250 Homework #1

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P1.1.4 [10 pts]

Let C be the set $\{0, 1, \ldots, 15\}$. Let D be a subset of C and define the number $f(D)$ as follows – $f(D)$ is the sum, for every element i of D, of 2^i. For example, if D is $1, 6$ then $f(D) = 2^1 + 2^6 = 66$.

(a) What are $f(\emptyset)$, $f(0, 2, 5)$, and $f(C)$?

(b) Is there a D such that $f(D) = 666$? If so, find it.

(c) Explain why, if D and E are any two subsets of C such that $f(D) = f(E)$, then $D = E$.

Solution:

(a)

(b)

(c)

*Collaborated with Nobody.

P1.2.5 [10 pts]

Let the alphabet C be $\{a, b, c\}$. Let the language X be the set of all strings over C with at least two occurrences of b. Let Y be the language of all strings over C that never have two occurrences of c in a row. Let Z be the language of all strings over C in which every c is followed by an a. (Recall that any string with no c's is thus in Z.)

(a) List the three-letter strings in each of X, Y, and Z. The easiest way to do this may be to first list all 27 strings in C^3 and then see which ones meet the given conditions.

(b) List the four-letter strings that are both in X and in Y, those that are both in X and in Z, those that are both in Y and in Z, and those that are in all three sets. How many total strings are in C^4?

(c) Are any of X, Y, or Z subsets of any of the others?

(d) Suppose u and v are two strings in X. Do we know that the strings u^R, v^R, uv, and vu are all in X? Either explain why this is always true, or give an example where it is not.

(e) Repeat the previous question for the languages Y and Z.

Solution:

(a)

(b)

(c)

(d)

(e)

P1.4.10 [10 pts]

Letting p denote "mackerel are fish" and q denote "trout live in trees", translate each of the following four statements into English: $\neg p \to q$, $\neg(p \to q)$, $\neg p \leftrightarrow q$, and $\neg(p \leftrightarrow q)$. Are any two of these four statements logically equivalent?

Solution:

P1.5.6 [10 pts]

Let Σ be the alphabet $a, b, c, \ldots, z$ and let U be the set Σ^3 of three-letter strings with letters from Σ. Let X be the set of strings in U whose first letter is c. Let Y be the set of strings whose second letter is a, and let Z be the set of strings whose last letter is t. Describe each of the following sets in English, and determine the number of strings in each set.

(a) $X \cap Y$

(b) $X \cap Y \cap Z$

(c) $Y \cup Z$

(d) $X \cap (Y \cup Z)$

Solution:

(a)

(b)

(c)

(d)

P1.7.6 [10 pts]

Suppose we substitute $a \oplus b$ for p and $a \wedge b$ for q in the contrapositive rule to get $((a \oplus b) \rightarrow (a \wedge b)) \leftrightarrow ((\neg a \wedge b) \rightarrow (\neg a \oplus b))$. Verify that this result is *not* a tautology. Why didn't our substitution lead to a valid tautology?

Solution:

P1.8.2 [10 pts]

A variant of the Proof By Cases rule is as follows: Given the premises $p \vee q$, $p \to r$, and $q \to r$, derive r.

Solution:

P1.8.7 [10 pts]

Prove that the compound propositions $p \wedge (q \rightarrow r)$ and $\neg(p \rightarrow (q \wedge \neg r))$ are equivalent by using the Equivalence and Implication Rule and constructing two deductive sequence proofs.

Solution:

P1.10.6 [12 pts]

We can define binary relations on the naturals for each of the five relational operators. Let $LT(x, y)$, $LE(x, y)$, $E(x, y)$, $GE(x, y)$, and $GT(x, y)$ be the predicates with templates $x < y$, $x \leq y$, $x = y$, $x \geq y$, and $x > y$ respectively.

(a) Show how each of the five predicates can be written using only LE and boolean operators. Use your constructions to rewrite $(LE(a, b) \oplus (E(b, c) \vee GT(c, a)) \rightarrow (LT(c, b) \wedge GE(a, c))$ in such terms.

(b) Express each of the five predicates using only LT and boolean operators, and rewrite the same compound statement in those terms.

Solution:

(a)

(b)

P2.3.9 [12 pts]

Let D be a set of dogs and let T be a subset of terriers, so that the predicate $T(x)$ means "dog x is a terrier". Let $F(x)$ mean "dog x is fierce" and let $S(x, y)$ mean "dog x is smaller than dog y". Write quantified statements for the following, using only variables whose type is D:

(a) There exists a fierce terrier.

(b) All terriers are fierce.

(c) There exists a fierce dog who is smaller than all terriers.

(d) There exists a terrier who is smaller than all fierce dogs, except itself.

Solution:

(a)

(b)

(c)

(d)

EC: P2.3.7 [10 pts]

Let D be a set of dogs, with R being the subset of retrievers, B being the subset of black dogs, and F being the subset of female dogs, with membership predicates $R(x)$, $B(x)$, and $F(x)$ respectively. Suppose that the three statements $\forall x : \exists y : R(x) \oplus R(y), \forall x : \exists y : B(x) \oplus B(y)$, and $\forall x : \exists y : F(x) \oplus F(y)$ are all true. What can you say about the number of dogs in D? Justify your answer.

Solution: