Announcements, Goals, and Reading

Announcements:

- HW11 due Tuesday 12/06
- HW12 due Monday 12/12
- MT2 solutions will be posted at end of *this* week.
- Forward FOCUS survey is now open—please provide feedback.

Goals for Today:

- Moment of Intertia
- Rolling

Reading (Physics for Scientists and Engineers 4/e by Knight)

- Chapter 12: Rotation of a Rigid Body

Center of mass is important. How do we determine it?

For a system of discrete point masses

N particles masses m_i $i = 1, 2, \ldots, N$

positions $\vec{x}_i$

$$\vec{x}_{cm} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2 + \cdots + m_N\vec{x}_N}{m_1 + m_2 + \cdots + m_N}$$

Weighted average of positions of particles

Positions of most massive particles will dominate

In terms of components…

$$\boxed{x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots + m_Nx_N}{m_1 + m_2 + \cdots + m_N}}$$

$$\boxed{y_{cm} = \frac{m_1y_1 + m_2y_2 + \cdots + m_Ny_N}{m_1 + m_2 + \cdots + m_N}}$$

$$\boxed{z_{cm} = \frac{m_1z_1 + m_2z_2 + \cdots + m_Nz_N}{m_1 + m_2 + \cdots + m_N}}$$

Center of mass of continuous distribution of mass

$$\sum_{i=1}^{N} m_i = M$$

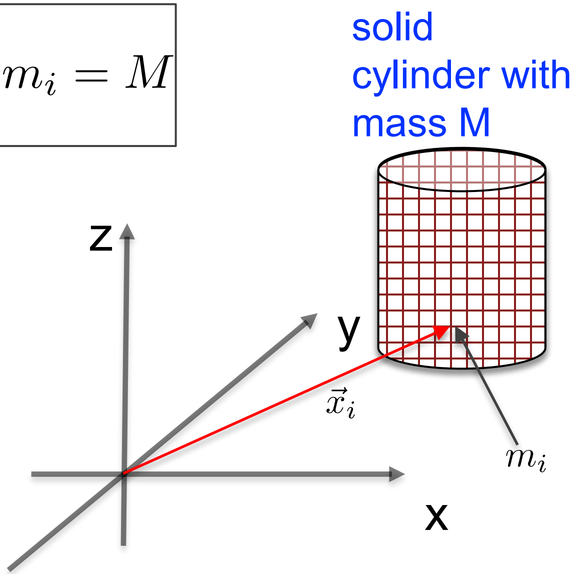
Divide object up into N small blocks

masses m_i $\quad i = 1, 2, \ldots, N$

positions $\vec{x}_i$

Center of mass $\vec{x}_{cm} = \sum_{i=1}^{N} m_i \, \vec{x}_i$ / M

To accurately model shape



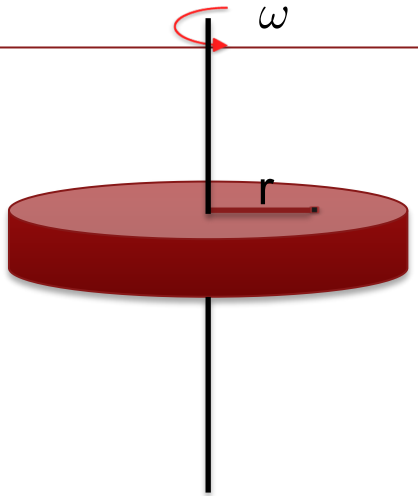
Keep making blocks smaller and smaller ➡ $N \to \infty$ ➡ Sum turns into integral over volume of object!

$$\boxed{\vec{x}_{cm} = \int \vec{x} \, \rho(\vec{x}) d^3x \ / \text{M}}$$

$\rho(\vec{x})$ = mass density = mass / volume

$\rho(\vec{x}) d^3x$ = m_i = mass in an infinitesimal volume

Need to determine *kinetic energy associated with rotation*
Recall rotational kinematics



ω angular velocity

all parts of a rigid body rotate with same angular velocity and same angular acceleration

$$\alpha = \frac{d\omega}{dt}$$ angular acceleration

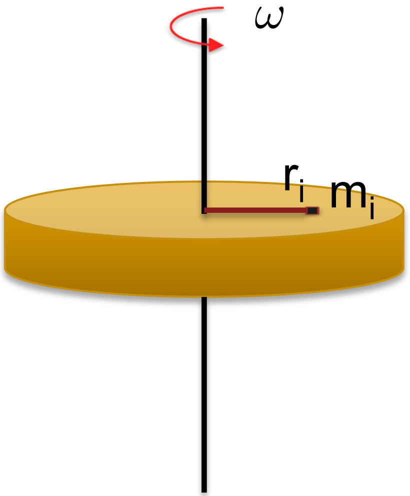
A point distance r from axis moves with

tangential velocity $v_t = \omega r$ tangential acceleration $a_t = \alpha r$

radial velocity $v_r = 0$ radial acceleration $a_r = -\omega^2 r$

centripetal acceleration

Important: What is kinetic energy of a rotating object?



All points in object rotate with same angular velocity ω

But may have different ***speeds*** v=ωr

Need to add up kinetic energies of all points at different distances from axis

For one small chunk of mass m_i a distance r_i from axis

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} (m_i r_i^2)\, \omega^2$$

Kinetic energy of all parts of object will have same factor of ω^2

Can write kinetic energy of whole object as:

$$\boxed{K = \frac{1}{2} I \omega^2}$$

"Moment of Inertia"

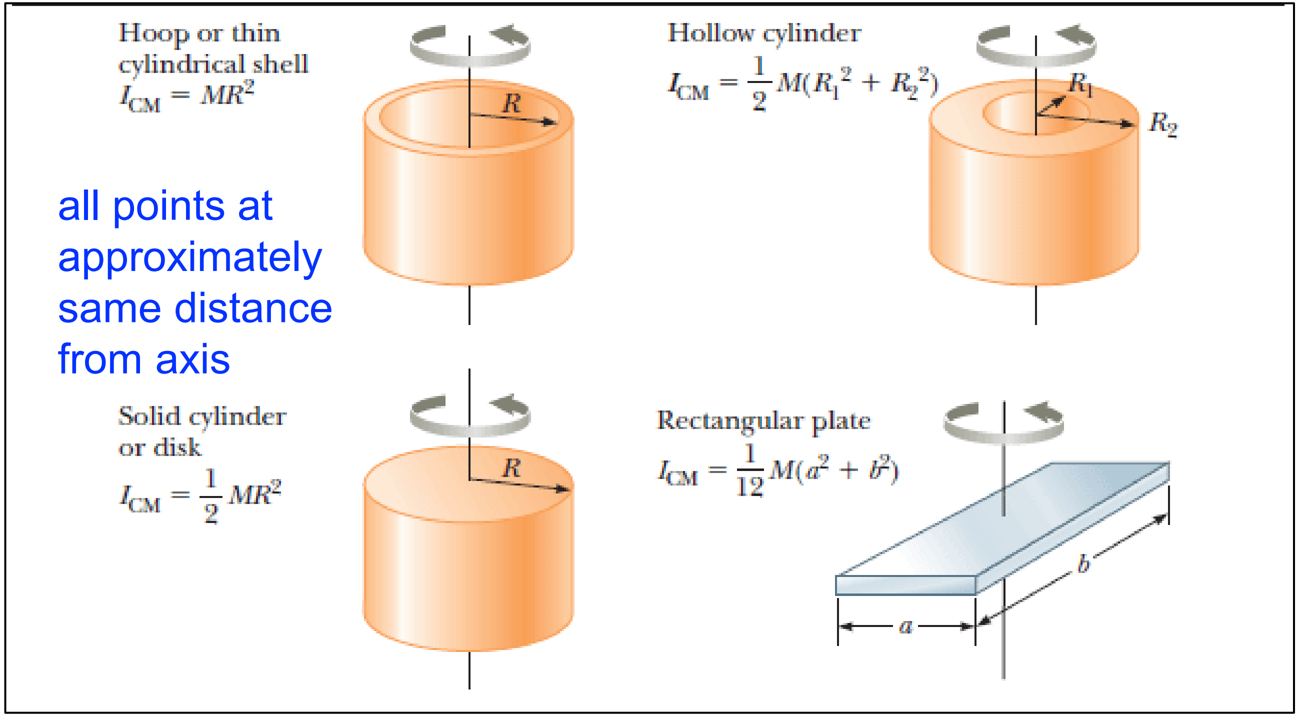
Moment of inertial is given by

$$\boxed{I = \sum_i m_i r_i^2}$$

or more precisely by an integral

Remember this result!

Moment of Inertia of different solid shapes



-You don't need to remember these moments of inertia

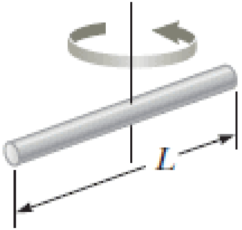
-They will be given to you on the final if necessary

-We will learn how to calculate ***I*** for simple shapes

Moment of Inertia of different solid shapes

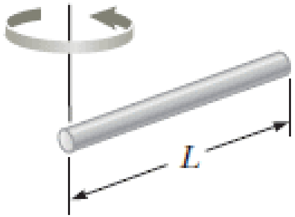
Long, thin rod with rotation axis through center

$$I_{CM} = \frac{1}{12} ML^2$$



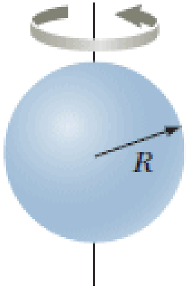
Long, thin rod with rotation axis through end

$$I = \frac{1}{3} ML^2$$



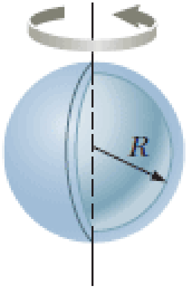
Solid sphere

$$I_{CM} = \frac{2}{5} MR^2$$



Thin spherical shell

$$I_{CM} = \frac{2}{3} MR^2$$



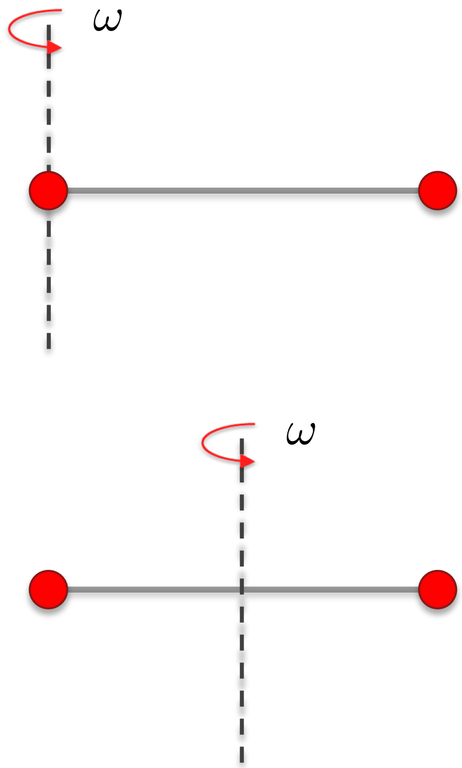
Important note ➡ **moment of inertia depends on choice of rotational axis**

Moment of Inertia: Depends on choice of rotational axis

Barbell consists of two 5 kg balls connected by a 1m massless bar

Compute the moment of inertia for two different choices of rotational axis

$$I = \sum_i m_i r_i^2$$



Moment of Inertia: Depends on choice of rotational axis

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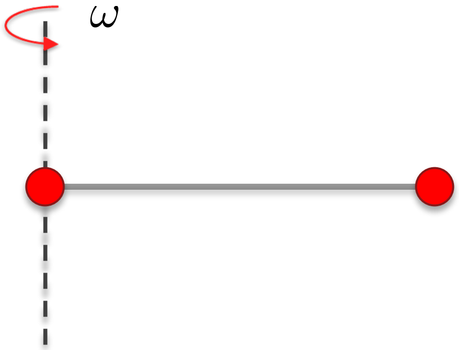
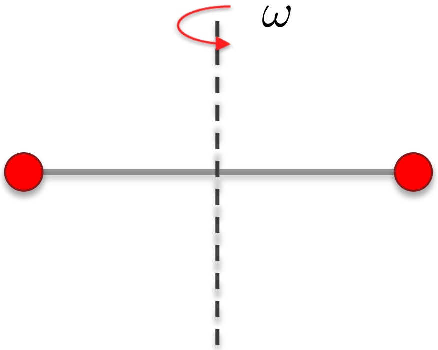


$$I = \sum_i m_i r_i^2$$

$$I_{end} = (5kg)(0)^2 + (5kg)(1m)^2$$

$$= 5\, kgm^2$$



$$I_{mid} = (5kg)(0.5m)^2 + (5kg)(0.5m)^2$$

$$= 2.5\, kgm^2$$

Example

An axe has L=0.5 m handle of mass M=0.25 kg attached to a head of mass m=1.5 kg

Find its moment of inertial ***about its handle end***

Moment of inertia gets contributions from axe handle and head

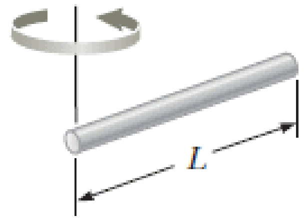
Head $I_1 = mL^2 = (1.5kg)(0.5m)^2$

$$= 0.38\, kgm^2$$

Handle $I_2 = \frac{1}{3}ML^2$

$$= \frac{1}{3}(0.25kg)(0.5m)^2$$

$$= 0.02\, kgm^2$$



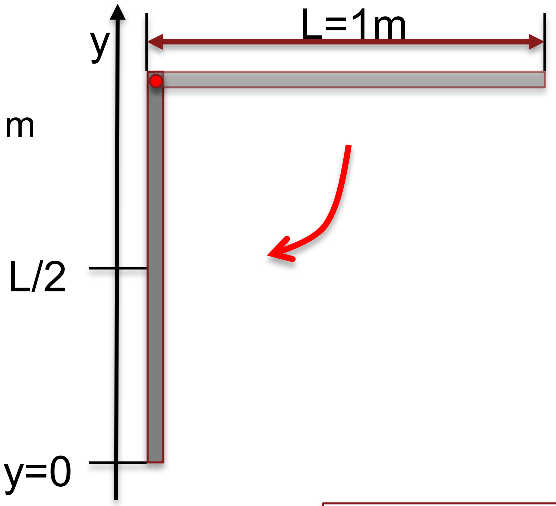
$$I = I_1 + I_2 = 0.40\, kgm^2$$

Kinetic energy example

Metal rod with m=0.5 kg and length L=1.0 m can pivot about a hinge at one end.

Released from horizontal position

What is its angular velocity as it passes through a vertical position?



Use conservation of energy

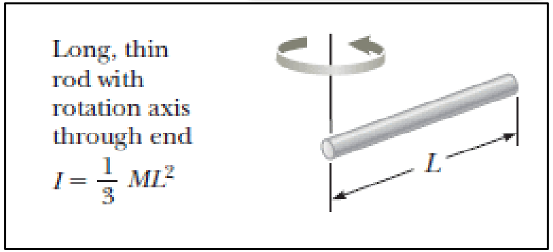
$$E = K + U$$

$$K = \frac{1}{2}I\omega^2 \qquad U = mgy_{cm}$$

vertical position of center of mass

moment of inertia

$$I = \frac{1}{3}mL^2$$



Initial energy $E_i = 0 + mgL$

Final energy $E_f = \frac{1}{2}I\omega_f^2 + mg\frac{L}{2}$

$$E_i = E_f \Rightarrow \frac{1}{2}I\omega_f^2 = mg\frac{L}{2}$$

Kinetic energy example

Metal rod with m=0.5 kg and length L=1.0 m can pivot about a hinge at one end

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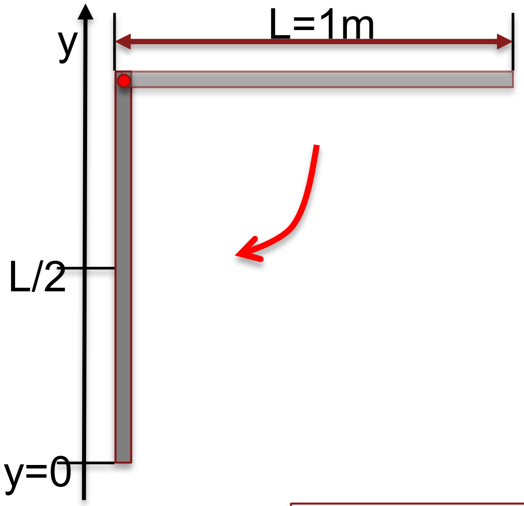


$$E_i = E_f \Rightarrow \frac{1}{2} I \omega_f^2 = mg\frac{L}{2}$$

moment of inertia

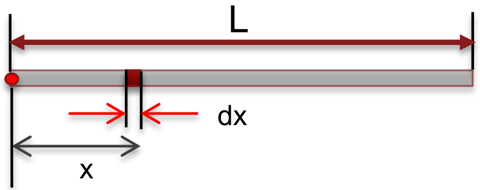
$$I = \frac{1}{3} mL^2$$

$$\Rightarrow \omega_f^2 = \frac{mgL}{I} = \frac{mgL}{\frac{1}{3}mL^2} = \frac{3g}{L}$$

$$\Rightarrow \omega_f = \sqrt{\frac{3g}{L}} = \sqrt{\frac{3(9.8m/s^2)}{1.0m}} = 5.4\, rad/s$$

Compute moments of inertia for some shapes

Thin rod of mass M and length L pivoted at one end

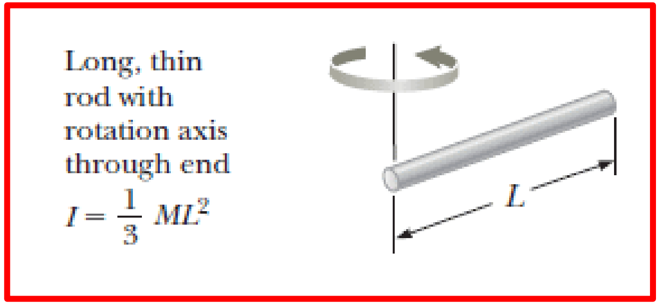


Break rod up into infinitesimal segments of width dx

Amount of mass in each segment

$$dm = \left(\frac{dx}{L}\right) M = \frac{M}{L} dx$$

mass per unit length



Moment of inertia of each segment

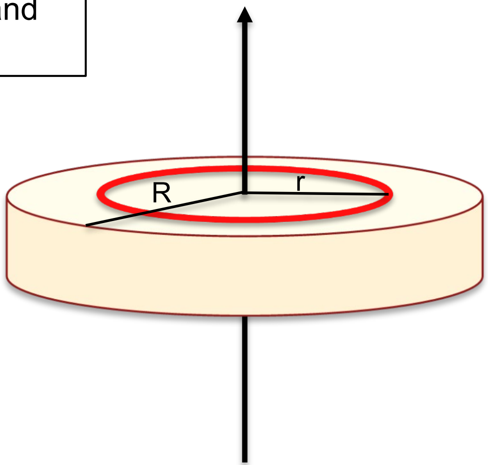
$$dI = x^2 dm = \left(\frac{M}{L}\right) x^2 dx$$

Add these all up

$$I = \int dI = \frac{M}{L} \int_{x=0}^{x=L} x^2 dx = \frac{M}{L}\left(\frac{L^3}{3}\right) = \frac{1}{3} M L^2 \checkmark$$

Determine moment of inertia of solid disk of mass M and radius R about axis through center of mass

Divide disk into infinitesimally thick cylindrical shells of thickness dr



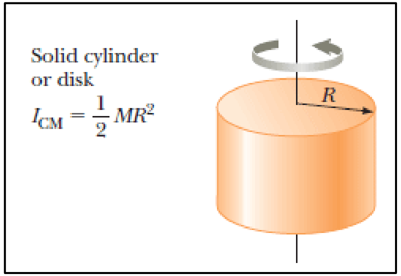
$A = \pi R^2$ area of entire disk

$\frac{M}{A} =$ mass per unit area of disk

$dA = 2\pi r\, dr$ area of cylindrical shell

$dm = \frac{M}{A} dA = \frac{M}{A}(2\pi r)\, dr$ mass in cylindrical shell

$dI = r^2 dm = \frac{2\pi M}{A} r^3\, dr$ moment of inertia of infinitesimal shell

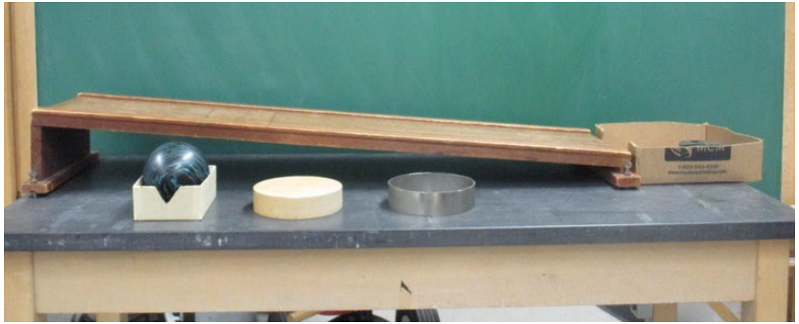


$$I = \int dI = \frac{2\pi M}{\pi R^2}\int_{r=0}^{r=R} r^3\, dr = \frac{2M}{R^2}\left(\frac{R^4}{4}\right) = \frac{1}{2}MR^2 \ \checkmark$$

16) A uniform disk, a uniform hoop, and a uniform solid sphere are released at the same time at the top of an inclined ramp. They all roll without slipping. In what order do they reach the bottom of the ramp?

A) disk, hoop, sphere
B) hoop, sphere, disk
C) sphere, disk, hoop
D) sphere, hoop, disk
E) hoop, disk, sphere

Rolling Sphere, Disk, Hoop Demo



$$I_{\text{sphere}} = \frac{2}{5}mR^2$$
$$I_{\text{disk}} = \frac{1}{2}mR^2$$
$$I_{\text{hoop}} = mR^2$$

Conservation of Energy

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2, \text{ where } v = R\omega,\ I = fmR^2$$
$$mgh = \frac{1}{2}fmR^2\frac{v^2}{R^2} + \frac{1}{2}mv^2$$
$$gh = \frac{1}{2}fv^2 + \frac{1}{2}v^2 \Rightarrow v^2 = \frac{2gh}{1+f}$$

Smaller is f, larger is v. f=2/5 for sphere, f=1/2 for disk, f=1 for hoop

Conservation of Energy

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2, \text{ where } v = R\omega, \ \ I = fmR^2$$

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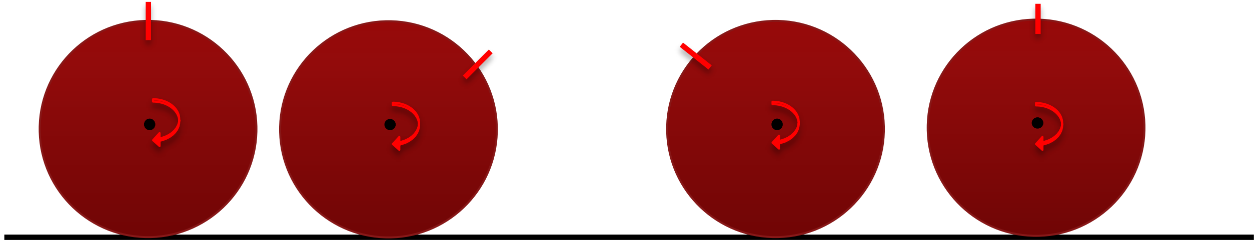
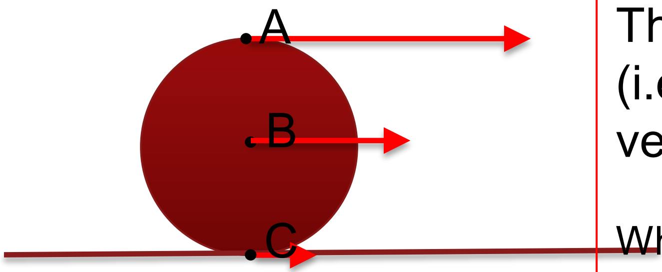
Smaller is f, larger is v. f=2/5 for sphere, f=1/2 for disk, f=1 for hoop

Why does v=Rω (the velocity of a point on the circumference) = velocity of ce



Center of mass will travel 1 full circumference in 1 period: v_{cm}=circumference

$$v_{cm} = \frac{2\pi R}{T} = 2\pi R\frac{1}{T} = 2\pi R\nu = 2\pi\nu R = \omega R = R\omega$$



The ball of radius R is rolling (i.e. not skidding) at angular velocity ω.

What are the velocities of points A, B, C?

A) $V_A = V_B = V_C = R\omega$

B) $V_A = 2R\omega$, $V_B = R\omega$, $V_C = 0$

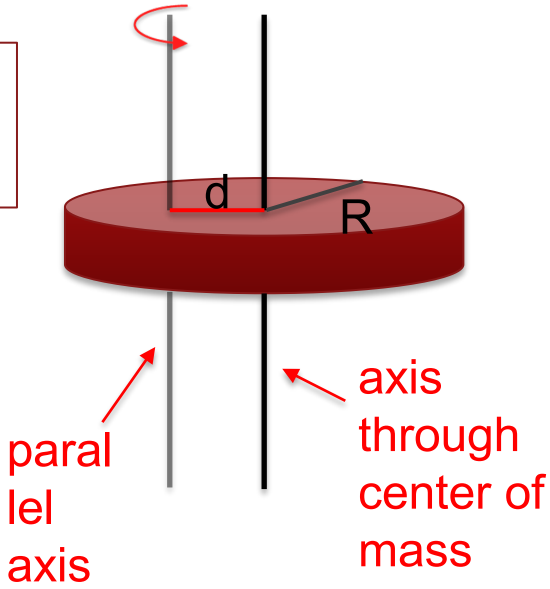
C) $V_A = R\omega$, $V_B = 0$, $V_C = -R\omega$

Parallel Axis Theorem

Moment of inertia depends both on the object *and the choice of axis of rotation*

Assume moment of inertia I_{cm} of an object of mass M is known for an axis that passes through the center of mass

Parallel axis theorem gives moment of inertia I_{par} through any parallel choice of axis



$$I_{par} = I_{cm} + Md^2$$

For disk $I_{cm} = \frac{1}{2}MR^2$

$$d = \frac{R}{2} \Rightarrow I_{par} = \frac{1}{2}MR^2 + M(\frac{R}{2})^2 = \frac{3}{4}MR^2$$

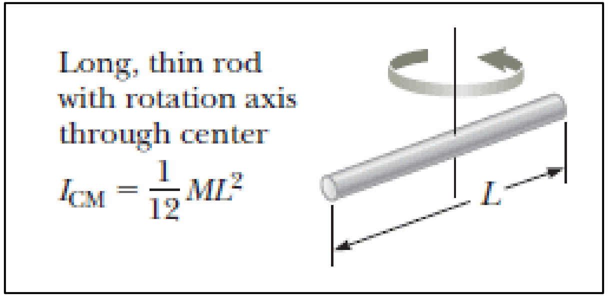
Parallel Axis Theorem

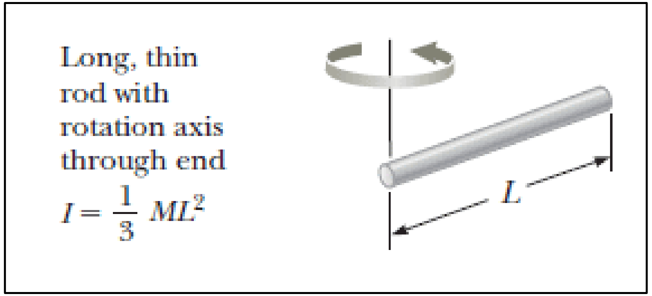
Parallel axis theorem relates two moment of inertia results for a thin rod

$$I_{end} = I_{cm} + M(\frac{L}{2})^2$$

$$= \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$

$$= \frac{1}{3}ML^2 \quad \checkmark$$



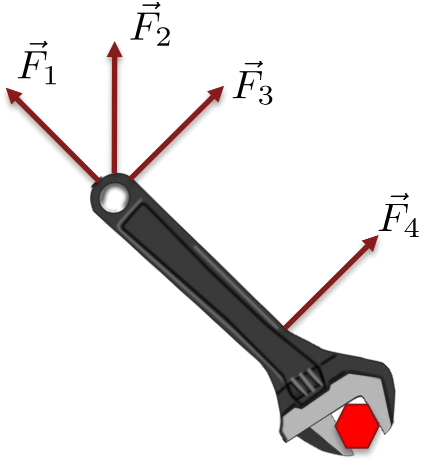


Torque: Chapter 10.6

Measure of the effectiveness of a force in causing rotation of an object

Try turning wrench and screw with different applications of the same magnitude force

$$|\vec{F}_1| = |\vec{F}_2| = |\vec{F}_3| = |\vec{F}_4|$$



Experience ➡ $\vec{F}_3$ is most effective

Longer "lever arm" and applied at right angle to wrench

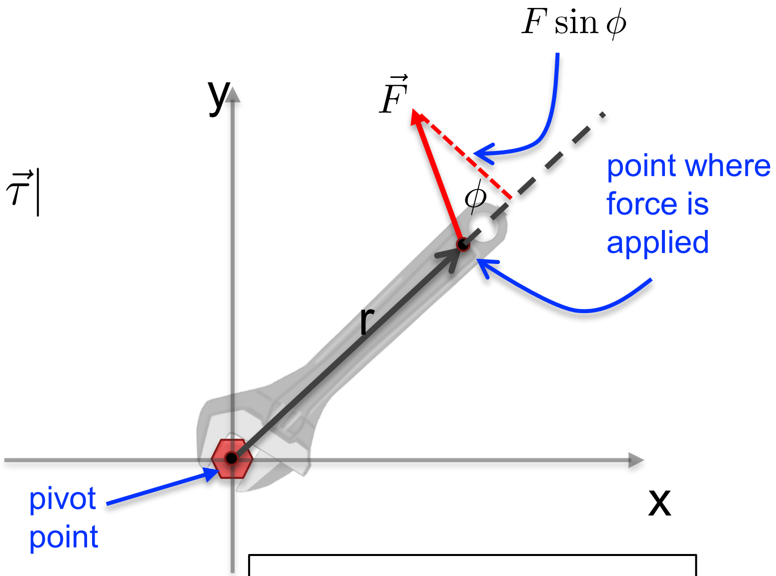
Component of force that pulls on wrench along its length, doesn't contribute to rotation

Computing Torque

Torque is a vector ➡ $\vec{\tau}$

First deal with its magnitude $\tau = |\vec{\tau}|$

$$\boxed{\tau = rF\sin\phi}$$



Torque depends on…

- Magnitude of applied force
- Distance from pivot point (rotational axis)
- Angle at which force is applied

only component of force orthogonal to radial direction contributes

SI Units of torque ➡ Newton-meters (force) x (distance)

English units ➡ Foot-pounds

Computing Torque

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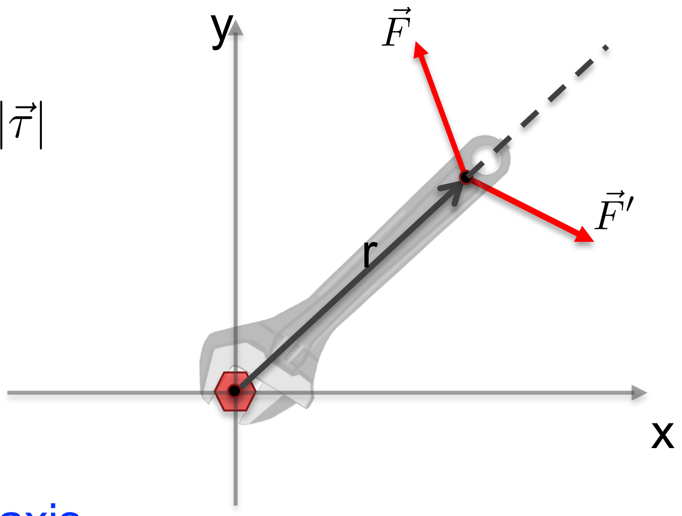


Can see directionality of torque by looking at direction of resulting rotation

$\vec{F}$ will cause bolt to rotate counterclockwise about z-axis

z-axis points out from slide

→ $\vec{\tau}$ points in + z direction

See how this comes out from formula for torque vector

$\vec{F}'$ will cause bolt to rotate clockwise about z-axis → $\vec{\tau}'$ points in – z direction

SI Units of torque → Newton-meters

English units → Foot-pounds

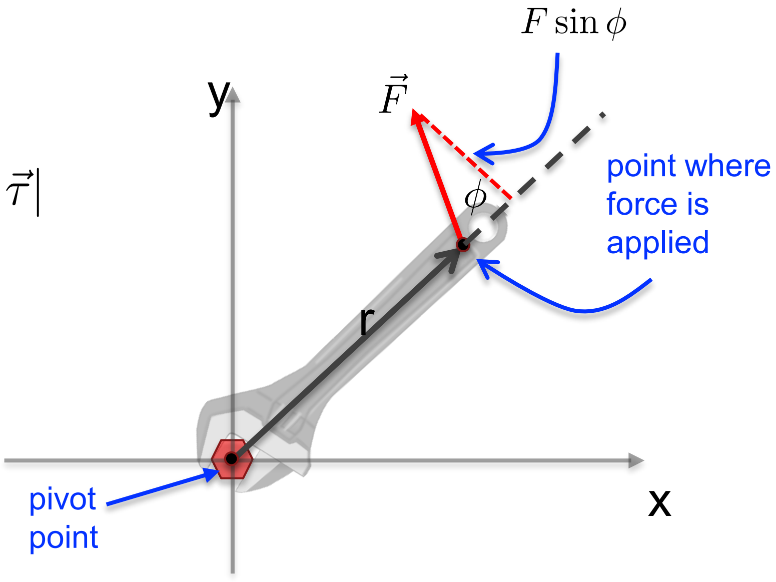
Torque wrench shows magnitude of torque applied

Computing Torque

Torque is a vector ➡ $\vec{\tau}$

First deal with its magnitude $\tau = |\vec{\tau}|$

$$\boxed{\tau = rF\sin\phi}$$



Example

Force of magnitude 100 N is applied to 20 cm long wrench at angle 75° with respect to radial vector

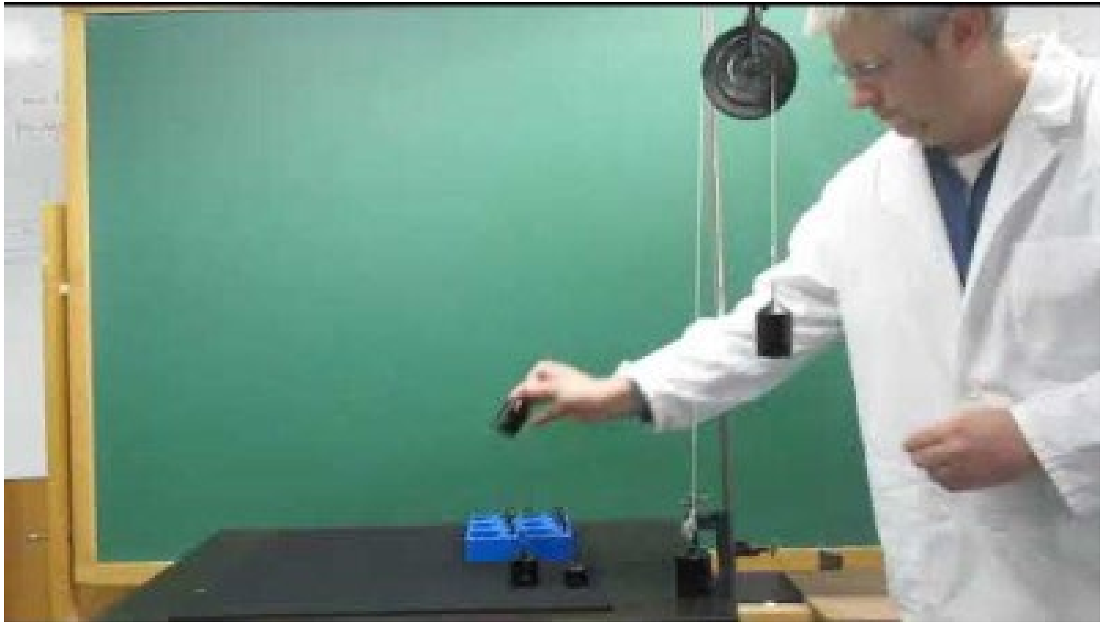
What is the magnitude of the torque exerted by the force?

$$\tau = (0.2m)(100N)\sin 75^o = 19Nm$$

Torque Works: Large Wrench Demo

Try with hands, then try with wrench $\tau = rF\sin\phi$

Torque Wheel Demo: How can small mass balance larger mass?

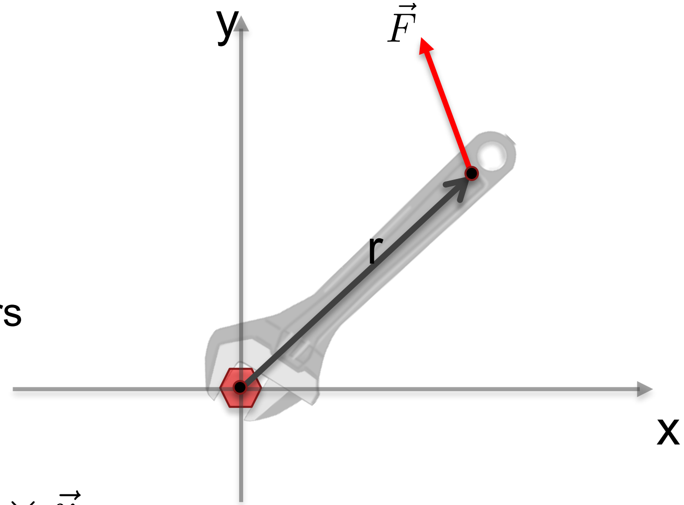


Small mass exerts same torque as larger mass if force from weight applied at larger radius

Computing Torque

Definition of **torque vector**

$$\vec{\tau} = \vec{r} \times \vec{F}$$



Based on “cross product” of vectors

Result of cross product is always orthogonal to original two vectors

Definition of cross product $\vec{w} = \vec{u} \times \vec{v}$

for arbitrary vectors

$$\vec{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$$

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$

$$\Rightarrow \quad \vec{w} = w_x\hat{x} + w_y\hat{y} + w_z\hat{z}$$

with

$$\begin{aligned} w_x &= u_y v_z - u_z v_y \\ w_y &= u_z v_x - u_x v_z \\ w_z &= u_x v_y - u_y v_x \end{aligned}$$

Cross Product – basic properties

$\vec{w} = \vec{u} \times \vec{v}$

$$\begin{aligned} w_x &= u_y v_z - u_z v_y \\ w_y &= u_z v_x - u_x v_z \\ w_z &= u_x v_y - u_y v_x \end{aligned}$$

Cross product is orthogonal to input vectors

$$\vec{u} \cdot \vec{w} = 0 = \vec{v} \cdot \vec{w}$$

Easy to show…

$$\begin{aligned} \vec{u} \cdot \vec{w} &= u_x w_x + u_y w_y + u_z w_z \\ &= u_x(\cancel{u_y} \cancel{v_z} - \cancel{u_z} \cancel{v_y}) + u_y(\cancel{u_z} \cancel{v_x} - \cancel{u_x} \cancel{v_z}) + u_z(\cancel{u_x} \cancel{v_y} - \cancel{u_y} \cancel{v_x}) \\ &= 0 \end{aligned}$$

Result of cross product is anti-symmetric in two input vectors

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Can see by inspection of definition

Implies that crossing a vector with itself gives zero

$$\vec{u} \times \vec{u} = -\vec{u} \times \vec{u} \Rightarrow \vec{u} \times \vec{u} = 0$$

Cross Product

$$\vec{w} = \vec{u} \times \vec{v}$$

$$\boxed{\begin{aligned} w_x &= u_y v_z - u_z v_y \\ w_y &= u_z v_x - u_x v_z \\ w_z &= u_x v_y - u_y v_x \end{aligned}}$$

Example $\vec{u} = 3\hat{x} - 4\hat{y} + 9\hat{z}$

$\vec{v} = 5\hat{x} + 5\hat{y} - 5\hat{z}$

Formula gives

$$w_x = (-4)(-5) - (9)(5) = -25$$

$$w_y = (9)(5) - (3)(-5) = 60$$

$$w_z = (3)(5) - (-4)(5) = 35$$

$$\vec{w} = -25\,\hat{x} + 60\,\hat{y} + 35\,\hat{z}$$

Result of cross product is orthogonal to original vectors

$$\begin{aligned} \vec{u} \cdot \vec{w} &= (3)(-25) + (-4)(60) + (9)(35) \\ &= 0 \ \checkmark \end{aligned}$$

$$\begin{aligned} \vec{v} \cdot \vec{w} &= (5)(-25) + (5)(60) + (-5)(35) \\ &= 0 \ \checkmark \end{aligned}$$

Cross Product

$$\vec{w} = \vec{u} \times \vec{v}$$

$$\boxed{\begin{aligned} w_x &= u_y v_z - u_z v_y \\ w_y &= u_z v_x - u_x v_z \\ w_z &= u_x v_y - u_y v_x \end{aligned}}$$

More important example

Cross product of basis vectors

$$\vec{u} = \hat{x} \; = 1\,\hat{x} + 0\,\hat{y} + 0\,\hat{z}$$

$$\vec{v} = \hat{y} \; = 0\,\hat{x} + 1\,\hat{y} + 0\,\hat{z}$$

$$\Rightarrow \quad \vec{w} = 0\,\hat{x} + 0\,\hat{y} + 1\,\hat{z} \; = \hat{z}$$

Altogether one finds…

$$\boxed{\begin{aligned} \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{x} &= \hat{y} \end{aligned}}$$

Another useful property of cross product

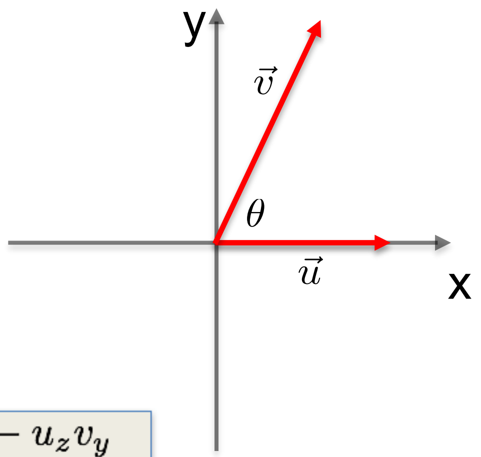
$$\boxed{\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}} \quad \Rightarrow \quad \boxed{\begin{aligned} \hat{x} \times \hat{x} &= 0 \\ \hat{y} \times \hat{y} &= 0 \\ \hat{z} \times \hat{z} &= 0 \end{aligned}}$$

Cross product

Useful property → Magnitude of cross product doesn't change if we rotate both input vectors in same way

Get standard expression for magnitude of cross product using this property

Rotate both vectors into xy-plane with one vector aligned in x-direction



Let $u = |\vec{u}|$ $\quad v = |\vec{v}|$

Component forms of vectors are then

$$\vec{u} = u\,\hat{x}$$
$$\vec{v} = v\cos\theta\,\hat{x} + v\sin\theta\,\hat{y}$$

$$w_x = u_y v_z - u_z v_y$$
$$w_y = u_z v_x - u_x v_z$$
$$w_z = u_x v_y - u_y v_x$$

$$\vec{w} = \vec{u}\times\vec{v}$$
$$= u\,v\,\sin\theta\,\hat{z}$$

→ $\boxed{|\vec{u}\times\vec{v}| = uv\sin\theta}$

Summary: Cross Product

$$\vec{w} = \vec{u} \times \vec{v}$$

$$\boxed{\begin{aligned} w_x &= u_y v_z - u_z v_y \\ w_y &= u_z v_x - u_x v_z \\ w_z &= u_x v_y - u_y v_x \end{aligned}}$$

Can show

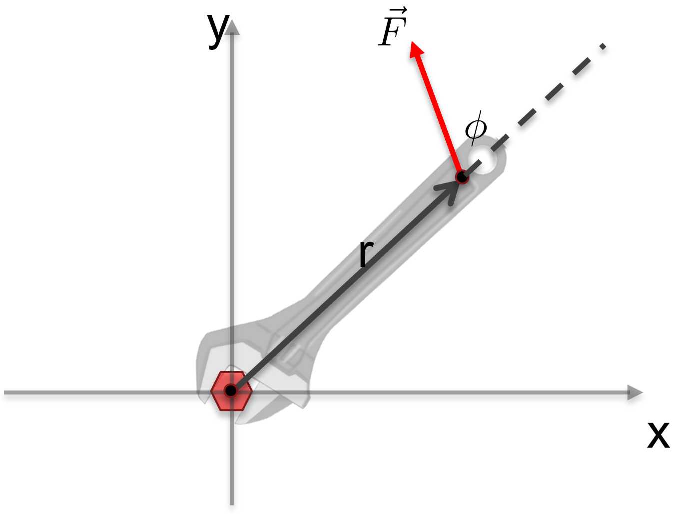
$$|\vec{w}| = |\vec{u}||\vec{v}| \sin\phi$$

angle between vectors

Direction of $\vec{w}$ given by "right hand rule"

Point fingers of right hand in direction of $\vec{u}$
and then wrap them in direction of $\vec{v}$
Thumb will then point in direction of $\vec{w}$

using left hand gives opposite direction!



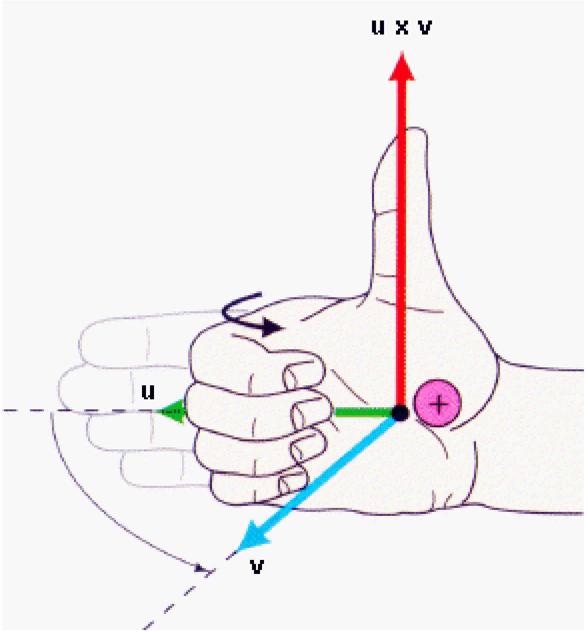
Back to wrench and force

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

$$\boxed{\tau = rF\sin\phi}$$

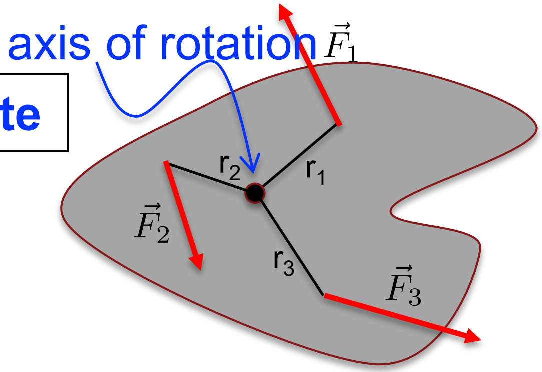
right hand rule gives torque pointing in + z direction

Right hand rule correctly applied



Rotational Dynamics

Torque causes things to rotate



What is relation between torque and rotational motion?

Can generally be some number of torques acting on object

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_i \qquad i = 1, 2, \ldots, N$$

Net torque on an object around a given axis is the sum of the individual torques

$$\vec{\tau}_{net} = \vec{\tau}_1 + \cdots + \vec{\tau}_N \qquad \tau_{net} = |\vec{\tau}_{net}|$$

angular analogue of

$$F_{net} = ma$$

Can show that Newton's second law

$$\boxed{\tau_{net} = I\alpha}$$

net torque

moment of inertia

angular acceleration

Rotational Dynamics

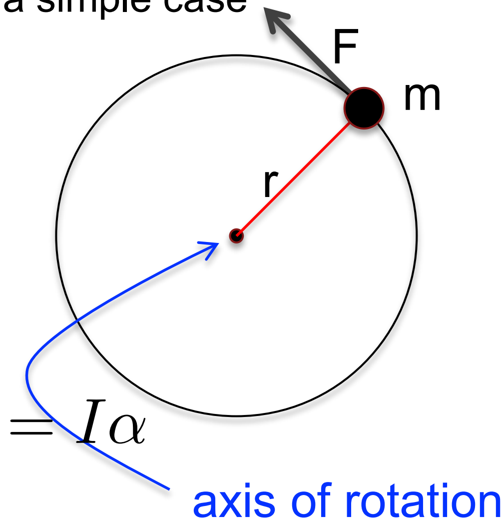
$$\boxed{\tau_{net} = I\alpha}$$

In the absence of a net torque, object will rotate with constant angular velocity (possibly zero)

Show this follows from Newton's 2nd law in a simple case

Mass m attached to massless string of length r with tangential force F

Tangential acceleration $F = ma_t$



Now say…

Torque $\tau = rF$

angular acceleration $a_t = r\alpha$

moment of inertia

$$\tau_{net} = I\alpha$$

$$\tau = mra_t = (mr^2)\alpha = I\alpha \ \checkmark$$

Rotational Dynamics: Summary

Torque causes things to rotate
Q: What is relation between torque and rotational motion?

$$\boxed{\tau_{net} = I\alpha}$$

net torque

moment of inertia

angular acceleration

$$\boxed{\tau = rF\sin\phi}$$

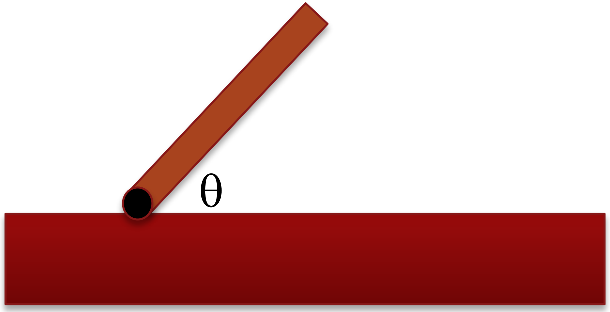
$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

Rotational Dynamics

$$\tau_{net} = I\alpha$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF\sin\phi$$



- Hold a rod of mass M, length L, at angle θ from the horizontal.
- The pivot point is fixed.
- We release the rod.
- What is initial angular acceleration?
- What do we expect when θ=0, θ=90?

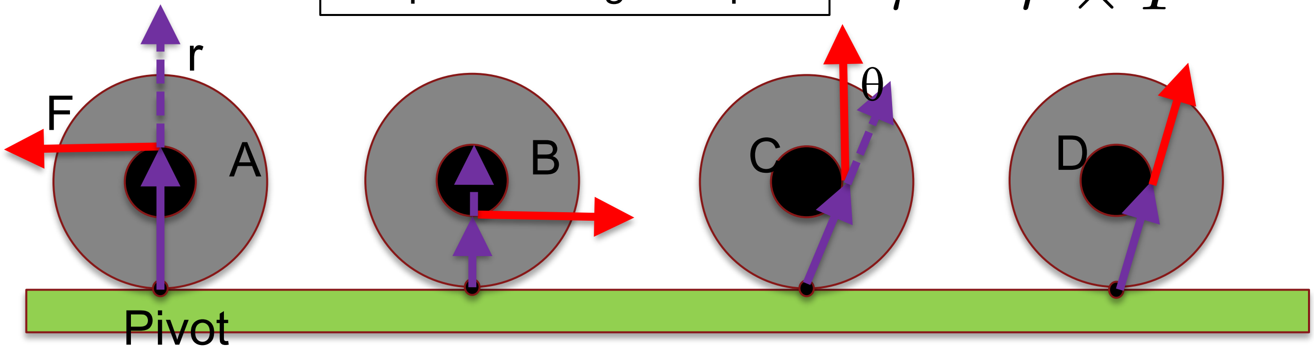
$$\begin{aligned}
\tau &= I\alpha = rF\sin\phi \\
&= r_{cm} mg \sin\bar{\theta} \quad \text{where} \quad \bar{\theta} = (90-\theta) \\
\therefore \alpha &= \frac{r_{cm} mg \sin\bar{\theta}}{I} = \frac{(L/2)mg\sin(90-\theta)}{(1/3)mL^2} \\
\alpha &= \frac{3g\sin(90-\theta)}{2L}
\end{aligned}$$

Rotational Motion: Walking the Spool

Which way does spool roll if I pull string from top of spool? Bottom?

Torque: Walking the Spool

$$\vec{\tau} = \vec{r} \times \vec{F}$$



In each scenario, A-D, **in what direction** is the torque about the point of due to the force F? Which way does the spool roll?

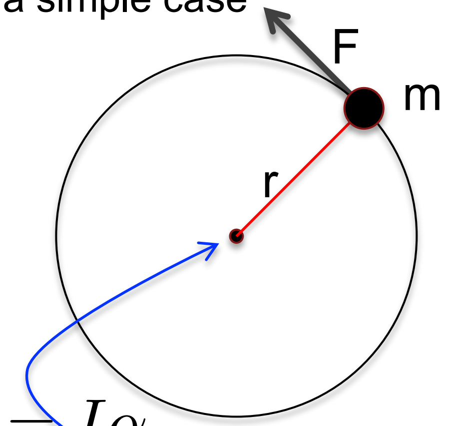
	A	B	C	D
Direction of Torque	Out of page	Into page	Out of page	None
Direction of Roll	CCW, left	CW, right	CCW, left	None

Rotational Dynamics

$$\boxed{\tau_{net} = I\alpha}$$

In the absence of a net torque, object will rotate with constant angular velocity (possibly zero)

Show this follows from Newton's 2nd law in a simple case



Mass m attached to massless string of length r with tangential force F

Tangential acceleration $F = ma_t$

Now say…

Torque $\tau = rF$

angular acceleration $a_t = r\alpha$

moment of inertia

$\tau_{net} = I\alpha$

axis of rotation

$$\tau = mra_t = (mr^2)\alpha = I\alpha \checkmark$$

Rotational dynamics example

A bicycle wheel has radius R=0.35m and mass M=0.44kg is initially spinning at 100rpm on a truing stand

Make approximation that all mass is at the rim

Torque comes from 0.8N force of ball bearings rubbing on edge of axle at r=0.0026m

How long does wheel take to come to rest?

Moment of inertia

$$I = MR^2 = (0.44kg)(0.35m)^2 = 0.054\,kgm^2$$

opposing rotational motion

Torque

$$\tau = Fr = -(0.8N)(0.0026m) = -0.0021\,Nm$$

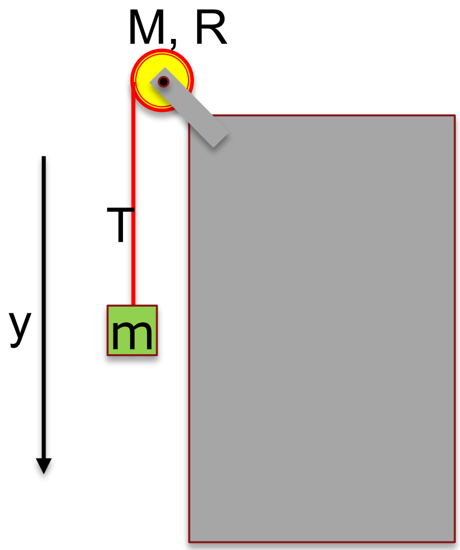
Find angular acceleration

$$\tau = I\alpha \Rightarrow \alpha = \frac{\tau}{I} = \frac{-0.0021\,Nm}{0.054\,kgm^2} = -0.038 rad/s^2$$

A solid disk with mass M=2.5kg and radius R=0.2m has massless rope wrapped around it

Block of mass m=1.2kg descends with rope as it unravels from disk

Find the acceleration of the block and the tension in the rope



Block → $mg - T = ma$

Disk → Feels torque from rope $\tau = TR$

Moment of inertia $I = \frac{1}{2}MR^2$

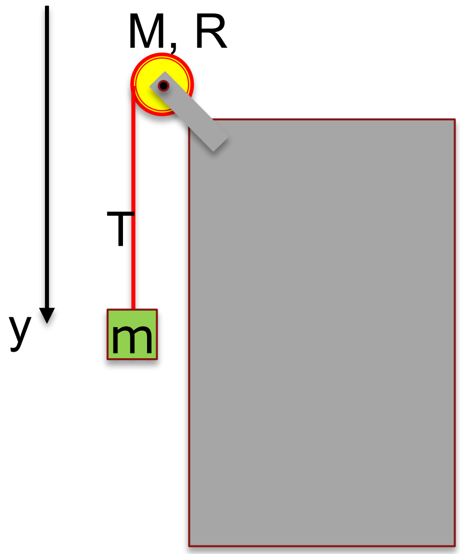
Angular acceleration $\alpha R = a$ → $\alpha = \frac{a}{R}$ as rope unspools, disk spins key point!

Subtlety Alert → Positive rotation is counterclockwi Made the y-axis point down so that a>0 coincides with α>0

A solid disk with mass M=2.5kg and radius R=0.2m has massless rope wrapped around it

Block of mass m=1.2kg hangs from rope as it unravels from disk

Find the acceleration of the block and the tension in the rope



Block $\Rightarrow$ $\boxed{mg - T = ma}$

Disk $\Rightarrow$

Feels torque from rope $\quad \tau = TR$

Moment of inertia $\quad I = \frac{1}{2}MR^2$

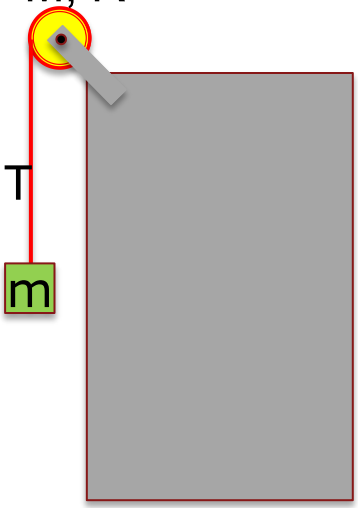
Angular acceleration $\quad \alpha R = a \Rightarrow \alpha = \frac{a}{R}$

$$\tau = I\alpha \Rightarrow TR = (\frac{1}{2}MR^2)(\frac{a}{R}) \Rightarrow \boxed{T = \frac{1}{2}Ma}$$

A solid disk with mass M=2.5kg and radius R=0.2m has massless rope wrapped around it

Block of mass m=1.2kg hangs from rope as it unravels from disk

Find the acceleration of the block and the tension in the rope



$$\boxed{mg - T = ma}$$

$$\boxed{T = \frac{1}{2}Ma}$$

2 equations with 2 unknowns

Solve to find…

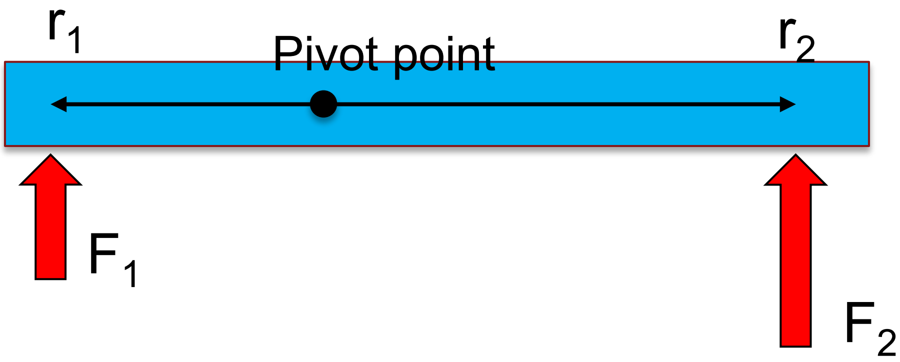
$$a = \frac{m}{m + \frac{1}{2}M}g$$

$$T = \frac{2mM}{2m + M}g$$

M=0 → Nothing restrains block from falling

→ $$a = g$$ $$T = 0$$

Torque and Static Equilibrium



-Apply a force F_1 at distance r_1 from a pivot point.
-Apply a force F_2 at distance r_2 from a pivot point.

Does the bar rotate?

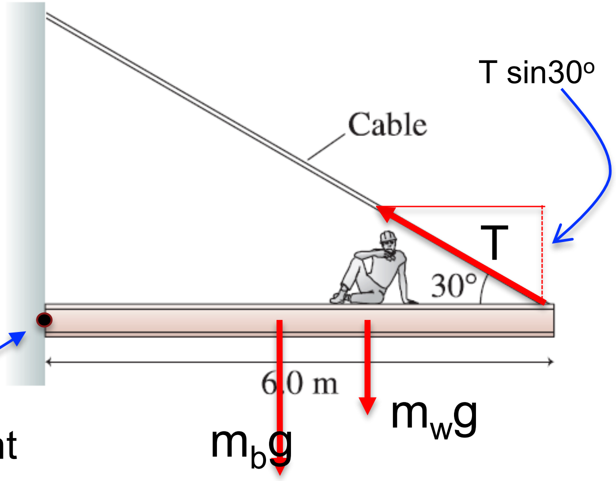
Static equilibrium (won't rotate) if sum of torques = 0

Torque and Static Equilibrium

An 80kg construction worker sits down 2m from the end of a 6m long 1450kg steel beam

The cable supporting the beam is rated to have a maximum tension of 15,000N

Should the worker be worried?



Consider sum of torques around pivot point

Beam

$$\tau_1 = -(m_b g)x_{cm} = -(1450kg)(9.8m/s^2)(3m)$$
$$= -42,600\,Nm$$

clockwise

Worker

$$\tau_2 = -(m_w g)x_{cm} = -(80kg)(9.8m/s^2)(4m)$$
$$= -3100\,Nm$$

Cable

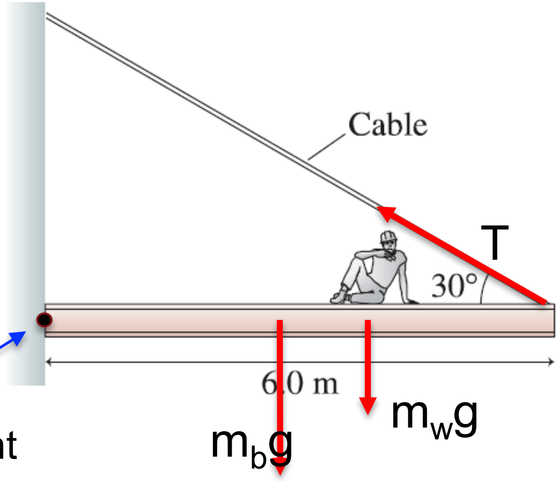
$$\tau_3 = +T\sin 30^o(6m) = T(3m)$$

1/2

Torque and Static Equilibrium

An 80kg construction worker sits down 2m from the end of a 6m long 1450kg steel beam

The cable supporting the beam is rated to have a maximum tension of 15,000N



Should the worker be worried?

Consider sum of torques around pivot point

$$\tau_1 = -42,600\ Nm \qquad \tau_2 = -3100\ Nm \qquad \tau_3 = T(3m)$$

Static ➡ Net torque must vanish

$$\tau_{net} = -42,600\ Nm - 3100\ Nm + T(3m) = 0$$

$$\Rightarrow \quad T = \frac{45,700\ Nm}{3m} = 15,200\ N > 15,000\ N$$

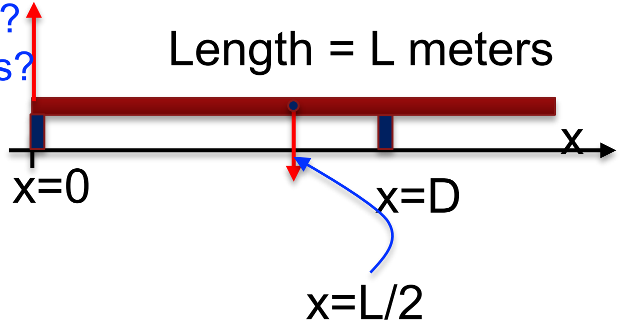
Yes!
He should be

A uniform beam L meters long and mass M kg rests on two posts.

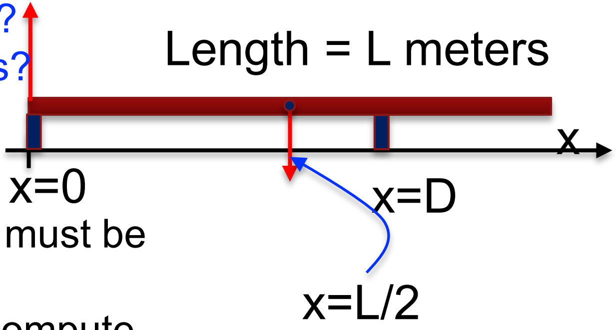
One post is at the left end x=0, the other is at x= D.

What is the force on each post, F_L and F_R?

What is the sum of the forces on the posts?



A uniform beam L meters long and mass M kg rests on two posts.
One post is at the left end x=0, the other is at x= D.
What is the force on each post, F_L and F_R?
What is the sum of the forces on the posts?



The beam is not rotating: sum of torques must be zero about *any axis/pivot point.*
Pick pivot point at right hand post, x=D, compute torques.

$$\tau = -F_L D + Mg(D - L/2) = 0, \text{ and } F_L + F_R = Mg$$

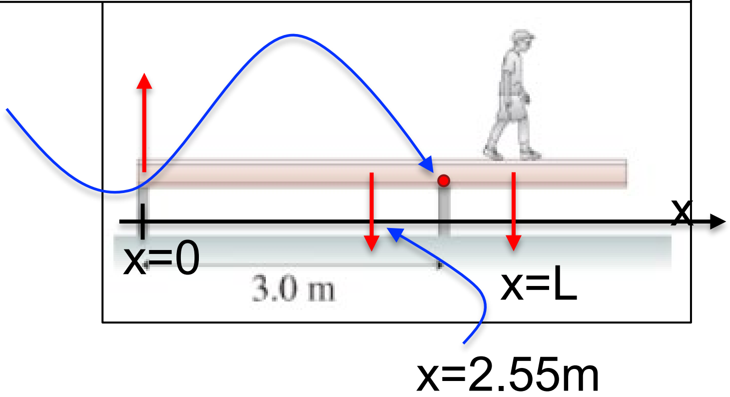
$$F_L D = Mg(D - L/2) \quad \Rightarrow \quad F_L = \frac{Mg}{D}(D - L/2) \; = \; Mg(1 - \frac{L}{2D})$$

$$F_R = Mg - F_L = Mg - Mg(1 - \frac{L}{2D}) \; = \; Mg\frac{L}{2D}$$

$$\text{Note when } D = L/2, \; F_R = Mg, \; F_L = 0$$

A 40 kg , 5.1-m-long beam is supported, but not attached to, the two posts in the figure (Figure 1) . A 20 kg boy starts walking along the beam.

How close can he get to the right end of the beam without it falling over?



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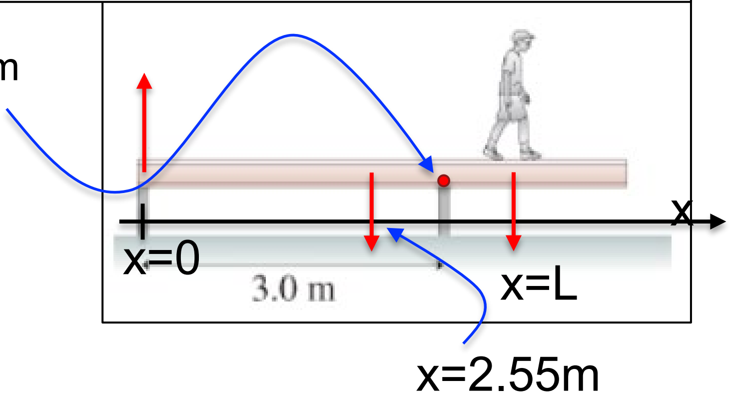


Compute torques from mass of beam and boy about axis above right support post

If net torque is positive (counterclockwise), it can be countered by negative (clockwise) torque from left support post

If net torque is negative (clockwise), beam will fall over

Gravity acts on beam at center of mass @ x=2.55m

Boy's center of mass @ x=L

$$\tau_{net} = +(3m - 2.55m)(40kg)g - (L - 3m)(20kg)g > 0$$

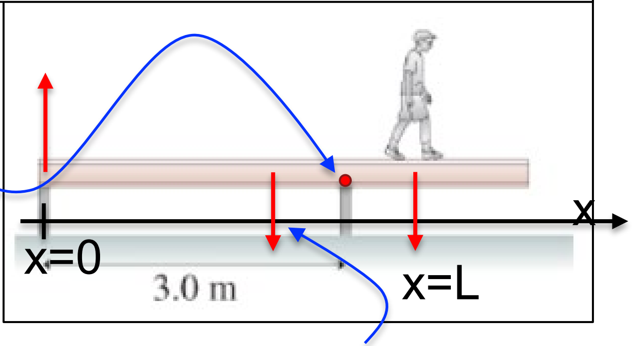
distance of beam cm from pivot point

distance of boy from pivot point

A 40 kg , 5.1-m-long beam is supported, but not attached to, the two posts in the figure (Figure 1) . A 20 kg boy starts walking along the beam.

How close can he get to the right end of the beam without it falling over?

Compute torques from mass of beam and boy about axis above right support post



$$\tau_{net} = +(3m - 2.55m)(40kg)\cancel{g} - (L - 3m)(20kg)\cancel{g} > 0$$

$$18\,kgm - (20kg)L + 60\,kgm > 0$$

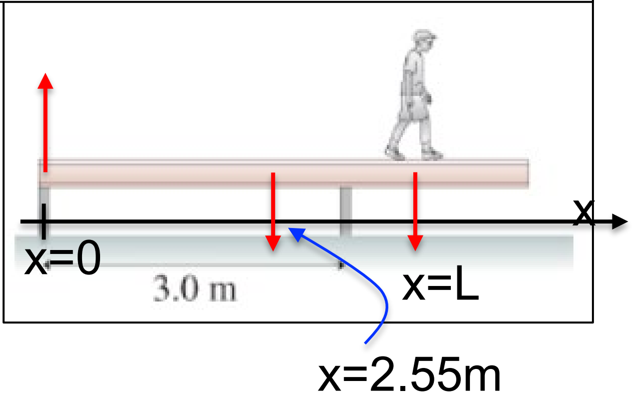
$$3.9\,m > L$$

Minimum safe distance from end

$$d = 5.1\,m - 3.9\,m = 1.2\,m$$

A 40 kg, 5.1-m-long beam is supported, but not attached to, the two posts in the figure (Figure 1). A 20 kg boy starts walking along the beam.

How close can he get to the right end of the beam without it falling over?



Slightly alternative reasoning

Find center of mass x_{cm} of combined beam and boy system

If x_{cm} < 3m then the torque around the right post will be positive and can be countered by torque from left post

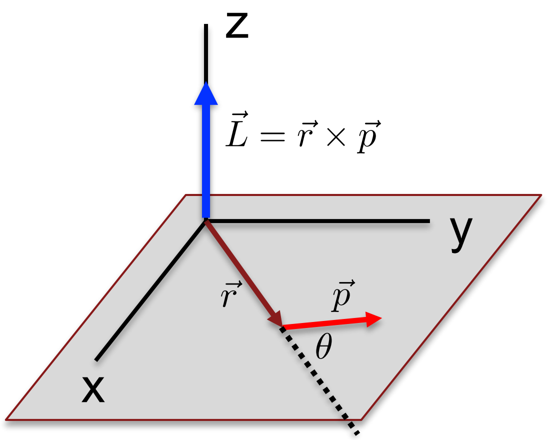
Angular momentum

Rotational analogue of momentum

Recall – if no net external forces act on a system, then momentum is conserved

Angular momentum is conserved if no net external torques act on system

For a particle at position $\vec{r}$ with momentum $\vec{p} = m\vec{v}$



Angular momentum is defined to be

$$\boxed{\vec{L} = \vec{r} \times \vec{p}} \Rightarrow |\vec{L}| = rmv\sin\theta$$

By "right hand rule" see that angular momentum is perpendicular to plane of motion

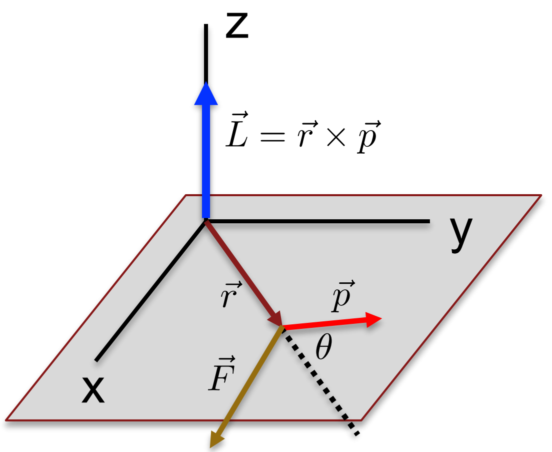
Counterclockwise rotation $\Rightarrow$ Angular momentum in + z-direction

Angular momentum

For a particle at position $\vec{r}$
with momentum $\vec{p} = m\vec{v}$

Angular momentum is defined to be

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$



If forces acts on the particle, can show from Newton's laws that

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

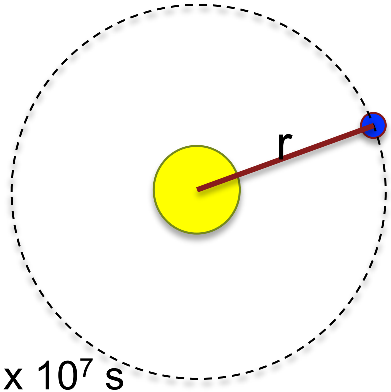
Rotational analogue of $\dfrac{d\vec{p}}{dt} = \vec{F}_{net}$

Net torque gives rate of change of angular momentum

Vanishin g net torque $\quad \vec{\tau}_{net} = 0 \Rightarrow \dfrac{d\vec{L}}{dt} = 0 \quad$ Angular momentum of particle is conserved

Angular momentum example

What is the angular momentum associated with the Earth's orbit around the sun?



Mass of earth $m = 6 \times 10^{24}$ kg

Radius of Earth's orbit $r = 1.5 \times 10^{11}$ m

Period of Earth's orbit $T = 1$ year $= 3.2 \times 10^{7}$ s

$$L = rp = rmv = rm(\frac{2\pi r}{T}) = (\frac{2\pi m r^2}{T}) = 2.7 \times 10^{40} Js$$

Angular Momentum of Rigid Body

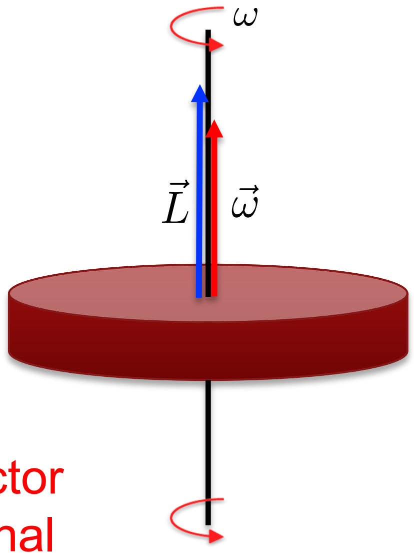
Need to add up the angular momenta of all parts of the body to get the total angular momentum

Result $\boxed{\vec{L} = I\vec{\omega}}$

mom
ent of
inertia

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$$

angular velocity vector
points along rotational
axis in direction given
and rule



Rotational analogue of $\vec{p} = m\vec{v}$

Relation between torque and angular momentum still holds for rigid bodies

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

For vanishing net torque angular momentum is conserved $\dfrac{d\vec{L}}{dt} = 0$

Example

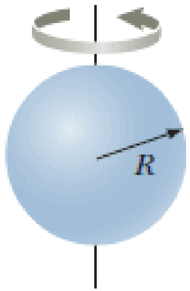
What is the angular momentum associated with the Earth's spin about its axis?

Mass of earth	$m = 6 \times 10^{24}$ kg
Radius of earth	$r = 6.4 \times 10^{6}$ m
Period of rotation	$T = 1$ day $= 86{,}000$ s

$$
\begin{aligned}
L &= I\omega \\
&= (\frac{2}{5}mr^2)(\frac{2\pi}{T}) \\
&= 7.2 \times 10^{33}\, Js
\end{aligned}
$$

Solid sphere

$I_{CM} = \frac{2}{5}MR^2$



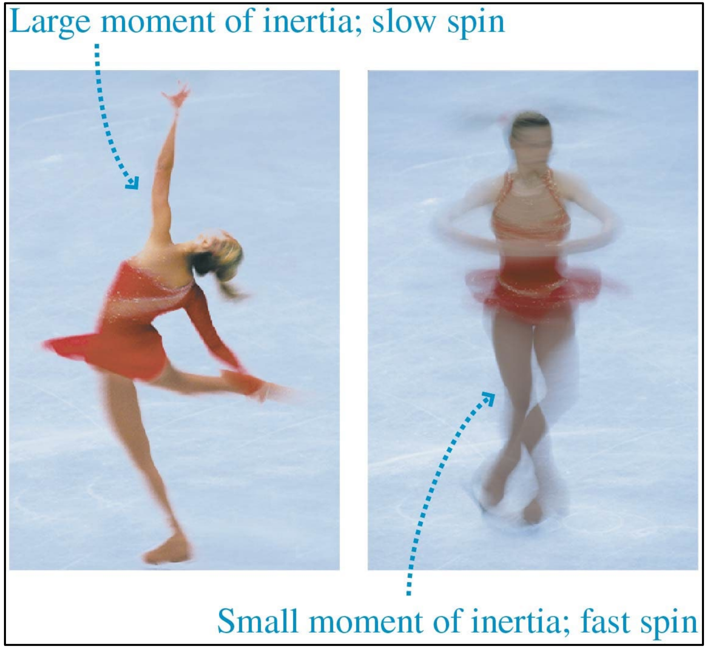
Demo: Conservation of Angular Momentum

Why does the angular velocity change as the weights are moved in/out?

Classic example: Conservation of Angular Momentum

As an ice skater spins, external torque is small, so her angular momentum is almost constant.

By drawing in her arms and legs to reduce her moment of inertia, she increases her angular velocity



$$L = I\omega$$

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow I_i\omega_i = I_f\omega_f$$

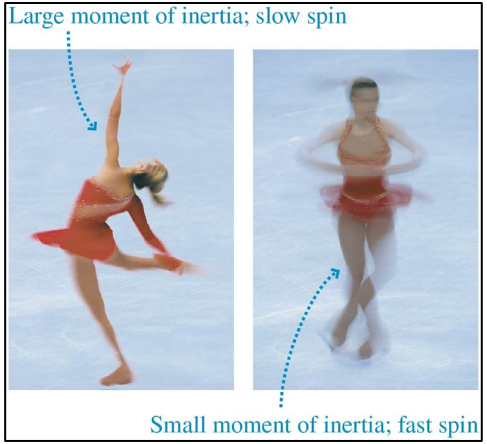
$$\omega_f = \omega_i\left(\frac{I_i}{I_f}\right)$$

$$I_f < I_i \Rightarrow \omega_f > \omega_i$$

Angular momentum example

A figure skater has moment of inertia I_i = 2kgm^2 when her arms are extended and I_f = 1kgm^2 when her arms are fully pulled in.

She is initially spinning at 20rpm with her hands out



What is her angular velocity when she pulls them in?

$$I_i\omega_i = I_f\omega_f \quad \Rightarrow \quad \omega_f = \omega_i\left(\frac{I_i}{I_f}\right)$$

$$20rpm = 2.1\,rad/s$$

$$\omega_f = (20rpm)\left(\frac{2\,kgm^2}{1\,kgm^2}\right) = 40rpm$$

Skater must do work to pull her arms in!

Does her kinetic energy change in this process?

$$K_i = \frac{1}{2}I_i\omega_i^2 = \frac{1}{2}(2\,kgm^2)(2.1\,rad/s)^2 = 4.4\,J$$

$$K_f = \frac{1}{2}I_f\omega_f^2 = \frac{1}{2}(1\,kgm^2)(4.2\,rad/s)^2 = 8.8\,J$$

$\Rightarrow$ Yes

Demo: Conservation of Angular Momentum: Your turn!

What should happen when the spinning wheel is slowly flipped over?

Bicycle Wheel Gyroscope

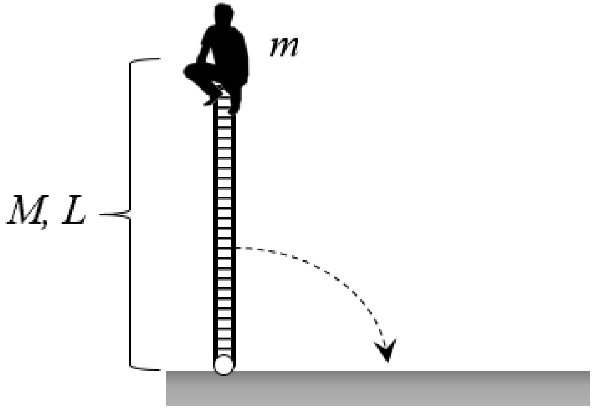
What is going on here? Is there a torque on this system?

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

Rotational Kinetic Energy

Bob is sitting (attached) on a ladder when it falls over (with the pivot point at the bottom of the ladder staying fixed)

How fast will Bob be moving when he hits the ground?



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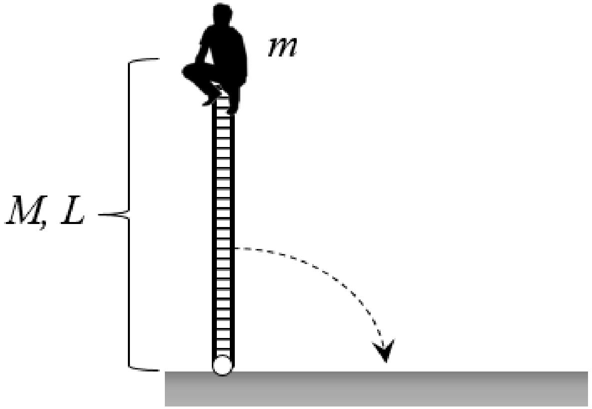


Use Conservation of Energy

Initial energy is all potential energy

$$E_i = Mg(L/2) + mgL$$

Final energy is all (rotational) kinetic energy

$$E_f = \frac{1}{2}I\omega^2$$

Want to find ω

Moment of inertia

$$I = I_{ladder} + I_{bob}$$

$$I_{ladder} = \frac{1}{3}ML^2$$

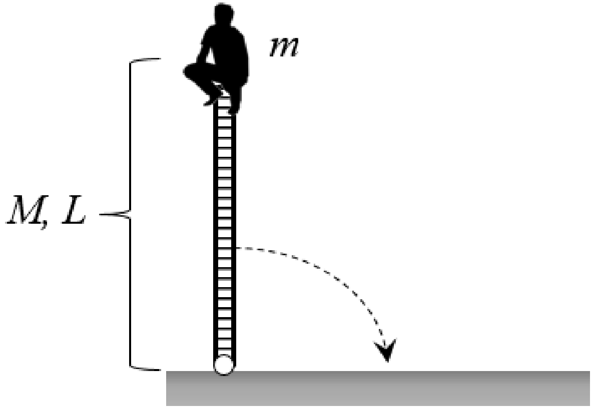
$$I_{bob} = mL^2$$

$$I = \frac{1}{3}ML^2 + mL^2$$

Rotational Kinetic Energy

Bob is sitting (attached) atop a ladder when it falls over (with the pivot point at the bottom of the ladder staying fixed)

How fast will Bob be moving when he hits the ground?



$$E_i = Mg(L/2) + mgL$$

$$E_f = \frac{1}{2} I \omega^2$$

$$I = \frac{1}{3} ML^2 + mL^2$$

$$E_i = E_f \Rightarrow \omega^2 = \frac{2(MgL/2 + mgL)}{I}$$

Plug in expression for I and solve for ω

Bob's speed $\Rightarrow$ $v = \omega L$

Torque

Typical torque for tightening a lug nut on the wheels of a Mustang is 95 ft-lbs (129 Nm). How hard would a mechanic need to push on the end of a 0.5 m torque wrench to achieve this? When doing this, what is the force applied from the wrench on the edge of a ½ inch (radius = 0.00635 m) lug nut?

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$$\begin{aligned} \tau &= rF\sin\phi \\ \text{where r} = 0.5\,\text{m},\ \phi &= 90^\circ \text{ for maximum torque} \\ \Rightarrow\ 129\,Nm &= 0.5\,m \times F \\ F &= 258\,N = 58\,lbs \end{aligned}$$

$$\begin{aligned} \tau &= rF \\ 129\,Nm &= (0.00635\,m)F \\ F &= 20,300\,N = 4,600\,lbs \end{aligned}$$

Consider a uniform solid sphere of radius R and mass M rolling without slipping. Which form of its kinetic energy is larger, translational or rotational?

A) Its translational kinetic energy is larger than its rotational kinetic energy.
B) Its rotational kinetic energy is larger than its translational kinetic energy.
C) Both forms of energy are equal.
D) You need to know the speed of the sphere to tell.

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$$\begin{aligned}
\mathrm{KE(translational)} &= \frac{1}{2}Mv^2 \\
\mathrm{KE(rotational)} &= \frac{1}{2}I\omega^2, \quad v = R\omega \\
&= \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2 \\
&= \frac{1}{5}Mv^2
\end{aligned}$$

A long thin rod of length L has a linear density $\lambda(x) = Ax$ where x is the distance from the left end of the rod.

(a) How far is the center of mass of the rod from the left end of the rod?
(b) What is the mass of the rod?

A long thin rod of length L has a linear density $\lambda(x) = Ax$ where x is the distance from the left end of the rod.

(a) How far is the center of mass of the rod from the left end of the rod?

$$\text{Mass} = \int_0^L \Lambda(x)dx = \int_0^L Ax\,dx = \frac{1}{2}AL^2$$

$$\text{Center of Mass} = \frac{\int_0^L \Lambda(x)x\,dx}{\text{Mass}} = \frac{\int_0^L Ax^2dx}{\text{Mass}} = \frac{\frac{1}{3}AL^3}{\text{Mass}}$$

$$= \frac{\frac{1}{3}AL^3}{\frac{1}{2}AL^2}$$

$$\text{Center of Mass} = \frac{2}{3}L$$