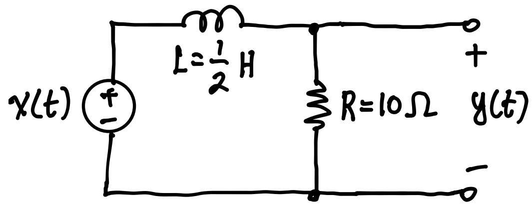
Example 5.3: Consider a periodic input signal[1]

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1+jn} e^{j4n\pi t}.$$

It is applied to the following circuit as input.



(a) Find the amplitude-phase version of the Fourier series for $x(t)$.

(b) Find the output $y(t)$ using phasor analysis.

Solution:
(a) First, we recognize

$$\omega_0 = 4\pi \text{ (rad/s)}, \ T_0 = \frac{2\pi}{\omega_0} = 0.5 \text{ (s)}. \tag{E1}$$

For the amplitude-phase representation,

$$c_0 = x_0 = \frac{1}{1+j0} = 1, \tag{E2}$$

$$c_n = 2|x_n| = \frac{2}{\sqrt{1+n^2}}, \tag{E3}$$

$$\phi_n = \angle x_n = -\tan^{-1} n. \tag{E4}$$

[1]Note that $x_{-n} = x_n^*$. $x(t)$ is a real-valued signal.

Therefore,

$$x(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{\sqrt{1+n^2}} \cos\left(4n\pi t - \tan^{-1} n\right). \quad \text{(E5)}$$

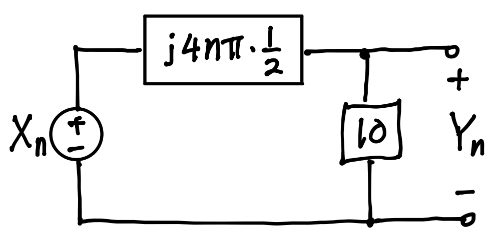
(b) First, (E5) can be put in phasor form:

$$x(t) = X_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left\{X_n e^{j4n\pi t}\right\}, \quad \text{(E6)}$$

where

$$X_0 = 1,\ X_n = \frac{2}{\sqrt{1+n^2}} e^{-j\tan^{-1} n}\ (n = 1, 2, \ldots). \quad \text{(E7)}$$

For the n-th component (at $\omega = 4n\pi$), the circuit has a phasor form below.



We find

$$Y_n = X_n \frac{10}{10 + j2n\pi}, \quad \text{(E8)}$$

from which the magnitude and phase are found to be

$$|Y_n| = |X_n| \frac{10}{\sqrt{100 + (2n\pi)^2}} = \frac{20}{\sqrt{1+n^2}\sqrt{100+(2n\pi)^2}}, \tag{E9}$$

$$\angle Y_n = \angle X_n + \angle \frac{10}{10 + j2n\pi} = -\tan^{-1} n - \tan^{-1} \frac{n\pi}{5}. \tag{E10}$$

Together with $Y_0 = X_0 = 1$, we find

$$\begin{aligned} y(t) &= Y_0 + \sum_{n=1}^{\infty} \mathrm{Re}\left\{ Y_n e^{j4n\pi t} \right\} \\ &= 1 + \sum_{n=1}^{\infty} \frac{20}{\sqrt{1+n^2}\sqrt{100+(2n\pi)^2}} \\ &\quad \times \cos\left(4n\pi t - \tan^{-1} n - \tan^{-1} \frac{n\pi}{5}\right) \end{aligned} \tag{E11}$$

Truncated series for the input and output with $N = 100$ are shown below.