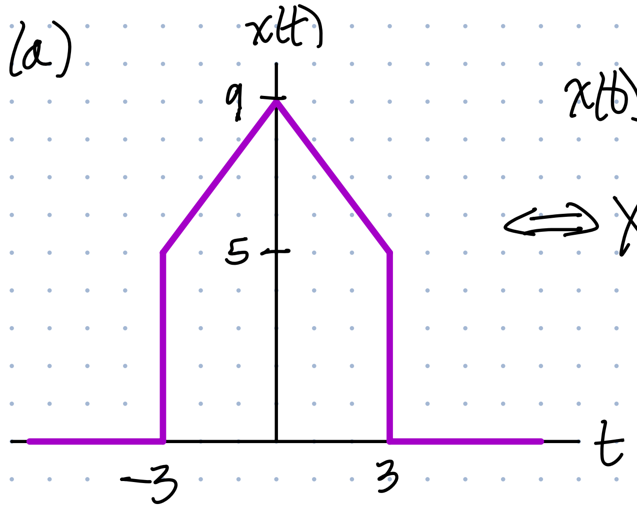
HW 10 Problem 1

(a)



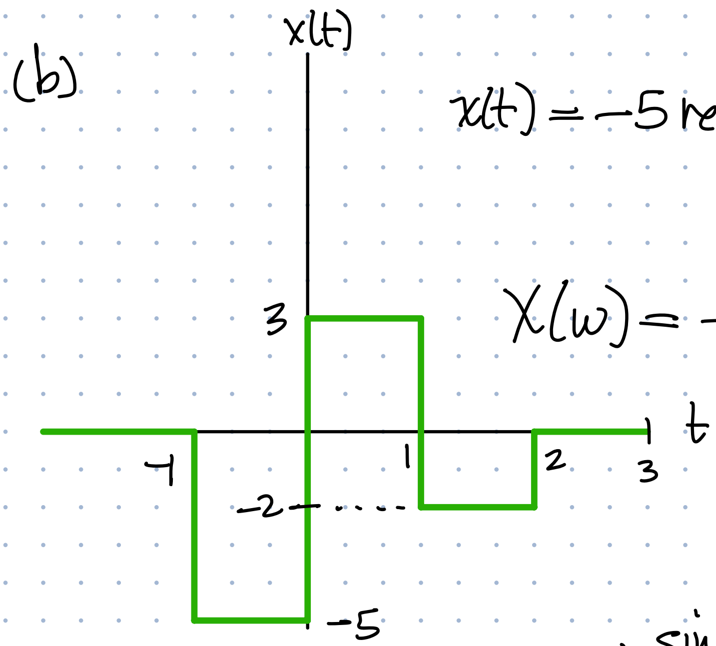
$$x(t) = 5\,\mathrm{rect}\left(\frac{t}{6}\right) + 4\Lambda\left(\frac{t}{3}\right)$$

$$\Leftrightarrow X(\omega) = 5\cdot 6\,\mathrm{sinc}\left(\frac{6\omega}{2\pi}\right) + 4\cdot 3\,\mathrm{sinc}^2\left(\frac{3}{2\pi}\cdot\omega\right)$$

$$= 30\,\frac{\sin(3\omega)}{3\omega} + 12\,\frac{\sin^2\left(\frac{3}{2}\omega\right)}{\left(\frac{3}{2}\omega\right)^2}$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

(b)



$$x(t) = -5\,\mathrm{rect}\left(t+\frac{1}{2}\right) + 3\,\mathrm{rect}\left(t-\frac{1}{2}\right) - 2\,\mathrm{rect}\left(t-\frac{3}{2}\right)$$

$$X(\omega) = -5\cdot e^{j\frac{1}{2}\omega}\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right) + 3e^{-j\frac{1}{2}\omega}\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right) - 2e^{-j\frac{3}{2}\omega}\,\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$= \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\left(-5e^{j\omega/2} + 3e^{-j\omega/2} - 2e^{-j3\omega/2}\right)$$

From the table...

$$\mathrm{rect}(t) \Leftrightarrow \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)}$$

$$g(t-t_0) \Leftrightarrow e^{-j\omega t_0}G(\omega) \quad \text{(time shift)}$$

1(c) $x(t) = e^{-10|t|} = e^{-10t}u(t) + e^{10t}u(-t)$

From the table...

$$e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}$$

time scaling: $g(bt) \Leftrightarrow \frac{1}{|b|}G\left(\frac{\omega}{b}\right) = G(-\omega)$ for $b=-1$

In this case, $b=-1$

$$e^{10t}u(-t) \Leftrightarrow \frac{1}{|-1|}\frac{1}{a-j\omega}$$

$$\Rightarrow X(\omega) = \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{1}{10+j\omega} + \frac{1}{10-j\omega}$$

1(d) We can show that...

$$g(t)\sin(\omega_0 t) \Leftrightarrow -\frac{j}{2}\left[G(\omega-\omega_0) - G(\omega+\omega_0)\right]$$

$$g(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}\left[G(\omega-\omega_0) + G(\omega+\omega_0)\right]$$

In both cases, write $\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

or $\sin(\omega_0 t) = \frac{1}{j2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$

frequency shift: $g(t)e^{j\omega_0 t} \Leftrightarrow G(\omega-\omega_0)$

$x(t) = \underbrace{e^{-10|t|}}_{g(t)} \underbrace{\sin(2t)}_{\sin(\omega_0 t)}$ $\quad G(\omega) = \frac{1}{10+j\omega} + \frac{1}{10-j\omega}$

$$\Rightarrow X(\omega) = -\frac{j}{2}\left(G(\omega-\omega_0) - G(\omega+\omega_0)\right)$$

$$= -\frac{j}{2}\left(\frac{1}{10+j(\omega-2)} + \frac{1}{10-j(\omega-2)} - \left(\frac{1}{10+j(\omega+2)} + \frac{1}{10-j(\omega+2)}\right)\right)$$

1(c) $x(t) = \underbrace{\operatorname{sinc}^2(2t)}_{g(t)} \underbrace{\cos(10t)}_{\cos(\omega_0 t)}$

$$G(\omega) = \frac{1}{2}\Lambda\left(\frac{\omega}{2\pi \cdot 2}\right)$$

$$\Rightarrow X(\omega) = \frac{1}{2}\left(G(\omega-\omega_0) + G(\omega+\omega_0)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\Lambda\left(\frac{\omega-10}{4\pi}\right) + \frac{1}{2}\Lambda\left(\frac{\omega+10}{4\pi}\right)\right)$$

HW 10 Problem 2

(a) $x(t) = te^{-3t}[u(t) - u(t-2)]$

$$= \underbrace{te^{-3t}u(t)}_{\text{fine}} - \underbrace{te^{-3t}u(t-2)}_{\text{not fine!! Needs to be "fixed"}}$$

$$= te^{-3t}u(t) - (t-2+2)\, e^{\overset{-3(t-2+2)}{-3(t-2)-6}} u(t-2)$$

$$= te^{-3t}u(t) - (t-2)e^{-3(t-2)-6}u(t-2) - 2e^{-3(t-2)-6}u(t-2)$$

$$= te^{-3t}u(t) - e^{-6}\cdot(t-2)e^{-3(t-2)}u(t-2) - 2e^{-6}e^{-3(t-2)}u(t-2)$$

$$\Rightarrow X(\omega) = \frac{1}{(3+j\omega)^2} - e^{-6}\frac{1}{(3+j\omega)^2}e^{-j2\omega} - 2e^{-6}\frac{1}{3+j\omega}e^{-j2\omega}$$

E.g. $g(t) = te^{-3t}u(t)$

$g(t-2) = (t-2)e^{-3(t-2)}u(t-2)$

2(b) $x(t) = \frac{\sin(8t)}{8t} \cdot \frac{\sin(2t)}{2t} = x_1(t)\, x_2(t)$

$$x_1(t)\, x_2(t) \Longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega) \quad \text{"convolution property"}$$

$\text{sinc}(Wt) \equiv \frac{\sin(\pi \cdot Wt)}{\pi \cdot Wt}$	$\frac{1}{W} rect\left(\frac{\omega}{2\pi W}\right)$

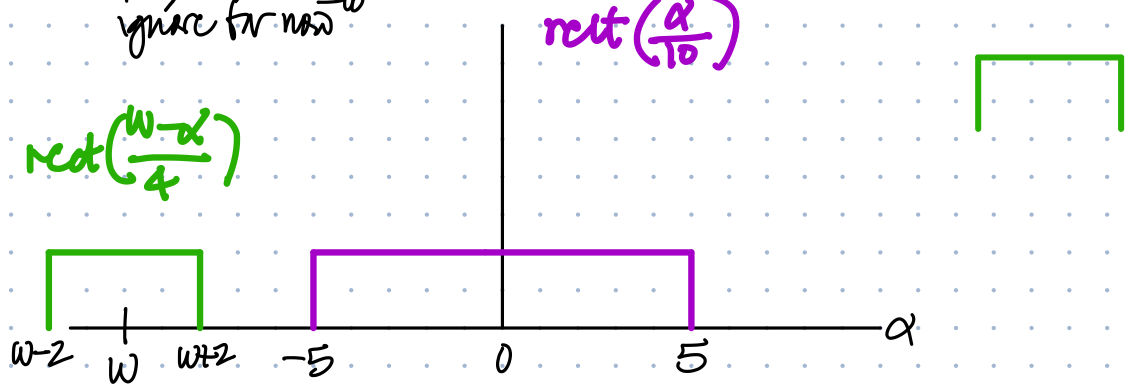
$$\text{sinc}\left(\frac{8t}{\pi}\right) = \frac{\sin(5t)}{5t} \Longleftrightarrow \frac{\pi}{5} \text{rect}\left(\frac{\omega}{2\pi \frac{5}{\pi}}\right) = \frac{\pi}{5} \text{rect}\left(\frac{\omega}{10}\right)$$

$$\text{sinc}\left(\frac{2t}{\pi}\right) = \frac{\sin(2t)}{2t} \Longleftrightarrow \frac{\pi}{2} \text{rect}\left(\frac{\omega}{2\pi \cdot \frac{2}{\pi}}\right) = \frac{\pi}{2} \text{rect}\left(\frac{\omega}{4}\right)$$

$$X(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{5} \text{rect}\left(\frac{\alpha}{10}\right) \frac{\pi}{2} \text{rect}\left(\frac{\omega - \alpha}{4}\right) d\alpha$$

$$= \frac{\pi^2}{2\pi \cdot 10} \int_{-\infty}^{\infty} \text{rect}\left(\frac{\alpha}{10}\right) \text{rect}\left(\frac{\omega - \alpha}{4}\right) d\alpha$$

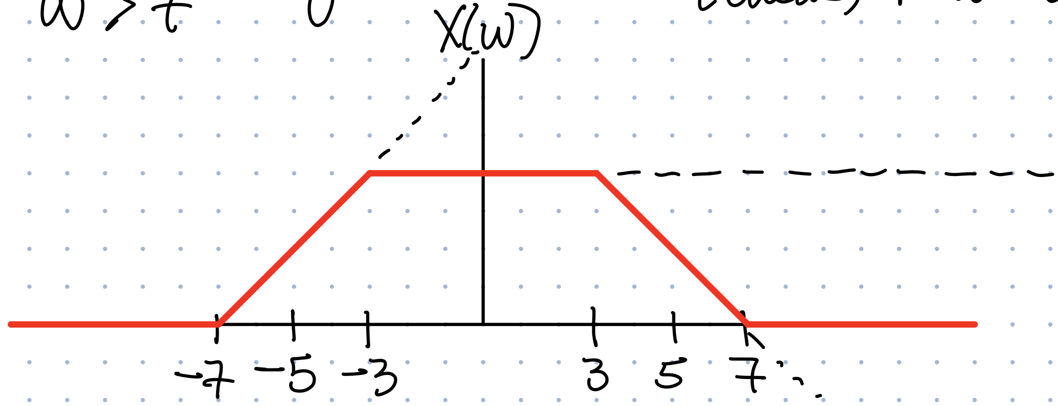
ignore for now



When $\omega + 2 < -5$, there is no "overlap" between green and purple $\Rightarrow$ area $= 0$

	$X(\omega)$		
$\omega < -7$	0		
$-7 < \omega < -3$	$\int_{-5}^{\omega+2} d\alpha$	$= \omega + 7$	(check, must $= 0$ at $\omega = -7$ ✓)
$-3 < \omega < 3$	4		(check, must $= 4$ at $\omega = -3$ ✓)
$3 < \omega < 7$	$\int_{\omega-2}^{5} d\alpha$	$= 7 - \omega$	(check, must $= 4$ at $\omega = 3$ ✓)

$\omega > 7 \quad 0$ (check, $7-\omega = 0$ at $\omega = 7$ ✓)



$$\frac{X(\omega)}{\pi/20} = r(\omega+7) - r(\omega+3) - r(\omega-3) + r(\omega-7)$$

$$X(\omega) = \frac{\pi}{20}\Big(\qquad \Big)$$

2(c) $x(t) = e^{-12t}\cos^2(5t)\,u(t)$

use trig identity: $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$$x(t) = e^{-12t}\left(\frac{1}{2} + \frac{1}{2}\cos(10t)\right)u(t)$$

$$= \underbrace{\frac{1}{2}e^{-12t}u(t)}_{g(t)} + \frac{1}{2}e^{-12t}\underbrace{\cos(10t)}_{\cos(\omega_0 t)}u(t)$$

$$X(\omega) = \frac{1}{2}\cdot\frac{1}{12+j\omega} + \frac{1}{2}\cdot\frac{1}{2}\left(G(\omega-\omega_0) + G(\omega+\omega_0)\right)$$

$$= \frac{1}{2}\,\frac{1}{12+j\omega} + \frac{1}{4}\left(\frac{1}{12+j(\omega-10)} + \frac{1}{12+j(\omega+10)}\right)$$

2(d) $x(t) = \text{sinc}\left(\frac{t-5}{12}\right)$

$$g(t) = \text{sinc}\left(\frac{t}{12}\right) \Leftrightarrow 12\,\text{rect}\left(\frac{12w}{2\pi}\right) = 12\,\text{rect}\left(\frac{6w}{\pi}\right)$$

$\text{sinc}(Wt) \equiv \dfrac{\sin(\pi \cdot Wt)}{\pi \cdot Wt}$	$\dfrac{1}{W}\,rect\left(\dfrac{\omega}{2\pi W}\right)$

"time shift" $g(t-t_0) \Leftrightarrow e^{-jwt_0} G(w)$

$$\Rightarrow X(w) = e^{-j5w} \cdot 12\,\text{rect}\left(\frac{6w}{\pi}\right)$$

2(e) $x(t) = [8 + \sin(2t)]\cos(3t)$

$$x(t) = 8\cos(3t) + \sin(2t)\cos(3t)$$

$$\sin(a)\cos(b) = \tfrac{1}{2}(\sin(a+b) + \sin(a-b))$$

check: $\frac{1}{2}(\sin a \cos b + \sin b \cos a$
$+ (\sin a \cos b - \sin b \cos a))$

$$x(t) = 8\cos(3t) + \tfrac{1}{2}(\sin(5t) + \sin(-t))$$
$$= 8\cos(3t) + \tfrac{1}{2}\sin(5t) - \tfrac{1}{2}\sin(t)$$

$\cos(\omega_0 t + \theta)$	$\pi\left[e^{j\theta}\delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t + \theta)$	$-j\pi\left[e^{j\theta}\delta(\omega-\omega_0) - e^{-j\theta}\delta(\omega+\omega_0)\right]$

$\theta = 0°$

$$\Rightarrow X(w) = 8\pi[\delta(w-3) + \delta(w+3)]$$
$$+ \tfrac{\pi}{2}(-j)[\delta(w-5) - \delta(w+5)]$$
$$- \tfrac{\pi}{2}(-j)[\delta(w-1) - \delta(w+1)]$$

HW 10 Problem 3

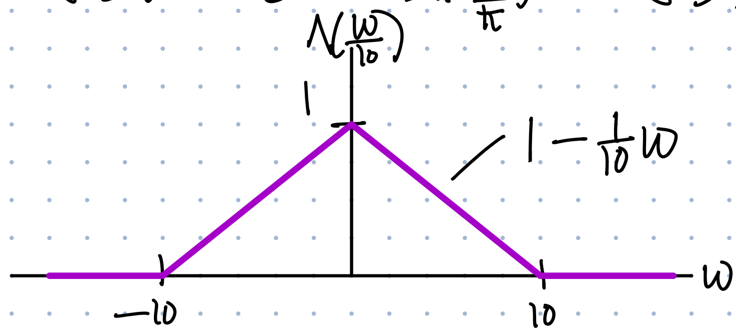
(a) $x(t) = \dfrac{\sin^2(5t)}{(3t)^2} = \left(\dfrac{5}{3}\right)^2 \dfrac{\sin^2(5t)}{(5t)^2} = \left(\dfrac{5}{3}\right)^2 \operatorname{sinc}^2\left(\dfrac{5t}{\pi}\right)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Let's transform $x(t)$ into $X(\omega)$...

$\operatorname{sinc}^2(Bt)$	$\frac{1}{B}\Lambda\left(\frac{\omega}{2\pi B}\right)$

$$X(\omega) = \left(\frac{5}{3}\right)^2 \cdot \frac{\pi}{5} \Lambda\left(\frac{\omega}{2\pi \cdot \frac{5}{\pi}}\right) = \left(\frac{5}{3}\right)^2 \frac{\pi}{5} \Lambda\left(\frac{\omega}{10}\right)$$



$$2 \cdot \frac{1}{2\pi}\int_0^{10} |X(\omega)|^2 \, d\omega = \frac{1}{\pi}\left[\left(\frac{5}{3}\right)^2 \frac{\pi}{5}\right]^2 \int_0^{10} \left(1 - \frac{1}{10}\omega\right)^2 d\omega$$

$$= \frac{1}{\pi}\left[\left(\frac{5}{3}\right)^2 \frac{\pi}{5}\right]^2 \int_0^{10} \left(1 - \frac{1}{5}\omega + \frac{1}{100}\omega^2\right) d\omega$$

$$= \left(\quad \right)\left[\omega - \frac{1}{10}\omega^2 + \frac{1}{300}\omega^3\right]_0^{10}$$

$$= \left(\quad \right)\left[10 - \frac{10^2}{10} + \frac{10^3}{300}\right] = \ldots = E$$

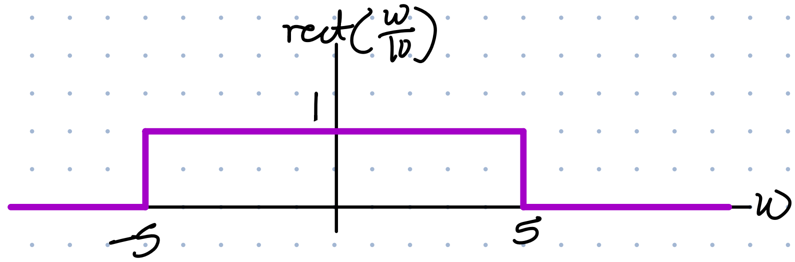
3(b) $x(t) = \frac{d^2 x_0(t)}{dt^2}$ $x_0(t) = \frac{\sin(5t)}{3t} = \frac{3}{5} \cdot \frac{\sin(5t)}{5t} = \frac{3}{5}\operatorname{sinc}\left(\frac{5t}{\pi}\right)$

time-differentiation $\frac{d^2 g(t)}{dt^2} \Leftrightarrow (jw)^2 G(w)$

$$X_0(w) = \frac{\pi}{5} \cdot \frac{3}{5} \operatorname{rect}\left(\frac{w}{2\pi \cdot \frac{5}{\pi}}\right) = \frac{3\pi}{25} \operatorname{rect}\left(\frac{w}{10}\right)$$

$\operatorname{sinc}(Wt) \equiv \frac{\sin(\pi \cdot Wt)}{\pi \cdot Wt}$	$\frac{1}{W} rect\left(\frac{\omega}{2\pi W}\right)$

$$X(w) = (jw)^2 X_0(w) = -w^2\left(\frac{3\pi}{25}\right)\operatorname{rect}\left(\frac{w}{10}\right)$$



$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-w^2 \cdot \frac{3\pi}{25}\right)^2 \operatorname{rect}\left(\frac{w}{10}\right) dw$$

$$= \frac{1}{2\pi}\left(\frac{3\pi}{25}\right)^2 \int_{-5}^{5} w^4\, dw = \frac{1}{2\pi}\left(\frac{3\pi}{25}\right)^2 \frac{w^5}{5}\Big|_{-5}^{5} \cdots\cdots$$

3(c) $x(t) = e^{-8t}u(t) * \frac{\sin(5t)}{3t}$

$$x_1(t) \quad * \quad x_2(t) = \frac{3}{5}\,\text{sinc}\left(\frac{5t}{\pi}\right)$$

$$X(\omega) = X_1(\omega)\cdot X_2(\omega) = \frac{1}{8+j\omega}\cdot\frac{3}{5}\cdot\frac{\pi}{5}\,\text{rect}\left(\frac{\omega}{10}\right)$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\cdot X^*(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{3\pi}{25}\cdot\frac{1}{8+j\omega}\cdot\frac{3\pi}{25}\cdot\frac{1}{8-j\omega}\,\text{rect}\left(\frac{\omega}{10}\right)d\omega$$

$$= \frac{1}{2\pi}\left(\frac{3\pi}{25}\right)^2\int_{-5}^{5}\frac{d\omega}{8^2+\omega^2}$$

$$\int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$\Rightarrow E = \frac{1}{2\pi}\left(\frac{3\pi}{25}\right)^2\frac{1}{8}\tan^{-1}\left(\frac{\omega}{8}\right)\Bigg|_{-5}^{5}$$

$$= \frac{1}{2\pi}\left(\frac{3\pi}{25}\right)^2\frac{1}{8}\left(\tan^{-1}\left(\frac{5}{8}\right)-\tan^{-1}\left(-\frac{5}{8}\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{3\pi}{25}\right)^2\frac{1}{8}\,2\tan^{-1}\left(\frac{5}{8}\right)$$

3(d) $x(t) = \underbrace{x_0(t) * x_0(t) * \cdots * x_0(t)}_{10}$ with $x_0(t) = 9\,\mathrm{sinc}(5t)$

$$X(\omega) = [X_0(\omega)]^{10} = \left[9 \cdot \frac{1}{5} \mathrm{rect}\left(\frac{\omega}{10\pi}\right)\right]^{10}$$

$\mathrm{sinc}(Wt) \equiv \dfrac{\sin(\pi \cdot Wt)}{\pi \cdot Wt}$	$\dfrac{1}{W} rect\left(\dfrac{\omega}{2\pi W}\right)$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \left(\left(\frac{9}{5}\right)^{10}\right)^2 \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{\omega}{10\pi}\right) d\omega$$

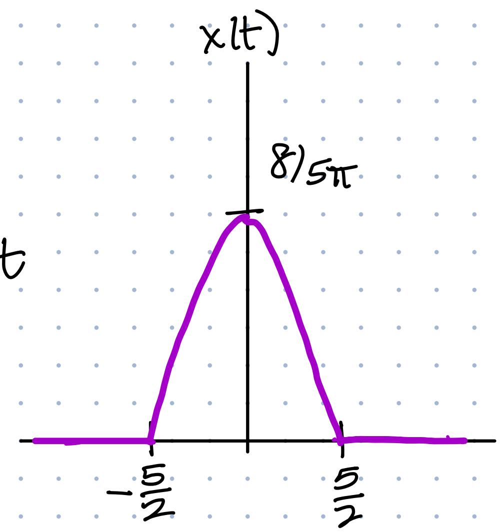
$$= \frac{1}{2\pi}\left(\left(\frac{9}{5}\right)^{10}\right)^2 10\pi$$

3(e) $$X(\omega) = \frac{4\cos(5\omega/2)}{(\pi/2)^2 - (5\omega/2)^2} = \underbrace{4 \cdot \frac{2}{\pi}}_{A} \cos\left(\underbrace{\frac{5\omega}{2}}_{\frac{\omega T}{2}}\right) \cdot \frac{\pi/2}{(\pi/2)^2 - (\omega T/2)^2}$$

From solution to D12 Examples...

$$\frac{A}{T}\cos\left(\frac{\pi t}{T}\right)\mathrm{rect}\left(\frac{t}{T}\right) \Longleftrightarrow A\cos(\omega T/2) \cdot \frac{\pi/2}{(\pi/2)^2 - (\omega T/2)^2}$$

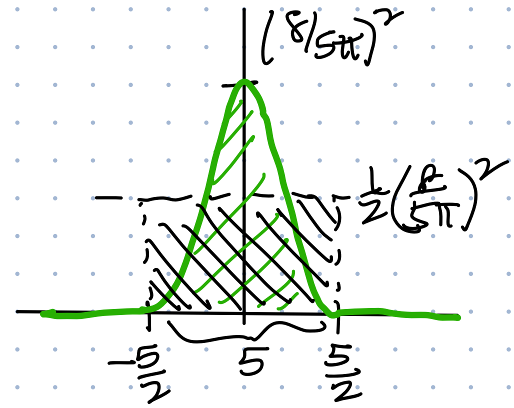
$$\Rightarrow x(t) = \frac{8}{\pi} \cdot \frac{1}{5}\cos\left(\frac{\pi t}{5}\right)\mathrm{rect}\left(\frac{t}{5}\right)$$



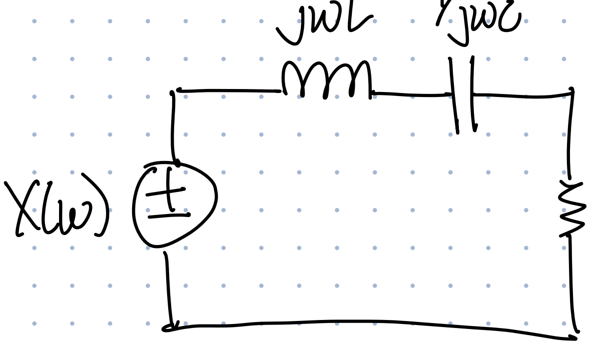
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \left(\frac{8}{5\pi}\right)^2 \int_{-\frac{5}{2}}^{\frac{5}{2}} \cos^2\left(\frac{\pi t}{5}\right) dt$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$E = \left(\frac{8}{5\pi}\right)^2 \cdot 5 \cdot \frac{1}{2}$$



HW 10 Problem 4



$+$

$Y(\omega) = X(\omega) \cdot \frac{R}{Z_{eq}}$, $Z_{eq} = R + jF(\omega)$

$-$

$$\Rightarrow H(\omega) = \frac{R}{R + jF(\omega)}$$

(b) $|H(\omega)| = 1 \Rightarrow F(\omega) = 0 \Rightarrow \omega_0 \Rightarrow f_0 = \frac{\omega_0}{2\pi}$

(c) $|H(\omega)| = \frac{1}{\sqrt{2}} \Rightarrow \left|\frac{R}{R+jF}\right| = \frac{R}{\sqrt{\underbrace{R^2+F^2}_{2R^2}}} = \frac{1}{\sqrt{2}} = \frac{R}{\sqrt{2R^2}}$

$\Rightarrow F^2 = R^2 \Rightarrow F = \pm R$

$F(\omega) = \pm R$

There will be a quadratic equation with $\underline{4}$ solutions.
Only 2 are positive. These are ω_1 and ω_2.

$f_1 = \frac{\omega_1}{2\pi}$ $f_2 = \frac{\omega_2}{2\pi}$