First-Order Differential Equations

We have three methods of solving first order Differential Equations:

1. **Integrating Factors for First-Order Linear Differential Equations**

   A first-order differential equation is called **linear** if it can be written as:

   $y'(t) + p(t)y = g(t)$ This is called standard linear form

   If $\mu(t) = e^{\left(\int p(t)dt\right)}$ then the solution is:

   $$y(t) = \frac{1}{\mu} \cdot \int \mu(t) \cdot g(t)dt$$

2. **Solving Separable First-Order Differential Equations**

   A first-order differential equation is called **separable** if it can be written as:

   $$N(y)\frac{dy}{dx} = M(x)$$

   If we have a separable differential equation, we can solve by integrating both sides with respect to x:

   $$\int N(y)dy = \int N(y)\frac{dy}{dx}dx = \int M(x)dx$$

   Note: Often times we cannot write our solutions to non-linear, separable equations explicitly in y. Thus, we often need to leave them as implicit solutions.

   Important Example: $\frac{dy}{dx} = ay$ where a is a constant has solution $y = ce^{ax}$

3. **Solving Exact First-Order Differential Equations**

   A differential equation of the form:

   $$M(x,y) + N(x,y) \cdot \frac{dy}{dx} = 0$$

   such that $M(x,y) = \psi_x(x,y)$ and $N(x,y) = \psi_y(x,y)$ for some function $\psi(x,y)$ is called an **exact differential equation**. The implicit solution is $\psi(x,y) = c$

   A differential equation of the form $M(x,y) + N(x,y) \cdot \frac{dy}{dx} = 0$ is exact if and only if $M_y = N_x$

   If the differential is exact, then we can find $\psi(x,y)$ by integrating $M(x,y)$ with respect to x to get $\psi(x,y)$ up to a function of y.

   That is $\psi(x,y) = \int Mdx + h(y)$

   Taking the partial derivative of this resulting function with respect to y and setting it equal to $N(x,y)$ to get:

   $N(x,y) = \psi_y = \frac{\delta}{\delta y}\left(\int Mdx\right) + h'(y)$

   This will determine $h'(y)$, which leads to $h(y)$ by integrating, and thus gives $\psi(x,y) = \int Mdx + h(y)$ (from above).

First-Order Autonomous Differential Equations and their Equilibrium Solutions

If a differential equation can be written as:

$$\frac{dy}{dt} = f(y)$$

then it is called **autonomous**

Autonomous differential equations are separable, and can be solved using our methods of separable differential equations. However, a lot of information can be found from autonomous differential equations analytically.

Equilibrium Solutions occur when $\frac{dy}{dt} = 0$

The long-term behavior of autonomous differential equations can be determined by the equilibrium solutions.

If solutions that start near an equilibrium solutions $y = K$ tend toward this equilibrium solution, we call it a **asymptotically stable solution**.

If solutions that start near an equilibrium solutions $y = K$ tend away from this equilibrium solution, we call it an **unstable solution**.

A **Phase Line** can be drawn to show the equilibrium solutions and the direction of $y(t)$ between each eq. sol.

Second-Order Differential Equations

1. Second-Order differential equations of the *Homogeneous* form:

$$ay'' + by' + cy = 0$$

have an associated **characteristic equation** $ar^2 + br + c = 0$ such that if γ is a solution of the characteristic equation then $y = e^{\gamma t}$ is a solution to the differential equation.

If y_1 and y_2 are solutions to a linear second-order differential equation of the form:

$$y'' + p(t)y' + q(t)y = 0$$

then $y = c_1 y_1(t) + c_2 y_2(t)$ is also a solution.

Moreover, if the Wronskian $W[y_1, y_2](t_o) = y_1(t_o)y_2'(t_o) - y_1'(t_o)y_2(t_o) \neq 0$ at the initial point $t = t_o$ then $y = c_1 y_1(t) + c_2 y_2(t)$ is the general solution.

Since the characteristic equation is a quadratic equation, it can have 2 distinct real roots, complex conjugate roots, or 1 repeated real root.

Case 1: The solutions to the characteristic equation $ar^2 + br + c = 0$ are distinct, real roots $r_1 \neq r_2$
The general solution of the differential equation: $ay'' + by' + cy = 0$ is: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Case 2: The solutions to the characteristic equation $ar^2 + br + c = 0$ are complex roots $r_{1,2} = \gamma \pm i\mu$
The general solution of the diff eq: $ay'' + by' + cy = 0$ is: $y(t) = c_1 e^{\gamma t} \cdot cos(\mu t) + c_2 e^{\gamma t} \cdot sin(\mu t)$

Case 3: The solutions to the characteristic equation $ar^2 + br + c = 0$ are a repeated, real root γ
The general solution of the differential equation: $ay'' + by' + cy = 0$ is: $y(t) = c_1 e^{\gamma t} + c_2 t e^{\gamma t}$