INFO-F420 - The Art Gallery Problem

Kaplan Oruç & Gailly Vincent & Dudziak Thomas

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1 Introduction

In this report, we cover the Art Gallery Problem and how it has been applied/used in our project. From defining the problem and its complexity class(es) belonging to implementing algorithms to illustrate the problem and its solutions, this paper is a summary of all our researches as well as our experiments and code.

2 The Art Gallery Problem

First, let us define what is the Art Gallery Problem, its real-life applications and why is it a such interesting problem.

2.1 Defining the problem

Let P be a polygon and $p, q \in P$, then p sees q (and conversely q sees p) if the segment pq is fully contained in P [1, p. 4]. A set of points $G \subseteq P$ is a guard set of the polygon P only if $\forall\ p \in P,\ \exists\ g \in G : g\ \textit{guards}\ P$. In other words, every point of the polygon must be seen by at least one point of the guard set [1, p. 4].

A guard set of a polygon P is minimal (or optimal) if its cardinality is the smallest possible for a guard set of P [1, p. 4].

The goal of the Art Gallery Problem is, from an integer k and a polygon P, to decide whether the polygon P has a guard set of cardinality k [1, p. 4].

2.2 Real-life applications

This problem has many applications. Starting with the problem itself, that is, to place guards or security cameras to cover a place, such as a museum, a bank or a prison, etc. But where its use is most widespread, is in video games. In particular in the programming of artificial intelligence, 3D rendering, lighting effects. And from this comes its use in robotics, special effects and simulation tools.

3 Art Gallery Problem is $\exists\mathbb{R}$-complete

In this section, we will cover the $\exists\mathbb{R}$ complexity class and explain how is the Art Gallery Problem linked to that complexity class.

3.1 Existential Theory of Reals

The Existential Theory of Reals problem, often called ETR, is a computational problem where the goal is to determine if a formula ϕ, containing variables $x_1, ..., x_n$ where $x_i \in \mathbb{R}\ \forall i \in \{1, ..., n\}$ can be satisfied with a valuation of variables. The formula ϕ is not only composed of real variables, but is also composed of logic operations and other mathematical symbols. In short, the alphabet of ϕ is the following [1, p. 4] :

$$\{x_1, x_2, ..., x_n, \forall, \exists, \wedge, \vee, \neg, 0, 1, +, -, \cdot, (,), =, <, \leq\}.$$

In other words, does there exist a valuation for the variables $x_1, ..., x_n$ such that ϕ is satisfied :

$$\exists(x_1, ..., x_n) : \phi(x_1, ..., x_n)$$

3.2 $\exists\mathbb{R}$ class

Knowing the ETR problem, the $\exists\mathbb{R}$ class is straightforward. Indeed, the $\exists\mathbb{R}$ is simply the class of problems that can be reduced to ETR in polynomial time.

As proved by *Mikkel Abrahamsen, Anna Adamaszek and Tillmann Miltzow*, we know that the Art Gallery Problem belongs to $\exists\mathbb{R}$ [1, p. 8]. The proof being long and non-trivial, it is not provided here.

Theorem 1. *The Art Gallery Problem belongs to the $\exists\mathbb{R}$ class.*

The importance of the $\exists\mathbb{R}$ resides in the fact that we have the following relation where we do not know if the containments are strict yet :

$$\mathcal{NP} \subseteq \exists\mathbb{R} \subseteq PSPACE$$

3.3 Art Gallery is Existential-$\mathbb{R}$ complete

$\exists\mathbb{R}$-completeness for a given problem, say A, means that for every other problem in the Existential Theory of Reals, there is a reduction to A, in polynomial time.

The Art Gallery Problem is at least as hard as deciding whether a system of equations and inequalities over $\mathbb{R}$ has a satisfiable valuation [1, p. 4]. The latter is called ETR-INV :

Theorem 2. *Problem ETR-INV is $\exists\mathbb{R}$-complete [1, p. 16].*

After that, *Mikkel Abrahamsen, Anna Adamaszek and Tillmann Miltzow* have proven the following with a very long and non-trivial proof :

Theorem 3. *Let ϕ be an instance of ETR-INV. The polygon $\mathcal{P}(\phi)$ has corners at rational coordinates, which can be computed in polynomial time. Also, there exist constants $d_1, ..., d_n \in \mathbb{Q}$ such that for any $x := (x_1, ..., x_n) \in \mathbb{R}^n$, x is a solution to ϕ if and only if there exists a guard set G of cardinality $g(\phi)$ containing guards at all the positions $(x_1 + d_1, 0), ..., (x_n + d_n, 0-$ [1, p.56].*

In other words, there exists a reduction from ETR-INV to the Art Gallery Problem. Then we can establish the following :

Theorem 4. *The Art Gallery Problem is $\exists\mathbb{R}$-complete [1, p.57].*

Proof. Todo : Write the proof of it. □

4 Well-known results

Since the art gallery problem is a popular and a well-studied problem, some results are already well-known and very useful

4.1 Chvátal's watchman theorem and Steve Fisk's short proof

An early result found by Chvátal results in the following theorem which allows us to obtain an upper bound for the number of guards needed. The proof is very short and was introduced by Steve Fisk :

Theorem 5. *Let P be a polygon with n vertices. Then, we can find a guard set $G \subseteq P$ of cardinality at most $\lfloor \frac{n}{3} \rfloor$ [2].*

Proof. Let P be a polygon with n vertices. First, triangulate P without adding any new vertex. Every triangulation has a 3-colouring with colors a, b and c. Now, we can define three subsets T_a, T_b, T_c of vertices of P where T_k is the set of vertices coloured with the colour k. It is clear that at least one of the three subsets T_a, T_b, T_c has a cardinality $\leq \lfloor \frac{n}{3} \rfloor$. Let us call that set T. We know every point q of the polygon P is in one of the triangles and that every triangle has a point of p of T on it. Since triangles are convex, we can deduce that $pq \subset P$. □

4.2 The orthogonal art gallery theorem

Kahn, Klawe, and Kleitman have established that any orthogonal polygon is convexly quadrilateralizable [3]. The proof is quite long but is logical and is also explained in Rourke's book [5, p. 45]. Following this, it is possible to prove that $\lfloor \frac{n}{4} \rfloor$ guards are sufficient to cover an orthogonal polygon, a bit like Fisk's proof.

Theorem 6. *Let P be a orthogonal polygon with n vertices. Then, we can find a guard set $G \subseteq P$ of cardinality at most $\lfloor \frac{n}{4} \rfloor$ [5, p. 46].*

Proof. Let P be an orthogonal polygon and G the quadrilateralization of P. By adding the diagonals to each square, we end up with a planar G graph. Since every planar graph is 4-colorable and the polygon P is exclusively composed of convex squares. Each guard completely covers at least one square of the quadrilateralization. As in Chvátal's Proof, we can define four subsets and deduce that one of the four subsets has a cardinality $\leq \lfloor \frac{n}{4} \rfloor$. □

5 Using irrational coordinates for guards

TODO : Present results and their consequences.

6 Illustration & implementation

In order to realize an algorithm allowing us to apply the Chvátal's watchman theorem we followed the simple demonstration of Steve Fisk's. In his demonstration, he talks about a coloring of the triangulation of the polygon. To allow this step we used a simple $\mathcal{O}(n)$ algorithm [4] that colors the vertices of a triangulated simple polygon or outerplanar graph with three colors.

TODO : More illustrations & implementations... (Illustrate orthogonal art gallery theorem, irrational positions of guards, ...)

7 Glossary

- Polygon : A polygon is a plane geometric figure where the consecutive segments forming a cycle are the edges. And where the ends of each edge are the vertices of the polygon.

- Simple polygon : A simple polygon is a polygon in which the segments forming the cycle do not intersect. That means, two consecutive segments are touching each other only by their vertices and the non-consecutive segments are not touching each other.

- Orthogonal polygon : An orthogonal polygon is a simple polygon whose edges are aligned with a pair of perpendicular axes. This means that the angles in the polygon are either 90° or 270°.

- Problem reduction : Reducing a problem A to a problem B is simply the fact of transforming an instance of the problem A in an instance of the problem B.

8 Bibliography

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