INFO-F420 - The Art Gallery Problem

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1 Introduction

In this report, we cover the Art Gallery Problem and how it has been applied/used in our project. From defining the problem and its complexity class(es) belonging to implementing algorithms to illustrate the problem and its solutions, this paper is a summary of all our researches as well as our experiments and code.

2 The Art Gallery Problem

First, let us define what is the Art Gallery Problem, its real-life applications and why is it a such interesting problem.

2.1 Defining the problem

Let P be a polygon and $p, q \in P$, then p sees q (and conversely q sees p) if the segment pq is fully contained in P [1, p. 4]. A set of points $G \subseteq P$ is a guard set of the polygon P only if $\forall\ p \in P,\ \exists\ g \in G : g\ guards\ P$. In other words, every point of the polygon must be seen by at least one point of the guard set [1, p. 4].

A guard set of a polygon P is minimal (or optimal) if its cardinality is the smallest possible for a guard set of P [1, p. 4].

The goal of the Art Gallery Problem is, from an integer k and a polygon P, to decide whether the polygon P has a guard set of cardinality k [1, p. 4].

2.2 Real-life applications

This problem has many applications. Starting with the problem itself, that is, to place guards or security cameras to cover a place, such as a museum, a bank or a prison, etc. But where its use is most widespread, is in video games. In particular in the programming of artificial intelligence, 3D rendering, lighting effects. And from this comes its use in robotics, special effects and simulation tools.

3 Art Gallery Problem is $\exists\mathbb{R}$-complete

In this section, we will cover the $\exists\mathbb{R}$ complexity class and explain why is the Art Gallery Problem linked to that complexity class.

3.1 $\exists\mathbb{R}$ class

TODO

3.2 Art Gallery is Existential-R complete

TODO

https://www.youtube.com/watch?v=fWrZmT4iN8k
https://arxiv.org/pdf/1704.06969.pdf - section 1.2

4 Well-known results

Since the art gallery problem is a popular and a well-studied problem, some results are already well-known and very useful

4.1 Chvátal's watchman theorem and Steve Fisk's short proof

An early result found by Chvátal results in the following theorem which allows us to obtain an upper bound for the number of guards needed. The proof is very short and was introduced by Steve Fisk :

Theorem 1. *Let P be a polygon with n vertices. Then, we can find a guard set $G \subseteq P$ of cardinality at most $\lfloor \frac{n}{3} \rfloor$ [2].*

Proof. Let P be a polygon with n vertices. First, triangulate P without adding any new vertex. Every triangulation has a 3-colouring with colors a, b and c. Now, we can define three subsets T_a, T_b, T_c of vertices of P where T_k is the set of vertices coloured with the colour k. It is clear that at least one of the three subsets T_a, T_b, T_c has a cardinality $\leq \lfloor \frac{n}{3} \rfloor$. Let us call that set T. We know every point q of the polygon P is in one of the triangles and that every triangle has a point of p of T on it. Since triangles are convex, we can deduce that $pq \subset P$. $\square$

5 Illustration & implementation

In order to realize an algorithm allowing us to apply the Chvátal's watchman theorem we followed the simple demonstration of Steve Fisk's. In his demonstration, he talks about a coloring of the triangulation of the polygon. To allow this step we used a simple $\mathcal{O}(n)$ algorithm [3] that colors the vertices of a triangulated simple polygon or outerplanar graph with three colors.

TODO

6 Glossary

TODO definition of all the things we use

- Polygon : A polygon is a plane geometric figure where the consecutive segments forming a cycle are the edges. And where the ends of each edge are the vertices of the polygon.

- Simple polygon : A simple polygon is a polygon in which the segments forming the cycle do not intersect. That means, two consecutive segments are touching each other only by their vertices and the non-consecutive segments are not touching each other.

7 Bibliography

[1] M. Abrahamsen, A. Adamaszek, and T. Miltzow. The art gallery problem is $\exists\smallsetminus$-complete. *CoRR*, abs/1704.06969, 2017.

[2] S. Fisk. A short proof of chvátal's watchman theorem. *Journal of Combinatorial Theory, Series B*, 24(3):374, 1978.

[3] A. Kooshesh and B. Moret. Three-coloring the vertices of a triangulated simple polygon. *Pattern Recognition*, 25:443, 04 1992.