Balanced Trees

Chapter 10

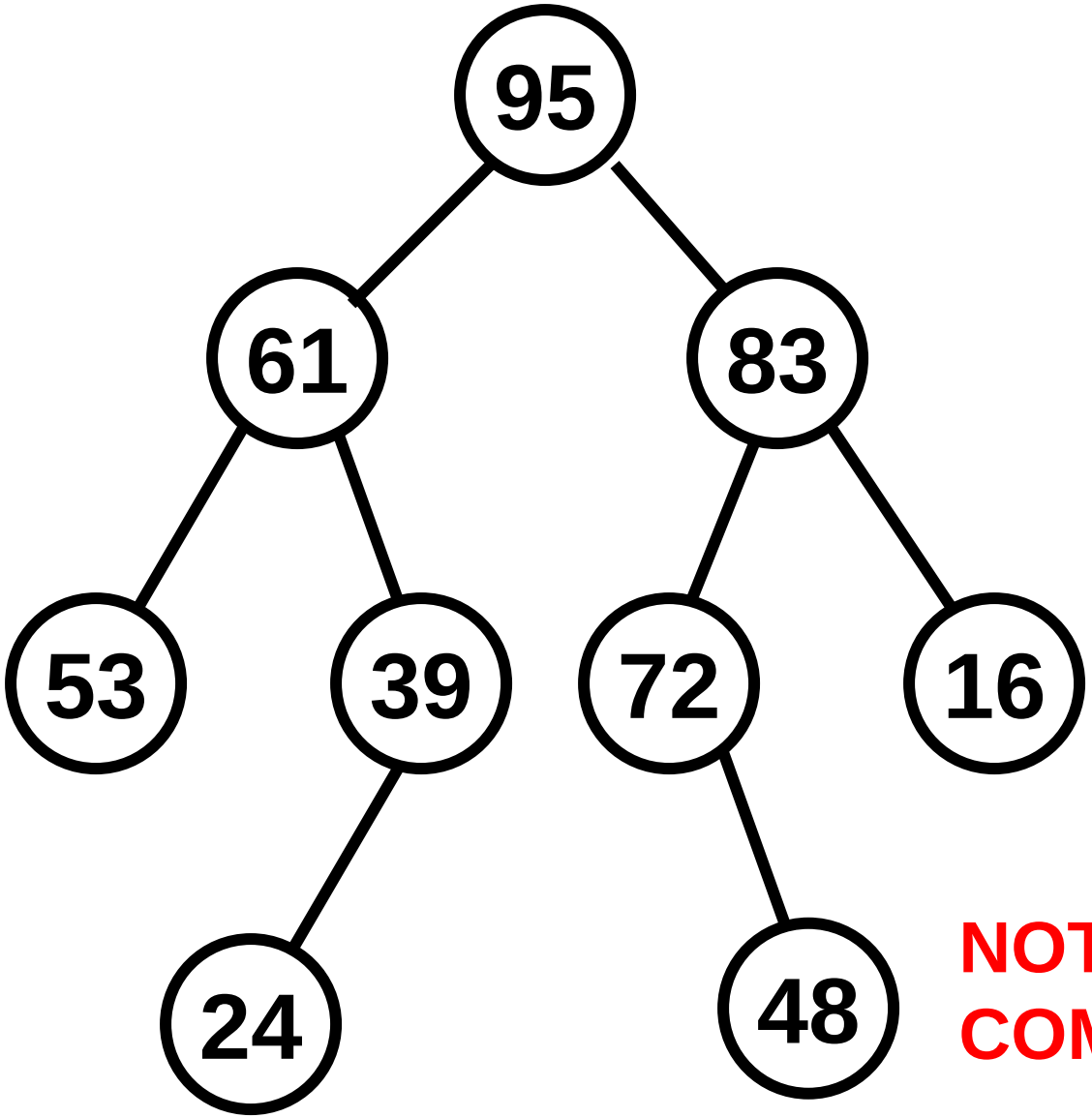
Heap

- A heap is a binary tree such that

  - the data contained in each node is greater than (or equal to) the data in that node's children.

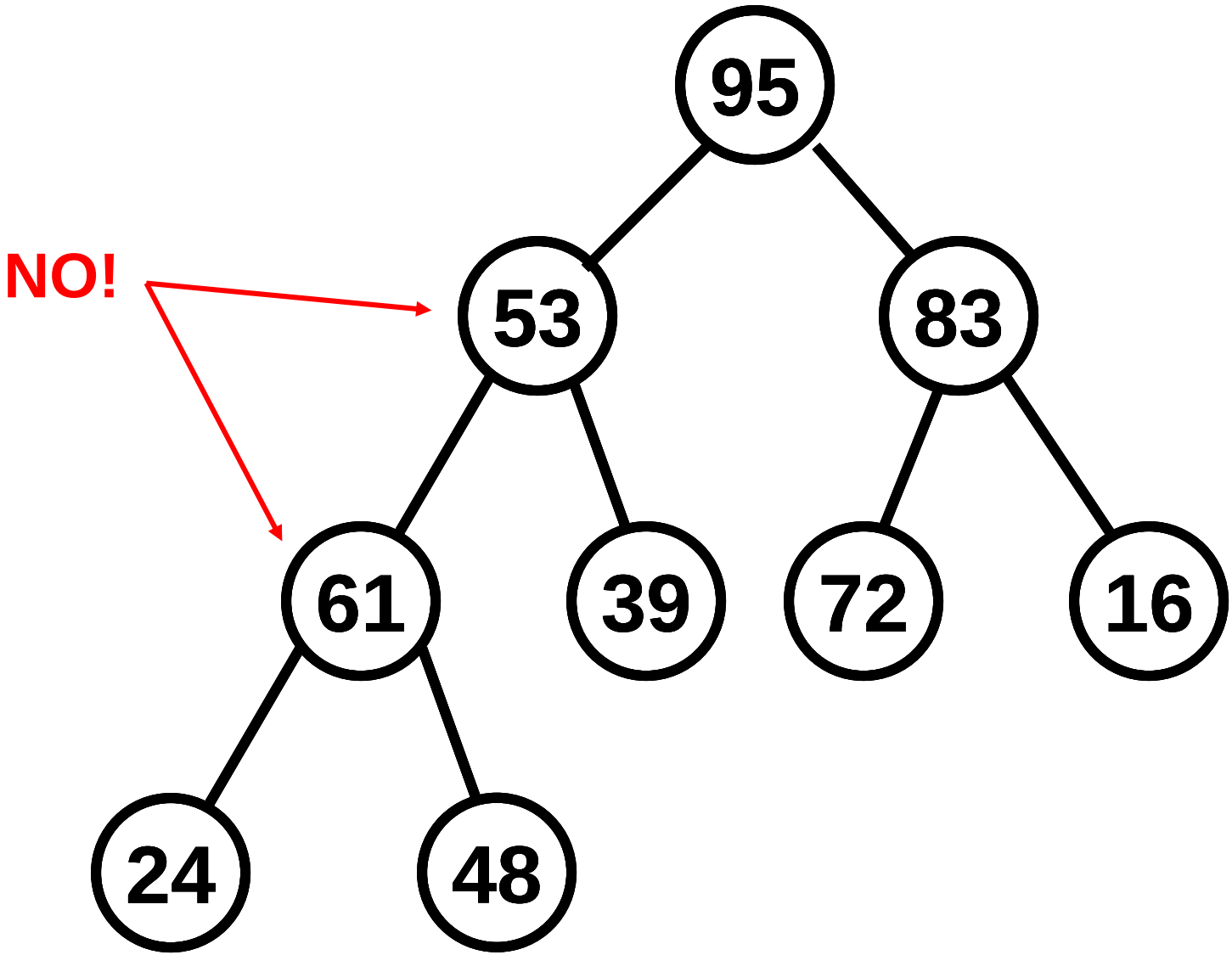
  - the binary tree is complete

Is it a heap?

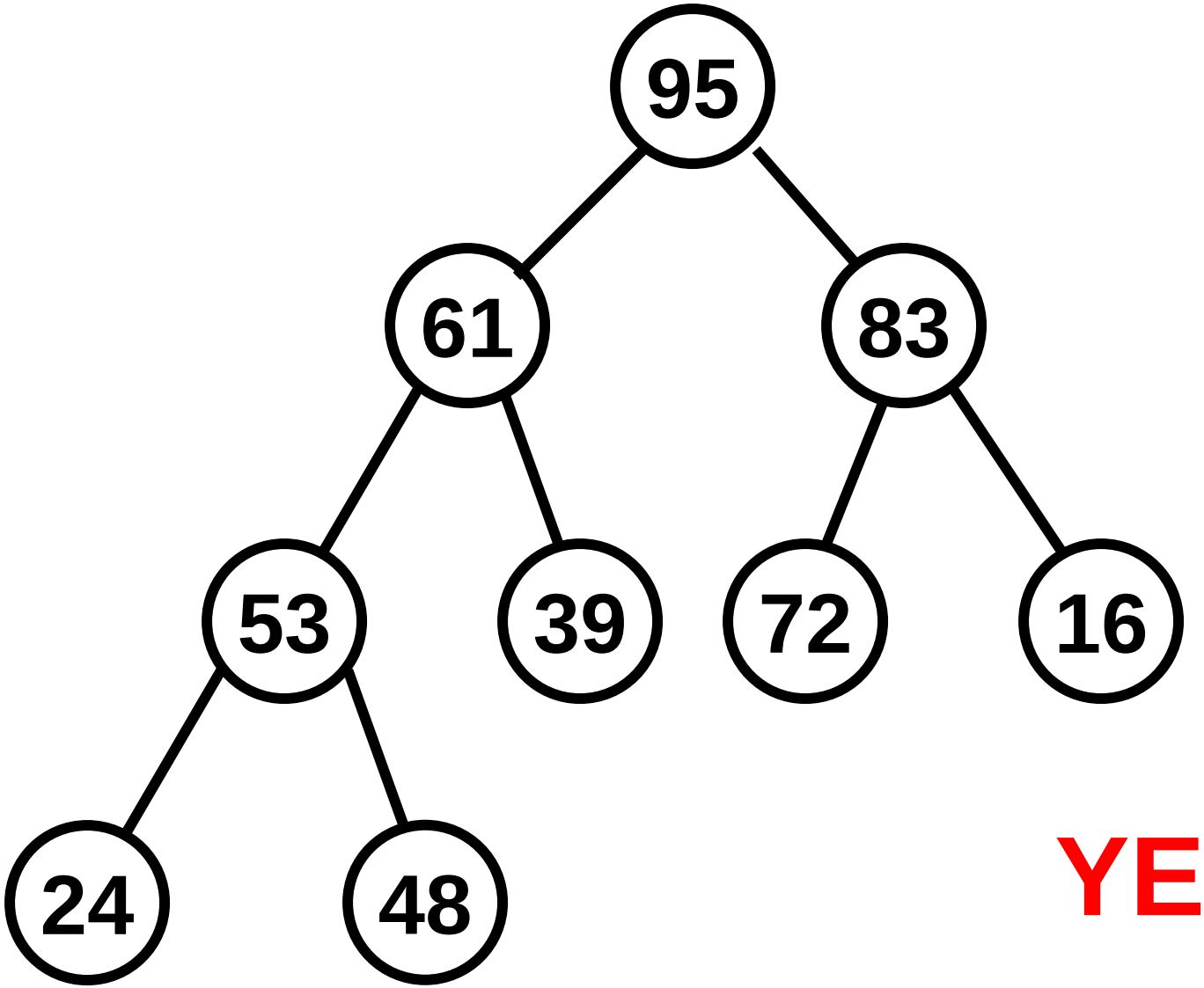


NOT COMPLETE!

Is it a heap?



Is it a heap?



YES!

Storage of a heap

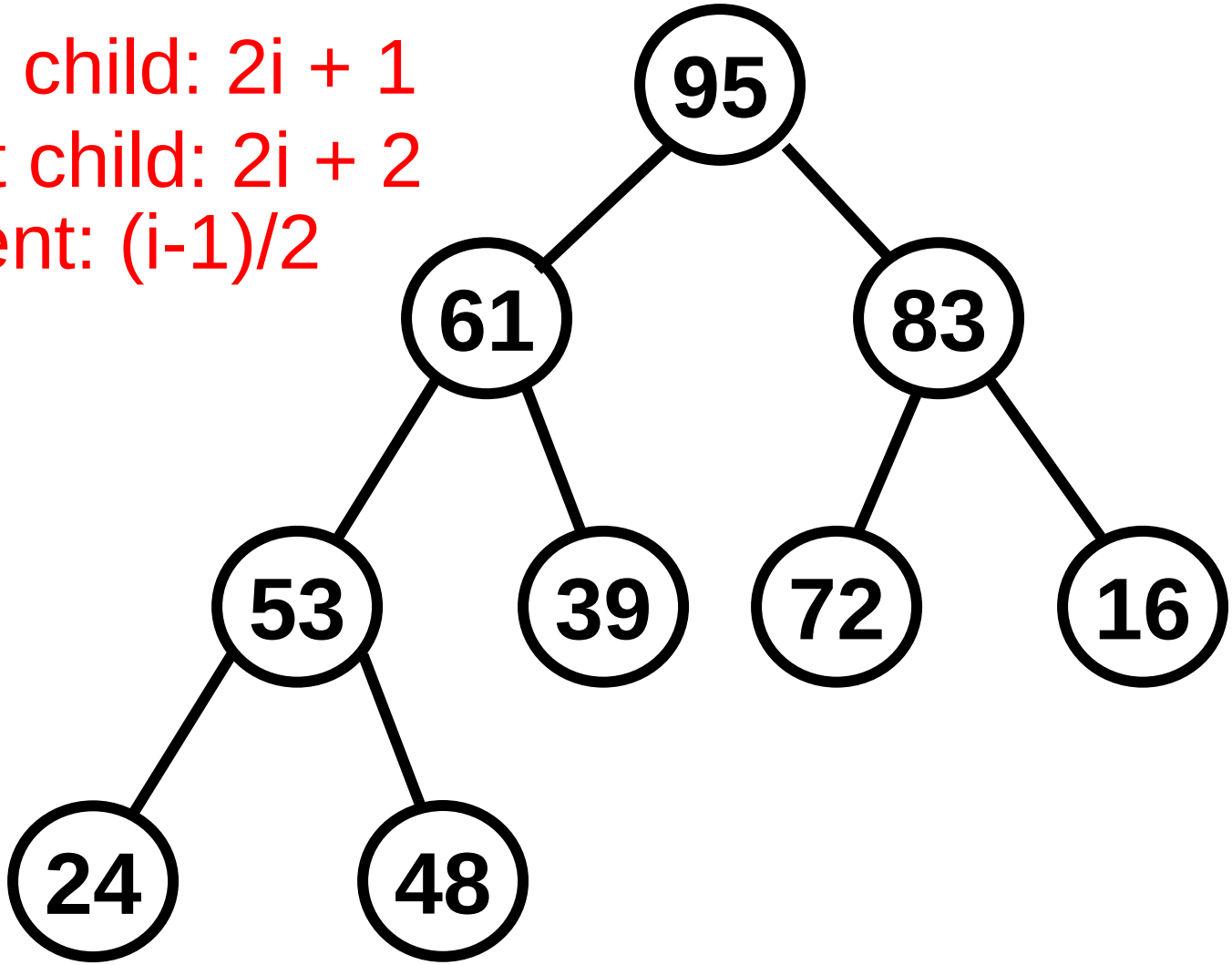
- Use an array to hold the data.
- Store the root in position 0.
- For any node in position i,
  - its left child (if any) is in position 2i + 1
  - its right child (if any) is in position 2i + 2
  - its parent (if any) is in position (i-1)/2

    (integer division)

left child: 2i + 1

right child: 2i + 2

parent: (i-1)/2



0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
95	61	83	53	39	72	16	24	48

Basic operations on a heap

- Create an empty heap (constructor).
- Insert a data element into a heap.
- Remove the maximum data element from the heap.

The heap data structure

```
public class Heap {
  private int[] data;
  private int heapSize;
  private int maxSize;
  public Heap(int maximumSize) {
     if (maximumSize < 1) maxSize = 100;
     else maxSize = maximumSize;
     data = new int[maxSize];
     heapSize = 0;
  }
  public boolean isEmpty()      // not
  public boolean isFull()       // shown
```

Inserting into a heap

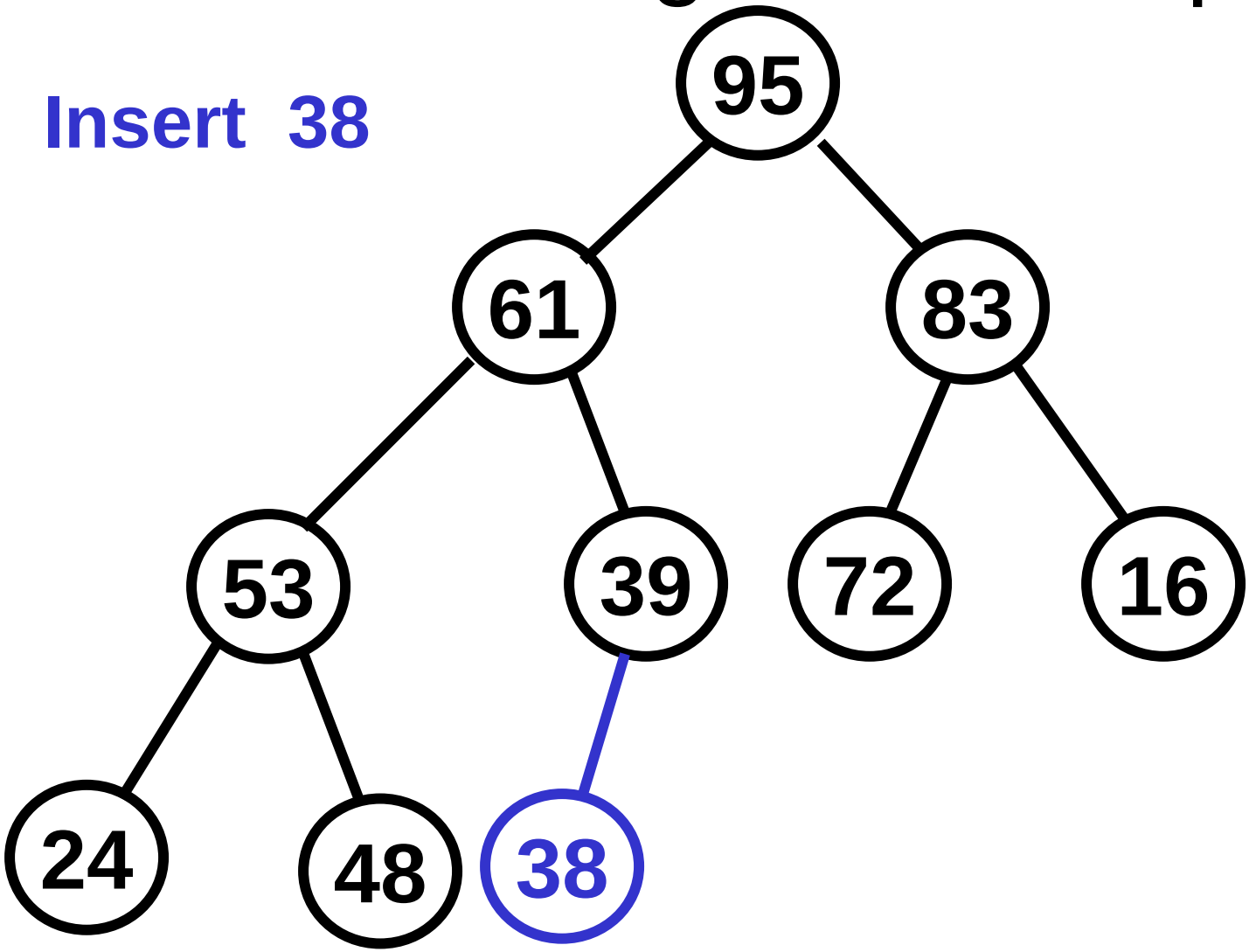
- Place the new element in the first available position in the array.
- Compare the new element with its parent. If the new element is greater, than swap it with its parent.
- Continue this process until either

  - the new element's parent is greater than or equal to the new element, or

  - the new element reaches the root

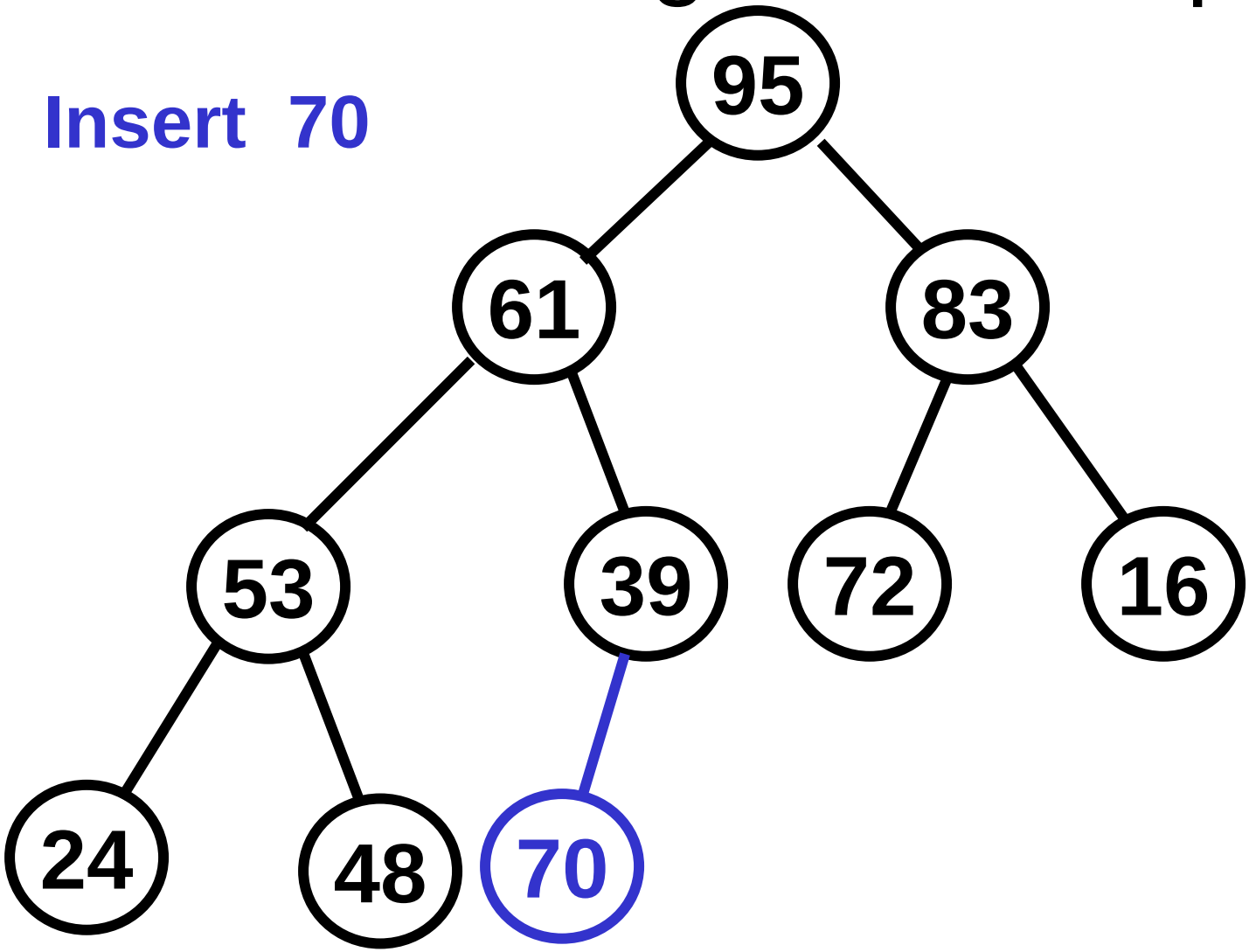
Inserting into a heap

Insert 38



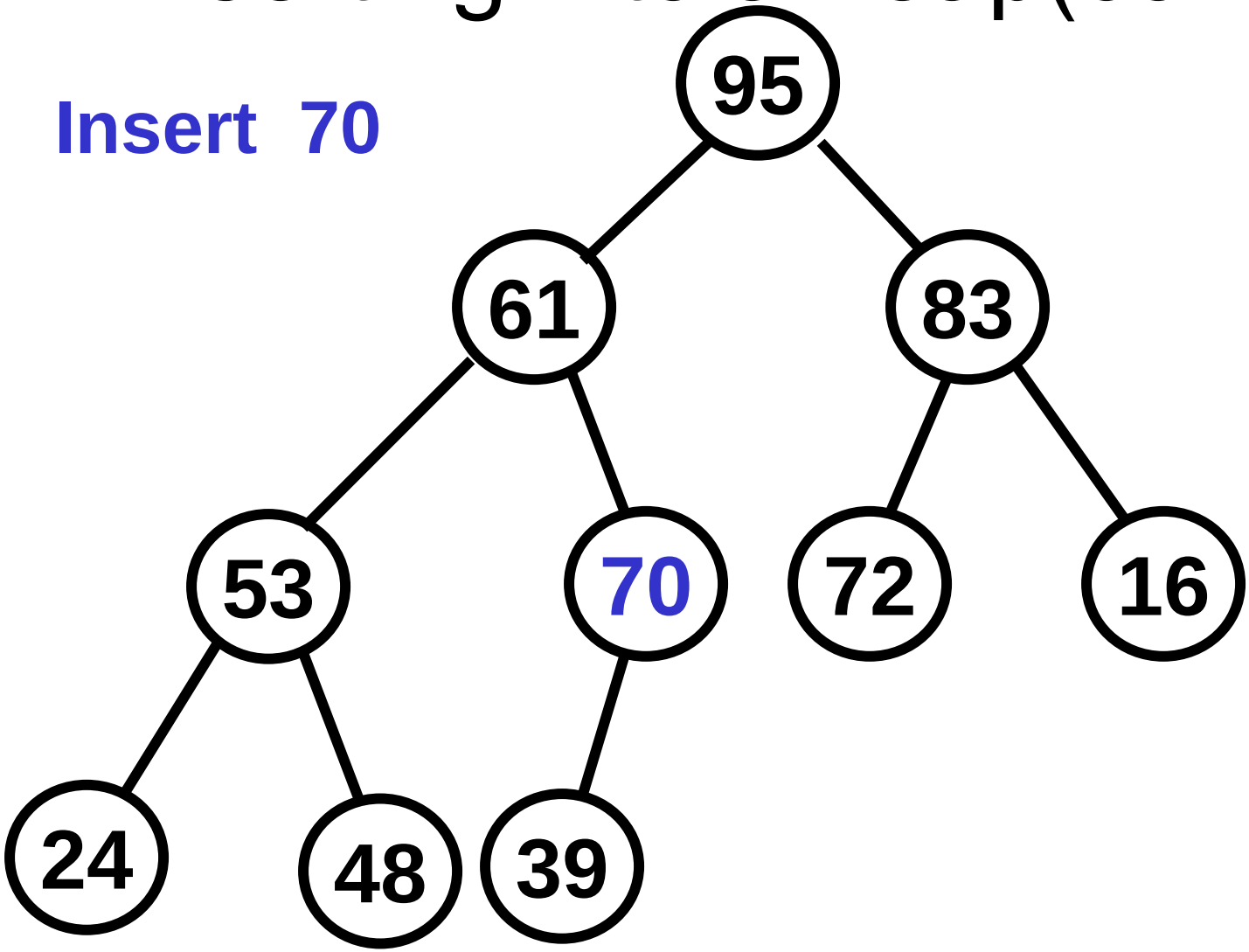
Inserting into a heap

Insert 70



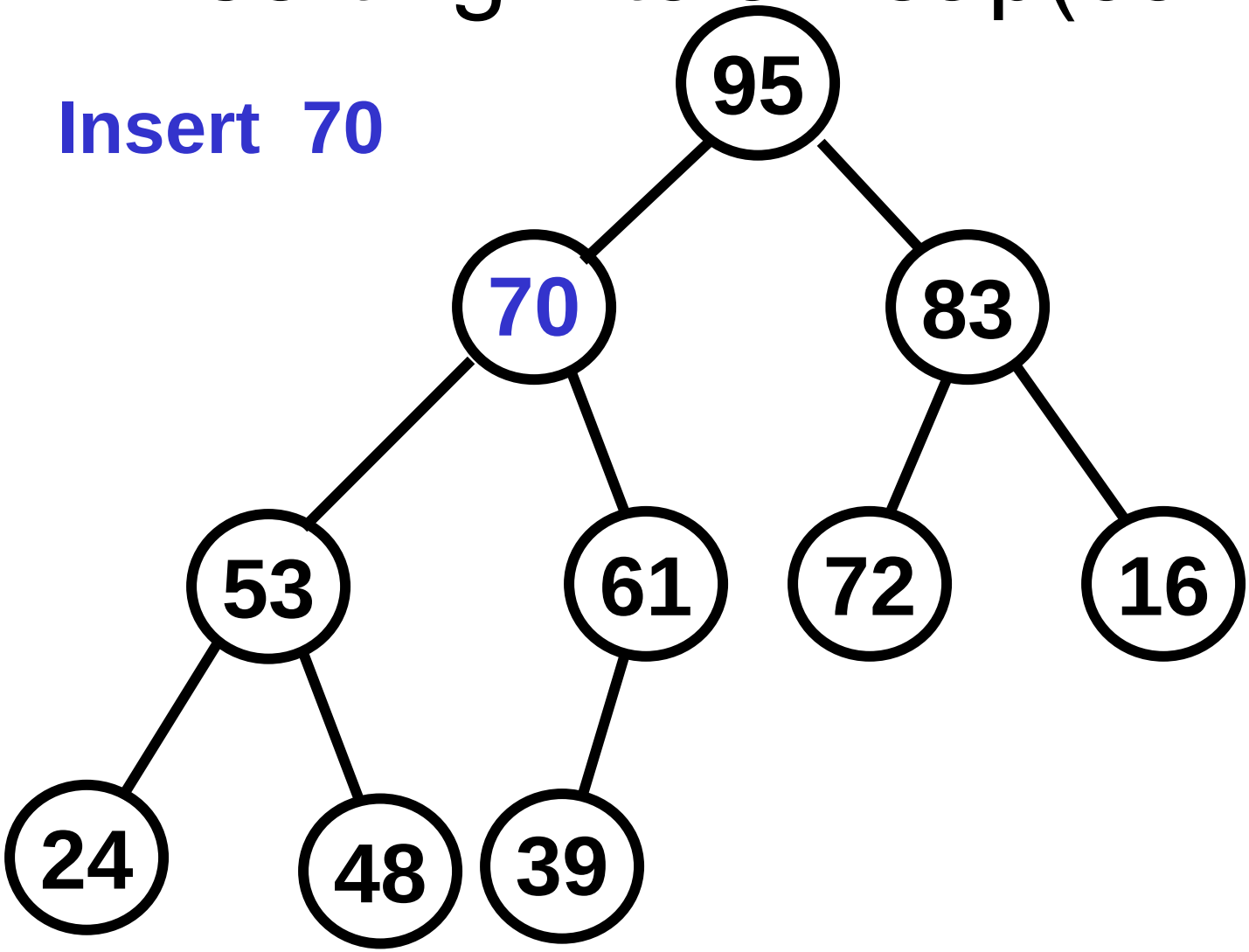
Inserting into a heap(cont'd)

Insert 70



Inserting into a heap(cont'd)

Insert 70



Inserting into a heap

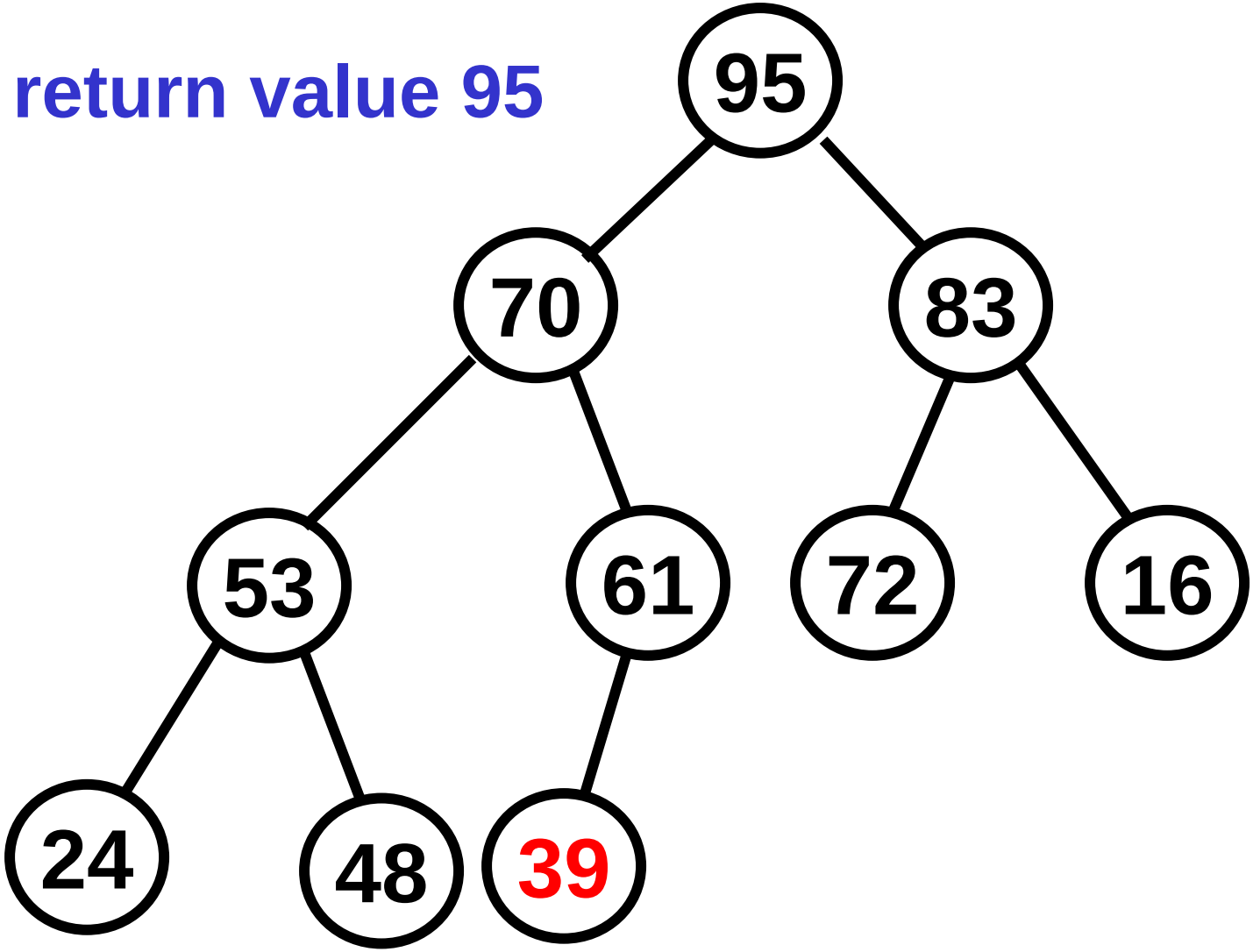
```
public void insert(int item) {
  int position;
  if (isFull()) throw new Exception();
  heapSize++;
  data[heapSize-1] = item;
  position = heapSize - 1;
  while (position > 0 &&
   data[position] > data[(position-1)/2])
  {
      swap(position, (position-1)/2);
      position = (position-1) / 2;
  }
}
```

Removing from a heap

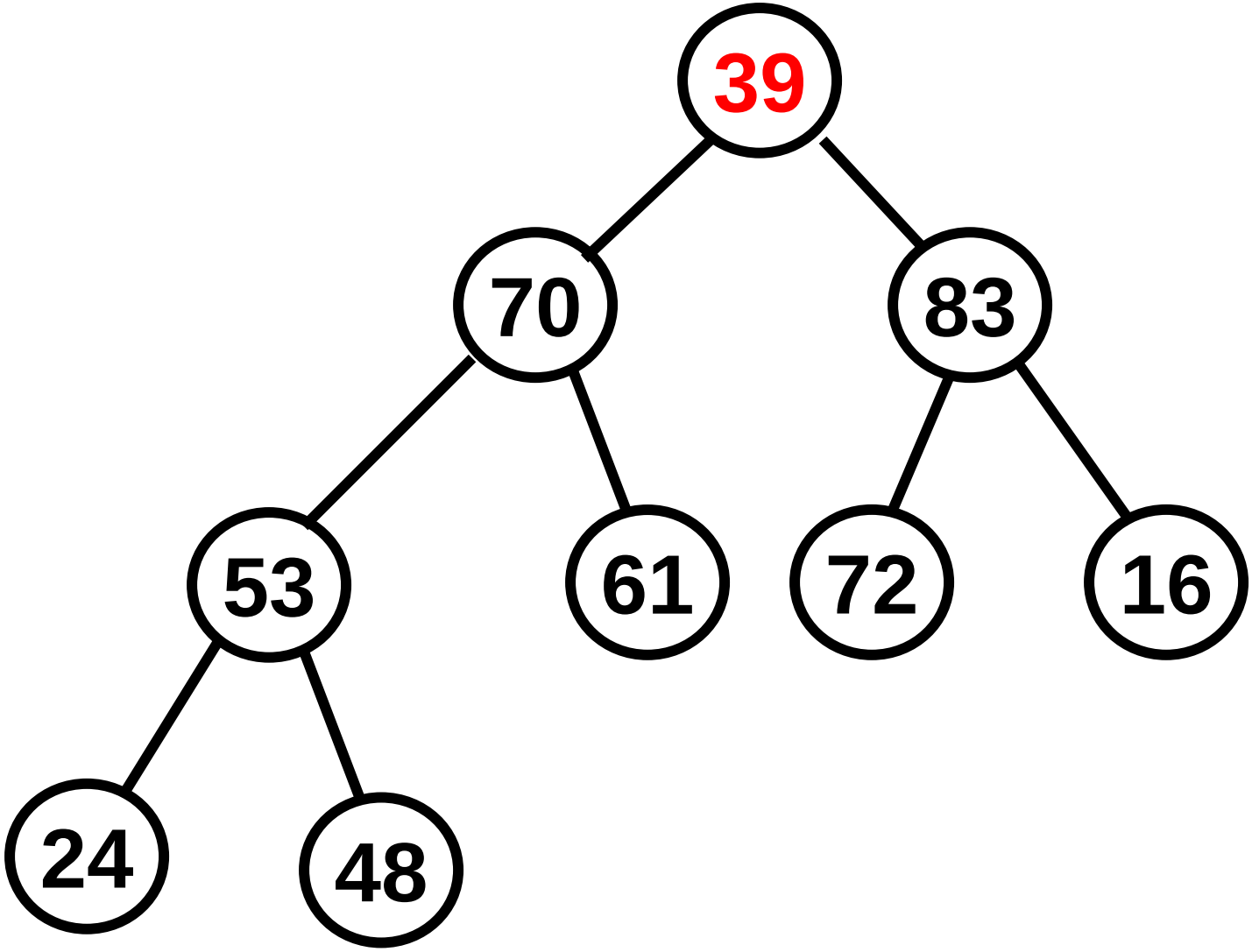
- Place the root element in a variable to return later.
- Move the last element in the deepest level to the root (reduce the size of the heap by 1).
- While the moved element has a value lower than one of its children, swap this value with the highest-valued child.
- Return the original root that was saved.

Removing from a heap

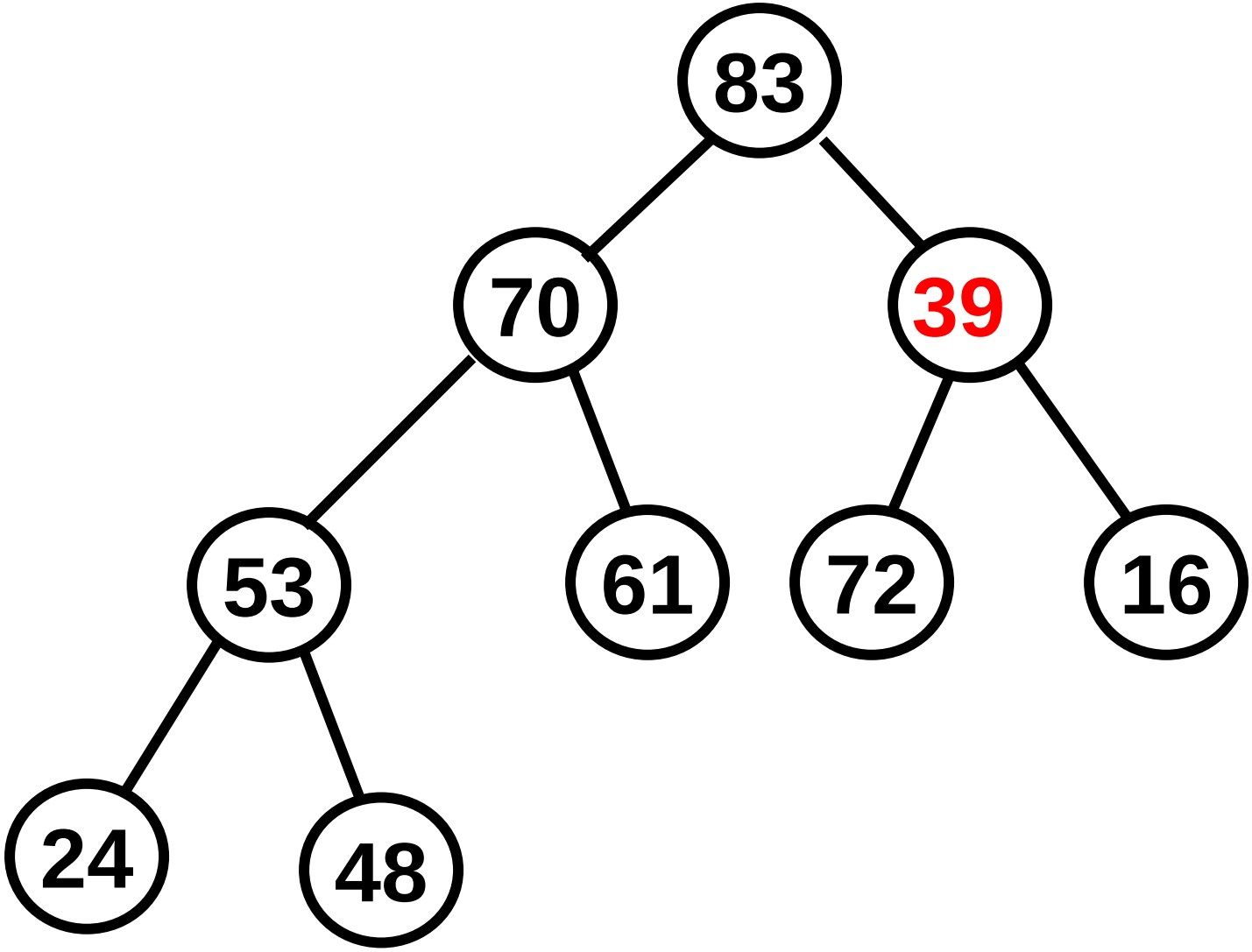
return value 95



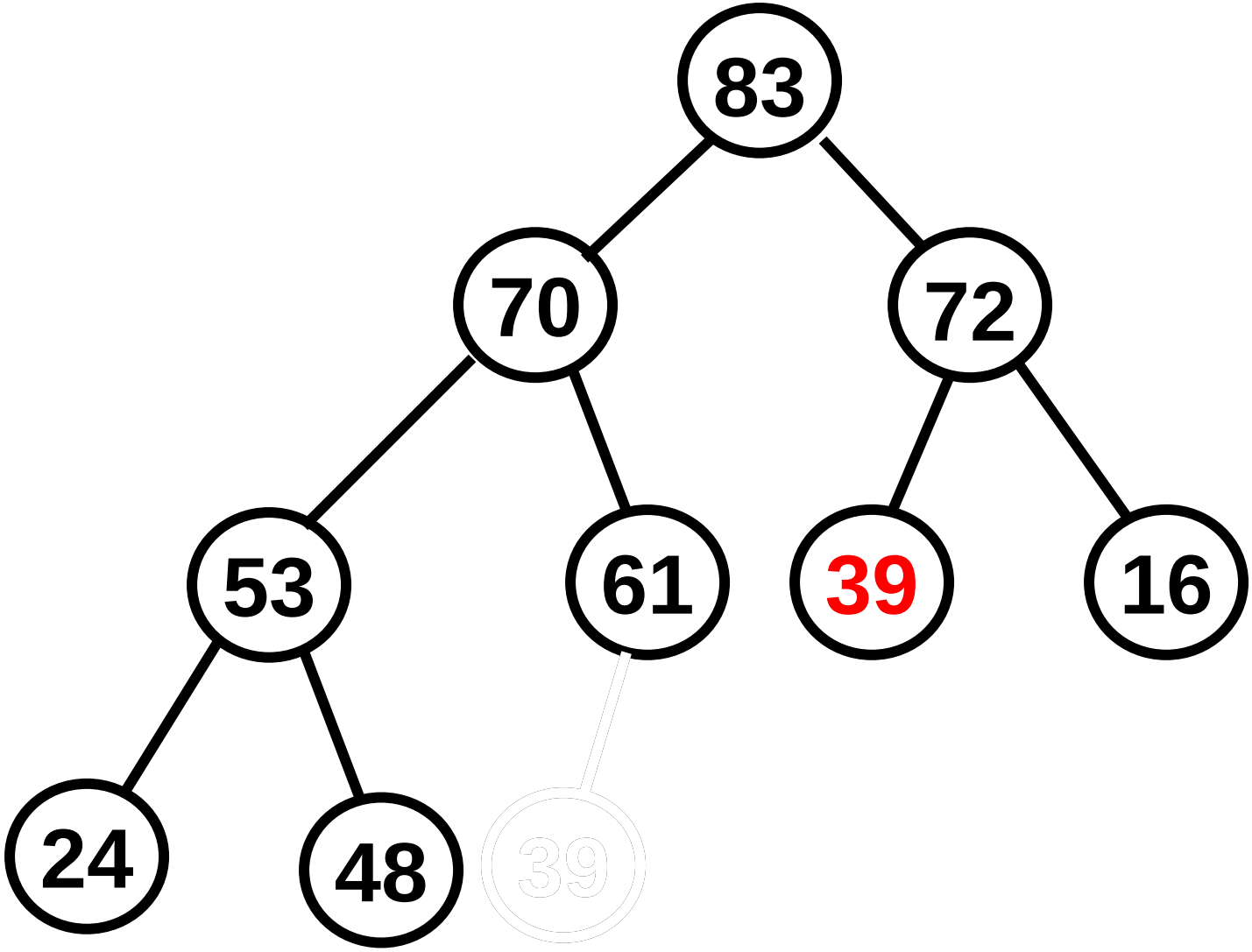
Removing from a heap(cont'd)



Removing from a heap(cont'd)



Removing from a heap



Removing from a heap

```
public int remove() {
  int answer;
  if (isEmpty()) throw new Exception();
  answer = data[0];
  data[0] = data[heapSize-1];
  heapSize--;
  fixheap();
  return answer;
}
```

Removing from a heap (cont'd)

```
private void fixheap() {
  int position = 0; int childPos;
  while (position*2 + 1 < heapSize) {
     childPos = position*2 + 1;
     if (childPos < heapSize-1 &&
       data[childPos+1] > data[childPos])
          childPos++;
     if (data[position]>= data[childPos])
          return;
     swap(position, childPos);
     position = childPos;
  }
}
```

Efficiency of heaps

Assume the heap has N nodes.

Then the heap has approximately $\log_2 N$ levels.

- Insert

  Since the insert swaps at most once per level, the order of complexity of insert is O(log N)

- Remove

  Since the remove swaps at most once per level, the order of complexity of remove is also O(log N)

Priority Queues

- A priority queue PQ is like an ordinary queue except that we can only remove the “maximum” element at any given time (not the “front” element necessarily).
- If we use an array for the PQ, enqueue takes O(1) time but dequeue takes O(n) time.
- If we use a sorted array for the PQ, enqueue takes O(n) time, while dequeue takes O(1) time.
- We can use a heap to implement a priority queue, so that enqueue and dequeue take O(log N) time.

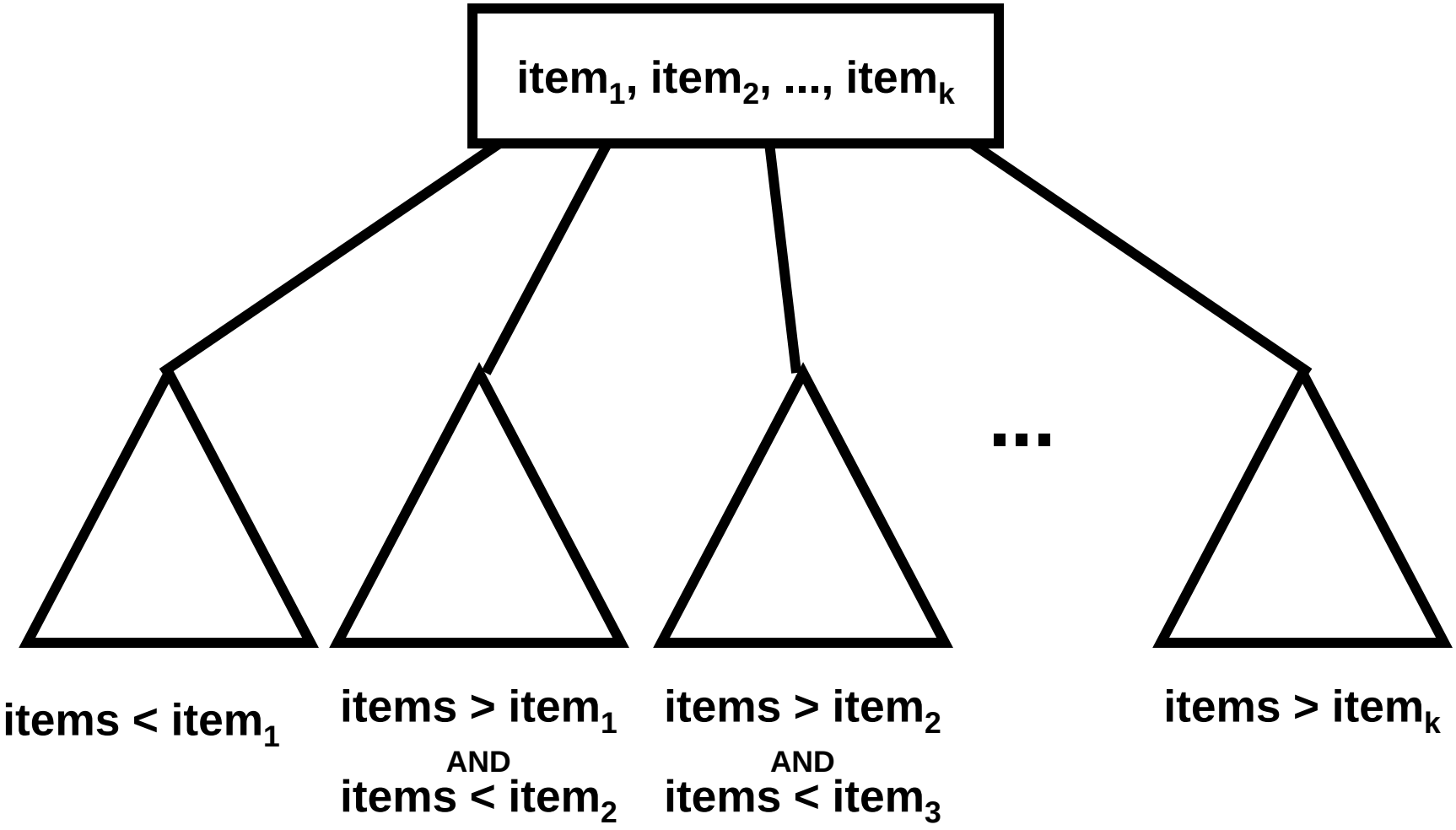
B-trees

- A B-tree is a balanced search tree.
- A B-tree is designed to work well on magnetic disks and other direct-access secondary storage devices by minimizing disk I/O operations.
- A node can hold more than one item.
- A node can have more than two children.

B-tree rules

- The root may have as few as one element. Every other node has at least MINIMUM elements.
- The maximum number of elements in a node is twice the value of MINIMUM.
- The elements of each B-tree node are stored in a partially filled array, sorted from the smallest element (in position 0) to the largest element.

General Idea



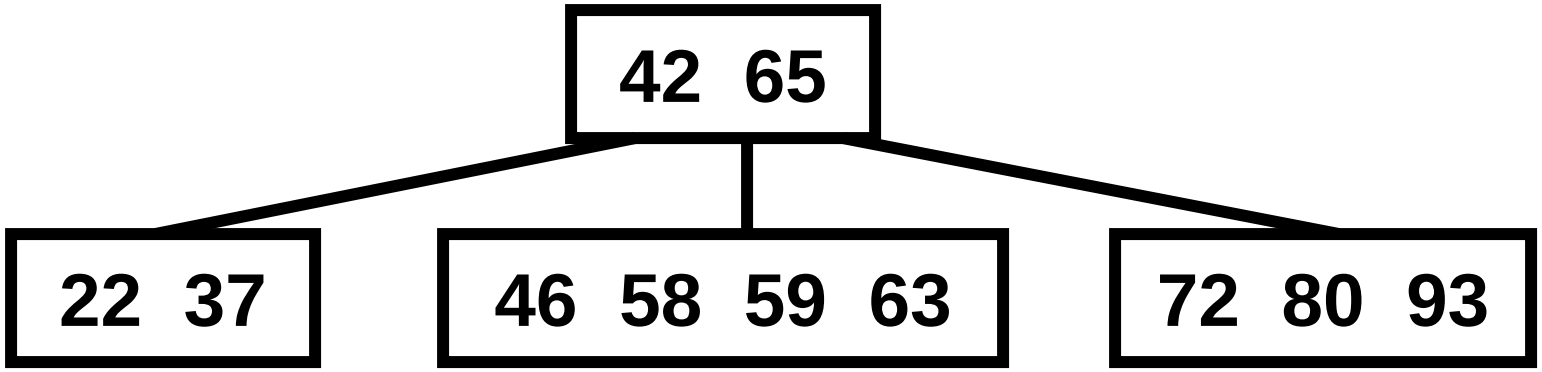
B-tree rules (cont'd)

- The number of subtrees below a non-leaf node is always one more than the number of elements in the node.
- For any non-leaf node, an element at index i is

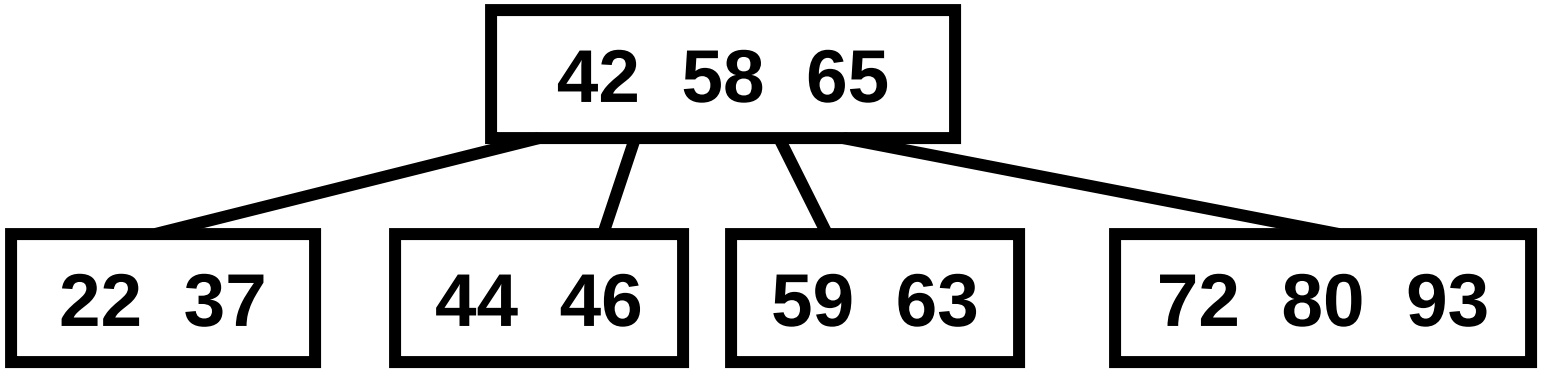
  (a) greater than all the elements in subtree i of the node, and
  (b) less than all the elements in subtree i+1 of the node
- Every leaf in a B-tree has the same depth.

Sample B-tree

MINIMUM = 2



Insert 44:



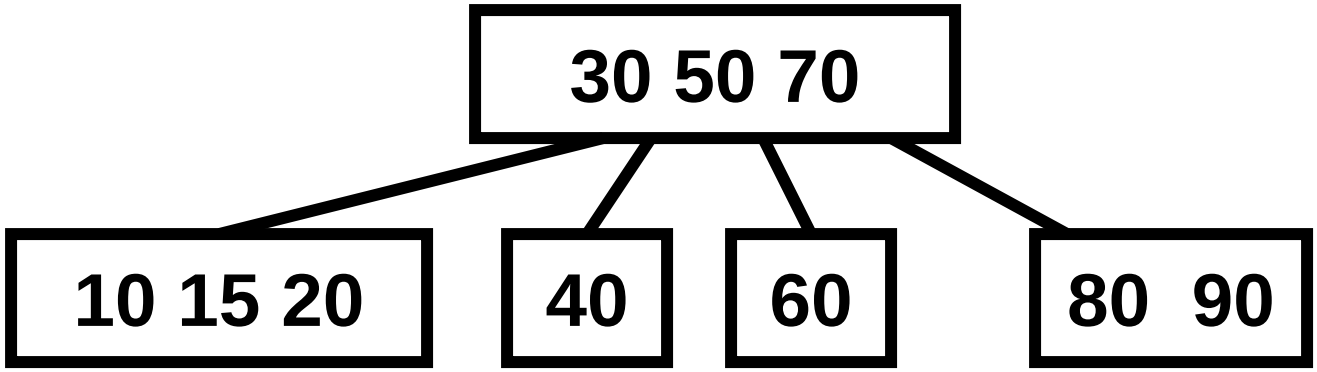
Searching in a B-tree

- Searching for an item in the current node:
  - Let k = number of data values in this node
  - Let i = the first index such that data[i] >= item. (If every data value in the node is < item, set i = k.)
  - If item = data[i], return true. (assume i < k)
  - Else if node has no children, return false
  - Else call search recursively on subtree[i]

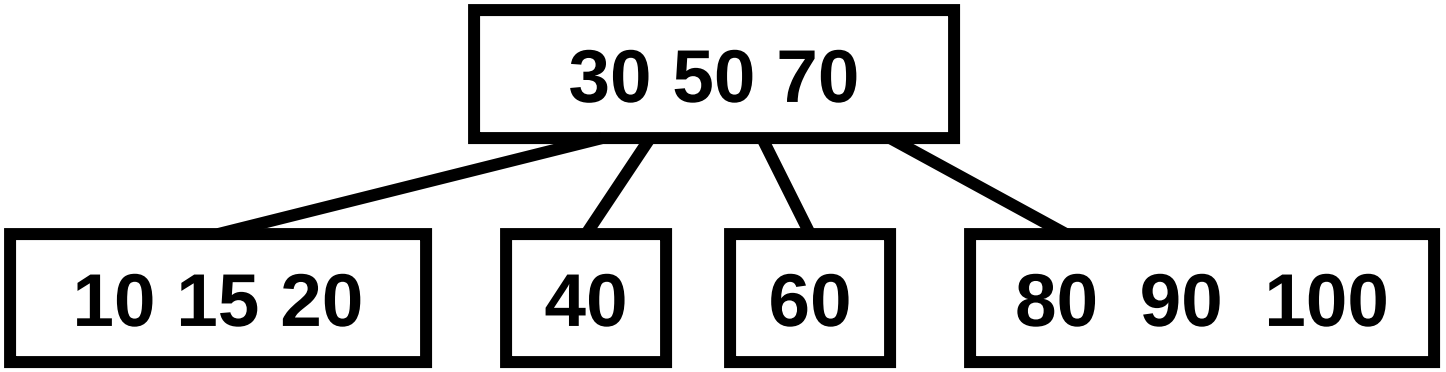
2-3-4-trees

- A 2-3-4 Tree is a tree in which each internal node (nonleaf) has two, three, or four children, and all leaves are at the same depth.

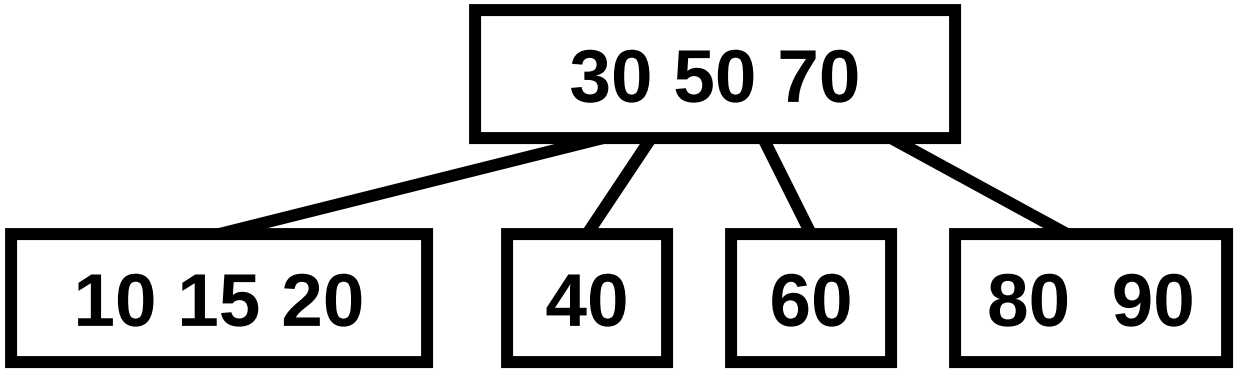
Sample 2-3-4-tree



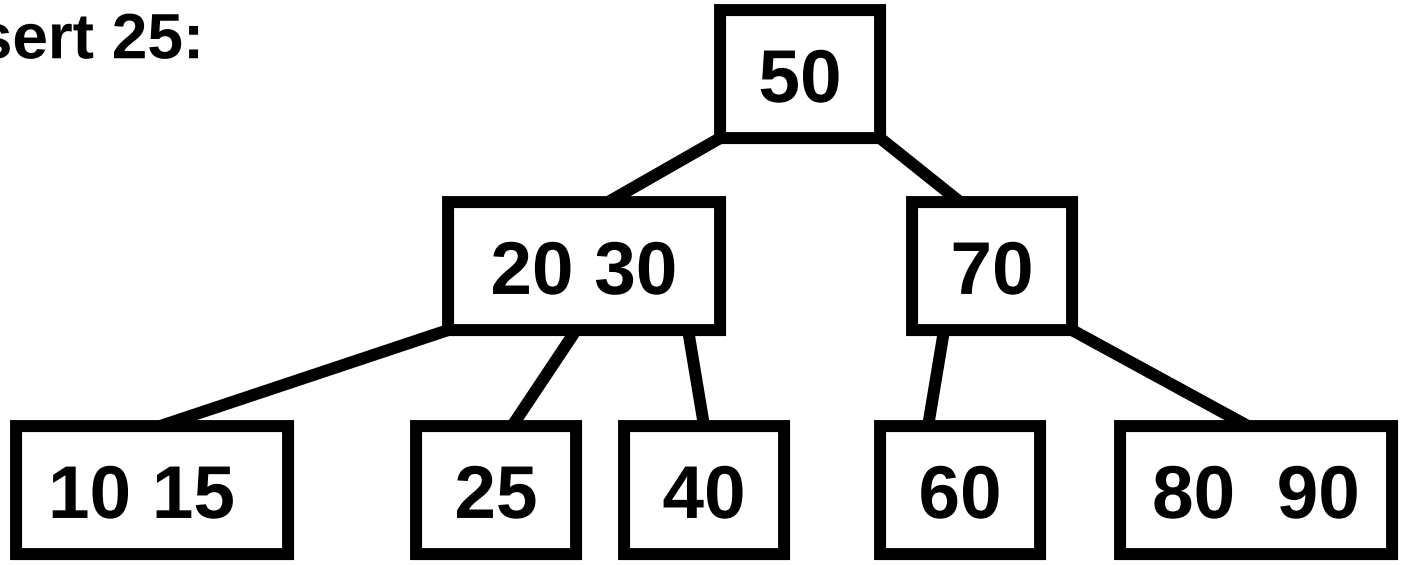
Insert 100:



Sample 2-3-4-tree



Insert 25:



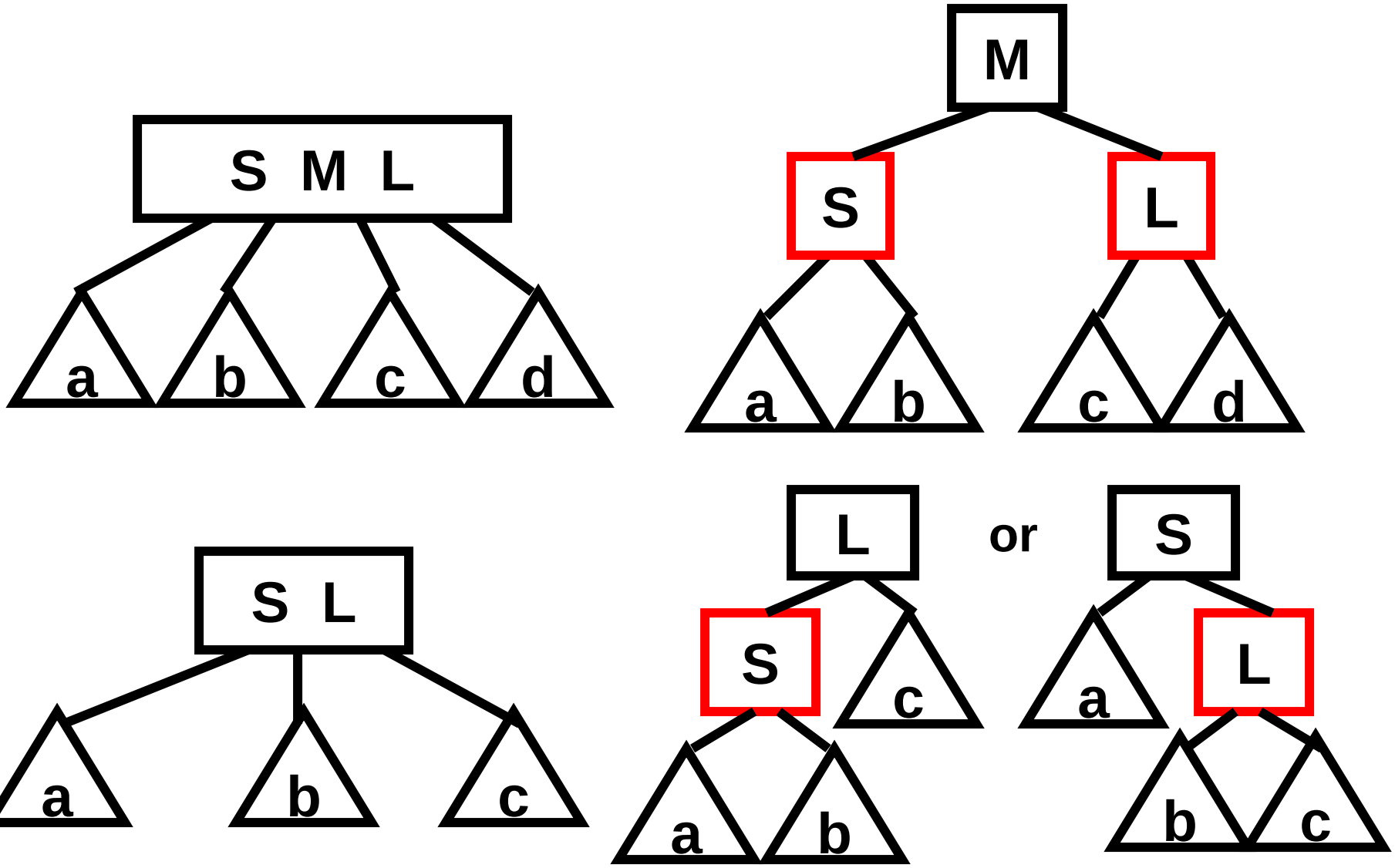
2-3-4 trees vs. B-trees

- The more data values in a node, the shorter the tree will be, which translates into less searches for an item.
- However, the more data in a node, the greater the number of comparisons to determine the subtree to examine.
- 2-3-4 trees good for trees in main memory.
- B-trees good for external memory searches.

Red-Black Trees

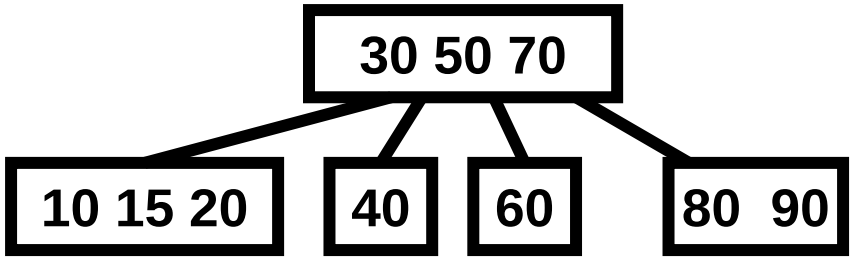
- A red-black tree has the advantages of a 2-3-4 tree but requires less storage.
- Red-black tree rules:
  - Every node is colored either red or black.
  - The root is black.
  - If a node is red, its children must be black.
  - Every path from a node to a null link must contain the same number of black nodes.

Red-Black Trees

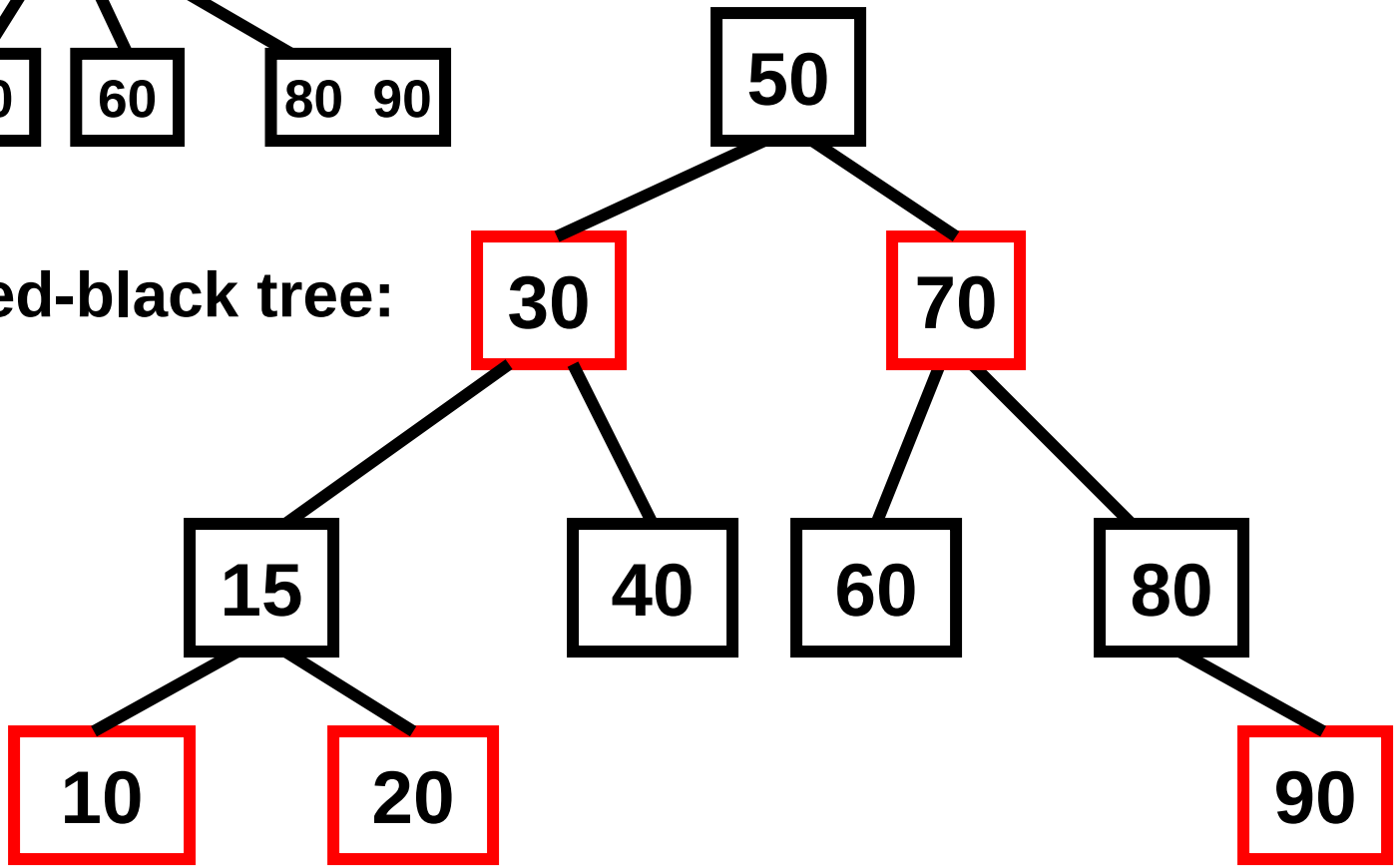


Sample Red-Black Tree

Original 2-3-4 tree:

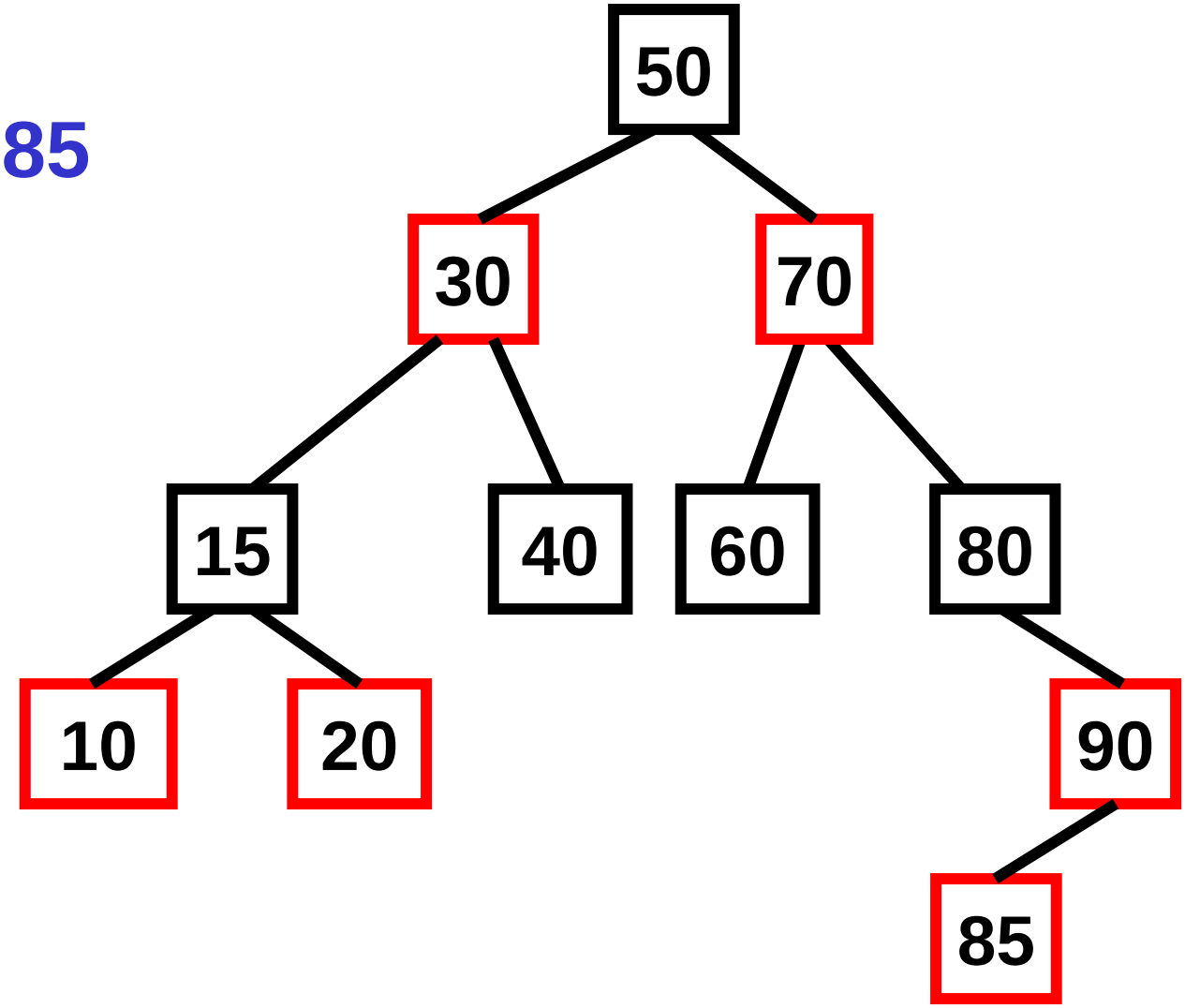


Equivalent red-black tree:



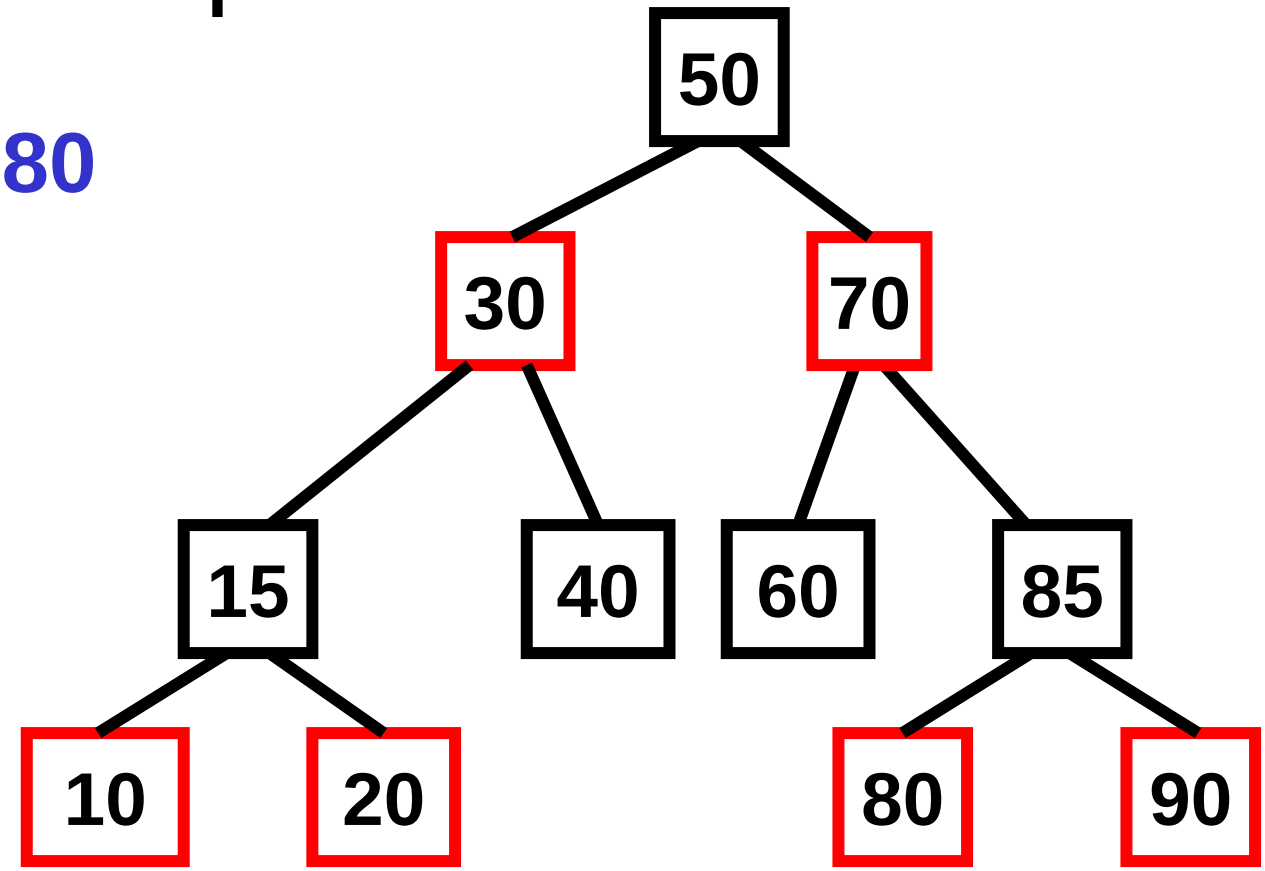
Insert in Red-Black Tree

Insert 85



Sample Red-Black Tree

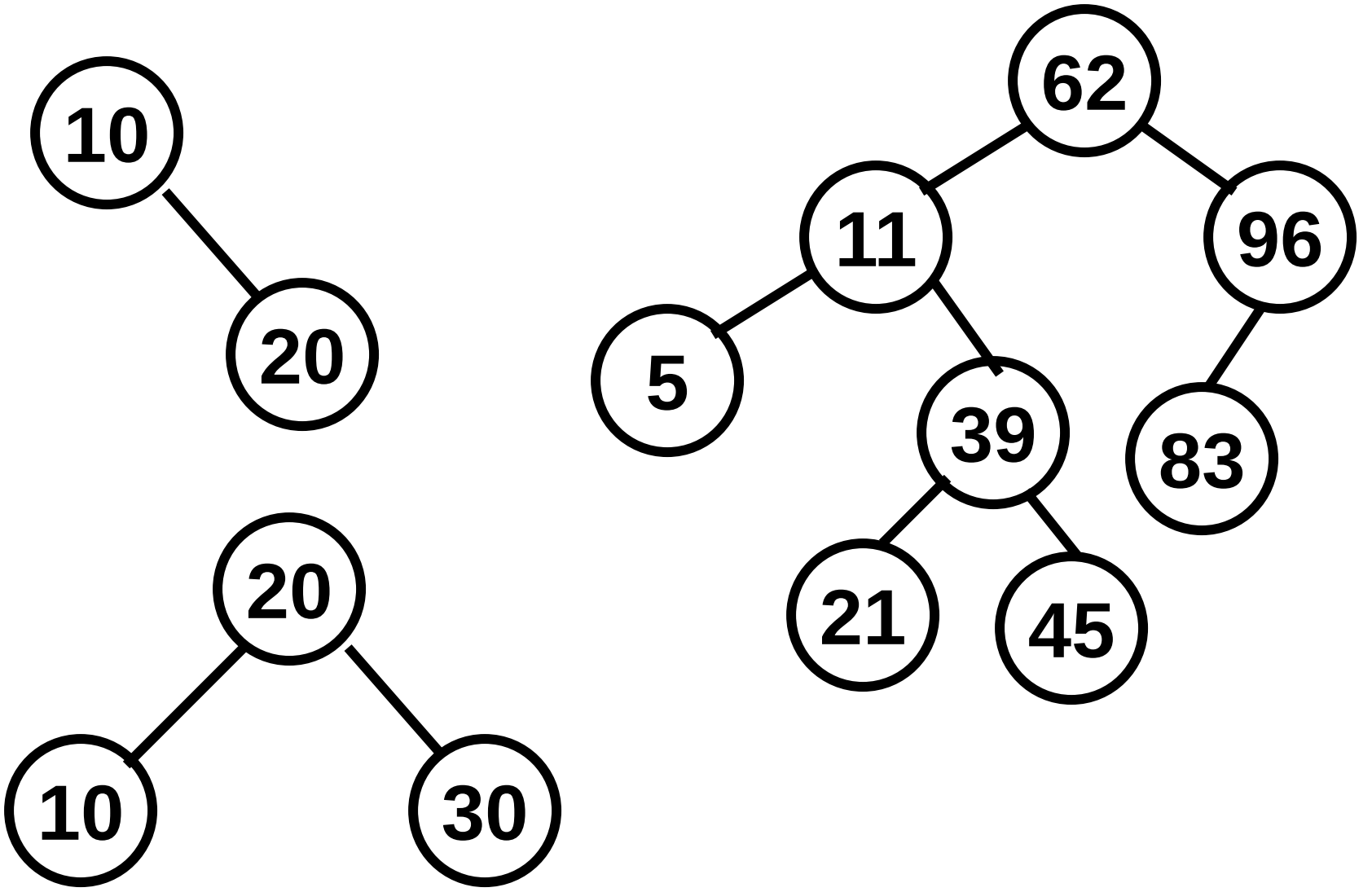
Insert 80



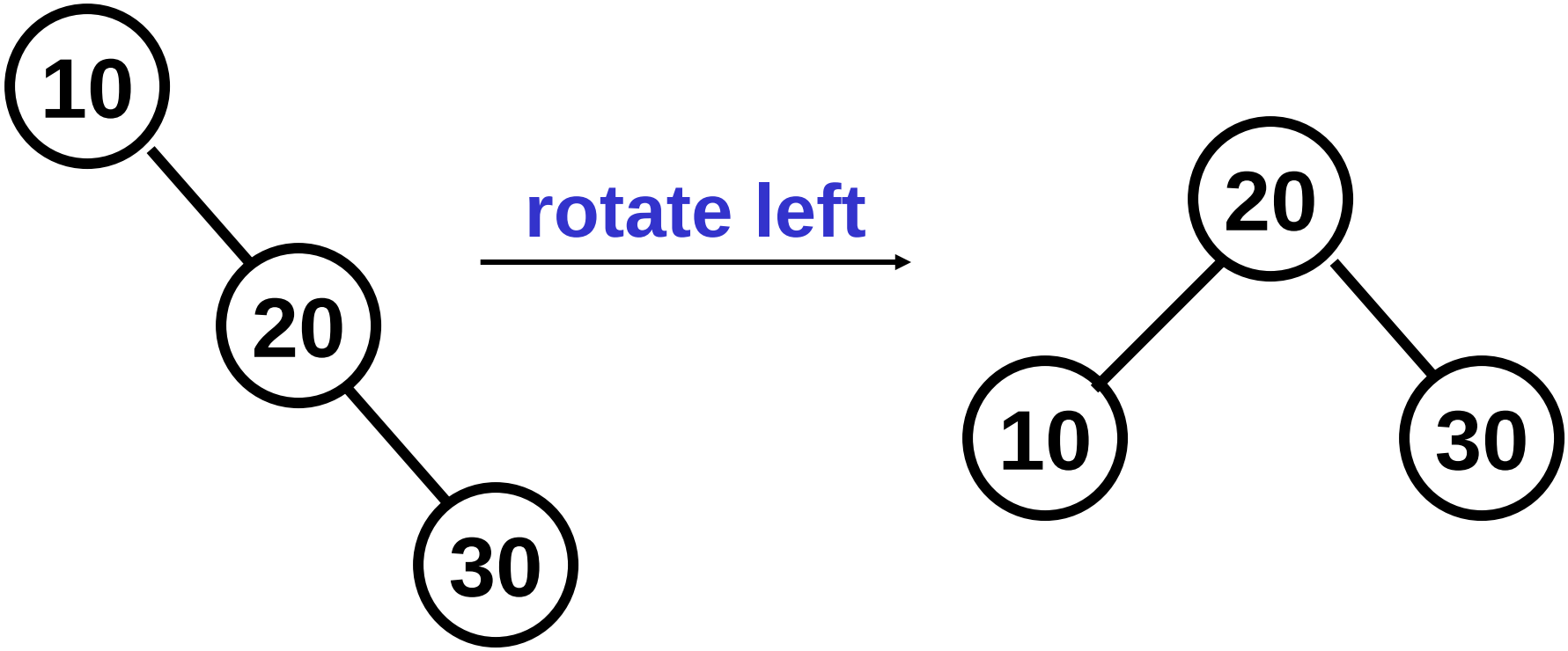
AVL Trees

- An AVL tree is a **balanced** binary search tree.(AVL comes from the inventors: Adelson, Velski, and Landis)
- The main idea is to maintain that for every node, and its subtrees. Depths differ by no more than 1.
- If the property above is violated, the tree is “rotated” to maintain its balance.
- The depth of an AVL tree with n nodes is O(log n)

AVL Examples

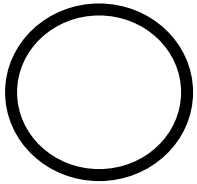


AVL Example



Counting Binary Trees

for n=1, count=1

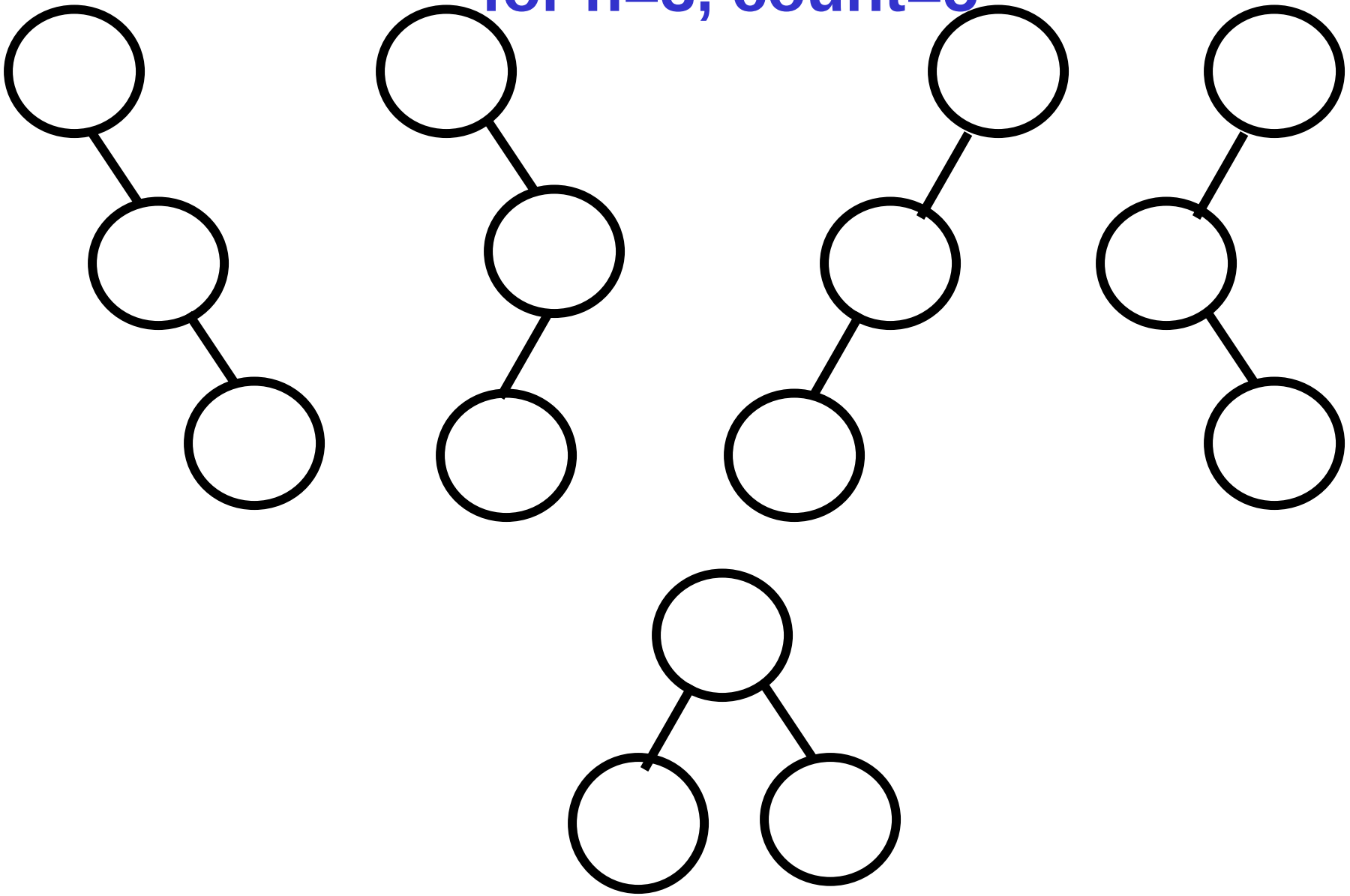


Counting Binary Trees(cont'd)

for n=2, count=2

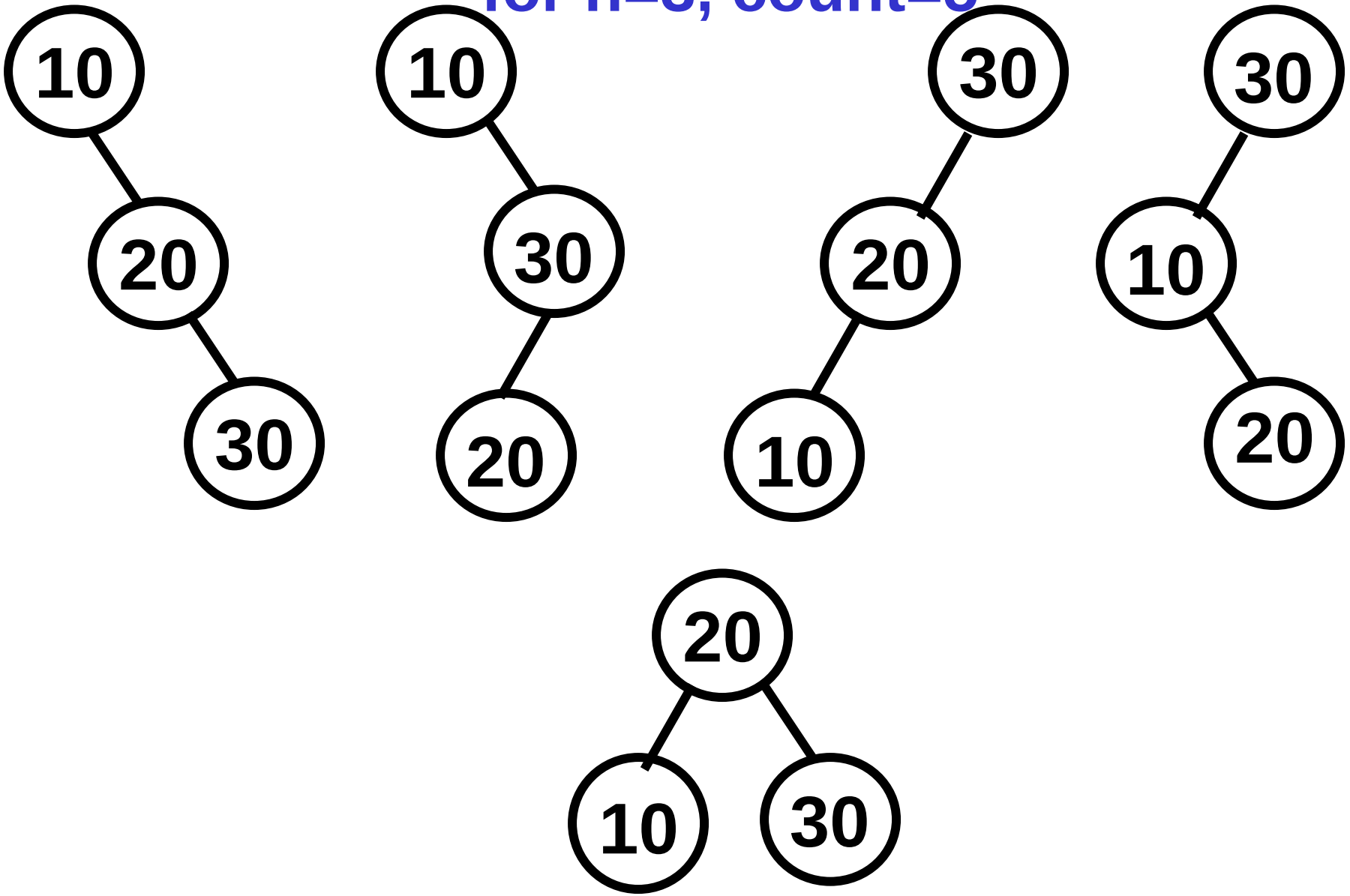
Counting Binary Trees (cont'd)

for n=3, count=5



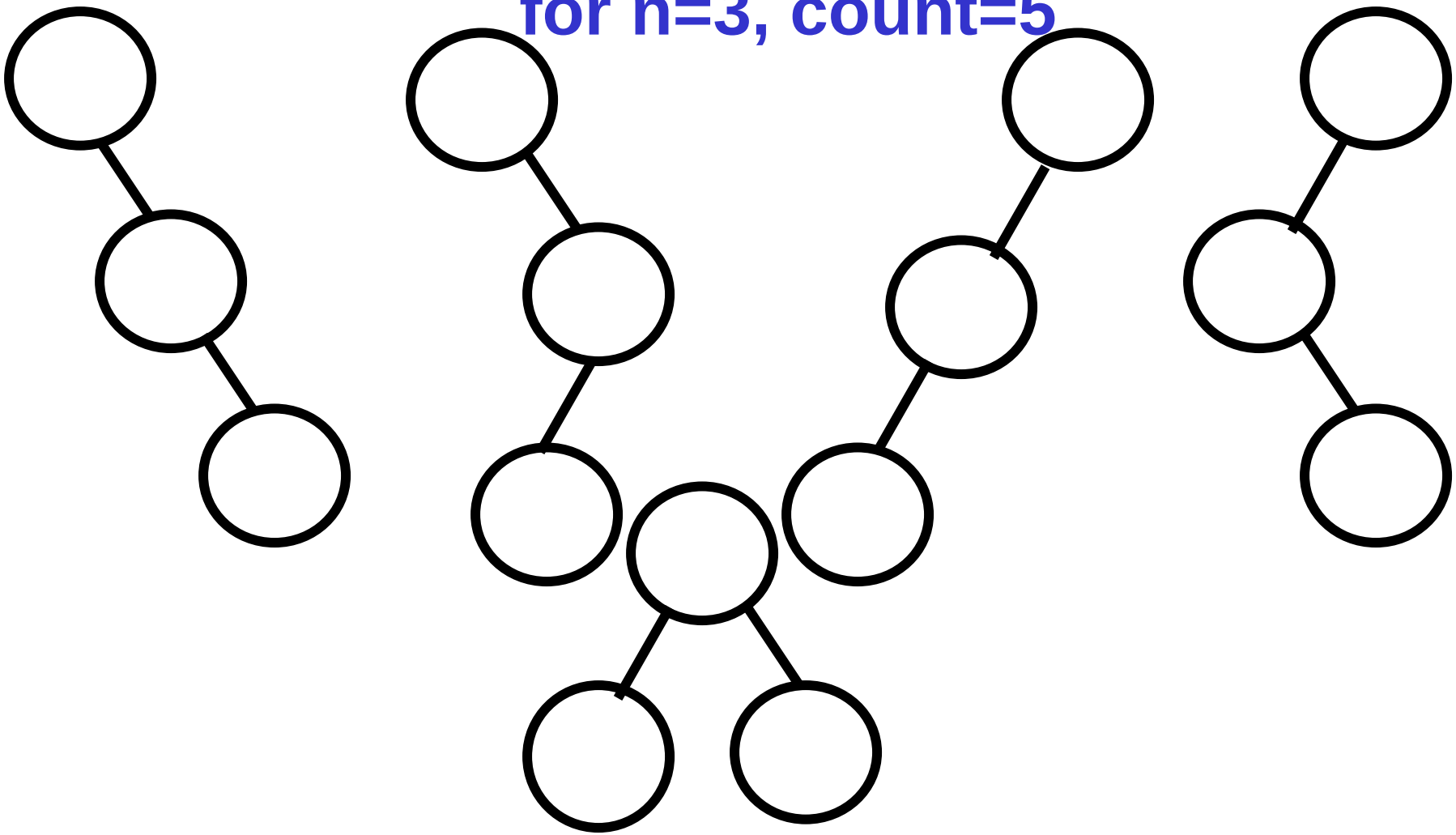
Counting Binary Trees (cont'd)

for n=3, count=5



Counting Binary Trees (cont'd)

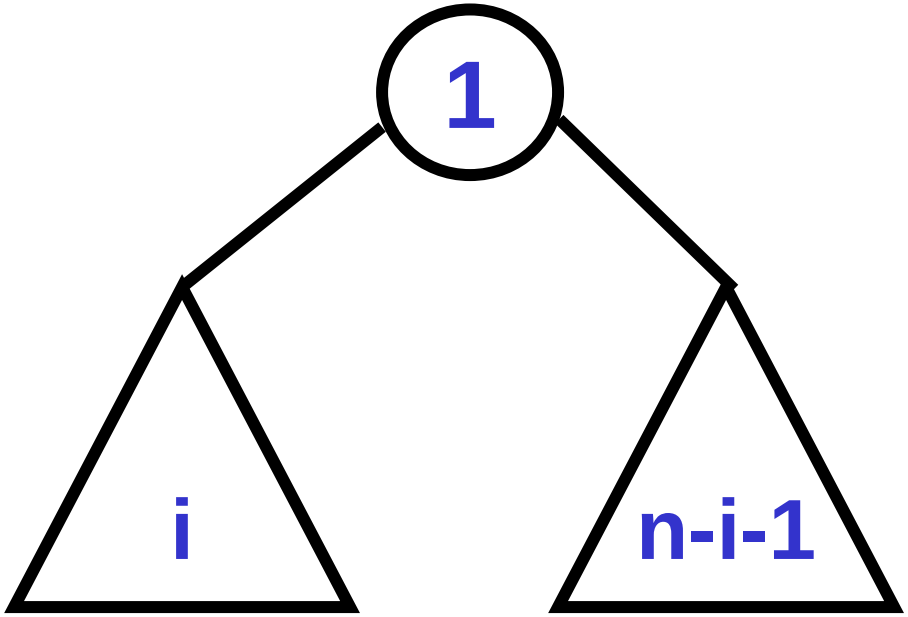
for n=3, count=5



$$b_3 = b_0 * b_2 + b_1 * b_1 + b_2 * b_0 = 1*2+1*1+2*1=5$$

Counting Binary Trees (cont'd)

General Form



$$b_n = b_0 * b_{n-i-1} + b_1 * b_{n-i-2} + \ldots\ldots + b_{n-i-1} * b_0$$