Examples of the Dynamics of Three Person Iterated Prisoners Dilemma

Terry Guan

May 21, 2023

Abstract

In Akin's paper [1], the dynamics of playing the prisoner's dilemma iteratively is explored. In particular, results on evolutionary dynamics among memory one strategies on two-person symmetric games with multiple strategies were developed. The goal of this project is to generate examples of these games and plot their behavior for the case when there are three players/strategies. This was done numerically using SageMath. Examples of games with dominating equilibrium and mixed equilibrium were found.

1 Introduction

The dynamics of the iterative prisoner's dilemma is explored in Akin's paper [1]. In particular, results on evolutionary dynamics among memory one strategies on two-person symmetric games with multiple strategies were developed. We restrict ourselves to only such games with three strategies. We will generate examples of such games with different set of strategies and initial conditions. We will then analyze the behavior of our example games by numerically solving the system of differential equations that describes the games' dynamics and plotting them. This will be achieved by our program which is written in Python with SageMath. We will show that our program has generated examples where, in the long time behavior, only one strategy remains; a particular strategy dominating the remaining strategies. Additionally, we have generated examples where two strategies survive in the long time behavior. We will present our example games' plots.

2 Background

We first define the properties of our game. Recall how the prisoners dilemma game is played. For two players X, Y, they have two possible moves: c for cooperate and d for defect. For any two combinations of moves from players X, Y, we have the following payoffs: $R = cc, S = cd, T = dc, P = dd$. That is, R is the payoff for X, Y both playing cooperative (c), etc. The game has to satisfy the following conditions: $T > R > P > S$ and $2R > T + S$. A player has a strategy parametrized as a tuple (α, β). We can determine β by our choice of α, Z where $P \leq Z \leq R$ and $\alpha + \beta = -\frac{1}{Z}$. For any pair of players X, Y with respective strategies $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$, we can determine their respective *payoff* s_X, s_Y. The payoffs can be calculated with the following equations:

$$\alpha_1 s_X + \beta_1 s_Y = -1 \tag{1}$$

$$\beta_2 s_X + \alpha_2 s_Y = -1 \tag{2}$$

Our games have three strategies $(\alpha_i, \beta_i), 1 \leq i \leq 3$. For any pair of strategies X, Y We can calculate the *payoff matrix* A using $A_{ij} = s_X, A_{ji} = s_Y$.

The dynamics of our games follow the differential equations for evolutionary dynamics. The initial conditions $\pi = (\pi_1, \pi_2, \pi_3)$ represent the starting population of each player. We normalize π so that it sums to 1. We will often to refer to different strategies or players by the their initial condition index. For example, we will refer to player 1 as π_1. We have the following equations

$$A_{i\pi} = \sum_{1 \leq j \leq 3} \pi_j A_{ij} \tag{3}$$

$$A_{\pi\pi} = \sum_{1 \leq j \leq 3} \pi_i A_{i\pi} \tag{4}$$

that we use to describe the differential equation of evolutionary dynamics.

$$\frac{d\pi_i}{dt} = \pi_i (A_{i\pi} - A_{\pi\pi}) \tag{5}$$

Definition 2.0.1. In the long term behavior of our dynamics, we say that the dynamical system has reached a *dominating equilibrium* when exactly one strategy or player remains.

Definition 2.0.2. In the long term behavior of our dynamics, we say that the dynamical system has reached a *mixed equilibrium* when more than one player remains.

We can characterize each game by its payoff matrix A and initial conditions π. The payoff matrix can be determined by the choice of our three strategies which is parametrized by each α, Z pair. Thus, in generating examples of games, we make choices of α, Z and π.

3 Methodology

Examples of games and their plots were generated using Python and SageMath. Each game in the program was differentiated by their strategies and initial conditions. A Python class `Game` was written to encapsulate any necessary functions for computing the game numerically and plotting. Each game object is then initiated with a length three list of 2-tuples, representing the three strategies (α, β) parameters, and a given initial condition. The (α, β) parameters are determined by our choice of α, Z. The initial conditions were chosen uniformly to be points on a grid, i.e equidistant from each other, and lying below or on the grid's diagonal. The motivation of the choice of our initial conditions is to maintain the same initial conditions between games, and to generate plots that capture the dynamics of a particular game on a uniformly distributed set of initial condition points. The program also allows for user-inputted strategies and initial conditions.

Given a list of strategies, the corresponding payoff matrix is computed as a SageMath Matrix object. The payoff matrix is then used to create a symbolic system of differential equations. The system of differential equations is computed using SageMath's `desolve_system_rk4()`, an implementation of the Runge-Kutta method. The computation runs up to time 135 with time-steps 0.2. The output is a collection of points which is fed into SageMath's plot functions to generate a plot of the system with respect to time and a plot with respect to the populations.

Different examples of games are generated by randomly generating three strategies, solving the system of differential equations, and plotting their dynamics. Through this process we were able to discover examples of dominating equilibria and mixed equilibria. The code is available here for viewing.

4 Results

By Theorem 4.11 [1], we know there exist games with dominating equilibrium. In particular, if our choice of α parameters satisfy that either all $\alpha < 0$ or $\alpha > 0$ and there is a unique Z parameter such that $|Z|$ is maximal, then the strategy with the maximal $|Z|$ parameter will dominate; that is it will wipe out the population of the other strategies. This is illustrated by Figure 1. This set of strategies also has plots with respect to time with specific initial conditions. Refer to Figure 2. We can clearly see that there is domination by π_3, here labeled as strat2.

We now move into unfamiliar territory and look at parameters that don't satisfy the conditions of Theorem 4.11. In particular, we have found that there are examples of domination equilibrium when these conditions are not satisfied. In Figure 3 and 4, we have games that have dominating behavior with a set of strategies that don't satisfy the conditions of Theorem 4.11. For Figure 4, we also show the time plots for two different initial conditions

Along with domination equilibria, we ask if there are examples of games where there is a mixed equilbirum. That is, we want two or more strategies to survive. Proposition 4.12 [1] tells us that such equilibrium exist in the two strategies case. By experimenting with different strategies we have discovered multiple examples of mixed equilibria in the three strategies case. Take for example Figure 6. Interestingly, this is an example of a game where domination equilbirium or mixed equilbirum occurs depending on the initial conditions.

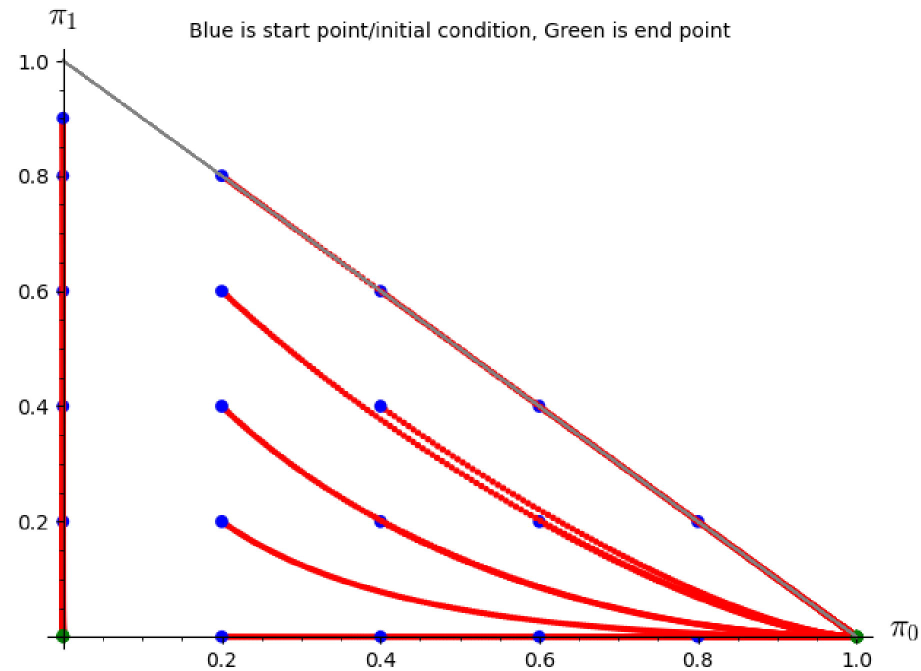
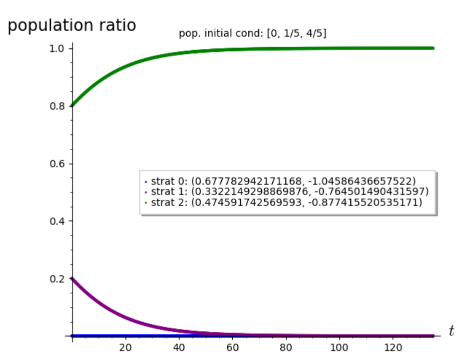
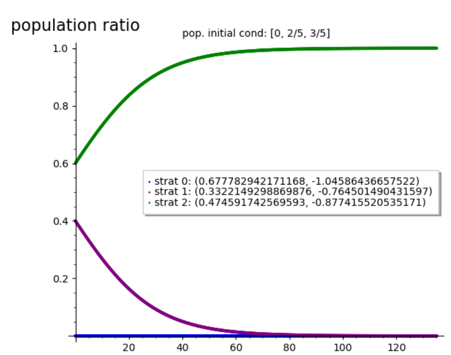


Figure 1: the parameters are $(\alpha, Z) \approx (0.677,\ 2.717)$, $(0.332,\ 2.313)$, $(0.474,\ 2.482)$. A domination equilibrium is illustrated by equilibrium at a vertex in the population plot. Here, when initial conditions have $\pi_0 = 0$, they have equilibrium at $\pi_3 = 1$. On all other initial conditions displayed, they have equilibrium at $\pi_0 = 1$



(a) Initial conditions: $[0, \frac{1}{5}, \frac{4}{5}]$



(b) Initial conditions: $[0, \frac{2}{5}, \frac{3}{5}]$

Figure 2: Time plots for $(\alpha, Z) \approx (0.677,\ 2.717)$, $(0.332,\ 2.313)$, $(0.474,\ 2.482)$ with different initial conditions

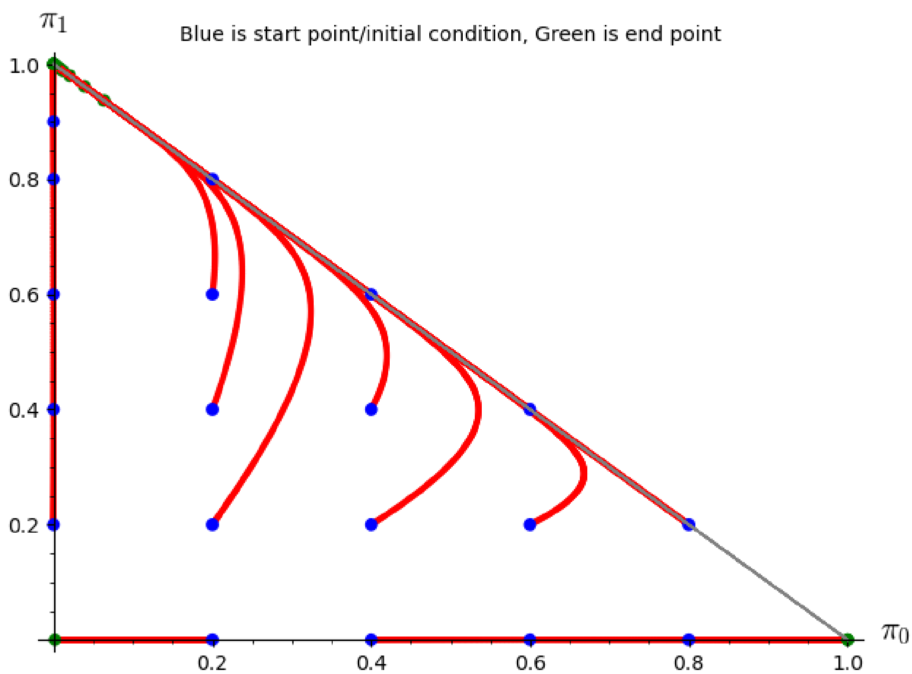


Figure 3: Parameters $(\alpha, Z) \approx (0.262, 2.027), (0.674, 2.999), (-0.102, 2.650)$. In this example, there is a mix of $\alpha > 0$ and $\alpha < 0$.

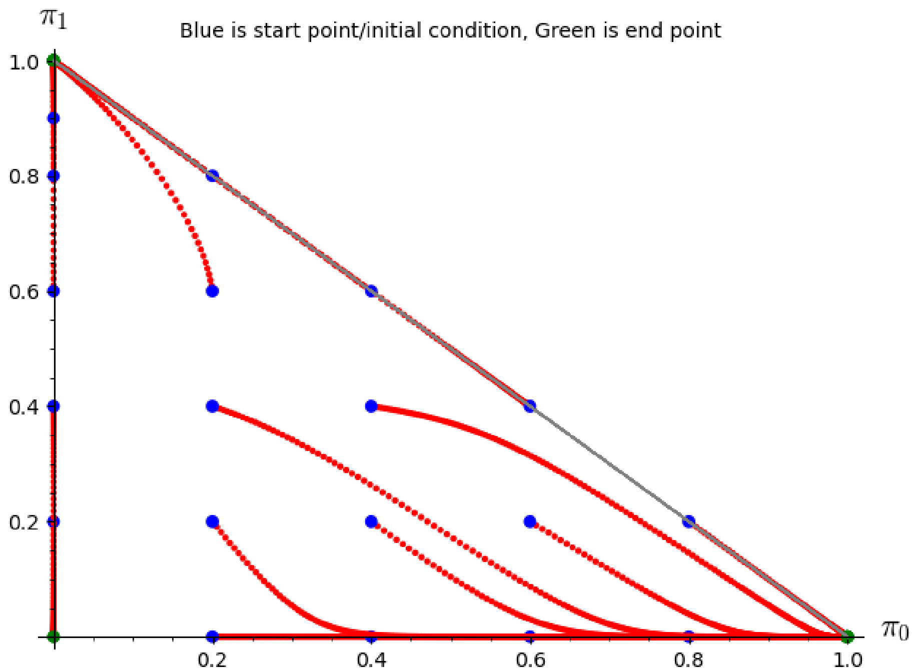
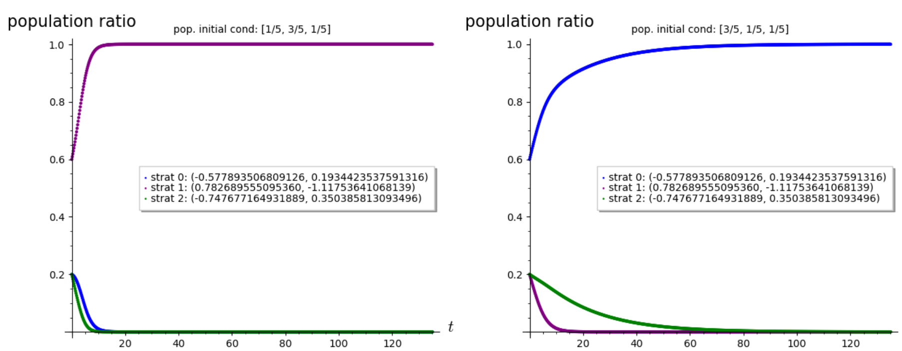


Figure 4: Parameters $(\alpha, Z) \approx (-0.578, 2.601), (0.783, 2.986), (-0.748, 2.517)$. There is a mix of $\alpha < 0$ and $\alpha > 0$.



(a) Initial conditions: $[\frac{1}{5}, \frac{3}{5}, \frac{1}{5}]$

(b) Initial conditions: $[\frac{3}{5}, \frac{1}{5}, \frac{1}{5}]$

Figure 5: Time plots for $(\alpha, Z) \approx (-0.578, 2.601), (0.783, 2.986), (-0.748, 2.517)$. There is a mix of $\alpha < 0$ and $\alpha > 0$ with different initial conditions. Note that different strategies have dominated depending on the initial condition. On the left, π_3 has dominated. On the right, π_1 has dominated. Here we say π_3 to refer to player 3 and π_1 to refer to player 1.

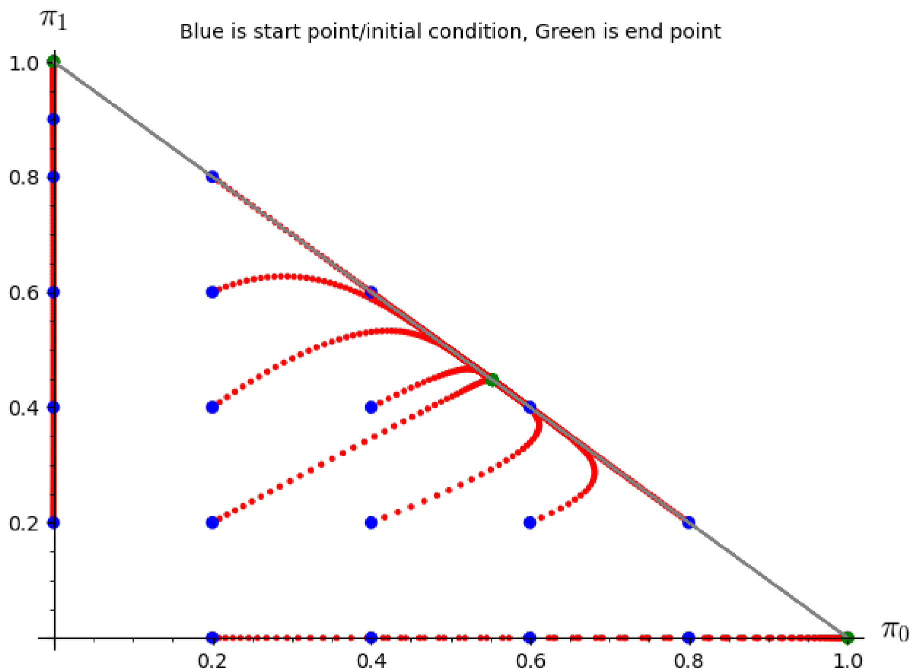
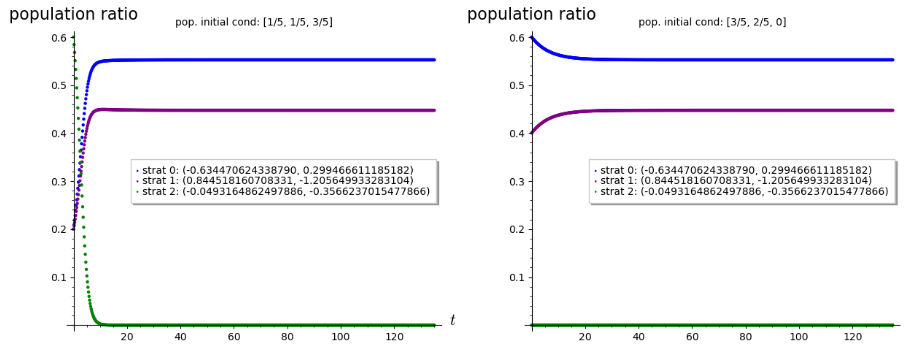


Figure 6: $(\alpha, Z) \approx (-0.634, 2.985), (0.845, 2.769), (-0.049, 2.463)$. An example of mixed equilibrium. Depending on the initial conditions, the game could play out to a mixed equilibrium or dominating equilibrium.

There are also examples of games with multiple mixed equilibrium points. Refer to Figure 8 and Figure 10.

5 Follow Up Questions

We have found examples of dominating equilibrium and mixed equilibrium of at most 2 players. The natural next question is does there exist an equilibrium where all players have non-zero population. This is referred to as an interior equilibrium and corresponds to an interior green point in the population plots. We came close to finding such a point. In Figure 12, the left diagram suggests a possible hyperbolic interior point. Unfortunately, after increasing the number of initial conditions and end time, it was discovered that there was no interior point, and that the behavior was to tend towards equilibrium points located on the axes. Interestingly, however, the right diagram shows that there are more than two mixed equilibrium. The natural next step in this project is to increase our search of games to find an example of a game with an interior equilibrium point, or to show that no such game exists.



(a) Initial conditions: $[\frac{1}{5}, \frac{1}{5}, \frac{3}{5}]$ (b) Initial conditions: $[\frac{3}{5}, \frac{2}{5}, 0]$

Figure 7: Time plots for $(\alpha, Z) \approx (-0.634, 2.985), (0.845, 2.769), (-0.049, 2.463)$ with different initial conditions. Notice that in the right figure $\pi_3 = 0$ and equilibrium between π_1, π_2 is quickly reached. In the left figure however, π_1, π_2 both grown exponentially at the beginning, eventually dominating together at equilibrium.

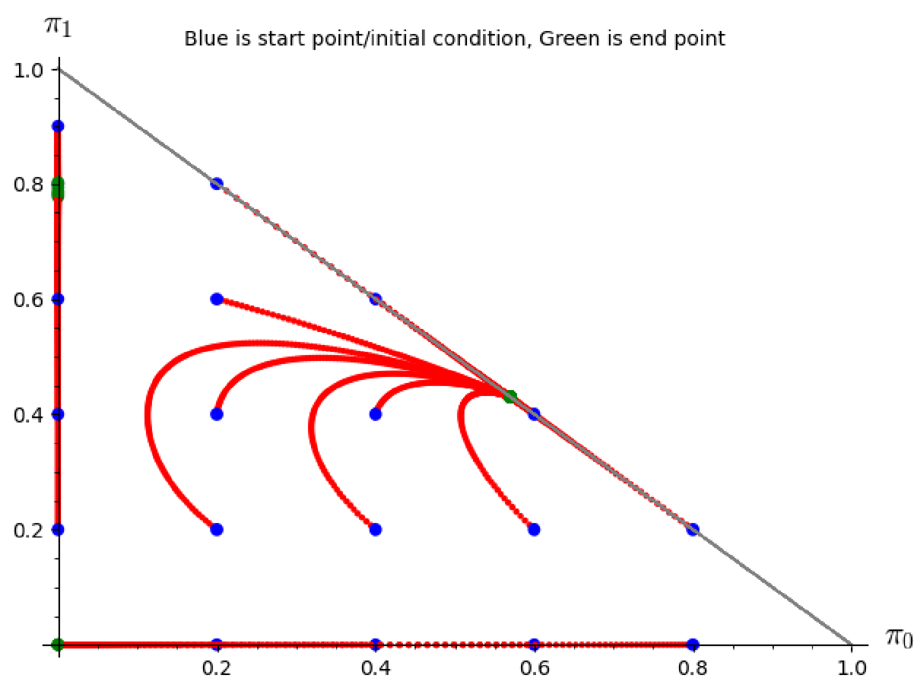
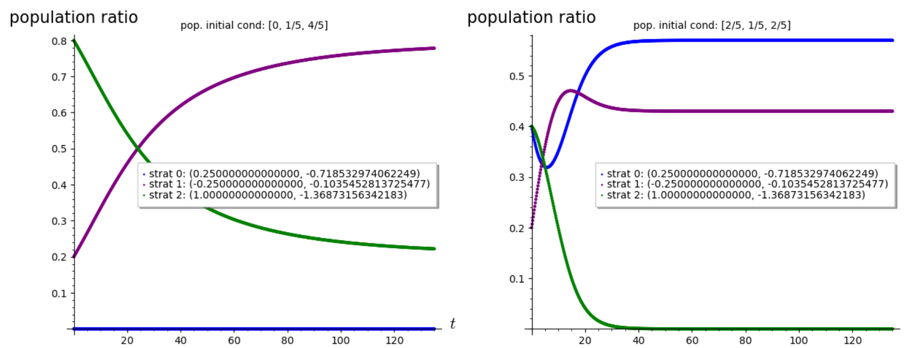


Figure 8: Parameters $(\alpha, Z) \approx (0.250, 2.134), (-0.25, 2.828), (1.000, 2.712)$. An example of a game with two different mixed equilibria: one at $\approx (0, 0.8)$ and one at $\approx (0.5, 0.5)$.



(a) Initial conditions: $[0, \frac{1}{5}, \frac{4}{5}]$ (b) Initial conditions: $[\frac{2}{5}, \frac{1}{5}, \frac{2}{5}]$

Figure 9: Time plots for $(\alpha, Z) \approx (0.250, 2.134), (-0.25, 2.828), (1.000, 2.712)$ with different initial conditions. The left figure goes towards the equilibrium point corresponding to point $\approx (0, 0.8)$ in Figure 8. The right figure corresponds to the equilibrium point at $\approx (0.5, 0.5)$.

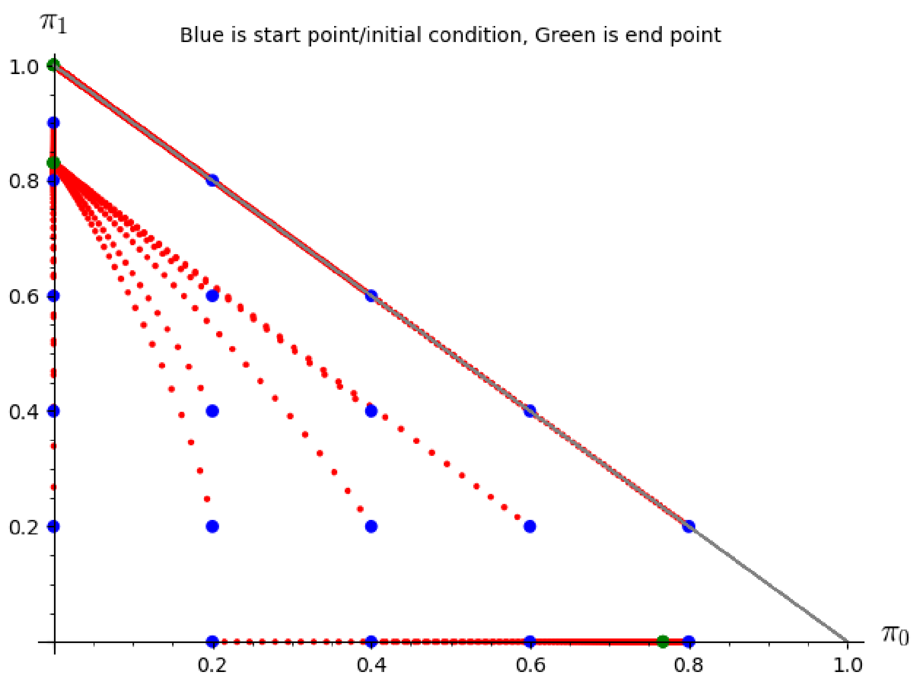
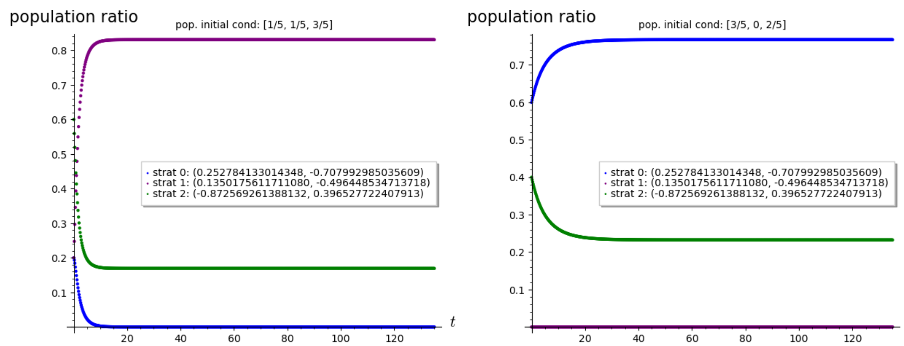
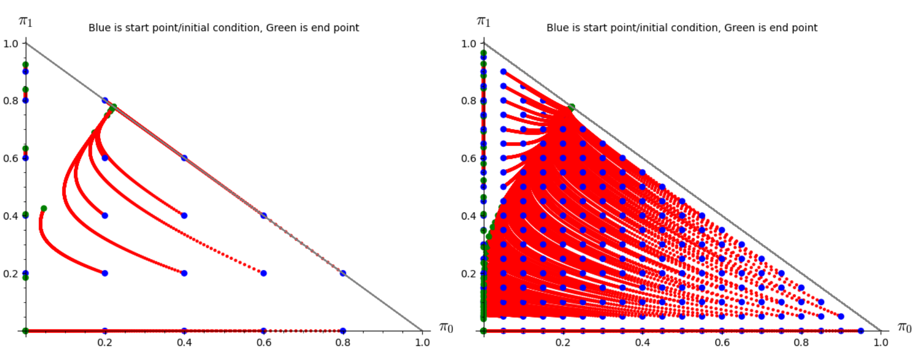


Figure 10: Parameters $(\alpha, Z) \approx (0.250, 2.197), (0.135, 2.767), (-0.873, 2.101)$. An example of multiple mixed equilibria: one at $\approx (0, 0.85)$ and another at $\approx (0.77, 0)$.



(a) Initial conditions: $[\frac{1}{5}, \frac{1}{5}, \frac{3}{5}]$ (b) Initial conditions: $[\frac{3}{5}, 0, \frac{2}{5}]$

Figure 11: Time plots for $(\alpha, Z) \approx (0.250, 2.134), (-0.25, 2.828), (1.000, 2.712)$ with different initial conditions. The left figure goes towards the equilibrium point corresponding to point $\approx (0, 0.85)$ in Figure 10. The right figure corresponds to the equilibrium point at $\approx (0.77, 0)$.



(a) (b)

Figure 12: Parameters $(\alpha, Z) = (0.624, 2.127), (-0.131, 2.886), (0.076, 2.892)$. The left figure has computation done with the same number of initial conditions and end time as the previous figures. The right figure has computations done with more initial conditions and with a larger end time.

References

[1] Ethan Akin. *The iterated Prisoner's Dilemma: good strategies and their dynamics*, pages 77–107. De Gruyter, Berlin, Boston, 2016.