Math 5610/6860: Assignment 3

Introduction to Numerical Analysis-1, Fall 2022

Due date: November, 18

Instructions:

1. *You are free to discuss the homework questions with your classmates. Sharing of solutions is not allowed.*
2. *Write your own codes, do not copy.*
3. *Submit your code files separately, but discuss the results in your main solution file.*
4. *Show the work done to obtain answers/solutions.*
5. *Questions marked with [G^*] are required for students registered for* 6860. *Others registered for* 5610 *need not answer the question.*

Problem 1 (20 points)

Consider the function $f(x) = \cos 2x^2 - x^2$ in the interval $0 < x < 1.5$. Use a tolerance of $1e - 10$ on the absolute error.

a) Using an initial guess of $x^{(0)} = 0.01$, use Newton's method to find the zero (root) of the function.
b) Plot the absolute error in each iteration of part a).
c) With numerical experiments, find the interval for $x^{(0)}$ over which the solution obtained from Newton's method converges to the root obtained in a).

Problem 2: (20 points)

State the theorem that states the convergence for fixed point iterations.

Given a function $f(x) = x^2 - x - 2$, consider the fixed point method $x^{(k+1)} = \phi(x^{(k)})$ to find the zeros $\alpha_1 = -1$ and $\alpha_2 = 2$. For the following iteration functions, analyze the convergence of the fixed point method for the two zeros of the function,

a) $\phi_1(x) = x^2 - 2$
b) $\phi_2(x) = \sqrt{2 + x}$
c) $\phi_3(x) = -\sqrt{2 + x}$
d) $\phi_4(x) = 1 + \dfrac{2}{x}$

Problem 3: (20 points)

State all the Karush-Kuhn-Tucker conditions for the following constrained optimization problem:

$$\begin{aligned} &\min_{x,y} f(x, y) \\ \text{subject to: } &h(x, y) = 0 \\ &g(x, y) \leq 0 \end{aligned} \tag{1}$$

Problem 4: (20 points)

Solve the following constrained optimization problem:

$$\begin{aligned} &\max_{x,y} xy \\ \text{subject to: } &x + y^2 \leq 2 \\ &x \geq 0 \\ &y \geq 0 \end{aligned} \tag{2}$$

a) Apply all the KKT conditions to this problems and state the resulting expressions clearly.
b) Solve the resulting equations and present your answer

Problem 5: (20 points)

Solve the following non-linear system of equations,

$$
\begin{aligned}
\frac{-1}{81}\cos x_1 + \frac{x_2^2}{9} + \frac{\sin x_3}{3} &= x_1 \\
\frac{1}{3}\sin x_1 + \frac{\cos x_3}{3} &= x_2 \\
\frac{-1}{9}\cos x_1 + \frac{x_2}{3} + \frac{\sin x_3}{6} &= x_3
\end{aligned}
\tag{3}
$$

using a fixed point iteration method $\boldsymbol{x}^{(k+1)} = \boldsymbol{\psi}(\boldsymbol{x}^{(k)})$, where $\boldsymbol{x} = [x_1, x_2, x_3]^T$ and $\boldsymbol{\psi}(\boldsymbol{x})$ is the left hand side of Equation (3).

a) Check if the function $\boldsymbol{\psi}(\boldsymbol{x})$ converges to compute the fixed point $\boldsymbol{\alpha} = [0, 1/3, 0]^T$

b) Numerically implement a fixed point iteration scheme to verify the solution given in part a) for your choice of initial guess.

c) Plot the solution in each iteration to observe convergence.

a) $J_G = \begin{bmatrix} \frac{1}{81}\sin(x_1) & \frac{2}{9}x_2 & \frac{1}{3}\cos(x_3) \\ \frac{1}{3}\cos(x_1) & 0 & -\frac{1}{3}\sin(x_3) \\ \frac{1}{9}\sin(x_1) & \frac{1}{3} & \frac{1}{6}\sin(x_3) \end{bmatrix}$

$\alpha = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$ $\quad J_G(\alpha) = \begin{bmatrix} 0 & \frac{2}{27} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ $\quad \rho(J_G(\alpha)) = 0.36 < 1$

converges

Problem 1 (20 points)

Consider the function $f(x) = \cos 2x^2 - x^2$ in the interval $0 < x < 1.5$. Use a tolerance of $1e-10$ on the absolute error.

a) Using an initial guess of $x^{(0)} = 0.01$, use Newton's method to find the zero (root) of the function.

b) Plot the absolute error in each iteration of part a).

c) With numerical experiments, find the interval for $x^{(0)}$ over which the solution obtained from Newton's method converges to the root obtained in a).

On notebook

Problem 2: (20 points)

State the theorem that states the convergence for fixed point iterations.

Given a function $f(x) = x^2 - x - 2$, consider the fixed point method $x^{(k+1)} = \phi(x^{(k)})$ to find the zeros $\alpha_1 = -1$ and $\alpha_2 = 2$. For the following iteration functions, analyze the convergence of the fixed point method for the two zeros of the function,

a) $\phi_1(x) = x^2 - 2$ $\quad 2, -2, 0, \sqrt{2}, \sqrt{\sqrt{2}-2} \quad -1, 1, \sqrt{3}, \sqrt{\sqrt{3}-2}$

b) $\phi_2(x) = \sqrt{2+x}$

c) $\phi_3(x) = -\sqrt{2+x}$

d) $\phi_4(x) = 1 + \dfrac{2}{x}$

Convergence of fixed point iterations.

Thm: Consider the sequence given by Iterative Scheme: $x^{(k+1)} = \phi(x^{(k)})$
Given $x^{(0)}$, assume:

a) $\phi : [a,b] \to [a,b]$

b) $\phi \in C^1([a,b])$ $\qquad f \in C^0$
[aka continuous and differentiable].

c) $\exists\, k<1: \quad |\phi'(x)| \le k \quad \forall x \in [a,b]$

Then ϕ has a unique fixed point α in $[a,b]$ & the sequence $\{x^{(k)}\}$ converges to α $\forall$ choice of $x^{(0)} \in [a,b]$

Moreover, $$\lim_{k\to\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha)$$

Thm 2: If $\phi \in C^{p-1}(J)$ (function has derivatives that is continuous up to p+1) for a neighborhood J of α and $p \geq 0$ ($\mathbb{Z}$) & if $\phi^{(i)}(\alpha) = 0$ for $i = 1, \ldots, p$ and $\phi^{(p+1)}(\alpha) \neq 0$

Then the fixed point iteration has order p+1 and

$$\lim_{k \to \infty} \frac{x^{(k+1)} - \alpha}{(x^{(k)} - \alpha)^{(p+1)}} = \frac{\phi^{(p+1)}}{(p+1)!} \qquad p \geq 0$$

a) $\phi_1(x) = x^2 - 2$

$\phi_1'(x) = 2x$

$|\phi_1'(x)| \leq 1 \Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}$

$\phi_1'(\alpha_1 = -1) = -2 \longrightarrow |-2| > 1$ divergent

$\phi_1'(\alpha_2 = 2) = 4 \longrightarrow |4| > 1$

b) $\phi_2(x) = \sqrt{2+x}$

$\phi_2'(x) = \frac{1}{2\sqrt{2+x}} \qquad |\phi_2'(x)| < 1$

$\phi_2'(-1) = \frac{1}{2} \quad \phi_2'(2) = \frac{1}{4}$ both points are convergent

c) $\phi_3(x) = -\sqrt{2+x}$

$\phi_3'(x) = -\frac{1}{2\sqrt{2+x}}$ $\quad |\phi_2'(x)| < 1$

$\phi_3'(-1) = -\frac{1}{2} \rightarrow |-\frac{1}{2}| < 1$ $\quad \phi_3'(2) = -\frac{1}{4} \rightarrow |-\frac{1}{4}| < 1$ both points are convergent

d) $\phi_4(x) = 1 + \frac{2}{x}$

$\phi_4'(x) = \frac{-2}{x^2}$ $\quad |\phi_2'(x)| < 1 \Rightarrow \sqrt{2} < x$ or $x < -\sqrt{2}$

$\phi_4'(-1) = -2 \rightarrow |-2| > 1$ not convergent to α_1

$\phi_4'(2) = -\frac{1}{2} \rightarrow |-\frac{1}{2}| < 1$ convergent to α_2 on bounds

Problem 3: (20 points)
State all the Karush-Kuhn-Tucker conditions for the following constrained optimization problem:

$$\begin{aligned} &\min_{x,y} f(x,y) \\ \text{subject to: } &h(x,y) = 0 \\ &g(x,y) \le 0 \end{aligned} \tag{1}$$

$$\mathcal{L}(\underline{x}, \underline{\mu}, \underline{\lambda}) = f(\underline{x}) + \underline{\mu}^T g(\underline{x}) + \underline{\lambda}^T h(\underline{x})$$

If f has a constrained local maxima at the point $\underline{x} = \underline{x}^*$, it is necessary that a vector $\underline{\lambda}^* \in \mathbb{R}^{m+n}$ exists s.t.,

$$\nabla_x \mathcal{L}(\underline{x}^*, \underline{\lambda}^*) \le 0$$

with strict equality for every component 'i' s.t. $x_i^* > 0$

KKT Condition (for this situation)

-1st Stationary

$$\nabla_{x,y} \mathcal{L}(x,y,\mu,\lambda) = 0$$

-2nd) Primal Feasibility

$h(x,y)=0 \quad g(x,y)\leq 0$

-3rd) Dual Feasibility

$\mu \geq 0$

-4th) Complementary Slackness

$\mu g(x,y)=0$

Problem 4: (20 points)
Solve the following constrained optimization problem:

$$\begin{aligned} &\max_{x,y} xy \\ \text{subject to: } & x + y^2 \leq 2 \\ & x \geq 0 \\ & y \geq 0 \end{aligned} \tag{2}$$

a) Apply all the KKT conditions to this problems and state the resulting expressions clearly.
b) Solve the resulting equations and present your answer

Find $\max\limits_{x,y} f(x) \qquad f(x)=xy$

$$\mathcal{L}(x,y,\mu_1,\mu_2,\mu_3) = -xy + \mu_1(x+y^2-2) + \mu_2(x) + \mu_3(y)$$

Stationary Check:

$$\frac{\partial \mathcal{L}}{\partial x} = -y + \mu_1 + \mu_2 = 0 \qquad \frac{\partial \mathcal{L}}{\partial y} = -x + 2y\mu_1 + \mu_3 = 0$$

Primal Feasibility: no $h(x,y)$ so skip

Dual Feasibility:

$\mu_1 \geq 0 \quad \mu_2 \geq 0 \quad \mu_3 \geq 0$

Complementary Slackness:

$\mu_1(x+y^2-2)=0 \quad \mu_2(x)=0 \quad \mu_3(y)=0$

Compute/Solve:

123 1) $\mu_1 = \mu_2 = \mu_3 = 0$

$$\frac{\partial L}{\partial x} = -y = 0 \qquad \frac{\partial L}{\partial y} = -x = 0$$

$(x,y) \to (0,0) \quad f(0,0) = 0$

0 2) $\mu_1 > 0 \quad \mu_2 > 0 \quad \mu_3 > 0$

Not possible $\to$ $\mu_2(x) = 0 \quad \mu_3(y) = 0$

$(x,y) = (0,0) \quad \mu_1(-2) = 0 \Rightarrow \mu_1 = 0$

1 3) $\mu_1 = 0$

Not possible $\to$ $-y + \mu_2 = 0 \qquad -x + \mu_3 = 0$

$\mu_2(x) = 0 \qquad \mu_3(y) = 0$

$(x,y) = (0,0) \qquad \Rightarrow m_2 = m_3 = 0$

2 4) $\mu_2 = 0$

$-y + \mu_1 = 0 \qquad -x + 2y\mu_1 + \mu_3 = 0$

$\mu_1(x + y^2 - 2) = 0 \qquad \mu_3(y) = 0$

1,2 4a)

$\mu_3 > 0 \Rightarrow y = 0, \ \mu_1 = 0, \ \mu_3 = x \qquad (x,0) \quad f(x,0) = 0$

2,3 4b) $\mu_1 > 0 \Rightarrow y = \mu_1, \ \mu_3 = 0,$

$y(x + y^2 - 2) = 0 \qquad -x + 2y \cdot y = 0$

$x + \frac{x}{2} - 2 = 0 \qquad y = \sqrt{\frac{x}{2}}$

$\frac{3}{2}x = 2 \Rightarrow x = \frac{4}{3} \qquad y = \frac{2}{3}$

$f\left(\frac{4}{3}, \frac{2}{3}\right) = \frac{8}{9} \qquad \mu_1 > 0$

1,3 5) $\mu_1 = \mu_3 = 0$

$\mu_2 = y \quad x = 0 \quad f(0,y) = 0$

, 6) $\mu_3 = 0$

$\mu_1(x+y^2-2)=0 \quad \mu_2(x)=0$

$y=\sqrt{2} \quad x=0 \quad f(0,\sqrt{2})=0$

$-y+\mu_1+\mu_2=0 \quad -x+2y\mu_1=0$

$\mu_1=0 \quad \mu_2=\sqrt{2}$

Solution: $f(\frac{4}{3},\frac{2}{3})=\frac{8}{9} \quad \mu_1=\frac{2}{3} \quad \mu_2=\mu_3=0$