Math 241: Notes

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Contents

Vectors and the Geometry of Space

12.4 Cross Product

The cross product of two vectors expresses a vector that is perpendicular to both the vectors.

Cross product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Notice also that the cross product of $\vec{a}$ and $\vec{b}$ can also be given by the determinant of the matrix

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

In other words,

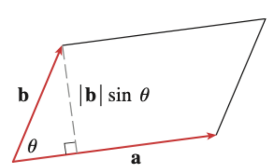
$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & b_3 \\ b_1 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Length of the cross product

If θ is the angle between $\vec{a}$ and $\vec{b}$ where $0 < \theta < \pi$ then the length of the cross product of vectors $\vec{a}$ and $\vec{b}$ is given by

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$$

The length of the cross product of $\vec{a}$ and $\vec{b}$ is also the area of the parallelogram composed by $\vec{a}$ and $\vec{b}$,

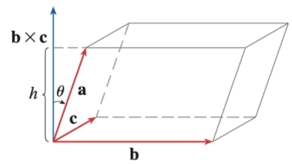


Note also that $\vec{a}$ and $\vec{b}$ are parallel only if,

$$\vec{a} \times \vec{b} = 0$$

The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by what is known as the scalar triple product.

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$



Notice that $|\vec{b} \times \vec{c}|$ is equal to the base area, and $|a|\cos(\theta)$ gives the height so V $=|\vec{b} \times \vec{c}||a|\cos(\theta) = \vec{a} \cdot (\vec{b} \times \vec{c})$. We need to apply the absolute sign to $\cos(\theta)$ because it is possible that $\theta > \frac{\pi}{2}$

12.5 Equations of lines and vectors

The position vectors of all points on a line can be traced out by a vector equation determined by a known point on a line and a direction vector parallel to the direction of the line. Mathematically this means,

$$\vec{r} = \vec{r_0} + t\vec{v}$$

where $\vec{r}, \vec{r_0}, t,$ and $\vec{v}$ are the position vector of any point on the line, the position vector of a known point on the line, a scalar that can be any real number, and the direction vector that is parallel to the line, respectively. We can also write $\vec{r}$ with its components,

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

Therefore,

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc$$

This is known as a parametric equation of a line. Since the variables are given in terms of the variable t, it is possible to eliminate the vector t by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is known as the symmetric or cartesian equation of a line. Notice that if the line passes through two points with position vectors $\vec{P_0} = \langle x_0, y_0, z_0\rangle$ and $\vec{P_1} = \langle x_1, y_1, z_1\rangle$, then we have

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

since the direction vector is given by

$$\vec{d} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{pmatrix}$$

If we wanted to find a line segment of a line and not the entire line, we can put the line into its vector equation with t bounded by 0 and 1 inclusive, and with endpoints having the position vectors $\vec{P_0} = \langle x_0, y_0, z_0\rangle$ and $\vec{P_1} = \langle x_1, y_1, z_1\rangle$

$$\vec{r} = \vec{r_0} + (\vec{r_1} - \vec{r_0})t$$

Realise that this makes $\vec{r} = \vec{r_0}$ when $t = 0$ and $\vec{r} = \vec{r_1}$ when $t = 1$

Equation of a plane

The equation of a plane is defined with respect to a vector orthogonal to the point and an known point on the plane. Given an arbitrary point on the plane with position vector $\vec{r} = \langle x, y, z \rangle$, a known point with position vector $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, and a normal vector $\vec{n} = \langle a, b, c \rangle$ we have

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

So an equation of a plane through $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ with $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

We can find the x, y, z intercepts by setting the two variables other from the one we are trying to find as 0. Two find the angle between two planes we can use the equation

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

where $\vec{n_1}$ and $\vec{n_2}$ are the normal vectors of the plane.

If we are asked to find the equation of the line of intersection of the plane we can find the cross product of the normal vectors of the two planes which will yield a direction vector that is perpendicular to both the normal vectors. In other words, we will get a direction vector that is parallel to both planes and a valid direction vector to form the vector equation of the line. For example, given the equations $x + y + z = 1$ and $x - 2y + 3z = 1$, the direction vector of the line will be given by

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\vec{i} - 2\vec{j} - 3\vec{k}$$

Then we can find a point on the line by setting one variable as 0. This choice of variable is dependent on which plane the line will intersect, and there will always be one plane that the line will intersect. In this case we notice that the line is not parallel with the xy plane, so the line will intersect the xy plane once. Therefore we can let $z = 0$ which give us the equations

$$x + y = 1 \text{ and } x - 2y = 1$$

solving this system of equation will yield the point $(1, 0, 0)$. Why does solving the plane gives us a point on the line? After all the intersection of the two planes will have a line of intersection with the planes of the graph. However, notice that solving a system of equations involving the **two** equations of the plane implies that the answer will be a point on the line, because the answer will be one that is on both the equations.

Distance from point to plane

The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

We can apply this to two skew lines by taking any point on one line as P_1 and the plane as the plane the other line is on.

12.6 Cylinders and Quadric Surfaces

Cylinders are surfaces that consist of all lines(**rulings**) that are parallel to a given line and pass through a given plane curve. $z = x^2$ is such a cylinder and its rulings are parallel to the y-axis.

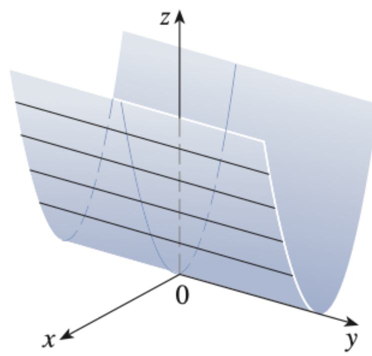


FIGURE 1
The surface $z = x^2$ is a parabolic cylinder.

If one of the variables x, y, and z is missing from the equation, then the surface is a cylinder and its rulings are parallel to the missing variable.
Traces are the cross sections of a curve, and the traces that is parallel to a given plane can be found by setting a variable as a constant k. For example, suppose that we have the quadric surface defined by the equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

To find the trace of the graph that is parallel to the $xy-$ plane, we could set the variable z to 0. In general, the trace of a quadric surface can be found by setting a particular variable to k depending on the plane which we want our traces to be parallel to. In the case of the function that we just saw, the horizontal trace in the plane $z = k$ is

$$x^2 + \frac{y^2}{9} + \frac{k^2}{4} = 1$$

Vector Functions

13.1 Vector valued functions and space curves

A vector valued function is a function that takes in a real number and outputs a vector. If we are interested in vector functions of three dimensions, the components of vector functions in three dimensiosn are also functions.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

The limit of a vector function $\vec{r}(t)$ is

$$\lim_{x \to a} \vec{r}(t) = \langle \lim_{x \to a} f(t), \lim_{x \to a} g(t), \lim_{x \to a} h(t) \rangle$$

A vector function is **continuous at a** if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$$

Space curves are the set of all points traced out by the tip of a vector function.

13.2 Derivatives and Integrals of Vector Functions

Derivative of a vector function

The derivative of a vector function is defined similarly to the derivative to a regular function. In fact

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(x) = \lim_{h \to 0} \frac{\vec{r}\,(t+h) - \vec{r}\,(t)}{h}$$

Furthermore

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\vec{i} + g'(t)\vec{j} + h(t)\vec{k}$$

$\vec{r}\,'(t)$ gives the tangent vector of the point at some point p and the tangent line at point p is defined to be the line that is parallel to the tangent vector through p. The unit tangent vector is the tangent vector multiplied by the reciprocal of the magnitude of itself just like finding the unit vector in the direction of any other vector

$$\vec{T(t)} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

If $\vec{r}(t) = c$, then $\vec{r}\,'(t)$ is orthogonal $\vec{r}(t)$ for all t.

The definite integral is defined in approximately the same way

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b \vec{g}(t)\,dt\right)\vec{j} + \left(\int_a^b \vec{r}(t)\,dt\right)\vec{k}$$

FTC could also be applied in the same way for a vector function, in fact

$$\int_a^b \vec{r}(t)\,dt = \left[\vec{R}(t)\right]_a^b = \vec{R}(b) - \vec{R}(a)$$

13.3 Arc Length and Curvature

Arc Length of a space curve

The arc length of a space curve is defined in a similar way as that of a plane curve. If we have the vector equation $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ and $a \le t \le b$. If the curve is traversed only once from a to b then

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

which can be conveniently written as

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

since

$$|\vec{r}\,'(t)| = |\langle f'(t), g'(t), h'(t)\rangle| = \sqrt{[f'(t)]^2, [g'(t)]^2, [h'(t)]^2}$$

The arc length function is given by

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$

It gives the length of the space curve from a to t.
The curvature of a curve is defined as

$$\kappa = \frac{|d\vec{T}|}{ds}$$

where $\vec{T}$ is the unit tangent vector. The curvature of a curve is the magnitude of the rate of change of the unit tangent vector with respect to arc length. The curvature of a somewhere can also be expressed as

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

and also

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

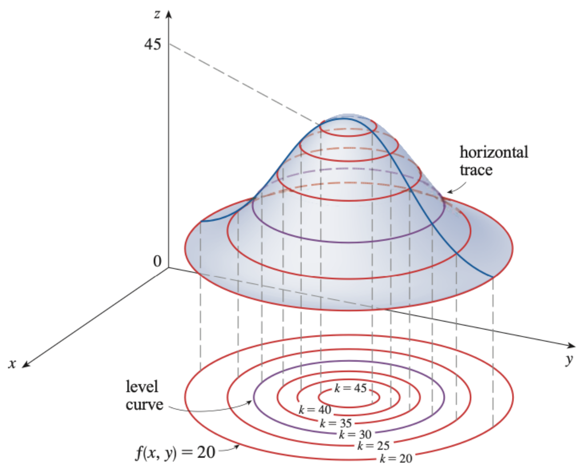
moreover

$$\kappa(x) = \frac{|f\,''(x)|}{[1 + (f'(x))^2]^{2/3}}$$

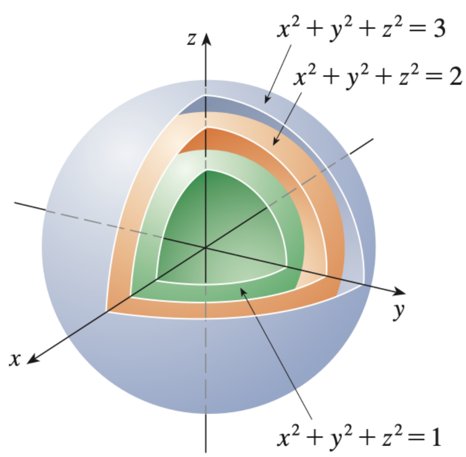
Partial Derivatives

14.1 Functions of several variables

If f is a function of two variables with domain D, then the graph of $f$is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D
A level curve of a function f are the curves with the equations $f(x, y) = k$, where k is a constant. It would graph all the x and y values that satisfies the function f when $z = k$ and project them onto the $x - y$ plane as is shown in the figure below.



Functions of three variables behave similarly as that of functions of two variables. Functions of three variables are still rules that assigns the triplets (x, y, z) that are elements of the set $\mathbb{R}^{\nvDash}$ to a unique real number denoted by $f(x, y, z)$. Its impossible to visualize the graphical representation of a function of three variables like that of functions of two variables. But it is possible to see that the **level surfaces** are the projections of the traces of the functions down to $\mathbb{R}^{\nvDash}$.
For example, suppose we have a function of three variables defined by the equation $f(x, y, z) = x^2 + y^2 + z^2$. If we let $f(x, y, z) = k$ we see that the level surfaces form concentric spheres with radius $\sqrt{k}$.



Similar concepts useful in functions of two variables can be extended to functions of n or more variables. A function of n variables would be a rule that assigns a number $z = f(x_1, x_2, ..., x_n)$ to an n tuple $(x_1, x_2, x_3, ..., x_n)$ that is an element of $\mathbb{R}^n$.

14.2 Limits and Continuity

In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to indicate that the values of $f(x,y)$ approach some number L as the point (x,y) approaches the point (a,b). To show that a limit does not exist at some point (a,b) for a function of two variables we can show that (x,y) approaches a different value for different paths. For example, if we are asked to show that the limit of

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

. We can show that we get different values when (x,y) approaches $(0,0)$ along the x axis and the y axis. If (x,y) approaches $(0,0)$ from the y axis we have $f(0,y) = \frac{x^2}{x^2} = 1$, on the other hand if (x,y) approaches $(0,0)$ from the x axis we have $f(0,y) = \frac{-y^2}{y^2} = -1$. So we have demonstrated that the limit does not exist by showing that the value of the function at a given point is different when approached from different directions.

When dealing with polynomial or rational functions of two variables we can compute the limit by direct substitution.

Continuity for functions of two variables

A function is continuous at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

in all the directions that x, y can approach a, b

14.3 Partial Derivatives

For a function f of two variables, if we keep one variable fixed, say y, and let the other vary, say x, then we are really considering a function of one variable $g(x) = f(x,b)$ and if the function g has a partial derivative at a, we call the derivative the partial derivative of f at a, b.

Partial derivatives of functions of two variables

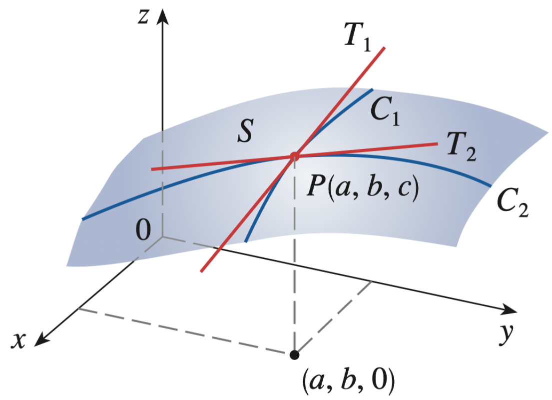
Formally, if f is a function of two variables, its partial derivatives are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to0} \frac{f(x,y+h) - f(x,yr)}{h}$$

$z = f(x,y)$ represents the surface S and the graph of $f(x,y)$, if $f(a,b) = c$, then the point $P(a,b,c)$ is a point on the surface. If we fix the y variable to be b and vary only the x variable, we would be finding the x values such that it is on the vertical plane $y = b$, and is intersecting the surface, a similar thing can be said when we are fixing x to a particular value. They form the traces of the surface in $y = b$ and in $x = a$ and they are represented by the figure below as C_1 and C_2 respectively. Notice that the curve C_1 is represented by the function $g(x) = f(x,b)$ and C_2 is represented by the function $G(y) = f(a,y)$. Consequently, we know that their slopes are given by the partial derivative with respect to x at (a,b) and the partial derivative of

y at (a, b).



The partial derivative of functions of three or more variables could be defined in a similar fashion as it is defined for functions of two variables. They revolve around the same central idea of measuring the change of a particular variable when keeping one variable constant.

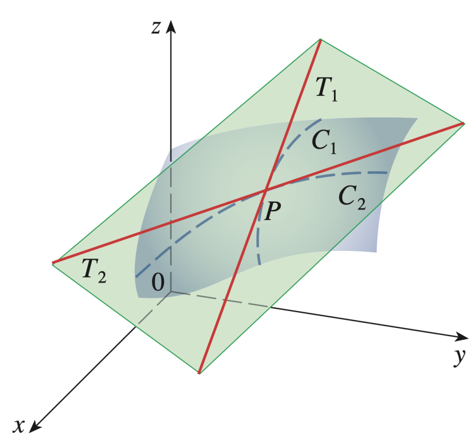
f_{xx} means the partial derivative of f with respect to x and then differentiated again with respect to x. In a similar logic, f_{xy} would denote the partial derivative with respect x and then to y.

Partial derivatives are incorporated in partial differential equations, which have important consequences in science and engineering. **Laplace's equation** after Pierre Laplace $(1749-1827)$, for example, play a role in problems such as heat conduction fluid flow, and electric potential.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

14.4 Tangent Planes and Linear Approximations

If we suppose that some surface S has the equation $z = f(x, y)$, and let $P(x_0, y_0, z_0)$ be a point on S. Then again we have two curves C_1 and C_2 which are the traces of the curve in $y = y_0$ and $x = x_0$. If T_1 and T_2 are tangent lines to the curve, then the **tangent plane** of the surface at P will be the plane that contains both T_1 and T_2.



From the previous sections we know that the equation of any plane is defined by the equation $A(x-x_0)+B(y-y_0)+C(z-z_)=0$, where A, B and C are x, y, z components of the normal vector respectively. If we divide both sides by C, we have $z-z_0=-\frac{A}{C}(x-x_0)-\frac{B}{C}(y-y_0)$, let $a=-\frac{A}{C}$ and $b=-\frac{B}{C}$, we have $z-z_0=-a(x-x_0)-b(y-y_0)$. The intersection of this plane with the plane $y=y_0$ will give us the equation of the tangent line T_1, so we must have $a=f_x(x_0,y_0)$, similarly we have $b=a=f_y(x_0,y_0)$

Equation of the tangent plane at some point

In fact an equation of the tangent plane to the surface $z=f(x,y)$ at the point $P(x_0,y_0,z_0)$ is given by the equation

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Linear approximation

The linear approximation of $f(x,y)$ near some point is simply the linear function that describes the tangent plane. Notice that $f(a,b)$ is simply z_0, expressed as the image of x and y.

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

The total differential or the differential dz is defined by the equation

$$dz=f_x(x,y)dx+f_y(x,y)dy$$

It gives us the change in the z value of the tangent plane instead of the actual point. However, when dx and dy are small, dz approximates the actual change in z value well.

14.5 The Chain Rule

The general case of the chain rule for partial

If the dependent variable u is a function of n intermediate variables $x_1, x_2, ..., x_n$, each of which is a function of m independent variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and we have

$$\frac{\partial u}{\partial t_i}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i}+...+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Consider the example where we have a dependent variable $w=f(x,y,z,t)$ and the intermediate variables $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$, and $t=t(u,v)$. Then,

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u}+\frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

The partial derivative of w with respect to s could be obtained in a similar fashion.

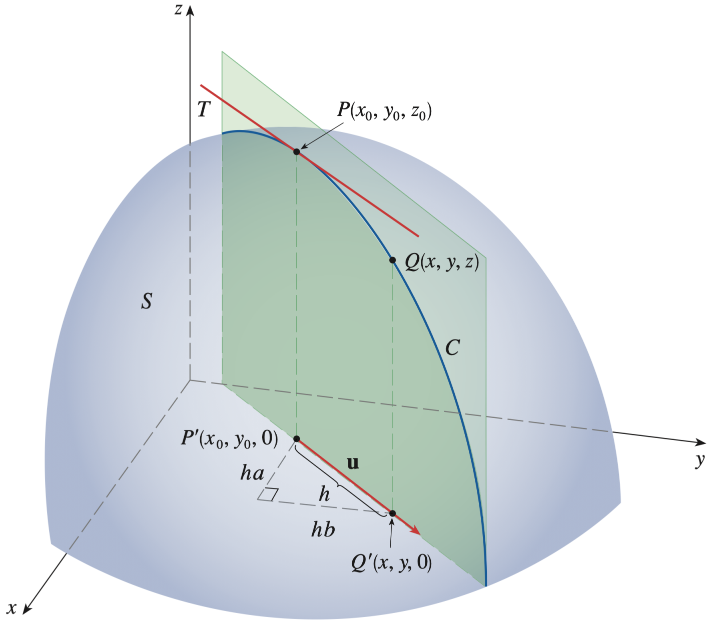
If we suppose that the function $F(x,y)=0$ defines y implicitly as a function of and if $y=f(x)$, then we can take the partial derivative of both sides and conduct algebraic manipulation to obtain a equation for the derivative of y with respect to x. In fact

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

14.6 Directional Derivatives and the Gradient Vector

The directional derivative can give us the rate of change of a function of two or more variables in any direction. Suppose that we want to find the rate of change of z at (x_0,y_0) in the direction of some vector

$\vec{u} = \langle a, b \rangle$. Consider the possibility that this point lies on the surface defined by $z = f(x, y)$ then we have $z_0 = f(x_0, y_0)$. So $P(x_0, y_0, z_0)$ lies on S. We would always be able to define a plane in the direction of $\vec{u}$ that contains the point P. Suppose there exist some other point $Q(x, y, z)$ on the curve of intersection between the surface and the plane. Then the projection onto the $x - y$ plane will be $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$.



Given any arbitrary two points on the plane, the line formed by their projection on the xy will always be perpendicular to the normal vector. Realize that the unit vector $\vec{u}$ is also perpendicular to the normal vector. Hence the line conjoined by the two points must be parallel to the unit vector $\vec{u}$. Therefore, we have $\overrightarrow{P'Q'} = \langle ha, hb \rangle$, that $\overrightarrow{P'Q}$ can be expressed as the scalar multiple of the unit vector. Finally, $x = x_0 + ha$ and $y = y_0 + ha$.

Now since, we can express x, y in terms of the increment of known positions in the direction of some vector $\vec{u} = \langle a, b \rangle$, we have

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

It is clear then that the directional vector of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Directional Derivatives

It is possible to prove the following equality for directional vectors

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$

where a, b are the components of the direction vector $\vec{u} = \langle a, b \rangle$

Notice that

$$\begin{aligned} D_u f(x,y) &= f_x(x,y)a + f_y(x,y)b \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \vec{u} \end{aligned}$$

The first vector of this dot product is known as the gradient of f.

Gradient of a function of two variables

If f is a function of two variables x and y, then the gradient f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$

and the directional derivative in the direction of the unit vector $\vec{u}$

$$D_f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

We can naturally extend the definition of the directional derivative to the three dimensions. The directional derivative of some function f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c\rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if the limit exists. By a similar approach to functions of two variables we can show that

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

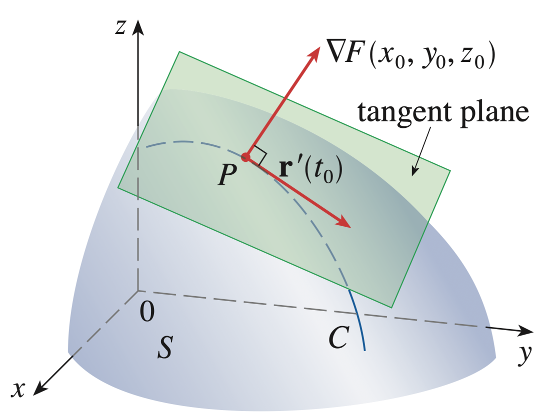
and that

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

Maximization of directional derivative

Since $D_u f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$. $D_u f$ is maximized when $\theta = 0$, in other words, when we are trying to find the directional vector in the direction of the gradient of f.

Let us suppose that we have a surface S with equation $F(x,y,z) = k$. It is the level surface of a function of three variables. Let some point $P(x_0, y_0, z_0)$ be on the surface and let C be some curve that is on the surface and is defined by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. Suppose that C passes P and there exists some parameter t_0 such that $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$.



Because any point on the curve must satisfy the equation $F(x, y, z) = k$, we have $F(x(t), y(t), z(t)) = k$. Assuming that F, f, x, y, z and t are all differentiable functions we have

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

Notice that $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$ So

$$\nabla F \cdot \vec{r}\,'(t) = 0$$
$$\nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$$

This means that the gradient vector at any point on the surface is always perpendicular to the direction of the tangent vector of the point on a curve that passes through the point.

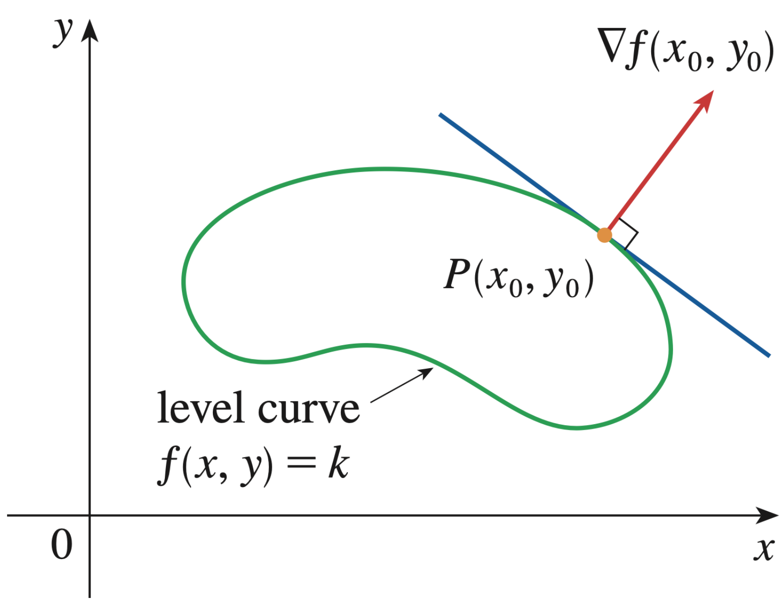
Tangent plane to level surface

In fact, we can defined the tangent plane to the level surface at some point $P(x_0, y_0, z_0)$ as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Astutely, we notice that this does not directly come from $\nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$. But the vector formed by some arbitrary point $P(x, y, z)$ on the tangent plane and a known point $P_0(x_0, y_0, z_0)$ will always be parallel to the tangent vector of some curve at (x_0, y_0, z_0). In other words, there will always be a curve passing through P_0 whose tangent vector at P_0 is parallel to the vector $\overrightarrow{P_0P}$ If the gradient is orthogonal to the tangent vector at a known point, then it must also be orthogonal to the vector which is parallel to the tangent vector. So it is safe to say that the gradient is indeed perpendicular to any line on the tangent plane and thus we can define the tangent plane as above.

If we consider a function of two variables and a point $P(x_0, y_0)$ in the domain. The gradient vector at P, $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P, in other words, its perpendicular to the tangent vector of the curve at P.



Properties of the Gradient Vector

Let f be a differentiable function of two or three variables and suppose that the gradient $\nabla f(x) \neq \vec{0}$ then we have the following three properties concerning the gradient vector at some point $P(x, y, z)$

- The directional derivative of f at P is given by $D_u f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$
- $\nabla f(x, y, z)$ points in the direction of the maximum rate of increase of x at P. In other words, the directional vector at P is maximized when we it is in the direction of the gradient
- $\nabla f(x, y, z)$ is perpendicular to the level curve or level surface of f through P

14.7 Maximum and minimum values

Local maximums and minimums

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). It has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b).

For single variable function we know that if a function f has a local minimum or maximum at c and $f'(c)$ exists, then $f'(c) = 0$. Similarly, we have the following theorem for functions of two variables.

Critical points

If f has a local maximum or minimum at (a, b) and the first order partial derivative exists there, we have $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

All local minima or maxima are critical points, but not all critical points are local maxima or minima. **If a critical point is neither a local maxima or minima, then it is a saddle point.** To determine whether a critical point is a local minimum or maximum or a saddle point. We have the second derivatives test.

Second derivatives test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2$$

It can also be defined as the determinant of a two by two matrix defined by the following

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
- If $D < 0$ then (a, b) is a saddle point of f

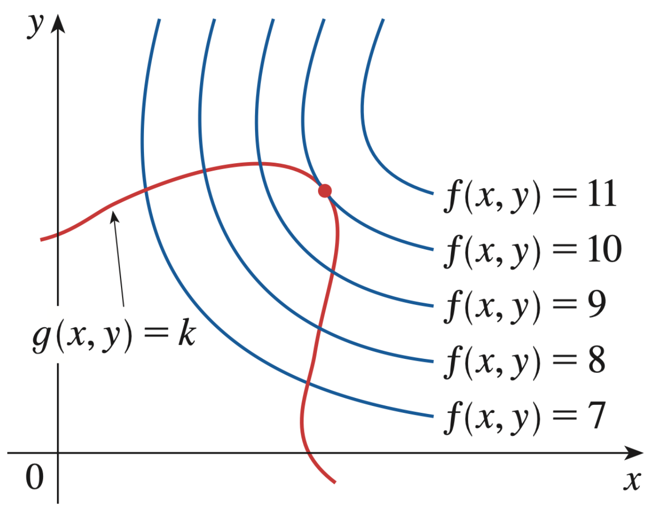
Absolute maximums and minimums

A function of two variables with domain D has a absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all $(x, y) \in D$ It has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all $(x, y) \in D$.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D, we need to find the critical points of f inside the bounded region, and then find the extreme values of f at the boundary, and finally we compare the values that we obtained to conclude the absolute maximum and minimum of the function.

14.8 Lagrange Multipliers

The method of the Lagrange multipliers is a way to maximize or minimize values.



Suppose we have a function $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. If we want to seek the extreme values of f we would want to seek the largest or smallest $f(x, y)$ while (x, y) satisfies $g(x, y) = k$. Notice that this will always occur when the level curve $g(x, y) = k$ touches the level curves of f. Consequently, we must have $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, because they would have gradient vectors that are parallel. This is extendable to functions of three variables which will follow a similar pattern of reasoning as that of functions of two variables. If the level surface of the constraint meet the extreme value of the function we are maximizing, it will always be such that their gradient vectors are parallel to each other.

Method of Lagrange Multipliers

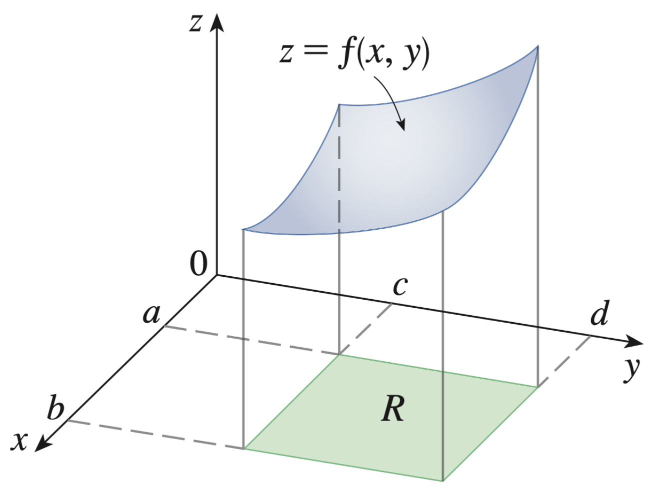
To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$

- We find all values of x, y, z and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x, y, z) = k$
- Evaluate f at all the points (x,y,z) that result from step1. The largest of these values is the maximum value of f and the smallest is the minimum value of f.

15 Multiple integrals

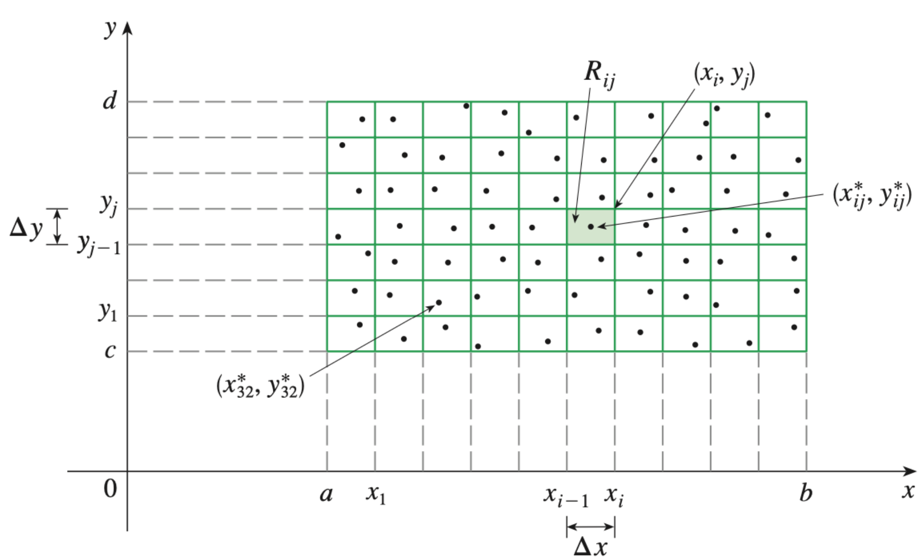
15.1 Double Integrals over Rectangles

Consider a function of two variables defined on a closed rectangle $R = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$



We shall follow a similar line of reasoning when finding the volume under a surface defined by two variables to that of finding the area under a curve. We shall divide the region R into sub rectangles each with area $\Delta A = \Delta x \Delta y$. If we choose some sample point in each sub-rectangle (x_{ij}^*, y_{ij}^*) then we can represent the volume under the surface and above each sub-rectangle as $f(x_{ij}^*, y_{ij}^*)\Delta A$ where $f(x, y)$ is the function defining the surface. The volume can thus be approximated by

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$



m and n would be the number of sub rectangles in the x direction and the y direction respectively. As the region is divided into mn sub-rectangles, we would expect the sub-rectangles to get smaller as m and n gets larger and $f(x_{ij}^*, y_{ij}^*)$ to reflect more accurately the height of the column above any sub-rectangles. Finally, we have

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

Double Integral Over A Region

The double integral of f over the rectangle R is

$$\iint\limits_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

if the limit exists and where (x_{ij}, y_{ij}) is any sample point inside the sub-squares.

We can evaluate these double integrals by expressing them as an iterated integral.

Double Integral As Interated Integral

The double integral of f over the rectangle R is

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

In the case that $f(x,y)$ can be written as the product of a function $g(x)$ only and $h(y)$ only we have

$$\iint\limits_R f(x,y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

where $R = [a,b]\times[c,d]$

Average value of a function of two variables

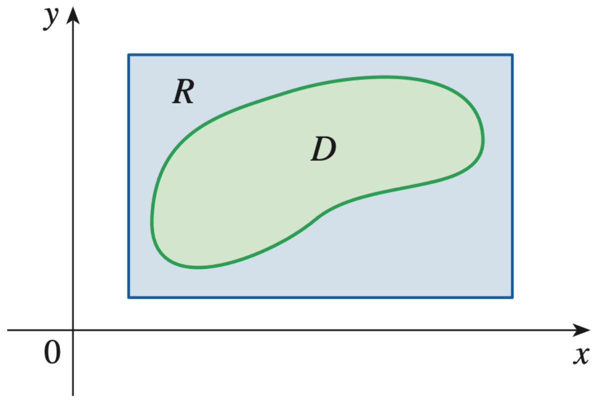
The average value of a function of two variables defined over a rectangle R can be defined as

$$f_{avg} = \frac{1}{A(r)} \iint\limits_R f(x,y)\,dA$$

15.2 Double Integrals over General Regions

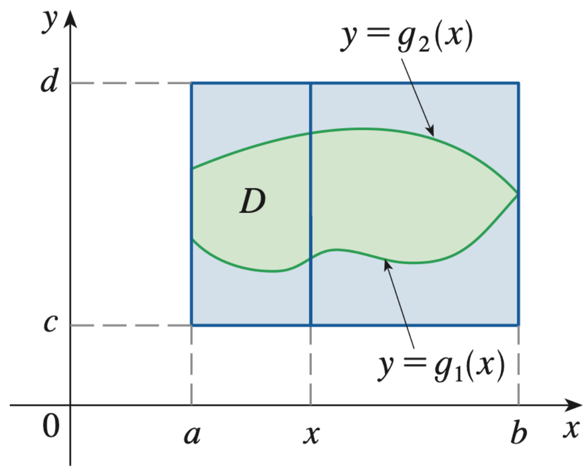
If we want to find the double integral of some function $f(x,y)$ over some general region D, we define a new function $F(x,y)$ which agrees with $f(x,y)$ in the region D but with domain R. In other words

$F(x, y) = f(x, y)$ for $(x, y) \in D$ and $F(x, y) = 0$ for $(x, y) \notin D$



Suppose now that the region D is a vertically simple region where the region lies between two functions of x. We choose a rectangle $R = [a, b] \times [c, d]$ that contains the region D and let $F(x, y)$ be defined as we defined it previously. By Fubini's theorem we have

$$\iint\limits_D f(x, y)\, dA = \iint\limits_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx = \int_a^b A(x)\, dx$$



Notice that beyond $g_2(x)$ and $g_1(x)$, $F(x, y) = 0$, so they contribute nothing to the area. So

$$A(x) = \int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

And since the region D would match the x bounds of the region R that contains it, we have

$$\iint\limits_D f(x, y)\, dA \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Double Integral over a vertically simple region

if f is continuous on a type I or vertically simple region D described by $D = \{(x,y)|\, a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

By similar argument we have, for a horizontally simple, or type II region we have

Double Integral over a horizontally simple region

if f is continuous on a type II or horiontally simple region D described by $D = \{(x,y)|\, h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

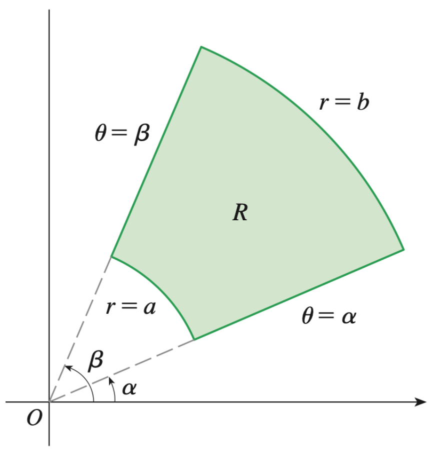
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

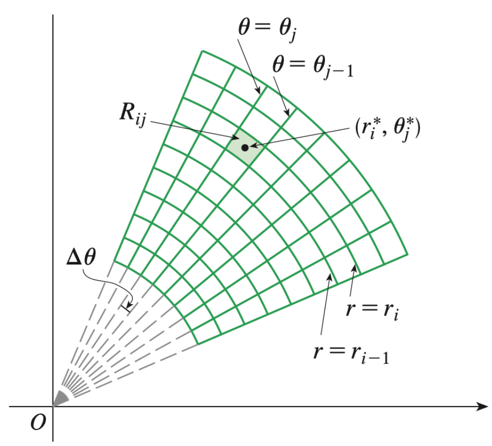
If we integrate the function $f(x,y) = 1$ over a region D, we obtain the area $A(D)$.

$$\iint_D f(x,y)\,dA = A(D)$$

15.3 Double Integrals in Polar Coordinates

In order to compute the double integral $\iint_R f(x,y)\,dA$ where R is a polar rectangle, we can divide the region into m sub-intervals from $r = a$ to $r = b$ with width Δr and we divide the region further by dividing the region from $\theta = \alpha$ to $\theta = \beta$ with width $\delta\theta$





Employing a similar approach to that of finding the double integral of a rectangular region, we find the area of each region and discover that it is given by

$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta \\ &= r_i^*\Delta r\Delta\theta\end{aligned}$$

So we have,

$$\begin{aligned}
\iint\limits_R f(x,y)\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A \\
&= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*,\theta_j^*)\Delta r\Delta\theta \\
&= \int_\alpha^\beta\int_a^b g(r,\theta)drd\theta \\
&= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,drd\theta
\end{aligned}$$

Double Integral over polar region

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \beta$ where $0 \le \beta - \alpha \le 2\pi$

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,drd\theta$$

Polar region where r is a function of θ

If f is continuous on a polar region of the form

$$D = \{(r,\theta)|\alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

$$\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\,drd\theta$$

In the particular case of $h_1(\theta) = 0$, the area of the region can be given by

$$\begin{aligned}
\iint\limits_D 1\,dA &= \int_\alpha^\beta\int_0^{h_2(\theta)} r\,drd\theta \\
&= \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{h_2(\theta)} drd\theta \\
&= \int_\alpha^\beta \frac{1}{2}\left[h(\theta)\right]^2\,d\theta
\end{aligned}$$

15.5 Surface Area

TODO

15.6 Triple Integrals

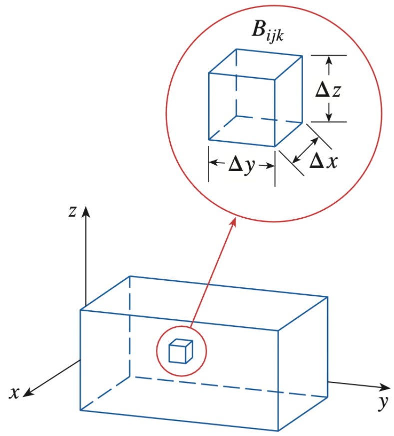
We can define triple integrals for functions of three variables just like we defined double integrals for functions of two variables. Suppose we have a function defined over the domain B, which is a rectangular box defined by

$$B = \{(x,y,z)|a \le x \le b, c \le y \le d, r \le z \le s\}$$

We then divide the box into sub boxes each of width Δx, Δy, and Δz. So each sub box has volume

$$\Delta V = \Delta x\Delta y\Delta z$$

While for functions of two variable, we can easily visualize that taking the double Riemann sum is equivalent to finding the volume of the volume under surface and above the region for which the function is defined, this is intuition is not readily available for functions of three variables. However, we can still think of the triple Riemann sum as analogous to that of double and single Riemann sums even though it is impossible for us to visualize it.



Triple integral over a box

The triple integral of f over the box B is

$$\iiint_B f(x, y, z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$$

if the limit exists and and where $(x_{ijk}, y_{ijk}, z_{ijk})$ is any sample point inside the sub-boxes.

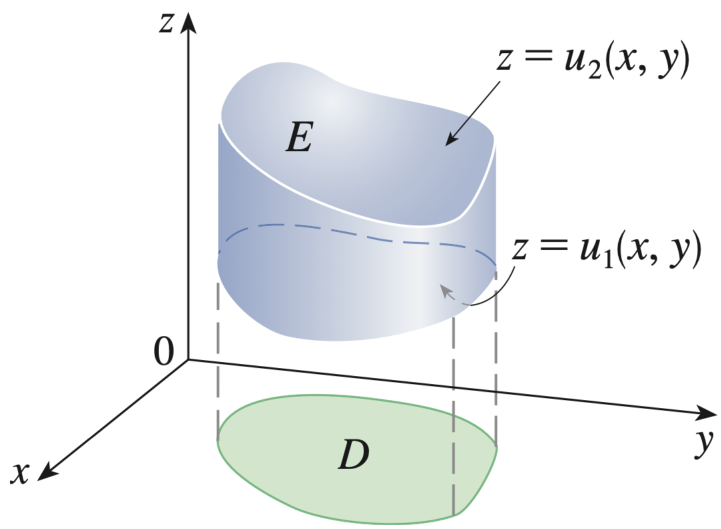
Fubini's Theorem for triple integrals

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then

$$\iiint_B f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y)\, dx\, dy\, dz$$

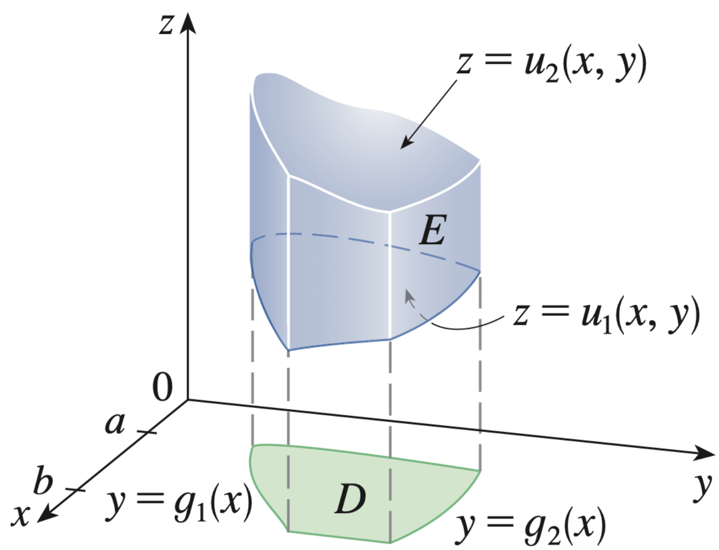
For a vertically simple solid region E where $u_1(x, y) \le z \le u_2(x, y)$ and $(x, y) \in D$ we have

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$



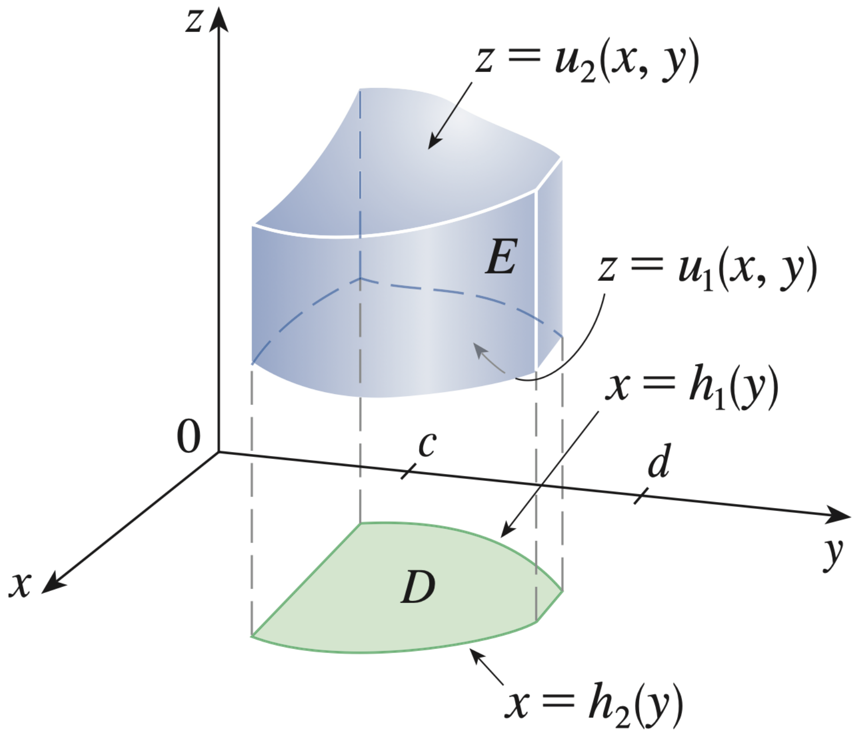
If the projection D of E onto the xy plane has $g_1(x) \leq y \leq g_2(x)$ then

$$\iiint_E f(x,y,z)dV = \int_r^s \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$



Naturally then, if the projection of E onto the xy plane has $h_1(x) \leq x \leq h_2(x)$ then

$$\iiint_E f(x,y,z)dV = \int_c^d \int_{h_1(x)}^{h_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dx\, dy$$



A type 2 region and a type 3 region are defined similarly as that of a type 3 region. A type 2 region is a region where the x values of the solid region is bounded by two functions which can be expressed as function of y and z, that is to say $u_1(y, z) \leq x \leq u_2(y, z)$. Similarly, for a type 3 region we have $u_1(y, z) \leq y \leq u_2(y, z)$. For each type of solid region, its projection onto the xy plane can either be vertically simple or horizontally simple. If $f(x, y, z) = 1$, then the volume of a solid region E can be given by the triple integral over the region E with respect to the volume of the region.

$$V(E) = \iiint\limits_E dV$$

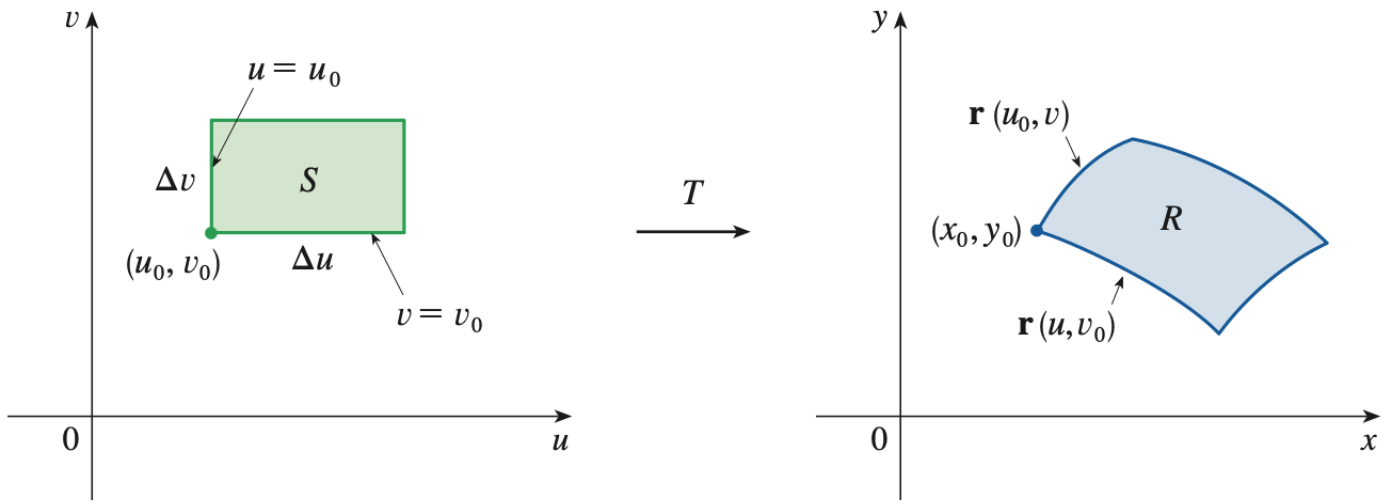
15.9 Change of variables in multiple integrals

In section 5, we considered u substitutions. By expressing a variable x in terms of some variable u, we were able to simplify integrands that were a function of x. More generally, we can consider a change of variables as a transformation from the uv plane to the xy plane:

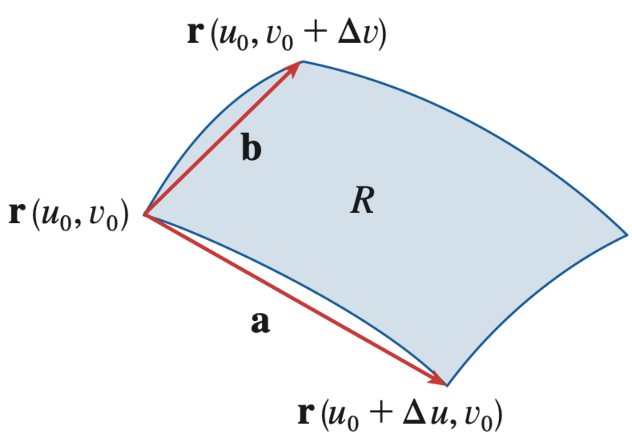
$$T(u, v) = (x, y)$$

where $x = x(u, v)$ and $y = y(u, v)$. If the transformation is one-to-one, there will be an inverse transformation. Consider a transformation T under which S is transformed into R. The vector

$$\boldsymbol{r}(u, v) = g(u, v)\boldsymbol{i} + h(u, v)\boldsymbol{j}$$



We can evaluate the area of a plane after transformation by a parallelogram



Notice that, however, that

$$\boldsymbol{r}_u = \lim_{\Delta u \to 0} \frac{\boldsymbol{r}(u_0 + \Delta u, v_0) - \boldsymbol{r}(u_0, v_0)}{\Delta u}$$

where $\boldsymbol{r}_u$ is the partial derivative of r with respect to u, hence we have

$$\boldsymbol{r}(u_0 + \Delta u, v_0) - \boldsymbol{r}(u_0, v_0) \approx \Delta u\, \boldsymbol{r}_u$$

By similar reasoning we have

$$\boldsymbol{r}(u_0, v_0 + \Delta v) - \boldsymbol{r}(u_0, v_0) \approx \Delta v\, \boldsymbol{r}_v$$

We know that the area of a parallelogram can be approximated by the cross product of two vectors that form adjacent sides of the parallelogram, so we have

$$|\Delta v\, \boldsymbol{r}_v \times \Delta u\, \boldsymbol{r}_u| = |\boldsymbol{r}_v \times \boldsymbol{r}_u|\, \Delta u\, \Delta v$$

. The cross product gives us,

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \boldsymbol{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \boldsymbol{k}$$

Jacobian of a transformation T

The **Jacobian** of a transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

16 Vector Calculus

16.1 Vector Fields

Vector field in space

Let E be a subset of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\vec{F}(x, y, z)$ where

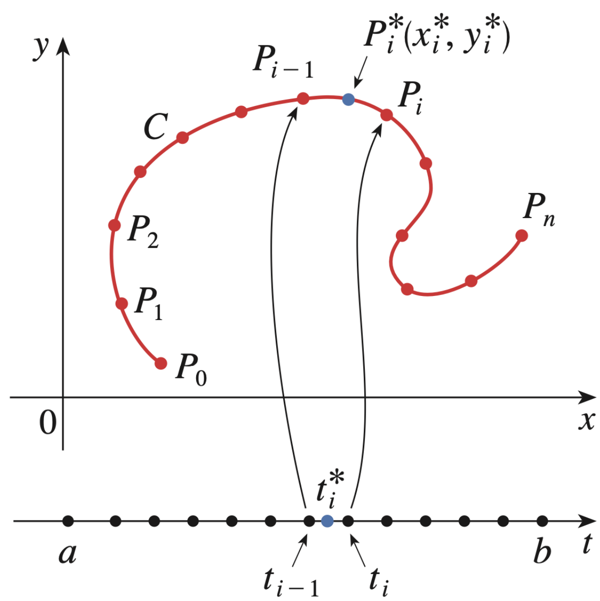
$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

The gradient ∇f of some function f is also, in fact, a vector field since it assigns to every point on the domain of f a vector.

A vector field $\vec{F}$ is called a conservative vector field if it is the gradient of some scalar function. In other words, if there exists a function f such that $\vec{F} = \nabla f$. f can be called the potential function of f.

16.2 Line Integrals

The line integral of some function f along some curve C can be defined along he similar line of thought of ordinary integrals. We can divide the curve into small sub arcs and by making the width of these sub arcs infinitely small we can find the value of a function summed over that region. Divide the parameter interval into n sub intervals $[t_{i-1}, t_i]$, the corresponding x and y values for all parameters t will divide the curve into n sub arcs or intervals. Taking the mid point of each interval and multiplying them by the length of the sub arc we obtain an approximate value for the function integrated over the region. The larger n gets the shorter the sub-arc gets and consequently, the more accurate our approximation of the actual value of the integral is.



Line integrals along C

If f is defined on a smooth curve C where $x = x(t)$, $y = y(t)$ and $a \le t \le b$, then the line integral of f along C is

$$\int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

and it can be shown that

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

If C is a piece-smooth curve, where C can be segmented into n smooth curves, then we have

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + ... + \int_{C_n} f(x,y)ds$$

We can also have line integrals with respect to x or with respect to y by replacinhg Δs_i with either $\Delta x_i = x_i - x_{i-1}$ or with $\Delta y_i = y_i - y_{i-1}$

Line integral with respect to x and y

The line integral with respect to x and y are given by

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

We can find parametrization of a line segment easily by the equation

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1}$$

where $\vec{r}(t)$ is the vector equation of the line and $\vec{r_0}$, $\vec{r_1}$ are the starting point and end point of the line segment respecitively.

In general, if $-C$ denotes the curve consisting of the same points as a curve C but with initial point B and end points A we have

$$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$$

This is also true for line integrals with respect to y. However, if we integrate with respect to the arc length we do not change the value of the integral.

$$\int_{-C} f(x,y)ds = \int_C f(x,y)ds$$

Similarly, we have if a smooth space curve is defined by some vector function $\vec{r}(t)$ we have

$$\int_C f(x,y,z) = \int_a^b f(x(t),y(t),z(t))ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|dt$$

We defined the work done $\boldsymbol{W}$ by the force field $\boldsymbol{F}$ as the limit of the Reimann sums

$$W = \int_C \boldsymbol{F}\cdot\boldsymbol{T}ds = \lim_{n\to\infty}\sum_{i=1}^{n} \boldsymbol{F}(x_i^*, y_i^*, z_i^*)\cdot\boldsymbol{T}(x_i^*, y_i^*, z_i^*)\Delta s_i$$

but

$$\boldsymbol{T} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

so

$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T} ds = \int_a^b \left[\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \right] |\boldsymbol{r}'(t)| dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt$$

This is often abbreviated As W =

$$\int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

We also have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C P dx + Q dy + R dz \text{ where } \boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$$

16.3 The Fundamental Theorem for Line Integrals

The Fundamental Tehorem for Line Integrals

Let C be a smooth curve given by the vector function $\vec{r}(t)$ for $a \leq t \leq b$. Let f be differentiable and a function of two or three variables whose gradient vector ∇f is continous on C. Then

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a))$$

This theorem tells us that if a vector field is a conservative vector field(it is the gradient vector field of some scalar function). Then its line integral is the net change of its potential function at the end points of the space curve. It also tells us that the line integral of a conservative function is independent of path. That is if C_1 and C_2 are two curves with the same end point then

$$\int_{C_1} \nabla f \cdot d\boldsymbol{r} = \int_{C_2} \nabla f \cdot d\boldsymbol{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Independence of path and closed paths

$\int_C \nabla f \cdot d\boldsymbol{r}$ is independent of path in D if and only if $\int_C \nabla f \cdot d\boldsymbol{r} = 0$ for every closed path C in D, where a closed path refers to a path that has the same end point as the start point.

Independence of path and conservative vector fields

It can be demonstrated that if $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of path in D then $\boldsymbol{F}$ is a conservative vector field on D for a open connected region.

Let F be some vector field defined by $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$ on an open connected region. If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then $\boldsymbol{F}$ is a conservative vector field. This is usually only the case for open connected regions. Generally, if a field is conservative then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ will hold by Clairaut's theorem.

16.4 Green's theorem

Green's theorem

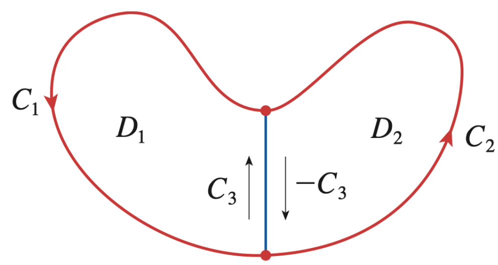
Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C Pdx + Qdy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$$

for some vector field $\boldsymbol{F} = \langle P, Q \rangle$

It's possible to employ Green's theorem in the calculation of areas, if we make the integrand $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. There are many selection of P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, here we shall use $P(x, y) = 0$ and $Q(x, y) = x$, so we have

$$A = \oint_C xdy = \iint_C \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_C 1dA$$



Green's theorem can be extended for regions which are finite unions of simple regions.

$$\oint_{C_1 \cup C_2} Pdx + Qdy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

16.5 Curl and Divergence

Curl

If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, and Q and R all exist, then the curl of F is the vector field defined by

$$\textbf{curl } \boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

The curl of a conservative vector field, a vector field which is the gradient vector of a scalar function is the $\vec{0}$ Therefore, if the curl of a vector field is not the 0 vector it is not conservative. If the curl is the 0 vector, it is conservative if it is defined everywhere in $\mathbb{R}^3$.

The curl of a vector field can be a measure of rotation if the vector field models the velocities of a fluid. If $\boldsymbol{F}(x, y, z) = \vec{0}$ for some point (x, y, z), then the fluid is irrotational at the point. The magnitude of the curl at that point is a measure of how fast the fluid moves around the axis.

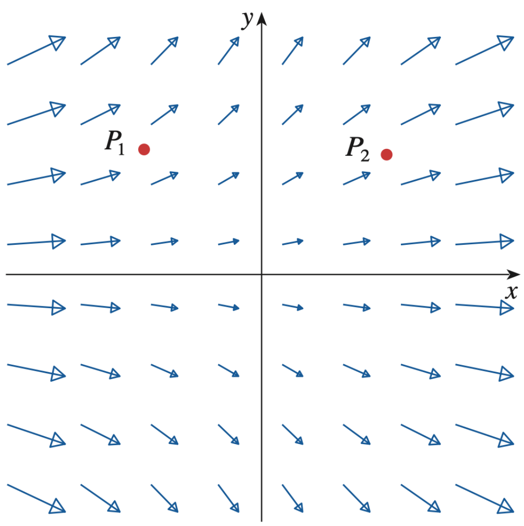
Divergence

If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, and Q and R all exist, then the divergence of F is the vector field defined by

$$\textbf{div F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \boldsymbol{F}$$

The divergence of curl $\boldsymbol{F}$ is div curl $\boldsymbol{F} = 0$. Notice also that the divergence of a vector field is a scalar field, whereas the curl of a vector field is a vector field.

If we think of a vector field F as modeling a fluid field, then div F models the tendency of the fluid to diverge from the point (x, y, z).



On this particular graph, for example, the divergence at point P_1 is negative because there is a net flow inwards as the arrows going out are smaller than those that are going in. The opposite is true for the other graph, where the divergence at P_2 is positive. With divergence and curl defined for vector fields, we can rewrite Green's theorem in their vector forms.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D (\text{curl } \boldsymbol{F}) \cdot \boldsymbol{k} \, dA = \iint_D \text{div } \boldsymbol{F}(x, y) \, dA$$

16.6 Parametric Surfaces and their areas

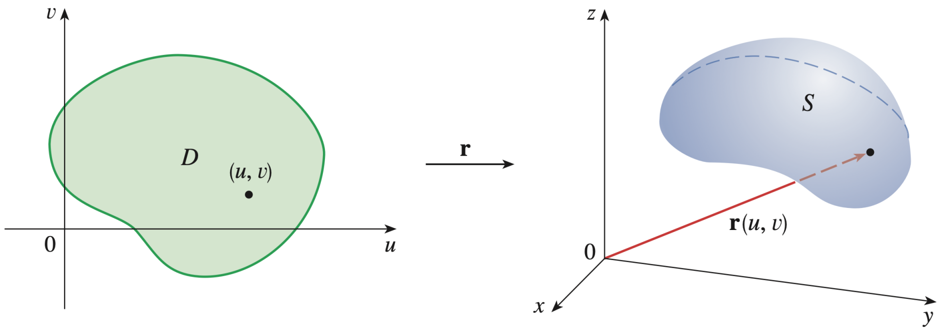
Similar to parametric curves in two dimensional space, we can parametrize a surface in three dimensional space using two variables. Particularly we can describe some surface S by a vector function

$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}$$

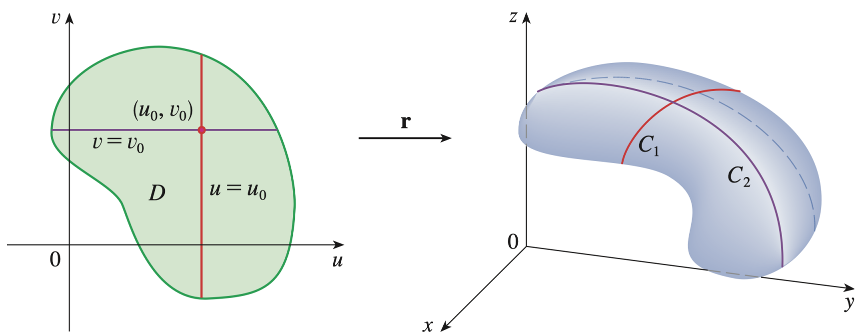
The set of all points (x, y, z) such that

$$x = x(u, v), y = y(u, v), z = z(u, v)$$

marks the surface S



If we fix on of the parameters of the function constant, what we obtain is a vector function of a single curve on the surface S



Following a similar line of approach to that of finding the value of a function of two variables summed over a region it is possible to find the surface area of a parametric surface.

Surface area of a parametric surface

If a smooth parametric surface S is given by the equation

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k}$$

ans S is covered just once as (u,v) ranges through the parameter domain then the surface area of S is

$$A(S) = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v|\, dA$$

For some surface with equation $z = f(x,y)$ with $x = x$ and $y = y$ as its paramters. It is trivial to show that its surface area is equivalent to

$$A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}$$

16.7 Surface Integrals

Surface integral

It can be shown that the surface integral of f also has an equivalent form of

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(\boldsymbol{r}(u,v))\left|\boldsymbol{r}_u \times \boldsymbol{r}_v\right| dA$$

Any surface S that can be defined by $x = x$, $y = y$ and $z = g(x,y)$ can be regarded as a parametric surface. It's possible to show that the surface integral of f over S is

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}dA$$

16.8 Stokes' Theorem

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\boldsymbol{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int\limits_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint\limits_S \text{curl } \boldsymbol{F} \cdot d\boldsymbol{S}$$