## Solutions to Examples from Discussion 6 (ECE 213, Spring 2024)

1. For each of the following LTI systems, determine if the system is (i) causal and/or (ii) BIBO stable.

   (a) The system is defined by $y(t) = \int_t^\infty x(\tau)\, e^{t-\tau}\, d\tau$.

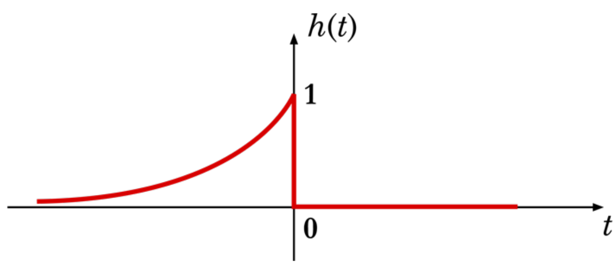
   (b) The impulse response is $h(t) = \frac{1}{(t+1)^2}\, u(t)$.

**ANSWERS:**

(a) The impulse response $h(t)$ is equal to…

$$h(t) = \int_t^\infty \delta(\tau)\, e^{t-\tau}\, d\tau = \int_{-\infty}^\infty u(\tau - t)\, \delta(\tau)\, e^{t-\tau}\, d\tau = u(-t)e^t$$

which is shown below…




Since $h(t)$ is non-zero for $t < 0$, this system is <u>**not causal**</u>.

To test for BIBO stability…

$$\int_{-\infty}^\infty |h(t)|\, dt = \int_{-\infty}^0 e^t\, dt = 1 = \text{finite}$$

Therefore, this system is <u>**BIBO stable**</u>.

(b) The impulse response is equal to 0 for $t < 0$, so this system is <u>**causal**</u>.

To test for BIBO stability…

$$\int_{-\infty}^\infty |h(t)|\, dt = \int_0^\infty \frac{dt}{(t+1)^2} = 1 = \text{finite}$$

Therefore, this system is also <u>**BIBO stable**</u>.

## Solutions to Examples from Discussion 6 (ECE 213, Spring 2024)

2. For each of the following non-LTI systems, determine if the system is (i) causal and/or (ii) BIBO stable.

   (a) $y(t) = \int_t^{2t} x(\tau)\, d\tau$

   (b) $y(t) = x(|t|)$

**ANSWERS:**

Because these systems are not LTI, causality and stability cannot be found as above. Instead, we will use the definitions.

(a) The output at time $t$ depends on the input between time $t$ and time $2t$. For $t > 0$, this time frame is in the future, so this system is **not causal**.

To test for BIBO stability, let's choose a simple bounded input $x(t) = C$, where $C$ is non-zero. The corresponding output is…

$$y(t) = \int_t^{2t} C\, d\tau = Ct$$

The magnitude of $y(t)$ goes infinite as $t$ goes infinite, so this system is **not BIBO stable**.

(b) For $t < 0$, the output depends on the input at a positive time, so this system is **not causal**.

To test for BIBO stability, assume that the input is bounded, i.e.,

$$|x(t)| < C$$

where $C > 0$. Therefore…

$$|y(t)| = |x(|t|)| < C$$

Since the output is bounded for a bounded input, this system is **BIBO stable**.

## Solutions to Examples from Discussion 6 (ECE 213, Spring 2024)

3. Consider the following system defined by an LCCDE:

$$\frac{d^2y}{dt^2} + 5\,\frac{dy}{dt} + 4\,y(t) = \frac{d^2x}{dt^2} + 3\,\frac{dx}{dt} + 6\,x(t)$$

   (a) Find the frequency response function $H(\omega)$.

   (b) Find $y(t)$ for $x(t) = 10\sin(2t)\,u(t)$.

**ANSWERS:**

As shown in class, using an input $x(t) = e^{j\omega t}$, the output is $y(t) = H(\omega)\,e^{j\omega t}$, where $H(\omega)$ is found by replacing the $n$th derivative of $x$ with $(j\omega)^n\,x(t)$ and replacing the $n$th derivative of $y$ with $(j\omega)^n\,y(t)$.

(a) Therefore, the LCCDE becomes…

$$(j\omega)^2\,y(t) + 5\,j\omega\,y(t) + 4\,y(t) = (j\omega)^2 x(t) + 3\,j\omega\,x(t) + 6\,x(t)$$

or…

$$y(t) = \frac{(j\omega)^2 + 3j\omega + 6}{(j\omega)^2 + 5j\omega + 4}\,x(t)$$

So, the frequency response function is…

$$H(\omega) = \frac{(j\omega)^2 + 3j\omega + 6}{(j\omega)^2 + 5j\omega + 4}$$

(b) Rewriting the input as a linear combination of complex exponentials…

$$x(t) = 10\sin(2t)\,u(t) = \frac{10}{j2}\left(e^{j2t} - e^{-j2t}\right)u(t)$$

… the response becomes…

$$y(t) = \frac{10}{j2}\left(H(2)\,e^{j2t} - H(-2)\,e^{-j2t}\right)u(t)$$

$$= \frac{10}{j2}\left(\frac{(j2)^2 + 3(j2) + 6}{(j2)^2 + 5(j2) + 4}\,e^{j2t} - \frac{(-j2)^2 - 3(-j2) + 6}{(-j2)^2 - 5(-j2) + 4}\,e^{-j2t}\right)u(t)$$

Note that this is a purely real-valued function! But it needs to be simplified (a lot) to see it. The result is…

$$y(t) = 6\sin(2t)\,u(t) - 2\cos(2t)\,u(t)$$

## Solutions to Examples from Discussion 6 (ECE 213, Spring 2024)

4. Consider an LTI system with impulse response $h(t) = 6\,e^{-2t}\,u(t)$.
   Find the output $y(t)$ when the input is (a) $x(t) = 8\,u(t)$ and (b) $x(t) = 10\sin 4t\;\,u(t)$.

**ANSWERS:**

First, find the frequency response function $H(\omega)$ using the following integral…

$$H(\omega) = \int_0^\infty h(t)\,e^{-j\omega t}\,dt = \int_0^\infty 6\,e^{-2t}\,e^{-j\omega t}\,dt = 6\int_0^\infty e^{(-2-j\omega)t}\,dt = \frac{6}{2+j\omega}$$

For (a), recognize that 8 can be written as $8\,e^{j0t}$, i.e., use $H(0)$ to find the output. The result is…

$$y(t) = 8\,H(0)\,e^{j0t}\,u(t) = \;8\,\frac{6}{2+j0}\,e^{j0t}\,u(t) = 24\,u(t)$$

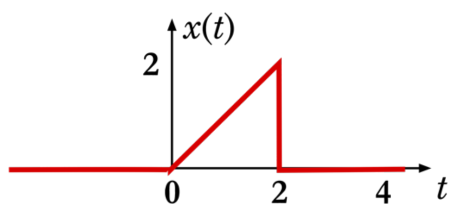
For (b), write $10\sin 4t$ in terms of complex exponentials, then find the output for each separately and combine them. The result is…

$$y(t) = \frac{10}{j2}\left(H(4)\,e^{j4t} - H(-4)\,e^{-j4t}\right)u(t) = \frac{10}{j2}\left(\frac{6}{2+j4}\,e^{j4t} - \frac{6}{2-j4}\,e^{-j4t}\right)u(t)$$

$$= \frac{10}{j2}\frac{1}{20}\left(6\,(2-j4)\,e^{j4t} - 6\,(2+j4)\,e^{-j4t}\right)u(t) = 6\sin 4t\;\,u(t) - 12\cos 4t\;\,u(t)$$

Note that this result is purely real, because the input is purely real.

5. A signal $x(t)$ is represented below. What is its Laplace Transform $X(s)$?




**ANSWER:**

First, rewrite $x(t)$ using unit step functions…

$$x(t) = r(t) - r(t-2) - 2u(t-2) = t\,u(t) - (t-2)\,u(t-2) - 2u(t-2)$$

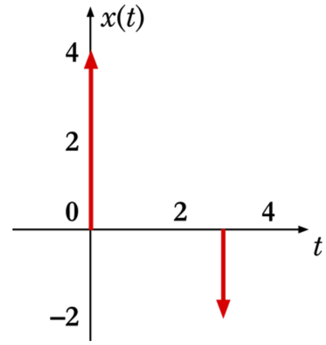
Next, transform each $u(t)$ and $t\,u(t)$, then use the time-shift property (of the Laplace Transform) to get…

$$X(s) = \frac{1}{s^2} - \frac{1}{s^2}\,e^{-2s} - \frac{1}{s}\,e^{-2s}$$

## Solutions to Examples from Discussion 6 (ECE 213, Spring 2024)

6. A signal is represented to the right. The arrows in the figure are impulse functions, with each value representing the area "below" it.

   What is its Laplace Transform $X(s)$?




**ANSWER:**

First, write $x(t)$ using impulse functions…

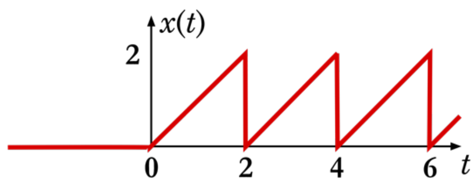
$$x(t) = 4\delta(t) - 2\delta(t-3)$$

Next, transform each $\delta(t)$, then use the time-shift property to get…

$$X(s) = 4 - 2\,e^{-3s}$$

7. For each of the following signals $x(t)$, find its Laplace Transform $X(s)$.

   (a) The causal periodic signal shown below.




**ANSWER:**

This is the same signal as in problem 5, repeated with a period of $T = 2$. For a causal periodic function, as here, $X(s)$ is the Laplace Transform of the function from 0 to $T$ divided by $1 - e^{-Ts}$. Therefore…

$$X(s) = \frac{\frac{1}{s^2} - \frac{1}{s^2}\,e^{-2s} - \frac{1}{s}\,e^{-2s}}{1 - e^{-2s}}$$

   (b) $x(t) = e^{-5t}\sin(2t)\,u(t)$

**ANSWER:**

Use the known Laplace Transform of $\sin(2t)\,u(t)$ and the frequency-shifting property to get…

$$X(s) = \frac{2}{(s+5)^2 + 2^2}$$

[continued]

# Solutions to Examples from Discussion 6 (ECE 213, Spring 2024)

7. [continued] For each of the following signals $x(t)$, find its Laplace Transform $X(s)$.

(c) $x(t) = 6\, e^{-2t}\, u(t-5)$

**ANSWER:**

First, rewrite the exponential, so that the time shift is more obvious…

$$x(t) = 6\, e^{-2(t-5)-10}\, u(t-5)$$

Next, simplify the expression…

$$x(t) = 6\, e^{-10}\, e^{-2(t-5)}\, u(t-5)$$

In other words, there is an overall factor of $6\, e^{-10}$, then we have a frequency shift of 2 and a time shift of 5. We do the frequency shift first, i.e., find the Laplace Transform of $e^{-2t}\, u(t)$, then add a factor of $e^{-Ts}$ for the time shift of $T = 5$ to get…

$$X(s) = 6\, e^{-10}\, \frac{1}{s+2}\, e^{-5s}$$

(d) $x(t) = t\cos(2t)\; u(t)$

**ANSWER:**

Let $x_1(t) = \cos(2t)\; u(t)$. The Laplace Transform of $t\, x_1(t)$ is found using the frequency-derivative property, that is…

$$X(s) = -\frac{dX_1}{ds} = -\frac{d}{ds}\left(\frac{s}{s^2+2^2}\right) = \frac{2s^2}{(s^2+4)^2} - \frac{1}{s^2+4}$$

(e) $x(t) = \frac{1-e^{-2t}}{t}\; u(t)$

**ANSWER:**

Let $x_1(t) = (1 - e^{-2t})\, u(t)$. The Laplace Transform of $x_1(t)/t$ is found using the frequency-integral property, that is…

$$X(s) = \int_s^{\infty} X_1(s')\, ds' = \int_s^{\infty} \left(\frac{1}{s'} - \frac{1}{s'+2}\right)\, ds' = \ln\left(\frac{s+2}{s}\right)$$

[continued]

7. [continued] For each of the following signals $x(t)$, find its Laplace Transform $X(s)$.

(f) $x(t) = \frac{d}{dt}[e^{-5t}\sin(2t)\ u(t)]$

**ANSWER:**

Let $x_1(t) = e^{-5t}\sin(2t)\ u(t)$. The Laplace Transform of the time-derivative of $x_1(t)$ is…

$$X(s) = sX_1(s) - x_1(0^-) = s\ \frac{2}{(s+5)^2+2^2} - 0 = \frac{2s}{(s+5)^2+2^2}$$

Note that this is **<u>much</u>** easier than taking the derivative of $e^{-5t}\sin(2t)\ u(t)$, then transforming it.

(g) $x(t) = \text{rect}\left(\frac{1}{2}(t-1)\right) * e^{-5t}\ u(t)$

**ANSWER:**

The Laplace Transform of $x_1(t) * x_2(t)$ is simply the product of the two individual Laplace Transforms, i.e., $X_1(s)\ X_2(s)$. (This is one of the reasons we LOVE the Laplace Transform!) Therefore, turn the time-scaled (signal-expanded), time-shifted rect function into a difference of two unit-step functions…

$$x_1(t) = \text{rect}\left(\frac{1}{2}(t-1)\right) = u(t) - u(t-2)$$

… then convolve them with $e^{-5t}\ u(t)$ to find the Laplace Transform…

$$X(s) = \left(\frac{1}{s} - \frac{1}{s}e^{-2s}\right) \cdot \frac{1}{s+5} = \frac{1-e^{-2s}}{s(s+5)}$$