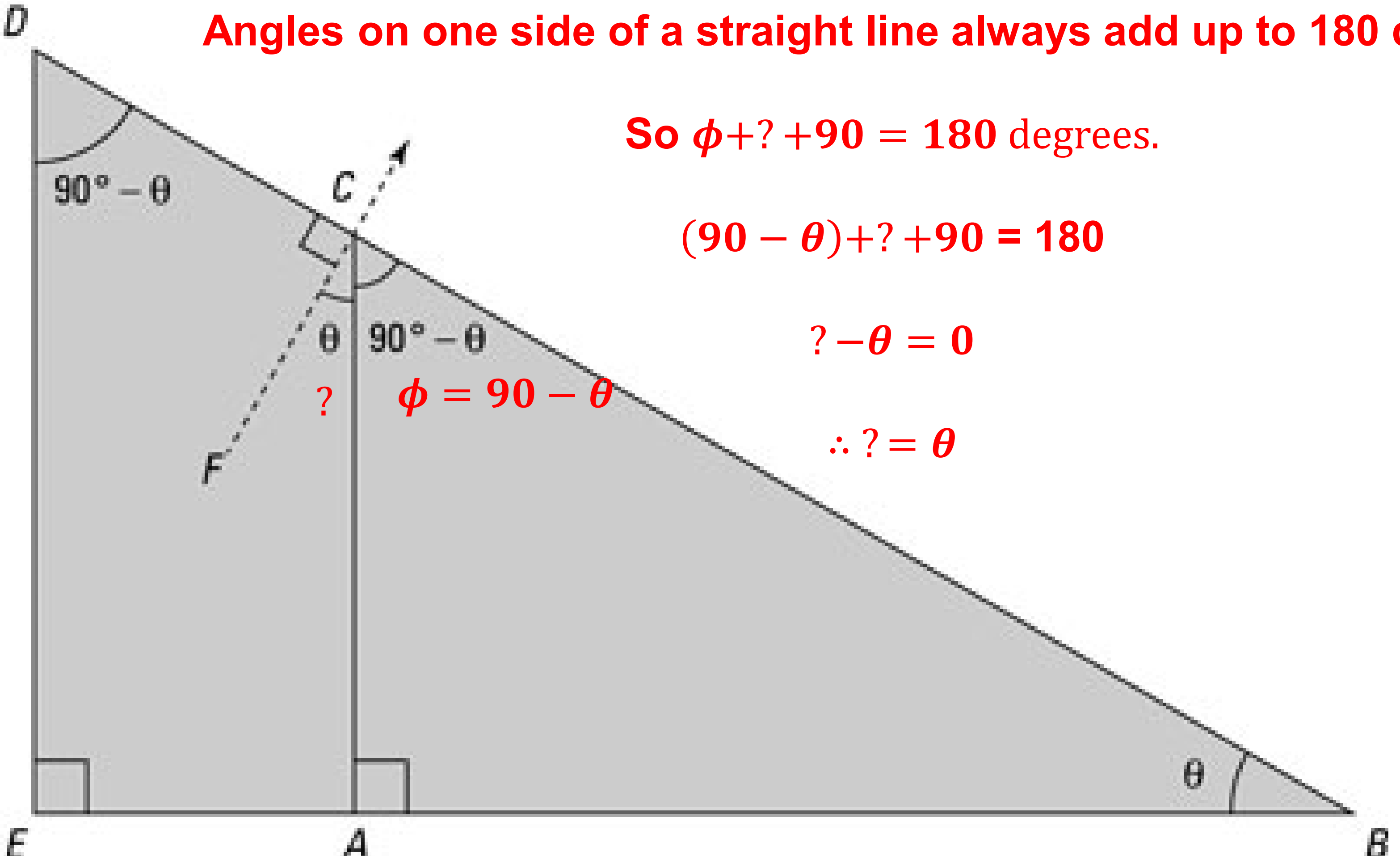
Note

$\theta = 0 \Rightarrow g \sin\theta = 0$

No acceleration for block on flat surface!

If a right triangle is drawn such that the hypotenuse is // to the side of the triangle opposite to $\boldsymbol{\theta}$, and the adjacent side of the new triangle is normal to the hypotenuse of the old triangle, and the hat is the unknown angle ? of this new triangle?

Angles on one side of a straight line always add up to 180 degrees.



So $\boldsymbol{\phi} + ? + \mathbf{90} = \mathbf{180}$ degrees.

$$(\mathbf{90} - \boldsymbol{\theta}) + ? + \mathbf{90} = \mathbf{180}$$

$$? - \boldsymbol{\theta} = \mathbf{0}$$

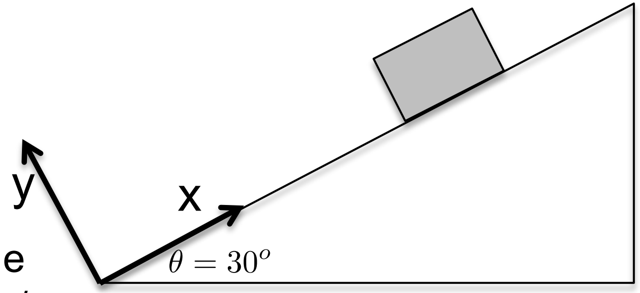
$$\therefore ? = \boldsymbol{\theta}$$

Block on Inclined Plane

Constant acceleration

$a = -g \sin\theta$

Work out example…



- Block starts out 100m up the plane moving upwards with velocity 25m/s
- How far up the plane does it go?
- How fast is it moving when it slides back down to the bottom of the plane?

Not interested in times…

$$D = x - x_0 = \frac{1}{2a}(v_1^2 - v_0^2)$$

Initial conditions $x_0 = 100m$

$$v_0 = 25m/s$$

Top $v_1 = 0$

Also need $\sin(30^o) = 0.5$

$$D = \frac{-(25m/s)^2}{2(-9.8m/s^2)(0.5)}$$

$$= 63m$$

$\sin 30^o$

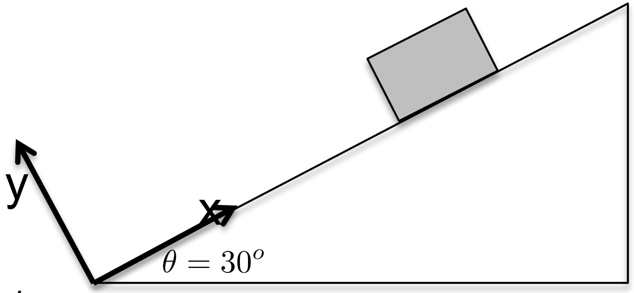
How much further block goes up plane

Block on Inclined Plane

$$\sin(30^o) = 0.5$$

Constant acceleration

$$a = -g\sin\theta$$



Work out example…

- Block starts out 100m up plane moving upwards with velocity 25m/s
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Not interested in times…

$$D = x - x_0 = \frac{1}{2a}(v_1^2 - v_0^2)$$

Initial conditions $x_0 = 100m$

$$v_0 = 25m/s$$

Bottom $x = 0$

Need to find v_1

Rearrange formula

$$v_1 = \sqrt{v_0^2 + 2a(x - x_0)}$$

$$= \sqrt{(25m/s)^2 + 2(-9.8m/s^2)(0.5)(0 - 100m)}$$

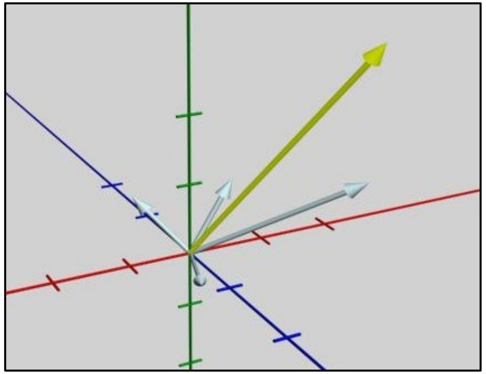
$$= 40m/s$$

$\sin 30^o$

Vectors in 3D

Have x, y and z components

Different representations



Graphical

Generally too awkward to be useful

Component form

$$\vec{v} = (v_x, v_y, v_z)$$

In terms of basis vectors

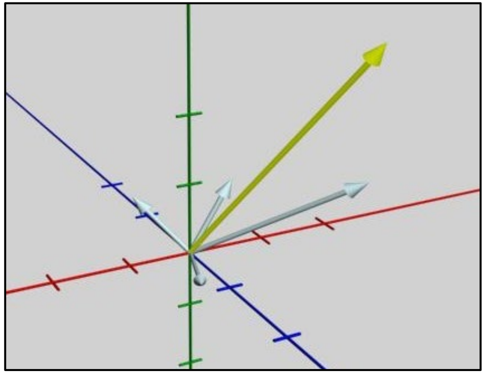
Unit basis vector in z-direction

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Magnitude of vector

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Adding vectors in 3D



If we have two vectors…

$$\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\begin{aligned}\vec{w} &= \vec{u} + \vec{v} \\ &= (u_x + v_x)\hat{i} + (u_y + v_y)\hat{j} + (u_z + v_z)\hat{k}\end{aligned}$$

$$\boxed{w_x = u_x + v_x, \quad w_y = u_y + v_y, \quad w_z = u_z + v_z}$$

Kinematics in Two Dimensions

Many common phenomena are essentially 2D

Planetary orbits stay in a fixed plane

Related to angular momentum; We'll cover this later..

Center of mass of projectile follows 2D path

Because of conservation of linear momentum (before touching the ground)

Motion in two dimensions

Position specified by a pair of coordinates as functions of time

$$x(t),\ y(t) \quad \Rightarrow \quad \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

position vector

Velocity also specified by a pair of functions

$$v_x(t) = \frac{dx}{dt},\ v_y(t) = \frac{dy}{dt} \quad \Rightarrow \quad \vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} = \frac{d\vec{r}}{dt}$$

velocity vector

Acceleration too…

$$a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

$$a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

$$\Rightarrow \quad \vec{a}(t) = a_x(t)\hat{i} + a_y(t)\hat{j} = \frac{d\vec{v}}{dt}$$

acceleration vector

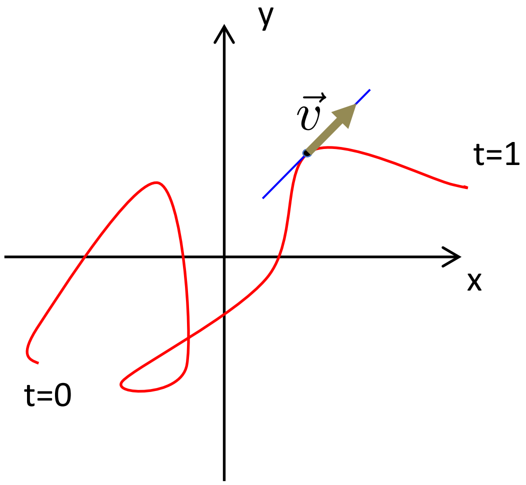
Motion in two dimensions

Can graph 2D motion

$$x(t),\ y(t)$$

For example between t=0 and t=1

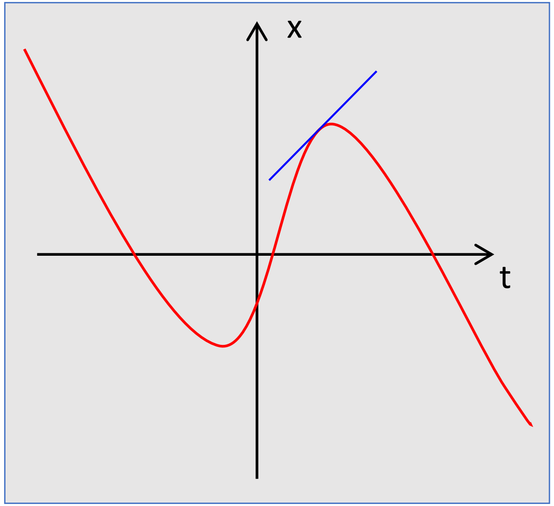
- Time is a parameter that runs along path of object
- Velocity vector is in direction of tangent to curve



Different from how we graphed 1D motion

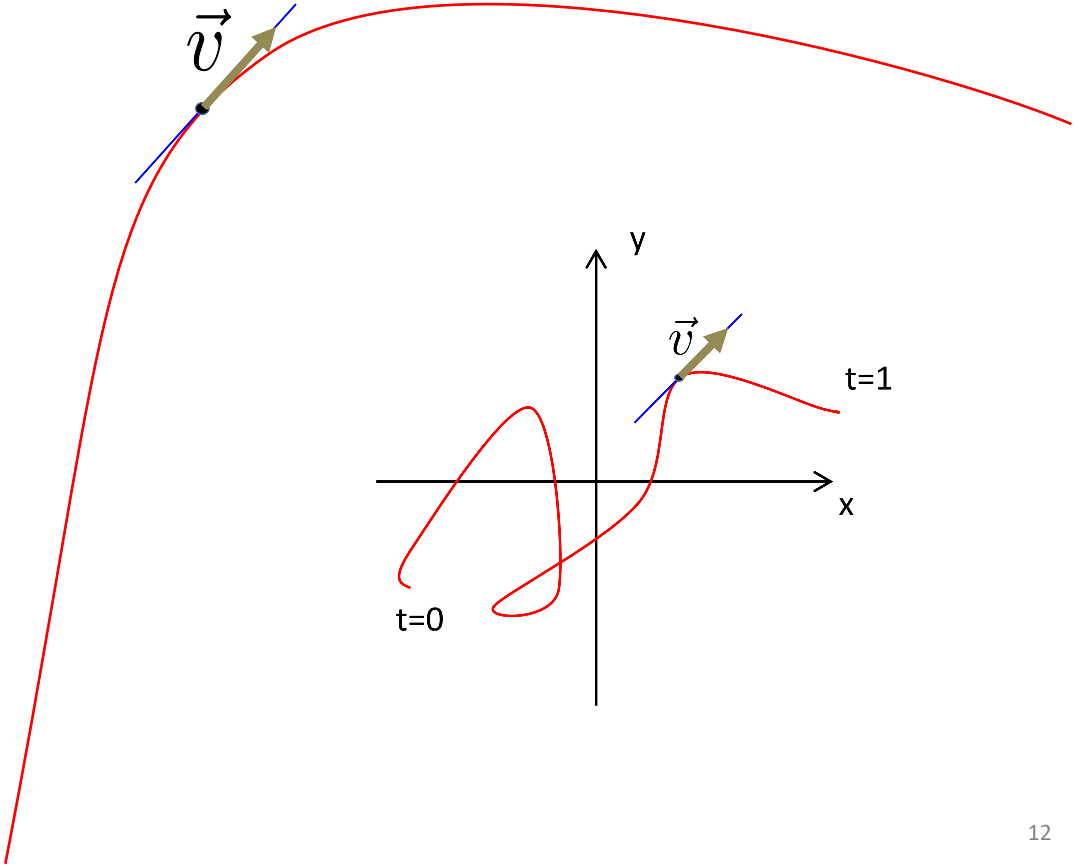
$$x(t)$$

- Time is one of the coordinates
- Velocity equals slope of tangent to curve



ZOOM in

We know the direction of velocity, but not the magnitude because this plot does not tell us the time spent

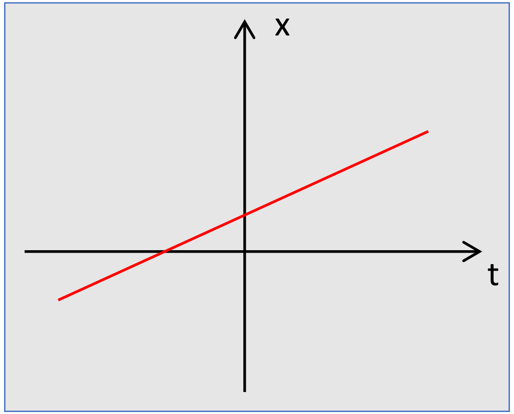


2D Motion with **Constant Velocity**

Recall in 1D

$$x(t) = x_0 + v_0 t$$ position

$$v(t) = \frac{dx}{dt} = v_0$$ velocity



In 2D…

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \vec{v}_0$$

$$\vec{r}_0 = x_0\hat{i} + y_0\hat{j}$$ position vector at t=0

$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$$ constant velocity vector

2D Motion with Constant Velocity

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

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$$\vec{r}_0 = x_0\hat{i} + y_0\hat{j}$$ position vector at t=0

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In components

$$\begin{aligned}\vec{r}(t) &= (x_0\hat{i} + y_0\hat{j}) + (v_{0x}\hat{i} + v_{0y}\hat{j})\,t \\ &= (x_0 + v_{0x}t)\,\hat{i} + (y_0 + v_{0y}t)\,\hat{j}\end{aligned}$$

$$\boxed{\begin{aligned}x(t) &= x_0 + v_{0x}t \\ y(t) &= y_0 + v_{0y}t\end{aligned}}$$

2D Motion with **Constant Velocity**

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \vec{v}_0$$

$$x(t) = x_0 + v_{0x} t$$

$$y(t) = y_0 + v_{0y} t$$

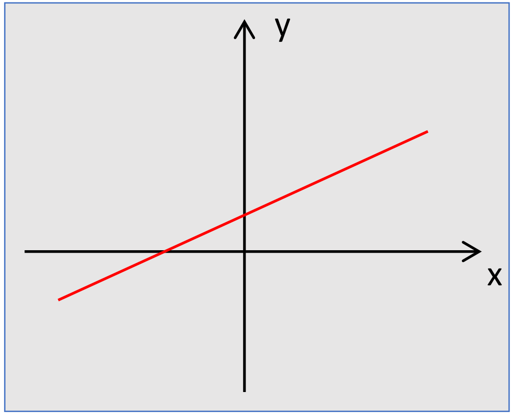
What does the y-x graph look like?

Express time in terms of x and plug into equation for y:

$$t = (x - x_0) / v_{0x}$$

$$y = y_0 + \frac{v_{0y}}{v_{0x}}(x - x_0)$$

$\tan\theta_v$



1. Object follows straight line
2. Slope is $\tan\theta_v$

2D Motion with Constant Acceleration. Vector form

Recall in 1D

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$ position

$$v = \frac{dx}{dt} = v_0 + at$$ velocity

$$a = \frac{dv}{dt} = a$$ constant acceleration

2D

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\, t + \frac{1}{2} \vec{a}\, t^2$$ position vector

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \vec{v}_0 + \vec{a}\, t$$ velocity vector

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \vec{a}$$ constant acceleration vector

2D Motion with Constant Acceleration: Vector form and Component form

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\, t + \frac{1}{2}\vec{a}t^2$$

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t$$

$$\vec{a}(t) = \vec{a}$$

Equations for 2D motion with constant acceleration: **vector form**

Plug in components of initial conditions to get expressions for x(t) and y(t)

$$\vec{r}_0 = x_0\hat{i} + y_0\hat{j}$$ Position vector at t=0

$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$$ Velocity vector at t=0

$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$ Constant acceleration vector

Equations for 2D motion with constant acceleration: **component form**

$$x(t) = x_0 + v_{0x}\, t + \frac{1}{2}a_x\, t^2$$

$$v_x(t) = v_{0x} + a_x t$$

$$y(t) = y_0 + v_{0y}\, t + \frac{1}{2}a_y\, t^2$$

$$v_y(t) = v_{0y} + a_y t$$

Motion along X and motion along Y expressed as separate time dependent equations…

Projectile Motion

Special case of 2D motion with constant acceleration

Acceleration due to gravity in vertical direction, no acceleration in horizontal

x = direction of horizontal motion

y = height above ground

$$a_x = 0$$
$$a_y = -g$$

Plug into the general equations for motion with constant acceleration

$$x(t) = x_0 + v_{0x}\, t + \frac{1}{2} a_x\, t^2$$

$$y(t) = y_0 + v_{0y}\, t + \frac{1}{2} a_y\, t^2$$

Equations for projectile motion

$$x(t) = x_0 + v_{0x} t$$
$$y(t) = y_0 + v_{0y} t - \frac{1}{2} g t^2$$
$$v_x(t) = v_{0x}$$
$$v_y(t) = v_{0y} - gt$$

Horizontal velocity stays constant

Projectile Motion

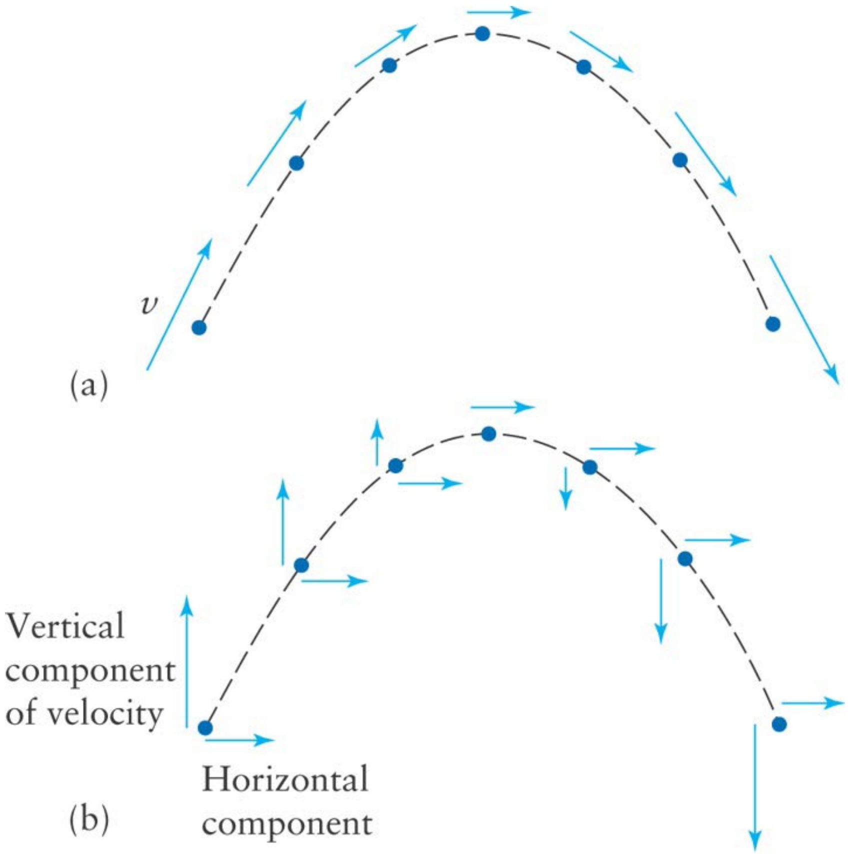
$$x(t) = x_0 + v_{0x}t$$

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Projectiles follow parabolic paths

Projectile Motion

- Curved path is a combination of motion in the horizontal and vertical directions
- We will ignore air resistance for this discussion



KEY POINT :

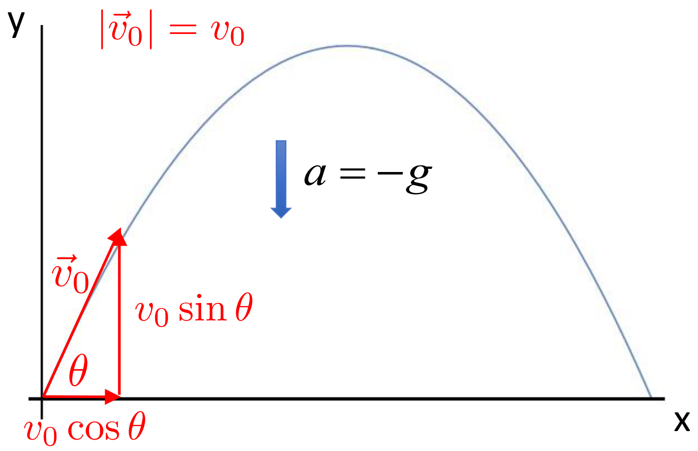
The horizontal and vertical motions are completely independent

- **Gravity changes *vertical* component of velocity**
- **Horizontal component of velocity is not affected - it doesn't change**

Questions about projectile motion

A projectile is launched with speed v_0 at angle θ with respect to the ground.

How high does it go?
How far does it go?
How long does it take to land?



$$x(t) = x_0 + v_{0x}t$$
$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

plug in initial conditions

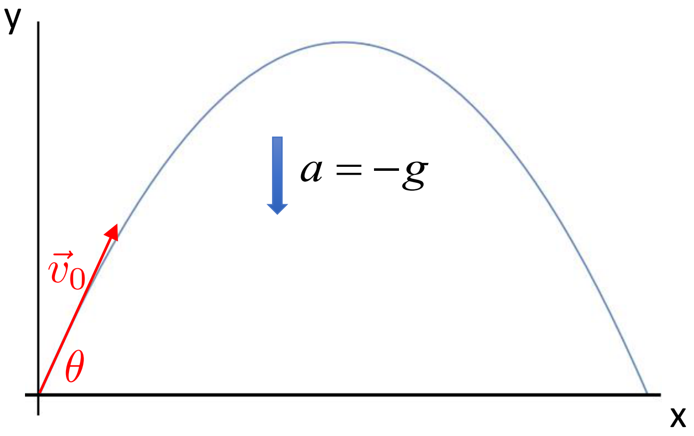
$x_0 = y_0 = 0$ Position & velocity at t=0

$v_{0x} = v_0\cos\theta$ $v_{0y} = v_0\sin\theta$

Arrive at…

$$x(t) = v_0\cos\theta\, t$$
$$y(t) = v_0\sin\theta\, t - \frac{1}{2}gt^2$$

$$x(t) = v_0 \cos\theta\, t$$
$$y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2$$



Can also express velocity components..

$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - gt$$

Assume projectile reaches top at time T

$$v_y(T) = v_0 \sin\theta - gT = 0$$

Solve to get $T = \dfrac{v_0 \sin\theta}{g}$

Plug into y(T) to get maximum height

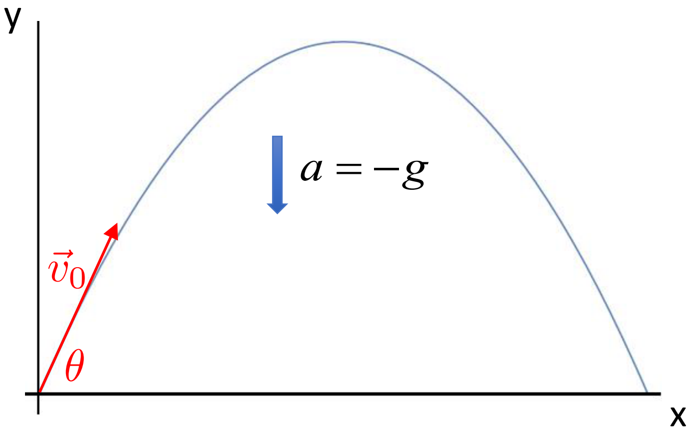
$$y(T) = v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2 = \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(t) = v_0 \cos\theta\, t$$
$$y(t) = v_0 \sin\theta\, t - \frac{1}{2} g t^2$$

$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - g t$$

$$T = \frac{v_0 \sin\theta}{g}$$ Time to top

$$y(T) = \frac{v_0^2 \sin^2\theta}{2g}$$ Max height



Check...

$$0 = v_0 \sin\theta\, t - \frac{1}{2} a t^2 \rightarrow t = \frac{2 v_0 \sin\theta}{a} = 2T$$

Distance travelled → "Range"

Total time for trajectory = 2T

Going down takes same time as going up!
Time reversal symmetry

$$x(2T) = v_0 \cos\theta \left(\frac{2 v_0 \sin\theta}{g} \right)$$
$$= \frac{2 v_0^2 \sin\theta \cos\theta}{g}$$
$$= \frac{v_0^2 \sin(2\theta)}{g}$$ Range

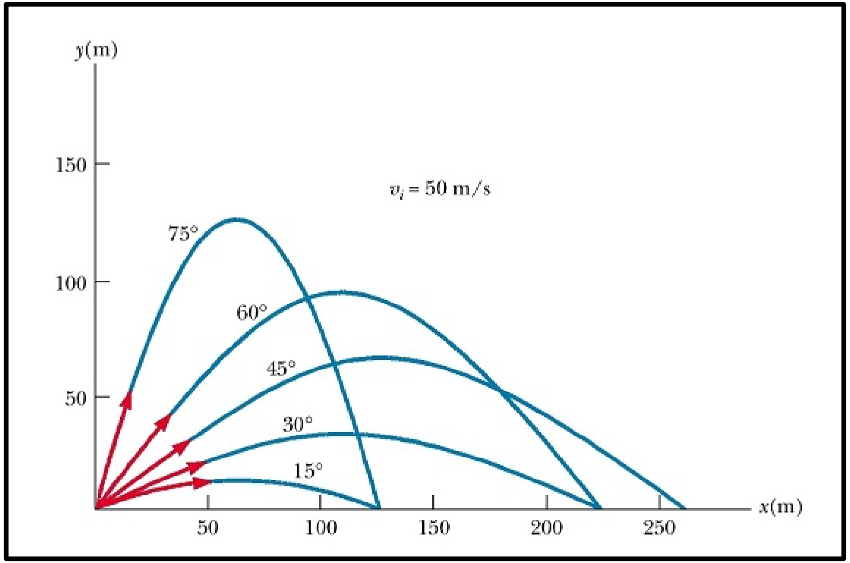
Using trigonometric identity

Range formula

$$x(2T) = \frac{v_0^2 \sin(2\theta)}{g}$$

If we fix v_0 and vary launch angle, what angle maximizes range?

Think about maximum value of sine function



$$\theta = 45^o \rightarrow \sin(2\theta) = 1$$

Optimum tradeoff between horizontal velocity and time of flight

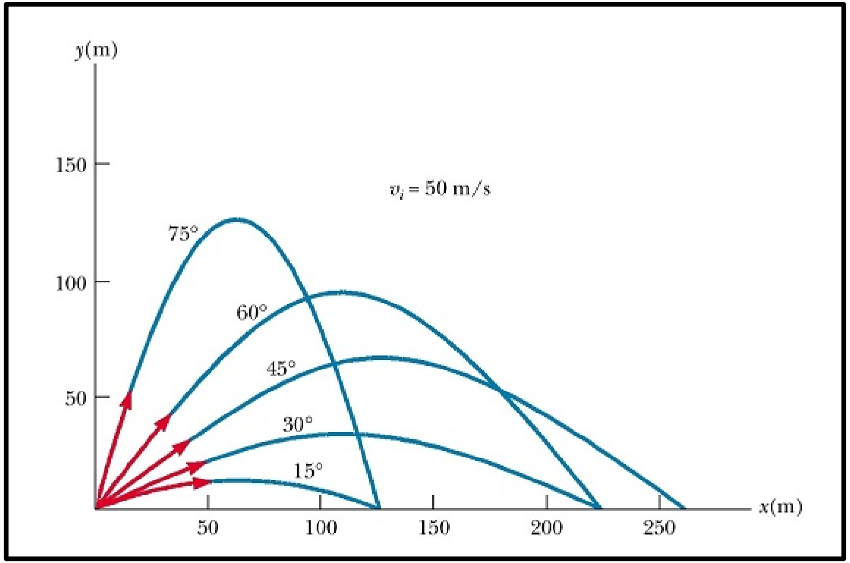
Gives maximum range $$x_{max} = \frac{v_0^2}{g}$$

Steeper angle ➔ More time in air, but not as much horizontal velocity

Shallower angle ➔ Larger horizontal velocity, but not as much time in air

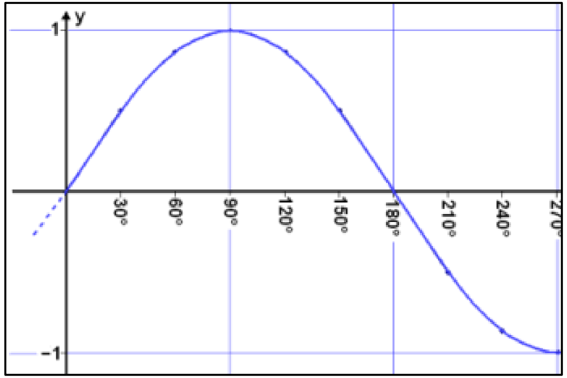
Range formula

$$x(2T) = \frac{v_0^2 \sin(2\theta)}{g}$$



Interesting feature can be seen from graph:
Can get same range for 2 different launch angles!

Range (90° - θ) = Range (θ)



Follow from property of Sine function

$$\sin(180^o - \phi) = \sin(\phi)$$

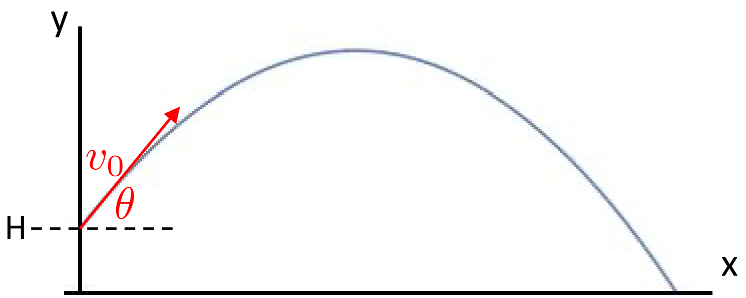
$$\phi = 2\theta \Rightarrow \sin(2(90^o - \theta)) = \sin(2\theta)$$

One possible variant of basic range problem

Launch projectile from height H, rather than from the ground

How far does it go?
What angle gives maximum range?

$$x(t) = x_0 + v_{0x}t$$
$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$
$$v_x(t) = v_{0x}$$
$$v_y(t) = v_{0y} - gt$$



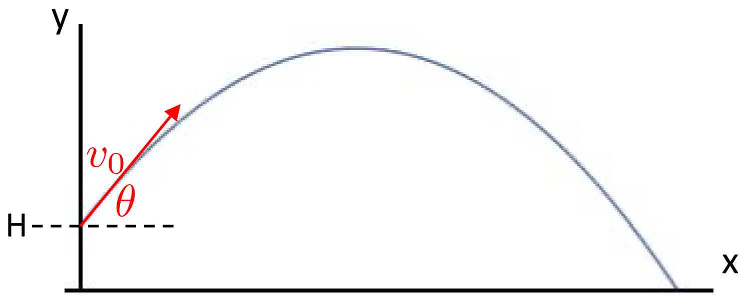
Plug in initial conditions…

$$x_0 = 0 \qquad v_{0x} = v_o \cos\theta$$
$$y_0 = H \qquad v_{0y} = v_o \sin\theta$$

$$x(t) = v_0 \cos\theta \, t$$
$$y(t) = H + v_0 \sin\theta t - \frac{1}{2}gt^2$$
$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - gt$$

Projectile Equations with these initial conditions

How far does it go?
What angle gives maximum range?



$$x(t) = v_0 \cos\theta\, t$$
$$y(t) = H + v_0 \sin\theta t - \frac{1}{2} g t^2$$
$$v_x(t) = v_0 \cos\theta$$
$$v_y(t) = v_0 \sin\theta - gt$$

Hitting ground $\longrightarrow y(T) = 0 \longrightarrow H + v_0 \sin\theta\, T - \frac{1}{2} g T^2 = 0$

Use quadratic formula: $\underset{-\frac{1}{2}g}{a}T^2 + \underset{v_0 \sin\theta}{b}T + \underset{H}{c} = 0 \longrightarrow T = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

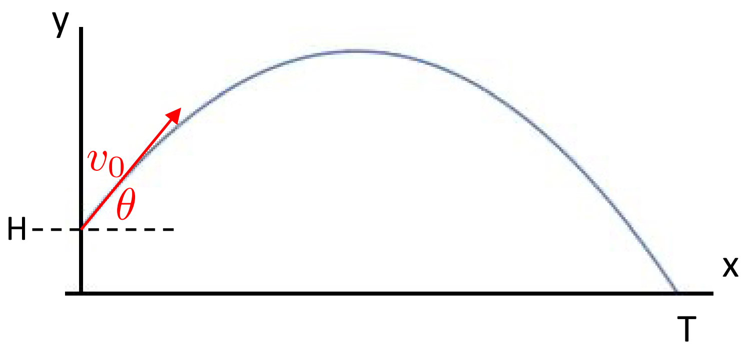
$$T = \frac{v_0 \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$

Other root gives T<0

How far does it go?
What angle gives maximum range?

$$x(t) = v_0 \cos\theta \, t$$

$$T = \frac{v_0 \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$



Range = x(T)

$$x(T) = \frac{v_0^2 \sin(2\theta)}{2g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$

Looks like a mess!
Check some basics…

$$H = 0 \Rightarrow x(T) = \frac{v_0^2 \sin(2\theta)}{g} \quad \checkmark$$

Gives back original range formula

Increasing H with fixed angle gives longer range ✓

Intuition about optimal angle

Longer free fall time makes larger horizontal velocity preferable

Larger H ➡ Smaller optimal angle

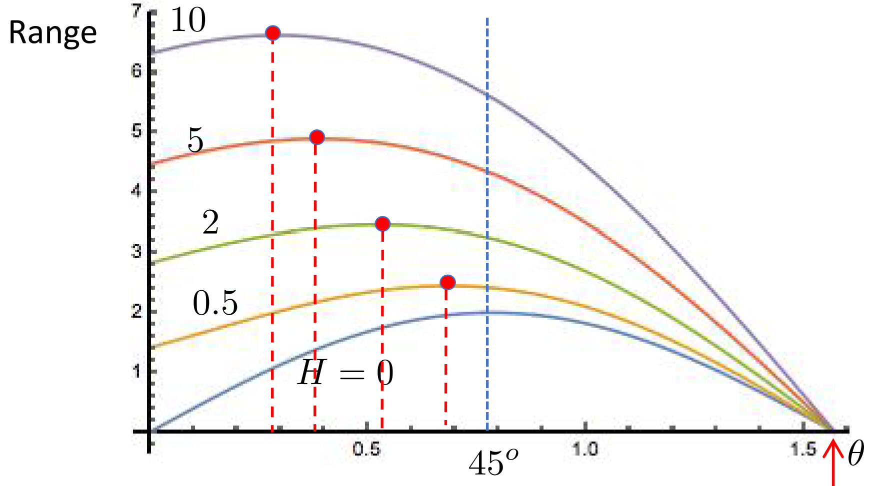
What angle gives maximum range?

$$x(T) = \frac{v_0^2 \sin(2\theta)}{2g}\left(1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2\theta}}\right)$$

Intuition about optimal angle

Longer free fall time makes Larger horizontal velocity will be preferable

Larger H ➡ Smaller optimal angle



$90^o = \pi/2\,\text{radians}$

Shooting straight up gives zero range

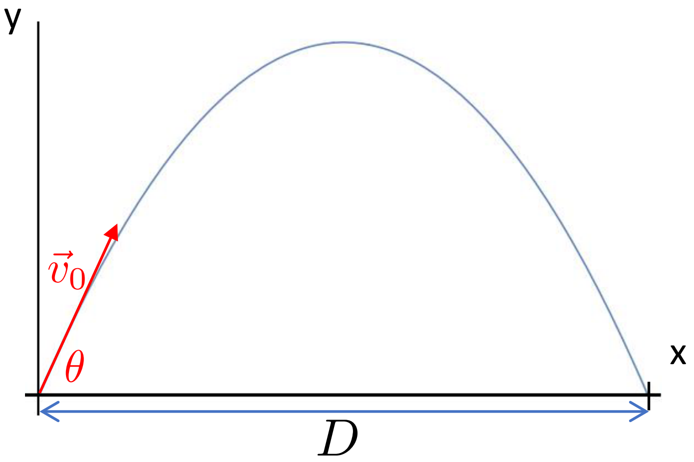
Plot range vs. Angle

Set $\dfrac{v_0^2}{2g} = 1m$

See that as H is increased, maximum range shifts to smaller angles

Back to range formula…

$$\boxed{D = \frac{v_0^2 \sin(2\theta)}{g}} \qquad v_0 = |\vec{v}_0|$$



Evel Knievel wants to jump over 100m worth of school buses

His motorcycle can go 50 m/s

What is the longest jump Evel can make with this motorcycle?

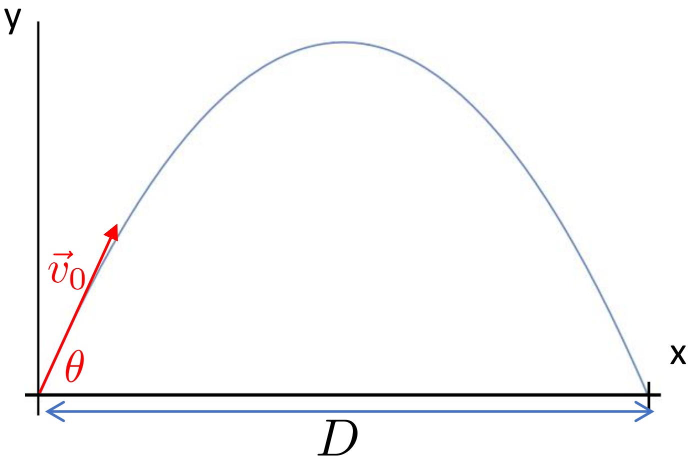
$$\theta = 45^o \Leftrightarrow \sin(2\theta) = 1$$

Gives maximum distance

$$D_{max} = \frac{v_0^2}{g} = \frac{(50m/s)^2}{9.8m/s^2} = 255m$$

Back to range formula…

$$\boxed{D = \frac{v_0^2 \sin(2\theta)}{g}} \qquad v_0 = |\vec{v}_0|$$



Evel Knievel wants to jump over 100m worth of school buses

His motorcycle can go 50 m/s

To what angle should his ramp be set?

Assuming no air resistance

$$\sin(2\theta) = \frac{gD}{v_0^2} = 0.39$$

$$\Rightarrow \theta = 11^o$$

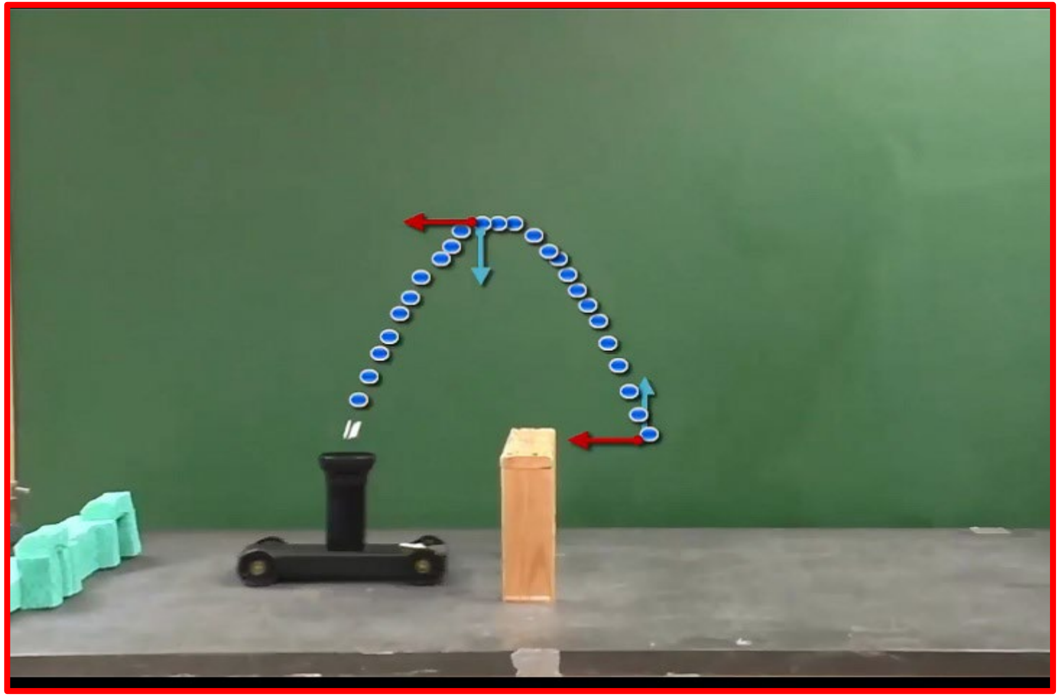
Second solution: $\theta = 90^0 - 11.5^0 = 78.5^0$ Check: D=50^2 x sin(2 x 78.5^0)/9.8 = 100 m

Projectile Motion: Cannon under the Bridge

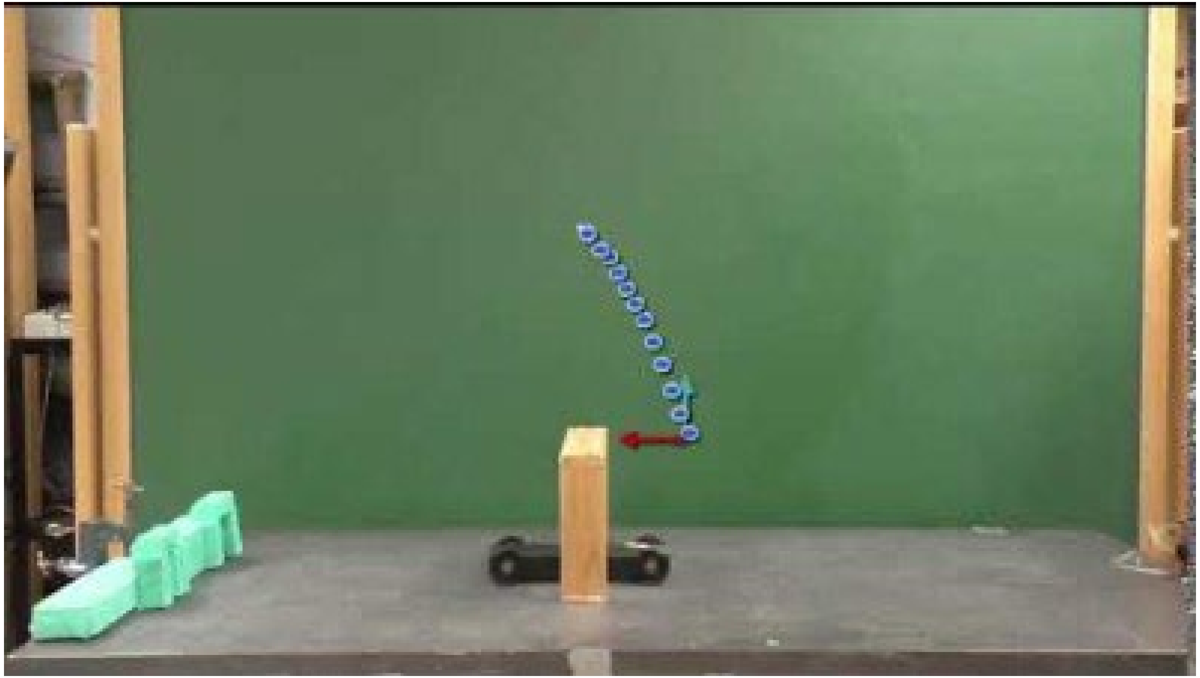
-Is horizontal motion of ball at constant velocity?

-Is horizontal motion of cannon at constant velocity?

-Are these velocities the same? Will ball land back in cannon?



Projectile Motion: Cannon under the bridge
Slow motion



Projectile Motion Demo: Which ball lands first?