MATH3027 Optimization 2021: Coursework 2

This is the second piece of assessed coursework for MATH3027 Optimization 2021. It is worth **20% of your final mark** for the module.

The deadline for submission is Thursday 16 December at 10am. Your solution should be submitted electronically as a pdf file via the MATH3027 Moodle page. Late submissions will be subject to the usual penalties.

Since this work is assessed, **your submission must be entirely your own work** (see the University's policy on Academic Misconduct). You can use, without acknowledgement, any of the material or code from the lecture notes or computer labs.

You must answer all of the questions. Your solution has to be a single pdf or html file that is less than 30 pages in total (no credit will be given for material that occurs after page 30). Your solution can be a combination of handwritten and typed solutions, but you must include the code you use to find your solution as well as the output of the code. The easiest way to do this is to use Rmarkdown to create a pdf file.

NOTE: The markers will not be able to run your code, so if you only submit raw code with no output, you are unlikely to get many marks for your answer.

This assignment will be marked out of 50.

Question 1 [15 marks]

Solve the following optimization problems. In each case clearly report the optimal value of the problem, and a value of $\mathbf{x}$ at which the optima is achieved (denoted as $\mathbf{x}^*$ in the lecture notes). You may check your answer numerically if you wish, but to get full marks, you will have to submit a mathematical solution explaining why you have found the global optima in each case.

(a)

$$\begin{aligned} \max_{\mathbf{x}} \quad & x_1^2 + 2x_2^2 - 3x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{a}$$

Hint: Is this a convex optimization problem?

(b)

$$\begin{aligned} \min_{\mathbf{x}} \quad & x_1^2 - x_2^2 - x_3^2 \\ \text{s.t.} \quad & x_1^4 + x_2^4 + x_3^4 \leq 1 \end{aligned} \tag{b}$$

Hint: You will need to read the section of the notes that discusses the KKT conditions for non-linear constraints.

Question 2 [35 marks]

In your first job after university, you find yourself working for a communications company who ask you to find the optimal location for a remote communication mast. The mast will be responsible for transmiting signals to m different locations, which have coordinates $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^2$. The area capable of receiving signals from a mast can be considered to be the interior of a circle centred at the mast. The cost of running the communications mast is proportional to the radius of the transmission region, i.e., to transmit to a larger area costs more than transmitting to a smaller area.

Your task is to find a location for the mast, $\mathbf{x}$ say, that minimizes the maximum distance between the mast and each of the given locations, i.e., you want to find $\mathbf{x}$ to minimimize $\max_i ||\mathbf{a}_i - \mathbf{x}||_2$. Another way of thinking about this problem is to think in terms of finding the ball $\mathcal{B}(\mathbf{x}, r)$ of smallest possible radius, with $\mathbf{a}_i \in \mathcal{B}(\mathbf{x}, r)$ for $i = 1, \ldots, m$. We can write the optimization problem as

$$\begin{aligned} \min_{\mathbf{x},\gamma} \quad & \gamma \\ \text{s.t.} \quad & ||\mathbf{x} - \mathbf{a}_i||_2^2 \leq \gamma \text{ for } i = 1, 2, \ldots, m \end{aligned}$$

where $\gamma = r^2$ is the square of the radius.

- Is this a convex optimization problem? Explain your reasoning.
- Write down the Lagrangian and KKT conditions for this problem.
- Show that the dual problem is

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & -||\mathbf{A}\boldsymbol{\lambda}||^2 + \sum \lambda_i ||\mathbf{a}_i||^2 \\ \text{s.t.} \quad & \sum_{i=1}^{m} \lambda_i = 1 \\ & \lambda_i \geq 0 \quad \text{for} \quad i = 1, \ldots, m \end{aligned} \tag{D1}$$

  where $\mathbf{A}$ is the matrix with columns equal to the locations $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

- Describe (i.e. give pseudocode for) a projected gradient descent algorithm that could be used to solve the dual problem. You do not need to write code at this point. Describe how the solution to the primal problem can then be found.
- Another way of writing the constraint on $\boldsymbol{\lambda} \in \mathbb{R}^m$ is that $\boldsymbol{\lambda} \in \Delta_m$, the m-dimensional simplex. In this question, you will write code to compute the orthogonal projection operator $P_{\Delta_m}(\mathbf{y}) = \arg\min_{\mathbf{x} \in \Delta_m} ||\mathbf{x} - \mathbf{y}||$.

  The Lagrangian is

$$L(\mathbf{x}, \mu) = ||\mathbf{x} - \mathbf{y}||^2 + 2\mu(\mathbf{e}^\top \mathbf{x} - 1) \quad \text{ for } \mathbf{x} \geq 0,$$

  where $\mathbf{e}$ is a vector of 1s and $\mathbf{x} \in \mathbb{R}^m_+$. Show that if we minimize this with respect to $\mathbf{x} \in \mathbb{R}^m_+$ we obtain the dual problem

$$\max_{\mu} g(\mu) = -\sum_{i=1}^{m} [y_i - \mu]_+^2 - 2\mu + ||\mathbf{y}||^2 \tag{D2}$$

  with the minimum achieved at

$$\begin{aligned} x_i^* &= \begin{cases} y_i - \mu & \text{when } y_i \geq \mu \\ 0 & \text{otherwise} \end{cases} \\ &= [y_i - \mu]_+. \end{aligned}$$

Write code to solve this dual problem (D2) using a bisection method. As a check, you should find that the projection of the point $\begin{pmatrix} -1 \\ 1 \\ 0.3 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ 0.85 \\ 0.15 \end{pmatrix}$.

Hint: Let $g(\mu) = -\sum[y_i - \mu]_+^2 - 2\mu + ||\mathbf{y}||^2$. (D2)is an unconstrained optimization problem. If we can show that g achieves its maximum, then we know that the optimal solution, μ^*, must satisfy $g'(\mu^*) = 0$. You may find it useful to revisit computer lab 8, where we used the bisection method to find the root of an equation.

- Write code to implement a projected gradient descent algorithm to solve the dual problem (D1). Use your code to find the optimal sensor location and radius r if the locations are

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -3 & -4 & 5 & 2 \\ -4 & 2 & 5 & -3 & 2 & -2 \end{pmatrix}.$$

Create a plot showing the sensor locations and the smallest encompassing ball. If you have found the smallest ball, you should find at least two of the locations are on the edge of the ball.

- Check your answer using CVXR (or the equivalent package in Python or MATLAB).