Hw1 Answer Key

Wednesday, February 22, 2023 11:46 AM

hw1

Math 235 Section 8 Exercise Sheet 1

February 17, 2023

1 Matrix Practice

For each of the following matrices, do everything we know how to do:

- Put it in reduced row echelon form
- Describe the possible outputs of the matrix both explicitly (as a span) and implicitly (by equations)
- Describe the kernel of the matrix both explicitly (as a span) and implicitly (by equations)

1.
$$\begin{pmatrix} 3 & 5 & 7 & -2 \\ 1 & 2 & 3 & 1 \end{pmatrix}$$

2.
$$\begin{pmatrix} 1 & -2 & -2 \\ 6 & -6 & -12 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3.
$$\begin{pmatrix} 0 \\ 5 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

2 Applications of linear algebra: Fraud

Alice, Bob, and Carol are in business together. They have an LLC with three joint ventures, named "Project 1", "Project 2", and "Project 3" respectively. Each project makes (or loses) some amount of money, and at year-end they split the profit (or loss) in each venture based on their ownership stake. They've

1.1. $A = \begin{pmatrix} 3 & 5 & 7 & -2 \\ 1 & 2 & 3 & 1 \end{pmatrix}$

RREF $\begin{pmatrix} 1 & 0 & -1 & -9 \\ 0 & 1 & 2 & 5 \end{pmatrix}$

Image

Explicit: $\boxed{\text{Span of columns of } A}$

Implicit: $\begin{pmatrix} 3 & 5 & 7 & -2 & y_1 \\ 1 & 2 & 3 & 1 & y_2 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 1 & 2 & 3 & 1 & y_2 \\ 3 & 5 & 7 & -2 & y_1 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 1 & 2 & 3 & 1 & y_2 \\ 0 & -1 & -2 & -5 & y_1 - 3y_2 \end{pmatrix}$

Always solvable

$$\boxed{\{\vec{y} \mid \vec{y} \in \mathbb{R}^2\}}$$

Kernel:

Explicit: RREF is $\begin{pmatrix} 1 & 0 & -1 & -9 \\ 0 & 1 & 2 & 5 \end{pmatrix}$

$$\Rightarrow \quad x_1 = x_3 + 9x_4$$
$$x_2 = -2x_3 - 5x_4$$

Turn into span:

Set $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{\text{Kernel}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ -5 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

Implicit: $\{\vec{x} \mid \vec{x} \in \mathbb{R}^4,\ A\vec{x} = \vec{0}\}$

1.2 $A = \begin{pmatrix} 1 & -2 & -2 \\ 6 & -6 & -12 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

RREF: $\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$

Image:

Explicit: $\boxed{\text{span of cols of } A}$

Implicit:

$$\begin{pmatrix} 1 & -2 & -2 & y_1 \\ 6 & -6 & -12 & y_2 \\ 1 & 1 & 0 & y_3 \\ 0 & 0 & 0 & y_4 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -2 & -2 & y_1 \\ 0 & 6 & 0 & y_2 - 6y_1 \\ 0 & 3 & 2 & y_3 - y_1 \\ 0 & 0 & 0 & y_4 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -2 & -2 & y_1 \\ 0 & 6 & 0 & y_2 - 6y_1 \\ 0 & 0 & 2 & y_3 - \frac{1}{2}y_2 + 2y_1 \\ 0 & 0 & 0 & y_4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & y_4 \end{array}\right)$$

Solvable iff $y_4 = 0$

$$\boxed{\text{Image}(A) = \left\{ \begin{pmatrix} y_1 \\ \vdots \\ y_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; y_4 = 0 \right\}}$$

Kernel:

Explicit: RREF is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

$$\Rightarrow \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$$\boxed{\text{Kernel}(A) = \{\vec{0}\}}$$

Implicit: $\boxed{\{\vec{x} \mid A \cdot \vec{x} = \vec{0}\}}$

1.3

$$A = \begin{pmatrix} 0 \\ 5 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

RREF: $\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$

Image:

Explicit:

$$\text{Span}\left\{ \begin{pmatrix} 0 \\ 5 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{\text{Span}\left\{\begin{pmatrix} 5 \\ -1 \\ 2 \\ 1 \end{pmatrix}\right\}}$$

Implicit:

$$\begin{pmatrix} 0 & y_1 \\ 5 & y_2 \\ -1 & y_3 \\ 2 & y_4 \\ 1 & y_5 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & y_5 \\ 0 & y_1 \\ 0 & y_2 - 5y_5 \\ 0 & y_3 + y_5 \\ 0 & y_4 - 2y_5 \end{pmatrix}$$

$$\boxed{\text{Image}(A) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \middle| \; y_1 = 0,\ y_2 - 5y_5 = 0,\ y_3 + y_5 = 0,\ y_4 - 2y_5 = 0 \right\}}$$

Kernel:

Explicit:

RREF is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, kernel is $\boxed{\text{Span}\{\vec{0}\}}$

Implicit:

$\boxed{\{\vec{x} \mid A\vec{x} = \vec{0}\}}$ or $\{\vec{x} \mid \vec{x} = 0\}$

negotiated different ownership stakes in each of the businesses (see below). Unfortunately, they despise each other. Alice wants to illegally alter the financial data so that she gets all of the money.

All together, the three projects made \$1,000,000 of profit this past year. How can Alice invent fake profits or losses for each of the projects so that the total remains \$1,000,000, but all \$1,000,000 goes to her and Bob and Carol get nothing?

- Project 1 is owned 30% by Alice, 30% by Bob, and 40% by Carol.
- Project 2 is owned 40% by Alice, 40% by Bob, and 20% by Carol.
- Project 3 is owned 10% by Alice, 5% by Bob, and 85% by Carol.

Hint: There is a function which takes as input the amount of money each project made or lost, and gives as output the total and the share for each of Alice, Bob, and Carol.

Further thought: Is this the best Alice can do? Could she make \$2,000,000, even though the projects only made \$1,000,000?

Don't commit financial crimes.

2.

Earnings by project

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \begin{array}{l} \text{Alice's share} \\ \text{Bob's} \\ \text{Carol's} \\ \text{Total} \end{array}$$

$$\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} \longrightarrow \begin{array}{c|l} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} & \begin{array}{l} \text{Carol's} \\ \text{Total} \end{array} \end{array}$$

Solve for $T(\vec{p}) = \begin{pmatrix} 1m \\ 0 \\ 0 \\ 1m \end{pmatrix}$

Matrix of T:

$$\begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.05 \\ 0.4 & 0.2 & 0.85 \\ 1 & 1 & 1 \end{pmatrix}$$

Write augmented matrix

$$\left(\begin{array}{ccc|c} 0.3 & 0.4 & 0.1 & 1m \\ 0.3 & 0.4 & 0.05 & 0 \\ 0.4 & 0.2 & 0.85 & 0 \\ 1 & 1 & 1 & 1m \end{array}\right)$$

and solve $\Rightarrow$ $\boxed{\begin{aligned} p_1 &= -66m \\ p_2 &= 47m \\ p_3 &= 20m \end{aligned}}$

Bonus:

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \xrightarrow{R} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{array}{l} \text{alice's share} \\ \text{total} \end{array}$$

Matrix of R:

$$\begin{pmatrix} 0.3 & 0.4 & 0.1 \end{pmatrix}$$

$$\begin{pmatrix} 0.3 & 0.4 & 0.1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solve for $R(\vec{p}) = \begin{pmatrix} 2m \\ 1m \end{pmatrix}$

$$\begin{pmatrix} 0.3 & 0.4 & 0.1 & 2m \\ 1 & 1 & 1 & 1m \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 1m \\ 0 & 0.1 & -0.2 & 1.7m \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 3 & -16m \\ 0 & 1 & -2 & 17m \end{pmatrix}$$

Free: p_3

Basic: $p_1 = -16m - 3p_3$

$p_2 = 17m + 2p_3$

or

$$\text{Solutions} = \begin{pmatrix} -16m \\ 17m \\ 0 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \right\}$$