

Modeling Market Downside Volatility

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Abstract

We propose a new methodology for modeling and estimating time-varying downside risk and upside uncertainty in equity returns and for assessment of risk-return trade-off in financial markets. Using the salient features of the binormal distribution, we explicitly relate downside risk and upside uncertainty to conditional heteroscedasticity and asymmetry through binormal GARCH (BiN-GARCH) model. Based on S&P500 and international index returns, we find strong empirical support for existence of significant relative downside risk, robust positive relationship between the relative downside risk and conditional mode, and evidence for positive expected value for the market price of risk, hence; a positive risk-return trade-off in market index returns.

Keywords: Binormal distribution, Downside risk, Intertemporal CAPM, GARCH, Relative downside volatility, Risk-return trade-off, Upside uncertainty.

JEL Classification: C22; C51; G12; G15.

1 Introduction

The idea of a systematic trade-off between risk and returns is fundamental to the modern finance theory. Merton (1973) intertemporal capital asset pricing (ICAPM) theory, asserts that there exists a positive and linear relation between the conditional variance

and expected excess market returns. Ghysels et al. (2005) describe this relationship as the “first fundamental law of finance”. Yet, as Rossi and Timmermann (2009) point out, after more than two decades of research, there is little agreement regarding the basic properties of this relationship. Empirical studies concerning this proposal report conflicting results. Campbell (1987), Nelson (1991), and more recently Brandt and Kang (2004), find a significantly negative conditional relationship. Harvey (1989) and Glosten et al. (1993) find both a positive and a negative relation depending on the method used. On the other hand, French et al. (1987), Baillie and DeGennaro (1990), and Campbell and Hentschel (1992) find a positive but mostly insignificant relation between the conditional variance and the conditional expected returns. Ghysels et al. (2005) and Ludvigson and Ng (2007) find a positive and significant relationship in the U.S. data. Bollerslev and Zhou (2006) find unambiguously positive relationship between returns and implied volatility, but they find that the sign of the relationship between contemporaneous returns and realized volatility depends on underlying model parameters.

Two important underlying assumptions in empirical risk-return trade-off literature are: a) a constant market price of risk, and b) a symmetric conditional distribution for returns. While the majority of these studies assume that the price of risk is time-invariant, time-varying market price of risk is widely accepted in the term structure of interest rate literature; see Dai and Singleton (2002) and Duffee (2002). The expected value of the market price of risk needs to be positive to support positive risk-return trade-off in the market returns. In our study, the market price of risk, which is the slope coefficient in the regression of excess returns on conditional volatility, is proportional to conditional skewness. The associated coefficient of proportionality is determined by the parameters of the relationship between conditional mode and conditional volatilities in down and up markets. Our empirical results imply that, on average, the value of the price of risk is positive for the plurality of markets studied, which supports the positive risk-return trade-off in the market returns.

At least since the 1980s, asymmetry in equity market returns and volatility has been recognized in the financial literature.¹ Christoffersen et al. (2006) document the presence of time-varying conditional skewness in financial time-series. Time-varying conditional skewness matters when it is negative. Under such conditions, extreme negative market realizations are more frequent than positive realizations. Jondeau and Rockinger (2003) document the evidence supporting the existence of negative skewness, both in major international equity market and in foreign exchange market returns. Harvey and Siddique (2000) show that conditional skewness also captures asymmetry in risk. Investors who operate in negatively asymmetric and volatile markets are more concerned with downside risk than upside uncertainty, and require compensation through appropriate risk premia to bear an increase in relative downside risk.² That is to say, when investors face down markets, modeling downside volatility is important since downside volatility is the pertinent measure of asset risk.

Based on these observations, we propose a new method to study risk-return trade-off in financial market returns. First, we derive a reduced-form equilibrium relationship between risk and equity returns for a representative investor with Gul (1991) disappointment aversion preferences in an endowment economy.³ This investor is aware of market relative downside risk, and hence demands compensation for relative downside volatility. This step is conceptually similar to the method of Ang et al. (2006). Second,

¹See Hansen (1994), Bekaert and Wu (2000), Brandt and Kang (2004), and Bollerslev et al. (2006) for a review of this literature.

²We define “downside risk” as the risk borne by the investor if the realized market return falls below a certain threshold. If the market return rises above the same threshold, we call it “upside uncertainty”. In addition, we define the difference between downside risk and upside uncertainty as the “relative downside risk” for each time period.

³Recently, there has been renewed interest in this class of preferences. See Routledge and Zin (2010) and Bonomo et al. (2010).

we argue that if the investor is aware of relative downside risk, then this should be reflected in equilibrium asset prices. To be consistent with this argument, we assume that in equilibrium, logarithmic returns follow a binormal distribution of Gibbons and Myloie (1973), explicitly disentangling downside and upside market volatilities. Under these conditions, we provide a detailed analysis of the risk-return tradeoff in equilibrium. Third, to empirically examine this risk-return trade-off, we introduce a new generalized autoregressive conditional heteroskedasticity (GARCH) model, which we call binormal GARCH (BiN-GARCH). We show that this model characterizes S&P500 and twenty six international financial market returns well. Finally, we show that under binormal dynamics and using the BiN-GARCH model, the relationship between conditional mode and relative downside risk is positive and significant.

This is the first paper to explicitly model upside and downside volatilities. Our empirical findings indicate that first, on average, annualized daily downside and upside volatilities over the sample period are 17.42% and 14.56% (an average relative downside volatility of almost 3%), respectively. These two measures are highly correlated, a correlation of 0.82, which suggests co-movements in the same direction. Second, our findings shed new light on the traditional “leverage effect” of Black (1976) and Christie (1982). Leverage effect states that negative return shocks today have larger impact on future volatility than positive return shocks of similar magnitude. In addition, we find that negative shocks today have a much smaller impact on asymmetry than positive shocks of similar magnitude, which is the opposite of what we typically observe in studies on asymmetry in volatility.

Our findings are instructive in understanding the conflicting empirical results on risk-return trade-off reported in the literature, since we tie these contradictory outcomes to market asymmetry and time-varying market price of risk. Our results suggest that the risk-return trade-off depends on the conditional skewness and the dynamics of the relationship between conditional mode and relative downside risk. Given sufficiently large

(small) values of the slope parameter in conditional mode-relative downside volatility relationship, if the conditional skewness is negative (positive), we have a positive risk-return trade-off. Otherwise, risk-return relationship is negative. Moreover, our results suggest that the amplitude of this trade-off increases with the value of the conditional skewness. In particular, our empirical results support the findings of Ghysels et al. (2005), Ludvigson and Ng (2007), and Rossi and Timmermann (2009).

Our work contributes to the literature on downside risks. Ang et al. (2006) show that the cross-section of stock returns reflects a premium for downside risk, and provide a methodology for estimating this downside risk-premium using daily data. We study downside risks in the time domain, model and estimate downside volatility through time, and study the relation between downside risk and measures of central tendency in asset returns. Barndorff-Nielsen et al. (2008) introduce measures of downside risk which they call “downside realized semivariances”. These measures are entirely based on downward moves, measured by using high frequency data. We rely on a GARCH framework to measure and estimate downside risk by maximum likelihood, using daily data.

Finally, our findings contribute to the literature on explicit modeling of conditional asymmetry and fat-tails in equity returns. Following the work of Hansen (1994), many studies have attempted modeling the conditional skewness and kurtosis. Harvey and Siddique (1999) introduce a methodology for estimating time-varying conditional skewness, using a maximum likelihood framework with instruments, and assuming a non-central Student- t distribution. They re-parameterize the standardized residuals’ conditional density in terms of skewness, and model the mean, the volatility and the skewness independently. Brooks et al. (2005) use a modified version of the Student- t distribution which allows for independent modeling of volatility and kurtosis, assuming that the skewness is zero. Leon et al. (2009) use a Gram-Charlier series expansion, and perform parameterizations which yield independent modeling of volatility, skewness and kurtosis. However, as they use the Gallant and Tauchen (1989) transformation, the interpretation

of their parameters as volatility, skewness and kurtosis is lost. We do not explicitly model conditional skewness, but the Pearson mode skewness, that explicitly relates to the relative downside volatility, which is a focus of this article.⁴

The remainder of the paper is organized as follows. In Section 2, we present a simple theoretical model for downside risk premium in a consumption-based equilibrium setting to motivate our empirical modeling study as the first step. A discussion of the properties of the binormally distributed returns in equilibrium follows. We close this section by reporting the results of a calibration study to analyze the equilibrium implications of our theoretical model. In Section 3, we introduce the binormal GARCH (BiN-GARCH) model. In Section 4, we present our empirical results and various robustness tests and model diagnostics. Section 5 concludes.

2 A Simple Equilibrium Model of Downside Risk

In this section, we have two goals. First, we show that in an equilibrium consumption-based setting, with an investor who recognizes relative downside risk through his preferences over consumption stream, the downside premium that he pays to avoid any increase in downside market risk, and the upside discount that he receives for any decrease in upside market uncertainty; are related. This first result does not depend on any distributional assumptions. We will view this novel interpretation of the usual intertemporal Euler equilibrium condition as indicative that there is a market downside risk and a market upside uncertainty that receive unequal treatment from the representative investor. Second, for such an economy, we further assume that in equilibrium, logarithmic returns follow a binormal distribution, explicitly disentangling measures of downside and upside

⁴Pearson mode skewness, defined as the difference between the mean and the mode divided by the standard deviation, is a more robust measure of asymmetry in comparison with conditional skewness. For a discussion, see Kim and White (2004).

uncertainties in the stock market. We then derive and analyze the implied intertemporal risk-return relationship and its sensitivities to investor's preferences. Subsequent sections empirically examine this new intertemporal risk-return trade-off using a large set of market index returns.

2.1 Downside Premium and Upside Discount in Equilibrium

We work with a rational disappointment aversion utility function (henceforth, DA) defined over the consumption flow, which embeds downside risk. Ang et al. (2006) use a similar setup to illustrate cross-sectional pricing of the downside risk in an equilibrium setting. However, in their model, the utility function depends on wealth and not on consumption. We construct our model using consumption-based preferences. Disappointment aversion preferences were introduced by Gul (1991) to be consistent with Allais (1979) paradox. They differ from expected utility by introducing an additional weight to outcomes that are below the certainty equivalent values. Routledge and Zin (2010) generalize disappointment aversion preferences and embed them in the recursive utility framework of Epstein and Zin (1989). Simple disappointment aversion of Gul (1991) is sufficient to motivate our study.

Formally, let V_t be the recursive intertemporal utility functional:

$$V_t = \left\{ (1 - \delta) C_t^{1 - \frac{1}{\psi}} + \delta [\mathcal{R}_t(V_{t+1})]^{1 - \frac{1}{\psi}} \right\}^{\frac{1}{1 - \frac{1}{\psi}}} \quad \psi > 0, \quad 0 < \delta < 1 \quad (1)$$

where C_t is the current consumption, δ is the time preference discount factor, ψ is the elasticity of intertemporal substitution and $\mathcal{R}_t(V_{t+1})$ is the certainty equivalent of the random future utility, conditional on time t information. We specialize this model by setting $\psi = \infty$. Hence, we obtain

$$V_t = (1 - \delta) C_t + \delta \mathcal{R}_t(V_{t+1}). \quad (2)$$

The recursion in Eq. (2) characterizes the Epstein and Zin (1989) recursive utility when the elasticity of intertemporal substitution is infinite, meaning that the represen-

tative agent perfectly substitutes out consumption through time.⁵ In DA preferences, the certainty equivalent function, $\mathcal{R}(\cdot)$, is implicitly defined by:

$$\frac{\mathcal{R}^{1-\gamma} - 1}{1-\gamma} = \int_{-\infty}^{\infty} \frac{V^{1-\gamma} - 1}{1-\gamma} dF(V) - (\alpha^{-1} - 1) \int_{-\infty}^{\mathcal{R}} \left(\frac{\mathcal{R}^{1-\gamma} - 1}{1-\gamma} - \frac{V^{1-\gamma} - 1}{1-\gamma} \right) dF(V). \quad (3)$$

The parameter α is the coefficient of disappointment aversion satisfying $0 < \alpha \leq 1$, and $F(\cdot)$ is the cumulative distribution function for the continuation value of the representative agent's utility. Several particular cases are worth mentioning. When α is equal to one, \mathcal{R} becomes the certainty equivalent corresponding to expected utility while V_t represents the Kreps and Porteus (1978) preferences. When $\alpha < 1$, outcomes lower than \mathcal{R} receive an extra weight $(1/\alpha - 1)$, decreasing the certainty equivalent. Thus, α is interpreted as a measure of disappointment aversion, and outcomes below the certainty equivalent are considered disappointing.⁶

DA preferences imply a stochastic discount factor given by:

$$S_{t,t+1} = \delta (\delta R_{t+1})^{-\gamma} \left(\frac{I(\delta R_{t+1} < 1) + \alpha I(\delta R_{t+1} \geq 1)}{\mathbb{E}_t[I(\delta R_{t+1} < 1)] + \alpha \mathbb{E}_t[I(\delta R_{t+1} \geq 1)]} \right). \quad (4)$$

where $I(\cdot)$ is the indicator function, and R_{t+1} is the return on an asset that yields aggregate consumption as payoff, which we call the market portfolio. It is clear that when there is no disappointment aversion ($\alpha = 1$), the expression above reduces to a particular case of the familiar Kreps and Porteus (1978) pricing kernel derived by Epstein and Zin (1989):

$$S_{t,t+1}^* = \delta (\delta R_{t+1})^{-\gamma}, \quad (5)$$

⁵In this study, our focus is on returns and volatility. Hence, this assumption eliminates the effect of consumption growth rate - it eliminates the possibility of future volatility feeding back into current consumption through precautionary savings. We thank the anonymous referee for this suggestion.

⁶Notice that the certainty equivalent, besides being decreasing in γ , is also increasing in α . Thus α is also a measure of risk aversion, but of a different type than γ .

the special case of an infinite elasticity of intertemporal substitution. This special case, also mentioned in Epstein and Zin (1989), is related to the CAPM since only the market portfolio returns matter for asset pricing.

We exploit the Euler equation $\mathbb{E}_t [S_{t,t+1} R_{t+1}] = 1$ and rewrite it as a model of expected returns and using the conditional expectation operator and the definition of covariance to yield:

$$\mathbb{E}_t [R_{t+1}^e] = Cov_t \left(-\frac{S_{t,t+1}}{\mathbb{E}_t [S_{t,t+1}]}, R_{t+1}^e \right) \quad (6)$$

where

$$R_{t+1}^e = R_{t+1} - R_{f,t+1} \quad (7)$$

is the market excess return over the risk-free rate, and where the following expressions

$$\delta^{-\gamma} (\delta R_{f,t+1}) = \frac{\mathbb{E}_t [I (\delta R_{t+1} < 1)] + \alpha \mathbb{E}_t [I (\delta R_{t+1} \geq 1)]}{\mathbb{E}_t [R_{t+1}^{-\gamma} I (\delta R_{t+1} < 1)] + \alpha \mathbb{E}_t [R_{t+1}^{-\gamma} I (\delta R_{t+1} \geq 1)]}, \quad (8)$$

$$F_{1,t} = \mathbb{E}_t [I (\delta R_{t+1} < 1)] \quad \text{and} \quad F_{2,t} = \mathbb{E}_t [I (\delta R_{t+1} \geq 1)] = 1 - F_{1,t},$$

respectively define the risk-free return $R_{f,t+1}$, the likelihood of down markets $F_{1,t}$, and the probability of up markets $F_{2,t}$.⁷

Using Eq. (4) and (8), we prove that the Euler equation (6) is also equivalent to

$$\mathbb{E}_t^u [R_{t+1}^e \mid \delta R_{t+1} \geq 1] = \left(\frac{1}{\alpha} \frac{F_{1,t} H_{1,t}}{F_{2,t} H_{2,t}} \right) (-\mathbb{E}_t^d [R_{t+1}^e \mid \delta R_{t+1} < 1]). \quad (9)$$

where

$$H_{1,t} = \mathbb{E}_t [R_{t+1}^{-\gamma} \mid \delta R_{t+1} < 1] \quad \text{and} \quad H_{2,t} = \mathbb{E}_t [R_{t+1}^{-\gamma} \mid \delta R_{t+1} \geq 1], \quad (10)$$

and where $\mathbb{E}_t^d [\cdot \mid \delta R_{t+1} < 1]$ and $\mathbb{E}_t^u [\cdot \mid \delta R_{t+1} \geq 1]$ are respectively the conditional expectation operators associated with the distorted downside and upside probability densities

⁷ $R_{f,t+1}$ is the return earned on the risk free asset between time t and time $t+1$. This value is known at time t .

D_{t+1} and U_{t+1} defined by:⁸

$$D_{t+1} = \frac{R_{t+1}^{-\gamma}}{\mathbb{E}_t [R_{t+1}^{-\gamma} \mid \delta R_{t+1} < 1]} \quad \text{and} \quad U_{t+1} = \frac{R_{t+1}^{-\gamma}}{\mathbb{E}_t [R_{t+1}^{-\gamma} \mid \delta R_{t+1} \geq 1]}. \quad (11)$$

For the representative investor, down markets in this model correspond to periods where the log return, $r_{t+1} = \ln R_{t+1}$, falls below the marginal rate of time preference, $-\ln \delta$. Also, since

$$\mathbb{E}_t^d [R_{t+1}^e \mid \delta R_{t+1} < 1] \leq \frac{1}{\delta} - R_{f,t+1} \leq \mathbb{E}_t^u [R_{t+1}^e \mid \delta R_{t+1} \geq 1], \quad (12)$$

then it is more likely that $\mathbb{E}_t^d [R_{t+1}^e \mid \delta R_{t+1} < 1] \leq 0$, so its opposite represents a paid premium. An increase in downside volatility, may be bad news for an investor, since the market return may become worse conditional on the already bad state. We can thus interpret $-\mathbb{E}_t^d [R_{t+1}^e \mid \delta R_{t+1} < 1] \geq 0$ as the premium the representative agent is willing to pay to avoid any increase in downside volatility, which represents the volatility conditional on being in unfavorable states where the market return falls below the disappointing threshold. Analogously, it is more likely that $\mathbb{E}_t^u [R_{t+1}^e \mid \delta R_{t+1} \geq 1] \geq 0$, representing a received discount. Investors may have a preference for an increase in upside volatility, since it reflects an increase in the possibility of realization of more positive excess returns. In this case, $\mathbb{E}_t^u [R_{t+1}^e \mid \delta R_{t+1} \geq 1] \geq 0$ may be interpreted as the discount the investor receives to compensate for a decrease in upside volatility, which represents the volatility conditional on being in favorable states where the market return is above the disappointing threshold.⁹ Eq. (9) states that in equilibrium, for an investor

⁸Since we have $\mathbb{E}_t [D_{t+1} \mid \delta R_{t+1} < 1] = 1$ and $\mathbb{E}_t [U_{t+1} \mid \delta R_{t+1} \geq 1] = 1$, then D_{t+1} can be thought of as distorting the downside probability distribution, and U_{t+1} as distorting the upside probability distribution. In a different context, Anderson et al. (2003) provide an excellent discussion of distorted beliefs and their impact on asset prices.

⁹Preference for upside uncertainty may come from the disappointing threshold being positive (good) return for the investor. Aversion to upside uncertainty may arise if the disappointing threshold is negative.

who is aware of downside risk, this market upside discount is proportional to the market downside premium, with a time-varying coefficient.¹⁰

The Euler equation written in the form of Eq. (9) demonstrates the existence of an upside and a downside risk having unequal perception and treatment by the representative investor. In the next section, we derive an equilibrium risk-return relationship that relates a measure of reward to measures of downside and upside volatilities in financial markets. In the standard setup, risk is measured by the total market volatility and rewarded through expected returns. To derive a new risk-return relationship which is consistent with the equilibrium implication that investors react differently to volatility in down and up markets, we assume that in equilibrium, the distribution of market returns explicitly disentangles market downside variance from upside variance with respect to a specific threshold.

2.2 Conditional Binormally Distributed Returns in Equilibrium

We use the binormal distribution introduced by Gibbons and Mylroie (1973) to model logarithmic returns in equilibrium. It is an analytically tractable distribution which accommodates empirically plausible values of skewness and kurtosis, and nests the familiar Gaussian distribution.¹¹ We assume that logarithmic returns, r_{t+1} , follow a binormal distribution with parameters $(m_t, \sigma_{1,t}, \sigma_{2,t})$ conditional on information up to time t . The

¹⁰In the special case of a risk-neutral investor, $\gamma = 0$, we have $D_{t+1} = U_{t+1} = 1$, meaning the downside and the upside probability densities are not distorted. We also have $H_{1,t} = H_{2,t} = 1$. In this case, the upside discount is $F_{1,t}/(\alpha F_{2,t})$ times greater than the downside premium.

¹¹See Bangert et al. (1986), Kimber and Jeynes (1987), and Toth and Szentimrey (1990), among others, for examples of using the binormal distribution in data modeling, statistical analysis and robustness studies.

conditional density function of r_{t+1} is given by:

$$f_t(x) = A_t \exp\left(-\frac{1}{2}\left(\frac{x - m_t}{\sigma_{1,t}}\right)^2\right) I(x < m_t) + A_t \exp\left(-\frac{1}{2}\left(\frac{x - m_t}{\sigma_{2,t}}\right)^2\right) I(x \geq m_t) \quad (13)$$

where $A_t = \sqrt{2/\pi} / (\sigma_{1,t} + \sigma_{2,t})$. We notice that m_t is the conditional mode, and up to a multiplicative constant, $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$ are interpreted as conditional variances of returns, conditional on returns being less than the mode (downside variance), and conditional on returns being greater than the mode (upside variance), respectively. More specifically,

$$\text{Var}_t[r_{t+1} \mid r_{t+1} < m_t] = \left(1 - \frac{2}{\pi}\right) \sigma_{1,t}^2 \quad \text{and} \quad \text{Var}_t[r_{t+1} \mid r_{t+1} \geq m_t] = \left(1 - \frac{2}{\pi}\right) \sigma_{2,t}^2. \quad (14)$$

We consider this property to be the most important characteristic of the binormal distribution, given our objectives in this project.

Binormal distribution can be parameterized by the mean μ_t , the variance σ_t^2 , the Pearson mode skewness, p_t , and the skewness, s_t as given by:

$$\begin{aligned} \mu_t &= m_t + \sigma_t p_t \\ \sigma_t^2 &= (1 - 2/\pi) (\sigma_{2,t} - \sigma_{1,t})^2 + \sigma_{1,t} \sigma_{2,t} \\ s_t &= p_t (1 - (\pi - 3) p_t^2) \\ p_t &= \sqrt{2/\pi} (\sigma_{2,t} - \sigma_{1,t}) / \sigma_t. \end{aligned} \quad (15)$$

It can be shown that the initial parameters $\sigma_{1,t}$ and $\sigma_{2,t}$ are expressed in terms of the total variance and the Pearson mode skewness as follows:

$$\begin{aligned} \sigma_{1,t} &= \sigma_t \left(-\sqrt{\pi/8} p_t + \sqrt{1 - (3\pi/8 - 1) p_t^2} \right) \\ \sigma_{2,t} &= \sigma_t \left(\sqrt{\pi/8} p_t + \sqrt{1 - (3\pi/8 - 1) p_t^2} \right), \end{aligned} \quad (16)$$

which implies a bound on the Pearson mode skewness: $|p_t| \leq 1 / \sqrt{\pi/2 - 1} \approx 1.3236$. Since return skewness is related to its Pearson mode skewness through the third equation

in (15), then, bounds on the Pearson mode skewness also imply bounds on the skewness: $|s_t| \leq 0.9953$. Also, conditional excess kurtosis is positive and less or equal to 3.8692.

Assuming that log returns are conditionally binormally distributed, we still need the conditional moment generating function $M_t(u) = \mathbb{E}_t[\exp(ur_{t+1})]$ as well as the conditional truncated moment generating function $M_t(u; x) = \mathbb{E}_t[\exp(ur_{t+1}) I(r_{t+1} \geq x)]$ of returns to be able to compute equilibrium quantities derived in the previous section, and in particular to explicitly expressed the Euler equilibrium restriction (6). These functions are given by:

$$\begin{aligned} M_t(u) &= \frac{2\sigma_{1,t}}{\sigma_{1,t} + \sigma_{2,t}} \exp\left(m_t u + \frac{\sigma_{1,t}^2 u^2}{2}\right) \Phi(-\sigma_{1,t} u) \\ &\quad + \frac{2\sigma_{2,t}}{\sigma_{1,t} + \sigma_{2,t}} \exp\left(m_t u + \frac{\sigma_{2,t}^2 u^2}{2}\right) \Phi(\sigma_{2,t} u) \end{aligned} \quad (17)$$

and

$$\begin{aligned} M_t(u; x) &= M_t(u) - \frac{2\sigma_{1,t}}{\sigma_{1,t} + \sigma_{2,t}} \exp\left(m_t u + \frac{\sigma_{1,t}^2 u^2}{2}\right) \Phi\left(\frac{x - m_t}{\sigma_{1,t}} - \sigma_{1,t} u\right) \quad \text{if } x < m_t \\ &= \frac{2\sigma_{2,t}}{\sigma_{1,t} + \sigma_{2,t}} \exp\left(m_t u + \frac{\sigma_{2,t}^2 u^2}{2}\right) \Phi\left(-\frac{x - m_t}{\sigma_{2,t}} + \sigma_{2,t} u\right) \quad \text{if } x \geq m_t, \end{aligned} \quad (18)$$

where Φ is the standard normal cumulative distribution function.

Notice that the Euler equation $\mathbb{E}_t[S_{t,t+1} R_{t+1}] = 1$, can also be represented by a nonlinear restriction, say

$$G(m_t, \sigma_{1,t}, \sigma_{2,t}) = 0, \quad (19)$$

on the parameters $(m_t, \sigma_{1,t}, \sigma_{2,t})$ of the conditional distribution of log returns. The mode is then derived as a function of downside and upside volatilities from Eq. (19). The equilibrium risk-free return, and the equilibrium probabilities of down and up markets, as defined in Eq. (8), are then given by:

$$\delta^{1-\gamma}(R_{f,t+1}) = \frac{1 + (\alpha - 1) M_t(0; -\ln \delta)}{M_t(-\gamma) + (\alpha - 1) M_t(-\gamma; -\ln \delta)}, \quad (20)$$

$$F_{1,t} = 1 - M_t(0; -\ln \delta) \quad \text{and} \quad F_{2,t} = M_t(0; -\ln \delta) = 1 - F_{1,t},$$

and the equilibrium equity premium obtains from:

$$\mathbb{E}_t[R_{t+1}] - R_{f,t+1} = M_t(1) - R_{f,t+1}. \quad (21)$$

The nonlinear function G is explicitly known and given by:

$$G(m_t, \sigma_{1,t}, \sigma_{2,t}) = \delta^{1-\gamma} \frac{M_t(1-\gamma) + (\alpha-1)M_t(1-\gamma; -\ln \delta)}{1 + (\alpha-1)M_t(0; -\ln \delta)} - 1. \quad (22)$$

The restriction (19) implies that the conditional mode is in fact an implicit nonlinear function of conditional downside and upside volatilities, $m_t = g(\sigma_{1,t}, \sigma_{2,t})$. The explicit function G and the implicit function g are both parameterized by the preference parameters δ , γ and α .

The equation $m_t = g(\sigma_{1,t}, \sigma_{2,t})$ defines a new risk-return relation that relates the conditional mode to the conditional downside and upside volatilities. To be able to deal with this new trade-off between risk and reward, we first-order linearize the nonlinear restriction (19) around the steady state values $(\bar{\sigma}_1, \bar{\sigma}_2)$ to obtain

$$m_t = g(\sigma_{1,t}, \sigma_{2,t}) \approx \lambda_0 + \lambda_1 \sigma_{1,t} + \lambda_2 \sigma_{2,t}, \quad (23)$$

where

$$\lambda_1 = -\frac{G_{\sigma_1}(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}, \quad \lambda_2 = -\frac{G_{\sigma_2}(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)} \quad (24)$$

and $\lambda_0 = \bar{m} - \lambda_1 \bar{\sigma}_1 - \lambda_2 \bar{\sigma}_2$,

and where $\bar{m} = g(\bar{\sigma}_1, \bar{\sigma}_2)$ and, $G_m(\cdot, \cdot, \cdot)$, $G_{\sigma_1}(\cdot, \cdot, \cdot)$ and $G_{\sigma_2}(\cdot, \cdot, \cdot)$ denotes the first-order partial derivatives of the function $G(\cdot, \cdot, \cdot)$ with respect to its arguments, m , σ_1 and σ_2 respectively.

Given the expression (23), the traditional risk-return trade-off that relates expected returns to the total variance may be expressed as:

$$\mu_t = m_t + \sigma_t p_t = \lambda_0 + \lambda_t^* \sigma_t \quad (25)$$

where

$$\lambda_t^* = \left(1 - (\lambda_1 - \lambda_2) \sqrt{\pi/8}\right) p_t + (\lambda_1 + \lambda_2) \sqrt{1 - (3\pi/8 - 1) p_t^2}. \quad (26)$$

The first equality in Eq. (25) follows by the definition of mean in binormal distribution, Eq. (15). The second equality in Eq. (25) and Eq. (26) follow from Eq. (16) and Eq. (23). Eq. (25) characterizes the traditional risk-return trade-off in this model, and shows that the price of risk depends on the asymmetry in returns. If $\lambda_2 \approx -\lambda_1$, then the mode is a function of the relative downside volatility, $\sigma_{1,t} - \sigma_{2,t}$, that is

$$m_t \approx \lambda_0 + \lambda_1 (\sigma_{1,t} - \sigma_{2,t}) \quad (27)$$

and the price of risk in the traditional risk-return trade-off simplifies to:

$$\lambda_t^* = \left(1 - \lambda_1 \sqrt{\pi/2}\right) p_t. \quad (28)$$

Notice that the coefficients λ_0 , λ_1 and λ_2 all depend on preference parameters. Next, for calibrated values of $(\bar{\sigma}_1, \bar{\sigma}_2)$, we vary the preference parameters δ , γ and α and evaluate their impact on the equilibrium quantities just derived.

2.3 Calibration Assessment

We calibrate the steady state values $(\bar{\sigma}_1, \bar{\sigma}_2)$ of the return distribution model to obtain a steady state annualized daily volatility of $\bar{\sigma} = 20\%$ and a steady state daily Pearson mode skewness of $\bar{p} = -0.125$. These values are close matches for daily sample volatility and Pearson mode skewness of the S&P500 returns from January 1980 to December 2009. We also calibrate the annualized daily time discount factor to $\delta = 0.96$, meaning that down markets correspond to periods where annualized daily log returns are below 4.08%. We consider two calibrations of the risk aversion parameter γ , corresponding to a case with $\gamma = 0.5$ where the representative investor is less risk averse than the myopic log utility investor, and a case with $\gamma = 1.5$ where the representative investor is more risk-averse than the myopic log utility investor.

2.3.1 Equilibrium Asset Pricing Implications

Given $\bar{\sigma}_1$, $\bar{\sigma}_2$, γ , δ and varying the disappointment aversion parameter α between 0 and 1, we compute the following quantities of interest. We first solve for the steady state equilibrium mode \bar{m} as the unique solution to the nonlinear equation $G(m, \bar{\sigma}_1, \bar{\sigma}_2) = 0$. We then compute the steady state equilibrium risk-free rate $R_f - 1$ and likelihood of down markets F_1 from Eq. (20), and finally the steady state equilibrium equity premium $\mathbb{E}[R] - R_f$ from Eq. (21). We plot these four quantities against α in Figure 1, restricted to values of α which yield a steady state equilibrium equity premium below 20%.¹²

The values of α which yield a steady state equity premium below 20% are between 0.90 and 1. The lowest effective risk-aversion in these scenarios corresponds to the combination $(\gamma = 0.5, \alpha = 1)$, and the highest effective risk aversion corresponds to $(\gamma = 1.5, \alpha = 0.90)$, leading to an effective risk-aversion that is smaller compared to values considered in the literature. To see this, we compare the level of effective risk aversion by plotting indifference curves for the same gamble for Kreps and Porteus (1978) preferences $(\gamma = 0.5, \alpha = 1)$ and $(\gamma = 5, \alpha = 1)$, and disappointment aversion preferences $(\gamma = 1.5, \alpha = 0.90)$. Figure 2 plots indifference curves for a hypothetical gamble with two equally probable outcomes for DA preferences with the highest effective risk aversion, and for Kreps-Porteus preferences with coefficient of relative risk aversion 0.5 (our lowest effective risk aversion case) and 5. The figure shows that DA preferences with highest effective risk aversion exhibit lower risk aversion than Kreps-Porteus preferences with relative risk aversion equal to 5.

For our lowest and highest levels of effective risk aversion, the corresponding equity premia are 2% and 19% respectively. The corresponding equilibrium modes are 43% and 58% respectively. Both the equity premium and the mode decrease as α increases, since the investor becomes less risk-averse. For the same reason, they decrease as γ decreases.

¹²As is seen in Table 1, values of equity premium within this range can be found in international equity market index returns.

The risk-free rate does not vary within this range of α , and decreases as γ increases. The annualized risk-free rate is 3% with $\gamma = 0.5$, and 1% with $\gamma = 1.5$. The likelihood of down markets does not vary either within this range of α , or across values of γ . This likelihood is just below 50%.

2.3.2 Equilibrium Implications for the Risk-Return Trade-off

Having shown that the model implications for asset pricing are consistent with empirical evidence and existing findings in the literature, we now assess the implications of our model for risk-return trade-off.¹³ In addition to quantities computed in Section 2.3.1, we compute the constant coefficients λ_0 , λ_1 and λ_2 of conditional mode in its linearized relationship with downside and upside volatilities from Eq. (24), and the equilibrium steady state value λ^* of the market price of risk in the traditional risk-return trade-off for market returns from Eq. (26). We plot these four quantities against α in Figure 3, restricting as before to values of α that yield a steady state equilibrium equity premium below 20%.

There are three main observations evident in Figure 3. First, Panel A shows that the loading of the conditional mode on downside volatility does not vary either within the range of α , or across values of γ . This loading is positive, implying that the conditional mode increases to compensate for an increase in downside volatility. Second, Panel B shows that the loading of the conditional mode on upside volatility does not vary within the range of α , or across values of γ . In contrast to the results shown in Panel A, this loading is negative; which implies that the conditional mode increases to compensate for a decrease in upside volatility. Third, the loading of the conditional mode on upside volatility is very close to the negative value of the loading on downside volatility. We

¹³We show that our model can generate equity premia which are consistent with 6% and above values often considered in the literature. See, for example, studies such as Mehra and Prescott (1985), Campbell and Cochrane (1999), and Bansal and Yaron (2004), among many others.

have discussed this case in the previous section and subsequently show, in Section 4, that this restriction is statistically supported by the data. Thus, only the relative downside volatility, $\sigma_{1,t} - \sigma_{2,t}$, seems to matter in equilibrium. An increase in relative downside volatility is compensated with an increase in the conditional mode, m_t .

In addition to these three critical observations, Panel C shows that the constant drift term in the linearized relation that relates the mode to the relative downside volatility does not vary within the permissible range of α , but decreases as γ increases. Finally, Panel D shows that the steady state market price of risk in the traditional risk-return trade-off is positive in general, but can become negative if effective risk-aversion is very low. We observe a negative λ^* for values of effective risk aversion corresponding to $(\gamma = 0.5, 0.98 < \alpha \leq 1)$.

In the next section, we empirically examine the risk-return relation represented by Eq. (23) using U.S. and international market index returns. The goal is to examine whether the three main implications of our reduced-form equilibrium model for the constant parameters in this relation are met in actual data. In order to estimate the constant parameters, we need dynamics of downside and upside volatilities.

3 Conditional Mode and Pearson Mode Skewness: the BiN-GARCH Model

Following the seminal work of Hansen (1994), many studies have provided theoretical and empirical evidence regarding time-varying asymmetry in returns. The importance of incorporating this time-varying asymmetry in asset pricing to capture salient features of financial data is well documented. Among many others, Harvey and Siddique (1999), Harvey and Siddique (2000), Jondeau and Rockinger (2003), and Brooks et al. (2005) have addressed this issue. The common theme in this literature is that central ten-

dency and asymmetry in returns are modeled through conditional mean and conditional skewness respectively. However, it is also known that these measures as well as excess kurtosis are very sensitive to outliers. Examples of such outliers are the crash of October 1987, the Asian financial crisis of 1997, the Russian debt default crisis in 1998, or the recent 2008-2009 credit crunch in the United States.

3.1 BiN-GARCH Model Specification

We allow for time variation in the return distribution. Specifically, we allow for heteroscedasticity dynamics similar to GARCH models, but we directly model the mode and the Pearson mode skewness of the conditional return distribution. This is where our model differs from existing competing models. We rely on conditional mode and Pearson mode skewness to model central tendency and asymmetry, since they are less sensitive to outliers than mean and skewness.¹⁴

We assume that, conditional on information up to time t , returns r_{t+1} follow a binormal distribution with mode m_t , variance σ_t^2 and Pearson mode skewness p_t . In line with the literature, we allow for the negative correlation between volatility and returns or the so-called “leverage effect”, where firms’ leverage increases with negative returns. We borrow our specification for heteroscedasticity from the NGARCH model of Engle and Ng (1993),

$$\sigma_{t+1}^2 = \omega + \beta\sigma_t^2 + \alpha\sigma_t^2 (z_{t+1} - \theta)^2, \quad (29)$$

where $z_{t+1} = (r_{t+1} - \mathbb{E}_t[r_{t+1}]) / \sigma_t$ are standardized residuals. As a result, our specification nests NGARCH.¹⁵ Christoffersen and Jacobs (2004) show that NGARCH has a better out-of-sample performance in option pricing compared to several alternative

¹⁴See Kim and White (2004) for a detailed discussion.

¹⁵Our empirical findings do not rely on NGARCH-type dynamics. Assuming EGARCH-type dynamics of Nelson (1991), we find very similar results.

GARCH models.

Given that the Pearson mode skewness is bounded ($|p_t| \leq 1 / \sqrt{\pi/2 - 1}$), and using the hyperbolic tangent transformation to guarantee the bounds, we assume that the Pearson mode skewness evolves following:

$$p_{t+1} = \sqrt{\frac{2}{\pi - 2}} \tanh \left(\delta_0 + \delta_1 z_{t+1}^* I(z_{t+1}^* \geq 0) + \delta_2 z_{t+1}^* I(z_{t+1}^* < 0) + \delta_3 p_t \right), \quad (30)$$

where $z_{t+1}^* = (r_{t+1} - m_t) / \sigma_t$. This nonlinear GARCH-type dynamics of the conditional Pearson mode skewness also features asymmetry in asymmetry. Asymmetries in the Pearson mode skewness are generated by deviations of realized returns from the conditional mode. We recall that dynamics of volatility and Pearson mode skewness lead to direct downside and upside volatility modeling through Eq. (16).

Following the linear approximation in Eq. (23), we specify the conditional mode as:

$$m_t = \lambda_0 + \lambda_1 \sigma_{1,t} + \lambda_2 \sigma_{2,t}. \quad (31)$$

This specification of the conditional mode is motivated by the equilibrium model of Section 2, and is analogous to the ARCH-in-Mean model of Engle et al. (1987) which relates expected returns to volatility. We recall from Section 2.2 that by definition,

$$\sigma_{1,t} = \sqrt{\frac{\pi}{\pi - 2} \text{Var}_t[r_{t+1} \mid r_{t+1} < m_t]} \text{ and } \sigma_{2,t} = \sqrt{\frac{\pi}{\pi - 2} \text{Var}_t[r_{t+1} \mid r_{t+1} \geq m_t]}. \quad (32)$$

The mode, similar to the mean, also characterizes the central tendency. Hence we assume that in Eq. (31), the future conditional mode has a linear relationship with upside or downside volatilities of returns, depending whether return realizations are above or below the current conditional mode.

3.2 BiN-GARCH and Risk-Return Trade-Off

Based on intertemporal capital asset pricing model (ICAPM) of Merton (1973), the vast majority of studies focus on verifying a positive (linear) relationship between the

conditional expected excess return of the stock market and the market's conditional variance through estimation of a time-invariant market price of risk. As we recall, the empirical evidence is quite mixed. In what follows, we propose an alternative to the conditional mean and conditional variance relationship as a measure for risk-return trade-off in empirical tests. As discussed above, for negatively asymmetric returns with outliers, and assuming time-varying market price of risk, we build our testing procedure for risk-return trade-off based on a relationship between the conditional mode and the conditional downside and upside variances. The basis of our proposal is the relationship between the conditional mode and the conditional mean in Eq. (25). The first equality in Eq. (25) follows from the definition of mean in binormal distribution, Eq. (15). The second equality results from the BiN-GARCH model specification or the equilibrium condition in Eq. (31) and property (16) of the binormal distribution. As we discussed earlier and based on what we present below, these equalities are particularly important for understanding the risk-return trade-off.

First, if both the conditional mode and the conditional Pearson mode skewness are constant, the first equality in Eq. (25) implies that they are respectively the drift and the slope of the linear regression of returns onto the conditional volatility. In this case, a negative Pearson mode skewness implies that expected returns fall in response to an increase in volatility. Consequently, the positive linear relationship between expected returns and volatility, as suggested by Merton (1973)'s ICAPM, would be inconsistent with the fact that both the conditional mode and the conditional Pearson mode skewness are constant and the latter is negative.

Second, based on Ang et al. (2006), it is clear from Eq. (32) that $\sigma_{1,t}$ and $\sigma_{2,t}$ are respectively the measures of market downside and upside volatilities using the conditional mode of returns as the cutoff point. Earlier, we argued that if equity is more volatile in a bear market than it is in a bull market, then investors require a compensation for holding it, since equity tends to have low payoffs when they feel poor and pessimist,

compared to when they feel wealthy and confident.¹⁶ This is in line with what Cochrane (2007) points out about the relationship between equity premium and business cycles. Thus, the relative downside volatility, $\sigma_{1,t} - \sigma_{2,t}$, should be compensated by appropriate returns. From Eq. (31), if $\lambda_2 = -\lambda_1$, then conditional mode is determined by relative downside volatility:

$$m_t = \lambda_0 + \lambda_1 (\sigma_{1,t} - \sigma_{2,t}). \quad (33)$$

When we present our empirical results, we discuss the implications of the BiN-GARCH model which imposes this restriction.

So far, we have shown that our theoretical results imply a positive relationship between the conditional mode and the relative downside risk. This result does not contradict the conventional risk-return equation used in the literature. As discussed in Section 2.2, we can rewrite the expected return as:

$$\mathbb{E}_t[r_{t+1}] = \lambda_0 + \lambda_t^* \sigma_t \quad \text{with} \quad \lambda_t^* = \left(1 - \lambda_1 \sqrt{\pi/2}\right) p_t, \quad (34)$$

which implies a time-varying price of risk that is proportional to the conditional asymmetry. This relationship is similar to the typical equation seen in the literature, for example in Ghysels et al. (2005), except for time-variation in market price of risk.

Since market price of risk in Eq. (34) is time-varying, then discussion of positive risk return trade-off boils down to the sign of unconditional expected value of λ_t^* . Positive $\mathbb{E}(\lambda_t^*)$ is possible when both terms in the right hand side relationship in Eq. (34) have the same sign. This condition can be summarized as:

$$\mathbb{E}(\lambda_t^*) > 0 \quad \text{if} \quad \begin{cases} \mathbb{E}(p_t) > 0 \text{ and } \lambda_1 < \sqrt{2/\pi}, & \text{or,} \\ \mathbb{E}(p_t) < 0 \text{ and } \lambda_1 > \sqrt{2/\pi}. \end{cases} \quad (35)$$

In Section 4.2, we show that both scenarios of Eq. (35) are observed in the data. Thus, on average, the traditional risk-return trade-off can be positive in this model.

¹⁶Explicitly, this statement translates into $\sigma_{1,t} > \sigma_{2,t}$.

Figure 4 illustrates the contribution of the binormal distribution in appropriate modeling of market returns, through visualizing the relationship between market volatility, asymmetry, downside risk and upside uncertainty. This is simply a graphical representation of the two functions defined in Eq. (16), for all possible values of the Pearson mode skewness and for values of annualized daily market volatility between 0 and 100%. Panel A of Figure 4 shows that downside risk is dominant in more volatile and negatively asymmetric markets. In contrast, Panel B shows that upside uncertainty dominates downside risk in more volatile and positively asymmetric markets.

4 Empirical Results

4.1 Data

In our study of risk-return trade-off, we study S&P500 index excess returns and MSCI daily market index excess returns for 26 developed, emerging, and frontier markets obtained from *Thomson Reuters Datastream*.¹⁷ All these series end in December 31, 2009. The start dates differ across markets, due to availability of the data. Table 1 reports summary statistics of the data used in the subsequent sections. Annualized return means and standard deviations in percentages are reported in the fourth and the fifth columns. We report unconditional skewness in column six. As it is seen, for the majority of market returns studied, with the exception of Greece, Indonesia, Korea, and the Philippines; we observe negative unconditional skewness. Yet, the value of skewness, positive or negative, is not small relative to the average daily returns. Thus, for all series in our study, the unconditional distributions of the returns are not symmetric and the use of binormal distribution for modeling returns is reasonable. All series seem to be highly fat-tailed, since they all have significant unconditional excess kurtosis. The

¹⁷We use USD-denominated MSCI indices in order to have comparable results across the markets.

reported p-values of Jarque and Bera (1980) normality test imply significant departure from normality in all series. Our proxy for risk free rate is the yield of 3-month constant maturity US Treasury Bill, which we obtained from Federal Reserve Bank of St. Louis FRED II data bank.

For over half of the international market returns series, crash of October 1987, Asian crisis of 1997, Russian default of 1998, and 2007-2009 credit crunch episodes are represented in the data. Data on seven additional markets does not include October 1987, but includes the rest of these significant global financial episodes. All data series include 2007-2009 credit crunch.

4.2 BiN-GARCH Model Estimation and Discussion

We now turn our attention to maximum likelihood estimation of the BiN-GARCH model, introduced in Section 3, and discuss the results shown in Tables 2 to 5. Our first step is to study the ability of different BiN-GARCH specifications in capturing the dynamics of the financial time series. We then perform extensive robustness and diagnostic testing. Thus, we first fit the S&P 500 returns using five BiN-GARCH specifications.¹⁸ We then use the best model to conduct the risk-return trade-off study. Our metrics for the best fit are Bayesian information criterion (BIC) and likelihood ratio tests against the benchmark model and the other specifications studied. In this study, the canonical NGARCH model of Engle and Ng (1993) is the benchmark for model comparison. Estimated parameters of NGARCH model are reported under Specification (I) in column 2 of Table 2. With NGARCH specification for returns, the conditional Pearson mode skewness is zero and the mode, which in this case is equal to the mean, is constant.¹⁹ As is seen in Table 2,

¹⁸As mentioned earlier, by setting Pearson mode skewness equal to zero, BiN-GARCH nests Engle and Ng (1993).

¹⁹To save space, we have summarized our findings in this document. An appendix with detailed estimation results is available on authors' SSRN page. For instance, volatility dynamics parameters in

likelihood ratio tests indicate that all other models studied are preferred to NGARCH. Similarly, BIC values reported in that table also indicate that all other models are preferred to the baseline NGARCH model.

We depart from the NGARCH model by allowing a constant, but non-zero, Pearson mode skewness in specification (II). Parameter estimates of this model are reported in column 3 of Table 2. The estimated value of the constant conditional Pearson mode skewness is -0.1138. Results for this specification confirms that S&P500 index returns are conditionally negatively skewed. The gain in likelihood resulting from the inclusion of a single parameter from (II) to (I), the associated likelihood ratio (LR) test statistic of 36.67 and the information criterion all indicate that the NGARCH with *i.i.d.* Gaussian standardized residuals is rejected in favor of the GARCH with constant skewness at 1% significance level or better. Estimates of constant conditional mode and Pearson mode skewness are respectively positive and negative and strongly significant. As discussed in Section 3, this leads to a negative relationship between expected returns and volatility. A positive risk-return relation would simply mean that either the mode or the Pearson mode skewness is misspecified, or both. This is an important result which underpins our study of risk-return trade-off based on GARCH-in-Mode estimations.

In specification (III), we keep the mode constant and allow the Pearson mode skewness to vary over time and follow the nonlinear autoregressive dynamics specified in Eq. (30). We report the estimated parameters of the specification (III) in the fourth column of Table 2. All parameters are strongly significant and the inclusion of three more parameters compared to specification (II) induces a substantial gain in likelihood. The corresponding likelihood ratio test statistic of 67.24 and information criterion also strongly reject the NGARCH model in favor of specification (III). Moreover, based on the difference in log-likelihoods between specifications (II) and (III), we find that LR test

Tables 3 to 5, are generally significant at 95% confidence level or better. Hence they are only reported in the Appendix.

statistic of 30.5696, which is statistically significant at 5% confidence level or better, along with BIC values for these two specifications, lead us to favor the GARCH-in-asymmetry specification over constant Pearson mode skewness.

Besides, these results suggest that realizations of returns relative to the conditional mode have different impacts on conditional asymmetry measured through the Pearson mode skewness. Estimates of δ_1 and δ_2 are both positive and δ_1 is three times higher than δ_2 . Thus, increases in the Pearson mode skewness due to realization of returns above the conditional mode are significantly larger than the reductions in the Pearson mode skewness due to realization of equal absolute value-sized returns below the conditional mode.

In specification (IV), we relax the fixed mode assumption maintained in specifications (I-III). Estimation results for specification (IV) are reported in column 5 of Table 2. In comparison with specification (III), there is only one meaningful restriction imposed on specification (IV): $\lambda_2 = -\lambda_1$. However, this linear restriction seems reasonably valid. This is due to the observation that first, likelihood ratio test statistic of 120.45 and BIC values imply that specification (IV) is statistically preferable to the baseline NGARCH model at 1% significance level or better. Second, in comparison with specification (III), specification (IV) is preferred since this model induces gains in likelihood which are not due to inclusion of additional parameters. This is attested by likelihood ratio test statistic of 53.21 which is statistically significant at 5% confidence level or better. Estimated parameters are all statistically significant at conventional confidence levels.

The estimated parameters of the full BiN-GARCH model, specification (V), are reported in column 6 of Table 2. Again, all estimated parameters are significant at conventional levels except for λ_0 , the drift in the conditional mode. As is seen in the table, this specification is readily preferable to the baseline NGARCH model based on LR test and BIC values. In comparison with specification (IV), first notice that while we have relaxed the $\lambda_2 = -\lambda_1$ restriction, the values of estimated λ_1 and λ_2 are reasonably

close and have opposite signs, thus confirming the validity of negative impact in time-varying conditional mode, explored in specification (IV).²⁰ Second, in comparison with specification (III), responses of the asymmetry to return realizations above and below the conditional mode increase when the conditional mode becomes time-varying. Notice that the estimates of δ_1 and δ_2 are more than twice their respective values when the mode is time-invariant. The likelihood ratio test at 1% level rejects specifications (I) to (III) in the table against specification (IV) and against the full BiN-GARCH specification. The same test rejects specification (IV) at the 5% level against specification (V), but not at the 1% level. Bayesian information criterion favors specification (IV) over the full BiN-GARCH.

The latter observation deserves more attention. It simply means that the data support predictability of excess returns, r_{t+1} , with relative downside volatility, $\sigma_{1,t} - \sigma_{2,t}$, as the predictor, in effect lending empirical support to Eq. (33). From column 5 of Table 2 we know that the estimated λ_1 for specification (IV) is both positive and significant at the 1% level.

Notice that the estimated value of λ_1 in specification (IV) is such that restriction (35) for positive market price of risk in presence of negative expected value of conditional Pearson mode skewness is violated. Hence, at this stage, we can not show a positive relationship between the expected returns and the conditional volatility. We believe that the estimation results presented subsequently, empirically substantiate our claim that the relevant measure for evaluating risk-return trade-off in a conditionally asymmetric market is conditional mode-relative downside volatility trade-off. This translates into a time varying, and on average positive, expected returns-volatility trade-off.

In Figure 5, we provide graphical representations for the contribution of relative

²⁰They are within the same order of magnitude, and the ratio of their absolute values is close to 0.88. Crucially, the absolute values of the estimated parameters are quite close, less than one standard error apart. Thus, one can reasonably infer that specifications (IV) and (V) are statistically equivalent.

downside risk for S&P500 index daily excess returns. Panel A of Figure 5 plots annualized expected daily excess returns. Conditional mode is plotted in Panel B. On average, annualized expected daily excess returns and conditional mode for the sample period are 3.12% and 39.36%, respectively. The two series are negatively correlated: excess returns fall following an increase in relative downside volatility, whereas the conditional mode goes up. The conditional mode is more sensitive to fluctuations in downside volatility, as predicted by the BiN-GARCH estimates. As discussed earlier, we interpret these observations to be supportive evidence of relative downside volatility being rewarded through an increase in the conditional mode, instead of expected returns.

The annualized daily volatility and the daily Pearson mode skewness are plotted in Panels C and D of Figure 5. On average, the annualized daily volatility for the sample period is 16.06%. The daily Pearson mode skewness is -0.1289 . This value closely matches the estimated parameter for the *i.i.d.* specification (II) in column 3 of Table 2. Fluctuations in the Pearson mode skewness show that, although the conditional asymmetry is centered to a negative value, stock returns can be positively skewed. This contrasts with the IG-GARCH model of Christoffersen et al. (2006) which imposes a negative conditional skewness over time. Instead, the direction of asymmetry in the BiN-GARCH model is determined by relative downside volatility. Returns are negatively skewed only if equity is more volatile in a declining market than in a rising market, and are positively skewed otherwise. Finally, we present the filtered downside and upside volatility series in Panels E and F of Figure 5. On average, annualized daily downside and upside volatilities over the sample period are 17.42% and 14.56% (an average relative downside volatility of almost 3%), respectively. These two measures are highly correlated, a correlation of 0.82, which suggests co-movements in the same direction.

We analyze the news impact curves resulting from the BiN-GARCH model. Panel A of Figure 6 shows reaction to return shocks of market, downside, and upside volatilities. The asymmetric pattern that emerges for market volatility is interesting and

corroborates existing findings.²¹ Positive and small negative return shocks lower market volatility. Large negative shocks, on the other hand, significantly increase market volatility in comparison with positive shocks of the same magnitude. This asymmetric pattern transmits to downside and upside volatilities too. Negative or small positive return shocks either do not change or slightly reduce upside volatility. On the other hand, positive return shocks increase upside volatility sharply. In contrast, while positive return shocks lower downside volatility, negative return shocks of the same magnitude cause noticeably larger increases in downside volatility.

Finally, Panel B of Figure 6 displays the reaction of market asymmetry to positive and negative return shocks. It is immediately obvious that this response is highly asymmetric and kinked at the origin. While negative return shocks cause linear reductions in market asymmetry, positive return shocks of the same magnitude significantly increase market asymmetry. These increases are arguably nonlinear. We can interpret this pattern as follows: a negative return shock today increases the likelihood of negative return shocks tomorrow. On the other hand, a positive return shock today increase the likelihood of positive returns tomorrow much more than a negative shock of a similar magnitude increase the possibility of future negative shocks. We find this result to be a nice confirmation of our “asymmetry in asymmetry” assertion.

4.3 Robustness Checks

4.3.1 BiN-GARCH in S&P500 Excess Returns Sub-Samples

So far we have shown that BiN-GARCH model characterizes S&P500 daily excess returns for 1980-2009 period. In this section, we perform robustness checks using sub-samples within the S&P500 sample and international data. As mentioned in the previous section, specification (IV) in Table 2 is statistically preferable to other BiN-GARCH specifica-

²¹See Bollerslev et al. (2006).

tions studied. Hence we use that specification for estimation in what follows. Estimation results are reported in Tables 3-5.

We break up the S&P500 excess returns sample into three sub-samples: 1980-1989, 1990-1999, and 2000-2009. Each sub-sample includes significant events or market activity periods. For example, crash of October 1987 and the oil shock of 1980 are in 1980-1989 sub-sample. The second sub-sample, 1990-1999, includes the dot-com boom of the late 1990s. The last sub-sample includes data on the 2007-2009 credit crunch. We fit the data in each sub-sample using maximum likelihood methodology and specification (IV) discussed in the previous section. We summarize and report these estimation results in Table 3.

The following regularities are observed in this table: First, estimated time-invariant Pearson mode skewness from Model (II) is statistically significant at 1% confidence level and has the expected negative sign across all three sub-samples. This observation implies that negative conditional skewness in the market excess returns is not due to a few outliers such as October 1987 crash or 2007-2009 financial crisis. It seems to be a feature of the market, even in the bullish 1990s.

Second, estimated values of time-invariant conditional mode from models (II) and (III) are statistically close (within one standard error of each other) both within each sub-sample and across the three sub-samples. This observation implies that time-invariant conditional mode is robust to significant outliers such as Crash of 1987, confirming the results of Kim and White (2004).

Third, we observe that once we allow for a dynamic formulation in conditional mode, we get significant variation in estimated parameters across sub-samples. Specifically, $\hat{\lambda}_1$ in model (IV) is much larger in 2000-2009 sub-sample in comparison with estimated parameters obtained from the other two sub-samples. Similarly, relaxing the linear restriction in model (IV), $\lambda_1 = -\lambda_2$, and estimating the full BiN-GARCH in model (V) does not alter this pattern. Estimated λ_1 and λ_2 parameters are statistically close in

the first two sub-samples, but they are significantly larger in the 2000-2009 sub-sample. We attribute this observation to the prolonged turmoil that engulfed financial markets during the 2007-2009 period. In contrast, events such as the crash of 1987 typically did not last as long. We believe that this lengthy duration of market instability altered the size of the parameters governing the dynamics of the model, but did not change the fundamental dynamics. All the reported estimated parameters for models (IV) and (V) are statistically significant across all sub-samples, and the absolute values of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are within one standard error of each other within all sub-samples and across the first two sub-samples.

Diagnostic measures of estimated models across the three sub-samples in Table 3 present a more complex picture. Based on LR test, all models are preferred to baseline NGARCH model. On one hand, BIC favors model (II) in almost every panel of the table. On the other hand, based on likelihood ratio tests statistics computed through comparing each model with the previous, for example model (IV) with model (III), we see that in all panels model (IV) is preferred to model (II).

Regarding traditional risk-return trade-off, we see that $\hat{\lambda}_1$ parameter in Panel C is large enough to support conditions required for an expected positive market price of risk specified in Eq. (35). These conditions are not supported by the estimated values of λ_1 for 1980-1989 and 1990-1999 periods.

4.3.2 BiN-GARCH in International Data

In the next step, to verify the ability of BiN-GARCH to characterize the excess return and risk-return trade-off dynamics beyond S&P500 index, we model daily MSCI index returns for twenty six international financial markets using BiN-GARCH. The sample includes major financial markets such as U.S., U.K., Japan, France, Germany, and Italy; important markets such as Switzerland, Singapore, and Hong Kong, and emerging and

frontier markets such as Korea, Russia, South Africa, Israel, Egypt, and Jordan.²² As mentioned in Section 4.1, we use MSCI country index values downloaded from *Thomson Reuter's Datastream* as our source. We concurrently estimate conditional mode, variance, and asymmetry using maximum likelihood methodology.

Table 4 presents empirical results for returns series of developed financial markets. As is seen, the crucial estimated λ_1 parameter is statistically significant at 1% confidence level for all markets. Point estimates of this parameter range between a low of 0.2225 for Singapore, to a high of 0.8506 for the Dutch market. Estimated GARCH parameters, are all significant at 1% level. To save space, these estimated parameters are not reported here, but are available in the online Appendix. Estimated parameters for Pearson mode skewness, specified in Eq. (30), are generally statistically significant at the usual confidence levels.

The results presented in Table 5, report our findings for ten emerging and frontier markets. Similar to what is reported for developed markets, estimated λ_1 parameter is statistically significant at 1% confidence level for all emerging and frontier markets. Point estimates of this parameter range between a low of 0.2660 for the Philippines, to a high of 0.6120 for Czech republic. Not surprisingly, estimated GARCH parameters are all significant at the 1% confidence level.²³ Based on statistical significance of the estimated parameters of the Pearson mode skewness, we can argue in favor of the dynamics specified in Eq. (30) in our sample of emerging and frontier markets.

It is immediately clear from the empirical evidence reported above, that our findings hold for a wide variety of international markets; including all the major markets and a sizeable sample of smaller developed, emerging, and frontier markets. We find such

²²In this exercise, we use MSCI U.S. index which includes mid-cap companies as well as large cap companies that comprise S&P500 index. MSCI U.S. index covers over 98% of U.S. equity market capitalization. Source: MSCI web site.

²³These estimates are available in the online Appendix.

broad statistical support to be very encouraging.

The empirical evidence overwhelmingly supports the main hypotheses in our study. First, we find significant asymmetry in asymmetry, as well as significant presence of relative downside risks in a large number of international equity markets. Second, we find statistically significant positive relationships between the conditional mode and relative downside risk.

To discuss the link between traditional risk-return trade-off and our model, which is specified through Eq. (34), we need expected values of conditional Pearson mode skewness and estimated values of λ_1 . As is seen in both Tables 4 and 5, $\mathbb{E}(p_t)$ assumes both positive and negative values. On the other hand, estimated λ_1 values range between 0.2225 and 0.8506. Given this wide range of estimates for λ_1 , the conditions specified in Eq. 35, and the value of $\sqrt{2/\pi} \approx 0.7979$, the observed mixed or negative relationship results between conditional volatility and expected returns documented in the literature, do not seem surprising.

In this study, we find positive expected market price of risk, and hence a positive traditional risk-return trade-off, in full sample of international excess index returns for 9 markets. They are Egypt, Germany, Greece, Indonesia, Japan, Jordan, Korea, the Philippines, and the Netherlands. As a robustness check, we re-estimate the model for 2000-2009 period for all international markets. These results are reported in Table 6. In this sub-sample, eleven markets show evidence of positive expected market price of risk. They include U.S. U.K. France, and Ireland, which did not support conditions (30) in the full sample, 1980-2009. Egypt, Greece, Indonesia, Japan, Jordan, Korea, and Philippines support conditions (30) in both full and 2000-2009 sub-sample. Thus, we believe that our findings are convincing in support of our claim that market price of risk, and hence the risk-return trade-off relationship, is time-varying.²⁴

²⁴The percentage of markets with positive expected market price of risk is larger than the anticipated variation under null hypothesis of no positive risk-return trade-off in the cross section of market returns,

Rossi and Timmermann (2009) point out that the common practice of studying conditional mean and conditional variance introduces two possible sources of bias: First, the empirical literature automatically assumes a linear relationship between conditional mean and conditional “total” variance. This formulation is not the only correct form.²⁵ Second, since neither conditional mean nor conditional variance of returns are observable, model mis-specification can not be ruled out. We argue that insistence on symmetric distributions for modeling the returns in the majority of empirical studies in the face of well-documented evidence of asymmetry in returns is a good example of such concerns.

Moreover, for samples that include substantial outliers such as October 1987 crash or 2007-2009 credit crunch, the conditional mean might not be a reliable or even a critical measure for risk compensation, see Kim and White (2004). If the return distribution is negatively skewed (with a negative Pearson mode skewness), we argue that the conditional mode of the return distribution may provide a better measure for risk compensation than the conditional mean. The BiN-GARCH model introduced in this paper provides a very good empirical alternative to many existing empirical studies for understanding the risk-return trade-off.

The estimated values for λ_1 fall between 0.2225 to 0.8506, a range of 0.6281, which is almost three times as large as the smallest value. A significant number of markets in our study, 64%, have estimated λ_1 values which are within one standard error from each other. In eight markets, these estimated parameters are more than one standard error apart from the rest. Czech Republic, Denmark, Egypt, Ireland, Poland, and U.K. are

with the usual confidence levels for both 1980-2009 and 2000-2009 samples.

²⁵The shape of the relationship between risk premium and conditional variance is essentially unrestricted. See Veronesi (2000) and Rossi and Timmermann (2009) for more details. We follow them in the belief that this relationship is non-linear and time-varying. We only derive a first order linear approximation for conditional mode-conditional volatility relation, and take a linear approximation for traditional risk-return relation as given.

within one standard error of the U.S. $\hat{\lambda}_1$, but larger than the rest. Dutch and German markets are bunched together, with $\hat{\lambda}_1$'s which are more than one standard error larger than the U.S. All estimated parameters are within two standard error from each other. This observation may seem indicative of one or more common factors driving the risk premia across markets. On the other hand, given that there is sizeable variation across some markets, for example between Dutch and Singapore markets, possibility of local factors affecting the risk premia can not be ruled out. However, this issue is beyond the scope of the current study. We address this interesting topic in future research.

Since λ_0 and λ_1 depend on structural parameters of the theoretical model, γ, α , and δ as discussed in Section 2.2, it is reasonable to expect that low estimated values of λ_1 indicate low effective risk-aversion. We study this very intriguing issue in future research.

Based on the estimation results presented so far, we find strong empirical support for equilibrium theoretical and calibration results presented in Sections 2.2 and 2.3. As we recall, our calibration results imply positive λ_1 and $\lambda_1 = -\lambda_2$. These two results are borne by the data in S&P500 sample and sub-samples, as well as in full sample and 2000-2009 sub-sample of international data. For example, given the estimated average value of Pearson's mode skewness and λ_1 for S&P500 in 2000-2009 sub-sample, which are equal to 1.0063 and -0.1253 respectively, annualized expected value of $\mathbb{E}(\lambda_t^*)$ is 0.0327. This in turn means that a 1% increase in annualized volatility, expected annualized excess return increases by 0.52%. Similarly, Japanese data in 1980-2009 period imply an expected annual market price of risk equal to 0.0015. Thus, 1% increase in annual volatility in the Japanese market increases expected returns by 0.02%. This observation is in line with the prolonged bear market in Japan since 1989. For Korea, 1980-2009 sample supports an expected annualized market price of risk equal to 0.0081, and hence; a 0.13% increase in annual expected returns in response to 1% increase in volatility.

4.4 BiN-GARCH Diagnostics

We perform several experiments to assess the ability of the BiN-GARCH model to fit the data. In the first step, we evaluate the performance of the BiN-GARCH model in generating accurate value at risk (VaR) and expected shortfall (ES) measures. In the Appendix, we show that for *i.i.d.* binormal innovations, VaR and ES measures exist in closed form and derive them. To test for the ability of the BiN-GARCH model to generate a correct conditional coverage, we follow Christoffersen (1998) who derives a test for correct conditional coverage as a joint test of unconditional coverage and independence. Recall that under the null hypothesis of unconditional coverage one has:

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T I(r_t > -VaR_{t-1}(1, \xi)) \right] = 1 - \xi \quad (36)$$

where $I(r_t > -VaR_{t-1}(1, \xi))$ follows a Bernoulli distribution with parameter $1 - \xi$. A likelihood ratio test of the null hypothesis uses the following statistic:

$$LR_{uc} = 2 \left[\ln \left(\pi_1^{T_1} (1 - \pi_1)^{T-T_1} \right) - \ln \left((1 - \xi)^{T_1} \xi^{T-T_1} \right) \right] \quad (37)$$

where T_1 is the number of observations such that $r_t > -VaR_{t-1}(1, \xi)$, T is the total number of observations and $\pi_1 = T_1 / T$. Under the null of unconditional coverage, LR_{uc} is χ^2 -distributed with one degree of freedom.

Since we are examining conditional VaR, Christoffersen (1998) also suggests testing for independence. The null hypothesis of independence is:

$$\begin{aligned} Prob[I(r_t > -VaR_{t-1}(1, \xi)) = j \mid I(r_{t-1} > -VaR_{t-2}(1, \xi)) = i] \\ = Prob[I(r_t > -VaR_{t-1}(1, \xi)) = j]. \end{aligned} \quad (38)$$

Similar to the test for unconditional coverage, a likelihood ratio test for independence uses the following statistic:

$$LR_{ind} = 2 \left[\ln \left(\pi_{01}^{T_{01}} (1 - \pi_{01})^{T_0-T_{01}} \pi_{11}^{T_{11}} (1 - \pi_{11})^{T_1-T_{11}} \right) - \ln \left((1 - \xi)^{T_1} \xi^{T-T_1} \right) \right], \quad (39)$$

where T_{ij} is the number of observations $I(r_{t-1} > -VaR_{t-2}(1, \xi)) = i$ followed by observations $I(r_t > -VaR_{t-1}(1, \xi)) = j$, and $\pi_{ij} = T_{ij}/T_i$. Under the null of independence, LR_{ind} is χ^2 -distributed with one degree of freedom.

The test for conditional coverage has the joint hypothesis (36)-(38) and uses the statistic $LR_{uc} + LR_{ind}$, which under the null, is χ^2 -distributed with two degrees of freedom. We perform the three tests using daily S&P500 index excess returns fitted by the BiN-GARCH model, both for 95 and 99% confidence levels, and we present results in Table 7.

The percentage π_1 of realized returns above the Value-at-Risk is high as expected. The unconditional coverage for the conditional VaR at 99% confidence level is not rejected at conventional levels of significance (the p -value is 0.9334). The same conclusion holds for independence and conditional coverage (p -values of 0.8392 and 0.9305, respectively). We also fail to reject unconditional coverage, independence and conditional coverage at conventional levels of significance for the conditional VaR at 5% confidence level (p -values are 0.9310, 0.7938, and 0.9139 respectively). Thus, the BiN-GARCH model is accurate for measuring conditional VaR at 1% or 5% levels for daily S&P500 index excess returns.

Finally, we evaluate the ability of the BiN-GARCH to fit, not only the “worst” events expected to be captured by VaR, but to forecast the conditional return distribution, that is, the complete pattern of conditional quantiles. Our test follows the work of Rosenblatt (1953), Diebold et al. (1998) and Bai (2003), who show that the transformed variables $u_{t-1} = F(r_t | \mathbb{I}_{t-1})$, $t = 1, \dots, T$; are *i.i.d.* uniform over $(0, 1)$ if and only if the forecasts $F(r_t | \mathbb{I}_{t-1})$ are correct. \mathbb{I}_{t-1} represents the information up to time $t-1$, and $F(\cdot | \mathbb{I}_{t-1})$ is the cumulative distribution function of returns r_t . This cumulative distribution function is binormal in the BiN-GARCH model and exists in closed-form. Specifically, it is given by:

$$F(r_t | \mathbb{I}_{t-1}) = 1 - M_{t-1}(0; r_t) \quad (40)$$

where $M_{t-1}(u)$ and $M_{t-1}(u; x)$, which refer to Eq. (17) and (18), are respectively the moment generating function and the truncated moment generating function of a binormal distribution with parameters $(m_{t-1}, \sigma_{1,t-1}, \sigma_{2,t-1})^\top$. The processes m_t , $\sigma_{1,t}$ and $\sigma_{2,t}$ are specified in Section 3.

Given Eq. (40), (17) and (18), our empirical findings in Section 4.2, and following Diebold et al. (1998) and Bai (2003); to evaluate density forecasts, we need to test for the uniformity and serial independence of the u_t 's. In this step, we rely on graphical tests. We plot the QQ plot of u_t against that of uniform distribution in Figure 7 (the quantiles of the series u_t are plotted against those of the uniform distribution over $(0, 1)$). This diagnoses if u_t follows a uniform distribution over $(0, 1)$. The plot reveals that the u_t 's are very close to the uniform distribution, despite small deviations observed around the 75% quantile.

Additionally, notice that if u_t is uniform over $(0, 1)$, then $\Phi^{-1}(u_t)$ is standard normal. Recall that Φ is the cumulative distribution function of the standard normal. We then plot the autocorrelation function of $\Phi^{-1}(u_t)$ in Figure 8. This step investigates whether u_t 's are serially independent. The graph suggests that autocorrelations are not statistically significant at 5% confidence level. Thus, based on these two graphical tests, we conclude that the binormal distribution is a suitable alternative for modeling conditional returns. The dependence of higher-order moments are well captured by the BiN-GARCH model through a time-varying Pearson mode skewness.

5 Concluding Remarks

In this paper, we introduce a new methodology for assessment and study of risk-return trade-off in S&P500 excess returns and international equity markets. We propose a discrete-time dynamic model of asset prices with binormal return innovations; the binormal GARCH (BiN-GARCH) model, which nests the canonical NGARCH model. Us-

ing an intuitive endowment and representative agent equilibrium model, we show that demand for relative downside risk compensation arises in familiar theoretical settings with only the assumption that the agent distinguishes between downside and upside risk in the market. We then test our theoretical model using annualized daily index excess returns from twenty six international equity markets and S&P500.

Our study suggests very strong numerical and empirical support for two main assertions in this paper, based on a calibration study and fitting a large sample of market data. We find that: First, there exists relative downside risk in equity markets, which is compensated through an increase in conditional mode of returns. Second, the relationship between relative downside risk and the conditional mode is positive. Furthermore, our empirical results support a positive expected value for market price of risk in many of the markets studied. This last result is due to estimated size of volatility spill-over parameters in the conditional mode and conditional market asymmetry measures which are mostly centered around negative values. In this sense, our findings provide additional support for studies such as Ghysels et al. (2005), Ludvigson and Ng (2007), and particularly, Rossi and Timmermann (2009).

These results are instructive in understanding the conflicting outcomes in the empirical literature on risk-return trade-off. We tie these contradictory outcomes to market asymmetry and time-varying market price of risk. More importantly, our results suggest that the amplitude of this trade-off increases with the value of the conditional skewness.

Compared to existing GARCH models, the BiN-GARCH explicitly relates downside risk and upside uncertainty to both conditional heteroscedasticity and conditional asymmetry. The model also relates the mode of conditional return distribution to downside and upside volatilities. Although previous studies assess risk compensations through expected return channels, our empirical findings strongly suggest that relative downside risks are compensated by an increase in the conditional mode. This finding is particularly appealing as it allows for other channels of risk compensation in equity markets.

Several experiments assess the performance of the BiN-GARCH model in different dimensions. In particular, back testing and density forecast evaluation of the model show that BiN-GARCH is very accurate in measuring conditional Value-at-Risk and in fitting conditional return distribution.

We have not studied nonparametric or semiparametric modeling and estimation of downside risk in this paper. Neither have we visited the existence of common or country specific factors across markets which influence risk premia. We have not studied the issue of differences in effective risk aversion across countries. We plan to address these questions in our future research. We also plan to study the implications of BiN-GARCH modeling of returns on daily index option data and measure downside and upside volatilities implicit in option prices. The resulting relative downside volatility may be used as return predictor, or to fit an empirical measure of the mode of conditional return distribution.

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Appendix: Value-at-Risk and Expected Shortfall Calculations

We now use the properties of the binormal distribution to derive closed-form solutions for Value-at-Risk (VaR) and expected shortfall (ES). We assume that under the physical measure, log returns, $R_t = \ln(S_t/S_{t-1})$, are *i.i.d.* binormal with parameters $(m, \sigma_1, \sigma_2)^\top$. The quantile function of a binormal distribution with parameters $(m, \sigma_1, \sigma_2)^\top$ is given by:

$$\begin{aligned} F^{-1}(\xi) = & m + \sigma_1 \Phi^{-1}\left(\xi \frac{\sigma_1 + \sigma_2}{2\sigma_1}\right) \mathbf{1}_{\{\xi < \frac{\sigma_1}{\sigma_1 + \sigma_2}\}} \\ & + \sigma_2 \Phi^{-1}\left(\xi \frac{\sigma_1 + \sigma_2}{2\sigma_2} + \frac{\sigma_2 - \sigma_1}{2\sigma_2}\right) \mathbf{1}_{\{\xi \geq \frac{\sigma_1}{\sigma_1 + \sigma_2}\}} \end{aligned}$$

where F is the cumulative distribution function of returns. Let $F_\tau^{-1}(\xi)$ be the quantile function, and let $M'_\tau(u)$ and $M'_\tau(u; x)$ be the first order derivatives with respect to u of moment generating function and truncated moment generating functions of a binormal distribution with parameters $(m_\tau, \sigma_{1\tau}, \sigma_{2\tau})^\top$. The conditional VaR and the conditional expected shortfall at $1 - \xi$ confidence level for horizon τ are given by:

$$VaR(\tau; \xi) = -F_\tau^{-1}(\xi) \quad \text{and} \quad ES(\tau; \xi) = \frac{M'_\tau(0) - M'_\tau(0; -VaR(\tau; \xi))}{1 - M_\tau(0; -VaR(\tau; \xi))}. \quad (41)$$

Eq. (41) can be viewed as a simple and tractable generalization of familiar VaR and ES formulas. Setting $p = 0$, yields the usual VaR and ES formulas with normally distributed returns. With normally distributed returns, VaR is a linear function of volatility as is seen here: $VaR(\tau; \xi) = -m_\tau - \sigma_\tau \Phi^{-1}(\xi)$ if $p = 0$. Once asymmetry is introduced, the above statement does not hold any more: the relationship between VaR and the volatility component may take a highly nonlinear form. However, our formulation shows that VaR increases for more volatile returns as well as for more negatively skewed payoffs.

Table 1: Summary Statistics of the Data

Return Series	Date	No. Obs.	Mean (%)	Std. Dev. (%)	Skewness	Kurtosis	J-B p-Value
S&P500	01/02/1980-12/31/2009	7,569	3.48	21.94	-1.24	31.87	0.01
Panel A: OECD Countries							
Australia	01/02/1980-12/31/2009	7,827	0.69	27.98	-3.36	71.32	0.01
Czech Republic	07/12/2004-12/31/2009	1,429	23.07	41.08	-0.19	16.31	0.01
Denmark	01/02/1980-12/31/2009	7,827	6.98	24.29	-0.34	10.66	0.01
France	01/02/1980-12/31/2009	7,827	2.93	26.20	-0.25	9.98	0.01
Germany	01/02/1980-12/31/2009	7,827	2.31	27.41	-0.23	9.49	0.01
Greece	01/12/1988-12/31/2009	5,733	3.48	36.26	0.02	8.46	0.01
Hong Kong	01/02/1980-12/31/2009	7,827	3.48	34.31	-2.02	43.22	0.01
Hungary	07/12/2004-12/31/2009	1,429	8.87	50.25	-0.07	11.66	0.01
Ireland	01/12/1988-12/31/2009	5,733	-4.48	29.42	-0.78	16.37	0.01
Israel	01/05/1992-12/31/2009	4,437	5.49	22.93	-0.32	4.43	0.01
Italy	01/02/1980-12/31/2009	7,827	2.41	28.90	-0.25	8.90	0.01
Japan	01/02/1980-12/31/2009	7,827	0.36	27.11	-0.09	11.07	0.01
Korea	01/05/1988-12/31/2009	5,738	1.56	44.10	0.24	17.14	0.01
The Netherlands	01/02/1980-12/31/2009	7,827	3.90	25.26	-0.20	10.57	0.01
New Zealand	01/12/1988-12/31/2009	5,733	-6.26	27.64	-0.41	10.21	0.01
Norway	01/02/1980-12/31/2009	7,827	1.30	31.95	-0.77	17.46	0.01
Singapore	01/02/1980-12/31/2009	7,827	1.39	26.85	-1.14	28.32	0.01
Switzerland	01/02/1980-12/31/2009	7,827	4.47	22.43	-0.24	9.27	0.01
U.K.	01/02/1980-12/31/2009	7,827	1.88	24.24	-0.32	12.33	0.01
U.S.*	01/02/1980-12/31/2009	7,827	2.91	21.55	-1.25	32.69	0.01
Panel B: Non-OECD Countries							
Egypt	07/12/2004-12/31/2009	1,429	37.10	36.97	-0.78	10.58	0.01
Jordan	07/12/2004-12/31/2009	1,429	1.68	28.50	-0.66	9.51	0.01
Indonesia	01/05/1988-12/31/2009	5,738	7.73	41.21	0.04	41.98	0.01
Philippines	01/05/1988-12/31/2009	5,738	0.29	32.86	0.37	13.89	0.01
Russia†	01/11/1995-12/31/2009	3,907	14.83	62.30	-0.37	12.62	0.01
South Africa‡	01/05/1993-12/31/2009	4,433	7.80	33.48	-0.39	9.38	0.01

This table reports summary statistics of excess returns. Calculation of the returns is based on subtracting daily 3-months U.S. Treasury Bill rate from the log difference of the market total return index in each country. Mean of excess returns and standard deviations are reported as annualized percentages. Excess kurtosis values are reported. The column titled “J-B p-Value” reports p-values of Jarque and Bera (1980) test of normality in percentages. † and ‡ indicate OECD accession candidate and enhanced engagement countries, respectively. * denotes MSCI U.S. index, which includes large and mid-cap companies. Source: Thomson Reuters Datastream, FRED II data bank at St. Louis Federal Reserve, and OECD.

Table 2: ML Estimation of Various BiN-GARCH Models Using S&P500 Daily Excess Returns.

Estimated Parameter	(I)	(II)	(III)	(IV)	(V)
m	$4.65E-05^*$ ($1.65E-05$)	0.0009^* (0.0002)	0.0008^* (0.0002)		
λ_0				$0.0002\ddagger$ (0.0001)	-0.0004 (0.0003)
λ_1				0.6756^* (0.0642)	0.6608^* (0.0585)
λ_2					-0.5805^* (0.0751)
ω	$1.56E-06^*$ ($2.11E-07$)	$1.15E-06^*$ ($2.02E-07$)	$1.18E-06^*$ ($2.02E-07$)	$1.35E-06^*$ ($1.96E-07$)	$1.62E-06^*$ ($2.60E-07$)
β	0.8841^* (0.0086)	0.8920^* (0.0081)	0.8835^* (0.0092)	0.8855^* (0.0086)	0.8796^* (0.0097)
α	0.0642^* (0.0051)	0.0628^* (0.0050)	0.0607^* (0.0050)	0.0623^* (0.0050)	0.0621^* (0.0050)
θ	0.7955^* (0.0661)	0.8078^* (0.0668)	0.9167^* (0.0828)	0.8370^* (0.0749)	0.8546^* (0.0777)
p		-0.1138^* (0.0187)			
δ_0			-0.0628^* (0.0158)	-0.0752^* (0.0209)	-0.0815^* (0.0197)
δ_1			0.0710^* (0.0167)	0.1449^* (0.0347)	0.1568^* (0.0318)
δ_2			$0.0187\ddagger$ (0.0105)	0.0725^* (0.0172)	0.0680^* (0.0167)
δ_3			$0.2603\ddagger$ (0.1043)	0.2903^* (0.0758)	0.2842^* (0.0737)
Diagnostic Measures					
Log-Lik	24,639.5362	24,657.8725	24,673.1573	24,699.7641	24,701.6897
LR Test Stat		36.6727*	67.2421*	120.4558*	124.3069*
BIC	-3.2506	-3.2507	-3.2491	-3.2515	-3.2506

This table presents maximum likelihood estimation results for different specifications of the binormal GARCH model for S&P500 daily excess returns. The sample spans continuously compounded value-weighted returns on the S&P500 Index from Jan. 2, 1980, to Dec. 31, 2009. Standard errors are reported below the estimates. *, † and ‡ denote statistical significance at 1%, 5%, and 10% levels, respectively. BIC represents Bayesian information criteria. Likelihood ratio test statistics are computed with respect to the benchmark NGARCH model of Engle and Ng (1993), represented as Model (I). Model (II) relaxes the symmetry assumption by allowing a non-zero p . Model (III) allows for time-varying p . Model (IV) imposes $\lambda_2 = -\lambda_1$ restriction. Model (V) allows for time-varying conditional mode and relaxes the $\lambda_2 = -\lambda_1$ restriction. Model (V) is the full BiN-GARCH model discussed in Section 3.

Table 3: ML Estimation of Various BiN-GARCH Models Using S&P500 Daily Excess Returns: Robustness Checks for 1980-1989, 1990-1999, and 2000-2009 Sub-Samples.

Panel A: 1980-1989							
	m	p	λ_1	λ_2	LR-Stat 1	LR-Stat 2	BIC
Model (II)	0.0011* (0.0003)	-0.1221* (0.0346)			12.3413*		-3.2262
Model (III)	0.0008† (0.0004)				44.4831*	32.1418*	-3.2232
Model (IV)			0.5493* (0.0713)		60.1193*	15.6362*	-3.2232
Model (V)			0.5829* (0.1107)	-0.3755* (0.1446)	63.1330	3.0137	-3.2207
Panel B: 1990-1999							
	m	p	λ_1	λ_2	LR-Stat 1	LR-Stat 2	BIC
Model (II)	0.0008* (0.0003)	-0.0937* (0.0332)			7.9804†		-3.3976
Model (III)	0.0006† (0.0003)				45.0953*	37.1149*	-3.3956
Model (IV)			0.5119* (0.0576)		69.1201*	24.0248*	-3.3973
Model (V)			0.5633* (0.0660)	-0.4308* (0.0747)	71.9855*	2.8654	-3.3948
Panel C: 2000-2009							
	m	p	λ_1	λ_2	LR-Stat 1	LR-Stat 2	BIC
Model (II)	0.0005‡ (0.0003)	-0.0941* (0.0238)			12.7519†		-3.1104
Model (III)	0.0008* (0.0003)				23.4448*	10.6930‡	-3.1031
Model (IV)			1.0063* (0.1250)		35.3552*	11.9104†	-3.1024
Model (V)			1.8367 (1.1681)	-2.1968 (1.5432)	41.6375*	6.2822	-3.1005

This table presents maximum likelihood estimation results for different specifications of the binormal GARCH model for S&P500 daily excess returns. The sample spans continuously compounded value-weighted returns on the S&P500 Index from Jan. 2, 1980, to Dec. 31, 2009, divided into three sub-samples. Standard errors are reported below the estimates. *, † and ‡ denote statistical significance at 1%, 5%, and 10% levels, respectively. BIC represents Bayesian information criteria. LR-Stat 1 represents likelihood ratio test statistics computed with respect to the benchmark NGARCH model of Engle and Ng (1993), which is not reported here. LR-Stat 2 represents likelihood ratio test statistics computed with respect to the preceding model. Refer to the notes in Table 2 for a discussion of the models studied here.

Table 4: Maximum Likelihood Estimation of the GARCH-in-Mode Model Using Developed Economies Equity Data.

Estimated Parameter	France	Germany	Italy	Japan	Netherlands	Switzerland	U.K.	U.S.
λ_0	0.0006† (0.0003)	-3.92E - 06 (0.0002)	0.0004† (0.0002)	-0.0001 (0.0002)	9.95E - 05 (0.0002)	0.0004* (0.0001)	0.0001 (0.0001)	0.0002 (0.0001)
λ_1	0.4947* (0.1465)	0.8267* (0.0917)	0.4819* (0.0797)	0.4287* (0.0871)	0.8506* (0.0811)	0.5315* (0.0783)	0.7584* (0.1044)	0.6693* (0.0662)
δ_0	-0.0521 (0.0385)	-0.0197 (0.0272)	-0.0339 (0.0272)	-0.0283‡ (0.0148)	-0.0248 (0.0243)	-0.0633‡ (0.0276)	-0.0463‡ (0.0239)	-0.0709* (0.0191)
δ_1	0.0154 (0.0382)	0.0319 (0.0461)	0.0582‡ (0.0277)	0.1253* (0.0239)	0.0295 (0.0306)	0.1203* (0.0359)	0.0709* (0.0281)	0.1387* (0.0331)
δ_2	0.1104* (0.0354)	0.1087* (0.0227)	0.1405* (0.0294)	0.0469* (0.0138)	0.0768* (0.0250)	0.0864* (0.0259)	0.0525† (0.0240)	0.0713* (0.0170)
δ_3	-0.0002 (0.1336)	0.3034† (0.1136)	-0.1229 (0.1012)	0.2351* (0.0725)	0.3595† (0.1606)	0.0591 (0.1160)	0.4364* (0.1658)	0.2848* (0.0744)
$E(p_t)$	-0.1262	-0.1255	-0.0819	0.0032	-0.1237	-0.0825	-0.1470	-0.1194
Diagnostic Measures								
LR Test Stat.	73.1995*	89.2277*	90.2512*	69.8978*	55.2475*	74.3838*	73.8098*	118.0030*
BIC	-3.0004	-2.9598	-2.8962	-2.9355	-3.083	-3.1327	-3.0871	-3.2669
Estimated Parameter	Australia	Denmark	Greece	Hong Kong	Ireland	New Zealand	Norway	Singapore
λ_0	0.0006* (0.0002)	0.0005* (0.0002)	-0.0002 (0.0003)	0.0010* (0.0002)	0.0004† (0.0002)	0.0005 (0.0003)	0.0007* (0.0002)	0.0006* (0.0002)
λ_1	0.4985* (0.0544)	0.5224* (0.1321)	0.3052* (0.1060)	0.3325* (0.0598)	0.5247* (0.1252)	0.3422‡ (0.1764)	0.3452* (0.1138)	0.2225* (0.0662)
δ_0	-0.0453‡ (0.0234)	-0.0079 (0.0197)	0.0299‡ (0.0272)	-0.0980* (0.0180)	0.0029 (0.0093)	-0.0944* (0.0309)	-0.0389† (0.0182)	-0.0900* (0.0155)
δ_1	0.1048* (0.0298)	0.0294 (0.0239)	0.1170* (0.0248)	0.1964* (0.0269)	0.0076 (0.0174)	0.1354* (0.0367)	0.0850* (0.0211)	0.2094* (0.0234)
δ_2	0.1674* (0.0190)	0.0785* (0.0192)	0.1064* (0.0192)	0.0275‡ (0.0153)	0.0371* (0.0121)	0.0218 (0.0304)	0.0850† (0.0150)	0.0658* (0.0095)
δ_3	0.0492 (0.0977)	0.3572* (0.1083)	0.1575‡ (0.0707)	0.3323* (0.0766)	0.6646* (0.0377)	0.0182 (0.1433)	0.0493 (0.1013)	0.0715 (0.0681)
$E(p_t)$	-0.1166	-0.0751	0.0654	-0.1087	-0.1099	-0.0779	-0.0624	-0.0654
Diagnostic Measures								
LR Test Stat.	235.9389*	55.6494*	103.3313*	203.6406*	39.9170*	47.7887*	119.1197*	262.4547*
BIC	-2.9864	-3.0500	-2.6871	-2.8071	-2.9549	-2.9071	-2.8440	-3.0482

This table reports maximum likelihood estimation results of the BiN-GARCH model for eight large developed markets' daily excess returns. The sample includes continuously compounded value-weighted returns on country indexes ending in December 31, 2009. Standard errors are reported below the estimated parameters. BIC represents Bayesian information criteria. Critical values for Likelihood ratio test statistic follow a χ^2 distribution with $df = 5$. *, †, and ‡ indicate statistical significance of the estimated parameters at 1%, 5%, and 10% levels, respectively. λ_i are as in $m_t = \lambda_0 + \lambda_1 \sigma_{1,t} + \lambda_2 \sigma_{2,t}$ with $\lambda_2 = -\lambda_1$ restriction imposed. δ_i are as in $p_t = \sqrt{2/(\pi - 2)} \tanh(\delta_0 + \delta_1 z_t^* \mathbf{1}_{\{z_t^* > 0\}} + \delta_2 z_t^* \mathbf{1}_{\{z_t^* < 0\}} + \delta_3 p_{t-1})$ with $z_t^* = (R_t - m_{t-1})/\sigma_{t-1}$. $E(p_t)$ is the sample average of conditional Pearson's mode skewness, p_t . Source: Thomson Reuters Datastream.

Table 5: Maximum Likelihood Estimation of the GARCH-in-Mode Model Using Emerging and Frontier Markets Equity Data.

	Czech Republic	Egypt	Hungary	Indonesia	Israel	Jordan	Korea	Philippines	Russia	South Africa
Estimated Parameter										
λ_0	0.0011* (0.0004)	0.0013* (0.0005)	0.0012† (0.0006)	5.45E-06 (0.0002)	0.0007* (0.0003)	0.0002 (0.0003)	-5.03E-05 (3.05E-04)	-0.0003 (0.0002)	0.0013* (0.0004)	0.0008* (0.0003)
λ_1	0.6120* (0.0656)	0.5756* (0.0500)	0.4031* (0.1730)	0.3897* (0.0627)	0.3143† (0.2103)	0.4835* (0.1252)	0.3526† (0.1547)	0.2660* (0.0386)	0.4039* (0.0860)	0.3086* (0.1318)
δ_0	-0.1081 (0.0624)	-0.0927* (0.0299)	0.0507 (0.0530)	0.0256* (0.0089)	-0.0133 (0.0135)	-0.0546* (0.0227)	-0.0127 (0.0167)	0.0422† (0.0168)	-0.0751* (0.0270)	-0.0685 (0.0719)
δ_1	0.3280* (0.0782)	0.3494* (0.0890)	-0.0446 (0.0599)	0.0763* (0.0098)	0.0303† (0.0182)	0.2092* (0.0608)	0.0868* (0.0245)	0.1194* (0.0242)	0.2111* (0.0410)	0.1217* (0.0281)
δ_2	0.1797* (0.0765)	0.0236 (0.0315)	0.2125* (0.0755)	0.1042* (0.0159)	0.0474* (0.0190)	0.0068 (0.0371)	0.0340 (0.0228)	0.1360* (0.0184)	0.0538* (0.0202)	0.0839* (0.0372)
δ_3	-0.2239† (0.1169)	0.5013* (0.0745)	-0.0350 (0.1619)	0.5063* (0.0338)	0.4500* (0.1224)	0.2646† (0.1197)	0.2286† (0.0920)	0.0023 (0.1018)	0.0968 (0.1483)	-0.0618* (0.0237)
$\mathbb{E}(p_t)$	-0.0651	0.0633	-0.0671	0.0808	-0.0718	0.0252	0.0146	0.0553	-0.0338	-0.0740
Diagnostic Measures										
LR Test Stat.	32.5851*	91.5614*	15.2339†	236.6578*	30.5994*	16.7032†	26.2606*	135.1989*	85.2035*	65.6521*
BIC	-2.6709	-2.5936	-2.4126	-2.6112	-2.8812	-2.8502	-2.5780	-2.7529	-2.2211	-2.7794

This table reports maximum likelihood estimation results of the BiN-GARCH model for eight large developed markets' daily excess returns. The sample includes continuously compounded value-weighted returns on country indexes ending in December 31, 2009. Standard errors are reported below the estimated parameters. BIC represents Bayesian information criteria. Critical values for Likelihood ratio test statistic follow a χ^2 distribution with $df = 5$. *, †, and ‡ indicate statistical significance of the estimated parameters at 1%, 5%, and 10% levels, respectively. λ_i are as in $m_t = \lambda_0 + \lambda_1 \sigma_{1,t} + \lambda_2 \sigma_{2,t}$ with $\lambda_2 = -\lambda_1$ restriction imposed. δ_i are as in $p_t = \sqrt{2/(\pi-2)} \tanh(\delta_0 + \delta_1 z_t^* \mathbf{1}_{\{z_t^* > 0\}} + \delta_2 z_t^* \mathbf{1}_{\{z_t^* < 0\}} + \delta_3 p_{t-1})$ with $z_t^* = (R_t - m_{t-1})/\sigma_{t-1}$. $\mathbb{E}(p_t)$ is the sample average of conditional Pearson's mode skewness, p_t . Source: Thomson Reuters Datastream.

Table 6: Maximum Likelihood Estimation of the GARCH-in-Mode Model for 2000-2009 Period.

Panel A								
Estimated Parameter	France	Germany	Italy	Japan	Netherlands	Switzerland	U.K.	U.S.
λ_0	-0.0002 (0.0003)	0.0002 (0.0004)	2.85E - 05 (0.0005)	-0.0009* (0.0004)	1.80804E - 05 (0.0003)	0.0002 (0.0002)	-0.0001 (0.0002)	-0.0007* (0.0003)
λ_1	0.8622* (0.1147)	0.7100* (0.1818)	0.7504* (0.2058)	1.1475* (0.1700)	0.7361* (0.1942)	0.5606* (0.1497)	0.8172* (0.1188)	0.9868* (0.1248)
$\mathbb{E}(p_t)$	-0.1286	-0.1296	-0.0832	-0.0004	-0.1287	-0.0834	-0.1516	-0.1268
Panel B								
Estimated Parameter	Australia	Denmark	Greece	Hong Kong	Ireland	New Zealand	Norway	Singapore
λ_0	0.0008† (0.0003)	0.0011† (0.0004)	0.0005 (0.0003)	0.0004 (0.0003)	-0.0003 (0.0005)	0.0007† (0.0004)	0.0009* (0.0004)	0.0004† (0.0002)
λ_1	0.5516* (0.0688)	0.4243† (0.1854)	0.2032 (0.2527)	0.4444* (0.0981)	0.8799* (0.1665)	0.5937* (0.1145)	0.6244* (0.1192)	0.6808* (0.0704)
$\mathbb{E}(p_t)$	-0.1150	-0.0758	0.0627	-0.1222	-0.1178	-0.0758	-0.0599	-0.0647
Panel C								
Estimated Parameter	Czech Republic	Egypt	Hungary	Indonesia	Israel	Jordan	Korea	Philippines
λ_0	0.0011* (0.0004)	0.0013* (0.0005)	0.0012† (0.0006)	0.0007† (0.0004)	0.0008 (0.0005)	0.0002 (0.0003)	0.0010† (0.0005)	-0.0008† (0.0004)
λ_1	0.6120* (0.0656)	0.5756* (0.0500)	0.4031* (0.1730)	0.4883* (0.0808)	0.4753† (0.2489)	0.4835* (0.1244)	0.5727* (0.1183)	0.3141* (0.0299)
$\mathbb{E}(p_t)$	-0.0651	0.0633	-0.0671	0.0803	-0.0716	0.0252	0.0169	0.0528
South Africa								
λ_0	0.0011† (0.0006)	0.0013* (0.0006)						
λ_1	0.6685* (0.1120)	0.3061 (0.2200)						
$\mathbb{E}(p_t)$	-0.0384	-0.0722						

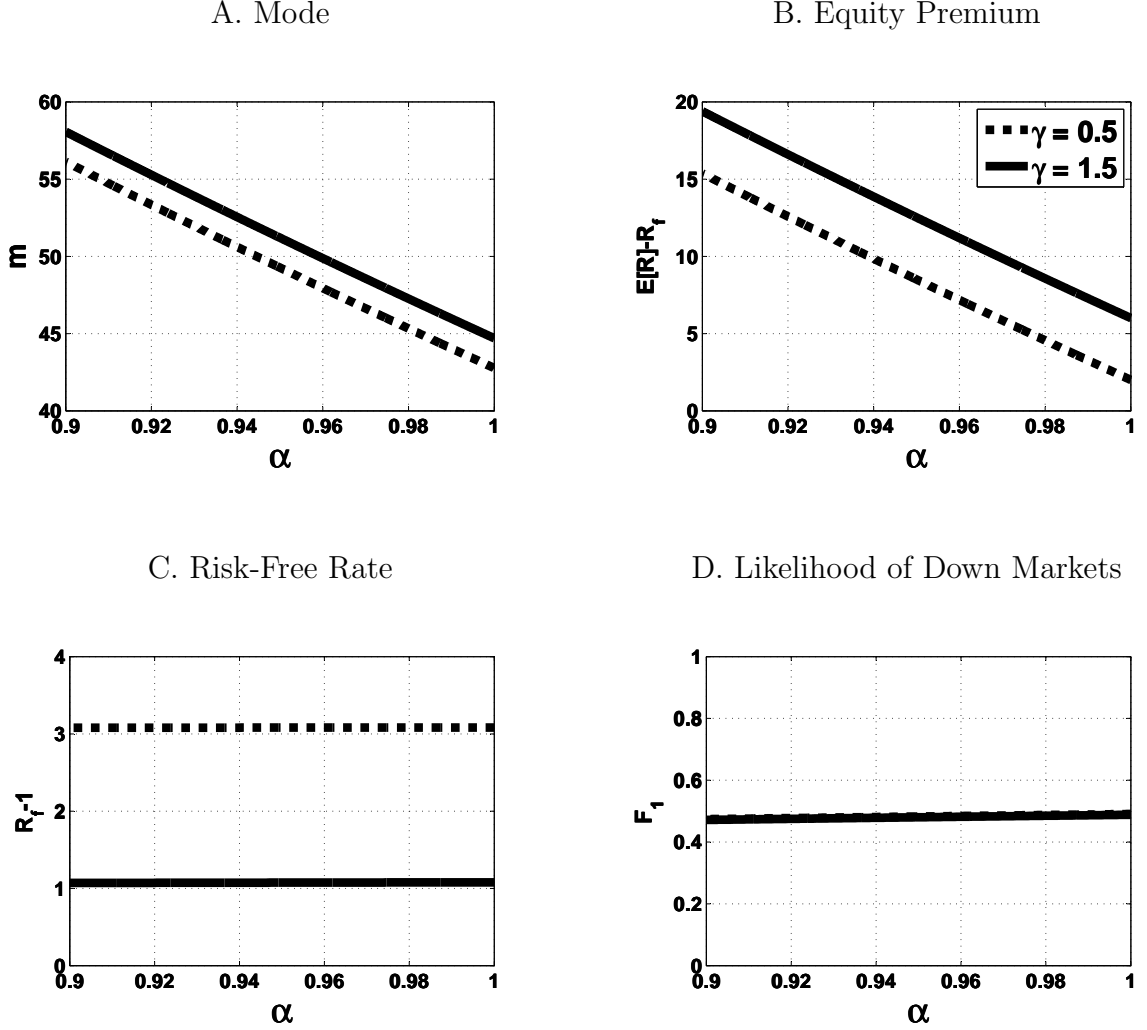
This table reports maximum likelihood estimation results of the BiN-GARCH model for international markets' daily excess returns. The sample includes continuously compounded value-weighted returns on country indexes ending in December 31, 2009. Standard errors are reported below the estimated parameters. *, †, and ‡ indicate statistical significance of the estimated parameters at 1%, 5%, and 10% levels, respectively. λ_i are as in $m_t = \lambda_0 + \lambda_1 \sigma_{1,t} + \lambda_2 \sigma_{2,t}$ with $\lambda_2 = -\lambda_1$ restriction imposed. $\mathbb{E}(p_t)$ represents sample average of conditional Pearson's mode skewness, $p_t = \sqrt{2/(\pi - 2)} \tanh(\delta_0 + \delta_1 z_t^* \mathbf{1}_{\{z_t^* > 0\}} + \delta_2 z_t^* \mathbf{1}_{\{z_t^* < 0\}} + \delta_3 p_{t-1})$ with $z_t^* = (R_t - m_{t-1})/\sigma_{t-1}$. Source: Thomson Reuters Datastream.

Table 7: **Test of Independence and Unconditional Coverage**

Parameter	$\xi = 1\%$	$\xi = 5\%$
π_{01}	0.9783	0.9593
π_{10}	0.0121	0.0457
π_1	0.9878	0.9545
LR_{uc}	3.3655 [0.9334]	3.3055 [0.9310]
LR_{ind}	1.9668 [0.8392]	1.5978 [0.7938]
$LR_{uc} + LR_{ind}$	5.3323 [0.9305]	4.9033 [0.9139]

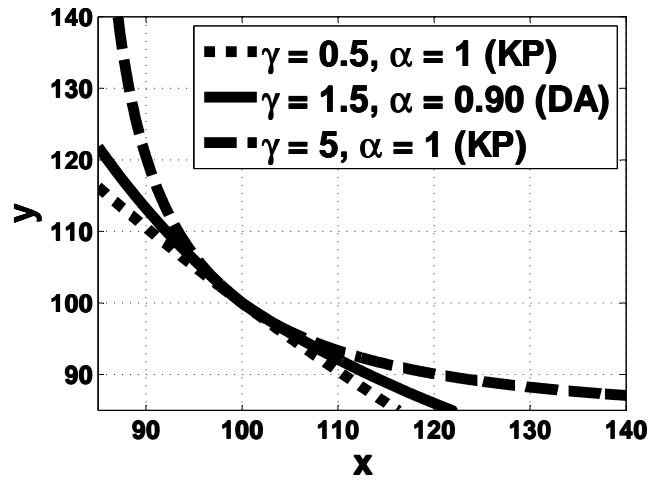
This table presents likelihood ratio tests for conditional coverage of the BiN-GARCH model, using S&P500 daily excess returns. $1 - \xi$ is the confidence level of the daily Value-at-Risk. The sample includes continuously compounded value-weighted returns on the S&P500 Index from January 2, 1980, to December 31, 2009, a total of 7,569 observations. p -Values are reported below the estimates.

Figure 1: Steady State Equilibrium Asset Pricing Implications



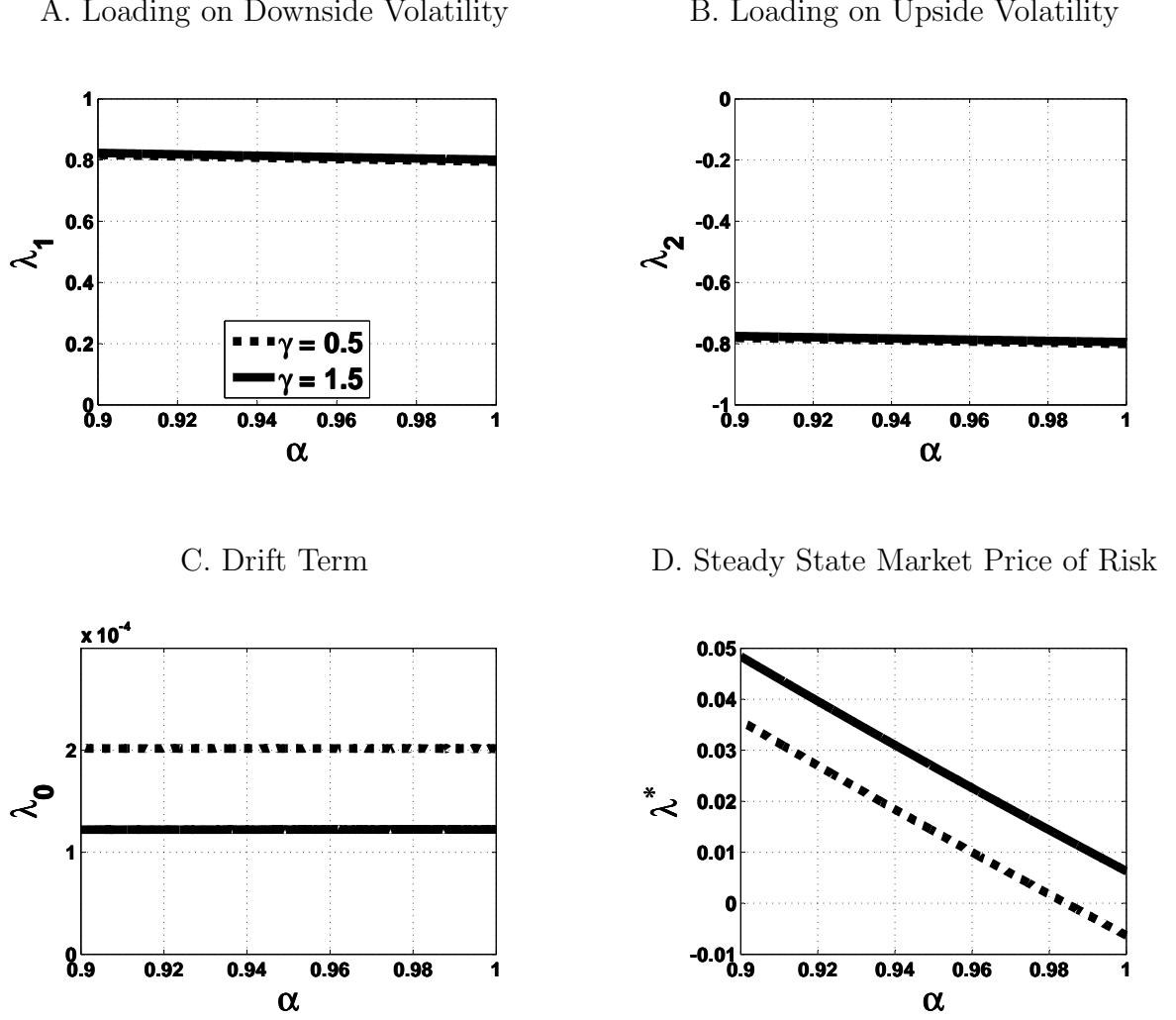
We calibrate the steady state values of downside and upside volatilities ($\bar{\sigma}_1, \bar{\sigma}_2$) to achieve an annualized steady state downside volatility of $\bar{\sigma} = 20\%$, and a steady state daily Pearson mode skewness of $\bar{p} = -0.125$. We calibrate the risk aversion parameter γ to 0.5 and 1.5 and the annualized daily subjective discount factor to $\delta = 0.96$. Given $\bar{\sigma}_1, \bar{\sigma}_2, \gamma, \delta$ and varying the disappointment aversion parameter $\alpha \in (0, 1]$, we solve for the steady state equilibrium mode \bar{m} as solution to the nonlinear equation $G(m, \bar{\sigma}_1, \bar{\sigma}_2) = 0$. We then compute the steady state equilibrium risk-free rate $R_f - 1$ and likelihood of down markets, $F_1 = 1 - M_t(0; -\ln \delta)$, and finally the steady state equilibrium equity premium $E[r] - r_f$ from $E_t[r_{t+1}] - r_{f,t+1} = M_t(1) - r_{f,t+1}$, where $M_t(u) = 2\sigma_{1,t}/(\sigma_{1,t} + \sigma_{2,t}) \exp(m_t u + (\sigma_{1,t}^2 u^2/2)) \Phi(-\sigma_{1,t} u) + 2\sigma_{2,t}/(\sigma_{1,t} + \sigma_{2,t}) \exp(m_t u + (\sigma_{2,t}^2 u^2/2)) \Phi(-\sigma_{2,t} u)$ and $M_t(u; x)$ is as in Eq. (17) and (18). We plot the four equilibrium quantities against α , restricting to values of α that yield a steady state equilibrium equity premium below 20%.

Figure 2: Effective Risk Aversion: Indifference Curves for DA Preferences



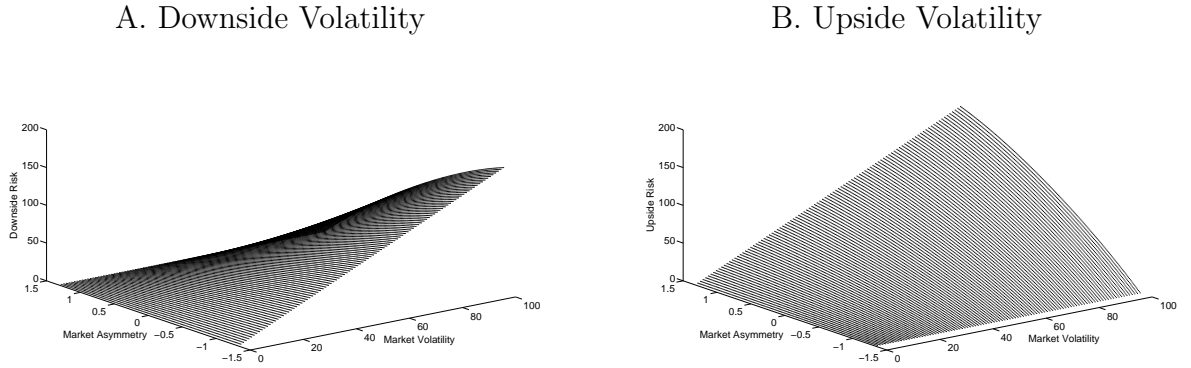
Indifference curves over two outcomes x and y with the fixed probability $p = Prob(x) = 1/2$. DA represents Gul (1991) disappointment aversion preferences and KP represents the familiar Kreps and Porteus (1978) preferences. Parameters of choice are γ and α . Disappointment aversion disappears when $\alpha = 1$. The values of α which yield a steady state equity premium below 20% are between 0.90 and 1. The lowest effective risk aversion case considered in our study implies $\gamma = 0.50$, and the highest effective case of risk aversion implies $\gamma = 5$. This figure shows that DA preferences with highest effective risk aversion parameter combination ($\alpha = 0.90, \gamma = 1.50$) exhibits lower risk aversion than KP preferences with $\gamma = 5$.

Figure 3: Equilibrium Risk-Return Tradeoff Implications



We calibrate the steady state values of downside and upside volatilities $(\bar{\sigma}_1, \bar{\sigma}_2)$ to achieve an annualized steady state downside volatility of $\bar{\sigma} = 20\%$, and a steady state daily Pearson mode skewness of $\bar{p} = -0.125$. We calibrate the risk aversion parameter γ to 0.5 and 1.5 and the annualized daily subjective discount factor to $\delta = 0.96$. Given $\bar{\sigma}_1, \bar{\sigma}_2, \gamma, \delta$ and varying the disappointment aversion parameter $\alpha \in (0, 1]$, we solve for the steady state equilibrium mode \bar{m} as solution to the nonlinear equation $G(m, \bar{\sigma}_1, \bar{\sigma}_2) = 0$. We then compute the constant coefficients λ_0, λ_1 and λ_2 of conditional mode in its linearized relationship with downside and upside volatilities from $\lambda_i = -\frac{G_{\sigma_i}(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}, i = 1, 2$, and finally the equilibrium steady state value λ^* of the market price of risk in the traditional risk-return trade-off from $\lambda_t^* = \left(1 - (\lambda_1 - \lambda_2) \sqrt{\pi/8}\right) p_t + (\lambda_1 + \lambda_2) \sqrt{1 - (3\pi/8 - 1) p_t^2}$. We plot the four quantities of interest against α , restricting to values of α that yield a steady state equilibrium equity premium below 20%.

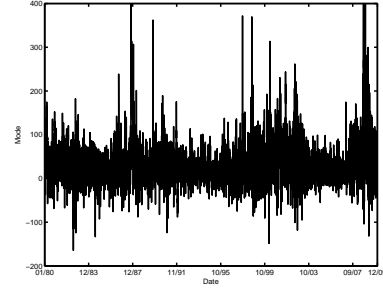
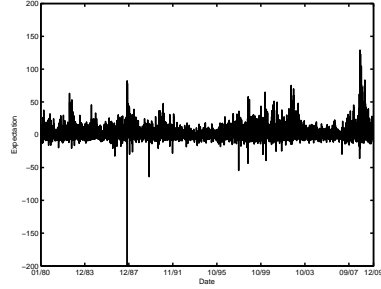
Figure 4: Downside and Upside Volatilities vs. Market Volatility and Market Asymmetry



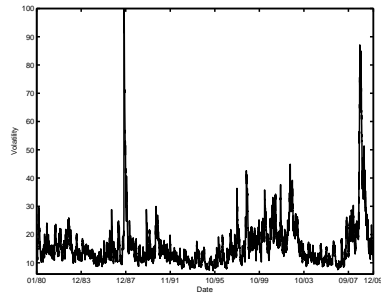
The above graphs are generated using $\sigma_t^2 = (1 - 2/\pi)(\sigma_{2,t} - \sigma_{1,t})^2 + \sigma_{1,t}\sigma_{2,t}$, $p_t = \sqrt{2/\pi}(\sigma_{2,t} - \sigma_{1,t})/\sigma_t$, and $\sigma_{i,t} = \sigma_t[(-1)^i \sqrt{\pi/8}p_t + \sqrt{1 - (3\pi/8 - 1)p_t^2}]$, where $i = 1, 2$. The values on the vertical axis in Panel A are conditional downside risk, conditioned on market volatility and asymmetry. The values on the vertical axis in Panel B are conditional upside risk, conditioned on market volatility and asymmetry. The range of values are $[0, 100]$ for market volatility, and $[-1.32, 1.32]$ for asymmetry.

Figure 5: **BiN-GARCH Conditional Moments for S&P500 Annualized Daily Excess Returns**

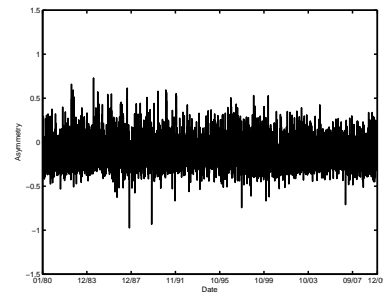
A. Annualized Expected Excess Returns B. Annualized Conditional Mode



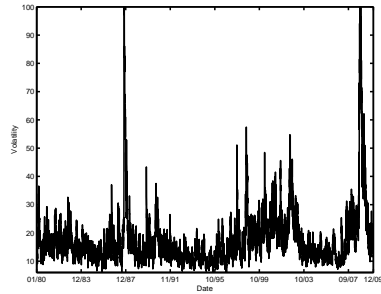
C. Annualized Market Volatility



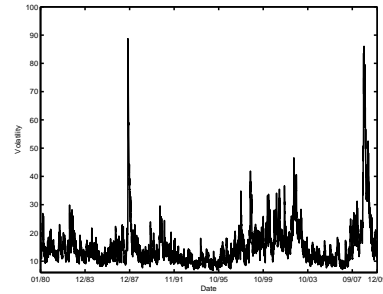
D. Pearson Mode Skewness



E. Annualized Downside Volatility

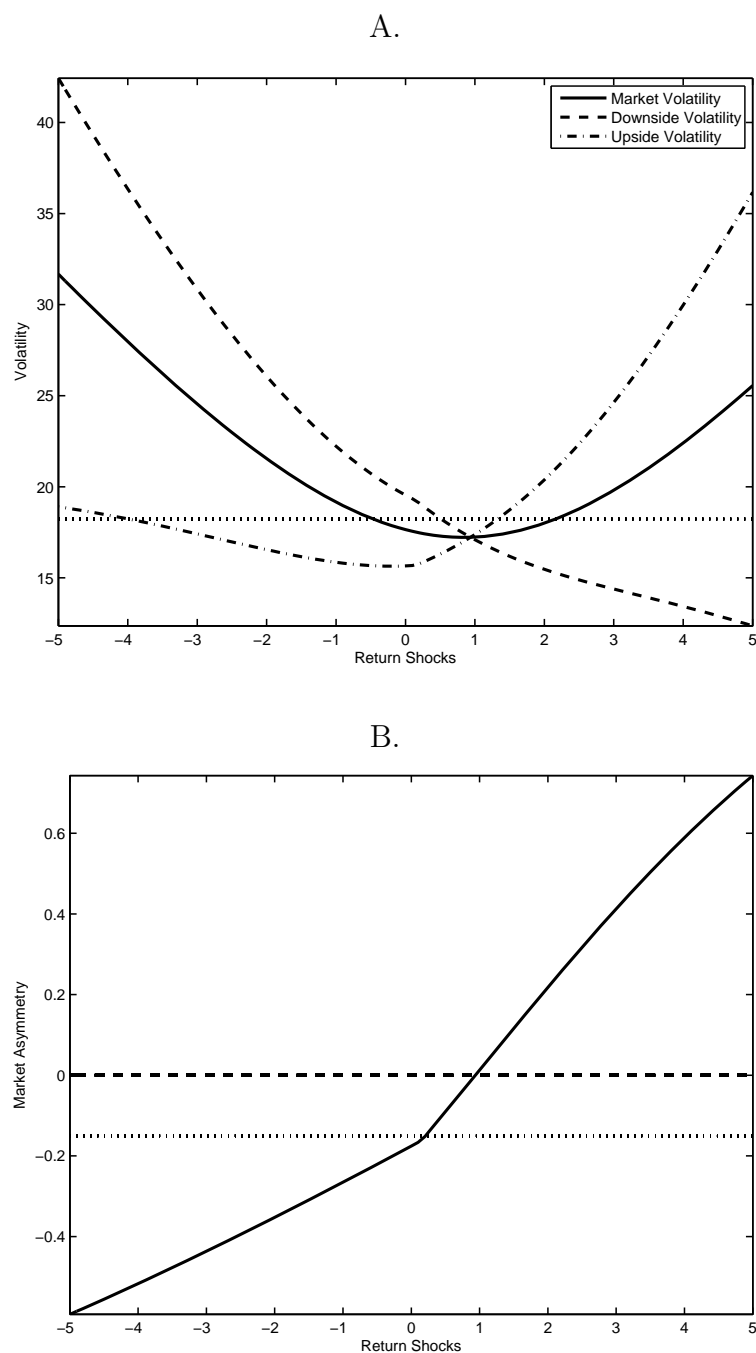


F. Annualized Upside Volatility



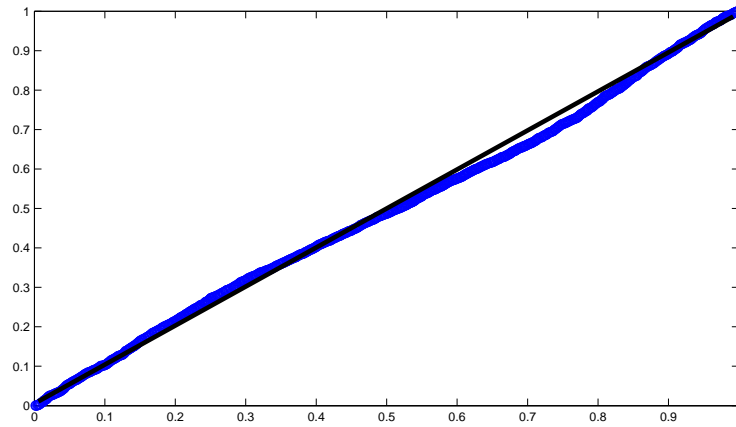
The figures above show conditional moments for annualized daily S&P 500 index excess returns. These values are filtered after fitting the returns series, using Specification (IV) in Table 2. In Panel A, we report annualized expected excess returns. Panel B reports annualized conditional mode. In Panel C, we report conditional “total” market volatility. Conditional Pearson mode skewness appears in Panel D. We report filtered annualized downside and upside market volatility measures in Panels E and F, respectively. Sampling period is Jan. 2, 1980, to Dec. 30, 2009. Source: Thomson Reuters’ Datastream.

Figure 6: **Volatility and Asymmetry News Impact Curves**



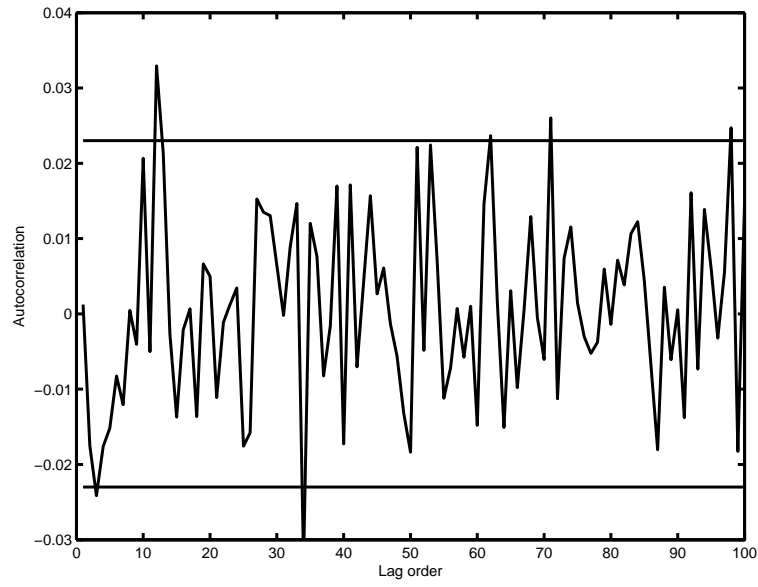
Panel A of this figure displays the response of market, upside, and downside volatility to negative and positive return shocks. Similarly, in Panel B, we observe the impact of positive and negative return shocks on market asymmetry. We view the response of market asymmetry to negative and positive return shocks as supportive of presence of a “asymmetry in asymmetry” effect in market returns data.

Figure 7: **Density Forecast of Standardized Residuals for S&P 500 Daily Returns**



This figure shows the QQ plot of transformed variables $u_{t-1} = F(R_t|I_{t-1})$, where $F(R_t|I_{t-1}) = 1 - M_{t-1}(0; R_t)$, for annualized daily S&P500 index returns on the vertical axis, against the QQ plot for a uniformly distributed random variable on the horizontal axis. Based on Diebold et al. (1998) and Bai (2003), if the forecasts are correct, then u_{t-1} should be uniformly distributed on $[0, 1]$ interval. The results are based on a BiN-GARCH fit of the S&P500 returns series. The sampling period is Jan. 2, 1980, to Dec. 30, 2009. Source: Thomson Reuters' Datastream.

Figure 8: **Autocorrelation Function of Standardized Residuals**



This figure plots autocorrelation function for standardized residuals from fitting annualized daily S&P500 index returns using the BiN-GARCH model. Based on uniform distribution of U_t 's, established in Section 4.4, we expect to observe standard Normal distributed $\Phi^{-1}(u_t)$. Φ is the standard Normal cumulative distribution function. The horizontal lines denote 95% Bartlett confidence intervals around zero. The sampling period is Jan. 2, 1980, to Dec. 30, 2009. Source: Thomson Reuters' Datastream.