250 Homework #2

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March 9, 2024

P2.1.4 [10 pts]

Let A, B, and C be sets defined in Exercise 2.1.1 above[1]. Explicitly list the direct products $(A \times B) \times C$ and $A \times (B \times C)$. (Note that an element of $A \times (B \times C)$ is a pair whose first element is in A and whose second element is in $B \times C$.) If X, Y, and Z are any sets, we generally do not distinguish among $X \times (Y \times Z)$, $(X \times Y) \times Z$, and the set of triples $X \times Y \times Z$. Explain how we may think of an element of each of these sets as an element of one of the others, by describing which elements of each set correspond to which elements of the others.

Solution:

$(A \times B) \times C = \{\langle 1, x, dog\rangle, \langle 2, x, dog\rangle, \langle 1, x, cat\rangle, \langle 2, x, cat\rangle, \langle 1, y, dog\rangle,$
$\langle 2, y, dog\rangle, \langle 1, y, cat\rangle, \langle 2, y, cat\rangle, \langle 1, z, dog\rangle, \langle 2, z, dog\rangle, \langle 1, z, cat\rangle, \langle 2, z, cat\rangle\}$

$A \times (B \times C) = \{\langle 1, x, dog\rangle, \langle 2, x, dog\rangle, \langle 1, x, cat\rangle, \langle 2, x, cat\rangle, \langle 1, y, dog\rangle,$
$\langle 2, y, dog\rangle, \langle 1, y, cat\rangle, \langle 2, y, cat\rangle, \langle 1, z, dog\rangle, \langle 2, z, dog\rangle, \langle 1, z, cat\rangle, \langle 2, z, cat\rangle\}$

$X \times (Y \times Z)$, $(X \times Y) \times Z$, and $X \times Y \times Z$ are considered the same set because the parentheses do not take precedence, whereas the order of the sets does. The elements of the sets $X \times (Y \times Z)$, $(X \times Y) \times Z$, and $X \times Y \times Z$ correspond with each other because the sets X, Y, Z are concatenated in the same order.

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[1] Let A be the set $\{1, 2\}$, B be the set $\{x, y, z\}$, and C be the set {cat, dog}

P2.1.6 [10 pts]

Let $P(x, y)$ be a binary predicate where the type of x is A and the type of y is B. We define the projection $\pi_1(P)$ to be the unary predicate on A defined as follows: $\pi_1(P)(x)$ is true if and only if there exists an element y of B such that $P(x, y)$ is true. Similarly, $\pi_2(P)$ is the unary predicate on B defined so that $\pi_2(P)(y)$ is true if and only if there exists an element x of A such that $P(x, y)$ is true.

(a) Let $A = B = \{0, 1, 2, 3\}$ and define $P(x, y)$ to mean "$x < y$". Describe $\pi_1(P)$ and $\pi_2(P)$ as both predicates and relations.

(b) What can you say about the predicate P if you know that $\pi_1(P)$ and $\pi_2(P)$ are each always false? What if they are each always true?

Solution:

(a) Below is a table of values at a given x with a set of elements $y \in B$ that fulfill the predicate $P(x, y)$, $x < y$.

$\pi_1(P)$:

x	B_y	$\pi_1(P)$
0	$\{1, 2, 3\}$	T
1	$\{2, 3\}$	T
2	$\{3\}$	T
3	$\emptyset$	F

The unary predicate $\pi_1(P)$:

- is not well-defined, as an input x can have more than one output y
- is not onto, as $y = 0$ does not have an input x
- is not total, as input $x = 3$ does not have an output y
- is not one-to-one, as an input x can have multiple outputs

Below is a table of values at a given y with a set of elements $x \in A$ that fulfill the predicate $P(x, y)$, $x < y$.

$\pi_2(P)$:

A_x	y	$\pi_2(P)$
$\emptyset$	0	F
$\{0\}$	1	T
$\{0, 1\}$	2	T
$\{0, 1, 2\}$	3	T

The unary predicate $\pi_2(P)$:

- is not well-defined, as an input x can have more than one output y
- is not onto, as $y = 0$ does not have an input x
- is not total, as input $x = 3$ does not have an output y
- is not one-to-one, as an input x can have multiple outputs

(b) Case for both being false: There $\forall x \in A : \nexists y \in B$ and $\forall y \in B : \nexists x \in A$ so that there are pairs (x, y) that satisfy P, so P is the $\emptyset$. Both $\pi_1(P)$ and $\pi_2(P)$ will both be false as their respective projections onto x and y will include no pairs.

Case for both being true: There $\forall x \in A : \exists y \in B$ and $\forall y \in B : \exists x \in A$ so that P is satisfied by every pair (x, y). Both $\pi_1(P)$ and $\pi_2(P)$ will both be true as their respective projections onto x and y will include pairs for every x and y.

P2.3.4 [10 pts]

The unique existential quantifier $\exists!$ has the following interpretation: $\exists! x : P(x)$ means "there is a unique x such that $P(x)$ is true", or equivalently "there is exactly one x such that $P(x)$ is true". Show how to write an expression with ordinary quantifiers and boolean operations (and the equals sign) that is equivalent to $\exists! x : P(x)$.

Solution:

$\exists x(P(x) \wedge (\forall y : P(y) \rightarrow y = x))$

P2.5.10 [10 pts]

It is possible for two different finite languages X and Y to have the same Kleene star, that is, for $X^* = Y^*$ to be true.

(a) Prove that $X^* = Y^*$ if and only if both $X \subseteq Y^*$ and $Y \subseteq X^*$.

(b) Use part (a) to show that $X^* = Y^*$ if $X = \{a, abb, bb\}$ and $Y = \{a, bb, bba\}$.

(c) Prove that if $X^* = Y^*$, $\lambda \notin X \cup Y$, w is any string of minimum length in X, and z is any string of minimum length in Y, then w and z have the same length.

Solution:

(a) If X is a subset of or equal to Y^*, that means the strings in X can be concatenated to make the strings in Y^*. If Y is a subset of or equal to X^*, that means the strings in Y can be concatenated to make the strings in X^*. X^* would therefore be equal to Y^* because if the individual strings of X make up Y^* and the individual strings of Y make up X^*, then all the concatenations that could be made up from the subsets X and Y would exist in Y^* and X^*.

(b) Y^* could contain all of X because Y could be concatenated so that $\{abb, bb, a\} \in Y^*$. X^* could contain all of Y because X could be concatenated so that $\{bba, bb, a\} \in X^*$. This means that $X \subseteq Y^*$ and $Y \subseteq X^*$. Since X^* could contain all concatenations of Y and Y^* could contain all concatenations of X that means Y^* contains X^* and X^* contains Y^* making $X^* = Y^*$.

(c) If $X^* = Y^*$ then both w and z are in X^* and Y^*. The string w cannot be shorter than z and z cannot be shorter than w as that would contradict the assertion that w is the shortest string in X and z is the shortest string in Y. Therefore, the strings w and z must be the same length in order to not contradict the assertions.

P2.6.8 [10 pts]

Let D be a set of dogs including the particular dog Nala (n) and let C be a set of conditions including the particular condition "Spanish Inquisition (SI)". Define the predicates $T(x)$ meaning "dog x is a terrier", $E(x, y)$ meaning "dog x expects condition y", and $F(x, y)$ meaning "dog x fears condition y". Translate the following statements into symbolic form, and use them to prove that Nala is not a terrier

- No terrier fears any condition.
- Nala fears any condition that she does not expect.
- No dog expects the Spanish Inquisition.

Solution:

$T(x) : D$: T when D is a terrier and F when D isn't a terrier
$E(x, y) : D \times C$: T when D expects C, F otherwise
$F(x, y) : D \times C$: T when D fears C, F otherwise

1. $T(x) \to \neg(\forall x : y \in C : F(x, y))$
2. $\neg E(n, y) \to \forall y \in C : F(n, y)$
3. $\forall x \in D : \neg E(x, y)$

Using statement 3, $\forall x \in D : \neg E(x, y)$ and $n \in D$, that means Nala doesn't expect the Spanish inquisition. Using statement 2, $\neg E(n, y) \to \forall y \in C : F(n, y)$, that means Nala fears any condition she doesn't expect, and she doesn't expect the Spanish inquisition, so she fears the condition "Spanish Inquisition". Using statement 1, $T(x) \to \neg(\forall x : y \in C : F(x, y))$, terriers don't fear any condition, yet Nala does so Nala is not a terrier.

P2.8.2 [10 pts]

Consider the following binary relations on the naturals (non-negative integers). Which ones are reflexive? Symmetric? Anti-symmetric? Transitive? Partial orders? Justify your claims.

(a) $A(x, y)$, defined to be true if and only if y is even.

(b) $B(x, y)$, defined to be true if and only if $x < y$.

(c) $C(x, y)$, defined to be true if and only if $x + 2 \geq y$.

(d) $D(x, y)$, defined to be true if and only if $x \neq y$.

(e) $E(x, y)$, defined to be true if and only if the English name of x comes no later than the name of y in alphabetical order. (So, for example, E(8, 81) is true because `eight` comes before `eighty-one`, and E(8, 8) is true because `eight` comes no later than `eight`.)

Solution:

(a)

(b)

(c)

(d)

(e)

P2.9.4 [10 pts]

If f is a function from a set A to itself, we can compose f with itself. We call the composition of f with itself k times the k'th iterate of f, and write it $f^{(k)}$.

(a) If $f(x) = x + 2$, what is the function $f^{(3)}$

(b) If $g(x) = x^2 + x + 1$, what is the function $g^{(3)}$?

(c) If i and j are any naturals, is it always true that $(f^{(j)})^{(k)}$ is equal to $f^{(jk)}$? Why or why not?

(d) How should we define $f^{(0)}$? Why?

Solution:

(a)

(b)

(c)

(d)

P2.9.6 [10 pts]

Let f be any bijection from the set $\{1, 2, 3\}$ to itself. Prove that the iterate $f^{(6)}$ (as defined in Problem 2.9.4) is the identity function.

Solution:

P3.1.3 [10 pts]

The greatest common divisor of two naturals x and y is the largest number that divides both x and y. For example, $gcd(8, 12) = 4$ because 4 divides both 8 and 12 and no larger natural divides both.

(a) What is the greatest common divisor of 60 and 339? How do you know?

(b) What are the possible greatest common divisors of p and x, if p is a prime number? How can you tell which is the correct one, given any x?

(c) What is the greatest common divisor of $2^3 \cdot 3^2 \cdot 5^4$ and $2^2 \cdot 3^4 \cdot 5^3$? Can you describe a method to find the greatest common divisor of any two numbers, given their factorization into primes? (You might not be able to prove that your method is correct, of course, without results from later in this chapter)

Solution:

(a)

(b)

(c)

EC: P2.7.1 [10 pts]

Here are a few practice statements to prove. For this exercise, you should go slightly overboard in justifying your steps. All the small-letter variables are of type "string in A^*" (where A is a non-empty finite set, and A^* is the set of all strings of letters in A) and all the capital-letter variables are of type "language over A" (or "subset of A^*")

2. $\forall L : \forall M : \forall N : L(M \cup N) = LM \cup LN$.
3. $\exists u : \forall v : \exists w : ((uv = w) \wedge (uw = v))$.
4. Assuming $\forall u : \exists v : (uv \in L)$ and $\exists a : \forall y : \forall z : (yaz \neq y)$, prove $\neg \exists w : \forall x : [(x \in L) \rightarrow \exists y : (w = xy)]$. Is this conclusion true without the first assumption? (The second assumption is true for strings over any nonempty alphabet.

Solution:

2.

3.

4.