Exam 3 Review

Reminders!

- Emphasis only on the third part of the course
- Your exam is on Tuesday, 5/23
- Covers everything you have done after 4.4

Cheat sheet!

- You should put things that you ideally that confuse you the most or theorems you have difficulty remembering.
- Particularly will be very helpful for true and false questions.
- We have already reviewed things you can put on your cheatsheet in the Mentimeter presentation, which I will upload after this session.
- The true and false questions are also on the worksheet./

1. (a) Show that the characteristic equation of the polynomial of matrix A $\begin{pmatrix} 5 & 0 & 4 \\ -2 & 3 & -4 \\ 2 & 0 & 7 \end{pmatrix}$ is $-(\lambda-3)^2(\lambda-9)$. $= 0$

cofactor expansion over column $\boxed{2}$.

$$\det(A-\lambda I) = \begin{bmatrix} 5-\lambda & 0 & 4 \\ -2 & 3-\lambda & -4 \\ 2 & 0 & 7-\lambda \end{bmatrix} = -0 + (3-\lambda)\det\begin{bmatrix} 5-\lambda & 4 \\ 2 & 7-\lambda \end{bmatrix} - 0$$

$$= (3-\lambda)\left((5-\lambda)(7-\lambda)-8\right) = (3-\lambda)(27-12\lambda+\lambda^2) = (3-\lambda)(\lambda-3)(\lambda-9)$$

$$= -(\lambda-3)^2(\lambda-9) = 0$$

$$(\lambda-3)(\lambda-3)(\lambda-9) = 0$$

(b)Find a basis of R^3 consisting of the eigenvectors of A

eigenvalues $\lambda = 3, 9$.

for $\lambda = 3$

$$\left[\begin{array}{ccc|c} 5-3 & 0 & 4 & 0 \\ -2 & 3-3 & -4 & 0 \\ 2 & 0 & 7-3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ -2 & 0 & -4 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array} = \begin{array}{l} x_1 + 0x_2 + 2x_3 = 0 \\ x_1 = -2x_3 \end{array} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 9$:

$$\left[\begin{array}{ccc|c} -4 & 0 & 4 & 0 \\ -2 & -6 & -4 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \underline{\underline{0}} & 0 \end{array}\right]$$

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{eigenvectors} = \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

c) Find the invertible matrix P and a diagonal matrix D such that the matrix A above satisfies

$$P^{-1}AP = D$$

$$A = \begin{bmatrix} 5 & 0 & 4 \\ -2 & 3 & -4 \\ 2 & 0 & 7 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{bmatrix} 1/3 & 1 & 2/3 \\ -1/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1 & 2/3 \\ -1/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ -2 & 3 & -4 \\ 2 & 0 & 7 \end{bmatrix} \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} =$$

show that this multiplication is D! and you'll be fine for the test.

2. Let W be the plane in R^3 spanned by u and v given below respectively.

$$= \underset{u}{\begin{pmatrix}1\\2\\1\end{pmatrix}} \underset{v}{\begin{pmatrix}1\\1\\0\end{pmatrix}} \qquad W^{\perp} = \mathrm{Nul}\begin{pmatrix}1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}$$

a) Find the length of u
b) Find the distance between u and v
c) Find a vector of length 1 which is orthogonal to W
d) Find the projection of v to the line spanned by u
e) Write v as the sum of a parallel vector to u and an orthogonal vector to u
f) Find an orthogonal basis for W

(a) $u = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1+4+1} = \sqrt{6}$

(b) $\|u - v\| = \|u\| - \|v\|$

$= \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

(c) $W^{\perp}$ is orthogonal to W

$W^{\perp} = \mathrm{Nul}\begin{pmatrix}1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}$

$\mathrm{Nul}\begin{pmatrix}1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix} \longrightarrow \mathrm{Nul}\underbrace{\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\end{pmatrix}}$: $a = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

$a = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\hat{a} = \frac{a}{|a|} = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix}1 & -1 & 1\end{bmatrix}$

(d) $\mathrm{proj} = \frac{v \cdot u}{u \cdot u} \cdot u$

$= \frac{\cancel{3}^{\,1}}{\cancel{6}_{\,2}} \cdot \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1/2\\1\\1/2\end{pmatrix}$

(e) write 'v' in terms of the sum of 2 things

→ a vector $\parallel^{\text{lel}}$ to u

→ a vector $\perp$ to u.

- subtract from v, its orthogonal projection to the line spanned to u.

$\perp$ to the line spanned by u. $\left\{ \quad v - \dfrac{v \cdot u}{u \cdot u} \cdot u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} \right.$

$$v = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(f) $\left\{ \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} \right]$

3. The vector v and u are eigenvectors of the matrix A (given below respectively)

$$\begin{bmatrix}1\\1\end{bmatrix} \rightarrow v$$

$$\begin{bmatrix}4\\-3\end{bmatrix} \rightarrow u$$

$$\begin{bmatrix}.6 & .4\\.3 & .7\end{bmatrix} \rightarrow A$$

Your task is to first

$\lambda = 1$ and $\lambda = .3$

a) Find the eigen~~vectors~~values of u and v
b) Find the coordinates of the following in the basis ~~{u,v}~~ $\{v, u\}$

$$\begin{bmatrix}1\\8\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}4\\-3\end{bmatrix}$$ equivalent to: $$\begin{bmatrix}1 & 4\\1 & -3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

① $(c_1 + 4c_2) = 1$

② $-(c_1 - 3c_2) = 8$

$7c_2 = -7$

$c_1 = 5$

$c_2 = -1$

c) $A^{20}\begin{bmatrix}1\\8\end{bmatrix}$ Compute this.
d) Replace 20 with n in the last question. Now, you need to find this vector as n becomes larger and larger

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(c) We know how to decompose $\begin{bmatrix}1\\8\end{bmatrix}$ into the sum of two vectors in the basis $\{v_1, v_2\}$

Also, use

$$A^n v = \lambda^n v \quad \text{for any eigenvector of a matrix}$$

$$A^{20}\begin{bmatrix}1\\8\end{bmatrix} = \lambda_1^{20} c_1 \begin{bmatrix}1\\1\end{bmatrix} + \lambda_2^{20} c_2 \begin{bmatrix}4\\-3\end{bmatrix} = (1)^{20}(5)\begin{bmatrix}1\\1\end{bmatrix} + (.3)^{20}(-1)\begin{bmatrix}4\\-3\end{bmatrix}$$

$$= 5\begin{bmatrix}1\\1\end{bmatrix} - .3^{20}\begin{bmatrix}4\\-3\end{bmatrix}$$

→ leaving it here would cost no penalty if the values are complex!

(d) Formula: $A^n \dot{v} = \lambda^n v$ so,

$$A^n\begin{bmatrix}1\\8\end{bmatrix} = \lambda_1^n c_1 \begin{bmatrix}1\\1\end{bmatrix} + \lambda_2^n c_2 \begin{bmatrix}4\\-3\end{bmatrix}$$

$$= (1)^n (5)\begin{bmatrix}1\\1\end{bmatrix} + (.3)^n (-1)\begin{bmatrix}4\\-3\end{bmatrix}$$

$$= 5\begin{bmatrix}1\\1\end{bmatrix} - (.3)^n\begin{bmatrix}4\\-3\end{bmatrix}$$

↳ this would approach 0!

So what would $A^n\begin{bmatrix}1\\8\end{bmatrix}$ approach?

(when n is large)

4(b) HINT:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1(1) = 1$
$c_2(1) = 3$
$c_3(1) = 1$

so what is $[\vec{u}]_B$ be? simply a vector with the c's.

4. Let W be a linear subspace of R^4 defined by the equation a+b+c+d=0 ← $x_1 + x_2 + x_3 + x_4 = 0$,
   a) Show that the vectors that I've given you below form a basis of W

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- pivot in each column
- linearly ind!
- given vectors are a basis.

   b) Compute $[\vec{u}]_B$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 1 \end{bmatrix},$$

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

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→ steps:
- 1) row reduce
- 2) general soln
3) find c's.

5) Find the eigenvectors and eigenvalues of A^2 if A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 2 & 4 & 2 \end{bmatrix}$

cofactor exp.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & -3-\lambda & -2 \\ 2 & 4 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)\det\begin{bmatrix} -3-\lambda & -2 \\ 4 & 2-\lambda \end{bmatrix} - 0 + 2\det\begin{bmatrix} 2 & 1 \\ -3-\lambda & -2 \end{bmatrix}.$$

$$= -\lambda(\lambda^2 - 1) = -\lambda(\lambda+1)(\lambda-1) = 0$$

eigenvalues: $-1, +1, 0.$
↳ A

eigenvalues of A^2: $\lambda \to (0)^2 = 0$
$(-1)^2 = 1$
$(1)^2 = 1$

for $\lambda = 0$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 0 \\ 2 & 4 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∘ take out gener

for $\lambda = 1$.

do the same as $\lambda = 0$, find eigenvectors