Math 331 Exam 1 Answer Sheet

March 2020

Question 1: Consider the differential equation

$$2xy + 1 + (x^2 + 1)\frac{dy}{dx} = 0$$

(a) (5 points) Test the exactness of the differential equation.

$$M(x, y) = 2xy + 1 \quad N(x, y) = (x^2 + 1)$$

$$M_y = 2x \quad N_x = 2x = M_y$$

so the equation is exact.
(b) (10 points) Find the general solution of the differential equation.
We want to find a function F(x,y) such that $F_x = M$, $F_y = N$. So then

$$\int F_x dx = F - h(y) = \int M dx = x^2 y + x$$

Where $h(y)$ is some unknown function of y. So then $F = x^2 y + x + h(y)$ and

$$F_y = x^2 + h'(y) = N = x^2 + 1$$

Thus $h'(y) = 1$ and $h(y) = y + C$. So the general solution is given by the integral equation

$$x^2 y + x + y + C = 0 \Leftrightarrow y = \frac{C - x}{x^2 + 1}$$

Question 2: Find the solution to the initial value problem

$$\frac{dy}{dx} = -6xy^2 + 2y^2$$

Rearranging the equation we get $\frac{dy}{dx} = y^2(2 - 6x)$ so by separation of variables we have

$$\frac{dy}{y^2} = (2 - 6x)dx, \quad \frac{-1}{y} = 2x - 3x^2 + C$$

Thus $y = \frac{1}{3x^2 - 2x + C}$ is the general solution. To solve the initial value problem note that $y(0) = 1 = \frac{1}{C}$ so $C = 1$. Thus the final solution is

$$y = \frac{1}{3x^2 - 2x + 1}$$

Question 3: Find the solution to the Initial Value Problem:

$$y'' + 3y' - 10y = 0 \quad y(0) = 3, \quad y'(0) = 5$$

The characteristic equation for the problem is $r^2 + 3r - 10 = 0$, which factors as $(r+5)(r-2) = 0$ so $r = 2$, $r = -5$ are the roots. Thus the general solution is $y = C_1 e^{2t} + C_2 e^{-5t}$. Note then that

$$y' = 2C_1 e^{2t} - 5C_2 e^{-5t}$$

$$y(0) = 3 = C_1 + C_2, \quad y'(0) = 5 = 2C_1 - 5C_2$$

Solving these equations we get that $C_1 = \frac{20}{7}$ $C_2 = \frac{1}{7}$. So the solution to the IVP is

$$y = \frac{20}{7} e^{2t} + \frac{1}{7} e^{-5t}$$

Question 4: Find the solution, $y(t)$, to the differential equation

$$y'' + \frac{y'}{t} = 3 + t$$

using the substitution $v(t) = y'(t)$. After the substitution the equation becomes:

$$v' + \frac{v}{t} = 3 + t$$

So this is a first order linear equation. Thus, we apply method of integrating factors.

$$\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln(t)} = t$$

$$v(t) = \frac{1}{t} \int (3+t) t dt = \frac{1}{t} \left(\frac{3}{2} t^2 + \frac{1}{3} t^3 + C_1 \right) = \frac{3}{2} t + \frac{1}{3} t^2 + \frac{C_1}{t}$$

So then, using the initial substitution we get

$$\int y' dt = y = \int v(t) dt = \frac{3}{4} t^2 + \frac{1}{9} t^3 + C_1 \ln(t) + C_2$$

Question 5: A population colony of duck billed platypi in Australia can be modelled by the logistic population model

$$\frac{dP}{dt} = 4P(1 - \frac{P}{2})$$

Here, P is given in hundreds of duck billed platypi and t is measured in years. (a) (10 points) Draw a phase line for this autonomous differential equation, and classifcy all equilibria as stable or unstable.

The equilibria are where $P' = 0$, so where $4P(1 - \frac{P}{2}) = 0$ This occurs at $P_1 = 0$ and at $P_2 = 2$. $P\prime(1) = 2 > 0$ and $P'(-1) = -4(\frac{3}{2}) < 0$ so the equilibrium at P_1 is unstable as P decreases when below it and increases when above it. $P'(3) = 12(-\frac{1}{2}) < 0$ so the equilibrium at P_2 is stable as P is decreasing above it and increasing below it.

The phase line should simply show arrows pointing away from 0 to the left and right, and an arrow pointing towards 2 from the right.

(b) (10 points) Sketch the graphs of the equilibrium solutions as well as the solutions given the starting populations $P_1(0) = 1, P_2(0) = 5$. Each graph should clearl show the solution's initial condition and long term behavior as $t \to \infty$.

The equilibrium lines are simply horizontal lines, and both starting conditions give lines which asymptotically approach the line $P = 2$. The exact shape of the lines is not important as long as the basic idea is there.

Question 6: A 5 gallon vat is full of pure water. At time $t = 0$ salt water is added to the vat through a pipe carrying water at a rate of 2 gallons per minute and a concentration of $\frac{1}{4}$ a pound per gallon. Water drains out of the vat at a rate of 2 gallon per minute, so that the level of the vat is always5 gallons. Assume that the salt is always evenly mixed throughout the vat. Let $S(t)$ denote the amount of salt in the vat at time t, and let t be measured in minutes

(a) (10 points) Set up the differential equation and initial condition for $\frac{dS}{dt}$ for the situation above.

$$\frac{dS}{dt} = 2(\frac{1}{4}) - 2\frac{S}{5}$$

(b) (5 points) Find $\lim_{t\to\infty} S(t)$. Justify your answer by classifying the equilibrium point.

Note that $\frac{dS}{dt} = 0$ only when $\frac{2}{5}S = \frac{1}{2}$. So the only equilibrium point is at $S = \frac{5}{4}$. $s'(0) = \frac{1}{2}$ and $S'(2) = \frac{1}{2} - \frac{4}{5} < 0$. Thus $S = \frac{5}{4}$ is a stable equilibrium point.

Because $S = \frac{5}{4}$ is the only equilibrium point, that means that

$$\lim_{t\to\infty} S(t) = \frac{5}{4}$$

For any starting condition.