Math 5610/6860: Assignment 2

Introduction to Numerical Analysis-1, Fall 2022

Due date: October, 28

Instructions:

1. *You are free to discuss the homework questions with your classmates. Sharing of solutions is not allowed.*
2. *Write your own codes, do not copy.*
3. *Submit your code files separately, but discuss the results in your main solution file.*
4. *Show the work done to obtain answers/solutions.*
5. *Questions marked with [G^*] are required for students registered for* 6860. *Others registered for* 5610 *need not answer the question.*

Problem 1 (10 points)

a) Show that if the Jacobi method is convergent, then the JOR method converges if $0 < \omega < 1$, where ω is the relaxation parameter.

b) ([G^*]) Show that for any $\omega \in \mathbb{R}$, the spectral radius of the iteration matrix is given by $\rho(B_{SOR}(\omega)) \geq |\omega - 1|$, and hence find the range of ω where SOR fails to converge.

Problem 2: (10 points)

Show that the iteration schemes for the Gauss-Seidl and SOR methods are consistent.

Problem 3: (10 points)

Consider the following 2 linear systems of the type $\boldsymbol{Ax} = \boldsymbol{b}$ and determine for each of them that, if the Jacobi method and Gauss-Seidl iteration methods are used for finding the numerical solution, will they converge to a solution? If they do, which one is slower or faster?

$$A_1 = \begin{bmatrix} -3 & 3 & -6 \\ -4 & 7 & -8 \\ 5 & 7 & -9 \end{bmatrix} \tag{1}$$

$$A_2 = \begin{bmatrix} 7 & 6 & 9 \\ 4 & 5 & -48 \\ -7 & 3 & 8 \end{bmatrix} \tag{2}$$

Problem 4: (20 points)

Consider the linear system,

$$\begin{bmatrix} 100 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 498 \\ 3 \end{bmatrix} \tag{3}$$

Starting from an initial guess $\boldsymbol{x}_0 = [10, 10]^T$,

a) Write a code to find the solution of the above linear system (3) using the Jacobi Over Relaxation (JOR). Use a suitable value of the relaxation parameter ω. Use the 2-norm of the residual as the stopping criteria with a tolerance $tol = 1e - 6$. Write down the solution once the iterations stop after reaching the stopping criteria.

b) Plot the 2-norm of the residual vs. number of iterations

c) What is the total number of iterations required to converge?

d) What is the total number of iterations required to converge if Jacobi method is used instead of JOR?

Problem 5: (40 points)

Solve the linear system in (3) by posing it as a minimization problem. In doing so, answer the following questions:

a) Write the function ϕ which can be minimized to find the solution of (3)

b) Implement the Gradient method, starting from the same initial guess $\boldsymbol{x}_0$ as in problem 4, and using the 2-norm of the residual vector as the stopping criteria with a tolerance of $1e - 6$. Write down the solution after reaching convergence and mention the number of iterations required.

c) Show the contour plot of the function ϕ over a suitable domain that includes the initial point and plot the successive iterates $\boldsymbol{x}^{(k+1)}$. This will show you the directions in which the iteration scheme proceeds.

d) Repeat the steps in b) and c), but now for the conjugate gradient method. Show the successive iterates on a separate plot.

e) Comment on the number of steps required for the gradient method vs. the conjugate gradient method.

Problem 1 (10 points)

a) Show that if the Jacobi method is convergent, then the JOR method converges if $0 < \omega < 1$, where ω is the relaxation parameter.

b) ($[G^*]$) Show that for any $\omega \in \mathbb{R}$, the spectral radius of the iteration matrix is given by $\rho(B_{SOR}(\omega)) \geq |\omega - 1|$, and hence find the range of ω where SOR fails to converge.

a) If JM is convergent, then

$$\rho(B_{JM}) < 1$$

$$B_{JOR} = \omega B_{JM} + (1-\omega) I$$

To converge, $\rho(B_{JOR}) < 1$

$$\rho(B_{JOR}) = \rho(\omega B_{JM} + (1-\omega)I)$$

$$\leq \rho(\omega B_{JM}) + \rho((1-\omega)I) \quad \text{Because JM converges, } \rho(B_{JM}) < 1$$

$$< |\omega| + |(1-\omega)| < 1 \quad \text{if } 0 < \omega \leq 1$$

so, $\rho(B_{JOR}) < 1$ and therefore JOR is convergent only if $0 < \omega \leq 1$

b) $$B_{SOR} = (I - \omega D^{-1}E)^{-1}[(1-\omega)I + \omega D^{-1}F]$$

Assume SOR is convergent, then $\rho(B_{SOR}) < 1$

First, $$\det[B_{SOR}] = \det[(I - \omega D^{-1}E)^{-1}((1-\omega)I + \omega D^{-1}F)]$$

$$= \det[(D - \omega E)^{-1}] \det[(1-\omega)D + \omega F]$$

$$= \frac{1}{\det[D - \omega E]} \det[(1-\omega)D + \omega F]$$

$$= \frac{1}{\det[D] - \det[wE]} \left(\det[(1-w)D] + \det[wF] \right)$$

because $\underline{\underline{E}}$ is lower triangular and $\underline{\underline{F}}$ is upper triangular w/ both diags are 0 AND w/ determinant of a triangular matrix being the product of the diag, then

$$= \frac{1}{\det[D]} \cdot \det[(1-w)D]$$

$$= \frac{1}{\det[D]} \cdot \det[(1-w)I] \det[D]$$

$$= \det[(1-w)I] = (1-w)^n$$

We can also express $\det[B_{SOR}]$ as $\prod_{i=1}^{n} \lambda_i$ where $\{\lambda_i\}$ are the eigenvalues of B_{SOR}, then we can say

$$|1-w|^n = |\det[B_{SOR}]| = \left| \prod_{i=1}^{n} \lambda_i \right| \leq \left(\max\{|\lambda_i|\} \right)^n = \left(\rho(B_{SOR}) \right)^n$$

$$\therefore |1-w|^n \leq \left(\rho(B_{SOR}) \right)^n$$

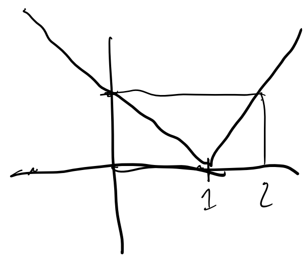
$$\Rightarrow |1-w| \leq \rho(B_{SOR})$$

And given $\rho(B_{SOR}) < 1$, then

$$|1-w| \leq \rho(B_{SOR}) < 1$$

$1 - w < 1 \Rightarrow 0 < w$

$w - 1 < 1 \Rightarrow w < 2$



SOR does not converge outside $0 < w < 2$

Problem 2: (10 points)

Show that the iteration schemes for the Gauss-Seidl and SOR methods are consistent.

GS)

$$
\begin{aligned}
x^{(k+1)} &= B_{GS}\, x^{(k)} + \underline{f} \\
&= (D-E)^{-1} F x + (D-E)^{-1} b \\
&= (D-E)^{-1} (D-E-A) x + (D-E)^{-1} b \\
&= \left(I - (D-E)^{-1} A\right) x + (D-E)^{-1} A x \\
&= I x - (D-E)^{-1} A x + (D-E)^{-1} A x \\
&= I x = x \quad \checkmark
\end{aligned}
$$

SOR)

$$
\begin{aligned}
x^{(k+1)} &= B_{SOR}\, x^{(k)} + \underline{f} \\
&= (D - wE)^{-1} \left[(1-w) D + wF\right] x + w (D - wE)^{-1} b \\
&= (D - wE)^{-1} \left[(1-w) D + w (D-E-A)\right] x + w (D - wE)^{-1} b \\
&= (D - wE)^{-1} \left[D - wE - wA\right] x + w (D - wE)^{-1} b \\
&= \left(I - w (D - wE)^{-1} A\right) x + w (D - wE)^{-1} b \\
&= I x - w (D - wE)^{-1} A x + w (D - wE)^{-1} A x = x \\
&\checkmark
\end{aligned}
$$

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$B_{Jm} = D^{-1}(E+F)$

$A = D - (E+F)$

$B_{GS} = (D-E)^{-1} F$

answers in Jupyter notebook

Problem 4: (20 points)

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answers in Jupyter Notebook