Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

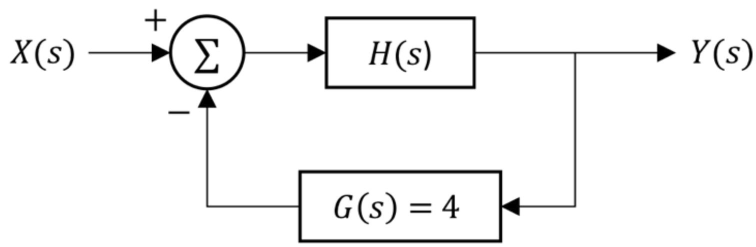
1. Consider an LTI system described by the (open-loop) transfer function…

$$H(s) = \frac{s}{s^2 - 2s + 5}$$

(a) Find the poles and zeros of $H(s)$.

(b) For this proportional feedback system with $G(s) = 4$, find the closed-loop transfer function $Q(s) = Y(s)/X(s)$.

(c) Determine if this feedback system is underdamped, critically damped, or overdamped.



ANSWERS:

(a) Writing $H(s)$ as $N(s)/D(s)$, $N(s) = s$, so there is one zero at $s = 0$, and $D(s) = s^2 - 2s + 5$, so there are two poles, at $s = 1 + j2$ and $s = 1 - j2$. This system is unstable, because the poles are on the right-half of the complex plane. We will need feedback to stabilize the system.

(b) The closed-loop transfer function can be written…

$$Q(s) = \frac{H(s)}{1 + G(s)\,H(s)}$$

… but it's usually more convenient in the form…

$$Q(s) = \frac{N(s)}{D(s) + G(s)\,N(s)}$$

In this case, with $G(s) = 4$, so…

$$Q(s) = \frac{s}{s^2 - 2s + 5 + 4s} = \frac{s}{s^2 + 2s + 5}$$

Note that the poles are different from before, but the zero is still at $s = 0$. This will remain true with proportional feedback, as here, and with PD feedback. That is, typically the zeros stay the same after stabilizing the system.

(c) Writing the denominator as $s^2 + 2\alpha s + \omega_0^2$, the poles can be written in the compact form…

$$p_{12} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

… which means the damping can be determined by comparing α to ω_0, that is…

$\alpha > \omega_0$	overdamped	2 negative, real poles
$\alpha = \omega_0$	critically damped	1 (repeated) negative, real pole
$\alpha < \omega_0$	underdamped	2 conjugate poles

In this case, $\alpha = 1$ and $\omega_0 = \sqrt{5}$, so $\alpha < \omega_0$, and the system is underdamped.

Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

2. Convert the following to the phasor domain…

   (a) $10\cos(20t)$ (b) $20\sin(40t)$

   (c) $30\cos(50t + 37°)$ (d) $40\sin(100t - 127°)$

ANSWERS:

We can show that the phasor $X = A\,e^{j\varphi}$ is associated with the function $x(t) = A\cos(\omega t + \varphi)$. In other words, from the defining equation…

$$x(t) = \mathrm{Re}\{X\,e^{j\omega t}\}$$

… we can write…

$$\mathrm{Re}\{A\,e^{j\varphi}\,e^{j\omega t}\} = \mathrm{Re}\{A\,e^{j(\omega t+\varphi)}\} = A\cos(\omega t + \varphi)$$

Further, we will adopt a more compact form using $\angle$, i.e., $A\,e^{j\varphi}$ will be written $A\angle\varphi$.

(a) The phase is 0°, so $10\cos(20t) \longrightarrow 10\angle 0° = 10 + j0 = 10$. (In general, $\cos(\omega t) \leftrightarrow 1$.)

(b) Convert sin to cos by subtracting π/2 or 90° from the phase, that is…

$$20\sin(40t) = 20\cos(40t - 90°)$$

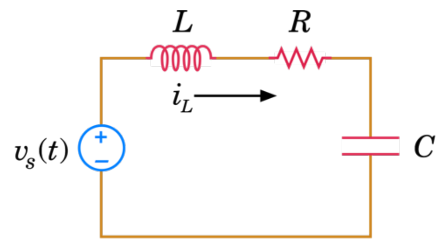
Therefore, the phase is –90°, which means $20\sin(40t) \longrightarrow 20\angle -90° = 0 - j20 = -j20$. (In general, $\sin(\omega t) \leftrightarrow -j$.)

(c) $30\cos(50t + 37°) \longrightarrow 30\angle 37°$.

(d) $40\sin(100t - 127°) = 40\cos(100t - 217°) \longrightarrow 40\angle -217° = 40\angle 143°$. Note that we typically "fix" the phase angle by adding or subtracting an integer multiple of 360°, as was done in the last step above.

Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

3. Consider this RLC circuit, with $R = 3\Omega$, $L = 30\text{mH}$, $C = 500\mu\text{F}$, and $v_s(t) = 500\text{mV}\cos(200t)$.



   (a) Convert the circuit to the phasor domain.

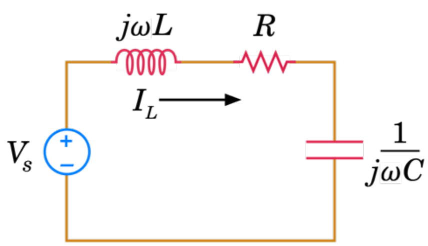
   (b) Find the phasor current I_L.

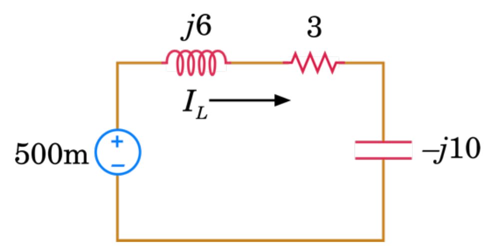
   (c) Compute the phasor voltages across R, L, and C, and verify that KVL is satisfied.

   (d) Find the unknown currents and voltages in the time domain.

ANSWERS:

(a) In the phasor domain, the impedances of R, L, and C are R, $j\omega L$, and $1/j\omega C$. Further, the source $v_s(t)$ becomes the phasor V_s, and the current $i_L(t)$ becomes the phasor I_L, as shown on the left...





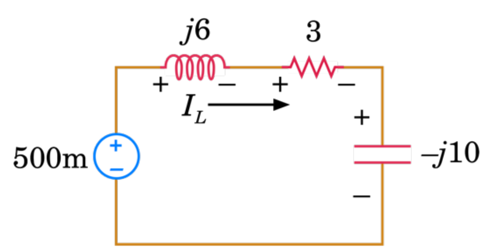
(b) The angular frequency is given indirectly inside the input signal, in this case, $\omega = 200\text{rad/s}$. Inserting known values for everything else, $\omega L = 6000\text{m}\Omega$, so $j\omega L = j6\Omega$. $R = 3\Omega$ stays the same. $\omega C = 100{,}000\mu\text{F/s} = 0.1\text{F/s}$, so $1/j\omega C = -j10\Omega$. (Recall that RC is a time constant, so Ω-F = s.) Finally, the voltage phasor for the source is $V_s = 500\text{mV}\angle 0° = 500\text{mV}$, as shown above on the right.

This circuit is solved like a purely resistive network, i.e., think about a voltage source connected to three resistors, all in series. The current is the voltage divided by the equivalent impedance of three elements in series, so...

$$I_L = \frac{V_s}{Z_{eq}} = \frac{500}{j6 + 3 - j10} = \frac{500}{3 - j4} \cdot \frac{3 + j4}{3 + j4} = \frac{500\,(3 + j4)}{25} = 20\,(3 + j4) = 60 + j80 \text{ mA}$$

... where we multiplied the numerator and denominator by $3 + j4$ to make the denominator real. Note also that the units are mA, because the source is in mV, and the impedances are all in Ω.

(c) Obeying the passive sign convention, we get the polarities shown to the right. The voltages across these three impedances are therefore...



$$V_L = j6\,I_L = j6\,(60 + j80) = -480 + j360 \text{ mV (+ on the left)}$$
$$V_R = 3\,I_L = 3\,(60 + j80) = 180 + j240 \text{ mV (+ on the left)}$$
$$V_C = -j10\,I_L = -j10\,(60 + j80) = 800 - j600 \text{ mV (+ on top)}$$

The sum of these is 500mV, which is the phasor value of the source, so KVL is satisfied.

[continued]

Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

(d) Converting these four phasors back to the time domain…

$$i_L(t) = 60\cos(200t) - 80\sin(200t)$$

$$v_L(t) = -480\cos(200t) - 360\sin(200t)$$

$$v_R(t) = 180\cos(200t) - 240\sin(200t)$$

$$v_C(t) = 800\cos(200t) + 600\sin(200t)$$

… where the current is in mA, and the voltages are all in mV.

Note that we could have done this entire calculation using amplitudes and phases by first converting Z_{eq} to polar form, that is…

$$Z_{eq} = 3 - j4 = 5\angle - 53.1°$$

Therefore…

$$I_L = \frac{V_s}{Z_{eq}} = \frac{500\angle 0°}{5\angle - 53.1°} = 100\text{mA}\angle 53.1°$$

Then…

$$V_L = 6\Omega\angle 90° \cdot 100\text{mA}\angle 53.1° = 600\text{mV}\angle 143.1°$$

$$V_R = 3\Omega\angle 0° \cdot 100\text{mA}\angle 53.1° = 300\text{mV}\angle 53.1°$$

$$V_C = 10\Omega\angle - 90° \cdot 100\text{mA}\angle 53.1° = 1000\text{mV}\angle - 36.9°$$

… and finally…

$$i_L(t) = 100\text{mV}\cos(200t + 53.1°)$$

$$v_L(t) = 600\text{mV}\cos(200t + 143.1°)$$

$$v_R(t) = 300\text{mV}\cos(200t + 53.1°)$$

$$v_C(t) = 1000\text{mV}\cos(200t - 36.9°)$$

Of course, it would have been impossible to check KVL using the polar form, so you will need to be comfortable using both forms.

Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

4. Consider this periodic signal $f(t)$.

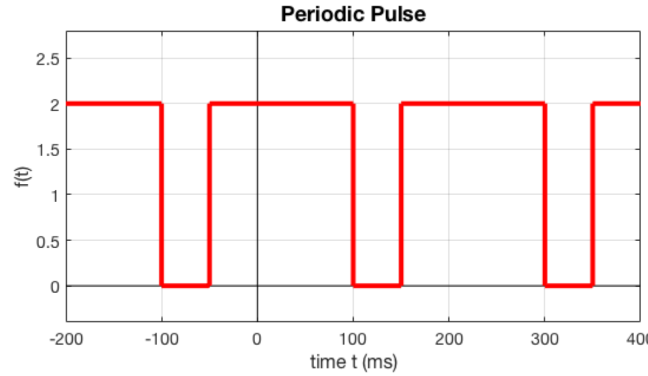
   (a) Find the fundamental period T_0.

   (b) Find the fundamental angular frequency ω_0.

   (c) Find the complex coefficients x_n of its Fourier series.

   (d) Compute x_n from n = –4 to n = 4.

   (e) Rewrite the complex coefficients as an amplitude c_n and phase φ_n.



   (f) Rewrite the complex coefficients as coefficients of $\cos n\omega_0 t$ and $\sin n\omega_0 t$, i.e., a_n and b_n.

ANSWERS:

(a) The fundamental period is the shortest time when the signal repeats. Sometimes, it helps to start at the leading or trailing edge of a pulse, then find the next corresponding edge. For instance, there is a trailing edge at t = –50ms, and the next trailing edge is at t = 150ms, so the fundamental period is T_0 = 200ms.

(b) The fundamental angular frequency ω_0 is equal to $2\pi/T_0$ = 10π rad/s. Note that it is common while doing Fourier analysis that every time you see either ω_0 or T_0, you should discover that it is paired with the other. In other words, the product $\omega_0 T_0$ = 2π is extremely useful to know, because it allows us to work symbolically. As you will see here, the final answers will not depend on either.

(c) The definition of the complex coefficient x_n is…

$$x_n = \frac{1}{T_0} \int_{\text{one full period}} x(t)\, e^{-jn\omega_0 t}\, dt$$

… where we can start at any time that is convenient, as long as we integrate over one complete period. In this case, it will be more efficient to start at $-T_0/4$, so…

$$x_n = \frac{1}{T_0} \int_{-T_0/4}^{3T_0/4} f(t)\, e^{-jn\omega_0 t}\, dt$$

The given function $f(t)$ is equal to 0 from $T_0/2$ until $3T_0/4$, and the value of the function otherwise is 2, so we can change the upper limit to $T_0/2$ and insert 2 for the value of the function…

$$x_n = \frac{1}{T_0} \int_{-T_0/4}^{T_0/2} 2\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\, 2 \frac{1}{-jn\omega_0}\, e^{-jn\omega_0 t}\Big|_{-T_0/4}^{T_0/2} = \frac{2j}{n\omega_0 T_0} \left(e^{-jn\omega_0 T_0/2} - e^{jn\omega_0 T_0/4}\right)$$

As mentioned above, we can replace each $\omega_0 T_0$ with 2π, and the final result is…

$$x_n = \frac{j}{n\pi} \left(e^{-jn\pi} - e^{jn\pi/2}\right)$$

… which does **not** depend on the time scale of the given signal.

[continued]

Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

(d) Note that the general expression from (c) does not actually work for $n = 0$, so we typically need to calculate that separately…

$$x_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$$

… because the exponential factor is equal to 1 when $n = 0$. x_0 is the average value of the function. In this case, the pulse is "on" for 3/4 of each period, so the average value is 3/4 of 2, or 1.5.

Let's make a table showing the values of x_n from $n = -4$ to $n = 4$…

n	x_n
–4	$\frac{j}{-4\pi}\left(e^{-j(-4)\pi} - e^{j(-4)\pi/2}\right) = \frac{j}{-4\pi}(1-1) = 0$
–3	$\frac{j}{-3\pi}\left(e^{-j(-3)\pi} - e^{j(-3)\pi/2}\right) = \frac{j}{-3\pi}(-1-j) = \frac{-1+j}{3\pi}$
–2	$\frac{j}{-2\pi}\left(e^{-j(-2)\pi} - e^{j(-2)\pi/2}\right) = \frac{j}{-2\pi}\left(1-e^{-j\pi}\right) = -\frac{j}{\pi}$
–1	$\frac{j}{-\pi}\left(e^{-j(-1)\pi} - e^{j(-1)\pi/2}\right) = \frac{j}{-\pi}(-1+j) = \frac{1+j}{\pi}$
0	1.5
1	$\frac{j}{\pi}\left(e^{-j\pi} - e^{j\pi/2}\right) = \frac{j}{\pi}(-1-j) = \frac{1-j}{\pi}$
2	$\frac{j}{2\pi}\left(e^{-j2\pi} - e^{j2\pi/2}\right) = \frac{j}{2\pi}\left(1-e^{j\pi}\right) = \frac{j}{\pi}$
3	$\frac{j}{3\pi}\left(e^{-j3\pi} - e^{j3\pi/2}\right) = \frac{j}{3\pi}(-1+j) = \frac{-1-j}{3\pi}$
4	$\frac{j}{4\pi}\left(e^{-j4\pi} - e^{j4\pi/2}\right) = \frac{j}{4\pi}(1-1) = 0$

Note that x_{-n} is equal to the conjugate of x_n in every case. This is generally true as well. Further, since $x_0 = x_0^*$, the value must be real, assuming the given signal is real. Also, every 4th coefficient is zero, because the pulse is "off" for 1/4 of each cycle. Finally, the Fourier representation of a signal…

$$x(t) = \sum_{n=-\infty}^{\infty} x_n\, e^{jn\omega_0 t}$$

… looks like the result might be complex, since we are multiplying a complex coefficient by a complex exponential, but it turns out that the result is perfectly real, because x_{-n} and x_n are conjugates of each other.

[continued]

Solutions to Examples for Discussion 10 (ECE 213, Spring 2024)

(e) The amplitude-phase representation of the Fourier series is…

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \varphi_n)$$

… where we have pulled out the $n = 0$ term, and the summation now starts at $n = 1$. This is made possible by combining each $x_n\, e^{jn\omega_0 t}$ for $n > 0$ with its conjugate $x_{-n}\, e^{-jn\omega_0 t}$, making the result purely real. Thus, $c_0 = x_0$, $c_n = 2|x_n|$, and $\varphi_n =$ phase of x_n, or $c_n \angle \varphi_n = 2x_n$. So, for example, $c_1 \angle \varphi_1 = 2x_1 = 2\frac{1-j}{\pi} = \frac{2\sqrt{2}}{\pi}\angle -45°$.

(f) The sine-cosine representation of the Fourier series is…

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

The first term is again renamed, so $a_0 = c_0 = x_0$. For the rest, it's easiest to go back to the complex exponential representation in order to derive expressions for the coefficients, that is, each term inside the sum is the sum of two terms from the first equation for $x(t)$…

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = x_n\, e^{jn\omega_0 t} + x_{-n}\, e^{-jn\omega_0 t}$$

Recognizing that $x_{-n} = x_n^*$, we get…

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = x_n\, e^{jn\omega_0 t} + x_n^*\, e^{-jn\omega_0 t}$$

Next, the two terms on the right-hand side are conjugates of each other. Whenever we add a complex number to its conjugate, we get twice the real part of the number, because the imaginary parts cancel. This is true also of complex expressions, as here, so…

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = 2\,\mathrm{Re}\{x_n\, e^{jn\omega_0 t}\} = 2\,\mathrm{Re}\{x_n\}\cos(n\omega_0 t) - 2\,\mathrm{Im}\{x_n\}\sin(n\omega_0 t)$$

Therefore, $a_n = 2\,\mathrm{Re}\{x_n\}$ and $b_n = -2\,\mathrm{Im}\{x_n\}$, or $a_n - jb_n = 2x_n$. For example, $x_1 = (1-j)/\pi$, so $a_1 = b_1 = 2/\pi$. (Or, if you prefer, you may instead use $a_n + jb_n = 2x_n^*$.)

Further, we can write expressions for $a_0, a_n,$ and b_n in terms of the given periodic signal $x(t)$…

$$a_0 = \frac{1}{T_0}\int_{\text{one full period}} x(t)\, dt$$

$$a_n = \frac{2}{T_0}\int_{\text{one full period}} x(t)\ \cos(n\omega_0 t)\ dt$$

$$b_n = \frac{2}{T_0}\int_{\text{one full period}} x(t)\ \sin(n\omega_0 t)\ dt$$

Finally, note that all of this is consistent with the work we have been doing with phasors, i.e., $a_n - jb_n$ in the phasor domain becomes $a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$ in the time domain, and $c_n \angle \varphi_n$ in the phasor domain becomes $c_n \cos(n\omega_0 t + \varphi_n)$ in the time domain.