$P = [x_P, y_P]$	Pixel Coordinate
$W = [\lambda, \phi, \rho]$	Geographic Coordinate
$U = [X_u, Y_u, Z_u]$	UTM Coordinate
$I = [W_I, H_I]$	Image Size (Width and Height)
N	Number of Ground-Control-Points

Our image has the following

4 Corner Transforms

First, determine the interpolation values t_x and t_y.

$$t_x = \frac{x_P}{W_I} \tag{1}$$

$$t_y = \frac{y_P}{H_I} \tag{2}$$

Next, apply the horizontal interpolation against the top and bottom rows.

$$P_{\texttt{utm}_1} = \begin{bmatrix} U_{x_{\text{TL}}} \\ U_{y_{\text{TL}}} \\ U_{z_{\text{TL}}} \end{bmatrix} \cdot (1 - t_x) + \begin{bmatrix} U_{x_{\text{TR}}} \\ U_{y_{\text{TR}}} \\ U_{z_{\text{TR}}} \end{bmatrix} \cdot t_x \tag{3}$$

$$P_{\texttt{utm}_2} = \begin{bmatrix} U_{x_{\text{BL}}} \\ U_{y_{\text{BL}}} \\ U_{z_{\text{BL}}} \end{bmatrix} \cdot (1 - t_x) + \begin{bmatrix} U_{x_{\text{BR}}} \\ U_{y_{\text{BR}}} \\ U_{z_{\text{BR}}} \end{bmatrix} \cdot t_x \tag{4}$$

Compute the vertical component.

$$P_{\texttt{utm}} = P_{\texttt{utm}_1} \cdot (1 - t_y) + P_{\texttt{utm}_2} \cdot (t_y) \tag{5}$$

GDAL Geo-Transforms

Given Geo-Transform parameters:

$$\bar{T} = (T_1, T_2, T_3, T_4, T_5, T_6) \tag{6}$$

Solve for the geographic coordinates (λ, ϕ)

$$\begin{bmatrix} \lambda, \\ \phi \\ 1 \end{bmatrix} = \begin{bmatrix} T_1, & T_2, & T_5 \\ T_3, & T_4, & T_6 \\ 0, & 0, & 1 \end{bmatrix} \cdot \begin{bmatrix} x_P, \\ y_P, \\ 1 \end{bmatrix} \tag{7}$$

Solving for a Geo-Transform
Representing our transform as a set of linear equations...
Longitude

$$\lambda = (T_1 \cdot x_P) + (T_2 \cdot y_p) + (T_5 \cdot 1) \tag{8}$$

Latitude

$$\phi = (T_3 \cdot x_P) + (T_4 \cdot y_p) + (T_6 \cdot 1) \tag{9}$$

Reconfigure to solve for Coefficients

$$\begin{aligned} A \cdot x_\lambda &= B \\ A \cdot x_\phi &= B \end{aligned} \tag{10}$$

Where

$$\begin{bmatrix} x_{P1} & y_{P1} & 1 \\ x_{P2} & y_{P2} & 1 \\ \cdots & \cdots & 1 \\ x_{PN} & y_{PN} & 1 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_N \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} x_{P1} & y_{P1} & 1 \\ x_{P2} & y_{P2} & 1 \\ \cdots & \cdots & 1 \\ x_{PN} & y_{PN} & 1 \end{bmatrix} \cdot \begin{bmatrix} T_3 \\ T_4 \\ T_6 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_N \end{bmatrix} \tag{12}$$

Solving

$$A^+ \cdot B = x \tag{13}$$

Solving Rational Polynomial Coefficients

The weights used in this analysis is are the size of the number of terms. They are initialized to the Identity matrix.

$$\bar{W} = I = \begin{bmatrix} w_1 & 0 & 0 & \cdots & 0 \\ 0 & w_2 & 0 & \cdots & 0 \\ 0 & 0 & w_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & w_n \end{bmatrix} \tag{14}$$

Establish the set of linear equations, this is an N x 40 matrix, with N defined as the number of Ground-Control-Points. X represents either the normalized x or y pixel values, depending on which set of coefficients are being solved.

$$f_{syseq}\left(X \rightarrow \{P_x, P_y\}\right) = \begin{bmatrix}
1_i, & 1, & \cdots, & 1 \\
L_i, & L_{(i+1)}, & \cdots, & L_n \\
P_i, & P_{(i+1)}, & \cdots, & P_n \\
H_i, & H_{(i+1)}, & \cdots, & H_n \\
L_i \cdot P_i, & L_{(i+1)} \cdot P_{(i+1)}, & \cdots, & L_n \cdot P_n \\
L_i \cdot H_i, & L_{(i+1)} \cdot H_{(i+1)}, & \cdots, & L_n \cdot H_n \\
P_i \cdot H_i, & P_{(i+1)} \cdot H_{(i+1)}, & \cdots, & P_n \cdot H_n \\
{L_i}^2, & {L_{(i+1)}}^2, & \cdots, & {L_n}^2 \\
{P_i}^2, & {P_{(i+1)}}^2, & \cdots, & {P_n}^2 \\
{H_i}^2, & {H_{(i+1)}}^2, & \cdots, & {H_n}^2 \\
L_i \cdot P_i \cdot H_i, & L_{(i+1)} \cdot P_{(i+1)} \cdot H_{(i+1)}, & \cdots, & L_n \cdot P_n \cdot H_n \\
{L_i}^3, & {L_{(i+1)}}^3, & \cdots, & {L_n}^3 \\
L_i \cdot {P_i}^2, & L_{(i+1)} \cdot {P_{(i+1)}}^2, & \cdots, & L_n \cdot {P_n}^2 \\
L_i \cdot {H_i}^2, & L_{(i+1)} \cdot {H_{(i+1)}}^2, & \cdots, & L_n \cdot {H_n}^2 \\
{L_i}^2 \cdot P_i, & {L_{(i+1)}}^2 \cdot P_{(i+1)}, & \cdots, & {L_n}^2 \cdot P_n \\
{P_i}^3, & {P_{(i+1)}}^3, & \cdots, & {P_n}^3 \\
P_i \cdot {H_i}^2, & P_{(i+1)} \cdot {H_{i+1}}^2, & \cdots, & P_n \cdot {H_n}^2 \\
{L_i}^2 \cdot H_i, & {L_{(i+1)}}^2 \cdot H_{(i+1)}, & \cdots, & {L_n}^2 \cdot H_n \\
{P_i}^2 \cdot H_i, & {P_{(i+1)}}^2 \cdot H_{(i+1)}, & \cdots, & {P_n}^2 \cdot H_n \\
{H_i}^3, & {H_{(i+1)}}^3, & \cdots, & {H_n}^3 \\
-X_i \cdot L_i & -X_{(i+1)} \cdot L_{(i+1)} & \cdots, & -X_i \cdot L_n \\
-X_i \cdot P_i & -X_{(i+1)} \cdot P_{(i+1)} & \cdots, & -X_i \cdot P_n \\
-X_i \cdot H_i & -X_{(i+1)} \cdot H_{(i+1)} & \cdots, & -X_i \cdot H_n \\
-X_i \cdot L_i \cdot P_i & -X_{(i+1)} \cdot L_{(i+1)} \cdot P_{(i+1)} & \cdots, & -X_i \cdot L_n \cdot P_n \\
-X_i \cdot L_i \cdot H_i & -X_{(i+1)} \cdot L_{(i+1)} \cdot H_{(i+1)} & \cdots, & -X_i \cdot L_n \cdot H_n \\
-X_i \cdot P_i \cdot H_i & -X_{(i+1)} \cdot P_{(i+1)} \cdot H_{(i+1)} & \cdots, & -X_i \cdot P_n \cdot H_n \\
-X_i \cdot {L_i}^2 & -X_{(i+1)} \cdot {L_{(i+1)}}^2 & \cdots, & -X_i \cdot {L_n}^2 \\
-X_i \cdot {P_i}^2 & -X_{(i+1)} \cdot {P_{(i+1)}}^2 & \cdots, & -X_i \cdot {P_n}^2 \\
-X_i \cdot {H_i}^2 & -X_{(i+1)} \cdot {H_{(i+1)}}^2 & \cdots, & -X_i \cdot {H_n}^2 \\
-X_i \cdot L_i \cdot P_i \cdot H_i & -X_{(i+1)} \cdot L_{(i+1)} \cdot P_{(i+1)} \cdot H_{(i+1)} & \cdots, & -X_i \cdot L_n \cdot P_n \cdot H_n \\
-X_i \cdot {L_i}^3 & -X_{(i+1)} \cdot {L_{(i+1)}}^3 & \cdots, & -X_i \cdot {L_n}^3 \\
-X_i \cdot L_i \cdot {P_i}^2 & -X_{(i+1)} \cdot L_{(i+1)} \cdot {P_{(i+1)}}^2 & \cdots, & -X_i \cdot L_n \cdot {P_n}^2 \\
-X_i \cdot L_i \cdot {H_i}^2 & -X_{(i+1)} \cdot L_{(i+1)} \cdot {H_{(i+1)}}^2 & \cdots, & -X_i \cdot L_n \cdot {H_n}^2 \\
-X_i \cdot {L_i}^2 \cdot P_i & -X_{(i+1)} \cdot {L_{(i+1)}}^2 \cdot P_{(i+1)} & \cdots, & -X_i \cdot {L_n}^2 \cdot P_n \\
-X_i \cdot {P_i}^3 & -X_{(i+1)} \cdot {P_{(i+1)}}^3 & \cdots, & -X_i \cdot {P_n}^3 \\
-X_i \cdot P_i \cdot {H_i}^2 & -X_{(i+1)} \cdot P_{(i+1)} \cdot {H_{(i+1)}}^2 & \cdots, & -X_i \cdot P_n \cdot {H_n}^2 \\
-X_i \cdot {L_i}^2 \cdot H_i & -X_{(i+1)} \cdot {L_{(i+1)}}^2 \cdot H_{(i+1)} & \cdots, & -X_i \cdot {L_n}^2 \cdot H_n \\
-X_i \cdot {P_i}^2 \cdot H_i & -X_{(i+1)} \cdot {P_{(i+1)}}^2 \cdot H_{(i+1)} & \cdots, & -X_i \cdot {P_n}^2 \cdot H_n \\
-X_i \cdot {H_i}^3 & -X_{(i+1)} \cdot {H_{(i+1)}}^3 & \cdots, & -X_i \cdot {H_n}^3
\end{bmatrix} \tag{15}$$