Maths 2 : Combinatorics
Basics

AGENDA

- Addition & Multiplication Rule
- Permutation basics
- Combination basics & properties
- Pascal Triangle
- Find N-th column title.

Example 1:- Given 10 girls & 7 boys. How many different pairs can be formed?

Note :- pair = 1 boy + 1 girl.

Ex:-

Girls	Boy
G_1	B_1
G_2	B_2
G_3	B_2
⋮	⋮
	B_7
G_{10}	

Total pairs = $7 * 10$

= **70 pairs** Ans

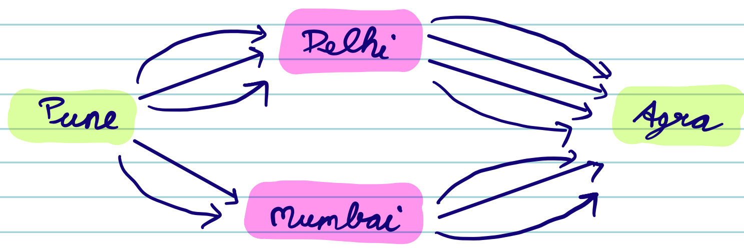
Example?

No. of ways to reach Agra from Pune via Delhi

APPROACH :- way to Reach Delhi from Pune * Way to reach Agra from Delhi

=> 3 * 2 => 6 ways

Quiz! No. of ways of reaching Agra from Pune?



APPROACH :- Way to Reach Agra from Pune via Delhi + Way to reach Agra from Pune via Mumbai

=> (3 * 4) (2 * 3)

=> 12 + 6 => 18 Ans

CONCLUSION

=> Multiplication (*) = AND: used when counting possibilities that occur together in sequence.

=> Addition (+) = OR: used when counting possiblities that occur in separate way

PERMUTATION

↳ Permutation is defined as the arrangement of objects.

↳ In Permutation, order Matters

i.e. → $(i, j) \neq (j, i)$

Eg :- i:> A B , B A

ii:> RGB, BGR, RBG, ...

Example 1 :- Given 3 distinct characters, in how many ways we can arrange them?

Ex :- S = " a b c "

Ans $\underline{3}\ \underline{2}\ \underline{1}$ ⇒ 6 [3x2x1] Arrangement

⇒ 3!

a { b → c, c → b }

b { a → c, c → a }

c { a → b, b → a }

⇒ S = " $\underbrace{\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}}$ "

⇒ 4 * 3 * 2 * 1 = 24 (4!)

Ans

Quiz 2 In how many ways n distinct characters can be arranged?

↳ $n * (n-1) * (n-2) * \ldots * 1$

⇒ $\boxed{n!\ \text{Ans}}$

Ques :- In how many way you can arrange 2 out of 4 characters

a b c d ⇒ $\underline{4}\ \underline{3}$ ⇒ $4 * 3$

(a,b,c,d) ↖ (except character selected for first pos)

⇒ 12

⇒ Generalizing -

Ques :- In how many way you can arrange 2 out of 3 characters

a b c → $\underline{3}\ \underline{2}$ ⇒ $3 * 2$ ⇒ 6 ← Arrangements

Ques :- In how many way you can arrange 2 out of 5 characters

a b c d e ⇒ $\underline{5}\ \underline{4}$ ⇒ $5 * 4$ ⇒ 20 ← Arrangement

Ques :- In how many way you can arrange 3 out of N characters

$(N, (N-1), (N-2) \ldots 1)$ ⇒ $\underline{N}\ \underline{(N-1)}\ \underline{(N-2)}$

⇒ $N * (N-1) * (N-2)$

Ques :- In how many way you can arrange x out of N characters

$$\underbrace{\underline{N}\ \underline{(N-1)}\ \underline{(N-2)}\ \underline{(N-3)} \ldots \ldots \underline{(N-(x-1))}}_{x\ \text{places}}$$

⇒ $N * (N-1) * (N-2) * (N-3) \ldots \ldots * (N-x+1)$

Multiply & Divide the above expression with

$$[(N-r) * (N-r-1) * (N-r-2) \ldots\ldots * 1]$$

$$\Rightarrow \frac{N * (N-1) * (N-2) * \ldots\ldots * (N-r+1) * (N-r) * (N-r-1) * \ldots\ldots * 1}{(N-r) * (N-r-1) * (N-r-2) * \ldots\ldots * 1}$$

$$\Rightarrow \boxed{\frac{N!}{(N-r)!} = {}^{N}P_{r}} \longrightarrow$$

Format for writing No. of ways to arrange r places from n distinct characters.

COMBINATIONS

↳ Combination is defined as the no. of ways to select something.

↳ In combination, order of selection doesn't matter

i.e. $(i, j) == (j, i)$

Eg :- ↳ RB = BR

↳ RGB = BGR

Example 1 :- Given 4 player, count the no. of ways of selecting 3 players.

Ex :- $\{P_1 \quad P_2 \quad P_3 \quad P_4\}$

Ways to select ⇒

$$\left.\begin{matrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_4 \\ P_1 & P_3 & P_4 \\ P_2 & P_3 & P_4 \end{matrix}\right\} \text{4 ways}$$

Example 2 :- Given 4 players, write the no. of ways to arrange players in 3 slots.

Ways to select ⇒

$$\begin{matrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_4 \\ P_1 & P_3 & P_4 \\ P_2 & P_3 & P_4 \end{matrix}$$

Ways to arrange the above 4 Selection

$P_1\ P_2\ P_3$	$P_1\ P_2\ P_4$	$P_1\ P_3\ P_4$	$P_2\ P_3\ P_4$
$P_1\ P_3\ P_2$	$P_1\ P_4\ P_2$	$P_1\ P_4\ P_3$	$P_2\ P_4\ P_3$
$P_2\ P_1\ P_3$	$P_2\ P_1\ P_4$	$P_3\ P_1\ P_4$	$P_3\ P_2\ P_4$
$P_2\ P_3\ P_1$	$P_2\ P_4\ P_1$	$P_3\ P_4\ P_1$	$P_3\ P_4\ P_2$
$P_3\ P_1\ P_2$	$P_4\ P_1\ P_2$	$P_4\ P_1\ P_3$	$P_4\ P_2\ P_3$
$P_3\ P_2\ P_1$	$P_4\ P_2\ P_1$	$P_4\ P_3\ P_1$	$P_4\ P_3\ P_2$

Observations

① First select 3 elements & then arranging will give us Permutation.

i.e. Permutation = Combinations + arrangement.

② For every selection = 6 arrangements

⇒ $\boxed{\text{Total No. of selections} * \text{No. of arrangement of each selection} = \text{Total No. of arrangement}}$

Example 3 :- Given N distinct elements, in how many ways we can select x elements such that

$$0 <= x <= n$$

Given n distinct element, arrange x element

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

if (x) $\rightarrow$ arrange x elements $= x!$

& No of selection for r distinct element $= 1$

$\therefore$ By using Unitary Math

selection of $x!$ arrangement $= 1$

" " 1 " $= \frac{1}{x!}$

" " $^{n}P_{x}$ " $= \frac{1}{x!} * {^{n}P_{r}}$

$= \frac{1}{x!} * \frac{n!}{(n-r)!}$

Selecting x out of N $= \frac{n!}{x!(n-r)!}$

$$\Rightarrow \boxed{^{n}C_{r} = \frac{n!}{r!(n-r)!}}$$

Imp Observation

arrangement

$$\boxed{{}^{n}C_{r} * r! = {}^{n}P_{r}}$$

Selection

=> Formula for selecting (Pair) 2 element out of N

Eg:- $\boxed{(a, b, c, d)}$

$$\hookrightarrow {}^{n}C_{2} = \frac{n!}{2!\,(n-2)!}$$

$$= \frac{n * (n-1) * \cancel{(n-2)!}}{2! * \cancel{(n-2)!}}$$

$$= \frac{n * (n-1)}{2}$$

$\boxed{\text{Break :- } 10:14 - 10:24}$

Properties of Combinations

① Property 1

↳ The No. of ways of selecting 0 items from N items, i.e. no. of ways to not select anything, will always be 1.

$$^nC_0 = \frac{N!}{(N-0)!\,0!} = 1$$

② Property 2

↳ The No. of ways of selecting N items from N items, i.e. no. of ways to select everything, will always be 1.

$$^nC_n = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = 1$$

③ Property 3

↳ No. of ways of selecting $(n-r)$ items from n:

$$^nC_{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!}$$

$$= \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,r!}$$

$$= {^nC_r}$$

So, $\boxed{^nC_{\underbrace{n-r}_{\text{Not Selecting } r}} = {^nC_r}}$

Eg :- Select 3 Boy out of 4.

Eg :- { B_1 B_2 B_3 B_4 }

Selection ⟹	Selecting 3	Not Selecting
	B_1 B_2 B_3	B_4
	⇒ B_1 B_2 B_4	B_3
	⇒ B_1 B_3 B_4	B_2
	B_2 B_3 B_4	B_1

★★★

④ **Property 4** [special property]

Given n distinct element, select r out of them

For each element, I have 2 option, either to select or skip (Not select)

_ _ _ _ _ _ _ _ _ _ _ n item

r places

Select → ${}^{n-1}C_{r-1}$

Not select → ${}^{n-1}C_{r}$

$$\boxed{{}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}}$$

Eg :- B_1 B_2 B_3 B_4 B_5

_ _ _ select or not select

$$\boxed{{}^5C_3 = {}^4C_2 + {}^4C_3}$$

Proof

$$\boxed{{}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$$

RHS

$$\frac{(n-1)!}{(r-1)!\,(n-1-(r-1))!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r+1)*(n-1-r)!} + \frac{(n-1)!}{r*(r-1)!\,(n-1-r)!}$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left[\frac{1}{n-\cancel{1}-r+\cancel{1}} + \frac{1}{r}\right]$$

$$\Rightarrow \frac{(n-1)!}{(r-1)!\,(n-1-r)!}\left[\frac{\cancel{r} + (n-\cancel{r})}{(n-r)*r}\right]$$

$$\Rightarrow \frac{n*(n-1)!}{[r*(r-1)!]*[(n-r)*(n-r-1)!]}$$

$$\Rightarrow \frac{n!}{r!\,(n-r)!} = {}^nC_r = L.H.S$$

$\therefore$ HP

Question : **PASCAL TRIANGLE**

Generate Pascal's triangle for given value of n.

Ex: $n = 4$

$^{0}C_0$

$^{1}C_0 \quad ^{1}C_1$

$^{2}C_0 \quad ^{2}C_1 \quad ^{2}C_2 \quad \Rightarrow$

$^{3}C_0 \quad ^{3}C_1 \quad ^{3}C_2 \quad ^{3}C_3$

$^{4}C_0 \quad ^{4}C_1 \quad ^{4}C_2 \quad ^{4}C_3 \quad ^{4}C_4$

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
```

Brute force

↳ 2 loops for going over rows & columns. Then one nested inner loop to calculate $^{n}C_r$

$$\boxed{\begin{array}{l} TC - O(N^3) \\ SC - O(1) \end{array}}$$

↳ OPTIMIZATION

IDEA (1) Can we use somehow use special property :-

i.e. $^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

Eg:- $^{3}C_2 = {}^{2}C_1 + {}^{2}C_2$

This is upper left — this is upper ele.

② Edge Case :- $^nC_0$ & $^nC_n = 1$

Eg:- $n = 4$

$^0C_0$

$^1C_0$ $^1C_1$

$^2C_0$ $^2C_1$ $^2C_2$

$^3C_0$ $^3C_1$ $^3C_2$ $^3C_3$

$^4C_0$ $^4C_1$ $^4C_2$ $^4C_3$ $^4C_4$

$\Rightarrow$

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

PSEUDO CODE

```
Pascal triangle ( n) {
        nCR [n+1] [n+1];
        for (i= 0; i<=n; i++) {
            nCR [i] [0] = 1 ;
            nCR [i] [i] = 1;
            for ( j =1 ; j <i ; j++) {
                nCR[i][j]= nCR[i-1][j-1]
                           +nCR[i-1][j];
            }
        }
}
```

$\boxed{\begin{array}{l} TC \rightarrow O(N^2) \\ SC \rightarrow O(N^2) \end{array}}$

Question :- EXCEL COLUMN TITLE

Find the n^{th} column title, the columns are titled as A, B, C....... & after Z, its AA, AB, AC........ & so on.

Given the column no., find the title of the column

Ex:

1	2	3		26	27	28	29		52	53	54
A	B	C	.....	Z	AA	AB	AC	....	AZ	BA	BB

$n = 3 \longrightarrow$ C

$n = 30 \longrightarrow$ AD

$n = 50 \longrightarrow$ AX

$n = 78 \longrightarrow$?? $\longrightarrow$ BZ (H.W.)

(1 — 26) $\longrightarrow$ A to Z

(27 — 52) $\longrightarrow$ AA to AZ

(53 — 78) $\longrightarrow$ BA to BZ

(79 — 104) $\longrightarrow$ CA to CZ

⋮

INTUTION / IDEA

Base 2 [0-1]	Base 8 [0-7]	Base 26 [1-26]
0	0	A → 1
1	1	B → 2
10	2	C → 3
11	3	⋮
100	4	Z → 26
	5	AA
	6	AB
	7	AC
	10	⋮
	11	AZ
	12	
	13	
	14	
	⋮	
	17	
	20	
	⋮	

So, this is simply Base 26 Conversion.

Decimal to Base 26

N = 1000

26	1000	12
26	38	12
26	1	1
	0	

(1, 12, 12) ⇒ $(ALL)_{26}$

Eg 2

N = 78

26	78	0
26	3	3
	0	

0 → Problem

3 → C

Way 1

↳ whenever you got remainder as 0 since it will be not possible so just take Ans as 1 less & keep remainder as 26

i.e.

26	78	26
26	2	2
	0	

26 → Z

2 → B

⇒ (BZ) Ans

Way 2

(A − Z) → (1 to 26)

or (A − Z) → (0 to 25)

Eg:-

N = 78 ← this is 1 based

N = 78 − 1 ← 0 based.

26	78-1=77	25	→ Z
26	2-1=1	1	→ B
	0		

=> (BZ)

eg:

26	1000-1= 999	11	→ L
26	38-1 = 37	11	→ L
26	1-1 =0	0	→ A
	0		

(ALL) Ans

PSEUDO CODE

```
void   columnTitle( int n) {
        ans = " ";
        while ( n > 0) {
            ans = (char)[(n-1)%26 +'A'] + ans;
                        // char + String
            n = (n-1)/26;
        }
        return ans;
```

$$\boxed{\begin{aligned} TC &= \log_{26}(N) \\ SC &= O(1) \end{aligned}}$$