Announcements, Goals, and Reading

Announcements:

- HW02 due Tuesday Sep 27th, 11:59 pm on Mastering Physics
- HW01 was due yesterday; now in grace period
- **Help Resources: See moodle**

Goals for Today:

- Finding position **x(t)** from velocity vs time **v(t)**
- Motion w/ constant acceleration **a**
- Free Fall

Reading (Physics for Scientists and Engineers 4/e by Knight)

- Chapter 2: Kinematics in One Dimension

Review: Calculus 'Special Move' #1: The Derivative

The derivative of a polynomial function $\boldsymbol{f(x) = cx^n}$ is given by…

This works for any polynomial function.

$$\frac{d}{dx}f(x) = \frac{d}{dx}(cx^n) = ncx^{n-1}$$

$$Example: \frac{d}{dt}(2t^2) = 4t$$

Finding Position from Velocity

We can find velocity from x(t) by taking a derivative. Can we infer x(t) from v(t)? The answer is yes, but we will again need to use calculus.

Velocity is derivative of position ⟺ Position is the integral of velocity

Fundamental theorem of calculus

(Math 131 spoiler warning)

In equations…

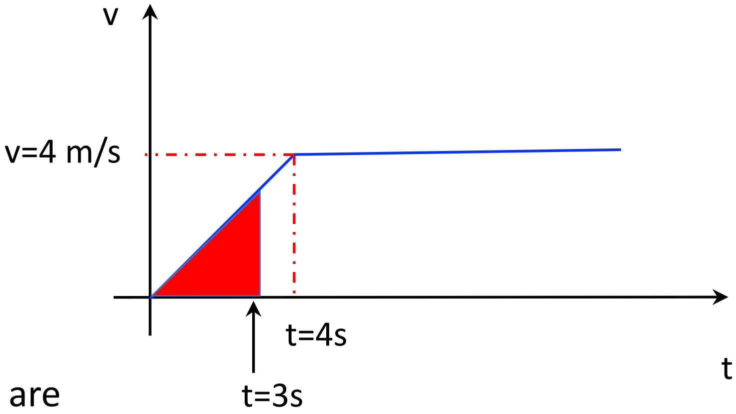
$$v(t) = \frac{dx}{dt} \iff x(t) = x_0 + \int_0^t v(t')dt'$$

What does this mean?

Displacement (change in position) is area under velocity curve

Example

Start with graph of velocity vs time



Object accelerates from rest to velocity 4 m/s in 4 s, then moves with constant velocity

Assuming object starts at x=0, what are positions at t =3s and t=7s?

Change in position is area under velocity graph

t = 3s ➡ area of triangle = ½ (base)(height)

slope 1 ➡ height = base = 3

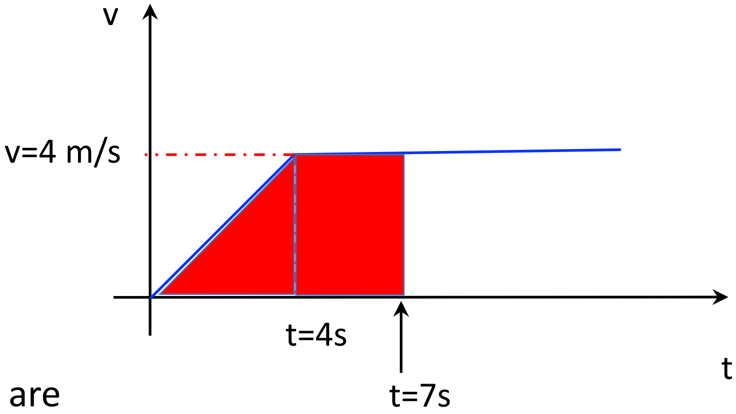
x (3s) = (1/2) x (3 s) x (3 m/s) = 4.5 m

If you are not familiar or comfortable yet with integrals

$$\int_0^t v(t)dt = \int_0^t at\ dt = \frac{at^2}{2}$$

Example

Start with graph of velocity vs time



Object accelerates from rest to velocity 4 m/s, then moves with constant velocity

Assuming object starts at x=0, what are positions at t =3s and t=7s?

Change in position is area under velocity graph

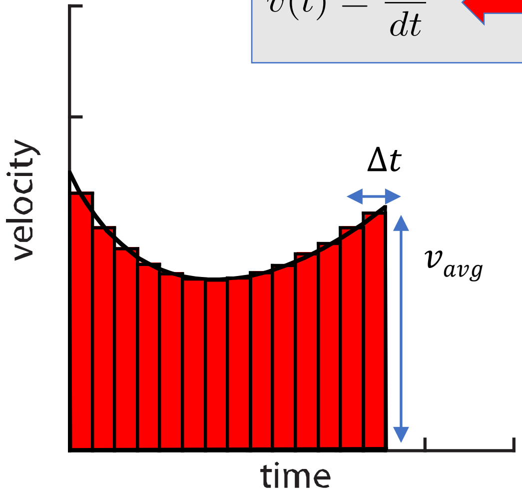
t = 7s ➡ area of triangle + area of rectangle

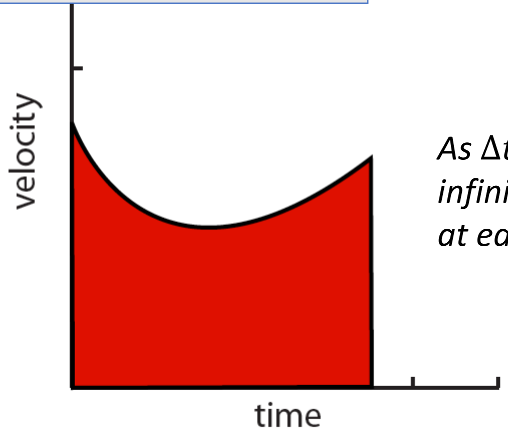
$$x\,(7s) = \underbrace{(1/2) \times (4s) \times (4\ m/s)}_{\text{triangle}} + \underbrace{3(s) \times (4\ m/s)}_{\text{rectangle}} = 8 + 12 = 20\ m$$

What if I want to find the area of a curve that doesn't have simple geometry?

More generally…

$$v(t) = \frac{dx}{dt} \Longleftrightarrow x(t) = x_0 + \int_0^t v(t')dt'$$





As $\Delta t \rightarrow 0$, Δt becomes the infinitesimally small "dt" at each point in v(t)

- Integral (area under curve) is approximated by by adding up area of rectangles
- Each rectangle acts like constant velocity motion over short time interval
- Exact integral is sum of infinite number of rectangles in limit $\Delta t \rightarrow 0$

Height of rectangle

$$\Delta x = v_{avg} \cdot \Delta t$$

Width of rectangle

*That's nice, but how do we **compute** this sum?*

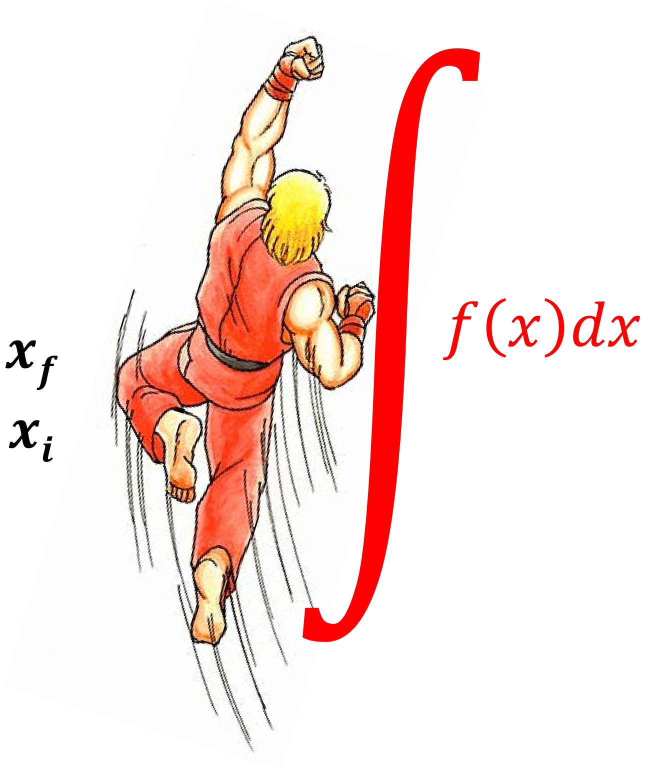
Calculus 'Special Move' #2: The Integral

The integral of a polynomial function
$f(x) = cx^n$ is given by…

$$\int_{x_i}^{x_f} f(x)dx = \int_{x_i}^{x_f} cx^n dx = \frac{cx^{n+1}}{n+1}\bigg|_{x_i}^{x_f}$$

$$= \frac{c{x_f}^{n+1}}{n+1} - \frac{c{x_i}^{n+1}}{n+1}$$

(For $n \neq -1$)



Example: Suppose we have a velocity v(t)=2t^2.
Starting at t=0, what is the displacement Δx traveled in 3 seconds?

$$\Delta x = \int_0^3 f(t)dt = \int_0^3 2t^2 dt = \frac{2t^3}{3}\bigg|_0^{3s} = \frac{2(3)^3}{3}m - 0m = 18m$$

We now know how to relate position and velocity using calculus.

$$v(t) = \frac{dx}{dt} \quad \Longleftrightarrow \quad x(t) = x_0 + \int_{t_0}^{t} v(t')\, dt'$$

…how does **acceleration** fit into this?

Instantaneous Acceleration

Recall... $$v(t) = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T}$$

Velocity is rate of change of position with time ➡ $v = \dfrac{dx}{dt}$

Acceleration is defined the same way in terms of velocity

$$a(t) = \lim_{T \to 0} \frac{v(t+T) - v(t)}{T}$$

Acceleration is rate of change of velocity with time ➡ $a = \dfrac{dv}{dt}$ $\left(= \dfrac{d^2x}{dt^2}\right)$

Acceleration gives **slope of tangent to v(t) curve** at time t
i.e. acceleration is the **derivative** of velocity (which is, in turn the **derivative** of position)
In other words, acceleration is the **second derivative** of position.

More practice with Instantaneous velocity and acceleration:

You won't usually use limits to calculate derivatives outside of math class

Use known results for derivatives of functions:

"Calculus Special Move #1"

$$\frac{d}{dt}t^n = nt^{n-1}$$

$$\frac{d}{dt}t^0 = 0$$

$$\frac{d}{dt}t^1 = 1$$

$$\frac{d}{dt}t^2 = 2t$$

$$x(t) = x_0 + v_0 t \quad \Rightarrow \quad v = \frac{dx}{dt} = v_0$$

Constant velocity motion

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \quad \Rightarrow \quad v = \frac{dx}{dt} = v_0 + at$$

Motion with constant acceleration

$$a(t) = \frac{dv}{dt} = a$$

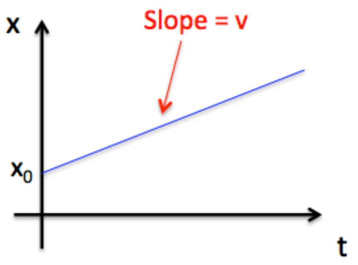
Acceleration

Recall…

Constant velocity

$$x = x_0 + vt$$

Position at t=0

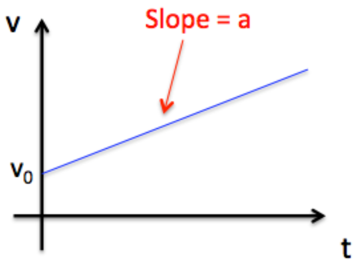


Similarly…

Constant acceleration

$$v = v_0 + at$$

Velocity at t=0



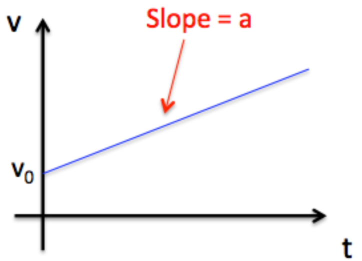
Check…

$$a = \frac{dv}{dt} = \frac{d}{dt}(v_0 + at) = a \quad \checkmark$$

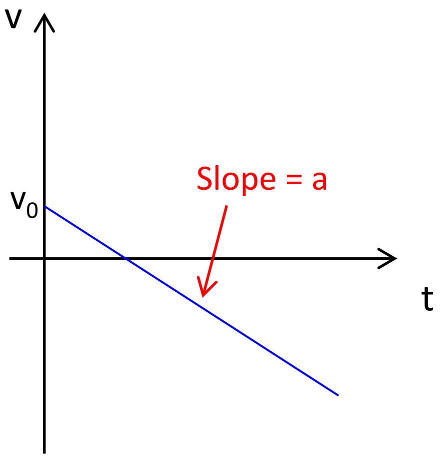
Constant acceleration basics…

$$v = v_0 + at$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(v_0 + at) = a$$



Slope **a** positive ➡ Velocity v increasing with time



Slope **a** negative ➡ Velocity decreasing with time

- When v>0, it is slowing down.
- When v<0, it is actually speeding up, but in the opposite direction

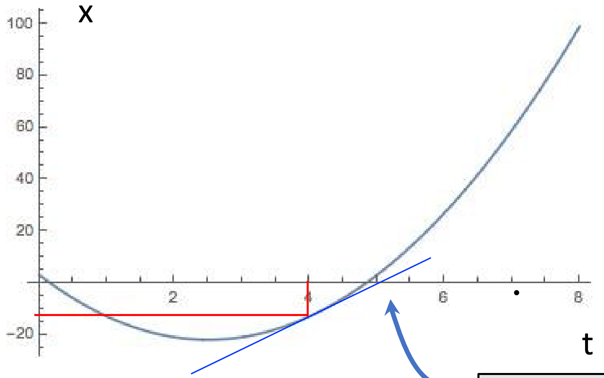
Motion with constant acceleration

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \quad \Rightarrow \quad v = \frac{dx}{dt} = v_0 + at$$

Example

Object starts at position x_0 = 3 m, moving with initial velocity v_0 = -20 m/s and accelerates at a = 8 m/s^2

What are its position and velocity at t = 4s?



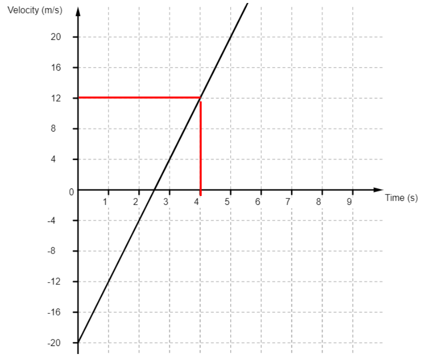
slope of tangent line is instantaneous velocity at t=4s

$$x(t) = 3 - 20t + 4t^2$$

$$x(4) = 3 - 20 \cdot 4 + 4 \cdot 4^2 = -13\,m$$

$$v(t) = -20 + 8t$$

$$v(4) = -20 + 8 \cdot 4 = 12\,m/s$$



Summary

Position x(t)	Velocity v(t)	Acceleration a(t)	Description
$x(t) = x_0$	$v(t) = 0$	$a(t) = 0$	Constant position
$x(t) = x_0+v_0\, t$	$v(t) = v_0$	$a(t) = 0$	Constant velocity
$x(t) = x_0 + v_0\, t + (1/2)\, a_0\, t^2$	$v(t) = v_0 + a_0\, t$	$a(t) = a_0$	Constant acceleration

$$a(t) = \frac{dv}{dt} \iff v(t) = v_0 + \int_{t_0}^{t} a(t')\, dt'$$

$$v(t) = v_0 + a(t - t_0)$$

$$v(t) = \frac{dx}{dt} \iff x(t) = x_0 + \int_{t_0}^{t} v(t')\, dt'$$

$$\boxed{x(t) = x_0 + v_0(t - t_0) + a(t - t_0)^2 / 2}$$

Closer look: Suppose some constant acceleration $\boldsymbol{a(t) = a_0}$.

Remember "Calculus Move #2":

$$\boldsymbol{\int_{x_i}^{x_f} cx^n dx = \frac{cx^{n+1}}{n+1}\bigg|_{x_i}^{x_f}}$$

$a(t) = \frac{dv}{dt}$, so $v(t) = v_0 + \int_0^t a(t')dt'$

$$= v_0 + a_0 \int_0^t dt'$$

$$= v_0 + a_0(t - 0)$$

$$\rightarrow \mathbf{v(t)} = \boldsymbol{v_0 + a_0 t}$$

$v(t) = \frac{dx}{dt}$, so $x(t) = x_0 + \int_0^t v(t')dt'$

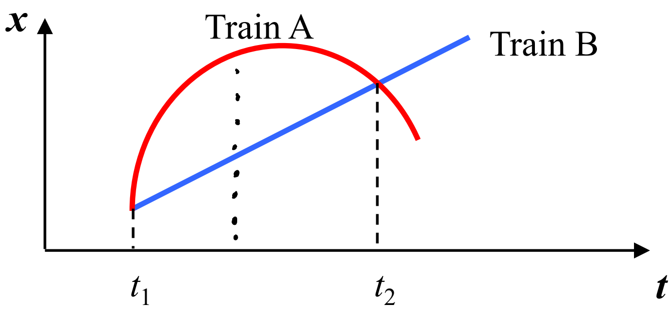
$$= x_0 + \int_0^t (v_0 + a_0 t')\, dt'$$

$$= x_0 + v_0 t + a_0 \int_0^t t'\, dt'$$

$\rightarrow \mathbf{x(t)} = \boldsymbol{x_0 + v_0 t + \frac{1}{2} a_0 t^2}$ under constant acceleration

Warm-Up: Two Trains

The graph below represents the position vs time reading for two trains moving on parallel tracks. Which of the following is true?



___At time $t = t_2$, both trains have the same instantaneous velocity.

___Both trains are accelerating all the time.

___Both trains have the same velocity at some time on the graph.

___After $t = t_1$, the trains never catch up to each other.

Position and constant acceleration

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$

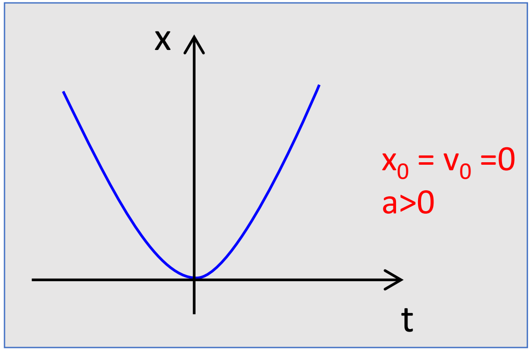
Position at t=0

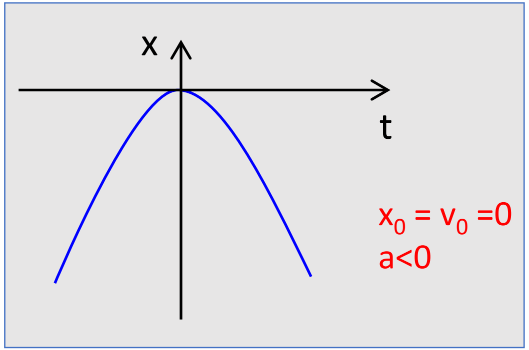
Velocity at t=0

Acceleration

Equation for a parabola

Draw some examples…



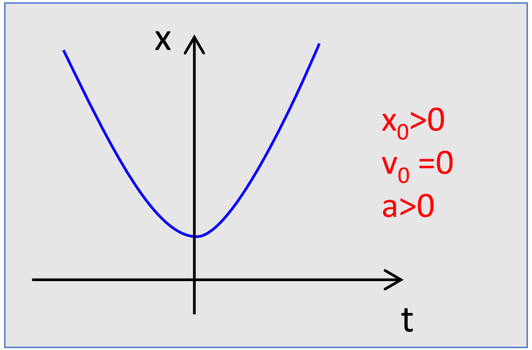


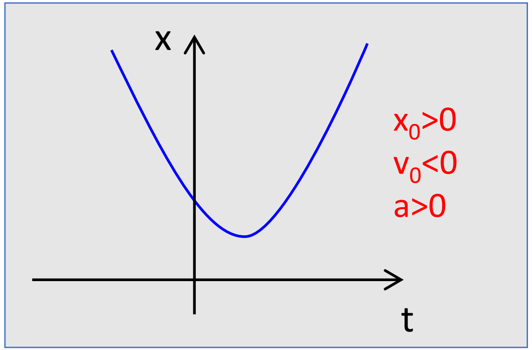
- Velocity starts out large and negative
- Slows down
- Comes momentarily to rest (v=0) at t=0
- Speeds up in positive x direction

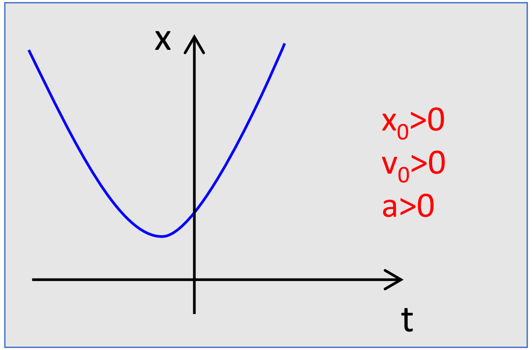
- Velocity starts out large and positive
- Slows down
- Comes momentarily to rest (v=0) at t=0
- Speeds up in negative x direction

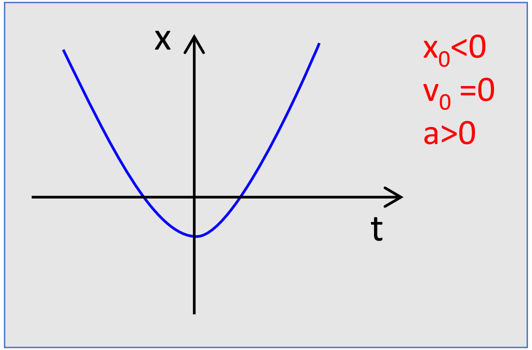
Draw some more examples…

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$









Increasing a ➡ Narrows opening of parabola

Decreasing a ➡ Widens opening of parabola

Motion with constant acceleration

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

Example…

- Alice is driving down the road at 30 m/s when she passes Bob, who is at rest in his car
- Bob accelerates at 0.1 m/s^2 in order to catch up and pass Alice
- How much time does it take Bob to catch up?
- How far has he travelled?
- How fast is he going when he catches up?

Write equations for positions

$$\boxed{x_A(t) = v_A t} \longleftarrow \boxed{\begin{array}{l} x_0 = a = 0 \\ v_A = 30 m/s \end{array}}$$

Alice

$$\boxed{x_B(t) = at^2 / 2} \longleftarrow \boxed{\begin{array}{l} x_0 = v_0 = 0 \\ a = 0.1 m/s^2 \end{array}}$$

Bob

- Alice is driving down the road at 30 m/s when she passes Bob, who is at rest in his car
- Bob accelerates at 0.1 m/s^2 in order to catch up and pass Alice
- How much time does it take Bob to catch up?
- How far has he travelled?
- How fast is he going when he catches up?

$$\boxed{x_A(t) = v_A t}$$

$$\boxed{x_B(t) = at^2/2}$$

Assume they meet at time T ➡ $$(30m/s)T = \frac{1}{2}(0.1m/s^2)T^2$$

➡ $$T = \frac{2(30m/s)}{(0.1m/s^2)} = 600s$$

Where they meet up

>Plug T into position formula $$x_A(600s) = \left(30\frac{m}{s}\right)600s = 18,000m$$

How fast Bob is going

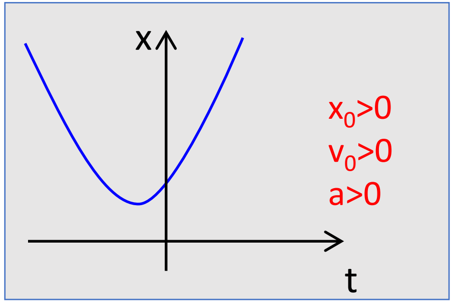
>Plug T into velocity formula

$$v_B(T) = aT$$

$$v_B(600s) = \left(0.1\frac{m}{s^2}\right)600s = 60\frac{m}{s}$$

Another type of problem…

- At **t=0** an object is at position $\mathbf{x_0}$, moving with velocity $\mathbf{v_0}$, and is accelerating at the constant rate **a.**
- **How far does it move before reaching velocity $\mathbf{v_1}$?**
- **Relevant for highway on-ramps, stopping distance, …**



Leads to a standard physics formula

Start with basic formulas for constant acceleration

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

$$v(t) = v_0 + at$$

Solve first for the time T at which this happens

$$v_1 = v_0 + aT \quad \Rightarrow \quad T = \frac{1}{a}(v_1 - v_0)$$

Plug this time into x(t) to see how far the object has moved

How far does the object move before reaching velocity v_1?

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

$$\boxed{T = \frac{v_1 - v_0}{a}}$$

Let $\Delta x = x(T) - x_0$

Plugging in T gives…

$$\Delta x = v_0\left(\frac{v_1 - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v_1 - v_0}{a}\right)^2$$

$$= \frac{1}{a}\left(\cancel{v_0 v_1} - v_0^2 + \frac{1}{2}v_1^2 - \cancel{v_0 v_1} + \frac{1}{2}v_0^2\right)$$

$$= \frac{1}{2a}(v_1^2 - v_0^2)$$

$$\boxed{\Delta x = \frac{v_1^2 - v_0^2}{2a}}$$

Commonly used by itself when not interested in how long process took. **Very useful formula!**

final velocity

Initial velocity

Example…

- A car can decelerate at a maximum rate of **$a = -5 m/s^2$**
- It is initially travelling at **$v_0 = 40 m/s$**
- How much distance is required for it to come to a stop?

$$\Delta x = \frac{v_1^2 - v_0^2}{2a}$$

final velocity

Initial velocity

Stop ➡ $v_1 = 0$ Final velocity is zero

$$\Delta x = \frac{0 - 40^2}{2(-5)} = \frac{-1600}{-10} = 160m$$

Similar example…

- A car can accelerate from rest to **50 m/s** in a distance of **100 m**.
- What is its acceleration, **a**?

$$\Delta x = \frac{v_1^2 - v_0^2}{2a}$$

final velocity

Initial velocity

Rearrange formula to solve for acceleration

$$a = \frac{v_1^2 - v_0^2}{2\Delta x}$$

Here

$$v_0 = 0$$
$$v_1 = 50m/s$$

Plug in to get… $$a = \frac{1}{2(100m)}\left((50m/s)^2 - (0m/s)^2\right) = 12.5m/s^2$$

- What car is it?

$$12.5\frac{m}{s^2}\frac{mile}{1609m}\frac{3600}{hour} \approx 28mph\frac{1}{s}$$

$$= 60mph\frac{1}{2.14s}$$

Fastest production cars by acceleration 0-60 MPH

2014 Nissan GT-R Track Edition: 2.7 s
2005 Bugatti Veyron 16.4: 2.7 s
2017 Audi R8 V10 Plus: 2.6 s
2018 Lamborghini Huracán Performante: 2.6 s
2015 Porsche 918 Spyder: 2.1 s
2020 Porsche 911 Turbo S: 2.2 s
…..

Motion with Constant Acceleration: Free Fall

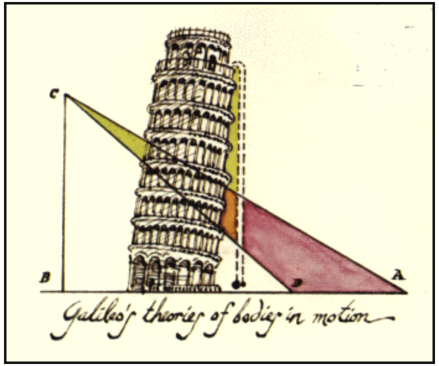
Acceleration due to gravity near Earth's surface happens at a very nearly constant rate.

...if we can ignore air resistance

$g = 9.8 m/s^2$ downwards

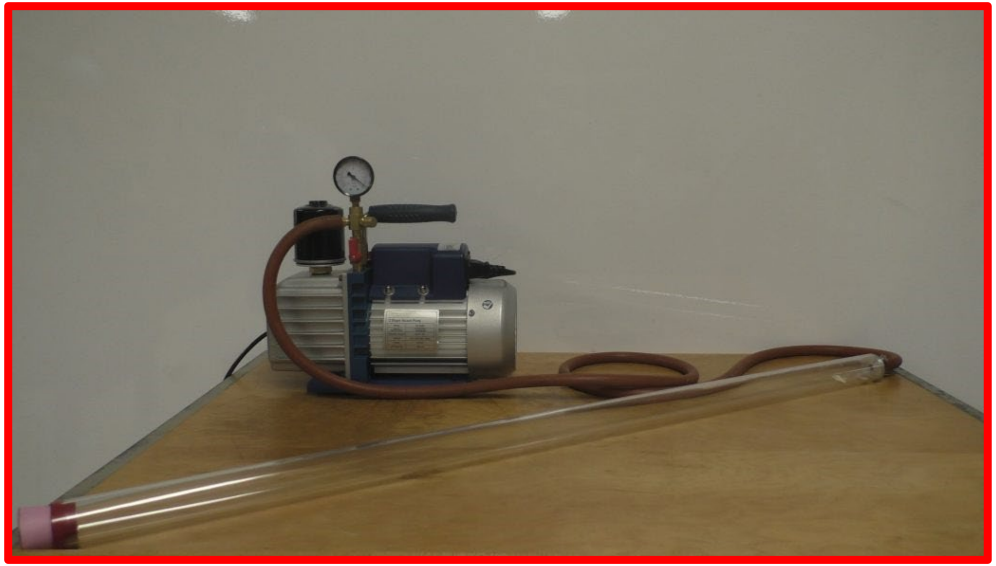
Everything falls at the same rate, regardless of its mass!

Demonstrated (according to lore) by Galileo dropping things off Leaning Tower of Pisa



For us...
Nice opportunity to use our constant acceleration formulas!

Motion with constant acceleration: Free Fall



Free Fall: Penny versus the Feather

https://youtu.be/XydJkXnklsI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