250 Homework #1

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P1.1.4 [10 pts]

Let C be the set $\{0, 1, \ldots, 15\}$. Let D be a subset of C and define the number $f(D)$ as follows – $f(D)$ is the sum, for every element i of D, of 2^i. For example, if D is $1, 6$ then $f(D) = 2^1 + 2^6 = 66$.

(a) What are $f(\emptyset)$, $f(0, 2, 5)$, and $f(C)$?

(b) Is th ere a D such that $f(D) = 666$? If so, find it.

(c) Explain why, if D and E are any two subsets of C such that $f(D) = f(E)$, then $D = E$.

Solution:

(a) $f(\emptyset) = \emptyset$ there is nothing in the set so you add nothing $f(0, 2, 5) = 2^0 + 2^2 + 2^5 = 37$ $f(C) = \Sigma_{n=0}^{15} 2^n = 65535$

(b) yes, using this base 2 system, you can define every number $f(1, 3, 6, 7, 9) = 666$

(c) in this function $f()$ there is only 1 unique output per input, therefore if the inputs are the same, the output would be the same, order doesn't matter in sets

*Collaborated with Nobody.

P1.2.5 [10 pts]

Let the alphabet C be $\{a, b, c\}$. Let the language X be the set of all strings over C with at least two occurrences of b. Let Y be the language of all strings over C that never have two occurrences of c in a row. Let Z be the language of all strings over C in which every c is followed by an a. (Recall that any string with no c's is thus in Z.)

(a) List the three-letter strings in each of X, Y, and Z. The easiest way to do this may be to first list all 27 strings in C^3 and then see which ones meet the given conditions.

(b) List the four-letter strings that are both in X and in Y, those that are both in X and in Z, those that are both in Y and in Z, and those that are in all three sets. How many total strings are in C^4?

(c) Are any of X, Y, or Z subsets of any of the others?

(d) Suppose u and v are two strings in X. Do we know that the strings u^R, v^R, uv, and vu are all in X? Either explain why this is always true, or give an example where it is not.

(e) Repeat the previous question for the languages Y and Z.

Solution:

(a) $X^3 = \{abb, bab, bba, bbb, bbc, bcb\}$
$Y^3 = \{aaa, aab, aac, aba, abb, abc, aca, acb, baa, bab, bac, bba, bbb,$
$bbc, bca, bcb, caa, cab, cac, cba, cbb, cbc\}$
$Z^3 = \{aaa, aab, aba, abb, baa, bab, bbb, bba, aca, bca, caa, cab, cac, cca\}$

(b) the total strings in C^4 is equivalent to 3^4 or 81.
$X^4 \cap Y^4 = \{aabb, abba, baab, bbaa, bbba, abbb, babb, bbab, bbbb, cabb,$
$acbb, cbba, abbc, bcab, bacb, bbca, bbac, bbbc, cbbb, bcbb, bbcb, cbbc\}$
$X^4 \cap Z^4 = \{bbbb, bbba, bbab, babb, abbb, bbca, bcab, cabb, aabb, baab, bbaa,$
$abab, baba\}$
$Y^4 \cap Z^4 = \{aaaa, aaab, aaba, abaabaaa, bbbb, bbba, bbab, babb, abbb,$
$bbaa, aabb, baab, abba, aaca, caaa, baca, caa, caab, caba, cabb, bcab,$
$aaca, acaa, caaa, baca, bcaa, caab, caba, cabb, bcab\}$

(c) as it shows, $X \subset Y$

(d) this is always true, the positions dont matter of the 2 b's as long as they're there it works. and reversing or concatenating does not affect the occurences of the elements in the set

(e) for Y this is not true, when $v = \{ac\}$ and $u = \{ca\}$ then $vu = \{acca\}$ which does not exist in Y
for Z this is not true, when $v = \{ca\}$ then v^r does not exist in Z

P1.4.10 [10 pts]

Letting p denote "mackerel are fish" and q denote "trout live in trees", translate each of the following four statements into English: $\neg p \rightarrow q$, $\neg(p \rightarrow q)$, $\neg p \leftrightarrow q$, and $\neg(p \leftrightarrow q)$. Are any two of these four statements logically equivalent?

Solution:

$\neg p \rightarrow q$ if mackerel arent fish, then trout live in trees
$\neg(p \rightarrow q)$ it is not the case if mackerel are fish, then trout live in trees
$\neg p \leftrightarrow q$ mackerel aren't fish, if and only if trout live in trees
$\neg(p \leftrightarrow q)$ it is not the case that mackerel are fish, if and only if trout live in trees
the truth table shows that $\neg p \leftrightarrow q$ and $\neg(p \leftrightarrow q)$ are equivalent

p	q	$\neg p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg p \leftrightarrow q$	$\neg(p \leftrightarrow q)$
0	0	0	0	0	0
0	1	1	0	1	1
1	0	1	1	1	1
1	1	1	0	0	0

P1.5.6 [10 pts]

Let Σ be the alphabet $a, b, c, \ldots, z$ and let U be the set Σ^3 of three-letter strings with letters from Σ. Let X be the set of strings in U whose first letter is c. Let Y be the set of strings whose second letter is a, and let Z be the set of strings whose last letter is t. Describe each of the following sets in English, and determine the number of strings in each set.

(a) $X \cap Y$

(b) $X \cap Y \cap Z$

(c) $Y \cup Z$

(d) $X \cap (Y \cup Z)$

Solution:

(a) $X \cap Y =$ the set of strings in U whose first first letter is c and the second letter is a

(b) $X \cap Y \cap Z = \{cat\}$

(c) $Y \cup Z =$ the set of strings in U that either has a as the second letter or t as the last letter, or both

(d) $X \cap (Y \cup Z) =$ the set of strings in U that have c as the first letter and either has a as the second letter or t as the last letter, or both a and t

P1.7.6 [10 pts]

Suppose we substitute $a \oplus b$ for p and $a \wedge b$ for q in the contrapositive rule to get $((a \oplus b) \rightarrow (a \wedge b)) \leftrightarrow ((\neg a \wedge b) \rightarrow (\neg a \oplus b))$. Verify that this result is *not* a tautology. Why didn't our substitution lead to a valid tautology?

Solution:

the reason this is not a valid tautology is because of the way that the $\neg$ was distributed does not follow the identity, the $(a \oplus b)$ should become $\neg(a \oplus b)$ and not $(\neg a \oplus b)$ and the same could be said for q

by checking the truth table we can find that they are definitely not equivalent

a	b	original	negated
0	0	0	1
0	1	1	0
1	0	1	1
1	1	0	1

P1.8.2 [10 pts]

A variant of the Proof By Cases rule is as follows: Given the premises $p \vee q$, $p \to r$, and $q \to r$, derive r.

Solution:

$p \vee q$, $p \to r$, $q \to r$
through definition of implication $\neg p \to q$, $p \to r$, $q \to r$
combining implications $\neg p \to r$, $p \to r$
tautology, r

P1.8.7 [10 pts]

Prove that the compound propositions $p \wedge (q \to r)$ and $\neg(p \to (q \wedge \neg r))$ are equivalent by using the Equivalence and Implication Rule and constructing two deductive sequence proofs.

Solution:

Proof 1 $p \wedge (q \to r) \to \neg(p \to (q \wedge \neg r))$
$p \wedge (q \to r)$ Given
$q \to r$ Given
$\neg q \vee r$ Implication rule
$p \wedge (\neg q \vee r)$ plug in
$\neg p \vee \neg(\neg q \vee r)$ implication rule
$\neg(p \to (q \neg r))$ equivalence / demorgans, since they are the same, they are equivalent
Proof 2 $\neg(p \to (q \wedge \neg r)) \to p \wedge (q \to r)$
$\neg(p \to (q \wedge \neg r))$given
$\neg p \vee \neg(\neg q \vee r)$ implication rule & demorgans
$\neg p \vee \neg(q \wedge \neg r)$ demorgans
$\wedge\ (q \to r)$ demorgans and implication rule they are logcially identical, therefore they are equivalent

P1.10.6 [12 pts]

We can define binary relations on the naturals for each of the five relational operators. Let $LT(x, y)$, $LE(x, y)$, $E(x, y)$, $GE(x, y)$, and $GT(x, y)$ be the predicates with templates $x < y$, $x \leq y$, $x = y$, $x \geq y$, and $x > y$ respectively.

(a) Show how each of the five predicates can be written using only LE and boolean operators. Use your constructions to rewrite $(LE(a, b) \oplus (E(b, c) \vee GT(c, a)) \rightarrow (LT(c, b) \wedge GE(a, c))$ in such terms.

(b) Express each of the five predicates using only LT and boolean operators, and rewrite the same compound statement in those terms.

Solution:

(a) $LT(x, y) = \neg LE(y, x)$
$LE(x, y) = LE(x, y)$
$E(x, y) = LE(x, y) \wedge LE(y, x)$
$GE(x, y) = LE(y, x)$
$GT(x, y) = \neg LE(x, y)$
the original equation in the new terms $LE(a, b) \oplus (LE(b, c) \wedge LE(c, b) \vee \neg LE(c, a)) \rightarrow (\neg(b, c) \wedge LE(c, a))$

(b) $LT(x, y) = LT(x, y)$
$LE(x, y) = \neg LT(y, x)$
$E(x, y) = \neg LT(x, y) \wedge \neg LT(y, x)$
$GE(x, y) = \neg LT(x, y)$
$GT(x, y) = LT(y, x)$
the original equation in the new terms $\neg LT(b, a) \oplus (\neg LT(b, c) \wedge \neg LT(a, c) \vee LT(a, c) \rightarrow (LT(c, b) \wedge \neg LT(a, c))$

P2.3.9 [12 pts]

Let D be a set of dogs and let T be a subset of terriers, so that the predicate $T(x)$ means "dog x is a terrier". Let $F(x)$ mean "dog x is fierce" and let $S(x, y)$ mean "dog x is smaller than dog y". Write quantified statements for the following, using only variables whose type is D:

(a) There exists a fierce terrier.

(b) All terriers are fierce.

(c) There exists a fierce dog who is smaller than all terriers.

(d) There exists a terrier who is smaller than all fierce dogs, except itself.

Solution:

(a) $\exists x \in D : F(x) \wedge T(x)$

(b) $\exists x \in D : F(x) \rightarrow T(x)$

(c) $\exists x \in D : \forall T : S(F(x, D)$

(d) $\exists x \in D : (T(x) \wedge \forall y \in D : (F(y) \wedge y \neq x) \rightarrow S(x, y))$

EC: P2.3.7 [10 pts]

Let D be a set of dogs, with R being the subset of retrievers, B being the subset of black dogs, and F being the subset of female dogs, with membership predicates $R(x)$, $B(x)$, and $F(x)$ respectively. Suppose that the three statements $\forall x : \exists y : R(x) \oplus R(y), \forall x : \exists y : B(x) \oplus B(y)$, and $\forall x : \exists y : F(x) \oplus F(y)$ are all true. What can you say about the number of dogs in D? Justify your answer.

Solution:

$\forall x : \exists y : R(x) \oplus R(y)$ - for every dog x, there is a dog y such that x is a retriever or y is a retriever, but not both
$\forall x : \exists y : B(x) \oplus B(y)$ - for every dog x there is a dog y such that x is black or y is black but not both
$\forall x : \exists y : F(x) \oplus F(y)$ - for every dog x there is a dog y such that x is female or y is female but not both from these we can conclude that for every dog, there is another dog that is a retriever, same for black dogs and female dogs we can see from this that the size of D must be at least 2x the size of either R, B, or F