Solutions to Examples for Discussion 8 (ECE 213, Spring 2024)

1. The response of the LTI system to input $x(t) = e^{-3t} \cos 4t \; u(t)$ is output $y(t) = 5\, e^{-3t} \cos(4t + 8) \; u(t)$.
   (a) Find $H(s)$.
   (b) Find the poles and the zeros.
   (c) Use $H(s)$ to write the system as an LCCDE.
   (d) Find $h(t)$.
   (e) Find the output $y(t)$ when the input is $x(t) = 10\, e^{-3t} \, u(t)$.

ANSWERS:

(a) First, rewrite $y(t)$ into terms that can be more easily transformed to the s-domain…

$$y(t) = 5\, e^{-3t} \cos(4t + 8) \; u(t) = 5\, e^{-3t} \left(\cos 4t \cos 8 - \sin 4t \sin 8\right) \; u(t)$$

Then, convert $x(t)$ and $y(t)$ to $X(s)$ and $Y(s)$, and compute $H(s) = Y(s)/X(s)$…

$$X(s) = \frac{s+3}{(s+3)^2 + 4^2}; Y(s) = \frac{5\,\big((s+3)\cos 8 - 4 \sin 8\big)}{(s+3)^2 + 4^2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{5\,((s+3)\cos 8 - 4 \sin 8)}{s+3}$$

(b) There is only one pole, at $s = -3$. There is only one zero, at $s = 4 \tan 8 - 3$.

(c) It helps to first rearrange the last equation, and then multiply out the right side…

$$(s+3)\, Y(s) = \Big(5\big((s+3)\cos 8 - 4 \sin 8\big)\Big) \; X(s) = (5s + 15 \cos 8 - 20 \sin 8)\, X(s)$$

Recognizing that each factor of s is the result of a time-derivative, the LCCDE must be…

$$\frac{dy}{dt} + 3\, y(t) = 5\, \frac{dx}{dt} + (15 \cos 8 - 20 \sin 8)\, x(t)$$

(d) The highest order of s in the numerator is the same as the highest order of s in the denominator, therefore, we must first rewrite $H(s)$ before we can use partial fractions…

$$H(s) = 5 \cos 8 - \frac{20 \sin 8}{s+3}$$

Therefore, $h(t) = 5 \cos 8 \;\, \delta(t) - 20 \sin 8 \;\, e^{-3t} \, u(t)$.

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(e) Find $X(s)$, then $Y(s) = H(s)\,X(s)$, which can be rewritten using partial fractions…

$$Y(s) = H(s)\,X(s) = \frac{5\left((s+3)\cos 8 - 4\sin 8\right)}{s+3} \cdot \frac{10}{s+3}$$

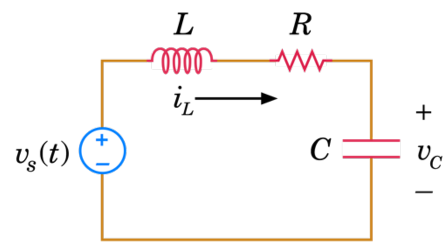
$$= \frac{50\left((s+3)\cos 8 - 4\sin 8\right)}{(s+3)^2} = \frac{A_1}{s+3} + \frac{A_2}{(s+3)^2}$$

In this case, we can simply split up the numerator into two terms…

$$Y(s) = \frac{50\cos 8}{s+3} - \frac{200\sin 8}{(s+3)^2}$$

Therefore, $y(t) = (50\cos 8 - 200\,t\,\sin 8)\,e^{-3t}\,u(t)$

2. Consider this RLC circuit, with $R = 3.2\Omega$, $L = 0.4$H, and $C = 0.1$F. Use $v_s(t)$ as the input signal and $v_C(t)$ as the output signal.



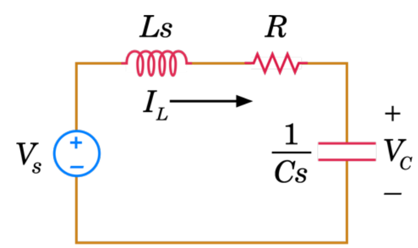
   (a) Find $H(s)$ using s-domain circuit analysis.

   (b) Use $H(s)$ to write the system as an LCCDE.

   (c) Find $h(t)$.

   (d) Find the step response $y_{step}(t)$.

ANSWERS:



(a) First, transform the circuit to the s-domain, as shown to the right, then use a voltage divider to find $V_C(s)$, and insert the impulse function as the input signal and simplify, so that the transfer function is a proper ratio of polynomials in s…

$$H(s) = V_s(s) \cdot \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = 1 \cdot \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

Making the coefficient of s^2 in the denominator equal to 1, then inserting known values…

$$H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{25}{s^2 + 8s + 25}$$

(b) Recognizing that $H(s) = Y(s)/X(s)$ and that each factor of s is the result of a time-derivative…

$$\frac{d^2y}{dt^2} + 8\,\frac{dy}{dt} + 25\,y(t) = 25\,x(t)$$

(c) The poles of $H(s)$ are complex, so $h(t)$ is a damped sinusoid…

$$H(s) = \frac{25}{(s+4)^2 + 3^2} = \frac{as + b\omega}{(s+4)^2 + 3^2}$$

In this case, $a = 0$ and $b = 25/3$, so $h(t) = 8.333\, e^{-4t} \sin 3t \;\, u(t)$.

(d) The input signal $x(t) = u(t)$, so $X(s) = 1/s$, and therefore…

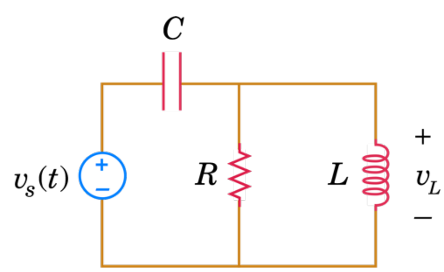
$$Y(s) = H(s)\, X(s) = \frac{25}{(s+4)^2 + 3^2} \cdot \frac{1}{s} = \frac{A_1}{s} + \frac{a(s+4) + b\omega}{(s+4)^2 + 3^2}$$

Using the method of residues, $A_1 = 1$. Subtracting $1/s$ from both sides and simplifying…

$$\frac{-s-8}{(s+4)^2 + 3^2} = \frac{a(s+4) + b\omega}{(s+4)^2 + 3^2}$$

Therefore, $a = -1$ and $b = -4/3$, so $y(t) = \left(1 - e^{-4t}(\cos 3t + 1.333\; \sin 3t)\right) u(t)$.

3. Consider this RLC circuit, with $R = 2\Omega$, $L = 5\text{H}$, and $C = 0.05\text{F}$.
Use $v_s(t)$ as the input signal and $v_L(t)$ as the output signal.



(a) Find $H(s)$ using s-domain circuit analysis.

(b) Find the frequency response function $H(\omega)$.

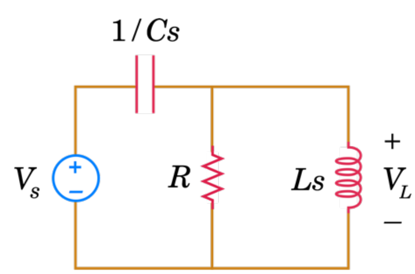
(c) Use $H(s)$ to write the system as an LCCDE.

(d) Find $h(t)$.

(e) Find the output $y(t)$ when the input is $x(t) = 2 \sin 3t \;\, u(t)$.

ANSWERS:

(a) First, transform the circuit to the s-domain, as shown to the right, then combine R and Ls in parallel, use a voltage divider to find $V_L(s)$, and insert the impulse function as the input signal…



$$H(s) = V_s(s) \cdot \frac{\dfrac{RLs}{R + Ls}}{\dfrac{1}{Cs} + \dfrac{RLs}{R + Ls}} = 1 \cdot \frac{RLs \cdot Cs}{R + Ls + RLs \cdot Cs}$$

Making the coefficient of s^2 in the denominator equal to 1, then inserting known values…

$$H(s) = \frac{s^2}{s^2 + \dfrac{1}{RC} s + \dfrac{1}{LC}} = \frac{s^2}{s^2 + 5s + 4}$$

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(b) Recall that the frequency response function $H(\omega)$ comes from the ratio of the output to the input when the input signal is a complex exponential $e^{j\omega t}$. Recall as well that the output is also proportional to a complex exponential, and every derivative in the time domain results in a factor of $j\omega$ multiplied by the original $x(t)$ or $y(t)$. This is similar to what happens in Laplace, as every derivative in the time domain produces a factor of s in the s-domain. Therefore (it turns out), we can find $H(\omega)$ by replacing every s with $j\omega$ in $H(s)$. The result is…

$$H(\omega) = \frac{(j\omega)^2}{(j\omega)^2 + 5(j\omega) + 4} = \frac{-\omega^2}{-\omega^2 + 5j\omega + 4}$$

(c) Recognizing that $H(s) = Y(s)/X(s)$ and that each factor of s is the result of a time-derivative…

$$\frac{d^2y}{dt^2} + 5\,\frac{dy}{dt} + 4\,y(t) = \frac{d^2x}{dt^2}$$

(d) Rewrite $H(s)$, then expand the second term into partial fractions…

$$H(s) = 1 - \frac{5s+4}{s^2+5s+4} = 1 - \frac{5s+4}{(s+1)(s+4)} = 1 - \left(\frac{A_1}{s+1} + \frac{A_2}{s+4}\right)$$

Using the method of residues, $A_1 = -1/3$ and $A_2 = 16/3$, so…

$$h(t) = \delta(t) + \left(\frac{1}{3}\,e^{-t} - \frac{16}{3}\,e^{-4t}\right)u(t)$$

(e) Transforming $x(t)$, constructing $Y(s)$, and rewriting using partial fractions…

$$Y(s) = H(s)\,X(s) = \frac{s^2}{(s+1)(s+4)} \cdot \frac{6}{s^2+9} = \frac{A_1}{s+1} + \frac{A_2}{s+4} + \frac{as + b\omega}{s^2 + 3^2}$$

Using the method of residues, $A_1 = 0.2$ and $A_2 = -1.28$. Solve for b next, by evaluating both sides at $s = 0$. The result is $b = 0.36$. Pick another convenient value of s to solve for a, e.g., $s = 1$. The result is $a = 1.08$. Therefore…

$$y(t) = (0.2\,e^{-t} - 1.28\,e^{-4t} + 1.08\cos 3t + 0.36\sin 3t)\,u(t)$$

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4. An LTI system is described by the following input-output LCCDE:

$$2\,\frac{d^3y}{dt^3} + 4\,\frac{dy}{dt} - y(t) = \frac{d^2x}{dt^2} + 5\,x(t)$$

   (a) Find $H(s)$ for this system.

   (b) Draw the Direct Form II implementation.

ANSWERS:

(a) Replacing each derivative with a factor of s, the differential equation becomes...

$$2\,s^3Y(s) + 4\,sY(s) - Y(s) = s^2X(s) + 5\,X(s)$$

$H(s) = Y(s)/X(s)$, so...

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 5}{2s^3 + 4s - 1}$$

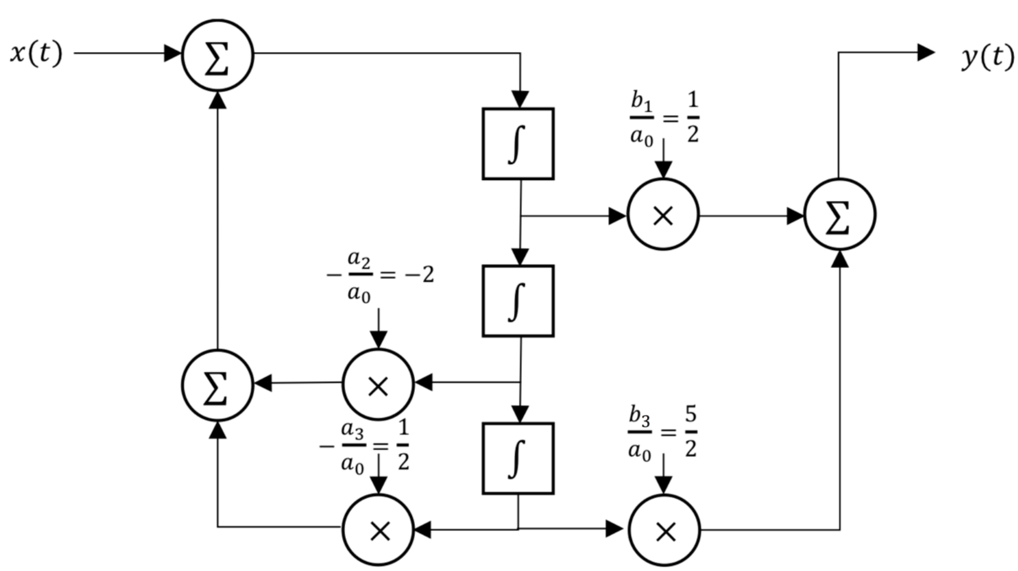
(b) Dividing the numerator and denominator by the highest order in s...

$$H(s) = \frac{Y(s)}{X(s)} = \left(\frac{1}{s} + 5\cdot\frac{1}{s^3}\right)\cdot\left(2 + 4\cdot\frac{1}{s^2} - 1\cdot\frac{1}{s^3}\right)^{-1} = H_1(s)\,H_2(s)$$

Therefore, using the conventions from class...

$$a_0 = 2;\; a_1 = 0; a_2 = 4; a_3 = -1;\; b_0 = 0; b_1 = 1; b_2 = 0; b_3 = 5$$

... and the resulting implementation is...



... where $H_1(s)$ is represented on the right, and $H_2(s)$ is represented on the left.

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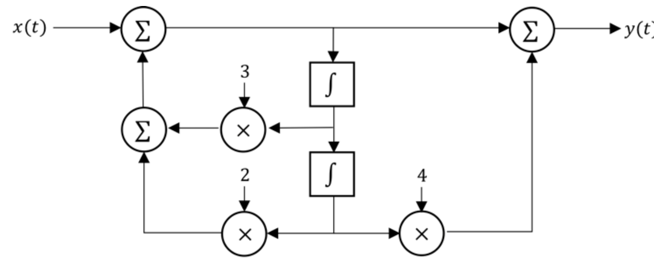
5. Write the input-output LCCDE for the system having the Direct Form II implementation shown to the right.



ANSWER:

The equation is second order, as there are only two "integrals" in the middle. Reading the values from the diagram, and assuming that $a_0 = 1$...

$$a_1 = -3; a_2 = -2;\ b_0 = 1; b_1 = 0; b_2 = 4$$

Therefore, the LCCDE associated with this LTI system is...

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 2\,y(t) = \frac{d^2x}{dt^2} + 4\,x(t)$$

NOTES:

- The values of a_1 and a_2 are negative, because the multipliers are positive, and they are understood to be $-a_1/a_0$ and $-a_2/a_0$.
- $b_0 = 1$, because there is a line on the top-right (to the left of the summation symbol near $y(t)$). In other words, there is no "multiplier" because it's equal to 1, so you should not assume that $b_0 = 0$. If that were true, then there would be no line there, as in Problem 4.