Solutions to Examples for Discussion 11 (ECE 213, Spring 2024)

1. Which of the following functions are odd? ... even? ... neither?

(a) $10 \sin(2t)$ (b) $20\, t \sin(5t)$ (c) $30 \cos(6t)$ (d) $40\, t \cos(8t)$

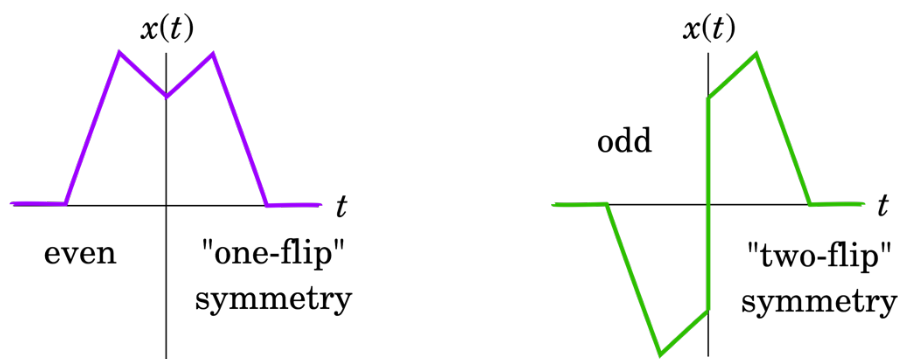
(e) $50 \sin|12t|$ (f) $50\, \mathrm{rect}(15\mathrm{t})$ (g) $60\, \mathrm{sinc}(20t)$ (h) $|e^{j3t}|$

ANSWERS:

Note the following...

- The functions t and $\sin(at)$ are odd functions, since $x(-t) = -x(t)$, i.e., "two-flip" symmetry.
- The functions a, $\mathrm{rect}(at)$, and $\cos(at)$ are even functions, since $x(-t) = x(t)$, i.e., "one-flip" symmetry.
- The product or ratio of an odd function and an even function is an odd function.
- The product or ratio of two even functions is an even function.
- ANY function with an even argument is an even function, since $x(-t) = x(t)$, e.g., $\exp(-t^2)$.
- The product or ratio of two odd functions is an even function.
- $\mathrm{sinc}(t) = \sin \pi t \,/\, \pi t$

For example, here are an even and an odd function...



Therefore, functions (a) and (d) are odd, and functions (b), (c), (e), (f), (g), and (h) are even.

Knowing the symmetry of a function will help simplify certain integrals. Symmetry is also useful when applying time-frequency duality.

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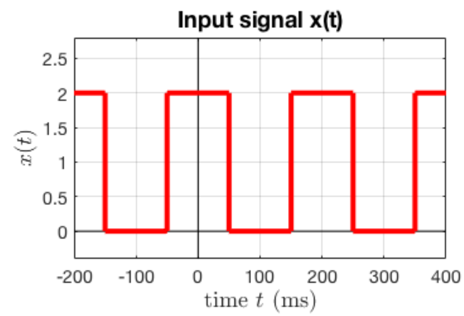
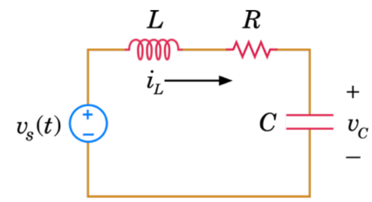
2. Consider this input signal $x(t)$.



   (a) Find the Fourier coefficients for the input, that is, write...

$$x(t) = X_0 + \sum_{n=1}^{\infty} \mathrm{Re}\{X_n\, e^{jn\omega_0 t}\}$$

Now, consider this RLC circuit, with R = 3Ω, L = 50mH, C = 10mF, and $v_s(t) = x(t)$, in V. Let the output $y(t) = i_L(t)$.



   (b) Find the Fourier coefficients for the output, that is, write...

$$y(t) = Y_0 + \sum_{n=1}^{\infty} \mathrm{Re}\{Y_n\, e^{jn\omega_0 t}\}$$

ANSWERS:

(a) X_0 is the average value of the signal. It's "on" for half the time, so the average value is 1. (Alternatively, find the area under the graph over one period and divide by T_0. In this case, the area is 2*100ms = 200ms, and the period T_0 = 200ms, so area/T_0 = 1.)

The input signal $x(t)$ is an even function, which means X_n is real. In other words, let $X_n = a_n - jb_n$, then rewrite the expression for $x(t)$ above...

$$x(t) = X_0 + \sum_{n=1}^{\infty} \mathrm{Re}\{(a_n - jb_n)\, e^{jn\omega_0 t}\} = X_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

There can be NO odd functions in the expansion of an even function, so ALL $b_n = 0$. (Likewise, if the signal is odd, there can be NO even functions in the expansion, so $a_0 = 0$ and ALL $a_n = 0$.)

The easiest way to solve for X_n is to double the integral over 1/2 the period, then "throw away" the imaginary part $(-jb_n)$, since integrating over the other half of the period would have canceled it anyway. Mathematically...

$$X_n = \mathrm{Re}\{\tilde{X}_n\} \qquad \text{[when } x(t) \text{ is even]}$$

... where...

$$\tilde{X}_n = 2\,\frac{2}{T_0} \int_{\text{half period}} x(t)\, e^{-jn\omega_0 t}\, dt$$

I choose to integrate from 0 to $T_0/2$. Inserting the given function, which is 0 from $T_0/4$ to $T_0/2$...

$$\tilde{X}_n = 2\,\frac{2}{T_0} \int_0^{T_0/4} 2\, e^{-jn\omega_0 t}\, dt = \frac{8}{T_0} \frac{1}{(-jn\omega_0)} \left[e^{-jn\omega_0 t}\right]_0^{T_0/4} = \frac{j8}{2\pi n} \left(e^{-\frac{jn\omega_0 T_0}{4}} - 1\right) = \frac{j4}{\pi n} \left(e^{-jn\pi/2} - 1\right)$$

The exponential factor can be written j^{-n}, since $j = e^{j\pi/2}$, so $\tilde{X}_n = 4/\pi n\,(j^{1-n} - j)$. Therefore, for even values of n, the exponential factor is ±1, so $\tilde{X}_n$ is purely imaginary (or 0), which means $X_n = 0$.

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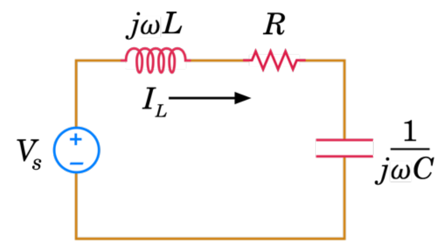
For odd values of n, the exponential factor is either j or $-j$, so the first term will be real and the second term will be imaginary. Specifically, using $(-1)^{-m} = (-1)^{m}$...

$$X_n = \mathrm{Re}\left\{\frac{4}{\pi n}\left(j^{1-n} - j\right)\right\} = \frac{4}{\pi n}\, j^{1-n} = \frac{4}{\pi n}\,(-1)^{(1-n)/2} = \frac{4}{\pi n}\,(-1)^{(n-1)/2} \quad [n = \text{odd only}]$$

NOTE: When $x(t)$ is an odd function, we can again compute $\tilde{X}_n$, but instead we throw away the real part, i.e., $X_n = \mathrm{Im}\{\tilde{X}_n\}$.

(b) This is similar to what was done in discussion last week, except now we need to write the current as a function of the frequency ω, so that it can be evaluated at ω_0, $2\omega_0$, $3\omega_0$, etc. The circuit in the phasor domain is shown to the right.



It will be MUCH easier to work symbolically, then simplify before inserting known values for R, L, and C...

$$I_L = \frac{V_s}{Z_{eq}} = \frac{V_s}{j\omega L + R + \frac{1}{j\omega C}} = \frac{V_s}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{V_s}{3 + j\left(\frac{\omega}{20} - \frac{100}{\omega}\right)}$$

The response Y_n at each particular frequency $n\omega_0$ is found by dividing X_n by the impedance Z_{eq} evaluated at $\omega = n\omega_0$.

In this case, we have $T_0 = 200\text{ms}$, so $\omega_0 = 10\pi$ rad/s. This is what the first few coefficients will be...

n	$\omega = n\omega_0$	X_n	Z_{eq}	$Y_n = X_n \,/\, Z_{eq}$
0	0	1	∞	0
1	10π	$\frac{4}{\pi}$	$3 + j\left(\frac{\pi}{2} - \frac{10}{\pi}\right)$	$0.329 + j\,0.177$
3	30π	$-\frac{4}{3\pi}$	$3 + j\left(\frac{3\pi}{2} - \frac{10}{3\pi}\right)$	$-0.056 + j\,0.0694$
5	50π	$\frac{4}{5\pi}$	$3 + j\left(\frac{5\pi}{2} - \frac{10}{5\pi}\right)$	$0.0125 - j\,0.03$

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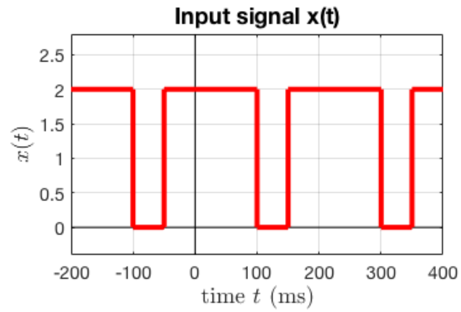
3. Consider a different input signal $x(t)$.
   (a) Find the Fourier coefficients for the input.
   (b) Using the same output as Problem 2, find the corresponding Fourier coefficients.



ANSWERS:

(a) X_0 is (again) the average value of the function, in this case, it's "on" for 3/4 of the time, so $X_0 = 3/4\ (2) = 1.5$.

X_n is found using…

$$X_n = \frac{2}{T_0}\int_{\text{one period}} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{2}{T_0}\int_{-T_0/4}^{3T_0/4} x(t)\, e^{-jn\omega_0 t}\, dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/2} 2\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{2}{T_0}\, 2\frac{1}{-jn\omega_0}\left[e^{-jn\omega_0 t}\right]_{-T_0/4}^{T_0/2} = \frac{j4}{n\omega_0 T_0}\left(e^{-jn\omega_0 T_0/2} - e^{jn\omega_0 T_0/4}\right)$$

Inserting $\omega_0 T_0 = 2\pi$…

$$X_n = \frac{j4}{n2\pi}\left(e^{-jn\pi} - e^{jn\pi/2}\right) = \frac{j2}{n\pi}\left[(-1)^n - j^n\right]$$

… since $e^{-j\pi} = 1$ and $e^{j\pi/2} = j$.

(b) From question 2…

$$I_L = \frac{V_s}{Z_{eq}} = \frac{V_s}{3 + j\left(\frac{\omega}{20} - \frac{100}{\omega}\right)}$$

The response Y_n at each particular frequency $n\omega_0$ is again found by dividing X_n by the impedance Z_{eq} evaluated at $\omega = n\omega_0$. Also as before, we have $T_0 = 200\text{ms}$, so $\omega_0 = 10\pi$ rad/s. This is what the first few coefficients will be…

n	$\omega = n\omega_0$	X_n	Z_{eq}	$Y_n = X_n\ /\ Z_{eq}$
0	0	1.5	∞	0
1	10π	$\frac{2}{\pi}\,(1-j)$	$3 + j\left(\frac{\pi}{2} - \frac{10}{\pi}\right)$	$0.253 + j\,0.0762$
2	20π	$\frac{j2}{\pi}$	$3 + j\left(\frac{2\pi}{2} - \frac{10}{2\pi}\right)$	$-0.0865 + j\,0.167$
3	30π	$\frac{2}{3\pi}\,(-1-j)$	$3 + j\left(\frac{3\pi}{2} - \frac{10}{3\pi}\right)$	$-0.0632 - j\,0.0062$
4	40π	0	$3 + j\left(\frac{4\pi}{2} - \frac{10}{4\pi}\right)$	0

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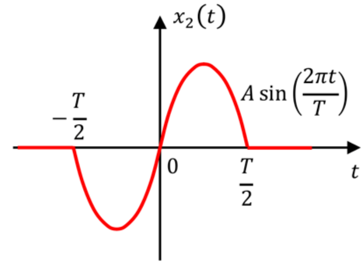
4. Determine the Fourier transform for the waveform shown to the right.



ANSWER:

The Fourier transform is…

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Inserting the given function…

$$X(\omega) = \int_{-T/2}^{T/2} A\sin\left(\frac{2\pi t}{T}\right)\, e^{-j\omega t}\, dt$$

Using a table of integrals…

$$\int \sin(ax)\; e^{bx}\, dx = \frac{e^{bx}}{a^2+b^2}\,[b\sin ax - a\cos ax] + C$$

Therefore, with $a = 2\pi/T$ and $b = -j\omega$, the result is…

$$X(\omega) = A\left[\frac{e^{-j\omega t}}{(2\pi/T)^2 + (-j\omega)^2}\left\{-j\omega\sin\left(\frac{2\pi t}{T}\right) - \frac{2\pi}{T}\cos\left(\frac{2\pi t}{T}\right)\right\}\right]_{-T/2}^{T/2}$$

$$= A\left[\frac{1}{(2\pi/T)^2 - \omega^2}\left(e^{-j\omega T/2}\frac{2\pi}{T} - e^{j\omega T/2}\frac{2\pi}{T}\right)\right]$$

$$= A\,\frac{2\pi/T}{(2\pi/T)^2 - \omega^2}\,(-j2)\sin\left(\frac{\omega T}{2}\right) = -\,j\pi AT\,\frac{\sin(\omega T/2)}{\pi^2 - (\omega T/2)^2}$$

… where we have multiplied numerator and denominator by $(T/2)^2$ to reach the final form.

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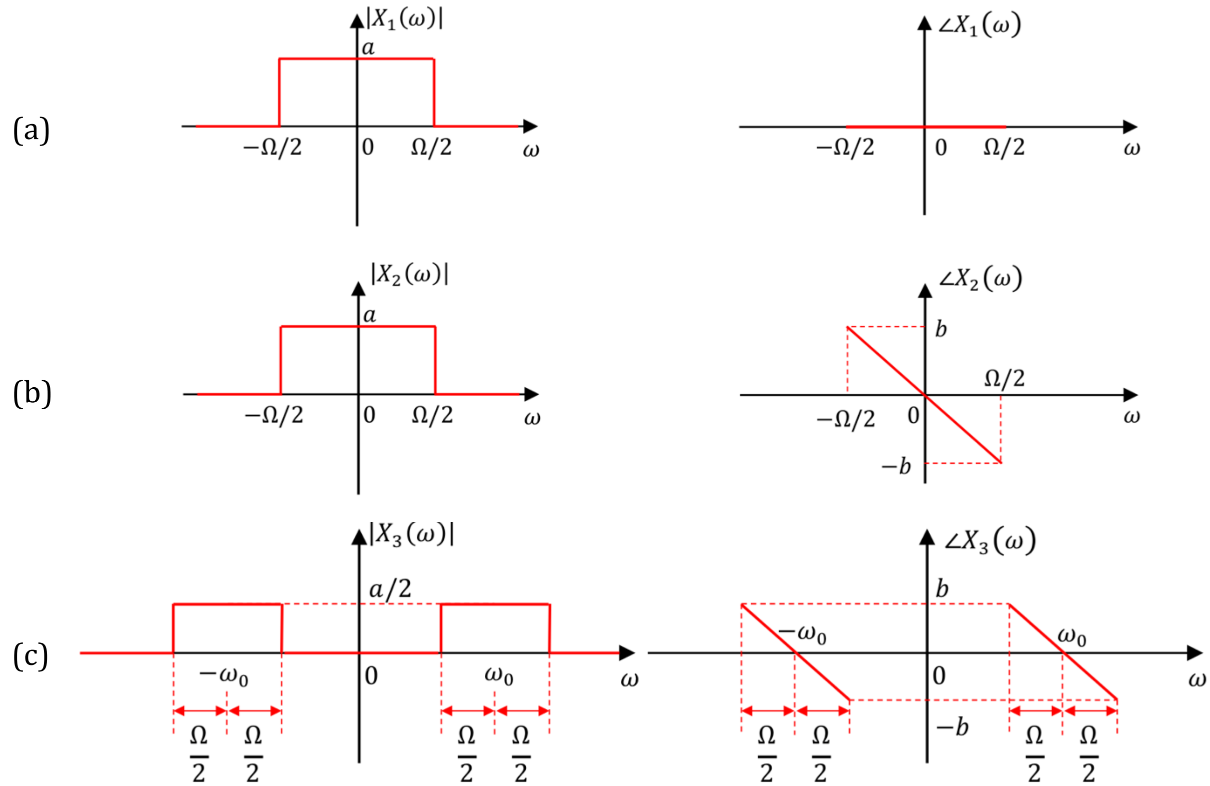
5. For each of the following Fourier transforms, find the inverse Fourier transform.



ANSWERS:

(a) Using time-frequency duality, we know that…

$$F(t) \Leftrightarrow 2\pi\, f(-\omega)$$

This means we can use the table of Fourier transform pairs in both directions! In other words, we can use the Fourier transform of the rectangle function to find the inverse transform of rectangle function. From the lecture…

$$\mathrm{rect}(t/T) \Leftrightarrow T\ \mathrm{sinc}(\omega T/2\pi)$$

Therefore, for $X_1(\omega) = a\ \mathrm{rect}(\omega/\Omega)$…

$$x_1(t) = \frac{1}{2\pi}\, a\Omega\ \mathrm{sinc}\left(\frac{\Omega t}{2\pi}\right)$$

… since $\mathrm{rect}(\omega/\Omega) = \mathrm{rect}(-\omega/\Omega)$, because it's an even function.

[continued]

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(b) This is the same as (a), except there is a phase. Specifically, the given function is…

$$X_2(\omega) = a\ \mathrm{rect}(\omega/\Omega)\ e^{-2jb\omega/\Omega}$$

… because the equation for the phase is $\angle X_2(\omega) = -2b\omega/\Omega$, since it's a straight line with an intercept of 0 and a slope of $-2b/\Omega$.

For this, we can use the time-shift property of the Fourier transform…

$$x(t - t_0) \Leftrightarrow e^{-j\omega t_0}\ X(\omega)$$

Therefore, using $t_0 = 2b/\Omega$…

$$x_2(t) = x_1(t - t_0) = \frac{1}{2\pi}\ a\Omega\ \mathrm{sinc}\left(\frac{\Omega}{2\pi}\left(t - \frac{2b}{\Omega}\right)\right) = \frac{a\Omega}{2\pi}\ \mathrm{sinc}\left(\frac{\Omega t}{2\pi} - b\right)$$

(c) We recognize that X_3 can be rewritten using X_2…

$$X_3(\omega) = \frac{1}{2}[X_2(\omega + \omega_0) + X_2(\omega - \omega_0)]$$

For this, we can use the modulation property of the Fourier transform…

$$x(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$$

Therefore…

$$x_3(t) = x_2(t)\cos(\omega_0 t) = \frac{a\Omega}{2\pi}\ \mathrm{sinc}\left(\frac{\Omega t}{2\pi} - b\right)\ \cos(\omega_0 t)$$