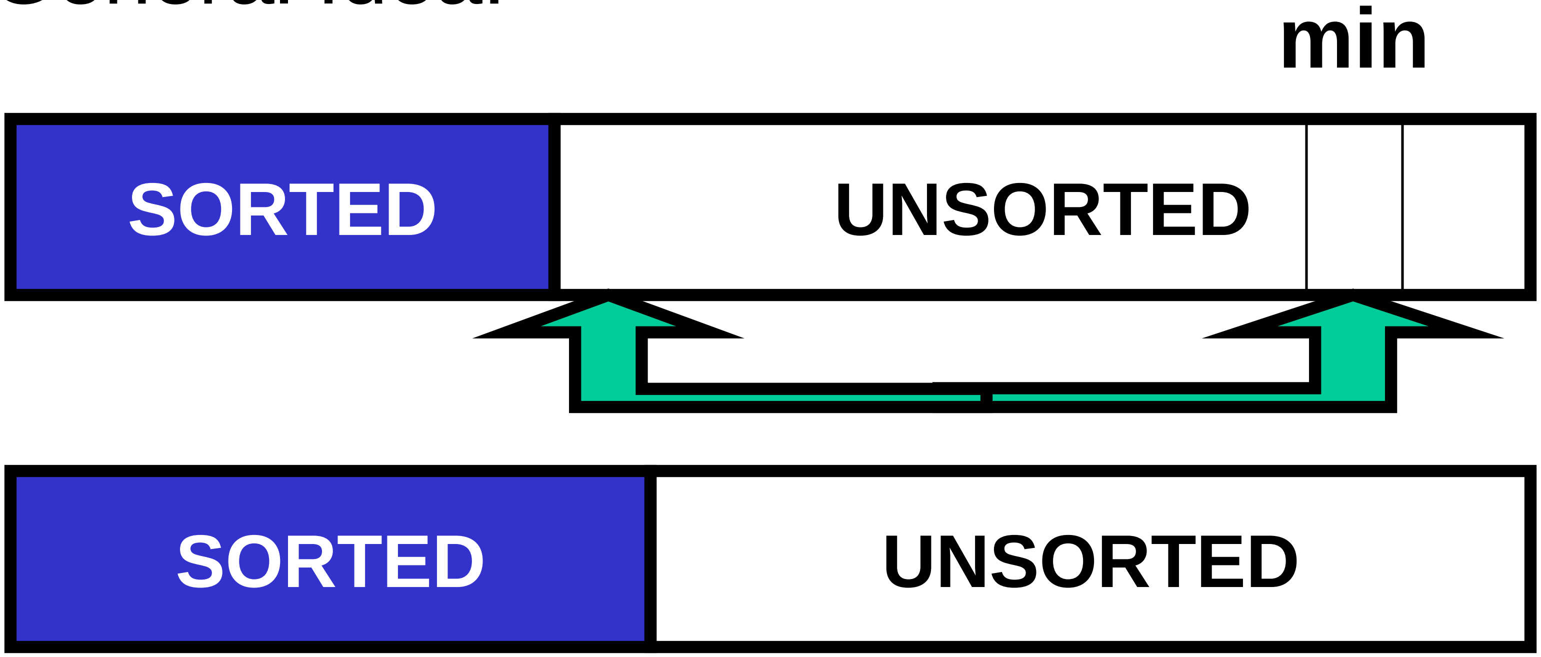
Sorting Algorithms

Chapter 12

Our first sort: Selection Sort

- General Idea:



What is the invariant of this sort?

Selection Sort

- Let A be an array of n ints, and we wish to sort these keys in non-decreasing order.
- Algorithm:

```
for i = 0 to n-2 do
    find j, $i \le j \le n-1$,
        such that $A[j] \le A[k], \forall k, i \le k \le n-1$.
    swap A[j] with A[i]
```

- This algorithm works in place, meaning it uses its own storage to perform the sort.

Selection Sort Example

66	44	99	55	11	88	22	77	33
11	44	99	55	66	88	22	77	33
11	22	99	55	66	88	44	77	33
11	22	33	55	66	88	44	77	99
11	22	33	44	66	88	55	77	99
11	22	33	44	55	88	66	77	99
11	22	33	44	55	66	88	77	99
11	22	33	44	55	66	77	88	99
11	22	33	44	55	66	77	88	99

Selection Sort

```
public static void sort
  (int[] data, int n)
{
  int i,j,minLocation;
  for (i=0; i<=n-2; i++) {
     minLocation = i;
     for (j=i+1; j<=n-1; j++)
       if (data[j] < data[minLocation])
               minLocation = j;
     swap(data, minLocation, i);
  }
}
```

Run time analysis

- Worst Case:

  Search for 1st min: n-1 comparisons

  Search for 2nd min: n-2 comparisons

  ...

  Search for 2nd-to-last min: 1 comparison

  Total comparisons:

  $$(n-1) + (n-2) + \ldots + 1 = O(n^2)$$

- Average Case and Best Case:

  $O(n^2)$ also! (Why?)

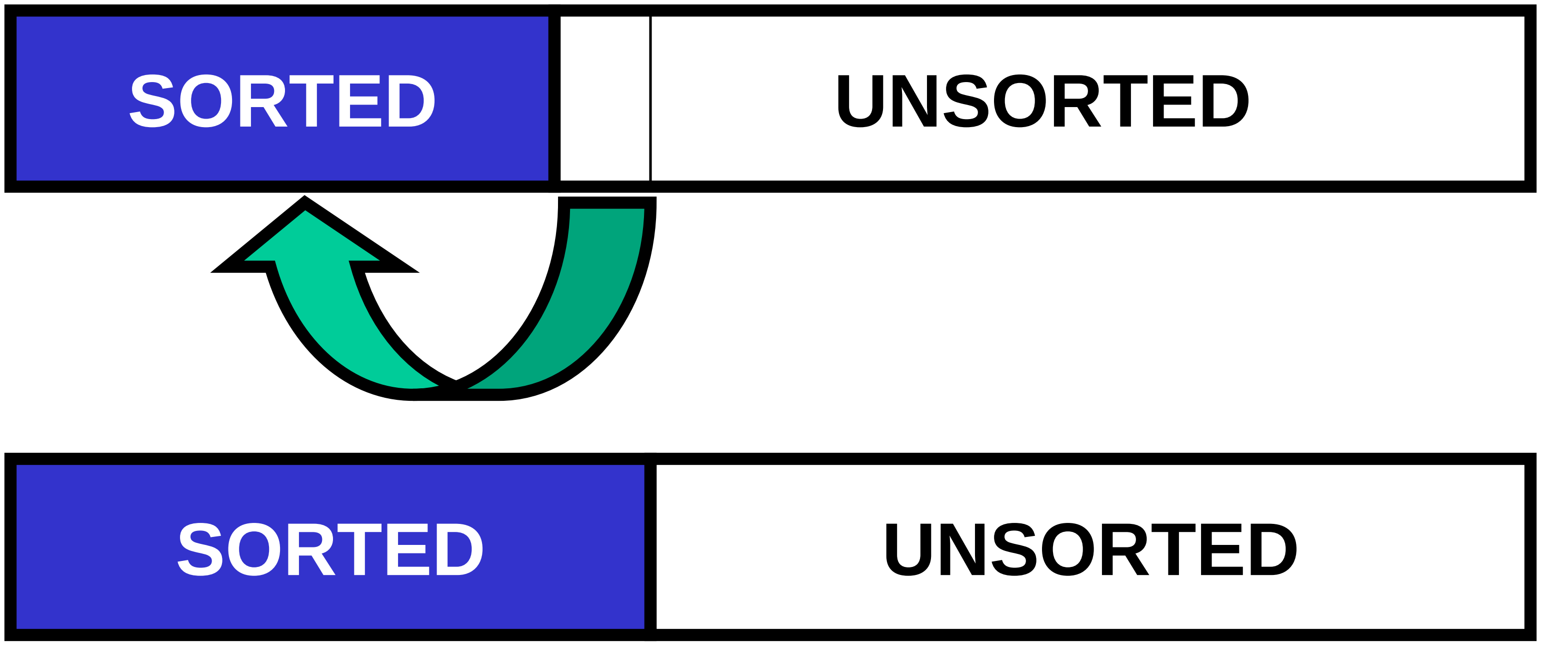
Selection Sort
(another algorithm)

```
public static void sort
  (int[] data, int n)
{
  int i, j;
  for (i=0; i<=n-2; i++)
     for (j=i+1; j<=n-1; j++)
        if (data[j] < data[i])
           swap(data, i, j);
}
```

Is this any better?

Insertion Sort

- General Idea:



What is the invariant of this sort?

Insertion Sort

- Let A be an array of n ints, and we wish to sort these keys in non-decreasing order.
- Algorithm:

```
for i = 1 to n-1 do
        item = A[i]
        shift A[j] to A[j+1], ∀j, j < i, where A[j] > item
        let k be the smallest j above, or k=i if no shifts
        A[k] = item
```

- This algorithm also works in place.

Insertion Sort Example

66	44	99	55	11	88	22	77	33
44	66	99	55	11	88	22	77	33
44	66	99	55	11	88	22	77	33
44	55	66	99	11	88	22	77	33
11	44	55	66	99	88	22	77	33
11	44	55	66	88	99	22	77	33
11	22	44	55	66	88	99	77	33
11	22	44	55	66	77	88	99	33
11	22	33	44	55	66	77	88	99

Insertion Sort

```
public static void sort(int[] data,
  int n) {
  int i,j,item;
  for (i = 1; i <= n-1; i++) {
     item = data[i];  j = i;
     while (j > 0 && data[j-1] > item) {
          data[j] = data[j-1];
          j--;
     }
     data[j] = item;
  }
}
```

Is insertion sort any better?

- Worst Case: *When does this occur?*

  Insert 2nd item: 1 shift

  Insert 3rd item: 2 shifts

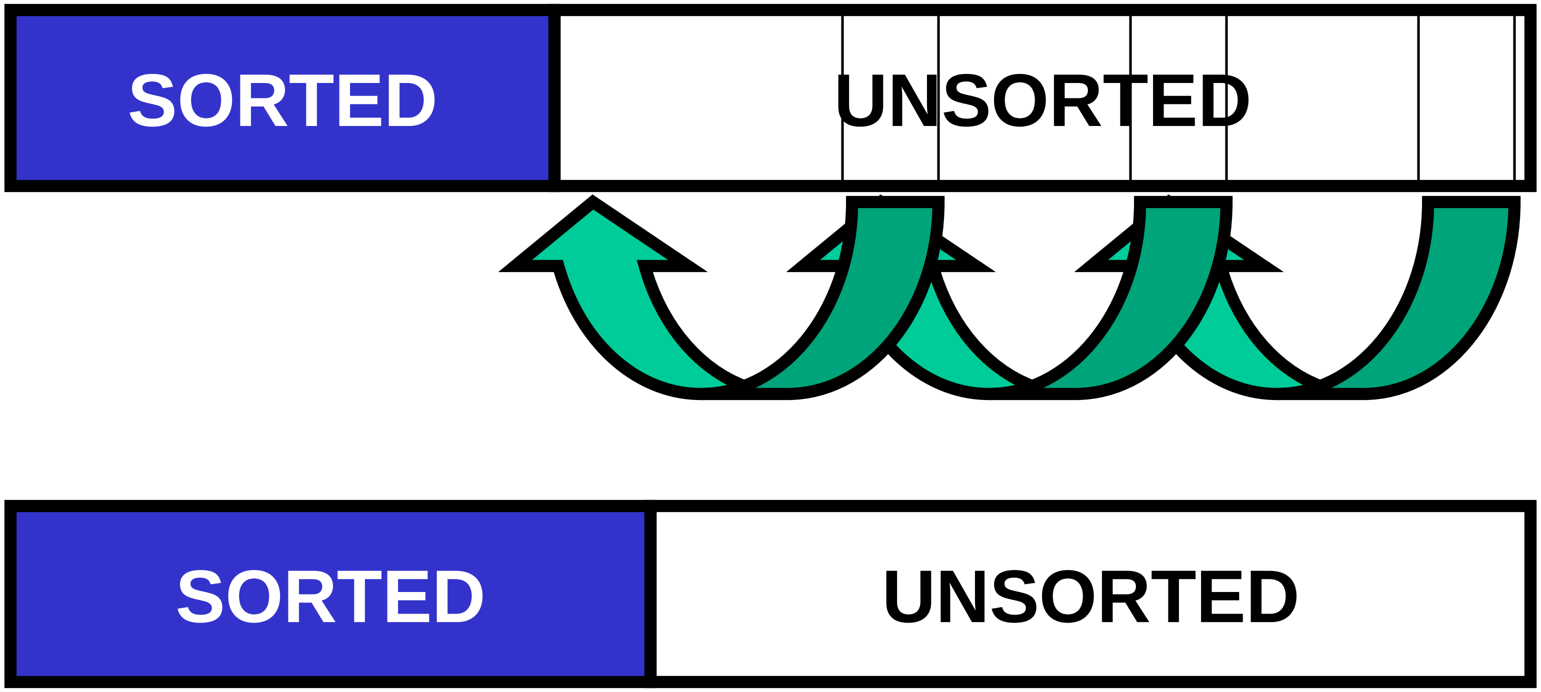
  ...

  Insert last item: n-1 shifts

  Total shifts: 1+2+...+(n-1) = $O(n^2)$
- Average Case: $O(n^2)$ also
- Best Case: $O(n)$

  This occurs if the array is already sorted.
  No shifts needed!

Bubble Sort (Exchange Sort)



What is the invariant of this sort?

Bubble Sort Example

66	44	99	55	11	88	22	77	33
11	66	44	99	55	22	88	33	77
11	22	66	44	99	55	33	88	77
11	22	33	66	44	99	55	77	88
11	22	33	44	66	55	99	77	88
11	22	33	44	55	66	77	99	88
11	22	33	44	55	66	77	88	99
11	22	33	44	55	66	77	88	99
11	22	33	44	55	66	77	88	99

Bubble Sort

```
public static void sort
  (int[] data, int n)
{
  int i, j;
  for (i=0; i <= n-2; i++)
     for (j=n-1; j > i; j--)
        if (data[j] < data[j-1])
           swap(data, j, j-1);
}
```

Bubble Sort Special cases

- Array is already sorted
- Array is partially sorted
- First element is out of place
- Last element is out of place

Quadratic Sorts

- Quadratic sorts have a worst-case order of complexity of $O(n^2)$
- Selection sort always performs poorly, even on a sequence of sorted keys!
- Insertion sort performs much better if the keys are sorted or nearly sorted.
- Bubble sort performs better on special cases.

Divide-and-Conquer Sorts

- Divide the elements to be sorted into two groups of approximately equal size.
- Sort each of these smaller groups.
- Combine the two sorted groups into one large sorted list.

Use recursion to sort the smaller groups.

Merge Sort

- Split the array into two “halves”.
- Sort each of the halves recursively using merge sort.
- Merge the two sorted halves to obtain a completely sorted array.
- Example:

66 44 99 55 11 88 22 77 33

sort the halves recursively...

44 55 66 99 11 22 33 77 88

Merge Sort (cont'd)

Then merge the two sorted halves into a new array:

44 55 66 99 11 22 33 77 88

___ ___ ___ ___ ___ ___ ___ ___ ___

44 55 66 99 11 22 33 77 88

11 ___ ___ ___ ___ ___ ___ ___ ___

44 55 66 99 11 22 33 77 88

11 22 ___ ___ ___ ___ ___ ___ ___

Merge Sort (cont'd)

44	55	66	99	11	22	33	77	88
11	22	33	___	___	___	___	___	___

44	55	66	99	11	22	33	77	88
11	22	33	44	___	___	___	___	___

44	55	66	99	11	22	33	77	88
11	22	33	44	55	___	___	___	___

Merge Sort (cont'd)

44	55	66	99	11	22	33	77	88
11	22	33	44	55	66	___	___	___

44	55	66	99	11	22	33	77	88
11	22	33	44	55	66	77	___	___

44	55	66	99	11	22	33	77	88
11	22	33	44	55	66	77	88	___

Merge Sort (cont'd)

Once one of the halves has been merged into the new array, copy the remaining element(s) of the other half into the new array:

44	55	66	99	11	22	33	77	88
11	22	33	44	55	66	77	88	99

Another Example:

43 42 21 44 45 26 27 25 23 24

sort the halves recursively...

21 42 43 44 45 23 24 25 26 27

then merge the two sorted halves ….

21 23 24 25 26 27 42 43 44 45

- Merge sort does not sort in place.

Merge Sort

```
public static void sort(int[] data,
     int first, int n) {
  int n1, n2;
  if (n > 1) {
     n1 = n / 2;  n2 = n - n1;
     sort(data, first, n1);
     sort(data, first+n1, n2);
     merge(data, first, n1, n2);
  }
}          // what is the stopping case?
```

Run-Time Analysis

- Let T(N) = number of operations to sort N elements using merge sort.
- How many operations does it take to sort half of the array?

  T(N/2)
- How many operations does it take to merge the two halves?

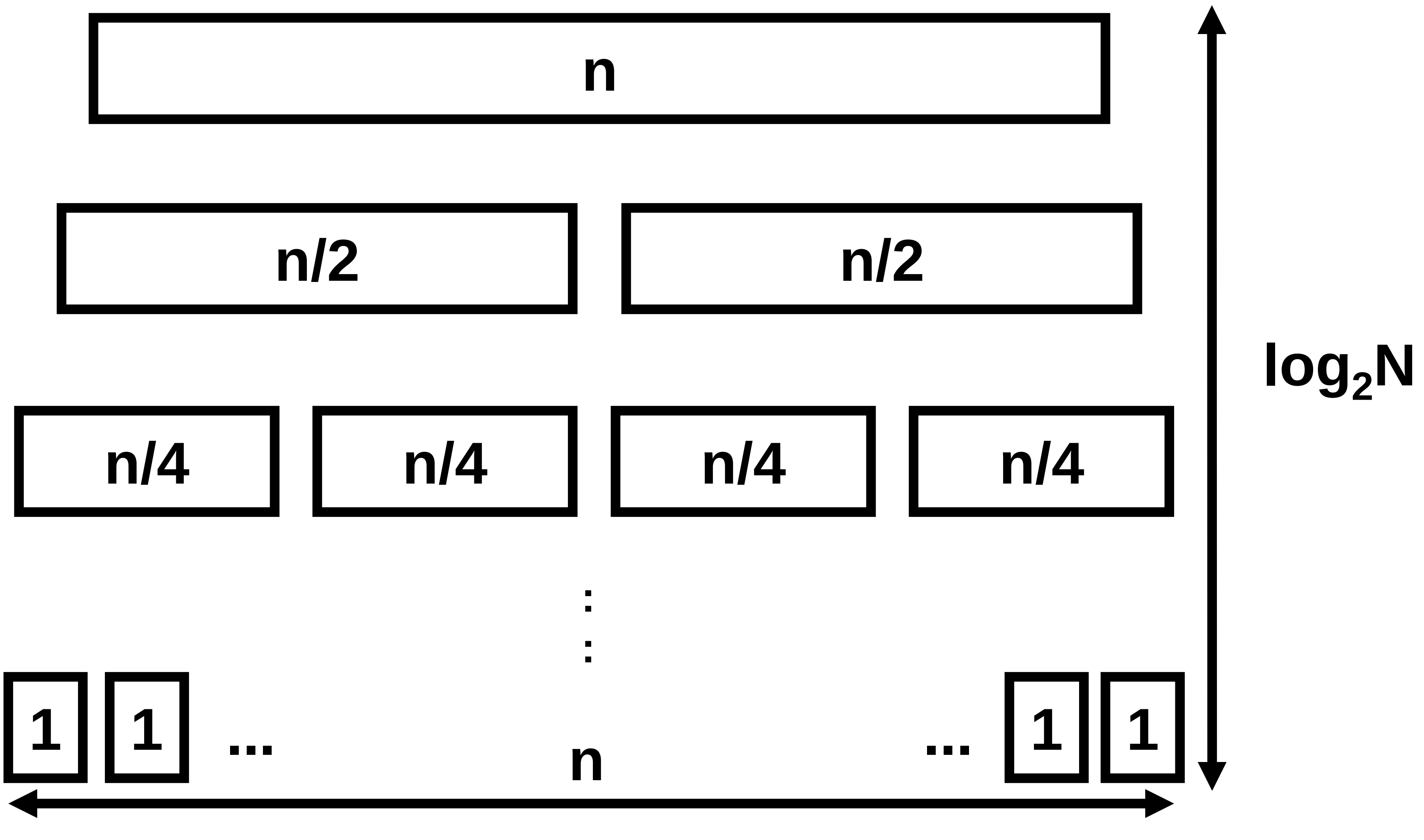
  2N
- T(N) = 2*T(N/2) + 2N

Run-Time Analysis (cont'd)

- What is the stopping case?

  $T(1) = 0$
- $T(N) = O(N \log N)$
- Note: $O(N \log N) < O(N^2)$

Another way to look at it



Quick Sort (Partition Sort)

- Choose a pivot element of the array.
- Partition the array so that
  - the pivot element is in the correct position for the sorted array
  - all the elements to the left of the pivot are less than or equal to the pivot
  - all the elements to the right of the pivot are greater than the pivot
- Sort the subarray to the left of the pivot and the subarray to the right of the pivot recursively using quick sort

Partitioning the array

Arbitrarily choose the first element as the pivot.

66 44 99 55 11 88 22 77 33

Search from the left end for the first element that is greater than the pivot.

66 44 99 55 11 88 22 77 33

Search from the right end for the first element that is less than (or equal to) the pivot.

66 44 99 55 11 88 22 77 33

Now swap these two elements.

66 44 $\underline{33}$ 55 11 88 22 77 $\underline{99}$

Partitioning the array (cont'd)

66 44 33 55 11 88 22 77 99

From the two elements just swapped, search again from the left and right ends for the next elements that are greater than and less than the pivot, respectively.

66 44 33 55 11 88 22 77 99

Swap these as well.

66 44 33 55 11 22 88 77 99

Continue this process until our searches from each end meet.

Partitioning the array (cont’d)

At this point, the array has been partitioned into two subarrays, one with elements less than (or equal to) the pivot, and the other with elements greater than the pivot.

66 44 33 55 11 22 88 77 99

Finally, swap the pivot with the last element in the first subarray section (the elements that are less than the pivot).

22 44 33 55 11 66 88 77 99

Now sort the two subarrays on either side of the pivot using quick sort recursively.

Quick Sort

```
public static void sort(int[] data,
     int first, int n) {
  int n1, n2, pivotIndex;
  if (n > 1) {
     pivotIndex = partition(data,first,n);
     n1 = pivotIndex - first;
     n2 = n - n1 - 1;
     sort(data,first,n1);
     sort(data,pivotIndex+1,n2);
  }
}
```

Run-Time Analysis

- Assume the pivot ends up in the center position of the array.
- Then, quick sort runs in O(N log N) time just like merge sort.
- However, what if the pivot doesn't end up in the center during partitioning?

  Example: Pivot is smallest element. Then we get two subarrays, one of size 0, and the other of size n-1 (instead of n/2 for each).
- Then, quick sort can perform as bad as $O(n^2)$.

  When does this occur?

Some Improvements to Quick Sort

- Choose three values from the array, and use the middle element of the three as the pivot.

66 44 99 55 11 88 22 77 33

  Of 11, 33, 66, use 33 as the pivot.

- As quick sort is called recursively, if a subarray is of “small size”, use insertion sort instead of quick sort to complete the sorting to reduce the number of recursive calls.

```
void quickSort(int[] data,int first,int last){
   int left, right, pivot;
   if (first >= last)  return;
   left = first; right = last;
   pivot = data[ (first + last)/2 ];
   do{
      while (data[left]  < pivot) left++;
      while (data[right] > pivot) right--;
      if (left <= right)
         swap(data, left++, right--);
   } while ( left <= right);
   quickSort(data, first, right);
}  quickSort(data, left , last);
```

Heap Sort

- We can use a heap to sort data.
- Convert an array to a heap.
- Remove the root from the heap and store it in its proper position in the same array
- Example:

HEAP

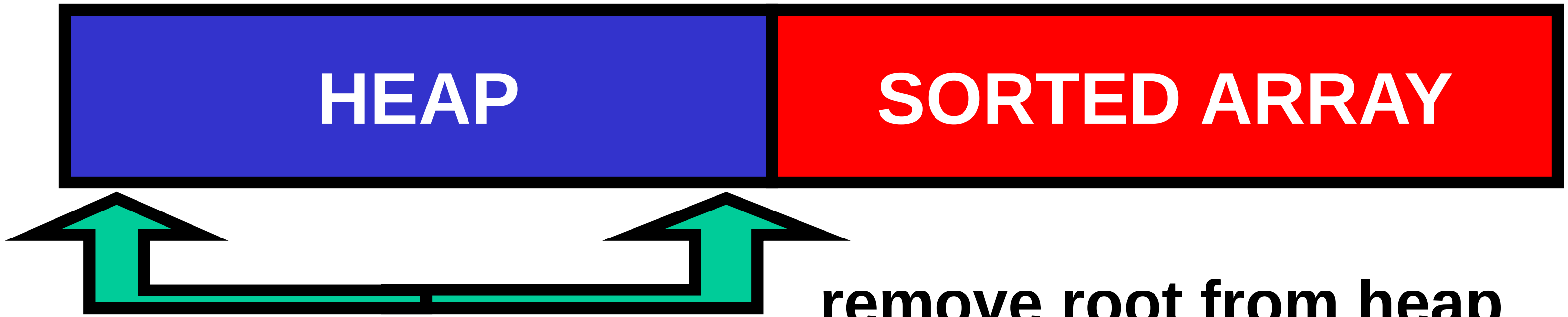
SORTED ARRAY

0	1	2	3	4	5	6	7	8
95	61	83	53	39	72	16	24	48

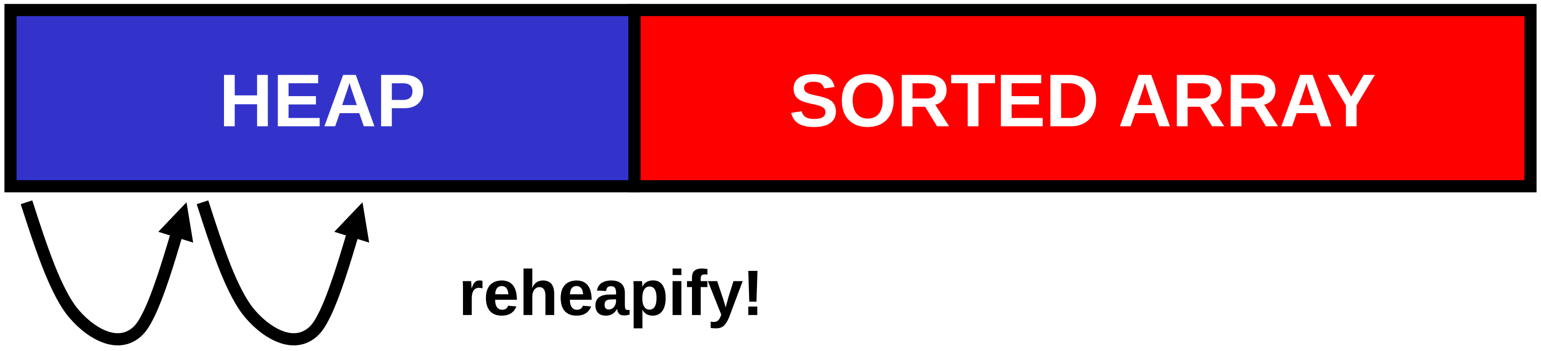
0	1	2	3	4	5	6	7	8
83	61	72	53	39	48	16	24	95

0	1	2	3	4	5	6	7	8
72	61	48	53	39	24	16	83	95

Sorting using the heap



remove root from heap



reheapify!

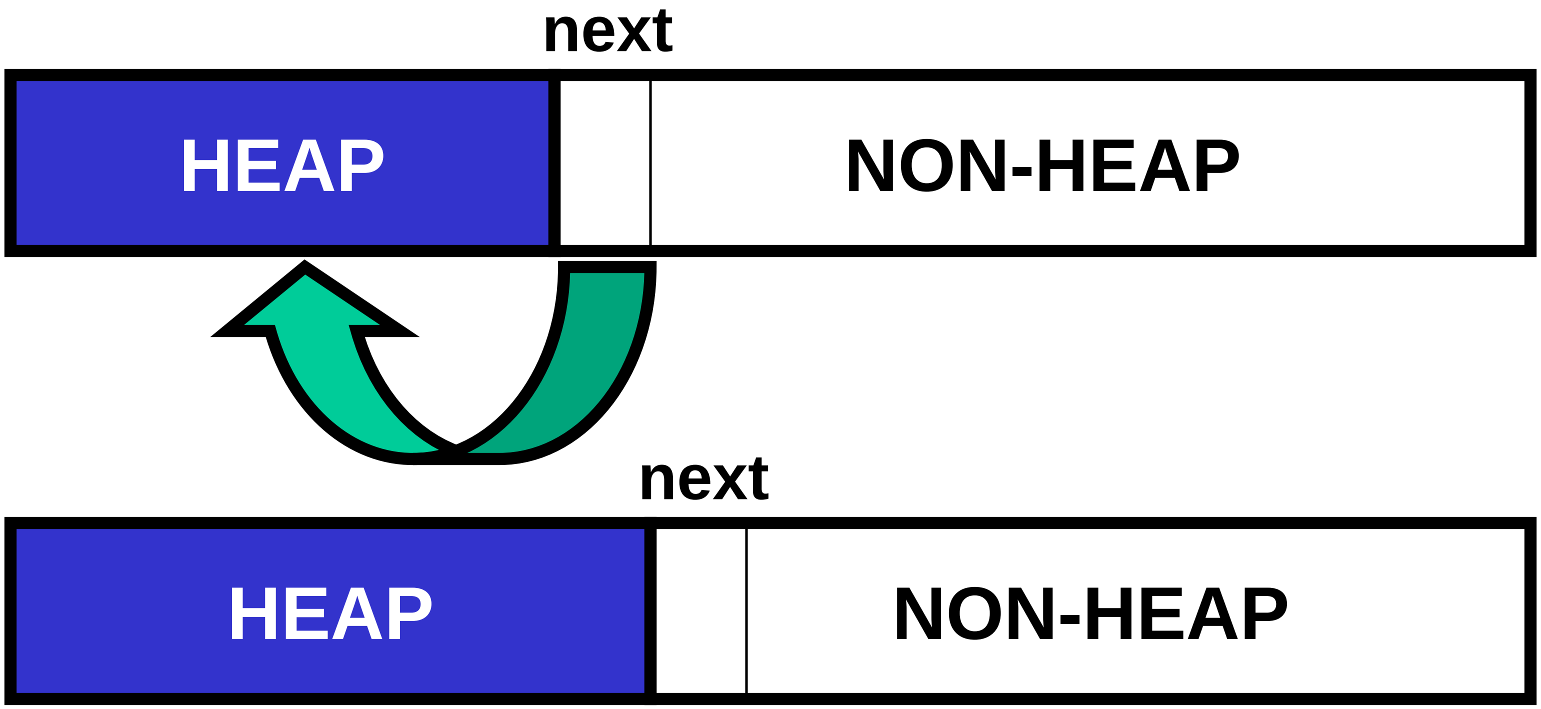
Heap Sort

```
public static void sort(int[] data,
  int n) {
  int lastHeapPosition;
  makeHeap(data, n);
  lastHeapPosition = n-1;
  while (lastHeapPosition > 0) {
    swap(data, 0, lastHeapPosition);
    reheapify(data, lastHeapPosition);
    lastHeapPosition--;
  }
}
```

Convert array to a heap.

Move the root and fix the remaining heap

Making the heap



```
public static void makeHeap
                  (int[] data, int n) {
  int i;
  int next;   // next element for heap
  for (i = 1; i <= n-1 ; i++) {
      next = i;
     //insert next into heap from 0..i-1
     while (next != 0 &&
        data[next] > data[parent(next)]) {
           swap(data, next, parent(next));
           next = parent(next);
     }
  }
}
```

Assumes a parent() method
to return index of parent of next.

Make Heap Example

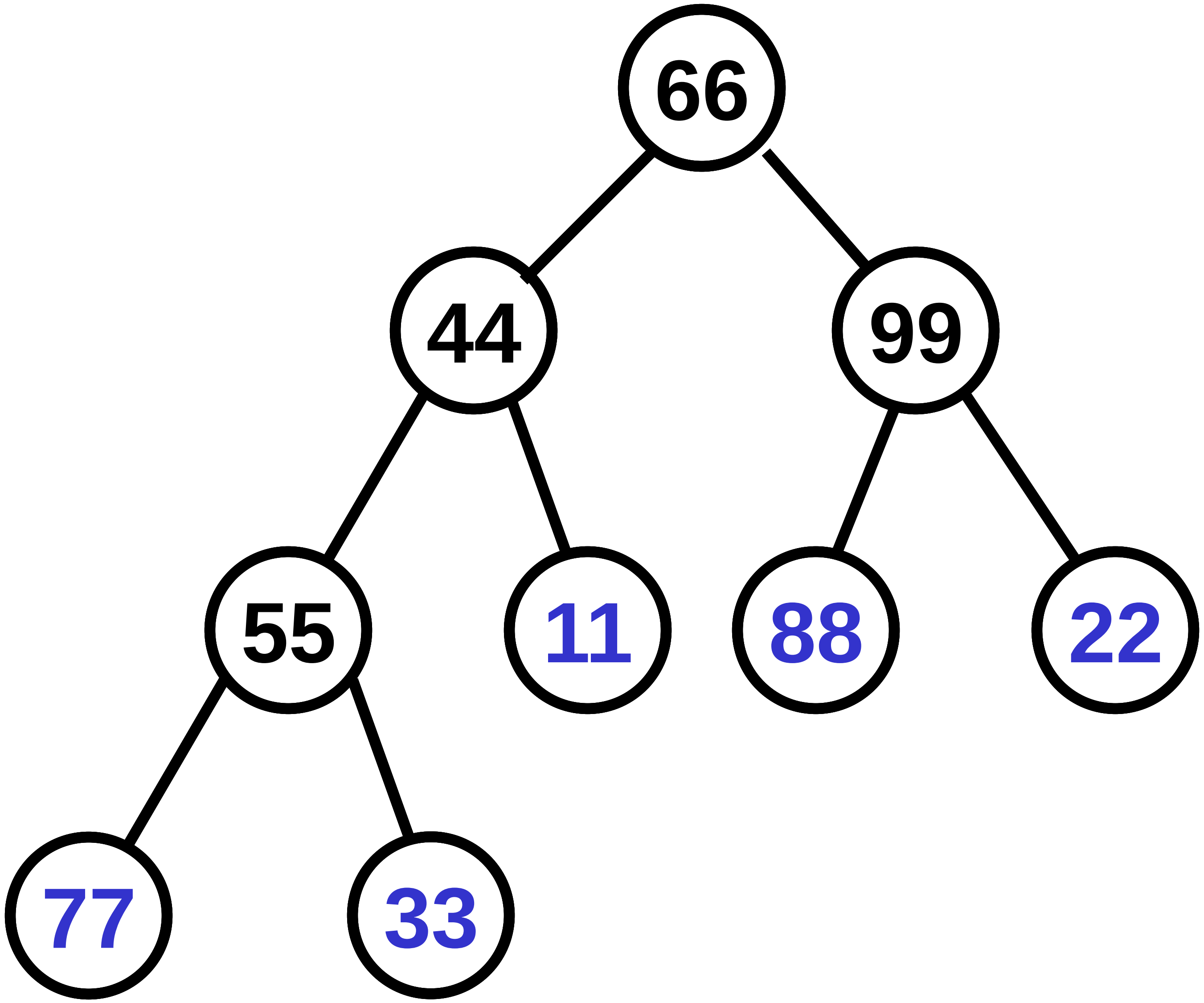
(heap in blue)

66	44	99	55	11	88	22	77	33
66	44	99	55	11	88	22	77	33
99	44	66	55	11	88	22	77	33
99	55	66	44	11	88	22	77	33
99	55	66	44	11	88	22	77	33
99	55	88	44	11	66	22	77	33
99	55	88	44	11	66	22	77	33
99	77	88	55	11	66	22	44	33
99	77	88	55	11	66	22	44	33

```
private static void reheapify
        (int[] data, int heapsize) {
   int position = 0; int childPos;
   while (position*2 + 1 < heapsize) {
      childPos = position*2 + 1;
      if (childPos < heapsize-1 &&
        data[childPos+1] > data[childPos])
           childPos++;
      if (data[position]<data[childPos]) {
           swap(data, position, childPos);
           position = childPos;
      }
      else return;
   }
}
```

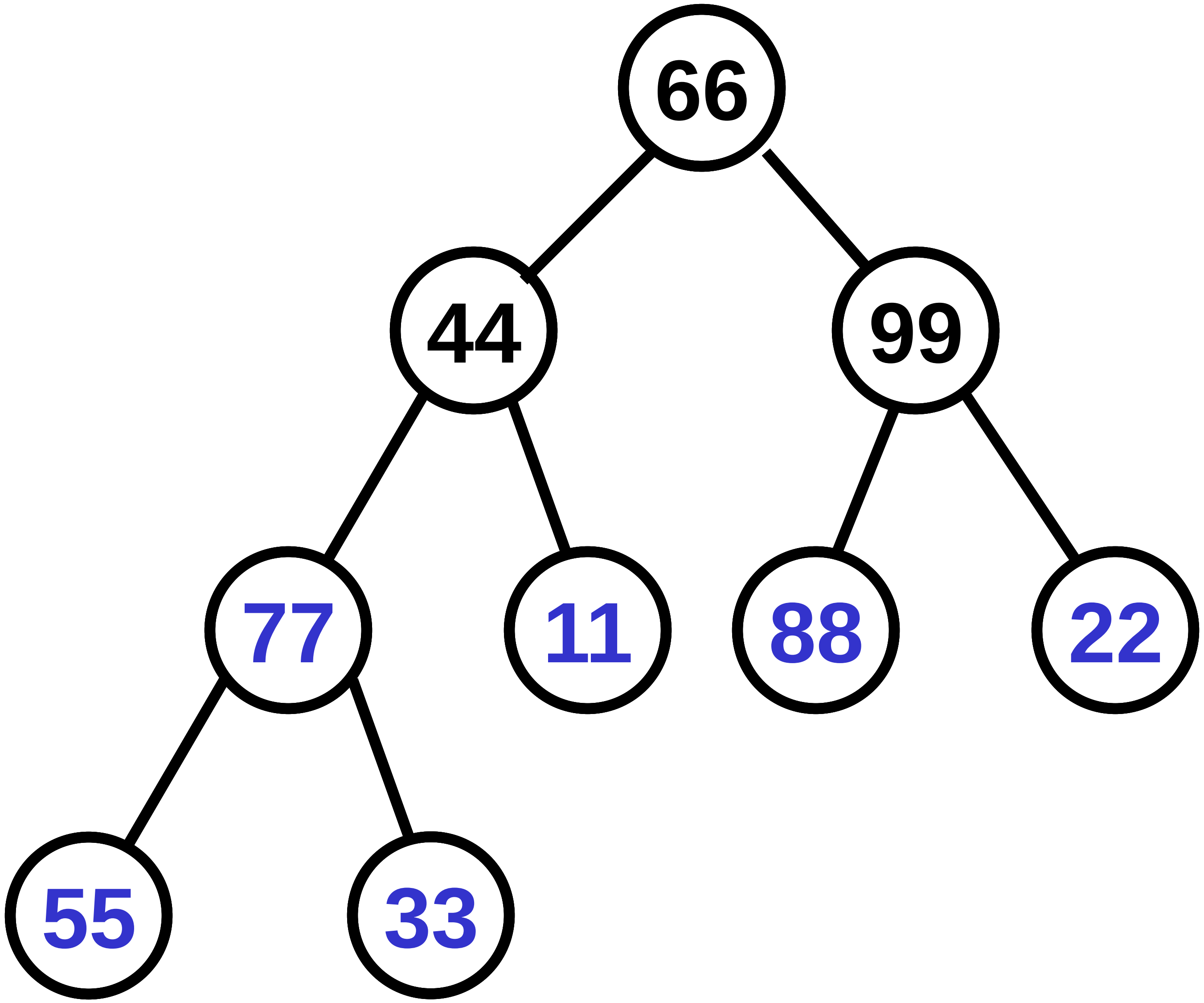
Make Heap - Another View

66 44 99 55 11 88 22 77 33



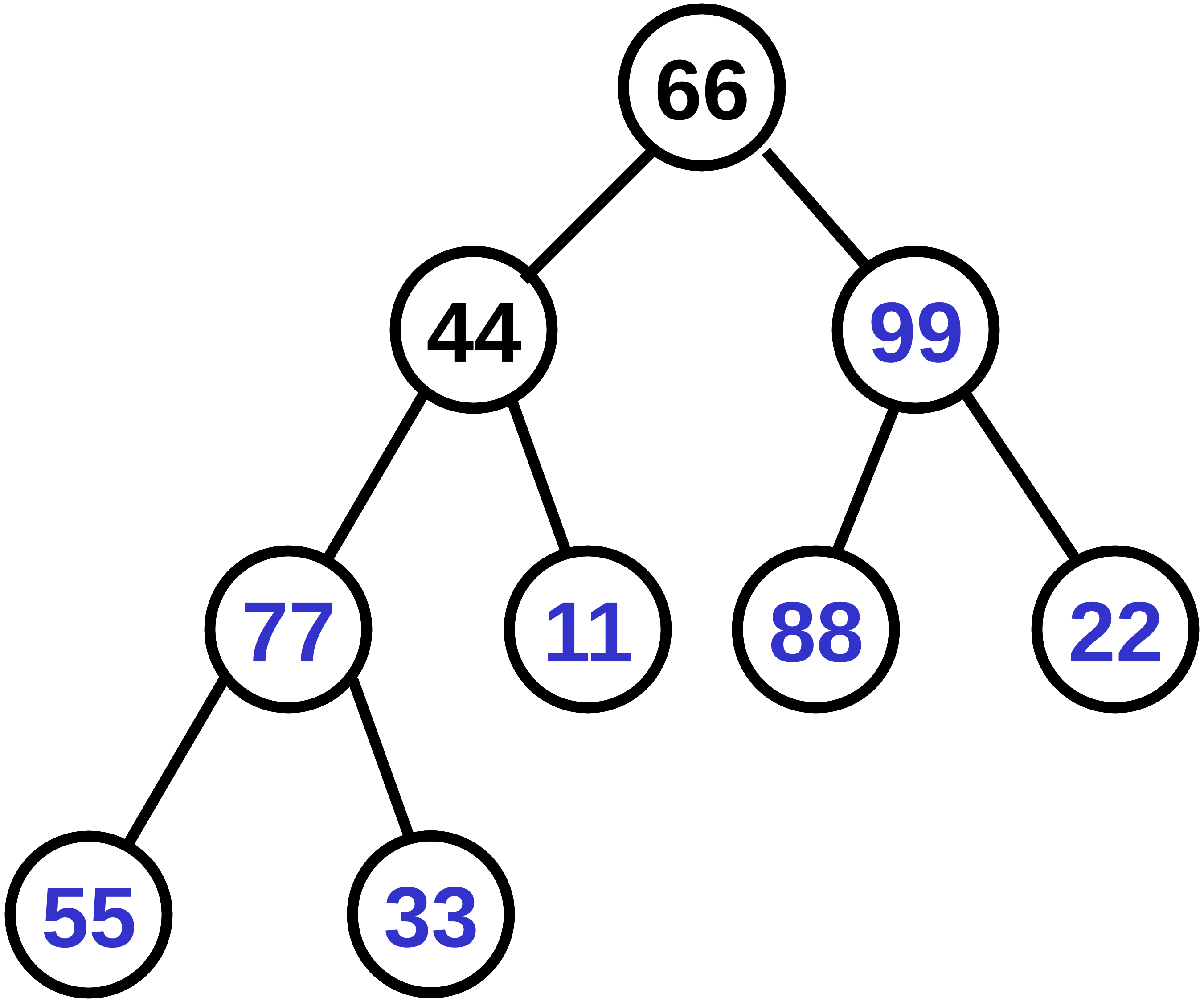
Make Heap - Another View

66 44 99 77 11 88 22 55 33



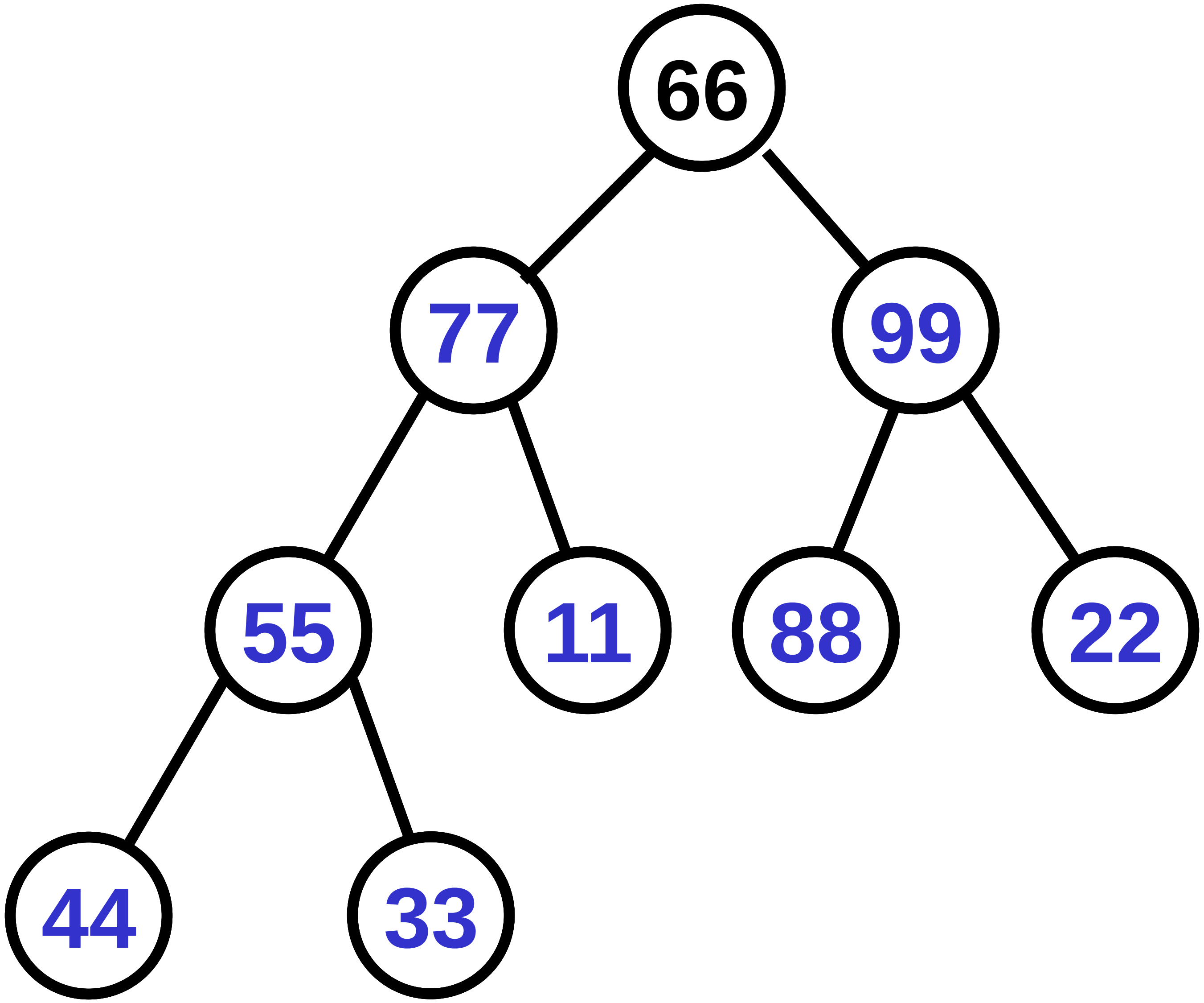
Make Heap - Another View

66 44 99 77 11 88 22 55 33



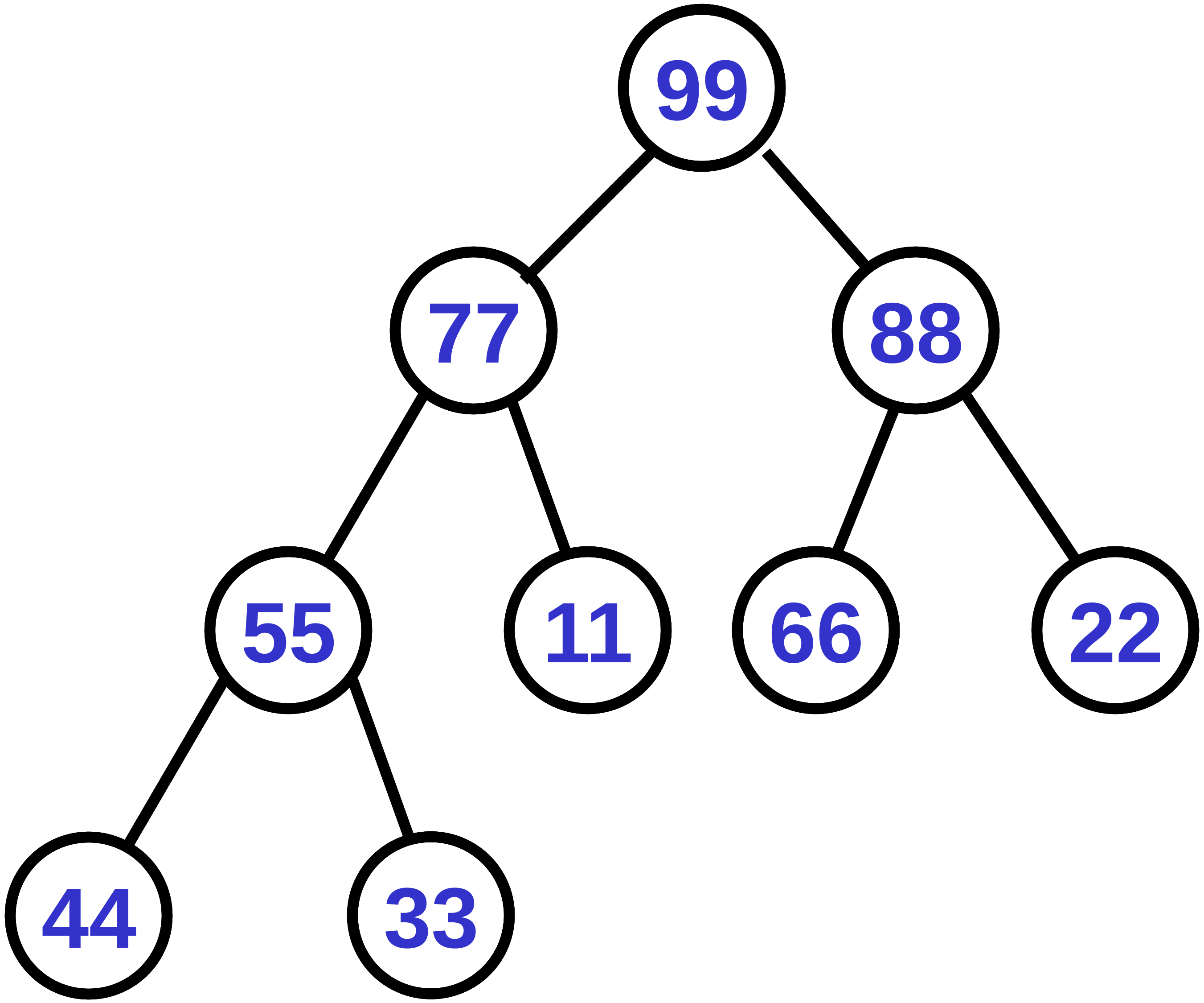
Make Heap - Another View

66 77 99 55 11 88 22 44 33



Make Heap - Another View

99 77 88 55 11 66 22 44 33



Make Heap Example

(organized heap in blue)

66	44	99	55	11	88	22	77	33
66	44	99	77	11	88	22	55	33
66	44	99	77	11	88	22	55	33
66	77	99	55	11	88	22	44	33
99	77	88	55	11	66	22	44	33

```
public static void efficientMakeHeap
                (int[] data, int n) {
  int i, child, position = 0;
  for (i =(n-2)/2; i >= 0 ; i--){
    position = i;
    while (position*2+1 < n){
       child = position*2+1;
       if (child < n-1  &&
              data[child+1]>data[child])
          child++;
       if(data[child] > data[position]){
          swap(data, child, position);
          position = child;}
       else break;
    }
  } // for
} //efficientMakeHeap
```

Heap Sort Run-Time Analysis

- Number of operations to convert an array into a heap: O(n)
- Number of operations to remove each element from a heap: O(log n)
- Number of operations to remove n elements from the heap: O(n * log n)
- Total number of operations:

  O(n) + O(n*log n) = O(n * log n)

Counting Sort(Linear Time)

- Input array has n integers in the range [0,k-1]
- The idea is that for each number x, how many elements are less than x.
- For example If there are 6 elements less than x, then x belongs in 7th position.
- If there are several elements with the same value we place them one after the other.
- We use following arrays for the sort:
  - data [0..n-1] – The input array
  - result[0..n-1] – The output array
  - count[0..k-1] & start[0..k-1] – Temp arrays

Counting Sort Example

	0	1	2	3	4	5	6	7	8	9	10
data	1	3	5	3	4	3	1	3	4	3	1

	0	1	2	3	4	5
count	0	3	0	5	2	1

	0	1	2	3	4	5
start	0	0	3	3	8	10

	0	1	2	3	4	5	6	7	8	9	10
result	1	1	1	3	3	3	3	3	4	4	5

Counting Sort Algorithm

```
for i = 0 to k-1 do                  ← O(k)
    count[i] = 0
for i = 0 to n-1 do                  ← O(n)
    count[data[i]]++
start[0]=0
for  i = 1 to k-1 do                 ← O(k)
    start[i] = count[i-1] + start[i-1]
for  i = 0 to n-1 do
                                     ← O(n)
    pos = start[ data[i] ]
    result[pos] = data[i]
    start[data[i]]++
```

Counting Sort (Cont'd)

- Run-Time Analysis
  - The first and third loops O(k)
  - The second and last loops O(n)
  - Total running time O(n+k)
  - If k = O(n), then total time will be O(n)
- Counting sort is stable
- This algorithm does not work in place.