Think and Answer

e-YRC#9

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Please answer all the questions given below. You are allowed to use figures or diagrams to support your answer. Since these questions test your understanding of the whole subject, please refrain from directly asking for answers on Piazza.

Section 1 - Simple Pendulum

Q1) Find the eigenvalues of Simple Pendulum at equilibrium point (0,0). Is the system stable or unstable at this point? (2)

To find Eigen values we have to find $|A - \lambda I|$ and solve for λ.

Matrix A at (0, 0) is given by, $A = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}$

$$|A - \lambda I| = |\begin{bmatrix} -\lambda & 1 \\ -\frac{g}{l} & -\lambda \end{bmatrix}|$$

$$\rightarrow \quad \lambda^2 = \frac{g}{l} \quad \rightarrow \quad \lambda = \pm i\sqrt{\frac{g}{l}}$$

Code - [eigen_value] = eig(A);

The system is stable at point (0, 0).

Q2) Can the Pendulum be balanced at an arbitrary point such as (2π/3, 0) using the Pole Placement or LQR controller? Why? Why Not? Justify your answer. (3)

We try to make a closed loop controller with overall eigen values on the left half of the complex plane irrespective of the eigen values of the open loop system, hence the system will be stable for any input hence the Pendulum can be balanced at an arbitrary point using both Pole Placement and LQR controller.

Proof :

Controllability is the ability to drive a state from any initial value to a final value in finite amount of time by providing a suitable input. If we have a system which is controllable, then we can place its eigenvalues anywhere in the left half plane by choosing appropriate gain matrix K.

For simple pendulum,

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \quad R = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} \frac{1}{ml^2} \\ 0 \end{bmatrix}$$

$$\text{So } R = \begin{bmatrix} 0 & \frac{1}{ml^2} \\ \frac{1}{(ml^2)} & 0 \end{bmatrix}$$

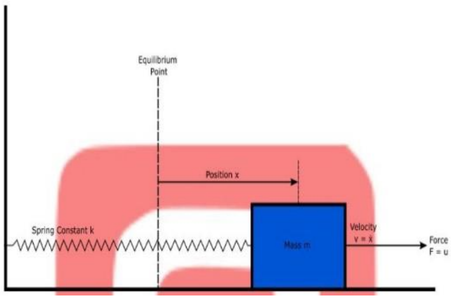
Since the rank of the matrix is two and so is the number of state variables, the system is controllable and therefore the system can be driven to any arbitrary point.

Section 2 - Mass Spring System

Q3) Derive the equations of Mass Spring system. (3)

Let m be the mass of the block, k be the spring constant, x and $\dot{x}$ be the displacement and velocity respectively.

We know that Langrangian L = K.E – P.E



K.E = $\frac{1}{2}m\dot{x}^2$

P.E = $\frac{1}{2}k\dot{x}^2$

L = $\frac{1}{2}m\dot{x}^2$ - $\frac{1}{2}k\dot{x}^2$

$$The\ \ Euler - Lagrange\ \ equation\ \ is\ \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F -- \quad (1)$$

$$\frac{d}{dt}\left(\ \frac{\partial L}{\partial \dot{x}}\ \right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x} \quad -- \ (2)$$

$$\frac{\partial L}{\partial x} = \ -kx \qquad\qquad -- \ (3)$$

$$from\ \ 2\ \ \&\ \ 3 \quad (1) \rightarrow \ \ m\ddot{x} + kx = F$$

$$\ddot{x} = -\frac{k}{m}x + \frac{F}{m}$$

So the states are given by, $x_1 = x \qquad x_2 = \dot{x}$

Derivative of state variables are given by, $\dot{x}_1 = \ x_2 \qquad \dot{x}_2 = \ddot{x} = -\frac{k}{m}x_1 + \frac{F}{m}$

Q4) Is the mass spring system a linear system or non-linear? Justify your answer. (1)

It is linear system since it can be represented in the standard form ẋ=Ax+Bu where x is the state variable and u is the input. The system is linear for all positions of x and no small regions is taken for the analysis. This is a linear system because all of its terms are single, first-degree variables with constant coefficients.

Q5) Can the mass spring system be driven to arbitrary state (0.8, 0) using pole placement controller? (Assuming 0.8 is the position and 0 is the velocity). (1)

Yes, pole placement controller can be used to drive the mass spring system to an arbitrary state since the state equations is linear for all values of the state variables. We try to make a closed loop controller with overall eigenvalues on the left half of the complex plane irrespective of the eigenvalues of the open loop system.

Proof:

For simple pendulum,

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad R = \ [\ B \quad AB]$$

$$AB = \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix}$$

So, $R = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{bmatrix}$

Since the rank of the matrix is two and so is the number of state variables, the system [...] system can be driven to any arbitrary point.

Section 3 - Simple Pulley

Q6) Under what conditions, will the system remain perfectly at rest? Justify your answer. (1)

For input = 0 and $m_1 = m_2$ there will be no net acceleration, so the system will be at rest.

Q7) How many equilibrium points does the system have? Are they stable or unstable? Justify your answer. (2)

For the simple pulley to be at equilibrium, T = 0

$$\dot{x}_1 = x_2 = 0 \;\; \& \;\; \dot{x}_2 = \frac{(m_1-m_2)g+\frac{T}{r}}{m_1+m_2} = 0 \rightarrow \; m_1 = m_2$$

Only condition for equilibrium is $m_1 = m_2$ which is irrespective of the position of the mass.

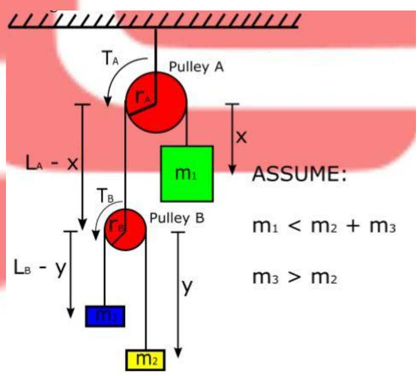
Its stable equilibrium,

$$proof \rightarrow A \;\; matrix = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Its eigenvalues are ± 0 which is stable.

Section 4 - Complex Pulley

Q8) Derive the equations of motion for the complex pulley system. (5)



Let v_1 be the velocity of mass m_1

v_2 be the velocity of mass m_1

v_3 be the velocity of mass m_3

Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$

$\rightarrow \dot{x}_1 = x_2, \quad \dot{x}_3 = x_4$

Assuming $m_1 < m_2 + m_3$ and $m_3 > m_2$

$v_1 = -\dot{x}$; $v_2 = \dot{x} - \dot{y}$; $v_3 = \dot{x} + \dot{y}$

K.E = $\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}-\dot{y})^2 + \frac{1}{2}m_3(\dot{x}+\dot{y})^2$

P.E = $-m_1gx - m_2g(L_A - x + y) - m_3g(L_A - x + L_B - y)$

We know that Lagrangian, L = K.E – P.E.

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}-\dot{y})^2 + \frac{1}{2}m_3(\dot{x}+\dot{y})^2 + m_1 g x + m_2 g(L_A - x + y) + m_3 g(L_A - x + L_B - y)$$

The $Euler - Lagrange$ equations for this system is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{T_A}{r_A} \quad -- \ (1) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \frac{T_B}{r_B} \quad -- \ (2)$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = m_1\dot{x} + m_2(\dot{x}-\dot{y}) + m_3(\dot{x}+\dot{y}) = (m_1+m_2+m_3)\dot{x} + (-m_2+m_3)\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_1+m_2+m_3)\ddot{x} + (-m_2+m_3)\ddot{y} \quad -- \ (3)$$

$$\frac{\partial L}{\partial x} = m_1 g - m_2 g - m_3 g \quad -- \ (4)$$

$$\frac{\partial L}{\partial \dot{y}} = -m_2(\dot{x}-\dot{y}) + m_3(\dot{x}+\dot{y}) = (-m_2+m_3)\dot{x} + (m_2+m_3)\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = (-m_2+m_3)\ddot{x} + (m_2+m_3)\ddot{y} \quad -- \ (5)$$

$$\frac{\partial L}{\partial y} = m_2 g - m_3 g \quad --(6)$$

$(3),(4)$ in (1) & $(5),(6)$ in $(2) \rightarrow$

$$(m_1+m_2+m_3)\ddot{x} + (-m_2+m_3)\ddot{y} = (m_1-m_2-m_3)g + \frac{T_A}{r_A} \quad -- \ (7)$$

$$(-m_2+m_3)\ddot{x} + (m_2+m_3)\ddot{y} = (m_2-m_3)g + \frac{T_B}{r_B} \quad -- \ (8)$$

Solving for $\ddot{x}$ & $\ddot{y}$ we get,

$$\boxed{\dot{x}_2 = \ddot{x} = \frac{(m_1m_2 + m_1m_3 - 4m_2m_3)g}{m_1m_2 + m_1m_3 + 4m_2m_3} + \frac{\left(\frac{T_A}{r_A}(m_2+m_3) - \frac{T_B}{r_B}(-m_2+m_3)\right)}{(m_1m_2 + m_1m_3 + 4m_2m_3)}} \quad -- \ (9)$$

$$\boxed{\dot{x}_4 = \ddot{y} = \frac{(2m_1m_3 - 2m_1m_2)g + \frac{T_A}{r_A}(-m_2+m_3) - \frac{T_B}{r_B}(m_1+m_2+m_3)}{(m_1m_2 + m_1m_3 + 4m_2m_3)}}$$

Q9) Derive the A and B matrices for the complex pulley system. Is the system linear or non linear? (4)

The system is non-linear since it doesn't obey superposition principle (it involves some constants which affects linearity).

From the equations derived above $\dot{x}_1 = x_2,\ \dot{x}_3 = x_4,\ \dot{x}_4\ \&\ \dot{x}_2\ refer\ (9)\ above$

$$A = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \frac{\partial \dot{x}_1}{\partial x_3} & \frac{\partial \dot{x}_1}{\partial x_4} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \frac{\partial \dot{x}_2}{\partial x_3} & \frac{\partial \dot{x}_2}{\partial x_4} \\ \frac{\partial \dot{x}_3}{\partial x_1} & \frac{\partial \dot{x}_3}{\partial x_2} & \frac{\partial \dot{x}_3}{\partial x_3} & \frac{\partial \dot{x}_3}{\partial x_4} \\ \frac{\partial \dot{x}_4}{\partial x_1} & \frac{\partial \dot{x}_4}{\partial x_2} & \frac{\partial \dot{x}_4}{\partial x_3} & \frac{\partial \dot{x}_4}{\partial x_4} \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial u_1} & \frac{\partial \dot{x}_1}{\partial u_2} \\ \frac{\partial \dot{x}_2}{\partial u_1} & \frac{\partial \dot{x}_2}{\partial u_2} \\ \frac{\partial \dot{x}_3}{\partial u_1} & \frac{\partial \dot{x}_3}{\partial u_2} \\ \frac{\partial \dot{x}_4}{\partial u_1} & \frac{\partial \dot{x}_4}{\partial u_2} \end{bmatrix} \rightarrow B = \frac{1}{m_1m_2+m_1m_3+4m_2m_3} \begin{bmatrix} 0 & 0 \\ \frac{m_2+m_3}{r_A} & \frac{m_2-m_3}{r_B} \\ 0 & 0 \\ \frac{m_2-m_3}{r_A} & \frac{m_1+m_2+m_3}{r_B} \end{bmatrix}$$

Q10) Under what conditions, will the system remain perfectly at rest? Justify your answer. (3)

When $m_1 = 2m_2 = 2m_3$

For rest condition,

$\dot{x},\ \dot{y},\ \ddot{x},\ \ddot{y},\ T_A, T_B\ = 0$

When this happens,

$(m_1m_2 + m_1m_3 - 4m_2m_3) = 0 - -(1)$

$(2m_1m_3 - 2m_1m_2) = 0 \qquad - -(2)$

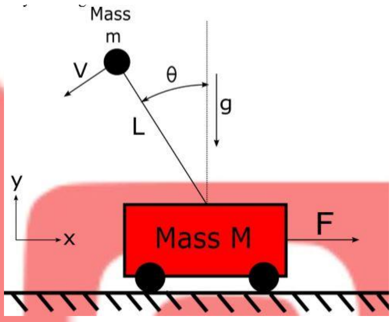
$from\ (2)\ m_2 = m_3 - -\ (3)$

$(2)\ in\ 1 \rightarrow\ m_1 = 2m_2$

Section 5 - Inverted Cart Pendulum

Q11) Derive the equations of motion for the inverted cart pendulum system. Is this system linear or non-linear? Why? (7)

This system is non-linear since the equations representing the dynamics of the system doesn't obey superposition principle, also from the equation 7 and 8 derived below we can see derivative of the state variables are sinusoidal functions of ϴ, we can't represent these type of equations in $A = Bx$ form .



The states are given by

$$x_1 = x \quad x_2 = \dot{x} \quad x_3 = \theta \quad x_4 = \dot{\theta}$$

Where x is position of the cart.

$\dot{x}$ is velocity of the cart.

θ is angular displacement of the pendulum from vertical.

$\dot{\theta}$ is the tangential angular velocity of the pendulum.

Let $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}$ be the derivative of state variable vector, where $\dot{x}_1 = x_2 \qquad \dot{x}_3 = x_4$

We know that Lagrangian L = K.E – P.E.

Let M - mass of the cart, m – mass of the pendulum, l = length of the pendulum.

Kinetic energy of the system = $\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{\theta}^2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{\theta}_x{}^2 + \dot{\theta}_y{}^2\right)$

$$. = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} - l\dot{\theta}\cos\theta)^2 + \frac{1}{2}m(-l\dot{\theta}\sin\theta)^2$$

$$= \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(-2\dot{x}l\dot{\theta}cos\theta + l^2\dot{\theta}^2)$$

Potential energy of the system = $-mgl\cos\theta$

Hence the Lagrangian is given by

$$L = \frac{1}{2}(m+M)\dot{x}^2 - m\dot{x}l\dot{\theta}cos\theta + \frac{1}{2}l^2\dot{\theta}^2 m + mgl\cos\theta \quad \text{--- (1)}$$

Equations that represent the dynamics of the system are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F \quad -- \ (2) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad -- \ (3)$$

$$(2) \ -> \quad \boxed{(m+M)\ddot{x} - ml\ddot{\theta}cos\theta + l\dot{\theta}^2 msin\theta = F} \quad -- \ (4)$$

$$(3) \ -> \quad \boxed{-m\ddot{x}\cos\theta + ml\ddot{\theta} + mg\sin\theta = 0} \quad -- \ (5)$$

$$from \ (5) \quad \ddot{\theta} = \frac{-g\sin\theta + \ddot{x}\cos\theta}{l} \quad -- \ (6)$$

$$(6) \ in \ (4) \rightarrow \ \dot{x}_2 = \ddot{x} = \frac{F - mg\sin\theta\cos\theta - ml\dot{\theta}^2 sin\theta}{\mathrm{M+m(1-\cos^2\theta)}} \quad -- \ (7)$$

$$(4) \ in \ (6) \rightarrow \ \dot{x}_4 = \ddot{\theta} = \frac{-(m+M)g\sin\theta + F\cos\theta - \frac{ml\dot{\theta}^2\sin 2\theta}{2}}{\mathrm{L(M+m(1-\cos^2\theta))}} \quad -- \ (8)$$

Q12) How many equilibrium points does the inverted cart pendulum system have? Categorize them as stable or unstable? (3)

There are two equilibrium points $i.(0 \ \ 0 \ \ 0 \ \ 0)$ $ii.(0 \ \ 0 \ \ \pi \ \ 0)$.

For the first point matrix $A - \lambda I$ is given by,

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & -\mathrm{m} * \frac{\mathrm{g}}{\mathrm{M}} & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -(\mathrm{m}+\mathrm{M})\frac{\mathrm{g}}{\mathrm{LM}} & -\lambda \end{bmatrix}$$

Its eigenvalues are,

$$\begin{bmatrix} 0 \\ -i\sqrt{\frac{(M+m)g}{LM}} \\ i\sqrt{\frac{(M+m)g}{LM}} \end{bmatrix}$$

For the second point matrix $A - \lambda I$ is given by,

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & -\mathrm{m} * \frac{\mathrm{g}}{\mathrm{M}} & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & (\mathrm{m}+\mathrm{M})\frac{\mathrm{g}}{\mathrm{LM}} & -\lambda \end{bmatrix}$$

Its eigenvalues are,

$$\begin{bmatrix} 0 \\ -\sqrt{\frac{(M+m)g}{LM}} \\ \sqrt{\frac{(M+m)g}{LM}} \end{bmatrix}$$

As shown above point $(0 \ \ 0 \ \ 0 \ \ 0)$ is stable equilibrium since its eigenvalues are on LHS of the S-plane, and $(0 \ \ 0 \ \ \pi \ \ 0)$ is an unstable equilibrium point since one its eigenvalues is on the RHS of the S-plane an also after a small change in the parameters around that point, the pendulum will not return to that point again unless there is an external force.